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ORIGINAL ARTICLE

A networking method to compare theories: metacognition in problem solving reformulated within the Anthropological Theory of the Didactic

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Abstract An important role of theory in research is to provide new ways of conceptualizing practical questions, essentially by transforming them into scientific problems that can be more easily delimited, typified and approached. In mathematics education, theoretical developments around 'metacognition' initially appeared in the research domain of Problem Solving closely related to the practical question of how to learn (and teach) to solve non-routine problems. This paper presents a networking method to approach a notion as 'metacognition' within a different theoretical perspective, as the one provided by the Anthropological Theory of the Didactic. Instead of trying to directly 'translate' this notion from one perspective to another, the strategy used consists in going back to the practical question that is at the origin of 'metacognition' and show how the new perspective relates this initial question to a very different kind of phenomena. The analysis is supported by an empirical study focused on a teaching proposal in grade 10 concerning the problem of comparing mobile phone tariffs.

1 The role of problems in the development of scientific theories

An important function of scientific theories is to transform problematic questions arising in different systems of the natural or social world into scientific problems formulated within a frame of notions, properties, statements and methodologies. A good formulation of a problem within a theoretical framework is always a first step to approach it and, thus, to explain, predict, control or modify the functioning of the system where the question or difficulty appeared. In this situation, theories represent an alternative to common sense. Instead of trying to directly handle problematic questions appearing in human activities with the means available at a given historical moment, theories propose to approach them by making a detour, apparently non-natural: considering a model that can sometimes be very far from the original system where difficulties came up and using the model to formulate a problem related to the initial question.[1] Moreover, a theory provides conceptual and procedural instruments not only to create the model but also to work with it in order to solve the initially formulated problems and eventually to come back to the system with appropriate solutions. An initial unexpected consequence of this work within the model—that also becomes a measure of the power and fecundity of the theory—is its capacity to formulate new questions about the system, which is inconceivable without the theory. The problems formulated in terms of the 'didactic contract' in the Theory of Didactic Situations (Brousseau, 1997) are a good example of this.[2]

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[1] In mathematics, a well-known example of distance between a system and the model used to solve problems appearing in this system is Galois' theory (using groups and fields) and the problem of solving polynomial equations by radicals. In classical mechanics, the modelling of the solar system, where planets appear as points provided with mass, represents another paradigmatic example of a model that is very distant from the modelled system. It has, however, been extremely efficient to formulate scientific problems related to the solar system and to increase our knowledge about it.

[2] See also Silver & Herbst (2007, p. 62).

We can thus talk about a dialectic between the evolution of problems and theories, which is made visible when considering the role of problems to develop theories and the way theories define and build new problems, introducing alternative ways of questioning reality. However, this dialectic is sometimes disregarded, maybe due to the pregnancy of a 'static vision' of problems as something that is always there, pre-existing theories without being modified by them. In opposition to this 'static vision' of problems, Thomas S. Kuhn (1962) considers that a fundamental characteristic to define a change in a theoretical perspective (what he calls a scientific revolution) is the change of the norms used by a community to determine what is considered an *admissible problem* and what a *legitimate solution* to it is. Theories can thus be seen as machines producing new questionings and, dually, problems can appear as a way to characterise theories because they determine research objects of study. At a given historical period or throughout history, differences between theoretical perspectives are often accompanied by differences in the kind of problems approached by them.

It seems clear that any strategy to compare, contrast or network theories has to take into account the way theories question reality and formulate problems about it. It may particularly be difficult to talk about how different perspectives deal with the same questions because *each theory defines and formulates its own problems* without there always being a clear 'translation' between them. In this situation, considering the problematic questions that were at the origin of these different problems seems a necessary detour for the comparison or contrast of the considered perspectives. The aim of this paper is to illustrate this kind of *networking methodology* based on the exercise of going back to the problematic questions that are at the origin of the considered problems, compare what kind of formulations are proposed by each theory and look at the differences between the admissible answers that can be provided. Roughly speaking, the strategy consists in considering the problematic question as a 'common denominator' of the theoretical developments that are to be compared. In this sense, this paper joins other studies of this ZDM issue, such as Prediger's and Bergsten's papers, but it proposes a more detailed analysis considering only the approach currently used by the authors—the Anthropological Theory of the Didactics (ATD)—and the way it can reformulate an original problem approached from another perspective—Problem Solving.

More concretely, this paper starts with the problematic question underlying the use of the term 'metacognition' in research on Problem Solving and crystallising in what we have called 'Pólya's problem' (Sect. 2). The formulation in terms of metacognition will be contrasted with the way the Anthropological Theory of the Didactic proposes to approach 'Pólya's problem' (Sects. 3 and 4). An experimentation carried out in the ATD frame is presented as a possible way of approaching the original question formulated (Sect. 5). As a collateral result, Sect. 6 proposes a possible interpretation of the difficulties in teaching metacognitive strategies in terms of the institutional restrictions hindering mathematical activities in the classroom and, more specifically, of the culturally established divide between *mathematical* and *didactic tasks*.

2 Metacognition and problem solving

We start from a teaching difficulty that has long occupied a central place in mathematics educational research and is at the origin of the importation of the notion of 'metacognition' to our research community. We have called it "Pólya's problem" and have formulated it as follows, using the common sense notions of school culture (Bosch & Gascón, 2005, p. 109, our translation):

> Once students master 'basic' or elementary techniques and have acquired the 'necessary' mathematical knowledge, how can we get them to build *complex strategies* to solve 'real' or 'creative' mathematical problems?

This question was at its very peak in the decade of 1980s due to the influence of the movement of Problem Solving based on Pólya's work in the middle of the twentieth century.[3] Pólya himself formulated it in the following terms (Pólya, 1981, pp. xi–xii):

> [I]n high school, as on any other level, we should impart, along with a certain amount of information, a certain degree of *know-how* to the student. What is know-how in mathematics? It is the ability to solve problems—not merely routine problems but problems requiring some degree of independence, judgment, originality, creativity. Therefore, the first and foremost duty of the high school in teaching mathematics is to emphasize *methodical work in problem solving*. This is my conviction; you may not go along with it all the way, but I assume that you agree that problem solving deserves some emphasis—and this will do for the present.

This problem can be formulated in different general ways including, for instance, a reference to the *modelling* activity. Mogens Niss formulates it as follows (Niss, 1999, p. 21):

[3] Schoenfeld (2007) provides good overview of research results about Problem Solving in the USA during the period 1970 to present.

