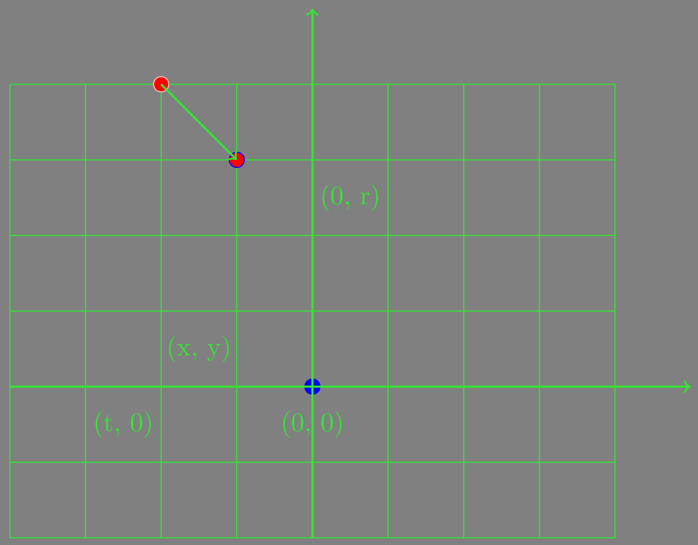
Given a point $p(1,1,1)$ and a vector $\vec{v}=(1,2,3)$.
A line can be defined as $u = p + t\vec{v}$ passing through $p(1,1,1)$ and perpendicular to $\vec{v}$ where $t \in \mathbb{R}$



$$\vec{w} = p'(x,y,z) - p(1,1,1) = \begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix}$$

$$[1,2,3] \cdot \vec{w} = 0$$

$$f(x,y,z) = x^2 + y^2 - z = 0$$

$$\nabla f = (\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}) = (2x, 2y, -1)$$

$$\text{at point } p(1,1,1)$$

$$\nabla f = (\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}) = (2, 2, -1)$$

$$\text{Normal at point } p(1,1,1)$$

$$\mathbf{n} = (2-1, 2-1, -1-1) = (1,1,-2)$$

$$\vec{u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ let } \vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = 0 = x^2$$

$$\lim_{h \to 0} \frac{0-0}{h} = 0$$

$$f(x) = y - x^2 = 0$$

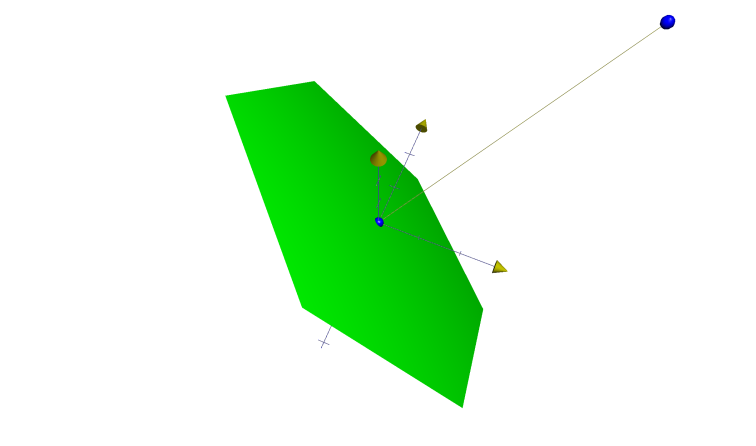
$$\text{The partial derivative of } f(x) = y - x^2 \text{ is } \frac{\partial f}{\partial x} = -2x$$

$$\frac{\partial f}{\partial y} = 1$$

$$\text{However, if } f(x) = 0, \Rightarrow \frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \Rightarrow \frac{\partial f}{\partial x} = 0$$

1 Plane equation in three dimensions

Given a function $x+y+z=0$ which is just a flat plane and perpendicular to vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$



Why the plane is perpendicular to vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$?

x + y + z = 0 can be written as $1 \cdot x + 1 \cdot y + 1 \cdot z = 0$ and it also can be written as dot product as following $[x, y, z] \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$, the dot product implies $[x, y, z]$ is perpendicular to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for any point (x, y, z) and $x + y + z = 0$ passes through point $(x, y, z) = (0, 0,)$.

