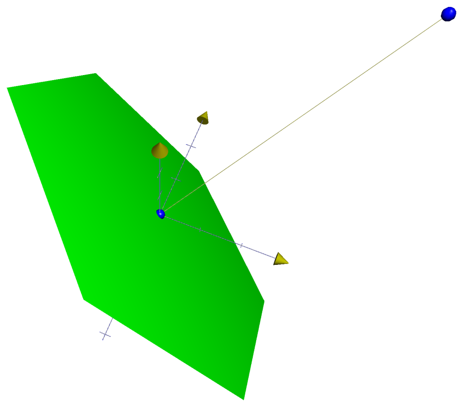
1 Plane equation in three dimensions

Given a function $x + y + z = 0$ which is just a flat plane and perpendicular to vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$



Why the plane is perpendicular to vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$?

x + y + z = 0 can be written as $1 \cdot x + 1 \cdot y + 1 \cdot z = 0$ and it also can be written as dot product as following $[x, y, z] \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$, the dot product implies $[x, y, z]$ is perpendicular to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for any point (x, y, z) and $x + y + z = 0$ passes through point $(x, y, z) = (0, 0,)$.

2 Derivative of $f(x) = x^2$

Find the minimum of $f(x) = x^2$, we start from point $(x, y) = (-1, 1)$, the derivative of $f(x)$ is $\frac{d}{dx} f(x) = 2x$

3 Gradient

Definition 1. *The gradient of a funciton $f(x, y)$ is defined as*

$$\nabla f(x, y) = \left\langle \frac{df}{dx}(x, y), \frac{df}{dy}(x, y) \right\rangle$$

Example 1. *For function $x + y + z = 0$, it can be written as $z = -x - y$ or $f(x, y) = -x - y$*
The gradient of $f(x, y) = -x - y$ is as following:

$$\begin{aligned} \frac{df}{dx} f(x, y) &= -1 \\ \frac{df}{dy} f(x, y) &= -1 \\ \nabla f(x, y) &= \langle -1, -1 \rangle \end{aligned} \tag{1}$$

The gradient of $f(x, y) = -x - y$ is constant, it means all the vectors on the surface have the **same direction** and **same magnitude**.

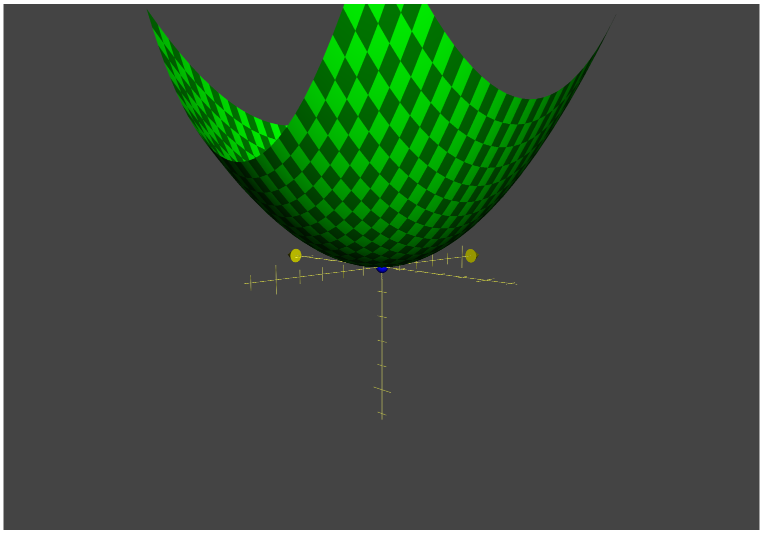
$$|\nabla f(x, y)| = 2 = C$$

Example 2. *Given a function $f(x, y) = e^x \cos y$, find the gradient of the function.*

$$
\begin{aligned}
f(x,y) &= e^x \cos y \\
\frac{df}{dx} &= e^x \cos y \\
\frac{df}{dy} &= -e^x \sin y \\
\nabla f(x,y) &= \left\langle \frac{df}{dx}, \frac{df}{dy} \right\rangle = \langle e^x \cos x, -e^x \sin y \rangle
\end{aligned} \tag{2}
$$

Example 3. *Given function $f(x,y) = 2x^2 + 3y^2$, find the following:*

- Compute the gradient of $f(x,y) = 2x^2 + 3y^2$
- Identify the level curve of $f(x,y) = C$ through the point $(x,y) = (1,1)$.
- Find the parameter equation $\vec{r}(t)$ of the level curve.
- Show $\frac{d}{dt}\vec{r}(t) \cdot \nabla f(x,y) = 0$ at point $(x,y) = (1,1)$.



Compute the gradient of $f(x,y) = 2x^2 + 3y^2$

$$
\begin{aligned}
\frac{df}{dx} f(x,y) &= 4x \\
\frac{df}{dy} f(x,y) &= 6y \\
\nabla f(x,y) &= \langle 4x, 6y \rangle
\end{aligned} \tag{3}
$$

Identify the level curve of $f(x,y)$

$$
\begin{aligned}
f(1,1) &= 2 + 3 = z \\
2x^2 + 3y^2 &= 5 \\
\frac{2}{5}x^2 + \frac{3}{5}y^2 &= 1 \\
\left(\sqrt{\frac{2}{5}}x\right)^2 + \left(\sqrt{\frac{3}{5}}y\right)^2 &= 1 \\
\cos t &= \sqrt{\frac{2}{5}}x \\
\sin t &= \sqrt{\frac{3}{5}}x \\
x &= \sqrt{\frac{5}{2}}\cos t \\
y &= \sqrt{\frac{5}{3}}\sin t \\
r(t) &= \left\langle \sqrt{\frac{5}{2}}\cos t, \sqrt{\frac{5}{3}}\sin t \right\rangle \\
\frac{d}{dt} r(t) &= \left\langle -\sqrt{\frac{5}{2}}\sin t, \sqrt{\frac{5}{3}}\cos t \right\rangle
\end{aligned} \tag{4}
$$

