2 Connectedness and Compactness

One of the classical aims of topology is to classify topological spaces by their topological type, or in other terms to find a complete set of topological invariants.
—Samuel Eilenberg (1949)

Introduction. In chapter 1, we discussed four main constructions of topological spaces: subspaces, quotients, products, and coproducts. In this chapter, we'll see how these constructions interact with three main topological properties: connectedness, Hausdorff, and compactness. That is, are subspaces of compact spaces also compact? Is the quotient of a Hausdorff space itself Hausdorff? Are products of connected spaces also connected? Is a union of connected spaces connected? We'll explore these questions and more in the pages to come.

Section 2.1 contains a survey of basic notions, theorems, and examples of connectedness. It also includes a statement and categorical proof of the one-dimensional version of Brouwer's well-known fixed-point theorem. Section 2.2 contains the Hausdorff property, though we'll keep the discussion brief. The Hausdorff property becomes much richer once it's combined with compactness, which is the content of section 2.3. The same section also introduces three familiar theorems—the Bolzano-Weierstrass theorem, the Heine-Borel theorem, and Tychonoff's theorem.

2.1 Connectedness

We'll begin with a discussion of the main ideas about connectedness. The definitions are collected up front and the main results follow. The proofs are mostly left as exercises, but they can be found in most any classic text on topology, such as Willard (1970), Munkres (2000), Kelley (1955), Lipschutz (1965).

2.1.1 Definitions, Theorems, and Examples

Definition 2.1 A topological space X is *connected* if and only if one of the following equivalent conditions holds:

(i) X cannot be expressed as the union of two disjoint nonempty open sets.

(ii) Every continuous function $f\colon X \to \{0, 1\}$ is constant, where $\{0, 1\}$ is equipped with the discrete topology.

Exercise 2.1 at the end of the chapter asks you to prove the equivalence of the two definitions. Even though they are equivalent, we prefer the second. We can define an equivalence relation $\sim'$ on X by declaring $x \sim' y$ if and only if there's a connected subspace of X that contains both x and y. Reflexivity and symmetry are immediate, while transitivity follows from theorem 2.3. The equivalence classes of $\sim'$ are called the *connected components* of X. But there is also a different—and richer—kind of connectedness.

Definition 2.2 A topological space X is said to be *path connected* if and only if for all $x, y \in X$ there is a *path* that connects x and y.

Recall that a path from x to y in a topological space X is a map $\gamma\colon I \to X$ with $\gamma 0 = x$ and $\gamma 1 = y$. There is an equivalence relation on X defined by declaring $x \sim y$ if and only if there is a path in X connecting x and y. The existence of the constant path shows $\sim$ is reflexive. To see that it is symmetric, suppose f is a path from x to y. Then g defined by $gt = f(1-t)$ is a path from y to x. For transitivity, we first define the product of paths. If f is a path from x to y and g is a path from y to z, the *product* $g \cdot f$ is the path from x to z obtained by first traversing f from x to y and then traversing g from y to z, each at twice the speed:

$$(g \cdot f)t = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \le t \le 1 \end{cases} \tag{2.1}$$

This shows that $\sim$ is transitive, and the equivalence classes of $\sim$ are called the *path components* of X. In essence, path components are homotopy classes of maps $* \to X$ since a point $x \in X$ is a map $* \to X$ and a path between two points $* \to X$ is a homotopy between the maps. We will denote the set of all path components in X by $\pi_0 X$.

Equipped with basic definitions, we now list some of the theorems. Commentary will be kept to a minimum as this section is meant to be a highlight of standard results. Do, however, take special notice of our frequent use of condition (ii) in lieu of condition (i) from definition 2.1.

Theorem 2.1 If X is (path) connected and $f\colon X \to Y$, then fX is (path) connected.

Proof. If fX is not connected, then there is a nonconstant map $g\colon fX \to \{0,1\}$, which implies the map $gf\colon X \to \{0,1\}$ is not constant. Now suppose X is path connected. Let $y, y' \in fX$ so that $y = fx$ and $y' = fx'$ for some $x, x' \in X$. By assumption, there is a path $\gamma\colon I \to X$ connecting x and x', and so $f\gamma$ is a path in Y connecting y and y'. □

Corollary 2.1.1 *Connected* and *path connected* are topological properties.

Since quotient maps are continuous surjections, we know quotients preserve (path) connectedness.

Corollary 2.1.2 The quotient of a (path) connected space is (path) connected.

With the right hypothesis, we can go the other way.

Theorem 2.2 Let X be a space and $f\colon X \to Y$ be a surjective map. If Y is connected in the quotient topology and if each fiber $f^{-1}y$ is connected, then X is connected.

