2 Connectedness and Compactness

One of the classical aims of topology is to classify topological spaces by their topological type, or in other terms to find a complete set of topological invariants.
—Samuel Eilenberg (1949)

Introduction. In chapter 1, we discussed four main constructions of topological spaces: subspaces, quotients, products, and coproducts. In this chapter, we'll see how these constructions interact with three main topological properties: connectedness, Hausdorff, and compactness. That is, are subspaces of compact spaces also compact? Is the quotient of a Hausdorff space itself Hausdorff? Are products of connected spaces also connected? Is a union of connected spaces connected? We'll explore these questions and more in the pages to come.

Section 2.1 contains a survey of basic notions, theorems, and examples of connectedness. It also includes a statement and categorical proof of the one-dimensional version of Brouwer's well-known fixed-point theorem. Section 2.2 contains the Hausdorff property, though we'll keep the discussion brief. The Hausdorff property becomes much richer once it's combined with compactness, which is the content of section 2.3. The same section also introduces three familiar theorems—the Bolzano-Weierstrass theorem, the Heine-Borel theorem, and Tychonoff's theorem.

2.1 Connectedness

We'll begin with a discussion of the main ideas about connectedness. The definitions are collected up front and the main results follow. The proofs are mostly left as exercises, but they can be found in most any classic text on topology, such as Willard (1970), Munkres (2000), Kelley (1955), Lipschutz (1965).

2.1.1 Definitions, Theorems, and Examples

Definition 2.1 A topological space X is *connected* if and only if one of the following equivalent conditions holds:

(i) X cannot be expressed as the union of two disjoint nonempty open sets.

(ii) Every continuous function $f\colon X \to \{0, 1\}$ is constant, where $\{0, 1\}$ is equipped with the discrete topology.