1 Derive Triangle Matrix

What elementary matrix can be used to convert a singular matrix to upper triangle matrix with Left Multiplication

$$
\begin{aligned}
&\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}\\
&=\begin{bmatrix}r_1\\ r_2\\ r_3\end{bmatrix}\begin{bmatrix}u_1 & u_2 & u_3\end{bmatrix}\\
&=\begin{bmatrix}\langle r_1,u_1\rangle & \langle r_1,u_2\rangle & \langle r_1,u_3\rangle\\ \langle r_2,u_1\rangle & \langle r_2,u_2\rangle & \langle r_2,u_3\rangle\\ \langle r_3,u_1\rangle & \langle r_3,u_2\rangle & \langle r_3,u_3\rangle\end{bmatrix}\\
&=\begin{bmatrix}b\langle r_1,u_1\rangle & b\langle r_1,u_2\rangle & b\langle r_1,u_3\rangle\\ a\langle r_2,u_1\rangle & a\langle r_2,u_2\rangle & a\langle r_2,u_3\rangle\\ \langle r_3,u_1\rangle & \langle r_3,u_2\rangle & \langle r_3,u_3\rangle\end{bmatrix}\\
&=\begin{bmatrix}b & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & a & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\langle r_1,u_1\rangle & \langle r_1,u_2\rangle & \langle r_1,u_3\rangle\\ \langle r_2,u_1\rangle & \langle r_2,u_2\rangle & \langle r_2,u_3\rangle\\ \langle r_3,u_1\rangle & \langle r_3,u_2\rangle & \langle r_3,u_3\rangle\end{bmatrix}\\
&=\begin{bmatrix}1 & 0 & 0\\ 1 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}b & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & a & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\langle r_1,u_1\rangle & \langle r_1,u_2\rangle & \langle r_1,u_3\rangle\\ \langle r_2,u_1\rangle & \langle r_2,u_2\rangle & \langle r_2,u_3\rangle\\ \langle r_3,u_1\rangle & \langle r_3,u_2\rangle & \langle r_3,u_3\rangle\end{bmatrix}\\
&=\begin{bmatrix}\langle br_1,u_1\rangle & \langle br_1,u_2\rangle & \langle br_1,u_3\rangle\\ \langle br_1,u_1\rangle-\langle ar_2,u_1\rangle & \langle br_1,u_2\rangle-\langle ar_2,u_2\rangle & \langle br_1,u_3\rangle-\langle ar_2,u_3\rangle\\ \langle r_3,u_1\rangle & \langle r_3,u_2\rangle & \langle r_3,u_3\rangle\end{bmatrix}
\end{aligned}\tag{1}
$$

2 From intuition to proof

$$
\begin{aligned}
&L_0A_0=A_1\\
&L_1A_1=A_2\\
&L_k\dots L_2L_1A_0=A_k=U\\
&L=L_k\dots L_2L_1\\
&LA=U\\
&\text{Apply LU factorization on } L \text{ again}\\
&L'_k\dots L'_1L'_0L=U'\\
&U' \text{ should be the identity}\\
&\Rightarrow L^{-1}=L'_k\dots L'_1L'_0
\end{aligned}\tag{2}
$$

3 Let's prove it

$$
L^{-1}=L'_k\dots L'_1L'_0
$$

$$
\begin{aligned}
&A_0=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\\
&\Rightarrow\begin{bmatrix}1 & 0 & 0\\ x_{21} & 1 & 0\\ y_{31} & 0 & 1\end{bmatrix}\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ 0 & a_{22} & a_{23}\\ 0 & a_{32} & a_{33}\end{bmatrix}\\
&\Rightarrow\langle r_2^*,v_1\rangle=\left\langle\begin{bmatrix}x_{21}\\ 1\\ 0\end{bmatrix},\begin{bmatrix}a_{11}\\ a_{21}\\ a_{31}\end{bmatrix}\right\rangle=0\\
&\Rightarrow a_{11}x_{21}+a_{21}=0\quad\Rightarrow x_{21}=\frac{-a_{21}}{a_{11}}\\
&\Rightarrow\langle r_3^*,v_1\rangle=\left\langle\begin{bmatrix}y_{31}\\ 0\\ 1\end{bmatrix},\begin{bmatrix}a_{11}\\ a_{21}\\ a_{31}\end{bmatrix}\right\rangle=0\\
&\Rightarrow a_{11}y_{31}+a_{31}=0\quad\Rightarrow y_{31}=\frac{-a_{31}}{a_{11}}\\
&\Rightarrow a_{kk}y_{3k}+a_{3k}=0\quad 1\to k\\
&\Rightarrow a_{kk}y_{ik}+a_{ik}=0\quad 3\to i\\
&\Rightarrow y_{ik}=\frac{-a_{ik}}{a_{kk}}\\
&\left\langle\begin{bmatrix}0\\ \vdots\\ y_{k,k}=1\\ \vdots\\ y_{i,k}\\ \vdots\\ 0\end{bmatrix},\begin{bmatrix}a_{1k}\\ \vdots\\ a_{kk}\\ \vdots\\ a_{ik}\\ \vdots\\ a_{mk}\end{bmatrix}\right\rangle=0\\
&\Rightarrow y_{i,k}=\frac{-a_{i,k}}{a_{k,k}}\quad i\in[k+1,\dots m]\\
&\text{Let } l_{i,k}=\frac{-a_{i,k}}{a_{k,k}}\quad i\in[k+1,\dots m]\\
&v_k=\begin{bmatrix}0\\ \vdots\\ 0\\ l_{k+1,k}\\ l_{k+2,k}\\ \vdots\\ l_{m,k}\end{bmatrix}\qquad\qquad l'_k=v_ke_k^*=v_k\otimes e_k\\
&l'_k=v_k\begin{bmatrix}0 & \cdots & 1 & \cdots\end{bmatrix}\\
&l'_k=\begin{bmatrix}0v_k & \cdots & 1v_k & \cdots\end{bmatrix}
\end{aligned}\tag{3}
$$