Proof. Let $g\colon X \to \{0,1\}$. Since the fibers of f are connected, g must be constant on each fiber of f. Therefore g factors through $f\colon X \to Y$, and there is a map $\overline{g}\colon Y \to \{0,1\}$ that fits into this diagram.

$$\begin{tikzcd} & X \arrow[d, "f"] \arrow[dl, bend right, "g"'] \\ \{0,1\} & Y \arrow[l, dashed, "\overline{g}"] \end{tikzcd}$$

But Y is connected and so $\overline{g}$ must be constant. Therefore $g = \overline{g}f$ is constant. □

Theorem 2.3 Suppose $X = \bigcup_{\alpha\in A} X_\alpha$ and that for each $\alpha \in A$ the space X_α is (path) connected. If there is a point $x \in \bigcap_{\alpha\in A} X_\alpha$ then X is (path) connected.

Proof. Exercise. □

Theorems 2.3 and 2.2 illustrate common strategies in mathematics. Theorem 2.3 involves a space decomposed into a collection of open sets. Information about each open set (they're connected) and information about the intersection (it's nonempty) provides information about the whole space (it's connected). On the other hand, theorem 2.2 involves a space X decomposed into fibers over a base space. Here, information about the base space (it's connected) and information about the fibers (they're connected) provides information about the total space (it's connected). This approach to extending knowledge of parts to knowledge of the whole appears over and over again in mathematics. Something else that commonly appears in mathematics is giving counterexamples to help illuminate definitions.

Example 2.1 The rational numbers $\mathbb{Q}$ are not connected as the continuous map $k\colon \mathbb{Q} \to \{0,1\}$ defined by $kx = 0$ if $x < \sqrt{2}$ and $kx = 1$ if $x > \sqrt{2}$ shows. In fact, the rationals are *totally disconnected*, meaning that the only connected subsets are singletons.

This prompts the question: Does $\mathbb{R}$ have *any* connected subsets? If we have a good definition of connected, then an interval ought to be connected. It turns out that there are no other connected subsets of $\mathbb{R}$.

Theorem 2.4 The connected subspaces of $\mathbb{R}$ are intervals.

Proof. Suppose A is a connected subspace of $\mathbb{R}$ that is not an interval. Then there exist $x, y \in A$ such that $x < z < y$ for some $z \notin A$. Thus

$$A = (A \cap (-\infty, z)) \cup (A \cap (z, \infty))$$

is a separation of A into two disjoint nonempty open sets.

Conversely, suppose I is an interval with $I = U \cup V$ where U and V are nonempty, open and disjoint. Then there exist $x \in U$ and $y \in V$, and we may assume $x < y$. Since the set $U' = [x, y) \cap U$ is nonempty and bounded above, $s := \sup U'$ exists by the completeness of $\mathbb{R}$. Moreover, since $x < s \leq y$ and I is an interval, either $s \in U$ or $s \in V$ and so $(s - \delta, s + \delta) \subseteq U$ or $(s - \delta, s + \delta) \subseteq V$ for some $\delta > 0$. If the former holds, then s fails to be an upper bound on U'. If the latter, then $s - \delta$ is an upper bound for U' which is smaller than s. Both lead to a contradiction. □

The completeness of $\mathbb{R}$ was essential in proving the above, which accounts for our use of part (i) of definition 2.1 in the proof in lieu of part (ii). Now that we've proved that the interval $I = [0, 1]$ is connected, we can prove a couple of nice general results about connectedness and path connectedness. Note that since I is connected, the image of any path is connected. That is, if $k\colon X \to \{0, 1\}$ is a continuous map from a space X, then k is constant along any path $\gamma\colon I \to X$. Here are a couple of immediate consequences.

Theorem 2.5 Path connected implies connected.

Proof. Suppose X is path connected, and let $k\colon X \to \{0, 1\}$ be a function. Choose any two points in X. There exists a path connecting them. Since k must be constant on that path, it takes the same value at these two points. Therefore k is constant. □

Theorem 2.6 *Connected* and *path connected* are homotopy invariants.

Proof. Suppose $f\colon X \to Y$ is a homotopy equivalence, and let $g\colon Y \to X$ and $h\colon Y \times I \to Y$ be a homotopy from fg to id_Y.

Suppose that X is connected. To show that Y is connected, let $k\colon Y \to \{0, 1\}$ be any map, and let $y, y' \in Y$. The map $kf\colon X \to \{0, 1\}$ must be constant since X is connected, and so $kfgy = kfgy'$. Observe that $h(y, -)\colon I \to Y$ is a path from $h(y, 0) = fgy$ to $h(y, 1) = y$, and so $kfgy = ky$. Also, $h(y', -)\colon I \to Y$ is a path from $h(y', 0) = fgy'$ to $h(y', 1) = y'$, and so $kfgy' = ky'$. Therefore, $ky = ky'$, which implies that k is constant.

