Loop Invariants

CS242

Lecture 13

Approaches to Proving Properties of Programs

Automatic, Low complexity	Automatic, High complexity	Automatic or Semi-automatic Often undecidable	Manual, Undecidable
Simply Typed Lambda Calculus	Static Analysis	Invariant Inference	Dependent Types

Notation: Hoare Triples

{ Precondition } P { Postcondition}

- Precondition and Postcondition are statements in logic
  - Over program variables

- P is a program

- Read: If the precondition holds on entry to P, then the postcondition holds on exit from P

Examples

$\{ x > 0 \}\ x := x + 1\ \{ x > 1 \}$

$\{ \text{true} \}\ \text{if } x \text{ then } y := 1 \text{ else } y := 0\ \{ y = 0 \lor y = 1 \}$

$\{ x = 1 \}\ \text{for } i = 1,k\ \{ x := x * k \}\ \ \{ x = k^k \}$

A Simple Example

```
X = 0
I = 0
while I < 10 do
        X = X + 1
        I = I + 1

assert(X == 10)
```

Loop Invariants

- To verify loops, it suffices to find a sufficiently strong *loop invariant*

- What is a loop invariant?
  - A predicate that holds on every loop iteration
  - (at the same point, usually at loop head)

- What is "sufficiently strong"
  - More in a minute …

Loop Invariant (1)

```
X = 0
I = 0
while I < 10 do
        { true }
        X = X + 1
        I = I + 1

assert(X == 10)
```

Loop Invariant (2)

```
Z = 42
X = 0
I = 0
while I < 10 do
        { Z = 42 }
        X = X + 1
        I = I + 1

assert(X == 10)
```

Loop Invariant (3)

```
Z = 42
X = 0
I = 0
while I < 10 do
        { I < 4327 }
        X = X + 1
        I = I + 1

assert(X == 10)
```

Loop Invariant (4)

```
Z = 42
X = 0
I = 0
while I < 10 do
        { X < 11 }
        X = X + 1
        I = I + 1

assert(X == 10)
```

Loop Invariant (5)

```
Z = 42
X = 0
I = 0
while I < 10 do
        { X = I && I < 11 }
        X = X + 1
        I = I + 1

assert(X == 10)
```

Comments

- Loop invariants aren’t hard to compute
    - If you don’t care about quality
    - true

- What we want is to prove the assertion at the end of the loop
    - Need an invariant strong enough to do this

Comments

- But how can we prove the assertion?

- We need a proof strategy
  - A process that we can apply to reason about *any* loop

Inductive Invariants

Pre

while (B)

I

{

… code …

Post

}

$Pre \Rightarrow I$

$I \wedge B$

{ code }

I

$I \wedge \neg B \Rightarrow$

$Post$

Inductive Invariants

- $Pre \Rightarrow I$

  The invariant holds initially

- $I \wedge B$ { code } I

  If the invariant and loop condition hold, executing the loop body re-establishes the invariant

- $I \wedge \neg B \Rightarrow Post$

  If the invariant holds and the loop terminates, then the post-condition holds

Loop Invariant (1)

```
X = 0
I = 0
while I < 10 do
        { true }
        X = X + 1
        I = I + 1

assert(X == 10)
```

Loop Invariant (2)

```
Z = 42
X = 0
I = 0
while I < 10 do
        { Z = 42 }
        X = X + 1
        I = I + 1

assert(X == 10)
```

Loop Invariant (3)

```
Z = 42
X = 0
I = 0
while I < 10 do
          { I < 4327 }
          X = X + 1
          I = I + 1

assert(X == 10)
```

Loop Invariant (4)

```
Z = 42
X = 0
I = 0
while I < 10 do
          { X < 11 }
          X = X + 1
          I = I + 1

assert(X == 10)
```

Loop Invariant (5)

```
Z = 42
X = 0
I = 0
while I < 10 do
        { X = I && I < 11 }
        X = X + 1
        I = I + 1

assert(X == 10)
```

First Question

- How do we decide whether these formulas are true?

