6 Paths, Loops, Cylinders, Suspensions, . . .

In certain situations (such as descent theorems for fundamental groups à la van Kampen) it is much more elegant, even indispensable for understanding something, to work with fundamental groupoids with respect to a suitable packet of base points....
—Alexander Grothendieck (1997)

Introduction. Chapter 0 introduced the categorical theme that *objects are completely determined by their relationships with other objects*. It has origins in theorem 0.1, a corollary of the Yoneda lemma stating that objects X and Y in a category are isomorphic if and only if the corresponding sets $\mathsf{Top}(Z, X)$ and $\mathsf{Top}(Z, Y)$ are isomorphic for all objects Z. This notion of gleaning information by "probing" one object with another is used extensively throughout algebraic topology where a famously useful probing object for topological spaces is the circle S^1 (and the sphere S^n, more generally).

A continuous map $S^1 \to X$ is a loop within the space X, so comparing $\mathsf{Top}(S^1, X)$ and $\mathsf{Top}(S^1, Y)$ amounts to comparing the set of all loops within X and Y. In practice, however, these are massively complicated sets. To declutter the situation, it's better to consider homotopy classes of loops, where loops aren't distinguished if one can be reshaped continuously to another. The question arises: "What are the most 'fundamental' loops in X, and do they differ from those in Y?" To simplify things further, it helps to consider only those loops that start and end at a fixed point in X. The set of such homotopy classes of loops forms a group called the *fundamental group* of X at the chosen point, which defines the object assignment of a functor $\mathsf{Top} \to \mathsf{Grp}$. And with this, the meaning of "algebraic topology" begins to come to life.

These ideas motivate a more general categorical study of pointed topological spaces and homotopy classes of maps between them. That is the goal of this chapter. Along the way, we'll encounter an interesting zoo of examples of such spaces; natural maps between them; and various adjunctions involving paths, loops, cylinders, cones, suspensions, wedges, and smashes. We open with a brief refresher in section 6.1 on homotopies and alternate ways of viewing them. In section 6.2 we motivate homotopy classes of based loops as a special case of a general construction called the fundamental groupoid. Focusing on "pointed" topological spaces in section 6.3 results in the fundamental group. We then analyze the pointed version of the product-hom adjunction, called the smash-hom adjunction, in section 6.4, and specializing yet further we obtain the suspension-loop adjunction in section 6.5. This adjunction and accompanying results on fibrations in section 6.6 provide a wonderfully short proof that the fundamental group of the circle is $\mathbb{Z}$. In section 6.6.3 we will showcase four