2 Gradient

Gradient is just a slope of curve if $f(x)$ is defined a curve. e.g. $f(x) = x^2$.
Given a curve $f(x, y) = C$ where $f(x, y) = x^2 - y = 0$ which is just parabolic. $y = x^2$.
Find the partical derivative of $f(x, y) = x^2 - y = 0$ respected to x, y at $p(1, 1)$.

$$
\begin{aligned}
f(x, y) &= x^2 - y \\
\frac{d}{dx} f(x, y) &= \frac{d}{dx}(x^2 - y) = 2x \\
\frac{d}{dy} f(x, y) &= \frac{d}{dy}(x^2 - y) = -1 \\
(\frac{df}{dx}, \frac{df}{dy}) &= (2x, -1) \\
(\frac{df}{dx}, \frac{df}{dy}) &= (2, -1) \qquad \text{where } p(1, 1) \\
\text{But f(x, y) = 0 } \Rightarrow (\frac{df}{dx}, \frac{df}{dy}) &= (0, 0)
\end{aligned}
$$

$$
\begin{aligned}
\text{Rewrite the equation as following:} \\
f(x) = x^2, \text{ the partial derivative of } f(x) \text{ is } \frac{\partial f}{\partial x} = 2x \\
\text{which is just a slope of } f(x)
\end{aligned}
$$

Find a line l passing through $p(1, 1)$ with slop $\frac{y}{x} = \frac{-1}{2}$

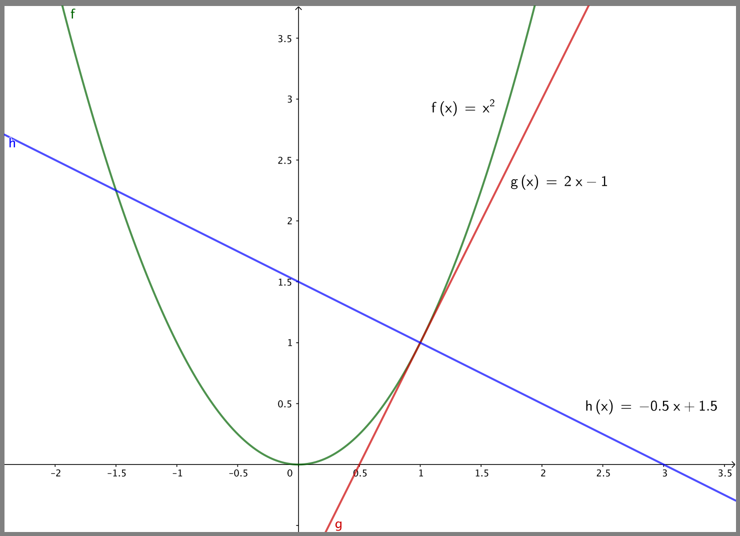
$$
\begin{aligned}
\frac{y - 1}{x - 1} &= \frac{-1}{2} \\
2y - 2 &= -x + 1 \\
y &= \frac{-1}{2} x + \frac{3}{2}
\end{aligned}
$$

∇f is the normal of curve at point $p(1,1)$

$$(x,y) = (1,1) + t\nabla f(1,1)$$
$$(x,y) = (1,1) + t(\frac{df}{dx}, \frac{df}{dy}) = (2,-1)$$
$$(x,y) = (1,1) + (2t, -1t)$$
$$x = 1 + 2t$$
$$y = 1 - t$$
$$\Rightarrow y = \frac{-1}{2}x + \frac{3}{2}$$

Find a line n passing through $p(1,1)$ that is perpendicular to l.

$$\frac{y-1}{x-1} = 2$$
$$y - 1 = 2x - 2$$
$$y = 2x - 1$$



The level curve of

Definition 1. *The gradient of a funciton $f(x,y)$ is defined as*

$$\nabla f(x,y) = \left\langle \frac{df}{dx}(x,y), \frac{df}{dy}(x,y) \right\rangle$$

Example 1. *For function $x+y+z=0$, it can be written as $z=-x-y$ or $f(x,y)=-x-y$*
The gradient of $f(x,y)=-x-y$ is as following:

$$\frac{df}{dx}f(x,y) = -1$$
$$\frac{df}{dy}f(x,y) = -1 \quad (1)$$
$$\nabla f(x,y) = \langle -1, -1 \rangle$$

The gradient of $f(x,y) = -x-y$ is constant, it means all the vectors on the surface have the **same direction** and **same magnitude**.

