6 Introduction to categories

6.1 The definition of a category

We have now seen many examples of representation theories and of operations with representations (direct sum, tensor product, induction, restriction, reflection functors, etc.) A context in which one can systematically talk about this is provided by **Category Theory**.

Category theory was founded by Saunders MacLane and Samuel Eilenberg around 1940. It is a fairly abstract theory which seemingly has no content, for which reason it was christened "abstract nonsense". Nevertheless, it is a very flexible and powerful language, which has become totally indispensable in many areas of mathematics, such as algebraic geometry, topology, representation theory, and many others.

We will now give a very short introduction to Category theory, highlighting its relevance to the topics in representation theory we have discussed. For a serious acquaintance with category theory, the reader should use the classical book [McL].

Definition 6.1. A category $\mathcal{C}$ is the following data:

(i) a class of objects $Ob(\mathcal{C})$;

(ii) for every objects $X, Y \in Ob(\mathcal{C})$, the class $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}(X, Y)$ of morphisms (or arrows) from X, Y (for $f \in \mathrm{Hom}(X, Y)$, one may write $f : X \to Y$);

(iii) For any objects $X, Y, Z \in Ob(\mathcal{C})$, a composition map $\mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$, $(f, g) \mapsto f \circ g$,

which satisfy the following axioms:

1. The composition is associative, i.e., $(f \circ g) \circ h = f \circ (g \circ h)$;

2. For each $X \in Ob(\mathcal{C})$, there is a morphism $1_X \in \mathrm{Hom}(X, X)$, called the unit morphism, such that $1_X \circ f = f$ and $g \circ 1_X = g$ for any f, g for which compositions make sense.

Remark. We will write $X \in \mathcal{C}$ instead of $X \in Ob(\mathcal{C})$.

Example 6.2. 1. The category **Sets** of sets (morphisms are arbitrary maps).

2. The categories **Groups**, **Rings** (morphisms are homomorphisms).

3. The category $\mathbf{Vect}_k$ of vector spaces over a field k (morphisms are linear maps).

4. The category $\mathrm{Rep}(A)$ of representations of an algebra A (morphisms are homomorphisms of representations).

5. The category of topological spaces (morphisms are continuous maps).

6. The homotopy category of topological spaces (morphisms are homotopy classes of continuous maps).

Important remark. Unfortunately, one cannot simplify this definition by replacing the word "class" by the much more familiar word "set". Indeed, this would rule out the important Example 1, as it is well known that there is no set of all sets, and working with such a set leads to contradictions. The precise definition of a class and the precise distinction between a class and a set is the subject of set theory, and cannot be discussed here. Luckily, for most practical purposes (in particular, in these notes), this distinction is not essential.