  $\text{Pre} \Rightarrow \text{I} \qquad \text{I} \wedge \text{B} \{ \text{code} \} \text{I} \qquad \text{I} \wedge \neg \text{B} \Rightarrow \text{Post}$

- Use SMT solvers
  - Satisfiability Modulo Theories
  - Tools that include decision procedures for a wide variety of logical theories relevant to program verification
  - Boolean satisfiability, theory or arrays, bitvectors, integers, ...

- Simply give an SMT a formula and it may
  - Report it is satisfiable (and give an assignment)
  - Report it is unsatisfiable (and give a counter example)
  - Report "I don't know"
  - Run forever

Second Question

Why focus on loop invariants?

First Answer

- Loop invariants are an important concept in everyday programming
- Why is my loop correct?
- You can break the problem into the three conditions stated above

Second Answer: Automated Verification

- Consider a loop-free program P
  - With conditionals
  - Memory references
  - Data structures
  - No function calls

- What is the computational complexity of verifying

{ Precondition } P { Postcondition}

Loops

- Now consider the same problem
  - Where P can have one loop
  - But still no function calls

- What is the computational complexity of verifying

{ Precondition } P { Postcondition}

Verification of Loops

- Verifying properties of loops is *the* hard problem
- Solve this, and everything else is much easier

Invariant Inference

- Find (infer) loop invariants automatically

- An old problem

- Many algorithms in the literature

- We will look at a simple approach

Invariant Inference

- Two ideas:
  1. Separate invariant inference from the rest of the verification problem

  2. Guess the invariant from executions

Why Use Data From Tests?

- Complementary to static reasoning

- "See through" hard analysis problems
  - functionality may be simpler than the code

- Possible to generate many, many tests

Outline

- Guess (many) invariants
  - Run the program
  - Discard candidate invariants that are falsified
  - Attempt to verify the remaining candidates

A Simple Program

```
s = 0;
y = 0;
while( * )
{
  print(s,y);
  s := s + 1;
  y := y + 1;
}
```

- Instrument loop head

- Collect the values of program variables on each iteration

Data Collection Example

```
s = 0;
y = 0;
while( * )
{
  print(s,y);
  s := s + 1;
  y := y + 1;
}
```

- Hypothesize
  - s = y
  - s = 2y
- Data

s	y
0	0

Data Collection Example

```
s = 0;
y = 0;
while( * )
{
  print(s,y);
  s := s + 1;
  y := y + 1;
}
```

- Hypothesize
  - s = y
  - ~~s = 2y~~
- Data

s	y
0	0
1	1

Data Collection Example

```
s = 0;
y = 0;
while( * )
{
  print(s,y);
  s := s + 1;
  y := y + 1;
}
```

- Hypothesize
  - s = y
  - ~~s = 2y~~
- Data

s	y
0	0
1	1
2	2
3	3

Another Approach

```
s = 0;
y = 0;
while( * )
{
  print(s,y);
  s := s + 1;
  y := y + 1;
}
```

- Data

s	y
0	0
1	1
2	2
3	3

Arbitrary Linear Invariant

`as + by = 0`

- Data

s	y
0	0
1	1
2	2
3	3

Observation

$$as + by = 0$$

s	y
0	0
1	1
2	2
3	3

$$\begin{bmatrix} \mathbf{w} \\ a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Observation

$$as + by = 0$$

$\{ w \mid Mw = 0 \}$

s	y
0	0
1	1
2	2
3	3

w
a
b

=

0
0

Observation

$$as + by = 0$$

NullSpace(M)

s	y
0	0
1	1
2	2
3	3

$$\begin{array}{c} \mathbf{w} \\ \begin{bmatrix} a \\ b \end{bmatrix} \end{array} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Linear Invariants

- Construct matrix M of observations of all program variables
- Compute NullSpace(M)
- All invariants are in the null space

Spurious "Invariants"

- All invariants are in the null space
  - But not all vectors in the null space are invariants

- Consider the matrix

s	**y**
0	0

- Need a check phase
  - Verify the candidate is in fact an invariant

An Algorithm

- Check candidate invariant
  - If an invariant, done
  - If not an invariant, get a *counterexample*
    - Counterexample can be guaranteed to satisfy all invariants
- Add new row to matrix
  - And repeat

Termination

- How many times can the solve & verify loop repeat?

