# 1 Introduction

Affine space is always mysterious for me and it is very confusing and not well defined in High school and University mathematical lecture. It makes your more miserable after you have learned Vector space

**Definition 1.** *Affine space*

*In Vector space, we don't have origin but we have zero Vector*

$$v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

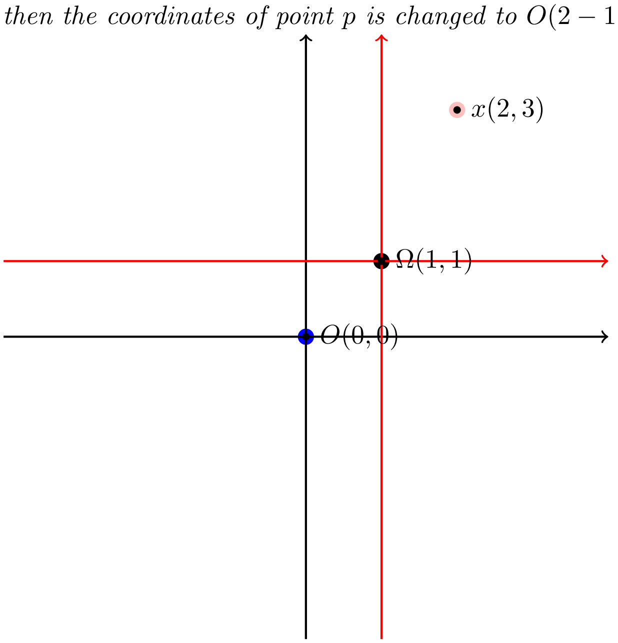
*One of the most important property of Vector Space is Linear Combination, any vector can be represent by linear combination of their basis*

$$v = \begin{bmatrix} 2 \\ 3 \end{bmatrix} v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} v = 2v_1 + 3v_2 \text{ where } \beta = \{v_1, v_2\} \text{ is called basis}$$

*In Vector Space, any vector can be represent by linear combination uniquely from above example.*
*How can we use Vector space concept and try to use similar "Linear Combination" in points*

*let's see a simple example from high school.*
*A point $p$ is something like $p = (2,3)$ and $(2,3)$ is called the coordinates of point $p$, how can we define the coordinates of point? There is something called basis vector for the coordinates system. For example, $\mathcal{B} = \{e_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, e_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$ is used in standard basis in coordinates system. The coordinates of point is the distance from Origin to the intersection which point $p$ is perpendicular to one of the basis vector: e.g. $e_1$ so coordinates system is defined in two things: Origin and $\mathcal{B} = \{e_1, e_2\}$*

*Time for example*
*Given an Origin $O(0,0)$ and basis vectors $\mathcal{B}$, point $p$ has coordinates $(2,3)$. If Origin is changed to $\Omega(1,1)$, then the coordinates of point $p$ is changed to $O(2-1, 3-1) = (1,2)$*



*It shows points are dependent on frame. If we change frame from $(O, (e_1, e_2))$ to $(\Omega, (e_1, e_2))$ then the coordinates of point $p$ is changed from $p = (2, 3)$ to $p = (1, 2)$.*
*In Vector space, normally there is no concept of Origin because Origin is referred to as point and there is no definition of point in Vector space.*
*Informal defintion of Vector Space $V$ has many vectors and with a field $\mathbb{K}$*

*Close addition and scalar multiplication.* $\vec{u}, \vec{v} \in V, \alpha \in \mathbb{K}$

$$\vec{v} + \vec{u} \in V$$
$$\alpha\vec{u} \in V$$

*Identity element: There is $\vec{0}$ such as for all vectors $\vec{u}$*

$$\vec{u} + \vec{0} = \vec{u}$$

*Commutivity*

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$

*For all vectors $\vec{v}$, there is unique vector $\vec{u}$ such as*

$$\vec{v} + \vec{u} = \vec{0} \quad \text{where } \vec{u} \text{ can be written as } \vec{-v}$$