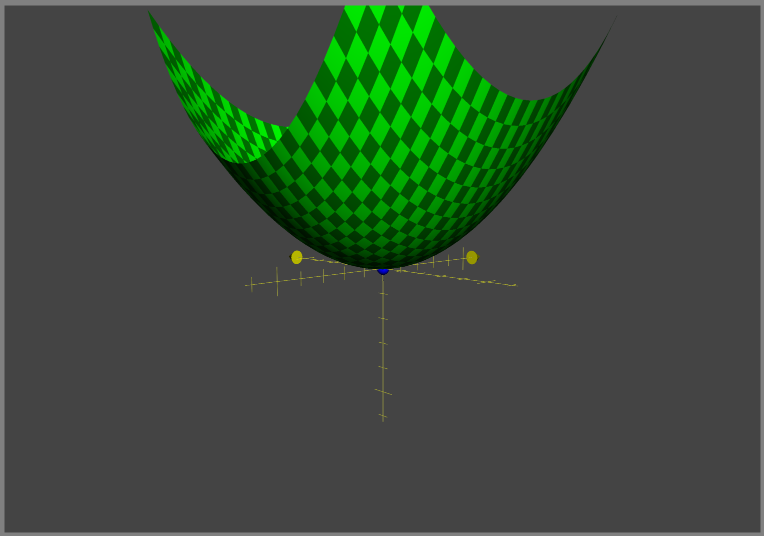
$$|\nabla f(x,y)| = 2 = C$$

Example 2. *Given a function $f(x,y) = e^x \cos y$, find the gradient of the function.*

$$
\begin{aligned}
f(x,y) &= e^x \cos y \\
\frac{df}{dx} &= e^x \cos y \\
\frac{df}{dy} &= -e^x \sin y \\
\nabla f(x,y) &= \left\langle \frac{df}{dx}, \frac{df}{dy} \right\rangle = \langle e^x \cos x, -e^x \sin y \rangle
\end{aligned}
\tag{2}
$$

Example 3. *Given function $f(x,y) = 2x^2 + 3y^2$, find the following:*

- Compute the gradient of $f(x,y) = 2x^2 + 3y^2$
- Identify the level curve of $f(x,y) = C$ through the point $(x,y) = (1,1)$.
- Find the parameter equation $\vec{r}(t)$ of the level curve.
- Show $\frac{d}{dt}\vec{r}(t) \cdot \nabla f(x,y) = 0$ at point $(x,y) = (1,1)$.



Compute the gradient of $f(x,y) = 2x^2 + 3y^2$

$$
\begin{aligned}
\frac{df}{dx} f(x,y) &= 4x \\
\frac{df}{dy} f(x,y) &= 6y \\
\nabla f(x,y) &= \langle 4x, 6y \rangle
\end{aligned}
\tag{3}
$$

Identify the level curve of $f(x, y)$

$$\begin{aligned}
f(1,1) = 2 + 3 = z \\
2x^2 + 3y^2 = 5 \\
\frac{2}{5}x^2 + \frac{3}{5}y^2 = 1 \\
\left(\sqrt{\frac{2}{5}}x\right)^2 + \left(\sqrt{\frac{3}{5}}y\right)^2 = 1 \\
\cos t = \sqrt{\frac{2}{5}}x \\
\sin t = \sqrt{\frac{3}{5}}x \\
x = \sqrt{\frac{5}{2}}\cos t \\
y = \sqrt{\frac{5}{3}}\sin t \\
r(t) = \left\langle \sqrt{\frac{5}{2}}\cos t, \sqrt{\frac{5}{3}}\sin t \right\rangle \\
\frac{d}{dt}r(t) = \left\langle -\sqrt{\frac{5}{2}}\sin t, \sqrt{\frac{5}{3}}\cos t \right\rangle
\end{aligned} \tag{4}$$