Now suppose that X is path connected. Since fX is path connected, we need only worry about its complement. But if $y \in Y \smallsetminus fX$, then $h_y\colon I \to Y$ is a path from fgy to y. In other words, any point in $Y \smallsetminus fX$ can be connected by a path h_y to a point in fX. Therefore Y is path connected. □

The connectedness of I has other nice consequences. We begin with a fun result we found in Nandakumar and Rao (2012) and Ziegler (2015).

Theorem 2.7 Every convex polygon can be partitioned into two convex polygons, each having the same area and same perimeter.

Proof. Let P be a convex polygon, and first observe that finding a line that bisects the area of P is not difficult. Simply take a vertical line and consider the difference of the area on the left and the right. As the line moves from left to right the difference goes from negative to positive continuously and therefore must be zero at some point. Of course, there was nothing special about a vertical line. There's a line in every direction which bisects P. So start with the vertical line, and consider the difference between the perimeter on the left and the perimeter on the right. Rotate this line in such a way that it always bisects the area of P, and note that the difference between the perimeters switches sign as the line goes halfway around. Therefore there exists a line that cuts P into two convex polygons, both with equal areas and equal perimeters. $\square$

The next result is a special case of *Brouwer's fixed-point theorem*, a landmark theorem in topology.

Theorem 2.8 Every continuous function $f\colon [-1,1] \to [-1,1]$ has a fixed point.

Proof. Suppose $f\colon [-1,1] \to [-1,1]$ is a continuous function for which $fx \neq x$ for all $x \in [-1,1]$. In particular we have $f(-1) > -1$ and $f1 < 1$. Now define a map $g\colon [-1,1] \to \{-1,1\}$ by

$$gx = \frac{x - fx}{|x - fx|}$$

Then g is continuous and $g(-1) = -1$ and $g1 = 1$. But this is impossible since $[-1,1]$ is connected. $\square$

We've just proved the $n = 1$ version of Brouwer's fixed-point theorem which states more generally that for all $n \geq 1$, any continuous function $D^n \to D^n$ must have a fixed point, where n denotes the n-dimensional disk. The result when $n = 2$ is proved in section 6.6.3, where we use a functor called the fundamental group. In fact, we can reprove the $n = 1$ case using a different but closely related functor, π_0.

2.1.2 The Functor π_0

As we hinted earlier in the chapter, there is an assignment $X \mapsto \pi_0 X$ that associates to a space X its set of path components $\pi_0 X$. Now suppose $f\colon X \to Y$ is continuous and $A \subseteq X$ is a path component of X. Then fA is connected and thus contained in a unique path component of Y. Therefore the function $\pi_0 f$ that sends A to the path component containing fA defines a function from $\pi_0 X \to \pi_0 Y$. These data assemble into a functor

$$\mathsf{Top} \xrightarrow{\pi_0} \mathsf{Set}$$

$$\begin{array}{ccc} X & & \pi_0 X \\ {\scriptstyle f}\big\downarrow & \mapsto & \big\downarrow{\scriptstyle \pi_0 f} \\ Y & & \pi_0 Y \end{array}$$

Now the fact that functors respect composition when applied to morphisms—often referred to as *functoriality*—makes them quite powerful. To illustrate, let's recast the proof of theorem 2.8 by using the functoriality of π_0. So suppose $f\colon [-1,1] \to [-1,1]$ is continuous. If $fx \neq x$ for any x, then the map $g\colon [-1,1] \to \{-1,1\}$ defined by

$$gx = \frac{x - fx}{|x - fx|} = \begin{cases} -1 & \text{if } x < fx \\ 1 & \text{if } x > fx \end{cases}$$

is continuous, assuming $\{-1,1\}$ is given the discrete topology. So we have a homeomorphism $\{-1,1\} \to \{-1,1\}$ that factors through $[-1,1]$, which is to say it can be written as a composition of the inclusion $i\colon \{-1,1\} \hookrightarrow [-1,1]$ with g.

$$\begin{tikzcd} \{-1,1\} \arrow[r, hook, "i"] \arrow[rr, bend left, "\mathrm{id}"] & {[-1,1]} \arrow[r, "g"] & \{-1,1\} \end{tikzcd}$$

Applying π_0, we get a diagram of sets

$$\begin{tikzcd} \{-1,1\} \arrow[r, hook, "\pi_0 i"] \arrow[rr, bend left, "\pi_0\,\mathrm{id}=\mathrm{id}"] & * \arrow[r, "\pi_0 g"] & \{-1,1\} \end{tikzcd}$$

But this is impossible! No map $\{-1,1\} \to *$ can be left invertible, nor can a map $* \to \{-1,1\}$ be right invertible.