> There is no automatic transfer from a solid knowledge of mathematical theory to the ability to solve non-routine mathematical problems, or the ability to apply mathematics and perform mathematical modelling in complex, extra-mathematical contexts.

And he concludes optimistically:

> For this to happen both problem solving and modelling have to be made object of explicit teaching and learning, and there is ample evidence that it is possible to design teaching settings so as to foster and solidify these abilities.

The *Problem Solving* movement initially pretended to answer Pólya's problem by ways of teaching 'general' *heuristic rules* supposedly useful to any kind of problem. However, this proposal was only a way of transposing the problem, because a heuristic is not easier to use than a theorem and, in fact, much more difficult to transform into a practical task.[4] It was in this searching for a solution, and after the work of Schoenfeld (1985a), that the notion of 'metacognition' entered the domain of mathematics education and, more particularly, the research area of Problem Solving.

The common use of the term 'metacognition' is originally based on Flavell's research in the field of developmental psychology (Flavell, 1976) which divided it into two different aspects: 'metacognitive knowledge' or the individual awareness of cognitive processes and the 'regulation' of cognitive processes, also called 'metacognitive experiences', which includes: planning, selecting and monitoring cognitive strategies; evaluating or checking the outcomes of those activities; and revising plans and strategies. Numerous authors emphasized that deficiencies in metacognitive aspects are a fundamental cause of students' failures when solving problems in general, and more especially so when it concerns mathematical problems (see, e.g., Goos, 1995; Kilpatrick, 1985; Lester and Garofalo, 1982; Schoenfeld, 1985a, 1985b, 1985c, 1987, 1992; Silver, 1985). Others stated the importance of metacognition in the development of effective mathematical thinking and problem solving (see, e.g., Adibnia & Putt, 1998; Clarke, Stephens & Waywood, 1992; Silver & Marshall, 1990). Some authors considered that difficulties in problem solving are strongly related to the students' inability to monitor and actively regulate their own cognitive processes (see, e.g., Garofalo & Lester, 1985; Lester & Garofalo, 1982; Schoenfeld, 1987; Fan & Zhu, 2007; Kaune, 2006), while others connected them to the difficulty of using previously acquired knowledge in a correct way and/or at the appropriate moment (see, e.g., McAfee and Leong, 1994).

[4] This is an idea developed by Brousseau in 1986. See Brousseau (1997, pp. 37–40).

Despite the existence of an agreement on the theoretical concept of metacognition, many questions remain unanswered, especially the ones focusing on what the term metacognition means in practice and on what could be valid and reliable strategies for monitoring and promoting metacognition in school teaching and learning processes (Wilson & Clarke, 2004). Numerous authors also comment on the difficulty in distinguishing between cognition and metacognition, given that most definitions of metacognition include both knowledge and strategy components. Given those theoretical difficulties, and with the aim of digging deeper into the effectiveness of the promotion of metacognitive activity at school, some new definitions of metacognition have been proposed. For instance, the last quoted authors extend the early definition of metacognition by Flavell considering three main functions: the awareness individuals have of their own thinking; their evaluation of that thinking; and their regulation of that thinking. In their own terms (Wilson & Clarke, 2004, p. 27): 'When thinking metacognitively, learners reflect on their existing knowledge of thought processes. Individuals may be aware of, evaluate and/or regulate their own thinking'.

According to Lester (1994), during the 1980s the Problem Solving research area was based on the notions of 'metacognition', 'relation of affects/beliefs to problem solving' and 'metacognition training'. In the decade of the 1990s these notions were replaced by other terms more related to socio-cultural and context-based factors such as 'social influences' and 'problem solving in context' or 'situated problem solving'. This paper focuses on what could be called 'classical Problem Solving' without taking into account the recent developments that, over the 1990s, have enlarged the empirical domain where Problem Solving activity is considered by introducing socio-cultural and context-based factors, and where the notion of 'metacognition' has lost its initial central role.

In this context, Pólya's problem can thus be formulated in the following terms (following Schoenfeld, 2007, pp. 539–541):

> What general heuristics strategies could be decomposed into families of more specific strategies and with what instruction could students learn to employ these strategies? What is the influence and importance of metacognition, especially of monitoring and self-regulation? What is the role of belief systems in shaping people's problem solving behaviour? What is the role of people's experiences with mathematics, both in and outside the classroom, to shape people's beliefs and practices when they are engaged in problem solving?

Despite the 'practical progress' concerning this issue during the period 1980–1990, Schoenfeld (2007, p. 541) highlights a slowdown at the theoretical level:

> On the theoretical side, there was still a fundamental unresolved issue. The research to date said what was important to look at when people were engaged in problem solving and it provided explanations for success and failure, but it did not explain how and why people made the choices they did. That is, it offered a framework for characterizing problem solving, but it did not yet offer a theory of problem solving.

After more than 50 years of research in Problem Solving, and using the terminology previously presented on the relation between theories and problems, it can be said that:

(a) Pólya's problem is still considered as a *nuclear problem* by the research community of mathematics education;
(b) The inclusion of the notion of 'metacognition' does not seem to have solved the problem in a satisfactory way. Evidence of this insufficiency is the turn given by Problem Solving research from metacognition to social and context-based factors;
(c) There is a need to dramatically enlarge the empirical basis of research on Pólya's problem in order to integrate a socio-cultural dimension into cognition and to take into account *mathematics teaching practices*. Again, Schoenfeld (2007, p. 541) accurately describes this evolution in the following terms:

> The core idea […] is that a teacher's actions can be modeled as a function of his or her knowledge, goals, and belief and value systems. A number of different teachers, with very different styles, have been modeled. The theory appears to be robust, and it appears likely that it will apply, without significant modification, to many if not all goal-directed behaviors, specifically to the characterization of mathematical problem solving (see Schoenfeld, 2006b). This, in itself, will not yield a complete theory of problem solving, but it will represent a new plateau. The next set of questions to be confronted will be related to the need to integrate the socio-cultural and the cognitive. How does context play into the choice of goals, the establishment of values and beliefs? How is identity shaped over time as a function of experience, and membership in various communities of practice (see, e.g., Wenger, 1998)? How are knowledge, beliefs, and values shaped over time? How does identity shape goal formation? And, how does all of this square with emerging research from neuroscience regarding the development and organization of knowledge?

The enlargement of the old perspective entails the emergence of new kinds of problems where the notion of metacognition stops playing a central role. Not only the empirical domain where Pólya's problem has to be approached is questioned, but also other more or less explicit criteria on what an *admissible problem* is in mathematics education.