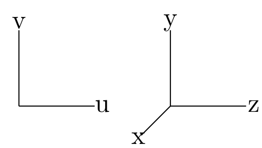
Show $\frac{d}{dt}r(t) \cdot \nabla f(x,y) = 0$ at point $(x,y) = (1,1)$

$$\begin{aligned}
x &= \sqrt{\frac{5}{2}}\cos t = 1 \Rightarrow \cos t = \sqrt{\frac{2}{5}} \\
y &= \sqrt{\frac{5}{3}}\sin t = 1 \Rightarrow \sin t = \sqrt{\frac{3}{5}} \\
\nabla f(1,1) &= \langle 6, 8 \rangle \\
\frac{d}{dt}r(t) &= \left\langle -\sqrt{\frac{5}{2}}\sin t, \sqrt{\frac{5}{3}}\cos t \right\rangle = \left\langle -\frac{15}{10}, \frac{10}{15} \right\rangle \\
\frac{d}{dt}r(t) \cdot \nabla f(1,1) &= \left\langle -\sqrt{\frac{15}{10}}, \sqrt{\frac{10}{15}} \right\rangle \cdot \langle 4, 6 \rangle = 0
\end{aligned} \tag{5}$$

4 Harmonic Function

$$f(x,y) = e^x \cos y$$

5 First Fundamental Form a surface



Cartesian Coordinate Equation

$$r^2 = x^2 + y^2 + z^2$$

Sphere parametric equation

$$\begin{aligned}
x &= r \sin\alpha \cos\theta \\
y &= r \sin\alpha \\
z &= r \cos\alpha \sin\theta
\end{aligned}$$

$$f(\alpha,\theta) = \begin{cases} x(\alpha,\theta) = r\sin\alpha\cos\theta \\ y(\alpha,\theta) = r\sin\alpha \\ z(\alpha,\theta) = r\cos\alpha\sin\theta \end{cases}$$

$$\begin{aligned}
J &= \begin{bmatrix} \frac{dx}{d\alpha} & \frac{dx}{d\theta} \\ \frac{dy}{d\alpha} & \frac{dy}{d\theta} \\ \frac{dz}{d\alpha} & \frac{dz}{d\theta} \end{bmatrix} = \begin{bmatrix} r\cos\theta\cos\alpha & -r\cos\alpha\sin\theta \\ r\cos\alpha & 0 \\ -r\sin\alpha\sin\theta & r\cos\alpha\cos\theta \end{bmatrix} \\
J^T &= \begin{bmatrix} \frac{dx}{d\alpha} & \frac{dy}{d\alpha} & \frac{dz}{d\alpha} \\ \frac{dx}{d\theta} & \frac{dy}{d\theta} & \frac{dz}{d\theta} \end{bmatrix} = \begin{bmatrix} -r\cos\alpha\sin\theta & 0 & r\cos\alpha\cos\theta \\ r\cos\theta\cos\alpha & r\cos\alpha & -r\sin\alpha\sin\alpha \end{bmatrix} \\
J^T J &= \begin{bmatrix} \frac{dx}{d\alpha} & \frac{dy}{d\alpha} & \frac{dz}{d\alpha} \\ \frac{dx}{d\theta} & \frac{dy}{d\theta} & \frac{dz}{d\theta} \end{bmatrix} \begin{bmatrix} \frac{dx}{d\alpha} & \frac{dx}{d\theta} \\ \frac{dy}{d\alpha} & \frac{dy}{d\theta} \\ \frac{dz}{d\alpha} & \frac{dz}{d\theta} \end{bmatrix} = \begin{bmatrix} x_\alpha & y_\alpha & z_\alpha \\ x_\theta & y_\theta & z_\theta \end{bmatrix} \begin{bmatrix} x_\alpha & x_\theta \\ y_\alpha & y_\theta \\ z_\alpha & z_\theta \end{bmatrix} \\
J^T J &= \begin{bmatrix} x_\alpha x_\alpha + y_\alpha y_\alpha + z_\alpha z_\alpha & x_\alpha x_\theta + y_\alpha y_\theta + z_\alpha z_\theta \\ x_\theta x_\alpha + y_\theta y_\alpha + z_\theta z_\alpha & x_\theta x_\theta + y_\theta y_\theta + z_\theta z_\theta \end{bmatrix} \\
J^T J &= \begin{bmatrix} -r\cos\alpha\sin\theta & 0 & r\cos\alpha\cos\theta \\ r\cos\theta\cos\alpha & r\cos\alpha & -r\sin\alpha\sin\alpha \end{bmatrix} \begin{bmatrix} r\cos\theta\cos\alpha & -r\cos\alpha\sin\theta \\ r\cos\alpha & 0 \\ -r\sin\alpha\sin\alpha & r\cos\alpha\cos\theta \end{bmatrix} \\
J^T J &= \begin{bmatrix} -r\cos\alpha\sin\theta r\cos\theta\cos\alpha + 0 + -r^2\cos\alpha\cos\theta\sin\alpha\sin\theta & a \\ b & c \end{bmatrix}
\end{aligned} \tag{6}$$