2.1.3 Constructions and Connectedness

In chapter 1, we worked through the constructions of new topological spaces from old ones. So far in this chapter, our discussion has centered on two topological properties: connectedness and path connectedness. We've already seen some interactions between these properties and the constructions, but let's systematically run through the the constructions and check whether they preserve connectedness. Quotients do, as stated in corollary 2.1.2. Subspaces don't preserve connectedness: it doesn't take much imagination to come up with an example. Neither do coproducts—the disjoint union of two connected spaces won't be connected—but remember theorem 2.3 if the union is not disjoint. Products, as the next theorem shows, do preserve connectedness.

Theorem 2.9 Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of (path) connected topological spaces. Then $X := \prod_{\alpha\in A} X_\alpha$ is (path) connected.

Proof. We'll prove the theorem for path connected spaces and will leave the rest as an exercise. Suppose X_α is path connected for every $\alpha \in A$, and let $a, b \in X$. Since each X_α is path connected, there exists a path $p_\alpha\colon [0,1] \to X_\alpha$ connecting a_α to b_α. By the universal property of the product topology, the unique function $p\colon [0,1] \to X$ defined by declaring that $\pi_\alpha p = p_\alpha$ for all α is continuous, and moreover p is a path from a to b. $\square$

So we've addressed the direct question for subspaces, quotients, coproducts, and products. Before we move on to other topological properties, there are a couple of other interesting things we can say about connectedness and coproducts.

Every topological space X is partitioned by its connected components $\{X_\alpha\}$. When viewed as a set, X is always equal to the disjoint union of its connected components

$$X = \coprod_\alpha X_\alpha$$

But if we view X as a topological space, it may or may not be homeomorphic to the coproduct of its connected components. For example, the connected components of the rationals $\mathbb{Q}$ are singletons $\{r\}$. But as a topological space, $\mathbb{Q}$ is *not* homeomorphic to $\coprod_{r\in\mathbb{Q}}\{r\}$ (why not?), which is a countable discrete space. A more positive result is the following, whose proof we leave as an exercise.

Theorem 2.10 The following are equivalent.

(i) A space X is the coproduct of its connected components

(ii) The connected components of X are open.

(iii) The quotient space $X/\sim$ of X by its connected components is discrete.

Recall from definition 2.1 that a space is connected if and only if the only maps from it to a two-point discrete space are constant. Let's make this a little more categorical. Observe that for any space X there is exactly one function $X \to *$. And let's think of a two-point discrete space as the coproduct $* \coprod *$. Now if X is connected, then there are precisely two maps $X \to * \coprod *$; namely, the two constant functions: X maps to the first point and X maps to the second point. So the set $\mathsf{Top}(X, * \coprod *)$ is the two-point set, which is canonically isomorphic to $\mathsf{Top}(X, *) \coprod \mathsf{Top}(X, *)$.

However, if X is not connected, then there are *more* than two maps $\mathsf{Top}(X, * \coprod *)$. For example, if $X = [0, 1] \cup [2, 3]$, then there are four functions $X \to * \coprod *$. So the set $\mathsf{Top}(X, * \coprod *)$ is not equal to $\mathsf{Top}(X, *) \coprod \mathsf{Top}(X, *)$. These observations motivate a definition of connectedness that makes sense in *any* category that has coproducts, including Top.

Theorem 2.11 A space X is connected if and only if the functor $\mathsf{Top}(X, -)$ preserves coproducts.

For more information, the categorically minded reader is encouraged to consult the entry on connectedness at the nLab (Stacey et al., 2019).

Wrapping up this brief excursion on constructions, we've seen that connectedness and path-connectedness are preserved by products and quotients but are not preserved by subspaces or coproducts. With that in mind, let's now turn our attention to a *local* version of connectedness.

2.1.4 Local (Path) Connectedness

Definition 2.3 A topological space is *locally connected* (or *locally path connected*) if and only if for each $x \in X$ and every neighborhood $U \subseteq X$ of x, there is a connected (or path connected) neighborhood V of x with $V \subseteq U$.

Example 2.2 Consider the graph of $fx = \sin(1/x)$ where $x > 0$ along with part of the y axis ranging from $(0, -1)$ to $(0, 1)$. This space, called the *topologist's sine curve*, is connected but not path connected. See figure 2.1.

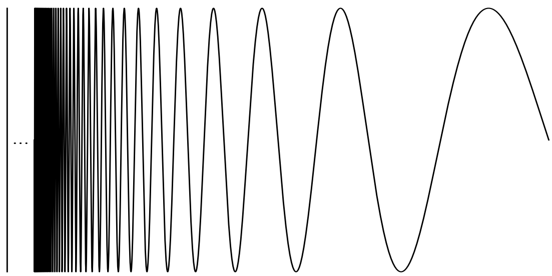


Figure 2.1 The topologist's sine curve

If a space X is locally connected, then the connected components are open, as can be easily verified. This has several consequences. For one, theorem 2.10 implies that locally connected spaces are the coproducts of their connected components. We also have the following.

