1 Introduction

Affine space is always mysterious for me and it is very confusing and not well defined in High school and University mathematical lecture. It makes your more miserable after you have learned vector Space. However, the confusing makes you wonder what is the difference between Affine Space and vector Space and what is the relationship between them. French mathematician said "An affine space is nothing more than a vector whose origin we try to forget about, by adding translation to linear map"
In vector space, we have zero vector as origin, e.g. in $\mathbb{R}^2$

$$v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

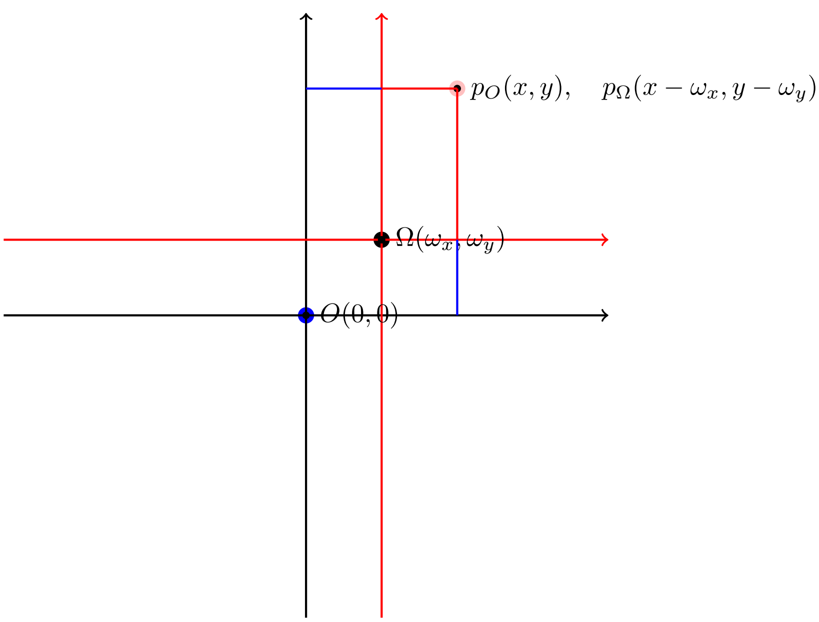
One of the most important property of vector Space is Linear Combination, any vector can be represent by linear combination of their basis, e.g.

$$v = \begin{bmatrix} 2 \\ 3 \end{bmatrix} v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} v = 2v_1 + 3v_2 \text{ where } \beta = \{v_1, v_2\} \text{ is called basis}$$

In vector Space, any vector can be represent by linear combination of basis vectors uniquely.
How can we use this concept and come up with similar concept, e.g. [linear combination] for points

Let's review the concept of coordinates system.
A point p represents as $p = (2,3)$ or $(2,3)$ is called the coordinates of point p, how can we define the coordinates of point? There is something called basis vector for the coordinates system. For example, $\mathcal{B} = \{e_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, e_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$ is stardard basis in $\mathbb{R}^2$. The coordinates of point is the distance from *Origin* to the intersection which point p is perpendicular to one of the basis vector: e.g. e_1 so coordinates system includes two things: *Origin* and $\mathcal{B} = \{e_1, e_2\}$

2 A picture is worth a thousand words



2.1 Coordinates system and frame

When we are talking about coordinates system which means two things: origin and basis vectors, e.g. $O(0,0)$ and $\mathcal{B} = \{e_1, e_2\}$. We call $(O, \mathcal{B})$ as standard *frame* in $\mathbb{R}^2$

2.2 All about linear combination

If we need something like linear combination in vector space. $\vec{v} = \lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2$
and we need similar thing for points, e.g. $p = \lambda_1 p_1 + \lambda_2 p_2$

2.3 From standard frame to new frame

let $p(x, y)$ in standard frame $(O, \mathcal{B})$ and $p_1, p_2 \in (O, \mathcal{B})$, similarly
let $p'(x', y')$ in other frame $(\Omega(\omega_x, \omega_y), \mathcal{B})$ and $p'_1, p'_2 \in (\Omega, \mathcal{B})$.

