3D Modeling
Parametric Curves & Surfaces

Shandong University

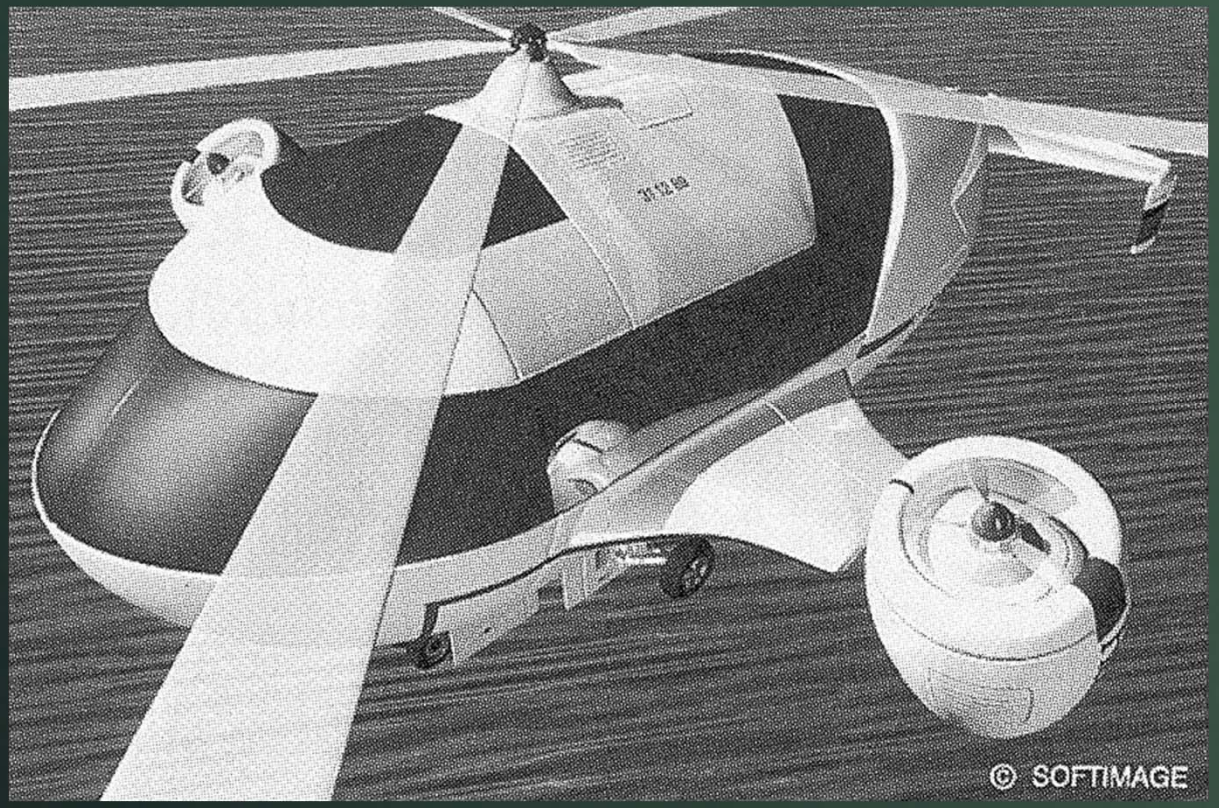
Spring 2013

3D Object Representations

- **Raw data**
  - Point cloud
  - Range image
  - Polygon soup
- **Surfaces**
  - Mesh
  - Subdivision
  - Parametric
  - Implicit
- **Solids**
  - Voxels
  - BSP tree
  - CSG
  - Sweep
- **High-level structures**
  - Scene graph
  - Skeleton
  - Application specific

Parametric Surfaces

- **Applications**
  - **Design of smooth surfaces in cars, ships, etc.**



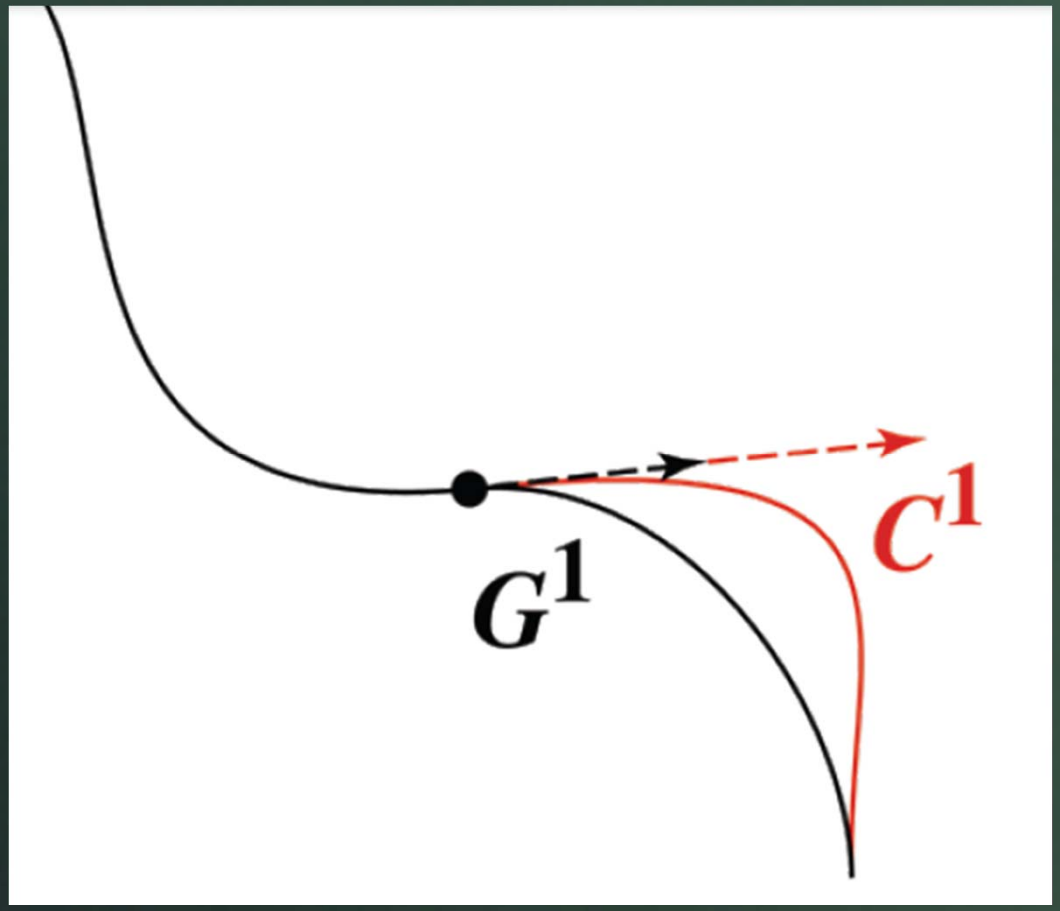
Continuity

- **When two curves are joined, we typically want some degree of continuity across the boundary (the knot)**
- ***Geometric Continuity***
  - **G^0: The curves touch at the join point.**
  - **G^1: The curves also share a common tangent direction at the join point.**
  - **G^2: The curves also share a common center of curvature at the join point.**

Continuity

- ***Parametric Continuity***
  - **C^{-1}: curves include discontinuities**
  - **C^{0}: curves are joined**
  - **C^{1}: first derivatives are continuous**
  - **C^{2}: first and second derivatives are continuous**
  - **C^{n}: first through n^{th} derivatives are continuous**

Continuity



Parametric (red) and Geometric (black) Continuity comparison

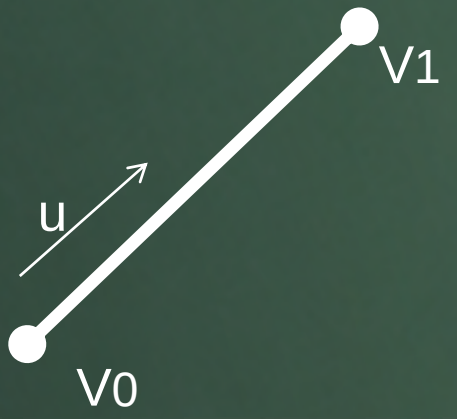
Parametric Curves

- **Boundary defined by parametric functions:**

$$x = f_x(u)$$
$$y = f_y(u)$$

- **Example: line segment**

$$f_x(u) = (1-u)x_0 + ux_1$$
$$f_y(u) = (1-u)y_0 + uy_1$$



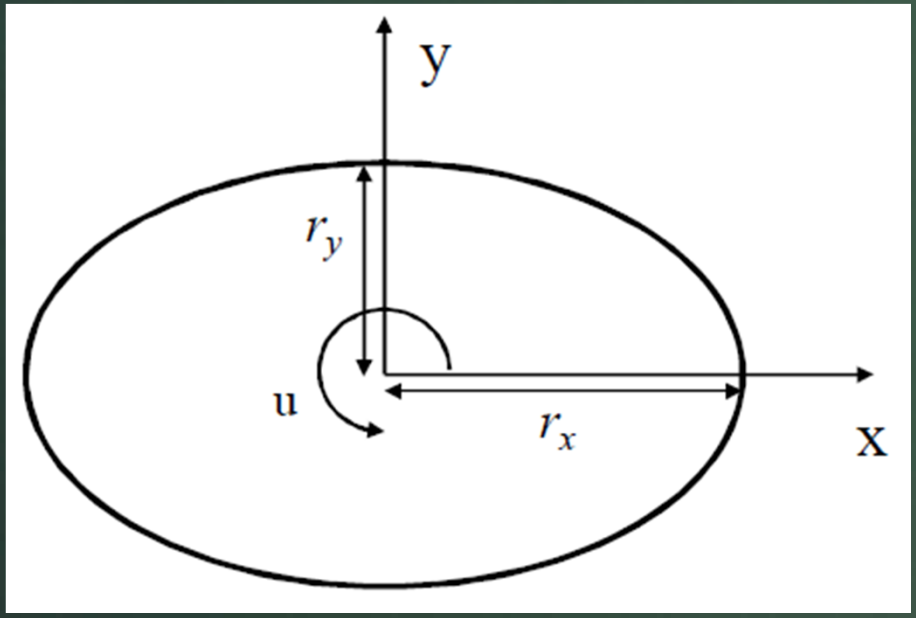
Parametric Curves

- **Boundary defined by parametric functions:**

$$x = f_x(u)$$
$$y = f_y(u)$$

- **Example: ellipse**

$$f_x(u) = r_x \cos u$$
$$f_y(u) = r_y \sin u$$



Parametric Curves

- **How can we define arbitrary curves?**

$$x = f_x(u)$$
$$y = f_y(u)$$

Parametric Curves

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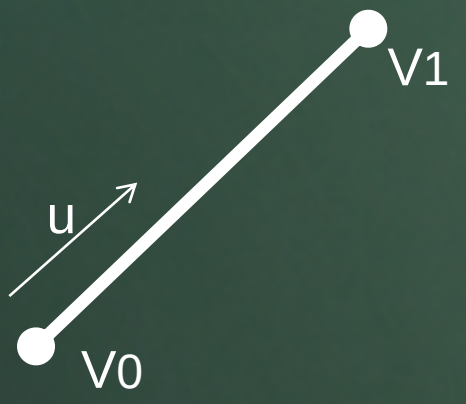