Theorem 2.12 In any locally path connected topological space, the connected components and path components are the same.

Proof. Exercise. □

Example 2.3 The topologist's sine curve from example 2.2, then, is connected but not locally connected. However, the space $[0, 1] \cup [2, 3]$ is locally connected but not connected.

The previous example illustrates that neither connectedness nor local connectedness implies the other, and the same is true if we replace "connected" with "path connected."

Example 2.4 Let $C = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$, and set $X = (C \times [0, 1]) \cup ([0, 1] \times \{0\})$. Then X, called the *comb space*, is path connected but not locally path connected. See figure 2.2.

On the other hand, the set $[0,1] \cup [2,3]$ in $\mathbb{R}$ with the subspace topology is locally path connected but not path connected.

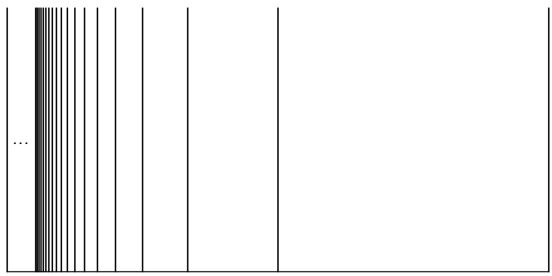
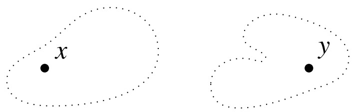


Figure 2.2 The comb space

2.2 Hausdorff Spaces

In the previous section, we discussed connectedness, which in a sense describes when a space can or cannot be separated into nonoverlapping "chunks." The next topological property arises when one seeks for separation at the level of individual points.

Definition 2.4 A space X is *Hausdorff* if and only if for every two points x and y, there exist disjoint open sets U and V with $x \in U$ and $y \in V$.



First, it's good to check that Hausdorff defines a topological property but not a homotopy invariant property. Then we might wonder which constructions preserve the Hausdorff property. One finds that subspaces of Hausdorff spaces are Hausdorff, products of Hausdorff spaces are Hausdorff, and coproducts of Hausdorff spaces are Hausdorff, but quotients of Hausdorff spaces are not necessarily Hausdorff. In fact, quotients of Hausdorff spaces are *the* source of non-Hausdorff spaces throughout the mathematical world. But quotients and Hausdorff spaces do interact well in the following sense.

Theorem 2.13 Every space X is the quotient of a Hausdorff space H.

Proof. Omitted. See Shimrat (1956). $\square$

Example 2.5 Metric spaces are Hausdorff. To see this, let x and y be points in a metric space. If $x \neq y$, then $d := d(x, y) > 0$ and $B(x, \frac{d}{2})$ and $B(y, \frac{d}{2})$ are disjoint open sets separating x and y.

Theorem 2.14 A space X is Hausdorff if and only if the diagonal map $\Delta\colon X \to X \times X$ is closed.

Proof. Exercise. □

The Hausdorff property interacts with other topological properties in some far-reaching ways. In particular, it gives rise to rich results when combined with compactness.

2.3 Compactness

In this section we introduce the notion of compactness, along with examples and theorems. Admittedly, the proofs in this section have a classical rather than a categorical flavor. But don't fret. Instead, we encourage you to eagerly anticipate chapter 5 where we'll revisit compact Hausdorff spaces in great categorical detail.

2.3.1 Definitions, Theorems, and Examples

Definition 2.5 A collection $\mathcal{U}$ of open subsets of a space X is called an *open cover* for X if and only if the union of sets in $\mathcal{U}$ contains X. The space X is *compact* if and only if every open cover of X has a finite subcover.

Theorem 2.15 If X is compact and $f\colon X \to Y$ is continuous, then fX is compact.

Proof. Exercise. □

Corollary 2.15.1 *Compactness* is a topological property.

One way to think of compact spaces is that they are somehow small—not in terms of cardinality but in terms of roominess. For example, if you squeeze an infinite set of points into the unit interval, they'll get cramped—for any $\varepsilon > 0$, there are two points that are less than ε apart. But it's easy to fit an infinite number of points in the real line so that they're all spread out. Indeed, the unit interval is compact while the real line is not. This idea is summarized in the next theorem. First, a piece of terminology. A point x is called a *limit point* of a space X if every neighborhood of x contains a point of $X \smallsetminus \{x\}$.

The Bolzano-Weierstrass Theorem Every infinite set in a compact space has a limit point.

