

1 Find the projection from w onto v .

Dot product of two vectors

$$w \cdot v = \|w\| \|v\| \cos \theta$$

Compute the magnitude of projection from w to v

$$\frac{w \cdot v}{\|v\|} = \|w\| \cos \theta$$

Multiply the unit vector of v

$$\begin{aligned} \text{proj}_v w &= \frac{w \cdot v}{\|v\|} \frac{v}{\|v\|} \\ \text{proj}_v w &= \frac{\langle w, v \rangle}{\langle v, v \rangle} v \end{aligned}$$

Given vectors: v_1, v_2, v_3 find the orthogonal basis for the three vectors.

$$\begin{aligned} a_1 &= v_1 = \frac{\langle a_1, v_1 \rangle}{\langle a_1, v_1 \rangle} a_1 & e_1 &= \frac{a_1}{\|a_1\|} \\ a_2 &= v_2 - \frac{\langle a_1, v_2 \rangle}{\langle a_1, a_1 \rangle} a_1 & e_2 &= \frac{a_2}{\|a_2\|} \\ a_3 &= v_3 - \left(\frac{\langle a_1, v_3 \rangle}{\langle a_1, a_1 \rangle} a_1 + \frac{\langle a_2, v_3 \rangle}{\langle a_2, a_2 \rangle} a_2 \right) & e_3 &= \frac{a_3}{\|a_3\|} \end{aligned}$$

$$\begin{aligned} \text{proj}_{w_1} v_2 &= \frac{\langle u_1, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 & \text{where } w_1 &= \text{Span}\{u_1\} \\ \text{proj}_{w_2} v_3 &= \frac{\langle u_1, v_3 \rangle}{\langle v_3, v_3 \rangle} v_3 + \frac{\langle u_2, v_3 \rangle}{\langle v_3, v_3 \rangle} v_3 & \text{where } w_2 &= \text{Span}\{u_1, u_2\} \\ v_1 &= \frac{\langle v_1, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 \\ v_2 &= a_2 + \frac{\langle v_1, v_2 \rangle}{\langle v_1, v_1 \rangle} v_1 \\ v_3 &= a_3 + \left(\frac{\langle v_1, v_3 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle v_2, v_3 \rangle}{\langle v_2, v_2 \rangle} v_2 \right) \end{aligned}$$

Project v_1, v_2, v_3 onto new **orthogonal basis**: $\{e_1, e_2, e_3\}$

$$\begin{aligned} v_1 &= \frac{\langle e_1, v_1 \rangle}{\langle e_1, e_1 \rangle} e_1 & \text{project } v_1 & \text{onto } e_1 \\ v_2 &= \frac{\langle e_1, v_2 \rangle}{\langle e_1, e_1 \rangle} e_1 + \frac{\langle e_2, v_2 \rangle}{\langle e_2, e_2 \rangle} e_2 & \text{project } v_2 & \text{onto } e_1, e_2 \\ v_3 &= \frac{\langle e_1, v_3 \rangle}{\langle e_1, e_1 \rangle} e_1 + \frac{\langle e_2, v_3 \rangle}{\langle e_2, e_2 \rangle} e_2 + \frac{\langle e_3, v_3 \rangle}{\langle e_3, e_3 \rangle} e_3 & \text{project } v_3 & \text{onto } e_1, e_2, e_3 \end{aligned}$$

Simplify a bit with $\langle e_i, e_i \rangle = 1$

$$\begin{aligned} v_1 &= \langle e_1, v_1 \rangle e_1 \\ v_2 &= \langle e_1, v_2 \rangle e_1 + \langle e_2, v_2 \rangle e_2 \\ v_3 &= \langle e_1, v_3 \rangle e_1 + \langle e_2, v_3 \rangle e_2 + \langle e_3, v_3 \rangle e_3 \end{aligned}$$

Write it in dot product form

$$\begin{aligned} v_1 &= \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} \langle e_1, v_1 \rangle \\ 0 \\ 0 \end{bmatrix} \\ v_2 &= \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} \langle e_1, v_2 \rangle \\ \langle e_2, v_2 \rangle \\ 0 \end{bmatrix} \\ v_3 &= \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} \langle e_1, v_3 \rangle \\ \langle e_2, v_3 \rangle \\ \langle e_3, v_3 \rangle \end{bmatrix} \end{aligned}$$

Combine it in matrix form

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} \langle e_1, v_1 \rangle & \langle e_1, v_2 \rangle & \langle e_1, v_3 \rangle \\ 0 & \langle e_2, v_2 \rangle & \langle e_2, v_3 \rangle \\ 0 & 0 & \langle e_3, v_3 \rangle \end{bmatrix}$$