- **Use functions that "blend" control points**

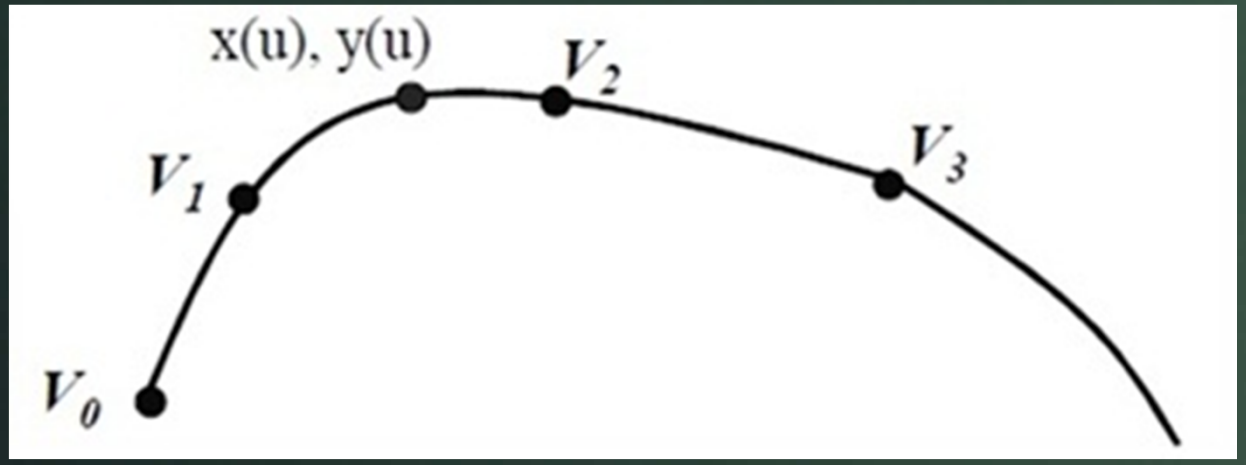
$$x = f_x(u) = (1-u)V_{0x} + uV_{1x}$$
$$y = f_y(u) = (1-u)V_{0y} + uV_{1y}$$

Parametric Curves

- **More generally:**

$$x(u) = \sum_{i=0}^{n} B_i(u) * V_{ix}$$

$$y(u) = \sum_{i=0}^{n} B_i(u) * V_{iy}$$

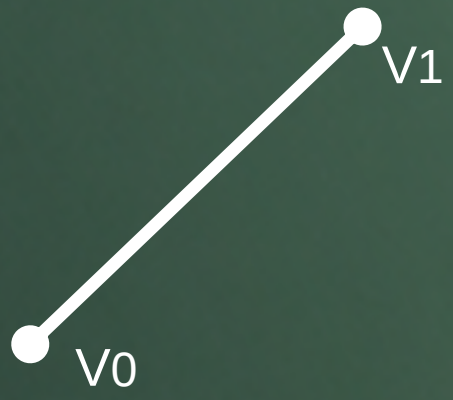


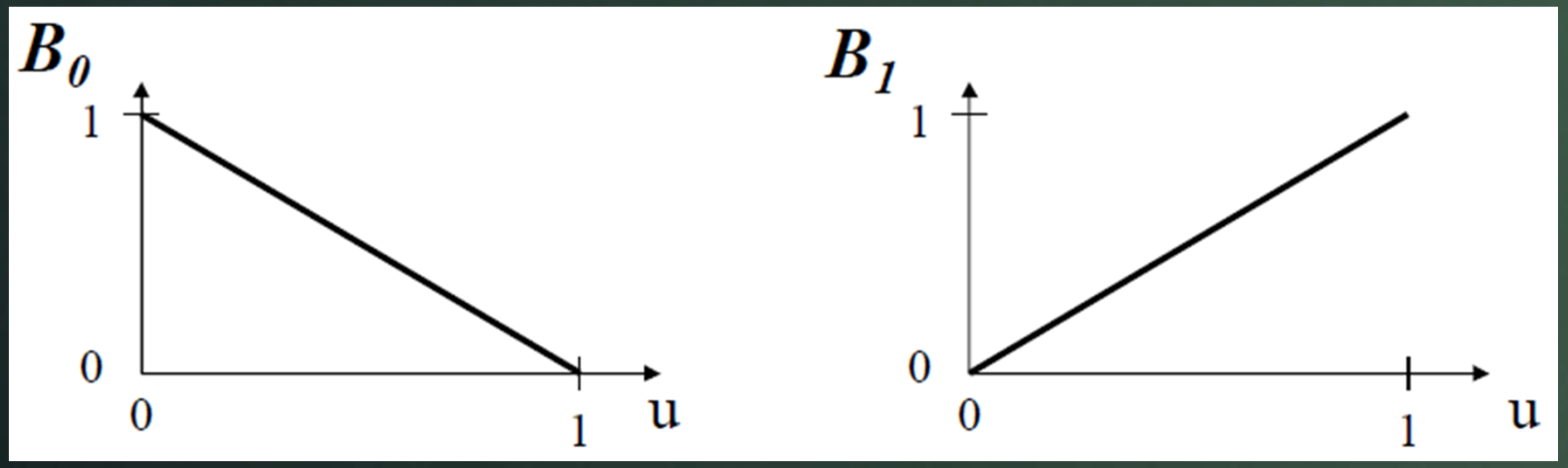
Parametric Curves

- **What B(u) functions should we use?**

$$x(u) = \sum_{i=0}^{n} B_i(u) * V_{ix}$$

$$y(u) = \sum_{i=0}^{n} B_i(u) * V_{iy}$$





Parametric Curves

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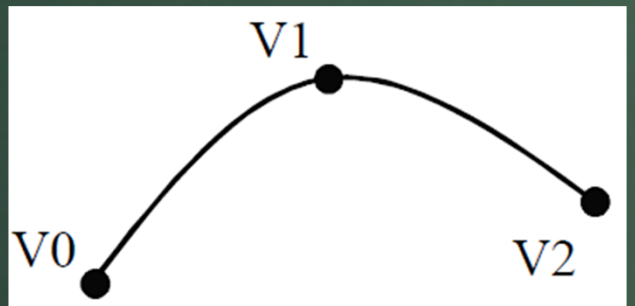
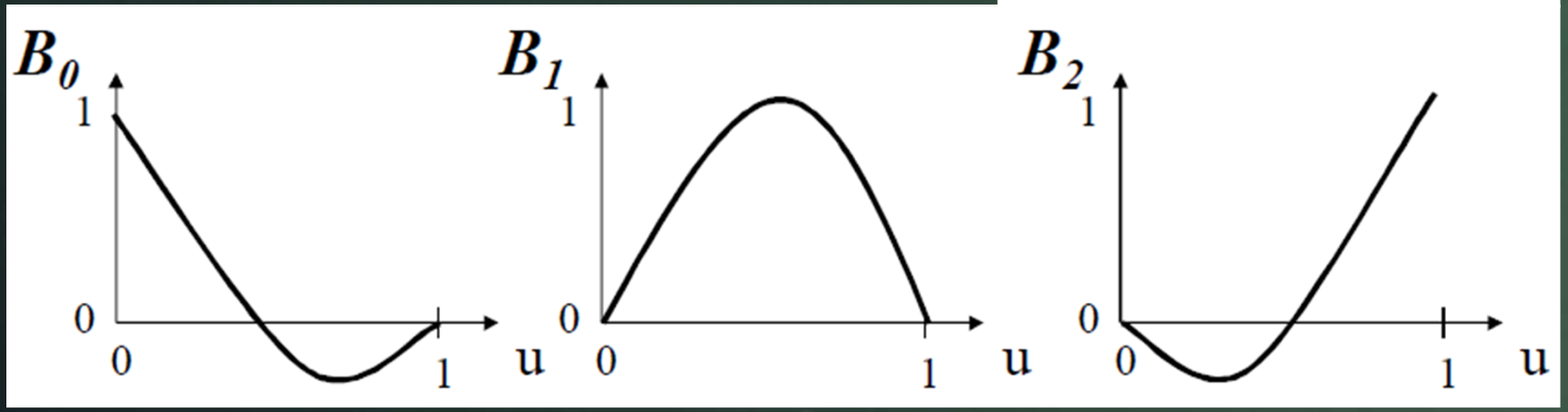
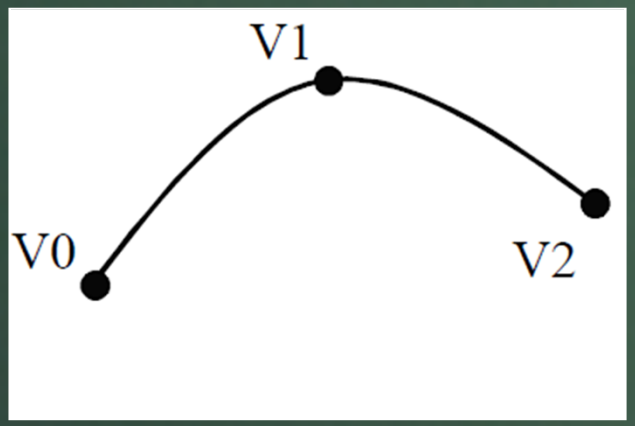




