4

GROUPS AND CATEGORIES

This chapter is devoted to some of the various connections between groups and categories. If you already know the basic group theory covered here, then this will give you some insight into the categorical constructions we have learned so far; and if you do not know it yet, then you will learn it now as an application of category theory. We will focus on three different aspects of the relationship between categories and groups:

1. groups in a category,
2. the category of groups,
3. groups as categories.

4.1 Groups in a category

As we have already seen, the notion of a group arises as an abstraction of the automorphisms of an object. In a specific, concrete case, a group G may thus consist of certain arrows $g : X \to X$ for some object X in a category $\mathbf{C}$,

$$G \subseteq \mathrm{Hom}_{\mathbf{C}}(X, X)$$

But the abstract group concept can also be described directly as an object in a category, equipped with a certain structure. This more subtle notion of a "group in a category" also proves to be quite useful.

Let $\mathbf{C}$ be a category with finite products. The notion of a group in $\mathbf{C}$ essentially generalizes the usual notion of a group in **Sets**.

Definition 4.1. A *group* in $\mathbf{C}$ consists of objects and arrows as so:

$$\begin{tikzcd} G \times G \arrow[r, "m"] & G & G \arrow[l, "i"'] \\ & 1 \arrow[u, "u"] & \end{tikzcd}$$