1 Complex Number 1

Addition

$$\begin{aligned}
\text{let } z_1 &= x_1 + y_1 i \\
\text{let } z_2 &= x_2 + y_2 i \\
z &= z_1 + z_2 \\
z &= (x_1 + x_2) + (y_1 + y_2)i
\end{aligned}$$

Multiplication

$$\begin{aligned}
z &= z_1 z_2 \\
z &= (x_1 + y_1 i)(x_2 + y_2 i) \\
z &= (x_1 y_2 + x_2 y_1) + (x_1 x_2 - y_1 y_2)i
\end{aligned}$$

Division

$$\begin{aligned}
z &= \frac{z_1}{z_2} \\
z &= \frac{x_1 + y_1 i}{x_2 + y_2 i} \\
z &= \frac{(x_1 + y_1 i)(x_2 - y_2 i)}{(x_2 + y_2 i)(x_2 - y_2 i)} \\
z &= \frac{(x_1 x_2 + y_1 y_2) + (-x_1 y_2 + x_2 y_1)i}{x_2^2 + y_2^2} \\
z &= \frac{(x_1 x_2 + y_1 y_2)}{x_2^2 + y_2^2} + \frac{(-x_1 y_2 + x_2 y_1)}{x_2^2 + y_2^2} i
\end{aligned}$$

Conjugation

$$\begin{aligned}
\overline{z} &= x_1 - y_1 i \\
z\overline{z} &= (x_1 + y_1 i)(x_1 - y_1 i) \\
z\overline{z} &= x_1^2 + y_1^2 \\
z\overline{z} &= |z|^2
\end{aligned}$$

Matrix represents Complex Number 1

$$
\begin{aligned}
\text{let } z_1 &= x_1 + y_1 i \\
\text{let } z_2 &= x_2 + y_2 i \\
z_1 z_2 &= (x_1 + y_1 i)(x_2 + y_2 i) \\
&= (x_1x_2 - y_1y_2) + (y_1x_2 + x_1y_2)i \\
(x_1x_2 - y_1y_2) &= \begin{bmatrix} x_1 & y_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \\
(y_1x_2 + x_1y_2) &= \begin{bmatrix} y_1 & x_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}
\end{aligned}
$$

Combine two dot products, we have

$$
\begin{aligned}
\mathcal{M} &= \begin{bmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{bmatrix} \\
\vec{V} &= \mathcal{M} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \\
(x_1x_2 - y_1y_2) + (y_1x_2 + x_1y_2)i &= \begin{bmatrix} 1 & i \end{bmatrix} \vec{V}
\end{aligned}
$$

Identity, $1 \in \mathbb{C}$ can be represented as identity matrix

$$
\begin{aligned}
x_1 + y_1 i &= 1, \quad \text{where } x_1 = 1, y_1 = 0 \\
1 = \mathcal{M} &= \begin{bmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

Imaginary unit, $i \in \mathbb{C}$ can be represented as matrix

$$
\begin{aligned}
x_1 + y_1 i &= i \quad \text{where } x_1 = 0, y_1 = 1 \\
i = \mathcal{M} &= \begin{bmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\end{aligned}
$$

Show $i^2 = -1$

$$
\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

Conjugate $\overline{z}$ corresponds to transpose of the matrix

$$
\begin{aligned}
z = x + yi &= \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \\
\overline{z} = x - yi &= \begin{bmatrix} x & -y \\ y & x \end{bmatrix}^T \\
x - yi &= \begin{bmatrix} x & y \\ -y & x \end{bmatrix}
\end{aligned}
$$

The square of absolute values of z corresponds to the determinant of the matrix

$$|z|^2 = \det \begin{bmatrix} x & -y \\ y & x \end{bmatrix} = x^2 + y^2$$

Polar form in Complex Number which has matrix form

$$\begin{aligned} e^{i\theta} &= \cos(\theta) + i\sin(\theta) \\ e^{-i\theta} &= \cos(\theta) + i\sin(-\theta) \\ e^{i\theta} + e^{-i\theta} &= 2\cos(\theta) \\ \frac{e^{i\theta} + e^{-i\theta}}{2} &= \cos(\theta) \end{aligned}$$

Polar form in Complex Number which has matrix form

$$\cos(\beta) + \sin(\beta)i \implies \begin{bmatrix} \sin(\beta) & -\cos(\beta) \\ \cos(\beta) & \sin(\beta) \end{bmatrix}$$