Parametric Polynomial Curves

- **Polynomial blending functions:**

$$B_i(u) = \sum_{j=0}^{m} a_j u^j$$



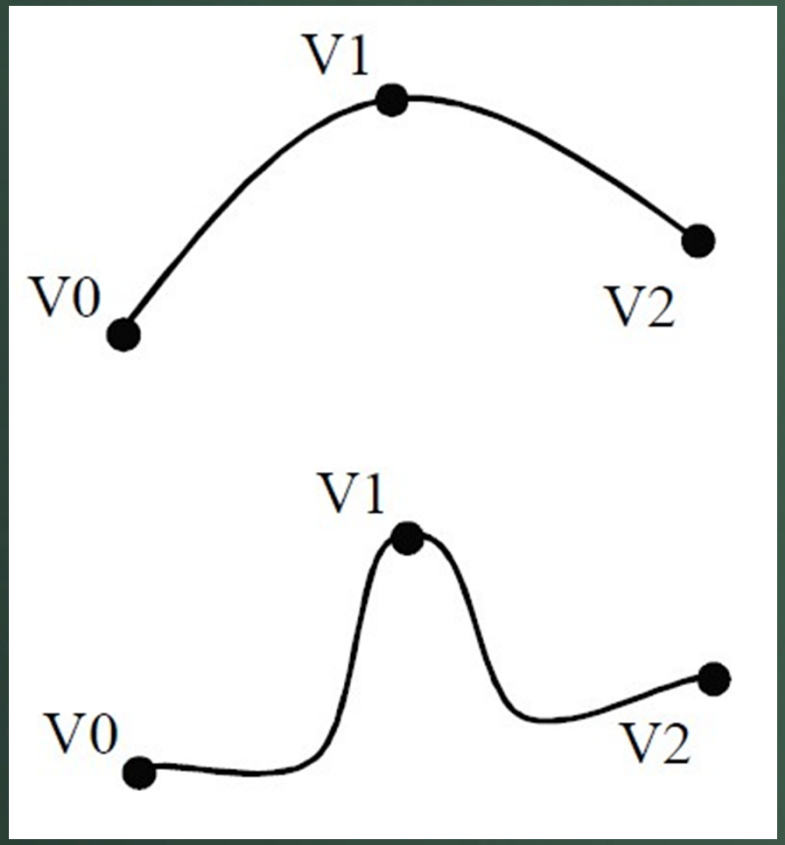
- **Advantages of polynomials**
  - **Easy to compute**
  - **Easy differentiation**
  - **Easy to derive curve properties**

Parametric Polynomial Curves

- **Polynomial blending functions:**

$$B_i(u) = \sum_{j=0}^{m} a_j u^j$$

- **What degree polynomial?**
  - **Easy to compute**
  - **Easy to control**
  - **Expressive**



Parametric Cubic Curves

- **Why cubic?**
  - **lower-degree polynomials give too little flexibility in controlling the shape of the curve**
  - **higher-degree polynomials can introduce unwanted wiggles and require more computation**
  - **lowest degree that allows specification of endpoints and their derivatives**
  - **lowest degree that is not planar in 3D**

Parametric Cubic Curves

- **General form:**

$$x(u) = a_x u^3 + b_x u^2 + c_x u + d_x$$

$$y(u) = a_y u^3 + b_y u^2 + c_y u + d_y$$

$$z(u) = a_z u^3 + b_z u^2 + c_z u + d_z$$

$$C = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix} \qquad T = [u^3 \quad u^2 \quad u \quad 1]$$

$$Q(u) = [x(u) \quad y(u) \quad z(u)] = T \cdot C$$

Major Types of Parametric Cubic Curves

- **Cubic Bézier**
  - **defined by two endpoints and two other points that control the endpoint tangent vectors**
- **Hermite**
  - **defined by two endpoints and two tangent vectors**
- **Splines**
  - **several kinds, each defined by four points**
  - **uniform B-splines, non-uniform B-splines, ß-splines**

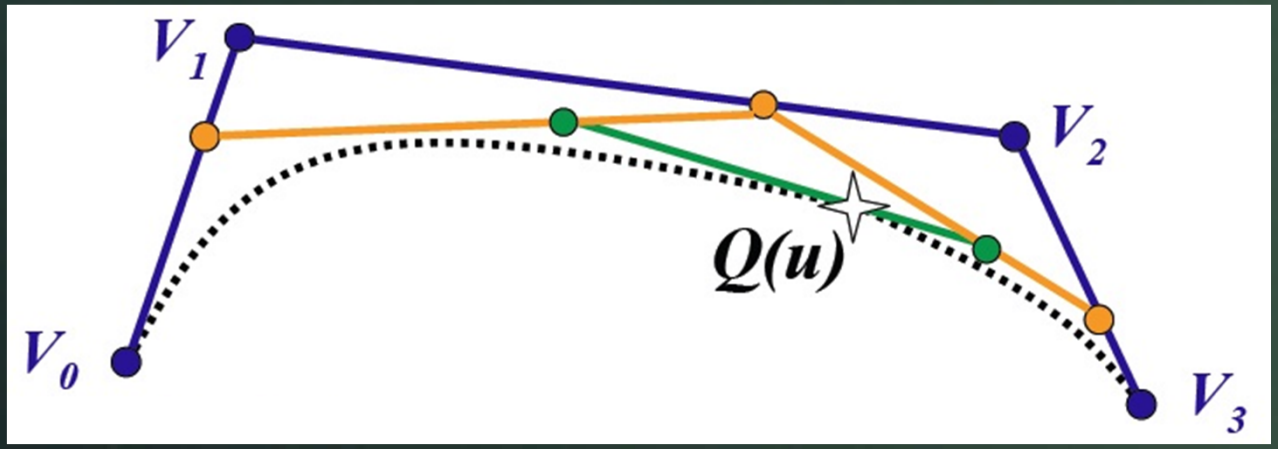
Bézier Curves

- **In 1962, Pierre Bézier, an engineer of French Renault Car company, proposed a new kind of curve representation, and finally developed a system UNISURF for car surface design in 1972.**

Bézier Curves

- **Two contributors**
  - **Pierre Bézier (at Renault)**
  - **Paul de Casteljau (at Citroen)**

- **Curve Q(u) is defined by nested interpolation:**



V_i's are control points

$\{V_0, V_1, \ldots, V_n\}$ is control polygon

Basic properties of Bézier curves

- **Endpoint interpolation:**

$$Q(0) = V_0$$
$$Q(1) = V_n$$

- **Convex hull:**
  - **Curve is contained within convex hull of control polygon**

- **Transformational invariance**

- **Symmetry**

  $Q(u)$ **defined by** $\{V_0, \dots, V_n\} \equiv Q(1-u)$ **defined by** $\{V_n, \dots, V_0\}$

More Properties

- **General case: Bernstein polynomials**

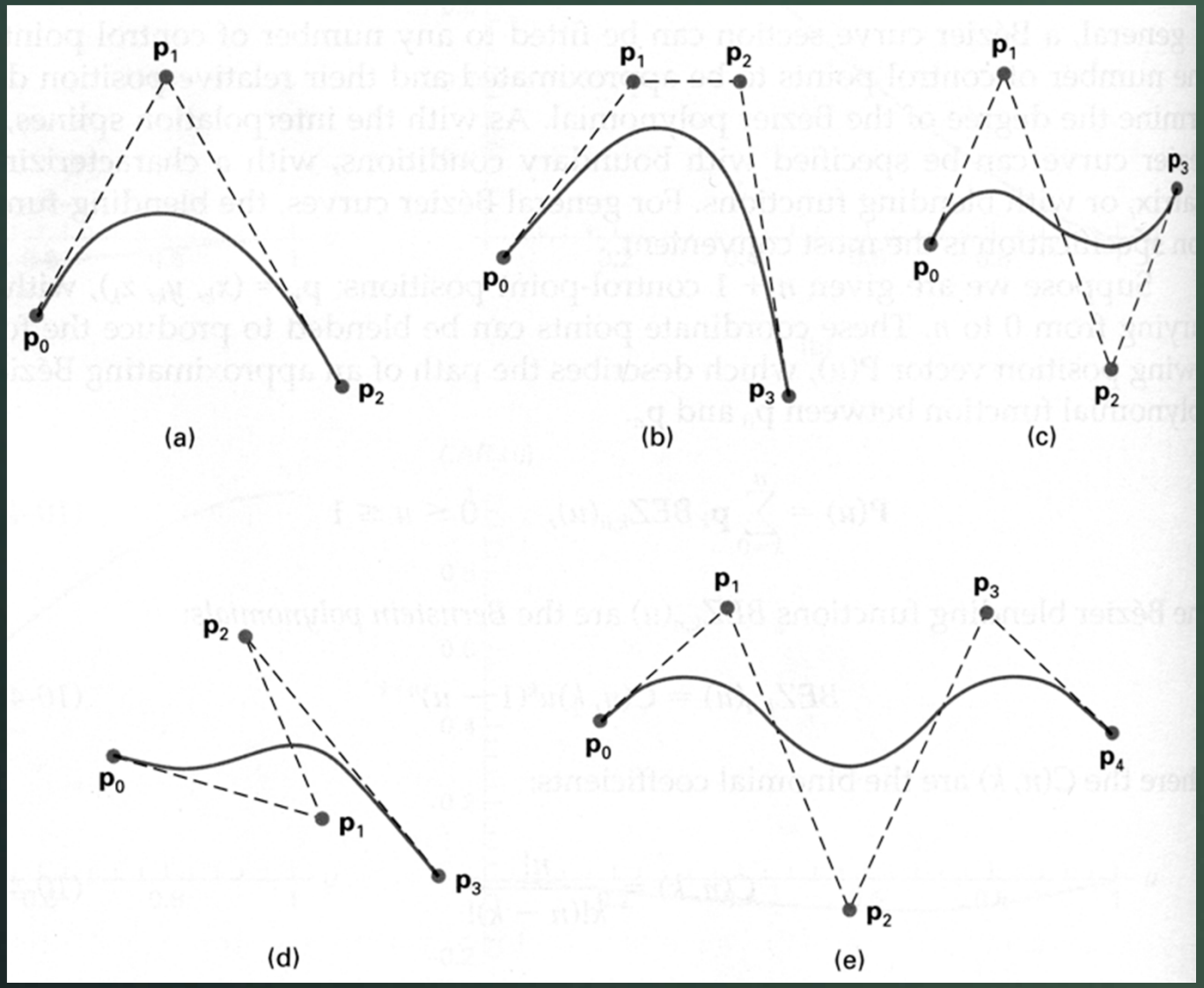
$$Q(u) = \sum_{i=0}^{n} \binom{n}{i} u^i (1-u)^{n-i} V_i$$

- **Degree: polynomial of degree n**

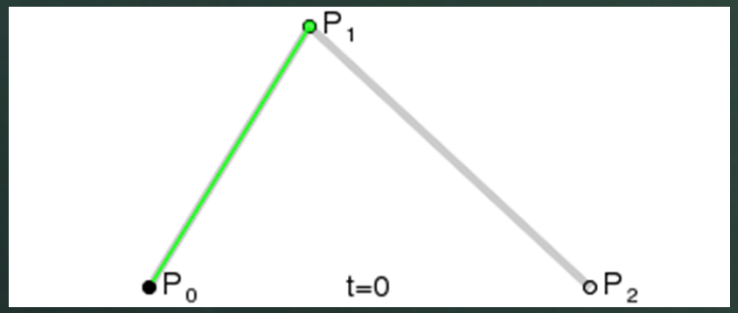
- **Tangents:**

$$Q'(0) = n(V_1 - V_0)$$
$$Q'(1) = n(V_n - V_{n-1})$$

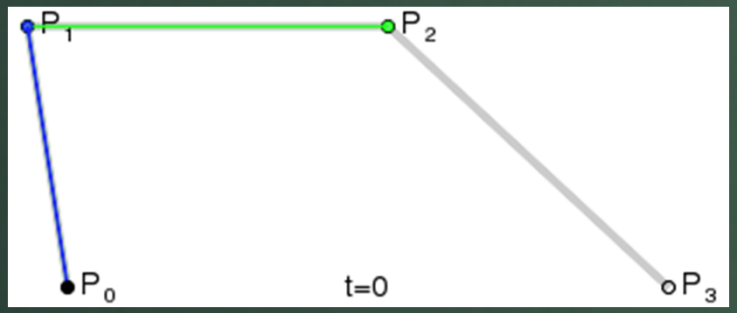
Some Bézier Curves



Some Bézier Curves



Animation of a quadratic Bézier curve, *t* in [0,1]



