Chapter 7

Delaunay Triangulation: Incremental Construction

In the last lecture, we have learned about the Lawson flip algorithm that computes a Delaunay triangulation of a given n-point set $P \subseteq \mathbb{R}^2$ with $O(n^2)$ Lawson flips. One can actually implement this algorithm to run in $O(n^2)$ time, and there are point sets where it may take $\Omega(n^2)$ flips.

In this lecture, we will discuss a different algorithm. The final goal is to show that this algorithm can be implemented to run in $O(n \log n)$ time; this lecture, however, is concerned only with the correctness of the algorithm. Throughout the lecture we assume that P is in general position (no 3 points on a line, no 4 points on a common circle), so that the Delaunay triangulation is unique (Corollary 6.17). There are techniques to deal with non-general position, but we don't discuss them here.

7.1 Incremental construction

The idea is to build the Delaunay triangulation of P by inserting one point after another. We always maintain the Delaunay triangulation of the point set R inserted so far, and when the next point s comes along, we simply update the triangulation to the Delaunay triangulation of $R \cup \{s\}$. Let $\mathcal{DT}(R)$ denote the Delaunay triangulation of $R \subseteq P$.

To avoid special cases, we enhance the point set P with three artificial points "far out". The convex hull of the resulting point set is a triangle; later, we can simply remove the extra points and their incident edges to obtain $\mathcal{DT}(P)$. The incremental algorithm starts off with the Delaunay triangulation of the three artificial points which consists of one big triangle enclosing all other points. (In our figures, we suppress the far-away points, since they are merely a technicality.)

Now assume that we have already built $\mathcal{DT}(R)$, and we next insert $s \in P \setminus R$. Here is the outline of the update step.

1. Find the triangle $\Delta = \Delta(p, q, r)$ of $\mathcal{DT}(R)$ that contains s, and replace it with the three triangles resulting from connecting s with all three vertices p, q, r; see Figure