Proof. $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

From Euler's formular

$$\cos\theta + i\sin\theta = e^{i\theta} \tag{1}$$
$$\cos\theta - i\sin\theta = e^{-i\theta} \tag{2}$$

Adding and substracting (1) and (2)

$$\begin{aligned} (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) &= e^{i\theta} + e^{i\theta} \\ (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta) &= e^{i\theta} - e^{i\theta} \\ \sin\theta &= \frac{e^{i\theta} - e^{i\theta}}{2i} \\ \cos\theta &= \frac{e^{i\theta} + e^{i\theta}}{2} \end{aligned}$$

□

Euler's formula can be represented in matrix form

$$\begin{aligned} e^{i\theta} &= \cos(\theta) + i\sin(\theta) \\ \exp(\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}) &= \begin{bmatrix} \sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix} \end{aligned}$$

Above matrix is one way to represent Complex number, other matrix such as

$$\mathcal{J} = \begin{bmatrix} p & q \\ r & -p \end{bmatrix}, \quad p^2 + rq + 1 = 0$$

has the properties that its square is the negative of the identity matrix: $\mathcal{J}^2 = -I$

Proof. $\mathcal{J}^2 = -I$

$$\begin{aligned} \mathcal{J}^2 &= \begin{bmatrix} p & q \\ r & -p \end{bmatrix} \begin{bmatrix} p & q \\ r & -p \end{bmatrix} \\ &= \begin{bmatrix} p^2 + rq & pq - pq \\ rp - rp & p^2 + rq \end{bmatrix} \\ &= -I \end{aligned}$$

□

2 Quaternion

2.1 Addition and Multiplication

Add two quaternions acts component-wise, let two quaternions $\mathbf{q}$ and $\mathbf{p}$

$$\begin{aligned} \mathbf{q} &= q_0 + q_1 i + q_2 j + q_3 k = q_0 + \vec{q} \\ \mathbf{p} &= p_0 + p_1 i + p_2 j + p_3 k = p_0 + \vec{p} \end{aligned}$$

We have

$$\begin{aligned} \mathbf{q} + \mathbf{p} &= (q_0 + q_1 i + q_2 j + q_3 k) + (p_0 + p_1 i + p_2 j + p_3 k) \\ &= (q_0 + p_0) + (q_1 + p_1)i + (q_2 + p_2)j + (q_3 + p_3 k) \end{aligned}$$

The product of two quaternions satisfies these fundamental rules introducted by Hamilton:

$$\begin{aligned} &\hat{i}^2 = \hat{j}^2 = \hat{k}^2 = \hat{i}\hat{j}\hat{k} = -1 \\ &\hat{i}\hat{j} = \hat{k}, \quad \hat{j}\hat{k} = \hat{i}, \quad \hat{k}\hat{i} = \hat{j} \\ &\mathbf{qp} = (q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{k})(p_0 + p_1\hat{i} + p_2\hat{j} + p_3\hat{k}) \\ &\quad = [q_0 + (q_1\hat{i} + q_2\hat{j} + q_3\hat{k})][p_0 + (p_1\hat{i} + p_2 j + p_3\hat{k})] \\ &\quad = q_0 p_0 + q_0(p_1\hat{i} + p_2\hat{j} + p_3\hat{k}) + p_0(q_1\hat{i} + q_2\hat{j} + q_3\hat{k}) + \\ &\qquad (q_1\hat{i} + q_2\hat{j} + q_3\hat{k})(p_1\hat{i} + p_2\hat{j} + p_3\hat{k}) \\ &\quad = (q_0 + \vec{q})(p_0 + \vec{p}) \quad \text{where} \quad \vec{q} = q_1\hat{i} + q_2\hat{j} + q_3\hat{k} \quad \vec{p} = p_1\hat{i} + p_2\hat{j} + p_3\hat{k} \\ &\quad = q_0 p_0 + q_0\vec{p} + p_0\vec{q} - \vec{q} \cdot \vec{p} + \vec{q} \times \vec{p} \end{aligned}$$

Conjugation

$$\mathbf{q}^* = q_0 - q_1 i - q_2 j - q_3 k$$

Unit Quaternion

$$\mathbf{q} = q_0 + q_1 i + q_2 j + q_3 k = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