Animation of a cubic Bézier curve, *t* in [0,1]

Cubic Bézier Curves

$$Q(u) = \sum_{i=0}^{n} \binom{n}{i} u^i (1-u)^{n-i} V_i$$

$$= (1-u)^3 V_0 + 3u(1-u)^2 V_1 + 3u^2(1-u)V_2 + u^3 V_3$$

$$= \left(u^3, u^2, u, 1\right) \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

Hermite Curves

- Given: two points and two tangent vectors
  - Similarity to cubic Bézier curves
  - Other two Bézier control points along those tangents
- Call the points P_1 and P_2, and the tangents R_1 and R_2
- So given two points and vectors, find the coefficients of $x(u) = a_x u^3 + b_x u^2 + c_x u + d_x$ etc.

Hermite Curves

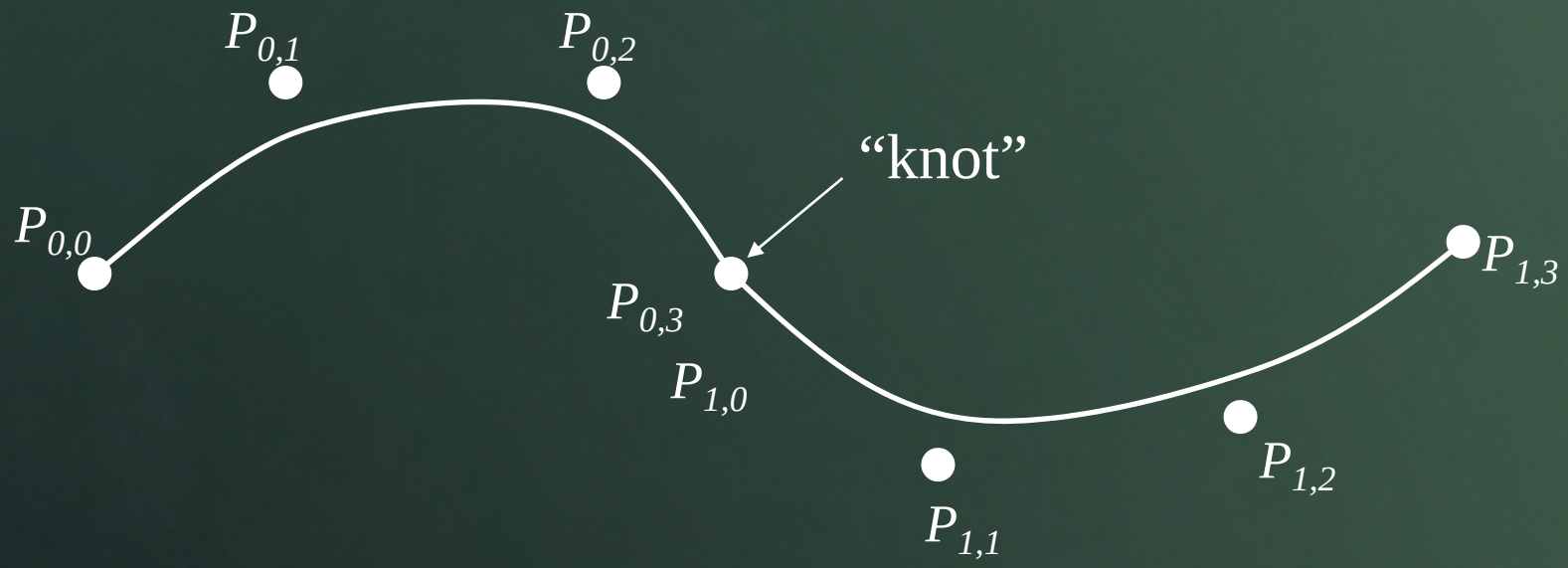
- **We can treat x in the mapping as a vector. Its components can be explicitly written as**

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 & x_0 & x_1' & x_0' \\ y_1 & y_0 & y_1' & y_0' \\ z_1 & z_0 & z_1' & z_0' \end{bmatrix} \begin{bmatrix} -2 & 3 & 0 & 0 \\ 2 & -3 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix}$$

Longer Curves

- A single cubic Bezier or Hermite curve can only capture a small class of curves
  - At most 2 inflection points
- One solution is to raise the degree
  - Allows more control, at the expense of more control points and higher degree polynomials
  - Control is not *local*, one control point influences entire curve
- Alternate, most common solution is to join pieces of cubic curve together into *piecewise cubic curves*
  - Total curve can be broken into pieces, each of which is cubic
  - *Local control*: Each control point only influences a limited part of the curve
  - Interaction and design is much easier

Piecewise Bézier Curve



Continuity

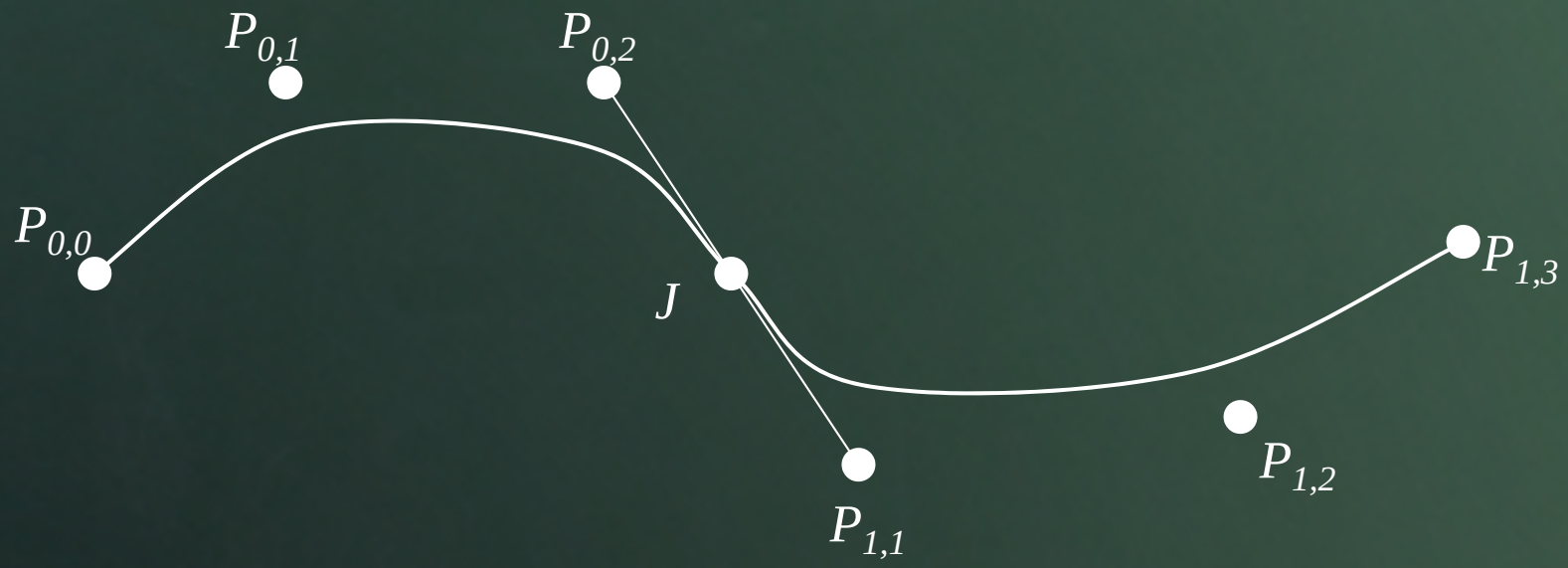
- **Question:**
  - **How do we ensure that two Hermite curves are C^1 across a knot?**

- **Question:**
  - **How do we ensure that two Bézier curves are C^0, or C^1, or C^2 across a knot?**

Achieving Continuity

- For Hermite curves, the user specifies the derivatives, so C^1 is achieved simply by sharing points and derivatives across the knot
- For Bézier curves:
  - **They interpolate their endpoints, so C^0 is achieved by sharing control points**
  - **The parametric derivative is a constant multiple of the vector joining the first/last 2 control points**
  - **So C^1 is achieved by setting $P_{0,3}=P_{1,0}=J$, and making $P_{0,2}$ and J and $P_{1,1}$ collinear, with $J-P_{0,2}=P_{1,1}-J$**
  - **C^2 comes from further constraints on $P_{0,1}$ and $P_{1,2}$**

Bézier Continuity



B-Spline Curves

- **Why to introduce B-Spline?**
  - **Bezier curve/surface has many advantages, but they have two main shortcomings:**
    - **Bezier curve/surface cannot be modified locally**
    - **It is very complex to satisfy geometric continuity conditions for Bezier curves or surfaces joining.**
- **Why not use lower degree piecewise polynomial with continuous joining?**
  - **that's Spline**

B-Spline Curves

- **Formula of B-Spline Curve.**

$$P(t) = \sum_{i=0}^{n} P_i N_{i,k}(t)$$

- **P_i (i=0,1,…,n) are control points.**
- **$N_{i,k}(t)$ (i=0,1,...,n) are the i-th B-Spline basis function of order k.**
- **B-Spline basis function is an order k (degree k -1) piecewise polynomial.**

B-Spline Curves

- **Definition of B-Spline Basis Function**

– de Boor-Cox recursion formula:

$$N_{i,1}(t) = \begin{cases} 1 & t_i < x < t_{i+1} \\ 0 & Otherwise \end{cases}$$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

– Knot Vector: a sequence of non-decreasing number

$$t_0, t_1, \cdots, t_{k-1}, t_k, \cdots, t_n, t_{n+1}, \cdots, t_{n+k-1}, t_{n+k}$$