Linear combination of $p_1(x_1, x_1), p_2(x_2, x_2)$ would be

$$\begin{aligned} p &= \lambda_1 p_1 + \lambda_2 p_2 \quad \text{in stardard frame } (O, \mathcal{B}) \\ p(x, y) &= (\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 y_1 + \lambda_2 y_2) \end{aligned} \tag{1}$$

If origin is changed from $O(0, 0)$ to $\Omega = (\omega_1, \omega_2)$, then the coordinates of p'_1, p'_2 are $(x_1 - \omega_x, y_1 - \omega_y)$ and $(x_2 - \omega_x, y_2 - \omega_y)$ or p_1, p_2 are $(x_1 + \omega_x, y_1 + \omega_y)$ and $(x_2 + \omega_x, y_2, +\omega_y)$ respectively. We have

$$\begin{aligned} p' &= \lambda_1 p'_1 + \lambda_2 p'_2 \\ (x - \omega_x, y - \omega_y) &= \lambda_1(x_1 - \omega_x, y_1 - \omega_y) + \lambda_2(x_2 - \omega_x, y_2 - \omega_y) \\ (x - \omega_x, y - \omega_y) &= (\lambda_1 x_1 + \lambda_2 x_2 - (\lambda_1 + \lambda_2)\omega_x, \lambda_1 y_1 + \lambda_2 y_2 - (\lambda_1 + \lambda_2)\omega_y) \end{aligned} \tag{2}$$

from (1) with new frame: $(\Omega, \mathcal{B})$, we have

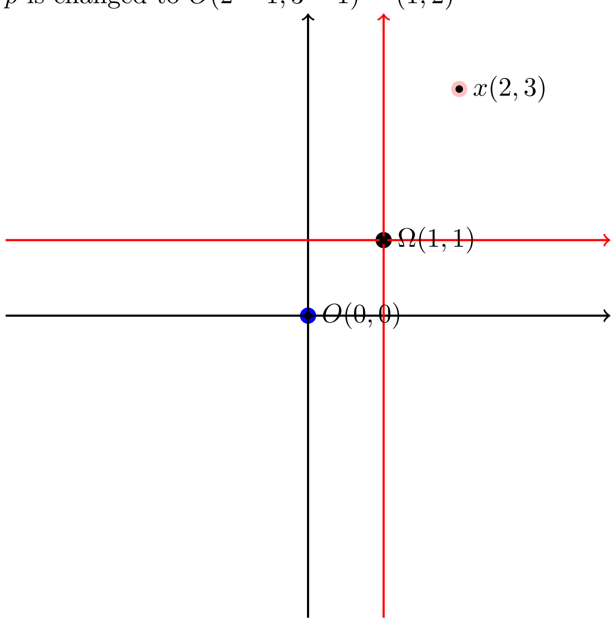
$$(x - \omega_x, y - \omega_y) = (\lambda_1 x_1 + \lambda_2 x_2 - \omega_x, \lambda_1 y_1 + \lambda_2 y_2 - \omega_y) \tag{3}$$

In order to make (2) and (3) are the same, then the following must be true

$$\lambda_1 + \lambda_2 = 1$$

3 Time for example

Given an *Origin* $O(0, 0)$ and basis vectors $\mathcal{B}$, point p has coordinates $(2, 3)$. If *Origin* is changed to $\Omega(1, 1)$, then the coordinates of point p is changed to $O(2 - 1, 3 - 1) = (1, 2)$



It shows points are dependent on frame. If we change frame from $(O, (e_1, e_2))$ to $(\Omega, (e_1, e_2))$ then the coordinates of point p is changed from $p = (2, 3)$ to $p = (1, 2)$.