- Each counterexample is linearly independent of previous entries in the matrix

- So at most N iterations
  - Where N is the number of columns
  - Upper bound on steps to reach a full rank matrix

Summary

- Superset of all linear invariants can be obtained by a standard matrix calculation

- Counter-example driven improvements to eliminate all but the true invariants
  - Guaranteed to terminate

What About Non-Linear Invariants?

```
s = 0;
y = 0;
while( * )
{
  print(s,y);
  s := s + y;
  y := y + 1;
}
```

Idea

- Collect data as before

- But add more columns to the matrix
  - For derived quantities
  - For example, y^2 and s^2

- How to limit the number of columns?
  - All monomials up to a chosen degree d

What About Non-Linear Invariants?

```
s = 0;
y = 0;
while( * )
{
  print(s,y);
  s := s + y;
  y := y + 1;
}
```

1	s	y	s^2	y^2	sy
1	0	0	0	0	0
1	1	1	1	1	1
1	3	2	9	4	6
1	6	3	36	9	18
1	10	4	100	16	40

Solve for the Null Space

$$a + bs + cy + ds^2 + ey^2 + fsy = 0$$

1	s	y	s^2	y^2	sy
1	0	0	0	0	0
1	1	1	1	1	1
1	3	2	9	4	6
1	6	3	36	9	18
1	10	4	100	16	40

w
a
b
c
d
e
f

=

0
0
0
0
0
0

Candidate invariant: $-2s + y + y^2 = 0$

Comments

- Same issues as before
  - Must check candidate is implied by precondition, is inductive, and implies the postcondition on termination
  - Termination of invariant inference guaranteed if the verifier can generate counterexamples
- Solvers do well as checkers!

Experiments

Name	#vars	deg	Data	#and	Guess time (sec)	Check time (sec)	Total time (sec)
Mul2	4	2	75	1	0.0007	0.010	0.0107
LCM/GCD	6	2	329	1	0.004	0.012	0.016
Div	6	2	343	3	0.454	0.134	0.588
Bezout	8	2	362	5	0.765	0.149	0.914
Factor	5	3	100	1	0.002	0.010	0.012
Prod	5	2	84	1	0.0007	0.011	0.0117
Petter	2	6	10	1	0.0003	0.012	0.0123
Dijkstra	6	2	362	1	0.003	0.015	0.018
Cubes	4	3	31	10	0.014	0.062	0.076
geoReihe1	3	2	25	1	0.0003	0.010	0.0103
geoReihe2	3	2	25	1	0.0004	0.017	0.0174
geoReihe3	4	3	125	1	0.001	0.010	0.011
potSumm1	2	1	5	1	0.0002	0.011	0.0112
potSumm2	2	2	5	1	0.0002	0.009	0.0092
potSumm3	2	3	5	1	0.0002	0.012	0.0122
potSumm4	2	4	10	1	0.0002	0.010	0.0102

Summary to This Point

- Algorithm for algebraic invariants
  - Up to a given degree

- Guess and Check
  - Hard part is inference done by matrix solve
  - Check part done by standard SMT solver
  - Simple and fast

What About Disjunctive Invariants?

- Disjunctions are expensive
  - In addition to conjunctions

- Invariant inference techniques tend to severely restrict disjunctions
  - E.g., to a template

What About Non-Numeric Invariants?

- Arrays?
- Lists?
- Other data structures?