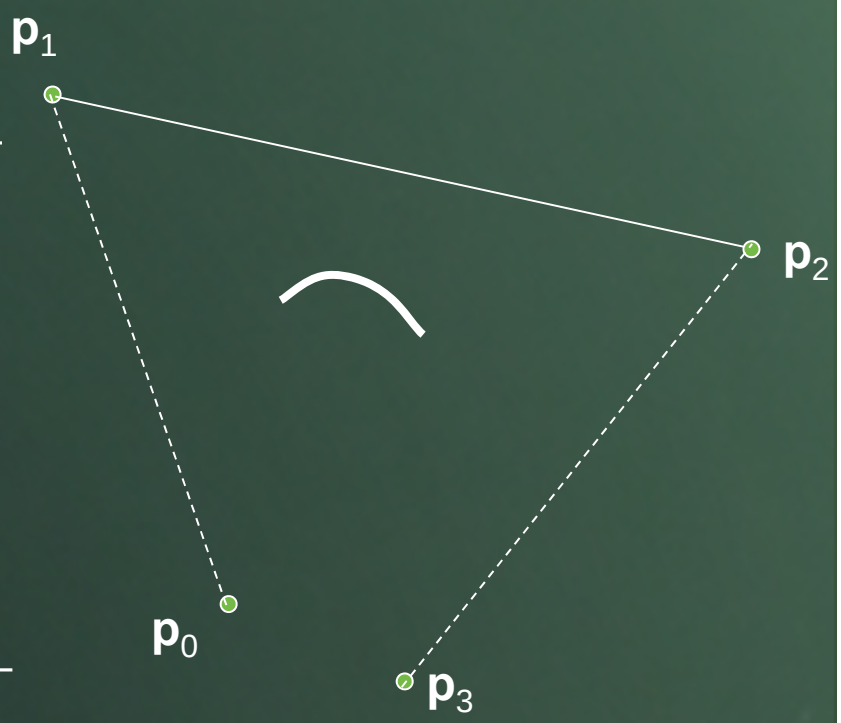
B-Spline Curves

Cubic B-Spline n=3, k=4

$$\mathbf{p}(u) = (-1/6\mathbf{p}_0 + 1/2\mathbf{p}_1 - 1/2\mathbf{p}_2 + 1/6\mathbf{p}_3)u^3 +$$
$$(\ 1/2\mathbf{p}_0 - \mathbf{p}_1 + 1/2\mathbf{p}_2\ )u^2 +$$
$$(-1/2\mathbf{p}_0 + 1/2\mathbf{p}_2\ )u +$$
$$1/6\mathbf{p}_0 + 2/3\mathbf{p}_1 + 1/6\mathbf{p}_2$$

but makes more sense as…

$$\mathbf{p}(u) = (-1/6u^3 + 1/2u^2 - 1/2u + 1/6)\mathbf{p}_0 +$$
$$(\ 1/2u^3 - u^2 + 2/3)\mathbf{p}_1 +$$
$$(-1/2u^3 + 1/2u^2 + 1/2u + 1/6)\mathbf{p}_2 +$$
$$(\ 1/6u^3\ )\mathbf{p}_3$$



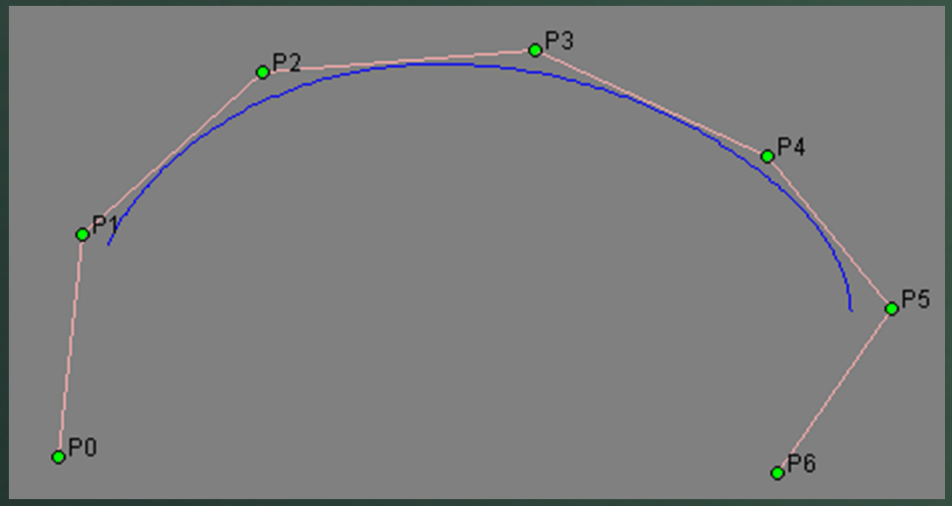
B-Spline Curves

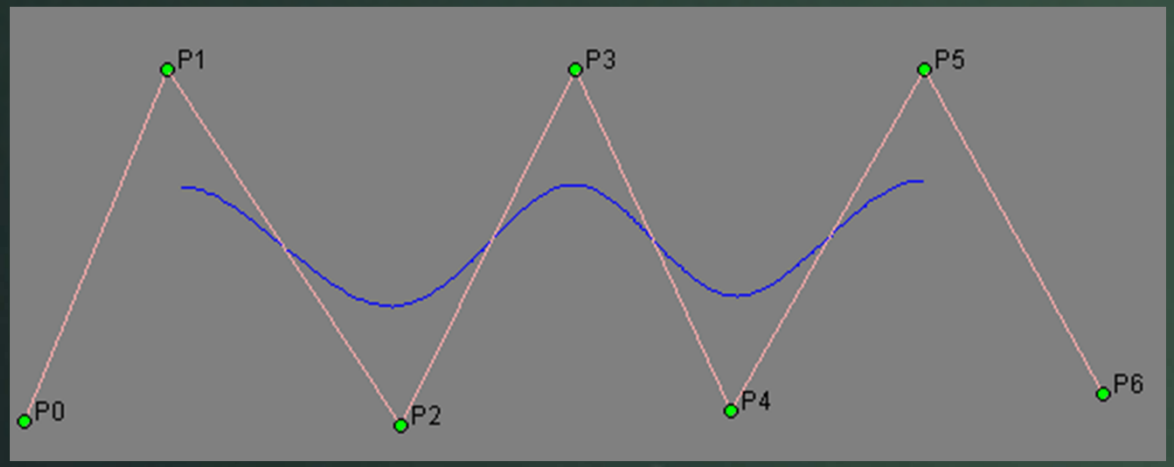
- **In matrix form**

$$Q(u) = \left(u^3, u^2, u, 1\right)\frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}\begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

B-Spline Curves

- **Examples:**





B-Spline Curves

- **Properties:**
  - **Local**
  - **Continuity**
    - P(t) is C^{k-1-r} continuous at a node of repetitiveness r.
  - **Convex hull**
  - **Piecewise polynomial**
  - **Geometry invariant**
  - **Affine invariant**
  - **Flexibility**

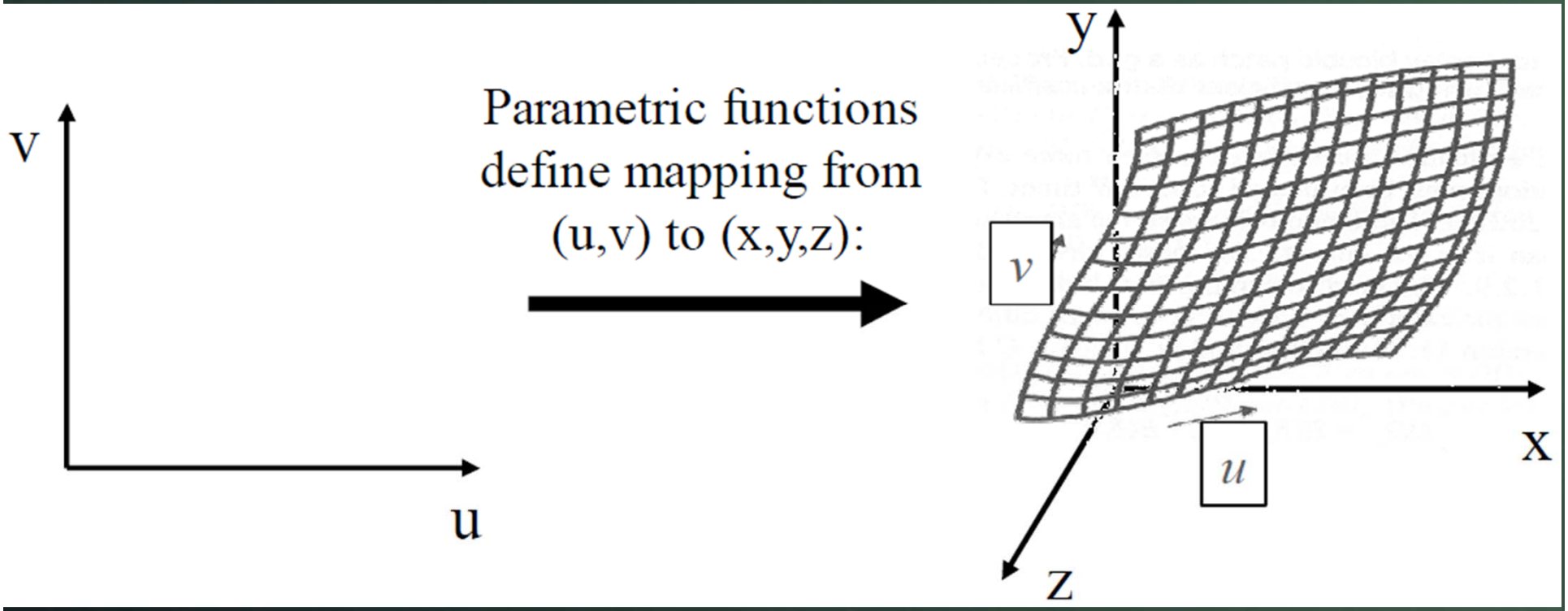
How to Choose a Spline

- **Hermite curves are good for single segments where you know the parametric derivative or want easy control of it**
- **Bézier curves are good for single segments or patches where a user controls the points**
- **B-splines are good for large continuous curves and surfaces**
- **NURBS are the most general, and are good when that generality is useful, or when conic sections must be accurately represented (CAD)**

Parametric Surfaces

- **Boundary defined by parametric functions:**

$$x = f_x(u, v)$$
$$y = f_y(u, v)$$
$$z = f_z(u, v)$$



Parametric Surfaces

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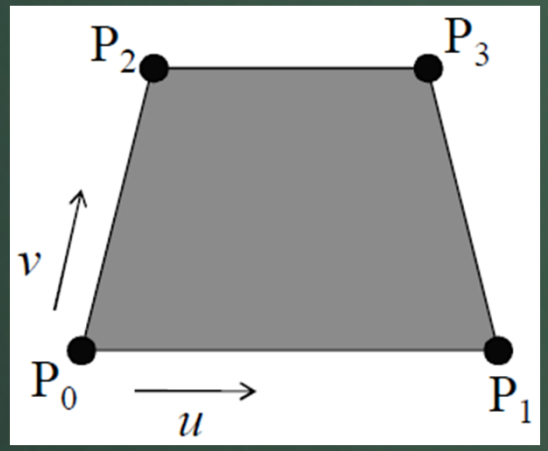