Inverse Quaternion

$$\mathbf{q}^{-1} = \frac{\mathbf{q}^*}{|\mathbf{q}|}$$

$$\text{If } |\mathbf{q}| = 1 \text{ then } \mathbf{q}^* = \mathbf{q}^{-1} \quad \mathbf{q}\mathbf{q}^* = 1$$

Proof. $\mathbf{q}\mathbf{q}^* = 1$

$$\begin{aligned}
|\mathbf{q}| = 1 &\implies |q_0|^2 + |\vec{q}|^2 = 1 \\
\mathbf{q}\mathbf{q}^* &= (q_0 + \vec{q})(q_0 - \vec{q}) \\
&= q_0 q_0 - 2q_0\vec{q} + \vec{q}\cdot\vec{q} + \vec{q}\times\vec{q} \\
&= |q_0|^2 + |\vec{q}|^2 \\
&= 1
\end{aligned}$$

□

Norm of Quaternion

$$\mathbf{q}\mathbf{q}^* = |\mathbf{q}|^2$$

Multiplication

$$\begin{aligned}
\mathbf{q}\mathbf{p} &= (q_0 + q_1 i + q_2 j + q_3 k)(p_0 + p_1 i + p_2 j + p_3 k) \\
&= [q_0 + (q_1 i + q_2 j + q_3 k)][p_0 + (p_1 i + p_2 j + p_3 k)] \\
&= q_0 p_0 + q_0(p_1 i + p_2 j + p_3 k) + p_0(q_1 i + q_2 j + q_3 k) + (q_1 i + q_2 j + q_3 k)(p_1 i + p_2 j + p_3 k) \\
&= (q_0 + \vec{q})(p_0 + \vec{p}) \\
&= q_0 p_0 + q_0\vec{p} + p_0\vec{q} - \vec{q}\cdot\vec{p} + \vec{q}\times\vec{p}
\end{aligned}$$

Multiplication table for $(1, i, j, k)$

	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

Multiplication table for $(p_0, p_1, p_2, p_3) \quad (q_0, q_1, q_2, q_3)$

	p_0	$p_1\mathrm{i}$	$p_2\mathrm{j}$	$p_3\mathrm{k}$
q_0	q_0p_0	$q_0p_1\mathrm{i}$	$q_0p_2\mathrm{j}$	$q_0p_3\mathrm{k}$
$q_1\mathrm{i}$	$p_0q_1\mathrm{i}$	$-q_1p_1$	$q_1p_2\mathrm{k}$	$-q_1p_3\mathrm{j}$
$q_2\mathrm{j}$	$p_0q_2\mathrm{j}$	$-q_2p_1\mathrm{k}$	$-q_2p_2$	$q_2p_3\mathrm{i}$
$q_3\mathrm{k}$	$p_0q_3\mathrm{k}$	$q_3p_1\mathrm{j}$	$-q_3p_2\mathrm{i}$	$-q_3p_3$

$\Longrightarrow p_0q_0 + p_0\vec{q} + q_0\vec{p} - \vec{q}\vec{p} + \vec{q} \times \vec{p}$

Multiplication table contains p_0q_0 and $p_0\vec{q}$, $q_0\vec{p}$

	p_0	$p_1\mathrm{i}$	$p_2\mathrm{j}$	$p_3\mathrm{k}$
q_0	q_0p_0	$q_0p_1\mathrm{i}$	$q_0p_2\mathrm{j}$	$q_0p_3\mathrm{k}$
$q_1\mathrm{i}$	$p_0q_1\mathrm{i}$			
$q_2\mathrm{j}$	$p_0q_2\mathrm{j}$			
$q_3\mathrm{k}$	$p_0q_3\mathrm{k}$			

$\Longrightarrow p_0q_0 + p_0\vec{q} + q_0\vec{p}$

Multiplication table contains inner product of $\vec{q}$ and $\vec{p}$

	1	i	j	k
1				
i		$-q_1p_1$		
j			$-q_2p_2$	
k				$-q_3p_3$

$\Longrightarrow -\vec{p} \cdot \vec{q}$

Multiplication table contains outer product of $\vec{p}$ and $\vec{q}$

	1	i	j	k
1				
i			$q_1p_2\mathrm{k}$	$-q_1p_3\mathrm{j}$
j		$-q_2p_1\mathrm{k}$		$q_2p_3\mathrm{i}$
k		$q_3p_1\mathrm{j}$	$-q_3p_2\mathrm{i}$	