The contribution of this paper is to propose a reformulation of Pólya's problem from the perspective of the Anthropological Theory of the Didactic (ATD), the basic assumptions of which are substantially different from those underpinning classical Problem Solving research (see Gascón 2003). We will see that this new interpretation introduces the *epistemological* and *institutional* dimensions as essential characteristics of the problem and as *primary research objects*. As a consequence of this methodological option, the ATD relates Pólya's problem to a set of facts and didactic phenomena that have not always been associated with it and, sometimes, have not even been supposed worth considering.

We will see that this reformulation, which also represents an important enlargement of the problem, becomes possible because of the epistemological model of mathematics provided by the ATD. This model makes it possible to integrate the activity named *Problem Solving* into what is considered by research as "doing mathematics" in a given institution. Then, we will propose and experimentally contrast a didactic organisation, based on this epistemological model, to establish the conditions that make this activity possible for the students. The general character of this way of doing—that is, the contrast between different formulations of a practical teaching question and the results obtained by each perspective—is thus presented as a method for comparing theories in mathematics education.

3 Basic assumptions of the ATD framework

3.1 The epistemological model of mathematical activity proposed by the ATD

The Anthropological Theory of the Didactic (ATD) puts forward a particular model of mathematical activity aimed to prevent researchers from assuming without any questioning the implicit conceptions or epistemological views of mathematics prevailing in all teaching institutions. In particular, this model intends to overcome the cultural separation between 'learning contents' and 'solving problems' or, using a newer terminology, between 'mastering a curriculum' and 'using or applying' it to new situations.[5]

[5] This is the formulation used by the OECD in the PISA programme: 'The assessment is forward-looking, focusing on young people's ability to use their knowledge and skills to meet real-like challenges, rather than just examining the extent to which they have mastered a specific school curriculum.'

The anthropological approach postulates that mathematics, like any institutionalised human activity, can be described in terms of *praxeologies* (Chevallard, 1999; Chevallard, Bosch & Gascón, 1997). Mathematical activities, as they are carried out at school, in research groups or in any other kind of situation, consist in (or can be described as) activating praxeologies. *A praxeology is the inseparable union of two main components*: the *praxis* or 'know-how', which includes certain types of tasks as well as the techniques to carry them out; and the *logos* or 'knowledge', which refers to the elements necessary to describe, explain and justify the techniques, including the 'technology' or discourse—*logos*—about the technique—*technè*—as well as the 'theory' or the formal argument that justifies the 'technology'.

In praxeologies, *praxis* and *logos* are inseparable, even if we can consider praxeologies with undeveloped *logos* (we know how to do it but cannot explain it) and theoretical discourses with undeveloped *praxis* (we can describe it but do not know what it is for). Solving a problem, searching the answer to problematic questions, meeting real-life challenges consists in the construction of praxeologies bringing up new ways of doing (new *praxis* or 'know-how') and new ways of describing, explaining and justifying them (new *logos* or 'knowledge'). Both the process of doing mathematics (solving problems and constructing new knowledge) and the mathematical knowledge produced by this activity are inseparable and equally described in terms of praxeologies. Talking about a specific piece of knowledge consists in metonymically referring to a part of a praxeology (its theoretical block), in the same way as a specific practice refers to the practical block of a praxeology whose theoretical block remains hidden.

It is important to notice that praxeologies are rarely individual: even if they are carried out by individuals, they are collective constructions shared by groups of human beings organised in institutions. In this sense, the ATD proposes an *institutional conception of cognition*: knowing in the sense of carrying out (or performing) a praxeology can be done by a single person as well as by an institution. We can say that an institution 'knows', 'learns' (or 'forgets') as well as a person or a group of persons (do).[6] In this frame, the process of creating, acquiring or helping others acquire new knowledge can be formulated in terms of 'bringing praxeologies into play' (Chevallard, 2006).

Mathematical praxeologies can be classified into *point, local*, *regional* and *global* ones: a *point* praxeology is generated by a unique type of problems and is often characterized by a unique technique to deal with them; a *local* praxeology is generated by the integration of several point praxeologies within the same technology; a *regional* praxeology is obtained by coordinating, integrating or linking several local praxeologies in a common mathematical theory and a *global* praxeology is a connection of some regional praxeologies. In a simplified way, we can say that what is learned and taught today at school are point praxeologies more or less articulated into local and regional ones. The *point*, *local*, *regional* and *global* character of a praxeology is *relative* and depends on the institution where the mathematical activity takes place (see Chevallard, 1999). Thus, for instance, what in an institution (e.g. tertiary education) can be considered as a general technique—like using the graphical representation of a function to solve an algebraic inequality—, and thus associated to a *point* praxeology, may appear in another institution (e.g. secondary education) as a *local* praxeology because it is treated as a set of linked *point* praxeologies (linear, quadratic, exponential, etc. inequalities) or even as a set of disconnected *point* praxeologies if no relation is established between the techniques used in each case ('algebraic transformations', 'drawing a parabola', 'applying logarithms', etc.).[7] This structuring of praxeologies into different levels will play an important role in the following developments of this paper (Sects. 4 and 5).

3.2 Structuring the process of study: didactic moments and didactic praxeologies

Mathematical praxeologies do not emerge suddenly; they do not have a definite form. On the contrary, they are the result of a complex and ongoing activity that is at the very heart of mathematical activity. There appear two indivisible aspects of this activity: the process of construction (or reconstruction) of mathematical praxeologies—the *process of study*—and the result of this construction—the *mathematical praxeologies*. Doing mathematics consists in using praxeologies to construct or reconstruct new mathematical praxeologies in order to solve problematic questions that, as we will see later, are at the origin of all processes of study.

Chevallard (1999, p. 237) places this *process of study* in a specific space characterised by six didactic stages or *moments*:[8] (1) *the moment of the first encounter* with a specific type of tasks, (2) *the moment of the exploration* of the type of tasks, (3) *the moment of the construction of the technological-theoretical environment*, (4) *the moment of working* on the technique (which provokes the evolution of

[6] With this assumption, the ATD joins some other visions of human knowledge as the one proposed by the British anthropologist Mary Douglas in her book 'How institutions think' (Douglas, 1987).

[7] For more details, see Barbé et al. (2005).

[8] The idea of *didactic moment* is defined not in a chronological or linear sense, but in the sense of different dimensions (o factors) of the mathematical activity.

the existing techniques and the creation of new ones), (5) *the moment of institutionalization* and (6) *the moment of evaluation* of the praxeology constructed.