- **Example: quadrilateral**

$$f_x(u, v) = (1 - v)\big((1 - u)x_0 + ux_1\big) + v((1 - u)x_2 + ux_3)$$
$$f_y(u, v) = (1 - v)\big((1 - u)y_0 + uy_1\big) + v((1 - u)y_2 + uy_3)$$
$$f_z(u, v) = (1 - v)\big((1 - u)z_0 + uz_1\big) + v((1 - u)z_2 + uz_3)$$

Parametric Surfaces

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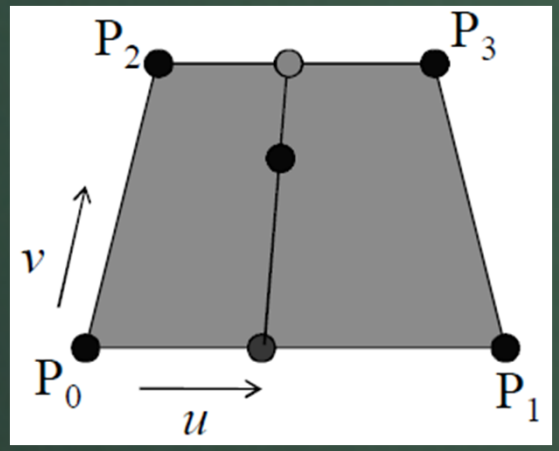


- **Example: quadrilateral**

$$f_x(u,v) = (1-v)\big((1-u)x_0 + ux_1\big) + v\big((1-u)x_2 + ux_3\big)$$
$$f_y(u,v) = (1-v)\big((1-u)y_0 + uy_1\big) + v\big((1-u)y_2 + uy_3\big)$$
$$f_z(u,v) = (1-v)\big((1-u)z_0 + uz_1\big) + v\big((1-u)z_2 + uz_3\big)$$

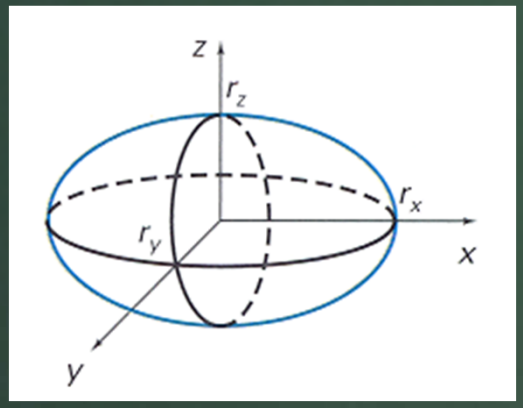
Parametric Surfaces

- **Boundary defined by parametric functions:**

$$x = f_x(u, v)$$
$$y = f_y(u, v)$$
$$z = f_z(u, v)$$

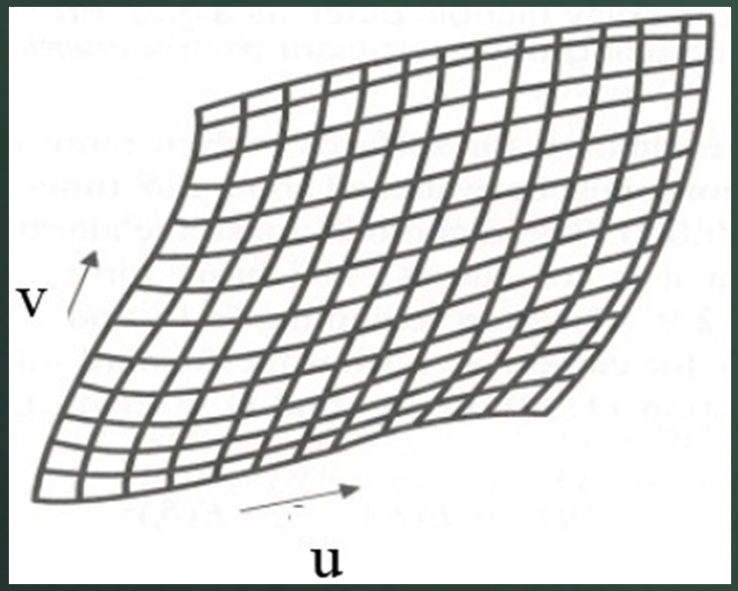
- **Example: ellipsoid**

$$f_x(u, v) = r_x \cos\varphi \cos\theta$$
$$f_y(u, v) = r_y \cos\varphi \sin\theta$$
$$f_z(u, v) = r_z \sin\varphi$$



Parametric Surfaces

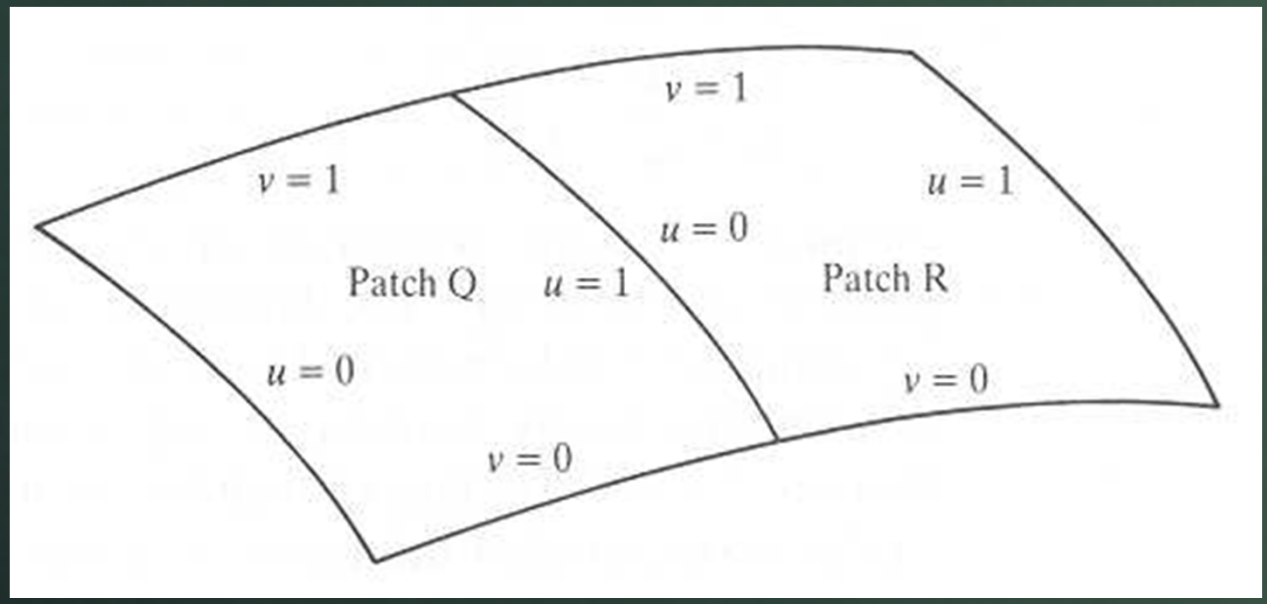
Advantage: easy to enumerate points on surface.



Disadvantage: need piecewise-parametric surface to describe complex shape.

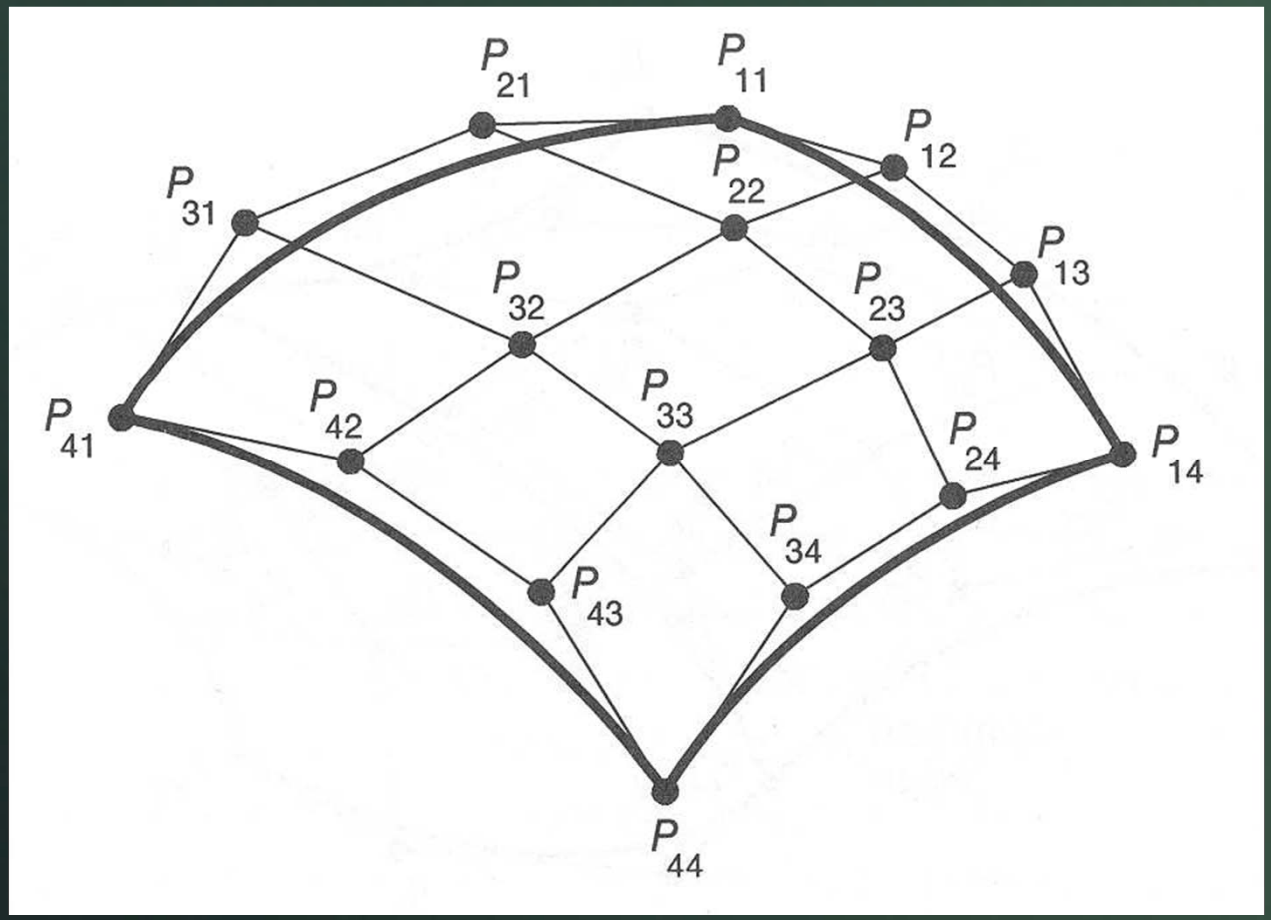
Piecewise Polynomial Parametric Surfaces

