Facts about Dual Spaces and Inner Products

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Throughout this note, V will be a vector space.

Definition 1. The **dual vector space** V^* is the vector space of linear maps

$$L : V \to \mathbb{R}.$$

Example 2. If we give $\mathbb{R}^n$ the standard basis $\vec{e}_1, \ldots, \vec{e}_n$ and write elements as column vectors $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ then the dual vector space $\mathbb{R}^{n*}$ is the vector space of row vectors $\begin{bmatrix} a'_1 & \ldots & a'_n \end{bmatrix}$.

Definition 3. Let $\{b_1, \ldots, b_n\}$ be a basis of V. Then we denote by $b_i^* \in V^*$ the linear transformation defined on the basis $\{b_1, \ldots, b_n\}$ by

$$b_i^*(b_j) = \delta_{ij}$$

where δ_{ij} is the *Kronecker delta function*:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Example 4. The dual basis for $\{\vec{e}_1, \ldots, \vec{e}_n\}$ is given by

$$\vec{e}_i^{\,*} = \begin{bmatrix} 0 & \ldots & 1_i & \ldots & 0 \end{bmatrix}.$$

Definition 5. Let

$$L : V \to W$$

be a linear map between two vector spaces. Then we may define a linear map

$$L^* : W^* \to V^*$$

given by

$$L^*(\varphi)(v) = \varphi(L(v)).$$

(here, $L^*(\varphi)$ is a map which takes $v \in V$ as input and spits out an element of $\mathbb{R}$). Furthermore, if

$$L : V \to W, M : W \to X$$

are two linear maps then

$$(M \circ L)^* = L^* \circ M^*.$$

The fact that $\vec{e}_i^* = \vec{e}_i^T$ when written in vector form is no accident. It follows from the following theorem (which we proved in class):

Theorem 1. *Let $B = \{b_1, \ldots, b_m\}$ be a basis of a vector space V and $B' = \{b'_1, \ldots, b'_n\}$ a basis of a vector space W. Then if*

$$L : V \to W$$

is given by the matrix $[a_{ij}]$ w.r.t. the bases B and B', then L^* is given by the matrix $[a_{ji}]$ w.r.t. the dual bases B'^* and B^*.

This explains easily why $(AB)^T = B^T A^T$. Given an inner product on V (see Definition 5.5.1) we get a map

$$D : V \to V^*$$

by

$$D(v)(w) = \langle v, w \rangle.$$