

# Cross Ratios

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## 1 Projective Geometry and Cross ratios

**Definition 1.** The *projective plane*  $\mathbb{P}^2$  is the set of lines through an observation point  $O$  in three dimensional space. A *projective line*  $l$  is a plane passing through  $O$ , and a *projective point*  $P$  is a line passing through  $O$ . If the line defining  $P$  is contained in the plane defining  $l$ , we say that  $P \in l$ .

If  $\mathbb{A}^2$  is an ordinary plane which does not pass through  $O$ , then we can identify most projective points of  $\mathbb{P}^2$  with ordinary points on  $\mathbb{A}^2$  by taking the intersection of the line defining the projective point with  $\mathbb{A}^2$ . The projective line which is defined by a plane passing through  $O$  and parallel to  $\mathbb{A}^2$  is called the *line at infinity*, or the *horizon line*. Projective points contained in the line at infinity are called *infinite points*.

If we take  $O = (0, 0, 0)$ , then we can put coordinates on the projective plane as follows. Every projective point  $P$  is a line through  $O$  and some other point  $(p, q, r)$ . Then every point on the line defining  $P$  is of the form  $(\lambda p, \lambda q, \lambda r)$  for some  $\lambda$ . We write  $P = [p : q : r]$ , where the colons indicate that we only care about the ratios of the coordinates. If  $\mathbb{A}^2$  is the plane  $z = 1$ , then the ordinary point on  $\mathbb{A}^2$  corresponding to  $P$  is  $(\frac{p}{r}, \frac{q}{r}, 1)$ , or if we ignore the  $z$ -coordinate it is just  $(\frac{p}{r}, \frac{q}{r})$ . If  $r = 0$ , then  $P$  is an infinite point with *slope*  $\frac{q}{p}$ .