- **Surface is partitioned into parametric patches:**



Parametric Patches

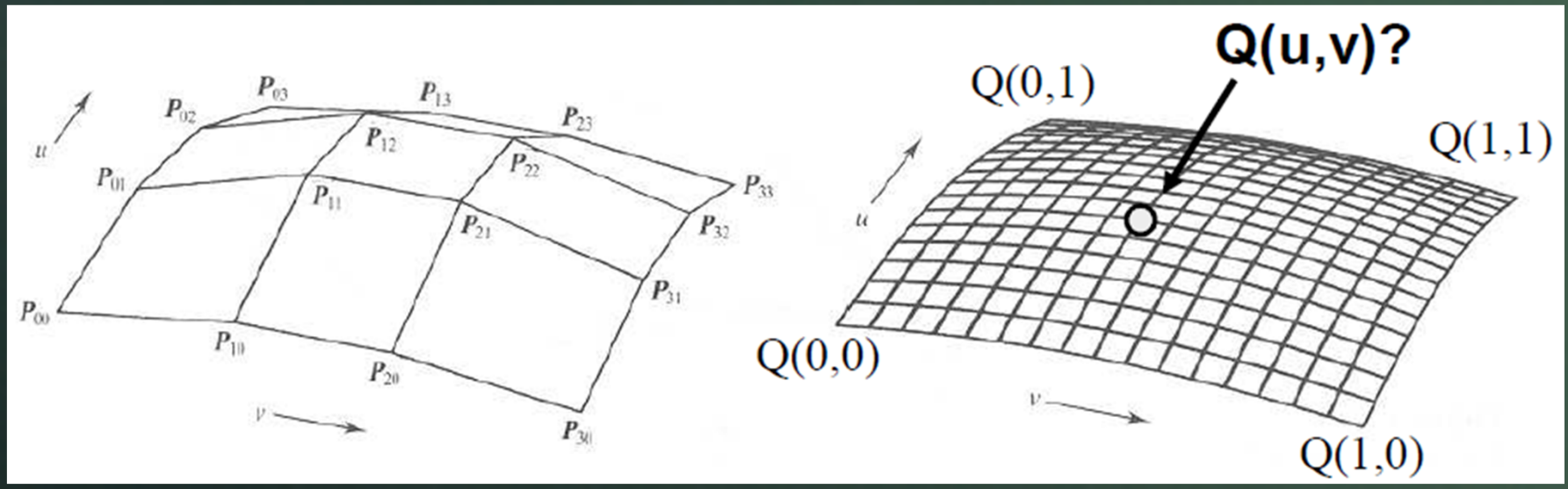
- **Each patch is defined by blending control points**



Same ideas as parametric curves!

Parametric Patches

- **Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points**



Parametric Bicubic Patches

- **Point Q(u,v) on any patch is defined by combining control points with polynomial blending functions:**

$$Q(u,v) = UM \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix} M^T V^T$$

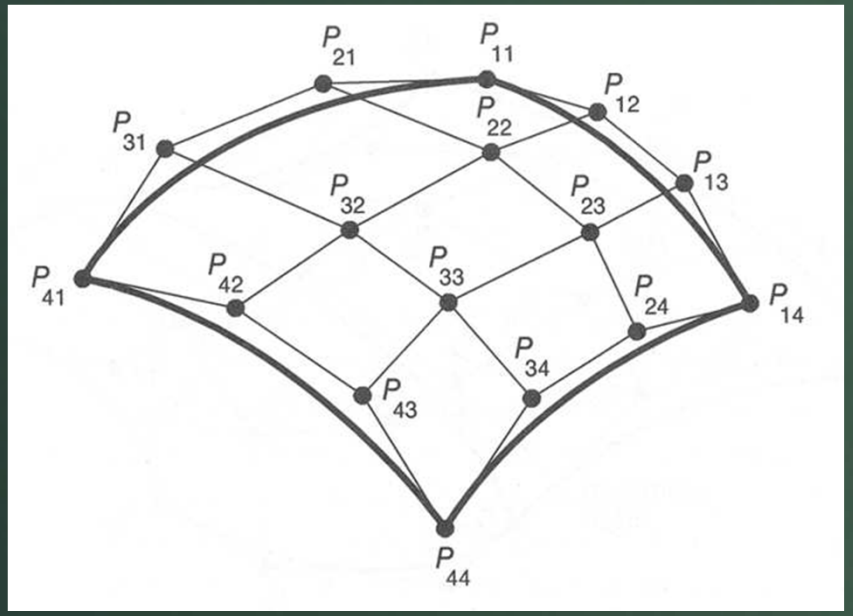
$$\boldsymbol{U = [u^3 \quad u^2 \quad u \quad 1]}, \mathbf{V} = [\boldsymbol{v^3 \quad v^2 \quad v \quad 1}]$$

Where M is a matrix describing the blending functions for a parametric cubic curve (e.g., Bézier, B-spline, etc.)

Bézier Patches

$$Q(u,v) = UM_{Bézier} \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix} M_{Bézier}{}^{T} V^{T}$$

$$M_{Bézier} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

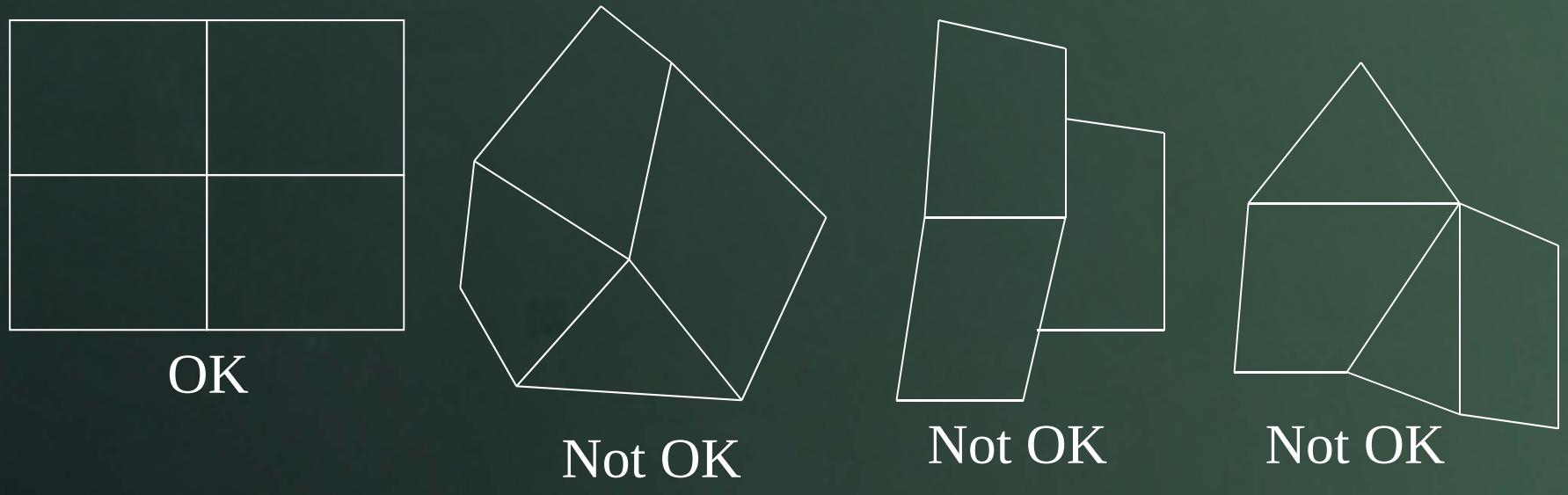


Bézier Patches

- **The patch interpolates its corner points**
  - Comes from the interpolation property of the underlying curves
- **The tangent plane at each corner interpolates the corner vertex and the two neighboring edge vertices**
- **The patch lies within the convex hull of its control vertices**
  - The basis functions sum to one and are positive everywhere

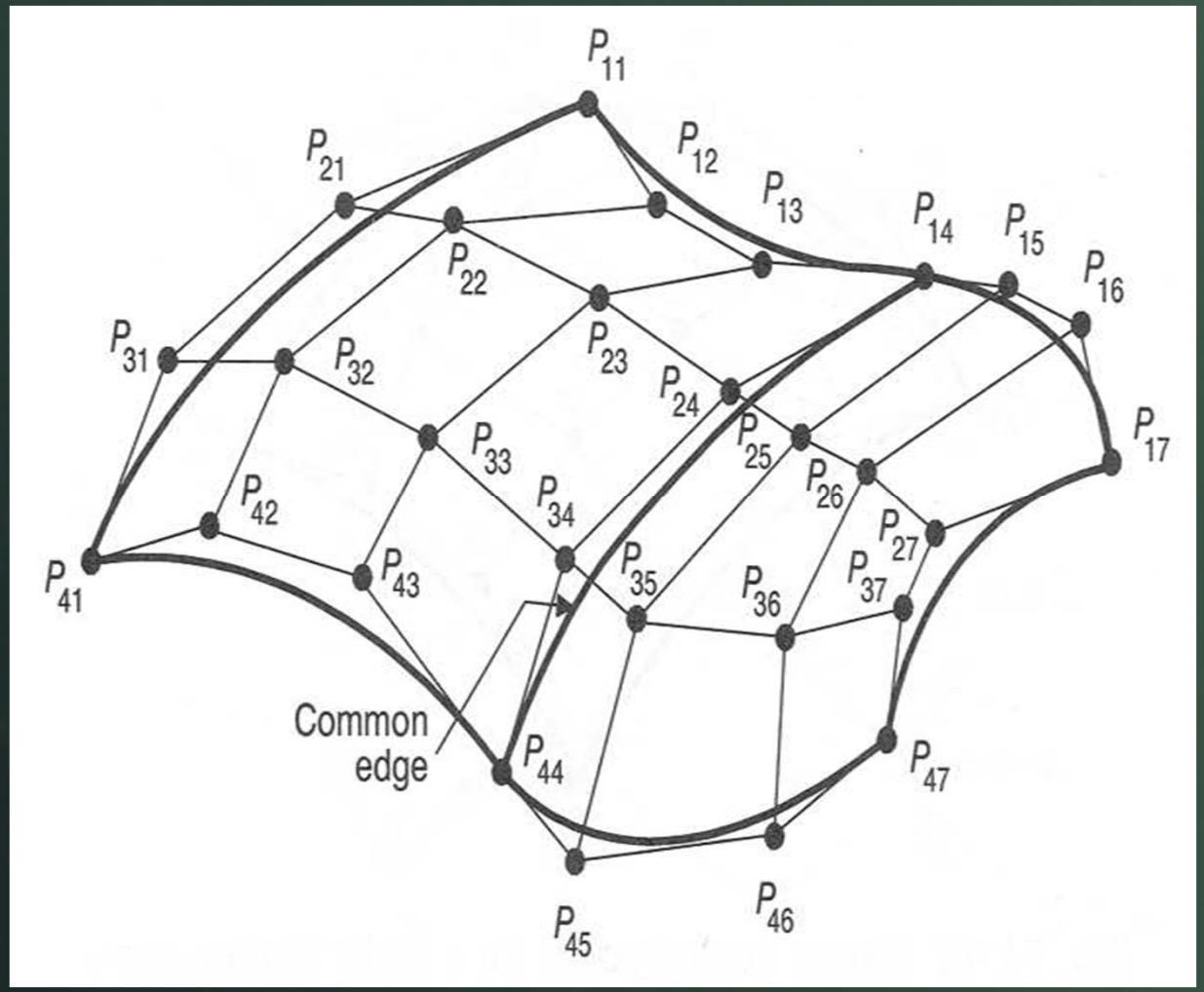
Bézier Patches

- **A patch mesh is just many patches joined together along their edges**
  - **Patches meet along complete edges**
  - **Each patch must be a quadrilateral**