- Invariant inference techniques tend to be specialized
  - Particularly to integer invariants

A Search-Based Approach

- All methods for finding invariants are heuristics
  - Can never be complete

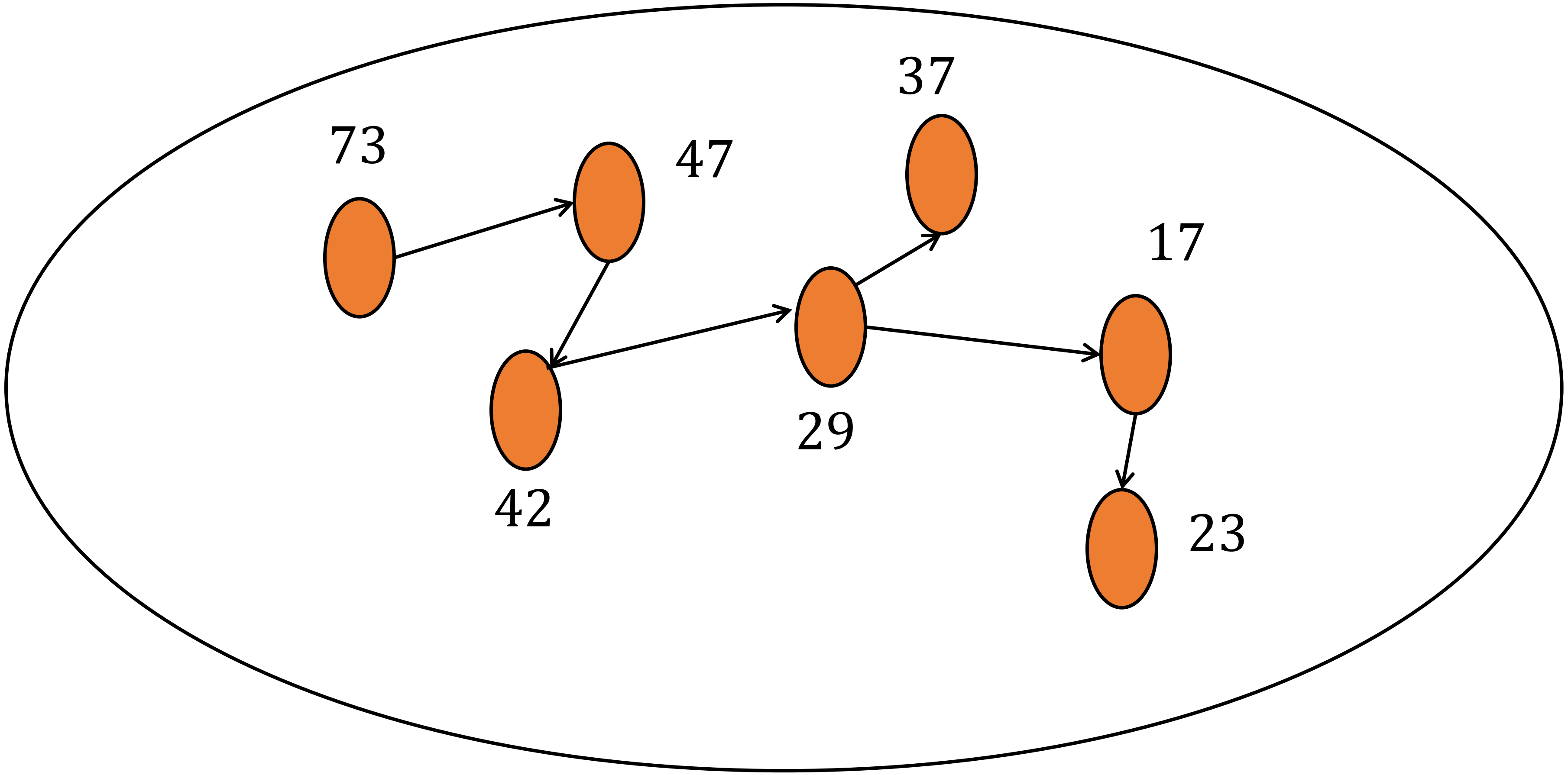
- So why not use general but incomplete techniques?