Once again, this process of study, like every human activity, can be modelled in terms of praxeologies, which are now called *didactical praxeologies* (Chevallard, 1999, p. 244). As every *praxeology*, didactical praxeologies include a set of problematic *didactic tasks*, *didactic techniques* (to tackle these tasks) and *didactic technologies and theories* (to justify, describe and explain these techniques and organise the tasks into types). A classical example of a didactic type of tasks is 'to set up a mathematical praxeology' under certain conditions in a given classroom at a given school. Depending on the mathematical praxeology considered, there exist different possible didactic techniques to carry out this important type of tasks and the ATD postulates that it is possible to mainly describe them in terms of the different didactic moments: how to manage the *first encounter* with the initial question of the process? What specific problems can help students carry out the *exploration moment*? How to introduce the *theoretical environment* in a functional way? What kind of *technical work* is necessary to foster the new mathematical techniques? How to organise and manage the mathematical work done in the classroom in order to progressively *institutionalize* the different components of the mathematical praxeologies in construction and their evolution? In the case of didactic praxeologies, it is clear that they are rarely personal creations of the teacher, but institutional and historical productions that teachers (and students) use to develop their activities. The *theoretical* block of didactic praxeologies (formed by didactic 'technologies' and 'theories') is then related to the way teaching and learning practices can be described, justified, organised and founded in the considered institution.[9]

It is important to notice a 'duality' between the structure of *local mathematical praxeologies* and the dimensions or *moments of the study process* in the following sense*:* a study process able to carry out the six didactic moments should normally end up in the construction of a *local* mathematical praxeology, generated by an initial question (*first encounter*) that leads to the construction of a technique (*exploration*). If the *technical work* is sufficiently developed, *technological and theoretical* questions will arise, leading to the making up of the theoretical block of the local praxeology, etc. This is the reason why *didactic processes related to local praxeologies* have been considered as the minimal unity of analysis for the ATD (Bosch & Gascón, 2005).

[9] For more details on didactic praxeologies and their structuring in didactic moments, see Barbé et al. (2005).

3.3 Functional genesis and development of praxeologies: the study and research courses

An important problem that mathematics education has to face today is the lack of sense of the kind of mathematical activities that are carried out at school. This lack of sense is correlated to what we can call today's 'monumentalism' in the teaching and learning of mathematics, that is, the study of praxeologies in themselves as the final goal of mathematical activity, in the same way as one can 'visit' a monument without knowing its current function or why it was once built. To avoid this situation, it is important to consider broader didactic processes going beyond the study (or reconstruction) of *local* mathematical praxeologies.

The recent developments of the ATD (Chevallard, 2004, 2006) put forward the consideration of *didactic processes* centred on the construction or reconstruction of *functional* mathematical praxeologies, the *raison d'être* of which is beyond the local level. As any research activity, and according to the dialectics between theories and problems mentioned at the beginning of this paper, these processes—called *Study and Research Courses (SRC)*—include a simple pattern based on the dynamics between *questions* and *answers* that Chevallard depicts as follows:

> In social life, a *question* is raised, in some institution, and persons in that institution try to do something in order to provide an *answer* to that question. The question is not intended to belong to any established field of study—it can be anything relating to any social practice. The answer that is being looked for has the structure of a praxeology, or of a fragment of a praxeology, or is a piece of a praxeological complex. Loosely speaking, therefore, the answer *is knowledge* in the broad sense I advocate. It should be clear that the question/answer pattern is the heart and soul of the social diffusion of knowledge—I mean, of praxeologies. (Chevallard, 2008)

The starting point of a Research and Study Course is thus a *question* $\boldsymbol{Q}$ that appears to a person or an institution as problematic, without any available answer and to which an answer or praxeology $\boldsymbol{A}$ has to be provided. When the person or institution decides to approach the question, a '*study community*' is formed, led by one or several 'study *supervisors*' (the teacher). The study community makes an effort to bring an answer $\boldsymbol{A}$ to $\boldsymbol{Q}$ based on its work with some pieces of available or previously established praxeologies. These praxeologies are always the answers that have been given before to other questions previously studied and without any necessary relation with $\boldsymbol{Q}$.

Considered in a school context, a SRC must be generated by the study of a question $\boldsymbol{Q}$ of real interest to the students (an 'alive' question) and it must be strong enough to generate many other sub-questions. The study of $\boldsymbol{Q}$ and the generated subsequent questions lead to the construction of a set of praxeologies that will outline a field of possible itineraries or 'courses' and their limits. An important feature of a SRC is that there is no previously given praxeology (or complex of praxeologies) towards which the study needs to be directed, that is, which the SRC proposes to build. The objective of a SRC is to study a problematic question, not to use the question as a means to build a previously determined body of knowledge. The answer to a problematic question will obviously be an answer in the form of a praxeology, but its characteristics, components, 'size', validity, etc., are to be detailed throughout the study process, without preceding it. Hence, we can talk about a bigger opening of the study process in contrast to the current ones. In spite of this openness, the initial question needs to be productive enough, that is, its resolution has to lead to the construction of a variety of kinds or tasks (or subquestions), and this productivity has to be checked (by the teacher or designer) before starting the study process. In any case, the intermediate questions or stages, which lead to a satisfactory answer, do not have to be determined beforehand with a lot of precision. It is also a concern of the study process to determine these intermediate questions, what a final satisfactory answer consists of and how to evaluate and diffuse it.

This openness of the SRC and, more specifically, the leading role attributed to the initial generative question, do not mean that a SRC is completely unrelated to any learning goal that the teacher (or the institution) might have. On the contrary, with the carrying out of a SRC learning goals are enlarged because students have to not only learn to provide an answer (construct a sequence of linked praxeologies) but also the way of building and validating such an answer. In order to preserve the genuine interaction between questions and answers, the answers have to maintain a certain degree of 'indetermination' and, more particularly, the course leading to the final construction cannot be univocally established. However, the SRC 'viability' has to be previously ensured through an *a priori analysis* that establishes different possible courses and issues the study of the question can lead to. More particularly, the a priori analysis has to provide the teacher with possible adequate interventions—by means of asking 'crucial questions'—in case the study process arrives at a dead end.

