1 Find the projection from w onto v.

Dot product of two vectors

$$w \cdot v = \|w\|\|v\| \cos\theta$$

Compute the scalar projection from w onto v

$$\frac{w \cdot v}{\|v\|} = \|w\| \cos\theta$$

Multiply the unit vector of v

$$proj_v w = \frac{w \cdot v}{\|v\|} \frac{v}{\|v\|}$$
$$proj_v w = \frac{\langle w, v \rangle}{\langle v, v \rangle} v$$

Given vectors: v_1, v_2, v_3 find the orthgonal basis for the three vectors.

$$\begin{aligned}
a_1 &= v_1 = \frac{\langle a_1, v_1 \rangle}{\langle a_1, v_1 \rangle} a_1 & e_1 &= \frac{a_1}{\|a_1\|} \\
a_2 &= v_2 - \frac{\langle a_1, v_2 \rangle}{\langle a_1, a_1 \rangle} a_1 & e_2 &= \frac{a_2}{\|a_2\|} \\
a_3 &= v_3 - \left(\frac{\langle a_1, v_3 \rangle}{\langle a_1, a_1 \rangle} a_1 + \frac{\langle a_2, v_3 \rangle}{\langle a_2, a_2 \rangle} a_2 \right) & e_3 &= \frac{a_3}{\|a_3\|}
\end{aligned}$$

$$\begin{aligned}
proj_{w_1} v_2 &= \frac{\langle u_1, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 \quad \text{where } w_1 = \text{Span}\{u_1\} \\
proj_{w_2} v_3 &= \frac{\langle u_1, v_3 \rangle}{\langle v_3, v_3 \rangle} v_3 + \frac{\langle u_2, v_3 \rangle}{\langle v_3, v_3 \rangle} v_3 \quad \text{where } w_2 = \text{Span}\{u_1, u_2\} \\
v_1 &= \frac{\langle v_1, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 \\
v_2 &= a_2 + \frac{\langle v_1, v_2 \rangle}{\langle v_1, v_1 \rangle} v_1 \\
v_3 &= a_3 + \left(\frac{\langle v_1, v_3 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle v_2, v_3 \rangle}{\langle v_2, v_2 \rangle} v_2 \right)
\end{aligned}$$

Project v_1, v_2, v_3 onto new **orthgonal basis**: $\{e_1, e_2, e_3\}$

$$\begin{aligned}
v_1 &= \frac{\langle e_1, v_1 \rangle}{\langle e_1, e_1 \rangle} e_1 && \text{project } v_1 \text{ onto } e_1 \\
v_2 &= \frac{\langle e_1, v_2 \rangle}{\langle e_1, e_1 \rangle} e_1 + \frac{\langle e_2, v_2 \rangle}{\langle e_2, e_2 \rangle} e_2 && \text{project } v_2 \text{ onto } e_1, e_2 \\
v_3 &= \frac{\langle e_1, v_3 \rangle}{\langle e_1, e_1 \rangle} e_1 + \frac{\langle e_2, v_3 \rangle}{\langle e_2, e_2 \rangle} e_2 + \frac{\langle e_3, v_3 \rangle}{\langle e_3, e_3 \rangle} e_3 && \text{project } v_3 \text{ onto } e_1, e_2, e_3
\end{aligned}$$

Simplify a bit with $\langle e_i, e_i \rangle = 1$

$$
\begin{aligned}
v_1 &= \langle e_1, v_1 \rangle e_1 \\
v_2 &= \langle e_1, v_2 \rangle e_1 + \langle e_2, v_2 \rangle e_2 \\
v_3 &= \langle e_1, v_3 \rangle e_1 + \langle e_2, v_3 \rangle e_2 + \langle e_3, v_3 \rangle e_3
\end{aligned}
$$

Write it in dot product form

$$
\begin{aligned}
v_1 &= \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} \langle e_1, v_1 \rangle \\ 0 \\ 0 \end{bmatrix} \\
v_2 &= \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} \langle e_1, v_2 \rangle \\ \langle e_2, v_2 \rangle \\ 0 \end{bmatrix} \\
v_3 &= \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} \langle e_1, v_3 \rangle \\ \langle e_2, v_3 \rangle \\ \langle e_3, v_3 \rangle \end{bmatrix}
\end{aligned}
$$

Combine it in matrix form

$$
\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} \langle e_1, v_1 \rangle & \langle e_1, v_2 \rangle & \langle e_1, v_3 \rangle \\ 0 & \langle e_2, v_2 \rangle & \langle e_2, v_3 \rangle \\ 0 & 0 & \langle e_3, v_3 \rangle \end{bmatrix}
$$