Proof. Suppose that F is an infinite subset with no limit points. If x is not a limit point of F and $x \notin F$, there is an open set U_x around x that misses F. If x is not a limit point of F and $x \in F$, then there is an open set U_x with $U_x \cap F = \{x\}$. Then $\{U_x\}_{x \in X}$

is an open cover of X. Notice that there can be no finite subcover $U_{x_1}, \ldots, U_{x_n}$ since $(U_{x_1} \cup \cdots \cup U_{x_n}) \cap F = \{x_1, \ldots, x_n\}$, and cannot contain the infinite set F. □

Example 2.6 Note that compactness is not necessary in the previous theorem, as there exist noncompact spaces for which every infinite subset has a limit point. For instance, take $\mathbb{R}$ with topology $\{(x, \infty) : x \in \mathbb{R}\}$ together with $\varnothing$ and $\mathbb{R}$. This space is not compact, but any set (infinite or not) has a limit point (infinitely many, in fact).

In general, directly checking if a space is compact can be tricky. The following definition sets the stage for an alternate criterion, as described in the next theorem.

Definition 2.6 Let $\mathcal{S}$ be a collection of sets. We say that $\mathcal{S}$ has the *finite intersection property* if and only if for every finite subcollection $A_1, \ldots, A_n \subseteq \mathcal{S}$, the intersection $A_1 \cap \cdots \cap A_n \neq \varnothing$. We abbreviate the finite intersection property by FIP.

Theorem 2.16 A space X is compact if and only if every collection of closed subsets of X with the FIP has nonempty intersection.

Proof. Exercise. □

Here's yet another way to check for compactness.

Theorem 2.17 Closed subsets of compact spaces are compact.

Proof. Let X be compact with $C \subseteq X$ closed and suppose $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ is an open cover of C. Then $X \smallsetminus C$ together with $\mathcal{U}$ forms an open cover of X. Since X is compact, there are finitely many sets $\{U_i\}_{i=1}^n$ in $\mathcal{U}$, possibly together with $X \smallsetminus C$, which covers X. Thus $\{U_i\}_{i=1}^n$ is a finite subcover for C. □

Now we're ready to see how compactness and the Hausdorff property interact. To start, compact subsets of Hausdorff spaces are quite nice—they can be separated from points by open sets.

Theorem 2.18 Let X be Hausdorff. For any point $x \in X$ and any compact set $K \subseteq X \smallsetminus \{x\}$ there exist disjoint open sets U and V with $x \in U$ and $K \subseteq V$.

Proof. Let $x \in X$ and let $K \subsetneq X$ be compact. For each $y \in K$, there are disjoint open sets U_y and V_y with $x \in U_y$ and $y \in V_y$. The collection $\{V_y\}$ is an open cover of K; hence there is a finite subcover $\{V_1, \ldots, V_n\}$. Let $U = U_1 \cap \cdots \cap U_n$ and $V = V_1 \cup \cdots \cup V_n$. Then U and V are disjoint open sets with $x \in U$ and $K \subseteq V$. □

This theorem quickly gives rise to two important corollaries.

Corollary 2.18.1 Compact subsets of Hausdorff spaces are closed.

Proof. Exercise. □

Corollary 2.18.2 If X is compact and Y is Hausdorff, then every map $f\colon X \to Y$ is closed.[1] In particular,

- if f is injective, then it is an embedding;
- if f is surjective, then it is a quotient map;
- if f is bijective, then it is a homeomorphism.

Proof. Let $f\colon X \to Y$ be a map from a compact space to a Hausdorff space, and let $C \subseteq X$ be closed. Then C is compact, so fC is compact, so fC is closed. □

As you'll recall from example 1.13, not every continuous bijection $f\colon X \to Y$ is a homeomorphism. The previous corollary guarantees us that such maps are homeomorphisms whenever X is compact and Y is Hausdorff.

2.3.2 Constructions and Compactness

As with our discussion on connectedness, we are also interested in the preservation of compactness under the four constructions: subspaces, quotients, products, and coproducts. Subspaces of compact spaces are not compact in general, but we saw in theorem 2.17 that closed subspaces of compact spaces are compact. You'll also realize that we've proved that quotients of compact spaces are compact. Coproducts of compact spaces are certainly not compact—just look at the coproduct of infinitely many copies of a point. But what about products? There are a few interesting things to explain here, and we'll start with *Tychonoff's theorem* and some of its corollaries.

Tychonoff's Theorem 1 The product of compact spaces is compact.

Proof. See section 3.4. □

Corollary 2.18.3 (Heine-Borel Theorem) A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof. Suppose that $K \subset \mathbb{R}^n$ is compact. Since the cover of K consisting of open balls centered at the origin of all possible radii must have a finite subcover, K must be bounded. Since $\mathbb{R}^n$ is Hausdorff and all compact subsets of a Hausdorff space must be closed, K is closed.