*Associativity: If $\vec{u}, \vec{v}, \vec{w} \in V$*

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$

*Distributivity on vector: $\alpha \in \mathbb{K}, \vec{u}, \vec{v} \in V$*

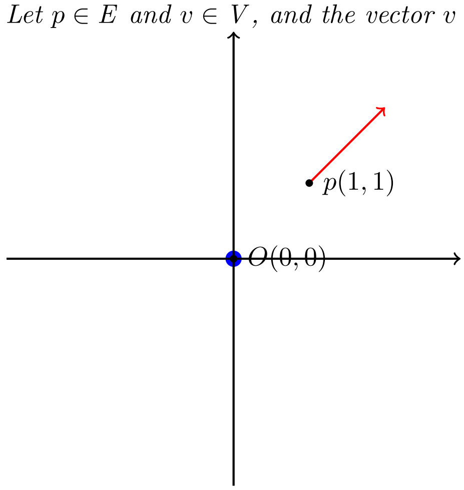
$$\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$$

*Distributivity on scalar: $\alpha, \beta \in \mathbb{K}, \vec{v} \in V$*

$$(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$$

*What is happening when people said given two points $a, b$ and a vector is defined as $\overrightarrow{ab} = b - a$. The concept is called Affine Space and IT IS NOT IN Vector Space*

*Affine Space: Given a set of points in $E$, there is no structure in $E$. And a Vector Space $V$.*
*Let $p \in E$ and $v \in V$, and the vector $v$ moves the point $p$ along $v$ direction to the tip of of vector $v$*



*Better Definition: Given a set of points $E$, a Vector Space $V$ and a action[translation] $+ : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ and $\forall a = (x, y) \in E$ and $\forall \vec{v} \in V$, satifying following conditions*

*There is unique point $b \in E$ such as*

$$a(x, y) + \vec{v}(u, v) = b(x + u, y + v)$$

*Zero vector acts on a point*

$$a(x, y) + \vec{v}(0, 0) = b(x + 0, y + 0)$$

*Associativity*

$$(a + \vec{v}) + \vec{u} = a + (\vec{v} + \vec{u})$$

*Chasles's Identity*
*Given any three points $a, b, c \in E$ From the conditions of Affine Space*

$$b = a + \overrightarrow{ab} \tag{1}$$
$$c = b + \overrightarrow{bc} \tag{2}$$
$$c = a + \overrightarrow{ac} \tag{3}$$

*from (1) and (2)*

$$c = (a + \overrightarrow{ab}) + \overrightarrow{bc}$$

*from (3)*

$$\begin{aligned} a + \overrightarrow{ac} &= a + \overrightarrow{ab} + \overrightarrow{bc} \\ a + \overrightarrow{ac} &= a + (\overrightarrow{ab} + \overrightarrow{bc}) \\ \implies \overrightarrow{ac} &= (\overrightarrow{ab} + \overrightarrow{bc}) \end{aligned}$$

*from Chasles' Identity, and let $a = c$*

$$\overrightarrow{aa} = \overrightarrow{ab} + \overrightarrow{ba} \tag{4}$$

*from defintion*

$$\begin{aligned} a + \vec{0} &= a \\ a + \overrightarrow{aa} &= a \\ \implies \overrightarrow{aa} &= \vec{0} \end{aligned}$$

*from (4)*

$$\overrightarrow{ab} + \overrightarrow{ba} = \vec{0}$$

*Given any four points $a, b, c, d \in E$, from Chasles' identity*

$$\begin{aligned} \overrightarrow{ab} + \overrightarrow{bc} &= \overrightarrow{ad} + \overrightarrow{dc} = \overrightarrow{ac} \\ \overrightarrow{ab} = \overrightarrow{ad} &\iff \overrightarrow{bc} + \overrightarrow{dc} \quad \square \end{aligned}$$