In terms of the didactic moments and the students' and teacher's didactic praxeologies, it is important to notice that in a SRC, when considering the construction of a single praxeology within the whole process, different sequences of moments are possible and the sharing of responsibilities within the study community can take different forms. Sometimes a crucial subquestion or a given 'subanswer' can be frontally introduced by the teacher, at other times it is up to the students to raise intermediate questions or to validate the answer constructed. An important set of research problems that are being considered today in the ATD refer to the institutional conditions under which SRC are 'viable' at current schools and to the means (for the teacher as well as for the students) that are necessary for their global design, management and evaluation.

We are now using the main notions introduced in this section to reformulate the problem approached by 'metacognition'—Pólya's problem—in terms of the ATD.

4 Formulating "Pólya's problem" within the ATD

In relation to the formulation of Pólya's problem in terms of 'metacognition', the reformulation within the ATD gives priority to the *institutional dimension* of the problem. In other words, we consider that current difficulties to teach how to solve non-routine problems do not have to be primarily found in the students' or teachers' characteristics but in the kind of mathematical activity that is possible to carry out in our teaching institutions. Without denying the evident personal differences between students' (and teachers') abilities in doing mathematics, the ATD postulates that what has to be studied primarily are the institutional conditions and restrictions that can explain why some kind of mathematical activities—praxeologies—can easily be set up in the classrooms and why others are so difficult to introduce.[10]

4.1 Epistemological dimension

The ATD considers that the activity of solving mathematical problems (especially non-routine ones) can only make sense in the study of questions leading to the construction of large enough mathematical praxeologies, local ones at least. In other words, the ATD postulates a permanent dialectic between the study of problematic questions and the progressive (re)construction of increasingly larger and complete mathematical praxeologies.

4.2 Didactic dimension

The model of didactic moments assumed by the ATD as a description of any teaching and learning process provides a duality between the functional integration of the dimensions of the study process and the construction of local praxeologies. Carrying out a process of study where all the

[10] An example of such institutional restrictions in the case of the teaching of limits of functions can be found in Barbé et al. (2005).

different moments take place is supposed to lead to the construction of local praxeologies. In the other sense, to construct local praxeologies it is necessary to carry out a process of study that, starting from a given and 'alive' problematic question, needs to pass through the six didactic moments considered.

Given this preliminary considerations, the formulation of Pólya's problem can now be stated in the following terms:

> In current teaching institutions, what conditions are required and what restrictions hinder the carrying out of didactic processes that, starting from 'alive' problematic questions, let students pass through the different didactic moments and, thus, generate (construct or re-construct) a sequence of local mathematical praxeologies as an answer to these questions?
> As a consequence, and given a concrete teaching institution, what characteristics of a didactic praxeology are necessary in order to carry out these kinds of didactic processes, creating the appropriate conditions and avoiding restrictions as far as possible?

From the ATD perspective, Pólya's problem has to be located beyond the point level and related to *local* mathematical praxeologies that are proposed to be learned, without neglecting the upper levels of regional or global praxeologies, the *raison d'être* of which curricular topics usually depends on.[11]

Given the new formulation of Pólya's problem, it is clear that the kind of 'admissible answer' in the ATD frame consists in proposing a *didactic praxeology* able to implement the necessary conditions to carry out a study process fulfilling the abovementioned characteristics. It is also clear that *Study and Research Courses* (SRC) appear as an appropriate tool for it. The research problem that emerges is *the study of the viability of SRC in current teaching institutions and the effective results obtained when they are experimentally introduced in the classroom*. The next section presents such an experimentation, the analysis of which brings new insight in both aspects of the problem. Finally, the results obtained will be related to those obtained from the study of Pólya's problem in terms of 'metacognition'.

[11] From this point of view, the new formulation of Pólya's problem takes into account the *minimal unity of analysis of didactic processes* postulated by the ATD: "To describe and interpret didactic phenomena, they have to be related to a sequence of the didactic process including, at least, the process of constructing a local mathematical praxeology" (Bosch & Gascón 2005).

5 Study and research courses: an experimentation

Two experimental Study and Research Courses (SRC) were carried out during the second terms of 2004 and 2005 with grade 11 students (16–17 years old) of two Spanish secondary schools. In both cases, all students were invited to participate in what was presented as a 'mathematical workshop' organised in 2-h sessions after class. A group of 10 to 14 volunteer students participated in each SRC. The total course lasted 18 sessions in both experimentations and the students did a lot of work outside the sessions. One of the researchers was the teacher of both workshops; she took notes of all the sessions, which were also video-recorded. When the students worked in small groups, each group was recorded in audio. In this way, the notes could later on be completed with the observation and transcript of both video and audio. Moreover, each group was asked to keep a portfolio including the progressive productions of the work done.

Both SRC started with the same initial question about the comparison of mobile phone tariffs. The main objective of the second SRC experimentation was to reproduce the experience at a different school and a different group of students. It also served to introduce more teaching strategies to let students assume more responsibilities in the management of the whole study process, especially in the planning of tasks, the organisation of the work in teams and the reformulations of the initial question. The purpose was for the teacher to avoid taking initiatives concerning these aspects of the study process, or at least making decisions explicit and 'negotiating' them with the students. We are now presenting the common main features of both SRC, only specifying their differences when necessary.

5.1 Studying a generative question: the open character of the RSC

The initial generating question—which has to be productive, 'alive' and relevant for the students—was specified in terms of 'investigate which mobile telephone company and tariff is best for each person'. It is obvious that the answer to this question might have consequences in the phone users' lives and was therefore not a mere opportunity for certain predetermined mathematical knowledge to appear. Given the fact that all the students were real phone-users, they were clearly interested in knowing whether the company tariff they were using was or not the most appropriate and, in case it was not, might change to a better company. The generating power of this question had previously been analysed by the research team during a course of mathematical modelling for first year university students of economics and business administration. It showed in what sense the comparison of more than two different tariffs needs to activate different local mathematical praxeologies learned by the students, like for instance the graphical representation of functions and the resolution of algebraic inequalities, which at this period appeared as completely different topics in Spanish secondary school syllabi.[12]

An important restriction for the viability of a SRC is the means needed by the students to start the study and deal with the initial and intermediate questions. In our case, the students had the necessary mathematical elements, as telephone tariffs are obtained from mathematical models based on straight lines or linear functions defined piecewise which the students had studied previously. Subsequently, the situation did allow the students to obtain 'good means', that is, elements for the evaluation of the solutions or intermediate answers proposed during the development of the study process. For example, the students could 'simulate' several possible situations (user's characteristics and tariffs) both with pen and paper (with the help of a calculator) and using Excel. As we will see later, other unexpected 'material means' appeared, like the students' own phone bills, which gave rise to start a statistical study of cases not initially foreseen by the research team.