Conversely (and this is the part that uses the Tychonoff theorem), suppose that $K \subset \mathbb{R}^n$ is closed and bounded. Since K is bounded, the projection of K onto the ith coordinate is bounded; that is, for each i there's an interval $[a_i, b_i]$ containing $\pi_i K$. Then $K \subseteq [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$. Since each set $[a_i, b_i]$ is compact, the Tychonoff theorem implies

[1] The map f is *closed* if fC is closed whenever $C \subseteq X$ is closed. See exercise 1.14 at the end of chapter 1.

that the product $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ is compact. Since any closed subset of a compact space is compact, we conclude that K is compact. □

Corollary 2.18.4 Continuous functions from compact spaces to $\mathbb{R}$ have both a global maximum and a global minimum.

Proof. Exercise. □

The characterization of compact subsets of $\mathbb{R}^n$ as closed and bounded may be familiar from analysis, but recall that *bounded* is *not* a topological property! For example, there is a homeomorphism of topological spaces $\mathbb{R} \cong (0, 1)$, yet $\mathbb{R}$ is not a bounded metric space while $(0, 1)$ is. It's also not a homotopy invariant, and neither is compactness.

Example 2.7 Like any space whose underlying set is finite, the one-point set $*$ is compact. Since $\mathbb{R}$ is not compact but is homotopy equivalent to $*$, we see that compactness is not a homotopy invariant.

Finally, we have the so-called *Tube Lemma*, which isn't a corollary of Tychonoff, but it does concern compact sets and products. First, here's an example.

Example 2.8 Let U be the interior of the triangle with corners $(0, 0)$, $(1, 0)$, and $(1, 1)$—

$$U := \{(x, y) \subset \mathbb{R}^2 \mid 0 < x < 1,\ 0 < y < x\}$$

—and consider the set $A \times \left\{\frac{1}{2}\right\}$ where A is the interval $A = \left(\frac{1}{2}, 1\right)$. Then $A \times \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right)$ is not contained in U for any $\varepsilon > 0$. But if A were compact. . . .

The Tube Lemma For any open set $U \subseteq X \times Y$ and any set $K \times \{y\} \subseteq U$ with $K \subseteq X$ compact, there exist open sets $V \subseteq X$ and $W \subseteq Y$ with $K \times \{y\} \subseteq V \times W \subseteq U$.

Proof. For each point $(x, y) \in K \times \{y\}$, there are open sets $V_x \subseteq X$ and $W_x \subseteq Y$ with $(x, y) \in V_x \times W_x \subseteq U$. Then, $\{V_x\}_{x \in K}$ is an open cover of K; take a finite subcover $\{V_1, \ldots, V_n\}$. Then $V = V_1 \cup \cdots \cup V_n$ and $W = W_1 \cap \cdots \cap W_n$ are open sets with $K \times \{y\} \subseteq V \times W \subseteq U$. □

We now close by briefly mentioning the local version of compactness.

2.3.3 Local Compactness

We will define local compactness by way of saying that "spaces that are *locally compact* are spaces whose neighborhoods look like neighborhoods of compact spaces."

Definition 2.7 A space X is *locally compact* if and only if for every point $x \in X$ there exists a compact set K and a neighborhood U with $x \in U \subseteq K$.

Example 2.9 Every compact space is locally compact, as is every discrete space. Also, $\mathbb{R}^n$ is locally compact; however, the real line with the lower limit topology $\mathcal{T}_{\text{ll}}$ (example 1.3) is not, as the reader can verify.

Note that the image of a locally compact space need not be locally compact. For example, consider the map $\mathrm{id}\colon (\mathbb{R}, \mathcal{T}_{\text{discrete}}) \to (\mathbb{R}, \mathcal{T}_{\text{ll}})$. Nonetheless, locally compact is a topological property, as one can verify. For Hausdorff spaces, local compactness is much stronger.

Theorem 2.19 Suppose X is locally compact and Hausdorff. Then for every point $x \in X$ and every neighborhood U of x, there exists a neighborhood V of x such that the closure $\overline{V}$ is compact and $x \in V \subseteq \overline{V} \subseteq U$.

Proof. This is a corollary of theorem 2.18 and the definition of local compactness. □

Lastly, we mention that the product and quotient topologies are not compatible in the sense of exercise 1.11 at the end of chapter 1, but the hypothesis of locally compact and Hausdorff makes the situation much better.

Theorem 2.20 If $X_1 \twoheadrightarrow Y_1$ and $X_2 \twoheadrightarrow Y_2$ are quotient maps and Y_1 and X_2 are locally compact and Hausdorff, then $X_1 \times X_2 \twoheadrightarrow Y_1 \times Y_2$ is a quotient map.

