Definition 1. *polynomial ring* $\mathbf{r}[x]$ *in* x *over the ring* $\mathbf{r}$ *is defined as set of expressions, called polynomials in* x*, of the form*

$$f(x) = a_0 + a_1x^1 + \cdots + a_mx^m$$

where $a_0, a_1, \ldots, a_n$*, the coefficients of* $p(x)$ *are elements of* $\mathbf{r}$*, and* x, x^2 *are symbols*

Definition 2. *let* f *be a field. by the ring of polynomial in the indeterminate,* x*, written as* $\mathbf{R}[x]$*, we mean the set of all symbols* $f(x) = a_0 + a_1x^1 + \cdots + a_mx^m$*, where* n *can be any nonnegative integer and where the coefficient* $a_0, a_1 + \cdots + a_n$ *are all in* f*. in order to make a ring out of* $\mathbf{f}[x]$*, we must be able to recognize when the two elements in it are equal, we must add and multiply element of* $\mathbf{f}[x]$ *so that the axiom defining the ring hold true for* $\mathbf{f}[x]$*.*

Definition 3. *if* $f(x) = a_0 + a_1x^1 + \cdots + a_mx^m$ *and* $g(x) = b_0 + b_1x^1 + \cdots + b_mx^m$ *are in* $\mathbf{f}[x]$*, then* $f(x) = g(x)$ *if and only if for every integer* $i \geq 0$*, such as* $a_i = b_i$

Definition 4. *if* $f(x) = \sum_{i=0}^{n} a_ix^i$ *and* $g(x) = \sum_{j=0}^{m} b_jx^j$*, then* $f(x) + g(x)$ *is equal*

$$\sum_{i=0}^{n} a_ix^i + \sum_{j=0}^{m} b_jx^j = \sum_{i=0}^{k} (a_i + b_j)x^k \qquad \textit{where } k = \max(n, m)$$

if $f(x)$ *or* $g(x)$ *do not contain the term* cx^t*, then assume* $c = 0, \quad k \geq t \geq 0$

Definition 5. *if* $f(x) = \sum_{i=0}^{n} a_ix^i$ *and* $g(x) = \sum_{j=0}^{m} b_jx^j$*, then* $f(x)g(x)$ *is equal*

$$\sum_{i=0}^{n} a_ix^i \sum_{j=0}^{m} b_jx^j = \sum_{i=0}^{n} \left(\sum_{j=0}^{m} a_ib_jx^{i+j} \right)$$

the definition say nothing more than: multiply two polynomials by multiplying out two symbols formally, use the relation $x^ix^j = x^{i+j}$ *and collect terms*

Definition 6. *the degree of nonzero polynomial is defined as the maximus power of a term with nonzero coefficients.*

Definition 7. *if* $f(x)$ *and* $g(x)$ *are nonzero polynomials in* $\mathbf{f}[x]$*, then*

$$\deg(f(x)g(x)) = \deg(f(x)) + \deg(g(x))$$

Proof. let $f(x) = \sum_{i=0}^{n} a_ix^i, a_n \neq 0$ and $g(x) = \sum_{j=0}^{m} b_jx^j, b_m \neq 0$ we have

$$\begin{aligned} \deg(f(x)) &= n \\ \deg(g(x)) &= m \end{aligned}$$

let $\alpha \in \{0 \ldots n\}, \alpha \neq n$ and $\beta \in \{0 \ldots m\}, \beta \neq m$

$$\begin{aligned} &\therefore \alpha < n \text{ and } \beta < m \\ &\implies \alpha + \beta < n + m \end{aligned}$$

from the definition of multiplication of two polynomials

$$f(x)g(x) = \sum_{i=0}^{n} a_ix^i \sum_{j=0}^{m} b_jx^j = \sum_{i=0}^{n} \left(\sum_{j=0}^{m} a_ib_jx^{i+j} \right)$$

we need to show $a_nb_m \neq 0$, from the definition

$$\begin{aligned} &a_n \neq 0 \\ &b_m \neq 0 \\ &\implies a_nb_m \neq 0 \quad \because f \text{ is a integral domain} \\ &\implies \text{ the maximus power of term is } a_nb_mx^{n+m} \\ &\implies \deg(f(x)g(x)) = n + m = \deg(f(x)) + \deg(g(x)) \end{aligned}$$

□

Proof. by induction □

Definition 8. *if* $f(x)$ *and* $g(x)$ *are nonzero element in* $\mathbf{f}[x]$*, then* $\deg(f(x)) \leq \deg(f(x)g(x))$