In the carrying out of both experimentations, two main stages may be considered: a first one based on the *comparisons of fictitious tariffs*, guided by the teacher, and a second one related to *real facts*, in which students assumed greater responsibility in finding the answer to the questions. During the first sessions, the teacher proposed the students to examine some easy cases in order to introduce the main tools for the comparison of tariffs: algebraic functional expressions, graphs and graphical resolution of inequalities. Different tariffs were considered, depending on the call cost rate per period of time (seconds or fractions of half minutes) and the cost of the connection charge. In the second stage, the students were asked to find the real tariffs by themselves and to start with the real comparisons. The students, helped by the teacher, soon became aware of the complexity of the situation because of the different tariffs and options to be considered, and the many variables intervening in the study: time of the call, duration, company of the receiver, payment options (card/contract), minimum call charge, etc. In the first SRC, and in a completely unexpected way, the students proposed to elaborate *different types of answers* depending on the amount of users' details required: a 'normal' version, a version for 'lazy people' and a version for 'extremely lazy people'. For instance, in the last version, the only required information was the period of the day and the duration of the calls, while the 'normal version' asked more detailed information about, for instance, the receivers' companies, the average number of calls in the morning, in the afternoon and during the weekend, etc.

The students then agreed to produce a *final answer* consisting in preparing three different Excel sheets (one for each version) with the user's personal data as input and the best tariff suggested as output. Once the 'form' of the final answer was decided upon, the students also suggested posting it on a web page so that a great amount of mobile phone users could take advantage of it. This is a sign, amongst others, of the students' deep appropriation of the generative question and their involvement in the study process. It also supposed a great effort for the students to summarise, organise and evaluate the whole work done throughout the study process and kept in the portfolios in order to present it in an easy way on the web. In the second SRC, only 'one final answer' was decided upon by the students to post on the web. However, impressed by the results obtained, they also decided to write a letter addressed to the Spanish ministry of industry and commerce to complain about the 'tariffs complexity' of the phone companies that disorientate consumers and avoid easy tariff comparisons.

5.2 Management of the study process and integration of the didactic moments

With regard to the running of the *didactic moments* through which the ATD structures the study process, we will only mention that finding the answer to the generating question made each moment appear in a relatively natural way. We noticed that, on some occasions, it was precisely the intervention of the teacher which limited this process, given her urge to make the study "progress". This may be clearly observed, for instance, when the students were carrying out the comparisons of all the companies' tariffs to elaborate the final report. Once the teacher considered they already knew how to use the comparison technique (using both graphical and algebraic models), she suggested that they could leave it and start another task. The students, surprised, replied: "How can we make the comparison without considering all the cases?!" Obviously, carrying out a comparative study means to compare everything with everything. The teacher did not realise that the *moment of the work of the technique* was emerging naturally and she was about to abort it! Concerning the *moment of institutionalisation*, which is currently carried out under the sole responsibility of the teacher, it here took a surprising form when the students proposed to design a website as a way to give a definite answer to the initial question. Determining which materials should be posted on the website and how to present them constituted an important device for the institutionalisation performing and was carried out in a complete cooperative way between the teacher and the

[12] More details about this experimentation are given in Rodríguez (2005).

students. Of course, the ‘public character’ of the agreed solution made the *moment of the evaluation* emerge in quite a natural way: the models and results obtained had to be carefully checked before posting them on the web.

In what concerns the assumption of responsibilities, it can be noticed that the work with the Excel files, checking the results using different techniques or between different groups of students, were examples of evaluation ‘gestures’ that appeared naturally during the SRC without any particular intervention on the teacher’s behalf. In some cases, however, the consideration of a specific device had to be provided by the teacher, for instance, when the students did not know how to deal with the tariffs ‘per fractions’ with calls having a duration of exactly 30, 60 or 90 s. When calculating the average price, how many calls of exactly 30, 60 or 90 s should be considered? The teacher suggested taking the students’ own phone bills as a *milieu* to contrast their hypotheses and see what proportion of each type of calls were considered more likely. This suggestion opened a new and unexpected way towards a *statistical study* of the length of the calls and their different proportions.

5.3 The teacher’s interventions and the institutional restrictions on the course of the study

The fact that the experimented SRC introduced so many new conditions to carry out the study compared to the ‘normal’ mathematical classes—after-class sessions, 2 h sessions, students working in groups, low time-pressure, etc.—did not avoid the presence of many current institutional restrictions that, in a more invisible way, are always hindering the management of the process of solving open problems. A restriction experienced in the first SRC was what we might call the ‘*temporary economy*’ of the teacher, who, urged by the need to advance in the study process, helped the students along with the answers or gave ‘hints’ instead of waiting and providing the correct means for the students to come up with the answers. There were also inverse situations in which the teacher knew how to put forward ‘crucial questions’ which meant a ‘U-turn’ in what had been covered until then, usually when getting to a deadlock situation—for instance, the abovementioned statistical study. In this case, although it could have been left up to the students, the help of the teacher may be considered appropriate.

In the first SRC the teacher also took a lot of decisions concerning many other aspects of the study management (planning the work to do, monitoring and evaluating the work done, etc), which are not normally of the students’ concern in the traditional didactic processes. It was not until the end of the first experimentation that we realized that this kind of responsibilities could also be assumed by the students. The only planning aspects up to the students had been very limited and they only referred to the prevision of usefulness of the techniques or to anticipating results. The students had not been responsible for the temporal and theme-related organization of the course nor for finding the answer to questions such as: where to start? what to deal with first?, how much time to spend on each case?, etc. This would be one of the main objectives of the second application of the SRC.

In the second SRC the teacher led the process in a more discrete way, trying to agree on all the decisions concerning the study process with the students. They had to decide on how to start with the study of the question, which variables were important to take into account, why and how to consider them, how to divide the study of the initial question into sub-questions, whether a considered sub-question had been answered sufficiently or whether it needed more time, etc. We then realised that the teacher did not have any special didactic resources to get the students to do the planning and scheduling of the work by themselves. She decided to draw up a sort of ‘planning sheet’ where, at each session, the students had to write the questions to be studied and those that should be considered in the next one. The sheet was also used for the regulation, checking every day if the SRC was going on as planned and for considering the deviations from the initially foreseen plan.