$$
\begin{aligned}
&L_k=I+v_ke_k^*=\begin{bmatrix}1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & l_{k+1,k} & & \\ & & \vdots & \ddots & \\ & & l_{m,k} & & 1\end{bmatrix}\\
&L_k^{-1}=I-v_ke_k^*=\begin{bmatrix}1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & -l_{k+1,k} & & \\ & & \vdots & \ddots & \\ & & -l_{m,k} & & 1\end{bmatrix}\\
&L_kL_k^{-1}=(I+v_ke_k^*)(I-v_ke_k^*)\\
&L_kL_k^{-1}=I-v_ke_k^*v_ke_k^*\\
&L_kL_k^{-1}=I\quad\text{where }\langle e_k,v_k\rangle=0\\
&L_kL_{k+1}=(I+v_ke_k^*)(I+v_{k+1}e_{k+1}^*)\\
&L_kL_{k+1}=I+v_ke_k^*+v_{k+1}e_{k+1}^*+v_ke_k^*v_{k+1}e_{k+1}^*\\
&L_kL_{k+1}=I+v_ke_k^*+v_{k+1}e_{k+1}^*\quad\because\langle e_k^*,v_{k+1}\rangle=0\\
&L_kL_{k+1}=\begin{bmatrix}1 & & & & \\ & 1 & & & \\ & -l_{k+1,k} & 1 & & \\ & \vdots & -l_{k+2,k} & \ddots & \\ & & \vdots & & \\ & -l_{m,k} & -l_{m,k} & & 1\end{bmatrix}\\
&\text{In general}\\
&L_1\dots L_m=\begin{bmatrix}1 & & & & & \\ -l_{2,1} & 1 & & & & \\ \vdots & -l_{k+1,k} & 1 & & & \\ & \vdots & -l_{k+2,k+1} & \ddots & & \\ & & \vdots & & & \\ -l_{m,1} & -l_{m,k} & -l_{m,k+1} & & -l_{m,m-1} & 1\end{bmatrix}\\
&\text{Try this}\\
&L_kL_{k-1}=I+v_ke_k^*+v_{k-1}e_{k-1}^*+v_ke_k^*v_{k-1}e_{k-1}^*\\
&\text{We can not go anywhere}\quad\because v_ke_k^*v_{k-1}e_{k-1}^*\neq 0\\
&\text{Let's try}\\
&L_{k-1}L_k=(I+v_{k-1}e_{k-1}^*)(I+v_ke_k^*)\\
&L_{k-1}L_k=I+v_{k-1}e_{k-1}^*+v_ke_k^*+v_{k-1}e_{k-1}^*v_ke_k^*\\
&L_{k-1}L_k=I+v_{k-1}e_{k-1}^*+v_ke_k^*\quad\because\langle e_{k-1},v_k\rangle=0\\
&L_{k-2}L_{k-1}L_k=(I+v_{k-2}e_{k-2}^*)(I+v_{k-1}e_{k-1}^*+v_ke_k^*)\\
&L_{k-2}L_{k-1}L_k=I+v_{k-2}e_{k-2}^*+v_{k-1}e_{k-1}^*+v_ke_k^*+v_{k-2}e_{k-2}^*v_{k-1}e_{k-1}^*+v_{k-2}e_{k-2}^*v_ke_k^*\\
&L_{k-2}L_{k-1}L_k=I+v_{k-2}e_{k-2}^*+v_{k-1}e_{k-1}^*+v_ke_k^*\quad\because\langle e_{k-2},v_{k-1}\rangle=0,\langle e_{k-2},v_k\rangle=0\\
&L_k^{-1}L_{k-1}^{-1}=(I-v_ke_k^*)(I-v_{k-1}e_{k-1}^*)\\
&L_k^{-1}L_{k-1}^{-1}=I-v_ke_k^*-v_{k-1}e_{k-1}^*+v_ke_k^*v_{k-1}e_{k-1}^*
\end{aligned}\tag{4}
$$

4 Code it up

`dog cat` α

```
mat vk(mat m, int k){
    mat id = identity(m.length);
    for(int kk=k + 1; kk<nrow; kk++){
        id.arr[kk][k] = m.arr[kk][k]/m[k][k];
    }
    return id;
}
```

5 Inverse of unit triangle matrix is also a triangle matrix