$\Longrightarrow \vec{p} \times \vec{q}$

$$(q_0 + q_1 i + q_2 j + q_3 k)(p_0 + p_1 i + p_2 j + p_3 k) = [1, i, j, k] \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$M_j = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$M_k = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$(\pm M_1, \pm M_i, \pm M_j, \pm M_k)(\pm 1, \pm i, \pm j, \pm k) \text{ forms a group}$$

3 Derive Tayor Series

Q: What problem that we try to solve?
A: We try to come up an equation to approximate $f(x) = e^x$
Q: What kind of equation that we can think of?
A: we can think about a polynomial e.g. $g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 [+ \cdots]$
Q: Approximate $g(x) \approx f(x) = e^x$? Sure:0
Q: What I should do next? come up a table and see what is on the table?

x	0	1	2	3
$f(x)$	e^0	e^1	e^2	e^3
$g(x)$	a_0	$a_0 + a_1 + a_2 + a_3$	$a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 + a_3 \cdot 2^3$	$a_0 + a_1 3 + a_2 \cdot 3^2 + a_3 \cdot 3^3$

We have four equations and four unknows and $a_0 = e^0 = 1$. Therefore, only a_1, a_2, a_3 need to be solved.

$$
\begin{aligned}
e^0 &= a_0 \\
e^1 &= a_0 + a_1 + a_2 + a_3 \\
e^2 &= a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 + a_3 \cdot 2^3 \\
e^3 &= a_0 + a_1 \cdot 3 + a_2 \cdot 3^2 + a_3 \cdot 3^3
\end{aligned}
$$

$$
\begin{aligned}
e^1 - a_0 &= a_1 + a_2 + a_3 \\
e^2 - a_0 &= a_1 \cdot 2 + a_2 \cdot 2^2 + a_3 \cdot 2^3 \\
e^3 - a_0 &= a_1 \cdot 3 + a_2 \cdot 3^2 + a_3 \cdot 3^3
\end{aligned}
$$

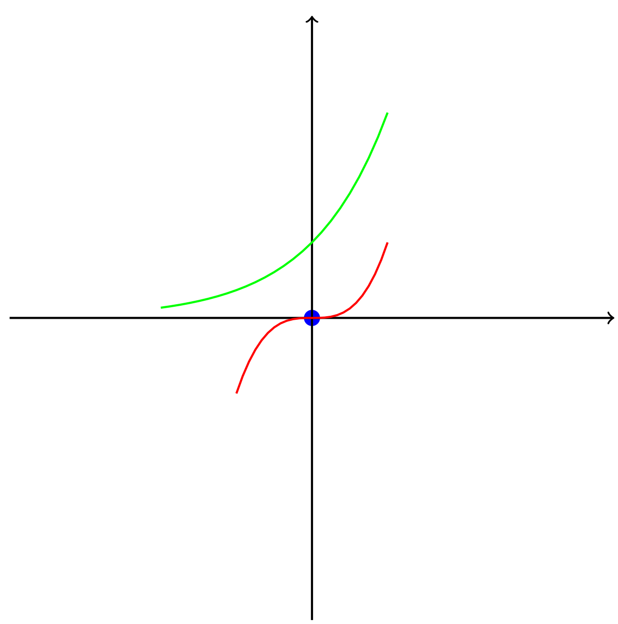
$$
\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2^2 & 2^3 \\ 3 & 3^2 & 3^3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} e^1 - a_0 \\ e^2 - a_0 \\ e^3 - a_0 \end{bmatrix}
$$

try to solve for a_0, a_1, a_2. However, polynomial $f(x)$ passes four points:

$$
(0, e^0) \quad (1, e^1) \quad (2, e^2) \quad (3, e^3)
$$

Assume we solve the matrix and find a_0, a_1, a_2, a_3

$$
f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3
$$



$$(x_1 + y_1 i)(x_2 + y_2 i) = x_1 x_2 + x_1 y_2 i + x_2 y_1 i - y_1 y_2 = x_1 x_2 - y_1 y_2 + (x_1 y_2 + x_2 y_1) i$$