MCMC

- Markov Chain Monte Carlo sampling

- The only known tractable solution method for high dimensional irregular search spaces

MCMC Overview



MCMC Sampling Algorithm for Invariants

1. Select an initial candidate

2. Repeat (millions of times)
   - Propose a random modification and evaluate cost
   - If (cost decreased)
     { accept }
   - If (cost increased)
     { with some probability accept anyway }

Recall

$$Pre \Rightarrow I$$

$$I(s) \Rightarrow I(t) \quad \text{if} \quad s\ \{body\}\ t$$

$$I \wedge \neg B \Rightarrow Post$$

Data

- Good states *G*
  - Reachable states
- Pairs *Z*
  - States (s,t) such that starting the loop body S in state s terminates in state t.
- Bad states *B*
  - States that lead to an assertion violation

Cost Function (Roughly)

- Penalize a candidate invariant C
  - 1 for each good state g in G where C(g) is false.
  - 1 for each bad state b in B where C(b) is true
  - 1 for each pair (s,t) in Z where C(s) and not C(t)

- The cost of C is the sum of the penalties

Overall Algorithm

- Run search until a 0-cost candidate C is found

- Use a decision procedure to verify that C is an invariant
  - If yes, done
  - If no, get a counterexample
    - A good state, bad state, or pair
    - Add to the data
    - Repeat

MCMC Sampling Algorithm for Invariants

1. Select an initial candidate

2. Repeat (millions of times)
   - Propose a random modification and evaluate cost
   - If (cost decreased)
     { accept }
   - If (cost increased)
     { with some probability accept anyway }

Numerical Invariants

- Find invariants of the form

$$\bigvee_{i=1}^{\alpha} \bigwedge_{j=1}^{\beta} \sum_{k=1}^{n} w_k^{(i,j)} x_k \leq d^{(i,j)}$$

Moves

- Replace a coefficient

- Replace a constant on the rhs

- Replace all coefficients and the constant in a single inequality

$$\bigvee_{i=1}^{\alpha} \bigwedge_{j=1}^{\beta} \sum_{k=1}^{n} w_k^{(i,j)} x_k \leq d^{(i,j)}$$

Results

Program	Z3-H	ICE	[50]	[30]	Pure	MCMC	Templ
cgr1 [27]	0.02	0.2	0.2	0.1	0.05	0.03	0.02
cgr2 [27]	0.03	2.1	?	?	0.68	1.49	1.17
ex7 [33]	0.02	1.1	0.4	?	0.08	0.05	0.04
ex11 [3]	0.03	0.5	0.2	0.1	0.04	0.03	0.05
ex14 [33]	0.01	0.2	0.2	?	0.05	0.03	0.02
ex23 [33]	?	7.3	?	?	0.16	0.13	0.11
fig1 [27]	0.02	1.0	?	?	4.42	0.95	1.44
fig3 [24]	0.01	0.5	0.1	0.1	0.23	0.04	0.04
fig9 [24]	0.02	0.9	0.2	0.1	0.01	0.02	0.01
monniaux	5.14	0.1	1.0	0.2	0.05	0.01	0.03
nested	0.02	?	1.0	0.04	5.21	0.29	2.12
tacas [34]	TO	4.8	0.5	0.1	0.75	0.52	0.08
w1 [27]	0.02	0.5	0.2	0.1	0.05	0.01	0.02
w2 [27]	0.02	0.4	0.1	0.1	0.09	0.03	0.05
array [3]	0.03	1.3	0.2	?	0.24	0.22	0.29
fil1 [3]	0.01	0.2	0.3	0.4	0.01	0.01	0.01
trex01 [3]	0.01	0.2	0.4	0.1	0.03	0.01	0.03

And More ...

- Can be extended to
  - Arrays, trees, lists, relations, ...
  - Any data structure for which a corresponding decision procedure exists

- Surprisingly robust and fast
  - But this probably says that the examples in the literature are too easy!

- Not the final word!
  - Invariant inference is an active area of research

Summary

- Loop invariants are an important concept in programming
  - Good to think about invariants for your code!
  - Even without a tool to check or infer invariants

- Automating loop invariant inference is challenging
  - Long-standing research problem
  - Used in practice is still limited