4 Informal defintion of *Vector Space*

Close addition and scalar multiplication. $\vec{u}, \vec{v} \in V, \alpha \in \mathbb{K}$

$$\begin{aligned} \vec{u} + \vec{v} &\in V \\ \alpha\vec{u} &\in V \end{aligned}$$

Identity element: There is $\vec{0}$ such as for all vectors $\vec{u}$

$$\vec{u} + \vec{0} = \vec{u}$$

Commutivity

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$

Associativity: If $\vec{u}, \vec{v}, \vec{w} \in V$

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$

Distributivity over vector addition: $\alpha \in \mathbb{K}, \vec{u}, \vec{v} \in V$

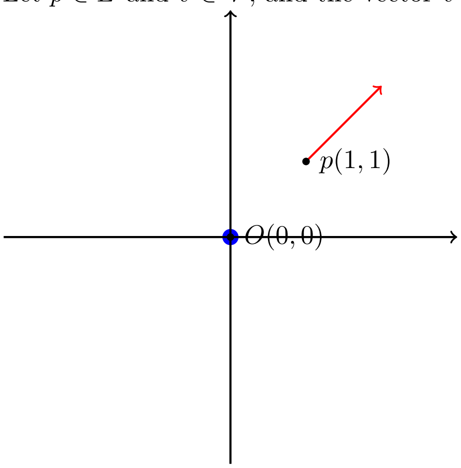
$$\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$$

Distributivity over scalar addition: $\alpha, \beta \in \mathbb{K}, \vec{u} \in V$

$$(\alpha + \beta)\vec{u} = \alpha\vec{u} + \beta\vec{u}$$

What is happening when people said given two points a, b and a vector is defined as $\overrightarrow{ab} = b - a$. The concept is called *Affine Space* and IT IS NOT IN *Vector Space* in general.

Affine Space: Given a set of points in E, there is no structure in E. Why? because there is NO operation is defined on those points. What about we are talking points subtraction, e.g. $p_1 - p_0 = \overrightarrow{p_0 p_1}$
And a Vector Space V.
Let $p \in E$ and $v \in V$, and the vector v moves the point p along v direction to the tip of of vector v



Better Definition: Given a set of points E, a Vector Space V and a action[translation] $+ : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ and $\forall a = (x, y) \in E$ and $\forall \vec{u}, \vec{v} \in V$, satifying following conditions

There is unique point $b \in E$ such as

$$a(x, y) + \vec{v}(u, v) = b(x + u, y + v)$$

Zero vector acts on a point

$$a(x, y) + \vec{v}(0, 0) = b(x + 0, y + 0)$$

Associativity

$$(a + \vec{v}) + \vec{u} = a + (\vec{v} + \vec{u})$$

5 Chasles's Identity

Given any three points $a, b, c \in E$, from the conditions of Affine Space

$$b = a + \overrightarrow{ab} \tag{1}$$
$$c = b + \overrightarrow{bc} \tag{2}$$
$$c = a + \overrightarrow{ac} \tag{3}$$

from (1) and (2)

$$c = (a + \overrightarrow{ab}) + \overrightarrow{bc}$$

from (3)

$$\begin{aligned} a + \overrightarrow{ac} &= a + \overrightarrow{ab} + \overrightarrow{bc} \\ a + \overrightarrow{ac} &= a + (\overrightarrow{ab} + \overrightarrow{bc}) \\ \implies \overrightarrow{ac} &= (\overrightarrow{ab} + \overrightarrow{bc}) \end{aligned}$$

from Chasles' Identity, and let $a = c$

$$\overrightarrow{aa} = \overrightarrow{ab} + \overrightarrow{ba} \tag{4}$$

from defintion

$$\begin{aligned} a + \vec{0} &= a \\ a + \overrightarrow{aa} &= a \\ \implies \overrightarrow{aa} &= \vec{0} \end{aligned}$$

from (4)

$$\overrightarrow{ab} + \overrightarrow{ba} = \vec{0}$$

Given any four points $a, b, c, d \in E$, from Chasles' identity

$$\begin{aligned} \overrightarrow{ab} + \overrightarrow{bc} &= \overrightarrow{ad} + \overrightarrow{dc} = \overrightarrow{ac} \\ \overrightarrow{ab} = \overrightarrow{ad} &\iff \overrightarrow{bc} + \overrightarrow{dc} \quad \text{[parallelogram law]} \quad \square \end{aligned}$$