The teacher also decided to adopt a secondary role at the starting point of the study, when some artificial cases were considered to introduce the mathematical tools needed for the comparison (algebraic formulae of the tariffs, corresponding graphs when fixing some parameters, etc.). During the first session it was not difficult to get the students to consider some simple examples they gave themselves and formulate some ‘ideal tariffs’ to start the comparison. The most interesting episode was that, during the first stage of the SRC (comparison of fictitious tariffs), once they had already looked for real information on phone tariffs, the students brought more complex examples—always fictitious—to cover the different situations. For instance, if at the first session they had only proposed to consider a case without connection charge cost and two different cases with such costs, at the second session they enriched the examples considering tariffs charging per 30-s fractions as they had seen in the real tariffs. They did not take the real tariffs as examples but used the information to make up new simpler fictitious examples. So it was the dynamics of the proper SRC that helped provide new ‘material’ for the regulation.

As a final comment on this quick summary of the two SRC experimentations, let’s only mention that, in order to evaluate the evolution of their individual knowledge at the end of the process, the students were asked to answer an individual written test on the comparison of *fixed* phone tariffs with some novelties such as a 'bonus' (pack of calls at a reduced price) and the payment per seconds during the first minute. The students' performance was really good, showing they were able to approach a question similar to the one previously studied, explain the process followed and use the comparison techniques constructed during the SRC in a flexible way.

6 Interpreting 'metacognition' in the ATD

When comparing the formulation of Pólya's problem between the classical Problem Solving tradition and the ATD, we pointed out a change of priorities between an essentially *personal (cognitive)* approach focused on the study of individual difficulties in the resolution of non-routine mathematical problems, and a primarily *institutional* perspective that highlights both the *epistemological* and *didactic* dimension of the problem, focusing on the institutional restrictions hindering the kind of mathematical activity that is possible to implement at school. A possible way to connect both approaches is to consider personal difficulties (in learning or teaching problem solving), and more particularly those related to metacognitive strategies, as the 'projection' of institutional restrictions upon the subjects of the institution considered. As we will see and illustrate with the results obtained during the experimentation of the SRC (section 5), this new interpretation may also bring some insights to open a way to progress in the effective resolution of Pólya's problem.

6.1 'Metacognition' at the epistemological level: structuring and linking praxeologies

At an epistemological level, most of the difficulties that are associated to non-routine problem solving in school institutions can be attributed to the usually *isolated* character of the mathematical problems considered and, thus, to the *point-character of the mathematical praxeologies* students are used to work with (at least in Spain). Students usually evolve in a mathematical world made of disconnected and isolated point-praxeologies (or at most local ones) and it is not easy for them to acquire the necessary means—and not even the necessity—to connect and contextualise the problems they study. We postulate that some difficulties related to what is considered as the "students' lack of metacognitive strategies" can be explained in terms of this *isolation and disconnection of mathematical praxeologies*. For instance:

(a) The *choice and planning of mathematical strategies* to solve a given non-routine problem strongly depends on the kind of different mathematical techniques that are available for the students which, in turn, depend on the possibility to locate the initially considered mathematical problem within *one or more connected local praxeologies*.

(b) Strategies to *control* the carrying out of mathematical strategies can be related to the *rationale of the mathematical activity* where the problem takes place or, in other terms, the *generative question* where the problem arises, the *means available* and the nature and extent of the mathematical praxeologies that have to be built to answer this question.

(c) The *monitoring* of the problem solving activity and its results, as well as the students' capacity to *revise plans and strategies* could also be connected to the *global evaluation* of mathematical praxeologies that are built during the study process, a *didactic moment* closely linked to the *moment of the construction of the theoretical block* of praxeologies and to the *moment of institutionalisation* of the 'valuable' knowledge produced.

To illustrate these general considerations with the conducted experimentations, we can refer to the capacity of SRC to create connections between different pieces of knowledge, or, in other words, between components traditionally belonging to disconnected mathematical praxeologies, thus overcoming their isolation. We can then talk about the development of the available mathematical praxeologies beyond their point-level (see García et al., 2006), which allows:

– To *give functionality to some contents* of the block 'functions and graphs' to validate the solutions obtained or to display information, such as the construction of the algebraic expression of a function and the use of graphs to solve inequalities. It is important to notice the main role played by the technological and theoretical components of the used praxeologies, considering their weak presence in current Spanish secondary education, where the students are rarely asked to describe and validate their problem solving procedures (see Rodríguez, 2005). This functional use of the theoretical block of praxeologies, that constitutes and example of the integration of the two blocks of a praxeology (the practical and the theoretical one), can be closely related to practices usually considered as 'metacognitive' such as the awareness of the procedure used and its validation.
– To *connect different blocks of contents*, such as 'statistics' and 'functions and graphs' (the former being used to build up and check the validity of an algebraic-functional model), and even to connect different knowledge areas, such as mathematics and 'new technologies' with the use of Excel, the search of

information on tariffs and the design of a web site to display the final results.

– To *globally evaluate* a praxeology needs to consider up to what point it answers the initial question that constitutes its *raison d'être*. The need to carry out this global evaluation (before posting the results on the web) provided strategies to evaluate the modelling techniques used by the students (algebraic formulae and graphs).

6.2 'Metacognition' at the didactic level: sharing responsibilities between the teacher and the students

At the didactic level, difficulties associated with 'metacognitive strategies' can be put in relation with the *distribution of responsibilities in the management of the teaching and learning process* classically established by current didactic contracts. The experimentation showed how SRC could promote a new distribution of responsibilities and what restrictions hindered the assumption of these responsibilities by the students as well as by the teacher. It is also interesting to observe what decisions need to be made and who makes them, when the study process gets rid of a great deal of restrictions, imposed by the school institution in a transparent way for the subjects.

In the current Spanish didactic contract, *the 'activation' of the theoretical block of a praxeology and the connection between different praxeologies* or blocks of contents are activities usually carried out by the teacher. This change in the sharing of responsibilities between the teacher and the students promoted by the SRC was not carried out without difficulties. We think that the 'strength' of the current didactic contract may explain the students' initial reluctance to assume responsibilities concerning the planning, regulation and evaluation of the study process, and how it gradually decreased throughout the study process (as soon as a new didactic contract was progressively established).