Show $\frac{d}{dt}r(t) \cdot \nabla f(x, y) = 0$ at point $(x, y) = (1, 1)$

$$\begin{aligned}
x = \sqrt{\frac{5}{2}}\cos t = 1 \Rightarrow \cos t = \sqrt{\frac{2}{5}} \\
y = \sqrt{\frac{5}{3}}\sin t = 1 \Rightarrow \sin t = \sqrt{\frac{3}{5}} \\
\nabla f(1,1) = \langle 6, 8 \rangle \\
\frac{d}{dt}r(t) = \left\langle -\sqrt{\frac{5}{2}}\sin t, \sqrt{\frac{5}{3}}\cos t \right\rangle = \left\langle -\frac{15}{10}, \frac{10}{15} \right\rangle \\
\frac{d}{dt}r(t) \cdot \nabla f(1,1) = \left\langle -\sqrt{\frac{15}{10}}, \sqrt{\frac{10}{15}} \right\rangle \cdot \langle 4, 6 \rangle = 0
\end{aligned} \tag{5}$$

3 Gradient Descent

Given a function $f(x_1, x_2, \ldots, x_n)$ and let $\mathbf{X} = \mathbf{X_0} + t\vec{u}$ where $\|\vec{u}\| = 1$, then $\alpha(t) = f(\mathbf{x_0} + t\vec{u})$
By the chain rule

$$\begin{aligned}
\alpha'(t) &= \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial t}(\mathbf{x_0} + t\vec{u}) \\
\alpha'(t) &= \frac{\partial f}{\partial x_1} \cdot \frac{\partial x_2}{\partial t} + \frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial t} + \cdots + \frac{\partial f}{\partial x_n} \cdot \frac{\partial x_n}{\partial t} \\
\alpha'(t) &= \nabla f(x_0 + t\vec{u}) \cdot \vec{u} \\
\alpha'(0) &= \nabla f(x_0) \cdot \vec{u} \quad \text{ where } t = 0 \\
\alpha'(0) &= \|\nabla f(x_0)\| \cos\theta
\end{aligned} \tag{6}$$

where θ is the angle between $\|\nabla f(x_0)\|$ and $\vec{u}$, When $\theta = \pi$, $\alpha'(0)$ is minizied.

$$\begin{aligned}
\alpha'(0) = \nabla f(x_0) \cdot \vec{u} = -\|\nabla f(x_0)\| \\
\alpha'(0) = -\|\nabla f(x_0)\| \\
\Rightarrow \nabla f(x_0) \cdot \vec{u} = -\|\nabla f(x_0)\| \\
\Rightarrow -\frac{\nabla f(x_0)}{\|\nabla f(x_0)\|} \cdot \vec{u} = 1 \\
\Rightarrow \vec{u} = -\frac{\nabla f(x_0)}{\|\nabla f(x_0)\|}
\end{aligned} \tag{7}$$

We try to minimize the slope of $f(\mathbf{x})$ which is $\alpha'(t)$

Example 4. *f(x) = (x-1)(x-2)(x-3)*

$$f(x) = (x-1)(x^2-5x+6)$$
$$f(x) = x^3-5x^2+6x-x^2+5x-6$$
$$f(x) = x^3-6x^2+11x-6$$

The derivative of $f(x)$ is

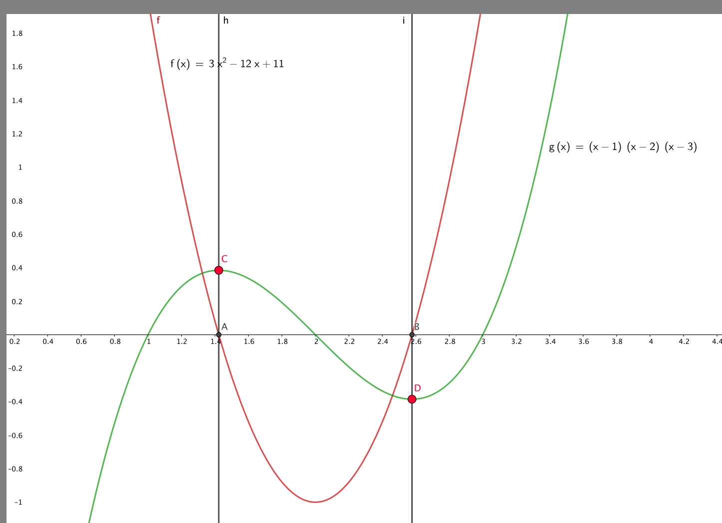
$$f'(x) = 3x^2-12x+11$$

solve $f'(x)=0$ using quadratic formula

$$0 = ax^2+bx+c$$
$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2b}$$

$$f'(x) = 3x^2-12x+11 = 0$$
$$x = \frac{12 \pm \sqrt{12^2-4 \cdot 3 \cdot 11}}{6}$$
$$x = \frac{12 \pm \sqrt{12}}{6}$$
$$x = \frac{12 \pm 2\sqrt{3}}{6}$$
$$x = \frac{6 \pm \sqrt{3}}{3}$$

From the graphic below, we graphic $f(x)$ in green and $f'(x)$ in red. When $x = \frac{6\pm\sqrt{3}}{3}$, $f'(x) = 0$. Therefore, $f(x)$ has two critical points which are in red dots.



