Dual Spaces

Definition 1 (Dual Space) Let V be a finite dimensional vector space.

(a) A linear functional on V is a function $\vec{u}^* : V \to \mathbb{R}$ that is linear in the sense that

$$\vec{u}^*(\vec{v}+\vec{w}) = \vec{u}^*(\vec{v}) + \vec{u}^*(\vec{w}) \qquad \text{and} \qquad \vec{u}^*(\alpha\vec{v}) = \alpha\vec{u}^*(\vec{v})$$

for all $\vec{u}, \vec{w} \in V$ and all $\alpha \in \mathbb{R}$.

(b) The dual space V^* of the vector space V is the set of all linear functionals on V. It is itself a vector space with the operations

$$\begin{aligned}(\vec{u}^* + \vec{v}^*)(\vec{w}) &= \vec{u}^*(\vec{w}) + \vec{v}^*(\vec{w}) \\ (\alpha\vec{v}^*)(\vec{w}) &= \alpha\vec{v}^*(\vec{w})\end{aligned}$$

(c) If V has dimension d and has basis $\{\vec{e}_1, \cdots, \vec{e}_d\}$ then, by Problem 1, below, V^* also has dimension d and a basis for V^* (called the dual basis of $\{\vec{e}_1, \cdots, \vec{e}_d\}$) is $\{\vec{e}_1^*, \cdots, \vec{e}_d^*\}$ where, for each $1 \le j \le d$

$$\vec{e}_j^*\big(\alpha_1\vec{e}_1 + \cdots + \alpha_d\vec{e}_d\big) = \alpha_j$$

That is,

$$\vec{e}_j^*(\vec{e}_k) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \ne k \end{cases}$$

Problem 1 Let $\{\vec{e}_1, \cdots, \vec{e}_d\}$ be a basis for the vector space V and define, for each $1 \le j \le d$, $\vec{e}_j^*$ as in Definition 1.c. Prove that $\{\vec{e}_1^*, \cdots, \vec{e}_d^*\}$ is a basis for V^*.

Problem 2 Let W be a proper subspace of the finite dimensional vector space V and let $\vec{e}$ be an element of V that is not in W. Prove that there exists a linear functional $\vec{w}^* \in V^*$ such that $\vec{w}^*(\vec{e}) = 1$ and $\vec{w}^*(\vec{w}) = 0$ for all $\vec{w} \in W$.

Problem 3 Let V be a finite dimensional vector space. Each $\vec{v} \in V$ has naturally associated to it the linear functional $\vec{v}^{**} \in (V^*)^*$ (i.e. $\vec{v}^{**}$ is a linear functional on the dual space V^*) defined by

$$\vec{v}^{**}(\vec{w}^*) = \vec{w}^*(\vec{v}) \qquad \text{for all } \vec{w}^* \in V^*$$

Prove that the map $\vec{v} \mapsto \vec{v}^{**}$ is an isomorphism (i.e. 1–1, onto and linear) between the vector spaces V and $(V^*)^*$. Consequently, for finite dimensional vector spaces, one usually thinks of V and $(V^*)^*$ as being the same.

Lemma 2 *Let V be a vector space of dimension $d < \infty$. Let $\vec{v}^* \in V^*$. Its kernel*

$$K = \big\{\ \vec{v} \in V \ \big|\ \vec{v}^*(\vec{v}) = 0\ \big\}$$

is a linear subspace of V. If $\vec{v}^*$ is not the zero functional, then the dimension of K is exactly $d-1$.

