1 Determinant function properties

0. $\det A = \det A^T$ that can be proved with $M = QR$ decomposition
where M is square matrix, Q is unitary matrix and R is upper triangle matrix with positive diagonal entries.
1. Exchange two different rows $\det A = (-1)\det A$

$$\det\left(\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \det\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \det\begin{bmatrix} b_1 & a_1 \\ b_2 & a_2 \end{bmatrix} = \det\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}(-1)$$

2. Multiply one column with non-zero scalar $\det A = \lambda \det A$

$$\det\begin{bmatrix} \lambda a_1 & b_1 \\ \lambda a_2 & b_2 \end{bmatrix} = \det\left(\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\right)\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\right)\lambda$$

3. Multiply one column with non-zero scalar and add it to other column $\det A = \det A$
Determinant function is linear function in any one column given that the rest of columns are fixed

$$\det\left(\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}\right) = \det\begin{bmatrix} a_1 + \lambda b_1 & b_1 \\ a_2 + \lambda b_2 & b_2 \end{bmatrix} = \det\left(\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\right)\det\left(\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}\right) = \det\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$