4 Harmonic Function

$$f(x, y) = e^x \cos y$$

5 First Fundamental Form a surface

Cartesian Coordinate Equation

$$r^2 = x^2+y^2+z^2$$

Sphere parametric equation

$$
\begin{aligned}
x &= r\sin\alpha\cos\theta \\
y &= r\sin\alpha \\
z &= r\cos\alpha\sin\theta \\
f(\alpha,\theta) &= \begin{cases} x(\alpha,\theta) = r\sin\alpha\cos\theta \\ y(\alpha,\theta) = r\sin\alpha \\ z(\alpha,\theta) = r\cos\alpha\sin\theta \end{cases} \\
\\
J &= \begin{bmatrix} \frac{dx}{d\alpha} & \frac{dx}{d\theta} \\ \frac{dy}{d\alpha} & \frac{dy}{d\theta} \\ \frac{dz}{d\alpha} & \frac{dz}{d\theta} \end{bmatrix} = \begin{bmatrix} r\cos\theta\cos\alpha & -r\cos\alpha\sin\theta \\ r\cos\alpha & 0 \\ -r\sin\alpha\sin\theta & r\cos\alpha\cos\theta \end{bmatrix} \\
\\
J^T &= \begin{bmatrix} \frac{dx}{d\alpha} & \frac{dy}{d\alpha} & \frac{dz}{d\alpha} \\ \frac{dx}{d\theta} & \frac{dy}{d\theta} & \frac{dz}{d\theta} \end{bmatrix} = \begin{bmatrix} -r\cos\alpha\sin\theta & 0 & r\cos\alpha\cos\theta \\ r\cos\theta\cos\alpha & r\cos\alpha & -r\sin\alpha\sin\alpha \end{bmatrix} \\
J^T J &= \begin{bmatrix} \frac{dx}{d\alpha} & \frac{dy}{d\alpha} & \frac{dz}{d\alpha} \\ \frac{dx}{d\theta} & \frac{dy}{d\theta} & \frac{dz}{d\theta} \end{bmatrix} \begin{bmatrix} \frac{dx}{d\alpha} & \frac{dx}{d\theta} \\ \frac{dy}{d\alpha} & \frac{dy}{d\theta} \\ \frac{dz}{d\alpha} & \frac{dz}{d\theta} \end{bmatrix} = \begin{bmatrix} x_\alpha & y_\alpha & z_\alpha \\ x_\theta & y_\theta & z_\theta \end{bmatrix} \begin{bmatrix} x_\alpha & x_\theta \\ y_\alpha & y_\theta \\ z_\alpha & z_\theta \end{bmatrix} \\
J^T J &= \begin{bmatrix} x_\alpha x_\alpha + y_\alpha y_\alpha + z_\alpha z_\alpha & x_\alpha x_\theta + y_\alpha y_\theta + z_\alpha z_\theta \\ x_\theta x_\alpha + y_\theta y_\alpha + z_\theta z_\alpha & x_\theta x_\theta + y_\theta y_\theta + z_\theta z_\theta \end{bmatrix} \\
J^T J &= \begin{bmatrix} -r\cos\alpha\sin\theta & 0 & r\cos\alpha\cos\theta \\ r\cos\theta\cos\alpha & r\cos\alpha & -r\sin\alpha\sin\alpha \end{bmatrix} \begin{bmatrix} r\cos\theta\cos\alpha & -r\cos\alpha\sin\theta \\ r\cos\alpha & 0 \\ -r\sin\alpha\sin\alpha & r\cos\alpha\cos\theta \end{bmatrix} \\
J^T J &= \begin{bmatrix} -r\cos\alpha\sin\theta r\cos\theta\cos\alpha + 0 + -r^2\cos\alpha\cos\theta\sin\alpha\sin\theta & a \\ b & c \end{bmatrix}
\end{aligned} \tag{8}
$$