Another important issue is to examine to what extent these decisions, despite being 'didactic' decisions (in the classical sense, i.e., affecting the running of the teaching and learning process), are *an integral part of the mathematical work*. We have seen that, at an epistemological level, 'metacognitive strategies' may be related to aspects of the mathematical work that go beyond the limits and the connection of the themes studied at school (that is, beyond the level of local praxeologies). At a didactic level, they could be related to the dimension of the mathematical work which, in the traditional didactic contract, is the sole responsibility of the teacher. This interpretation may also explain why these strategies are not considered 'cognitive' (mathematical) ones, but 'metacognitive'. 'Metacognition' can thus in a sense be connected to the decisions which, in the traditional teaching and learning processes, are usually the exclusive responsibility of the teacher: planning the study, deciding on its chronology and the distribution of tasks, synthesising and evaluating the results, choosing easier questions or particular cases to start with, formulating new problems, looking for information, analyzing and developing the available knowledge, etc.

It can be said that, in the teaching and learning of mathematics, *the divide between cognition and metacognition is not independent of the different sharing of responsibilities between students and teachers in the way the process of study is usually conduced at school.* Certainly due to the dominance of a narrow vision of mathematics in our teaching institutions, activities such as planning, assessing or revising strategies, or selecting and connecting topics, have long been considered as 'didactic gestures' and thus put under the exclusive responsibility of the teacher. In other words, organising the tasks and fixing the time devoted to each one, deciding what question has to be asked and when, formulating new problems and summarising and validating the results obtained are generally not seen as proper mathematical tasks but as what has to be done to organise mathematical learning. The conception of mathematical activity provided by the ATD can help overcome this situation through the 'mathematisation' of what is usually seen as 'metacognitive practices', which surprisingly seems to correspond to practices that are up to the teacher.

In general, we postulate that the inclusion of 'metacognitive regulation' in school mathematical activity needs a serious transformation of the current didactic contract to overcome the strict separation between what is commonly considered as 'the mathematic' and 'the didactic'. More concretely, it has to give students the possibility to be more and more co-responsible for different aspects of the study process (of the problematic question), including those traditionally assigned only to the teacher such as, for instance, the planning, regulation and evaluation of the study process, or the location of an isolated problematic question in a chosen mathematical praxeology. We thus propose a gradual transfer of mathematical and didactic responsibilities from primary to tertiary education.[13] Obviously this 'transfer of responsibilities' cannot take place in a spontaneous and natural way but requires more research and

[13] This proposal is coherent with the evolution of the 'didactic contract' as described by Brousseau: "The didactic contract involves the project of its own dissolution. It is understood from the beginning of the didactical relationship that a moment must arrive when it will be broken. At that moment, at the end of the teaching, the taught system will, with the help of the learned knowledge, be assumed to be capable of facing systems without didactical intentions." (Brousseau, 1997, p. 57).

new didactic proposals in order to achieve that the study of questions becomes the 'driving force' of the learning of mathematics. Our present research efforts on the integration and 'viability' of Study and Research Courses as a normalised activity at secondary and university level show the strong institutional restrictions this new didactic contract has to overcome (see, e.g., García et al., 2006; Barquero el al., 2008; Ruiz et al., 2008). Furthermore, it seems that overcoming these restrictions will even lead us to question the cultural distinction between what is considered as mathematical activity—doing mathematics—and the activity of organising its learning—the study of mathematics. It can be postulated that the 'mathematisation' of what is considered as 'didactic' in a teaching institution is a necessary step to transfer 'metacognitive' responsibilities to the students.

7 Conclusion: Towards a networking method for comparing theories

The research here presented originated from the study of the notion of 'metacognition' within the ATD frame (see Rodríguez 2005, Rodríguez, Bosch & Gascón 2008). It started with a first attempt to formulate this notion directly in terms of the model of (institutional) cognition provided by the ATD, that is, in terms of the mathematical and didactic praxeologies available in a given didactic institution. This first step of the research was hindered by a first difficulty: the notion of 'metacognition' belongs to a specific theoretical approach, psychology first and, in mathematics education, what has here been roughly designated as 'classical Problem Solving'. As this last approach, like any research perspective, constructs its own research problems and delimits the empirical fact considered worth studying in its own way, the 'translation' into the ATD frame did not seem to be directly practicable. It was clearly seen that a *research problem* formulated within a theoretical approach was not always necessarily meaningful in another perspective. However, the *problematic question* which was at the origin of the research problem that motivated the introduction of 'metacognition' in mathematics education—we have called it 'Pólya's problem'—did seem to be 'reformulable' in terms of institutional mathematical and didactic praxeologies. Furthermore, the new formulation of 'Pólya's problem' obtained could even be directly linked to a new kind of teaching proposal recently introduced in the ATD (the 'Study and Research Courses') in order to test the conditions under which a possible partial solution could be obtained. A teaching experimentation was then conducted and analysed, and it gave the idea that some of the difficulties classically interpreted in terms of 'metacognition' (like monitoring, planning, assessing, self-regulation, etc.) could also be described in terms of institutional didactic and mathematical praxeologies, by the different sharing of responsibilities between students and teachers in the way the study process of open problems is usually conduced at school. More generally, it could also be related to what, in current teaching institutions, is considered as corresponding to the 'mathematical work' (up to the students) in opposition to the 'didactic work' (the teacher's tasks); letting thus confronting the dichotomy 'cognition/metacognition' with the cultural dichotomy 'mathematical/didactic' activity (or doing mathematics/teaching and learning to do mathematics). This new formulation of the problem opens the way to new research within the ATD frame on the integration and 'viability' of new didactic praxeologies (like the Study and Research Courses) taking into account the strong institutional restrictions related to current didactic contracts that seems to hinder them.

The methods to compare or contrast research frameworks in mathematics education do not only have to overcome the difficulty of dealing with different conceptualisations of reality that not necessarily match each other. They also have to take into account that each frame is engaged in its own priorities and ways of questioning educational reality, which implies strong differences in the way research problems are formulated, research methodologies are constructed and 'admissible answers' searched. Facing this inevitable difficulty, it always seems possible to find an initial problematic question—a 'practical problem' which can be formulated in the words and culture of the considered educational institutions—, that can be meaningful in the different approaches since they are appearing as part of the reality modelled by them. This problematic question can thus be taken as a 'common base' for the comparison of approaches by looking at how each frame transforms it into a research problem and what kind of admissible answer can be provided. It is important to notice that this method does not stop at the level of the research results that can be given by each approach but may/should consider a step further: the capacity of each framework to provide practical and effective solutions to the problematic question considered, even if these solutions may include the necessity of 'deconstructing' the practical problem by introducing an evolution of educational cultures or by just pointing out other problematic issues. The efforts initiated by our research community in networking theories will be successful if they can both overcome the difficulties of dealing with not easily commensurable theoretical frameworks and reach the level of the important problematic questions that practitioners regularly raise.

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