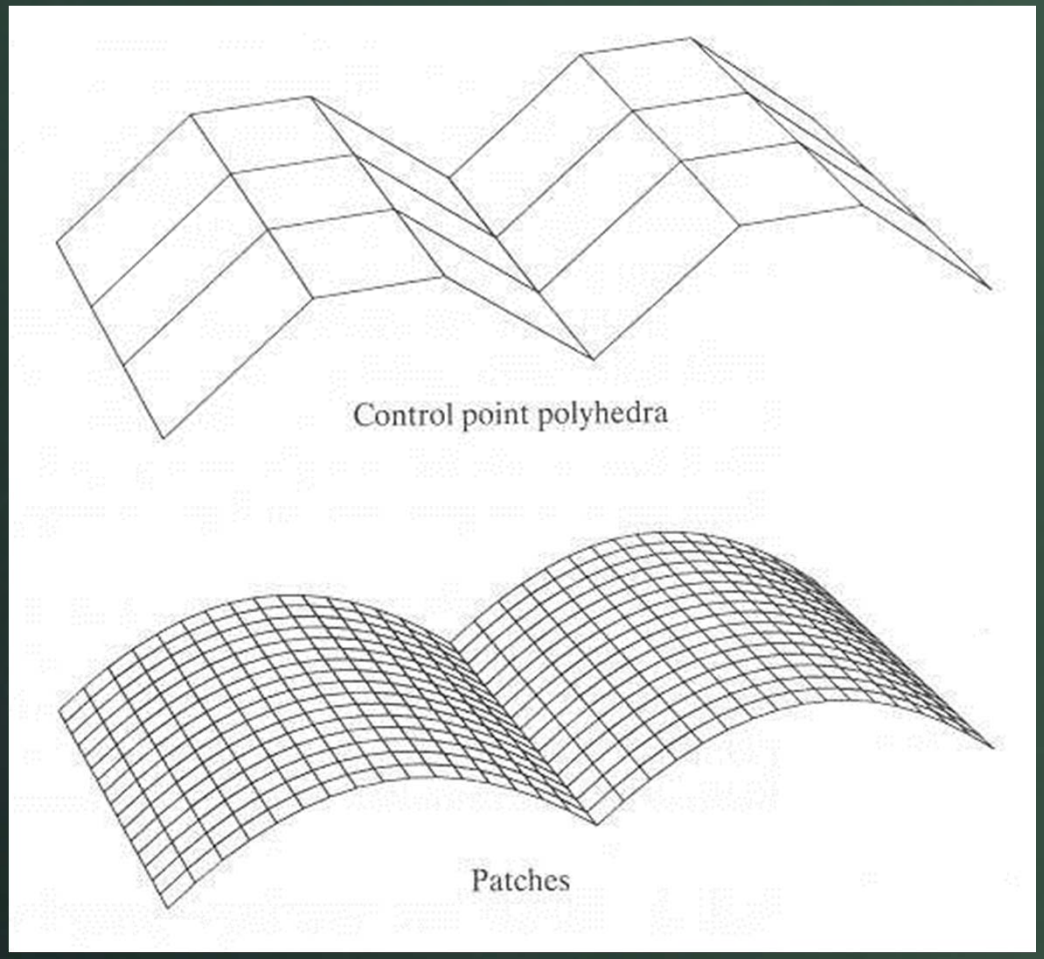
Bézier Surfaces

- **Continuity constraints are similar to the ones for Bézier splines**



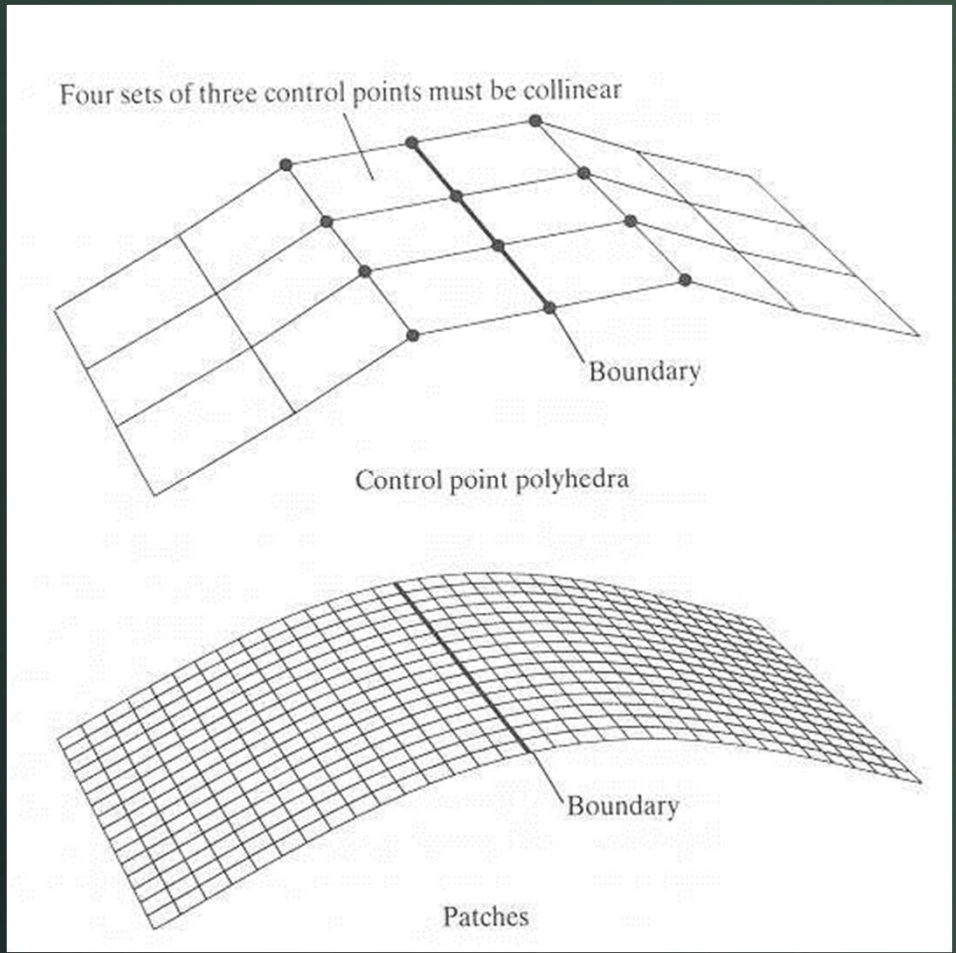
Bézier Surfaces

- **C^0 continuity requires aligning boundary curves**



Bézier Surfaces

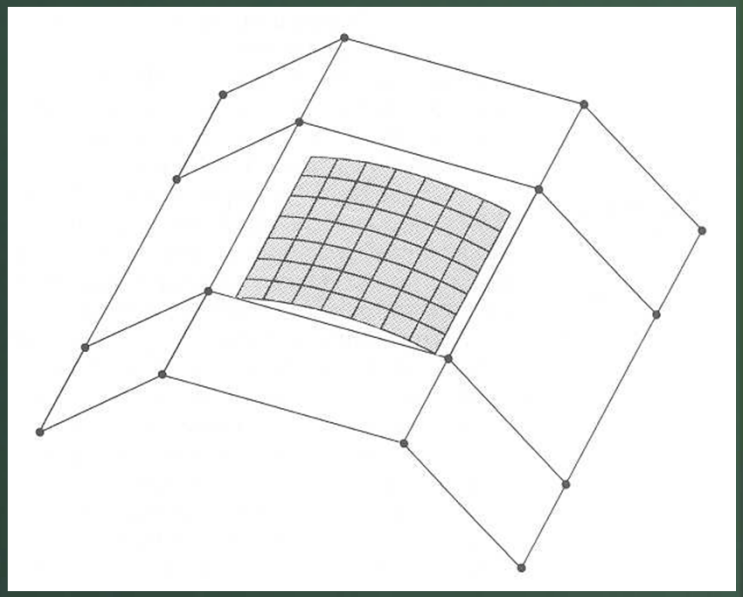
- **C^1 continuity requires aligning boundary curves and derivatives**



B-Spline Patches

$$Q(u,v) = UM_{B-Spline} \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix} M_{B-Spline}{}^{T} V^{T}$$

$$M_{B-Spline} = \begin{bmatrix} -1/6 & 1/2 & -1/2 & 1/6 \\ 1/2 & -1 & 1/2 & 0 \\ -1/2 & 0 & 1/2 & 0 \\ 1/6 & 2/3 & 1/6 & 0 \end{bmatrix}$$



Parametric Surfaces

- **Advantages:**
  - **Easy to enumerate points on surface**
  - **Possible to describe complex shapes**

- **Disadvantages:**
  - **Control mesh must be quadrilaterals**
  - **Continuity constraints difficult to maintain**
  - **Hard to find intersections**

Summary

Feature	Polygon Mesh	Subdivision Surface	Parametric Surface
Accurate	No	Yes	Yes
Concise	No	Yes	Yes
Intuitive specification	No	No	Yes
Local support	Yes	Yes	Yes
Affine invariant	Yes	Yes	Yes
Arbitrary topology	Yes	Yes	No
Guaranteed continuity	No	Yes	Yes
Natural parameterization	No	No	Yes
Efficient display	Yes	Yes	Yes
Efficient intersections	No	No	No