Proof. We postpone the proof until theorem 5.7 in chapter 5. □

Exercises

1. Prove that the two items in definition 2.1 are indeed equivalent.

2. A map $X \to Y$ is *locally constant* if for each $x \in X$ there is an open set U with $x \in U$ and $f|_U$ constant. Prove or disprove: if X is connected and Y is any space, then every locally constant map $f \colon X \to Y$ is constant.

3. Show that every countable metric space with at least two points must be disconnected. Construct a topological space with more than two elements that is both countable and connected.

4. In a variation of the topology on $\mathbb{Z}$ in example 1.5, consider the natural numbers $\mathbb{N}$ with topology generated by the basis

$$\{ak + b \mid k \in \mathbb{N} \text{ and } a, b \in \mathbb{N} \text{ are relatively prime}\}$$

Prove that $\mathbb{N}$ with this topology is connected (Golomb, 1959).

5. Let $\{X_\alpha\}_\alpha$ be a collection of spaces. Prove that $\pi_0 \prod X_\alpha \cong \prod \pi_0 X_\alpha$. Note: the special case $\pi_0 X_\alpha = *$ for all α is the statement that the product of path connected spaces is path connected.

6. Provide a proof of theorem 2.10.

7. Prove that a space X is connected if and only if the functor $\mathsf{Top}(X, -)$ preserves coproducts.

8. Show that $\mathbb{Q} \subseteq \mathbb{R}$ with the subspace topology is not locally compact.

9. Prove that the product of two locally compact Hausdorff spaces is locally compact Hausdorff.

10. Define a space X to be *pseudocompact* if and only if every real valued function on X is bounded. Prove that if X is compact, then X is pseudocompact, and give an example of a pseudocompact space that is not compact.

11. Give examples showing that locally compact is not preserved by subspaces, quotients, or products.

12. Let $\mathcal{U}$ be an open cover of a compact metric space X. Show that there exists an $\varepsilon > 0$ such that for every $x \in X$, the set $B(x, \varepsilon)$ is contained in some $U \in \mathcal{U}$. Such an ε is called a *Lebesgue number for* $\mathcal{U}$.

13. Show that $\mathbb{Z}$ endowed with the arithmetic progression topology of example 1.5 is not locally compact.

14. Suppose (X, d) is a compact metric space and $f \colon X \to X$ is an isometry; that is, for all $x, y \in X$, $d(x, y) = d(fx, fy)$. Prove f is a homeomorphism.

15. Let X be a space and suppose $A, B \subseteq X$ are compact. Prove or disprove:

a) $A \cap B$ is compact.

b) $A \cup B$ is compact.

If a statement is false, find a sufficient condition on X which will cause it to be true.

16. Let $B = \left\{\{x_n\} \in l^2 \mid \sum_{n=1}^{\infty} x_n^2 \leq 1\right\}$ be the closed unit ball in l^2, where l^2 is the space defined in example 1.8 of chapter 1. Show that B is not compact.

17. Prove that if Y is compact, then for any space X the projection $X \times Y \to X$ is a closed map. Give an example of spaces X and Y for which the projection $X \times Y \to X$ is not closed.

18. Show that the product of Hausdorff spaces is Hausdorff. Give an example to show that the quotient of a Hausdorff space need not be Hausdorff.

19. If X is any set and Y is Hausdorff, then a subset $A \subseteq \mathsf{Top}(X, Y)$ has compact closure in the product topology if and only if for each $x \in X$, the set $A_x = \{fx \in Y \mid f \in A\}$ has compact closure in Y.

20. For any map $f: X \to Y$, the set $\Gamma = \{(x, y) \in X \times Y \mid y = fx\}$ is called the *graph of f*. Suppose now that X is any space and Y is compact Hausdorff. Prove that Γ is closed if and only if f is continuous. Is the compactness condition necessary? (This is called the *closed graph theorem*.)

21. Let X be a Hausdorff space with $f: X \to Y$ a continuous closed surjection such that $f^{-1}y$ is compact for each $y \in Y$. Prove that Y is Hausdorff.

22. Prove or disprove: if $f: X \to Y$ is a continuous bijection and X is Hausdorff, then Y must be Hausdorff.

23. Prove or disprove: X is Hausdorff if and only if

$$\{(x, x, \ldots) \in X^{\mathbb{N}} \mid x \in X\}$$

is closed in $X^{\mathbb{N}}$.

24. Topologies that are compact and Hausdorff are nicely balanced. Take for an example $[0, 1]$.

a) Prove that if $\mathcal{T}$ is any topology on $[0, 1]$ finer than the ordinary one, then $[0, 1]$ cannot be compact in the topology $\mathcal{T}$.

b) Prove that if $\mathcal{T}$ is any topology on $[0, 1]$ coarser than the usual one, then $[0, 1]$ cannot be Hausdorff in the topology $\mathcal{T}$.