2 Show all the diagonal entries in R are positive if M is singular matrix

$$v_k = a_k + \langle e_1, v_k \rangle e_1 + \cdots + \langle e_{k-1}, v_k \rangle e_{k-1} = a_k + \sum_{j=1}^{k-1} \langle e_j, v_{k-1} \rangle e_j$$

$$\langle e_k, v_k \rangle = \left\langle e_k, a_k + \sum_{j=1}^{k-1} \langle e_j, v_k \rangle e_j \right\rangle$$

$$\langle e_k, v_k \rangle = \langle e_k, a_k \rangle + \left\langle e_k, \sum_{j=1}^k \langle e_j, v_k \rangle e_j \right\rangle \quad \because \text{linearity of inner product}$$

$$\langle e_k, v_k \rangle = \langle e_k, a_k \rangle \quad \because \langle e_i, e_j \rangle = 0 \text{ if } i \neq j$$

$$\langle e_k, a_k \rangle = \left\langle \frac{a_k}{\|a_k\|}, a_k \right\rangle$$

$$\langle e_k, a_k \rangle = \frac{1}{\|a_k\|} \langle a_k, a_k \rangle \quad \because \text{linearity on each component}$$

$$\langle e_k, a_k \rangle = \|a_k\|$$

From above, we show all the diagonal entries are positive in R if M is singular matrix.

3 QR decomposition

Given none singular matrix $A \in M$, there exists unique pair of unitary matrix $Q \in M$ and upper triangle $R \in M$ with positive diagonal entries such that

$$A = QR$$

Need a proof, Why there are all positive diagonal entries in the upper triangle matrix R ?

4 Gram Schmidt Process

Find the orthgonal basis to represent v_1, v_2, v_2

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix} \right\}$$
$$u_1 = v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$u_2 = v_2 - \text{proj}_{w_1} v_2 = \frac{\langle u_1, v_2 \rangle}{\langle v_1, v_1 \rangle} v_1 \quad (1)$$

$$= \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\rangle}{\left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\|} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad (2)$$

$$= \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \frac{4 + 10 + 18}{14} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad (3)$$

$$= \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \frac{32}{14} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad (4)$$

$$= \begin{bmatrix} 4 - \frac{32}{14} \\ 5 - \frac{64}{14} \\ 6 - \frac{96}{14} \end{bmatrix} = \begin{bmatrix} \frac{56}{14} - \frac{32}{14} \\ \frac{70}{14} - \frac{64}{14} \\ \frac{84}{14} - \frac{96}{14} \end{bmatrix} = \begin{bmatrix} \frac{24}{14} \\ \frac{6}{14} \\ \frac{-12}{14} \end{bmatrix} \quad (5)$$

$$u_3 = v_3 - (\text{proj}_{w_1} v_3 + \text{proj}_{w_2} v_3) \quad (6)$$

$$= u_3 - \left(\frac{\langle u_1, v_3 \rangle}{\|u_1\|} \cdot \frac{u_1}{\|u_1\|} + \frac{\langle u_2, v_3 \rangle}{\|u_2\|} \cdot \frac{u_2}{\|u_2\|} \right) \quad (7)$$

$$= \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix} - \left(\frac{\left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix} \right\rangle}{\left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\|} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \frac{\left\langle \begin{bmatrix} \frac{24}{14} \\ \frac{6}{14} \\ \frac{-12}{14} \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix} \right\rangle}{\left\| \begin{bmatrix} \frac{24}{14} \\ \frac{6}{14} \\ \frac{-12}{14} \end{bmatrix} \right\|} \cdot \begin{bmatrix} \frac{24}{14} \\ \frac{6}{14} \\ \frac{-12}{14} \end{bmatrix} \right) \quad (8)$$

$$= \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix} - \left(\begin{bmatrix} \frac{53}{14} \\ \frac{53.2}{14} \\ \frac{53.3}{14} \end{bmatrix} + \begin{bmatrix} \frac{64}{21} \\ \frac{16}{21} \\ \frac{-32}{21} \end{bmatrix} \right) \text{ where } \text{lcm}(14, 50) = 2 \cdot 7 \cdot 5 \cdot 5 \quad (9)$$

$$= \begin{bmatrix} \frac{1}{6} \\ \frac{-1}{3} \\ \frac{1}{6} \end{bmatrix} \quad (10)$$

$$\left\{ u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, u_2 = \begin{bmatrix} \frac{24}{14} \\ \frac{6}{14} \\ \frac{-12}{14} \end{bmatrix}, u_3 = \begin{bmatrix} \frac{1}{6} \\ \frac{-1}{3} \\ \frac{1}{6} \end{bmatrix} \right\} \quad (11)$$

Finally, we can normalize each vectors: u_1, u_2, u_3 to form an orthogonal matrix.