2 Show all the diagonal entries in M are positive if M is singular matrix

$$
v_k = a_k + \langle e_1, v_k \rangle e_1 + \cdots + \langle e_{k-1}, v_k \rangle e_{k-1} = a_k + \sum_{j=1}^{k-1} \langle e_j, v_{k-1} \rangle e_j
$$

$$
\begin{aligned}
\langle e_k, v_k \rangle &= \left\langle e_k, a_k + \sum_{j=1}^{k-1} \langle e_j, v_k \rangle e_j \right\rangle \\
\langle e_k, v_k \rangle &= \langle e_k, a_k \rangle + \left\langle e_k, \sum_{j=1}^{k} \langle e_j, v_k \rangle e_j \right\rangle \quad \because \text{ linarity of inner product} \\
\langle e_k, v_k \rangle &= \langle e_k, a_k \rangle \quad \because \langle e_i, e_j \rangle = 0 \text{ if } i \neq j \\
\langle e_k, a_k \rangle &= \left\langle \frac{a_k}{\|a_k\|}, a_k \right\rangle \\
\langle e_k, a_k \rangle &= \frac{1}{\|a_k\|} \langle a_k, a_k \rangle \quad \because \text{ linearity on each component} \\
\langle e_k, a_k \rangle &= \|a_k\| \quad \square
\end{aligned}
$$

3 Determinate whether a matrix is singular

From the previous computation $\{a_1, a_2, \ldots a_k\}$ one of $a_i \quad i \in [1 \ldots k]$ must be zero if a matrix is singular it means $\{|a_1|^2, |a_2|^2, \ldots |a_k|^2\}$ must be contains zero

4 QR decomposition

Given none singular matrix $A \in M$, there exists unique pair of unitary matrix $Q \in M$ and upper triangle $R \in M$ with positive diagonal entries such that

$$A = QR$$

Proof from above.

5 Gram Schmidt Process

Find the orthgonal basis to represent v_1, v_2, v_2

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix} \right\}$$

$$u_1 = v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$u_2 = v_2 - proj_{w_1} v_2 = \frac{\langle u_1, v_2 \rangle}{\langle v_1, v_1 \rangle} v_1 \tag{1}$$

$$= \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\rangle}{\left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\|} \cdot \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\|} \tag{2}$$

$$= \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \frac{4 + 10 + 18}{14} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{3}$$

$$= \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \frac{32}{14} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} 4 - \frac{32}{14} \\ 5 - \frac{64}{14} \\ 6 - \frac{96}{14} \end{bmatrix} = \begin{bmatrix} \frac{56}{14} - \frac{32}{14} \\ \frac{70}{14} - \frac{64}{14} \\ \frac{84}{14} - \frac{96}{14} \end{bmatrix} = \begin{bmatrix} \frac{24}{14} \\ \frac{6}{14} \\ \frac{-12}{14} \end{bmatrix} \tag{5}$$

$$u_3 = v_3 - (proj_{w_1} v_3 + proj_{w_2} v_3) \tag{6}$$

$$= u_3 - \left(\frac{\langle u_1, v_3 \rangle}{\|u_1\|} \cdot \frac{u_1}{\|u_1\|} + \frac{\langle u_2, v_3 \rangle}{\|u_2\|} \cdot \frac{u_2}{\|u_2\|} \right) \tag{7}$$

$$= \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix} - \left(\frac{\left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix} \right\rangle}{\left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\|} \cdot \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\|} + \frac{\left\langle \begin{bmatrix} \frac{24}{14} \\ \frac{6}{14} \\ \frac{-12}{14} \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix} \right\rangle}{\left\| \begin{bmatrix} \frac{24}{14} \\ \frac{6}{14} \\ \frac{-12}{14} \end{bmatrix} \right\|} \cdot \frac{\begin{bmatrix} \frac{24}{14} \\ \frac{6}{14} \\ \frac{-12}{14} \end{bmatrix}}{\left\| \begin{bmatrix} \frac{24}{14} \\ \frac{6}{14} \\ \frac{-12}{14} \end{bmatrix} \right\|} \right) \tag{8}$$

$$= \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix} - \left(\begin{bmatrix} \frac{53}{14} \\ \frac{53.2}{14} \\ \frac{53.3}{14} \end{bmatrix} + \begin{bmatrix} \frac{64}{21} \\ \frac{16}{21} \\ \frac{-32}{21} \end{bmatrix} \right) \text{where } \operatorname{lcm}(14, 50) = 2 \cdot 7 \cdot 5 \cdot 5 \tag{9}$$

$$= \begin{bmatrix} \frac{1}{6} \\ \frac{-1}{3} \\ \frac{1}{6} \end{bmatrix} \tag{10}$$

$$\left\{ u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, u_2 = \begin{bmatrix} \frac{24}{14} \\ \frac{6}{14} \\ \frac{-12}{14} \end{bmatrix}, u_3 = \begin{bmatrix} \frac{1}{6} \\ \frac{-1}{3} \\ \frac{1}{6} \end{bmatrix} \right\} \tag{11}$$

Finally, we can normalize each vectors: u_1, u_2, u_3 to form an orthgonal matrix.