1 Schur Decomposition

If matrix n by n matrix with complex entries, then

$$A = QUQ^{-1}$$

where Q is unitary matrix and U is upper triangle matrix

If
$$A^*A = AA^*$$

Then $A = P^*\Lambda P$

Proof

A is normal matrix

$$\Rightarrow \mathbf{A} = \mathbf{P}^* \Delta \mathbf{P} \tag{1}$$

 Δ is upper triangle matrix (Schur decomposition)

P is unitary matrix

First we need to show if **A** is **normal** then Δ is **normal** (2)

$$\mathbf{A} = \mathbf{P}^* \Delta \mathbf{P}$$

$$\Rightarrow \mathbf{A}^* = (\mathbf{P}^* \Delta \mathbf{P})^*$$

$$\Rightarrow \mathbf{A}^* = \mathbf{P}^* (\mathbf{P}^* \Delta)^*$$

$$\Rightarrow \mathbf{A}^* = \mathbf{P}^* (\Delta^* \mathbf{P})$$
from (2) and (3)

$$\mathbf{A}\mathbf{A}^* = (\mathbf{P}^*\Delta\mathbf{P})(\mathbf{P}^*\Delta^*\mathbf{P})$$

$$\Rightarrow \mathbf{A}\mathbf{A}^* = \mathbf{P}^*\Delta(\mathbf{P}\mathbf{P}^*)\Delta^*\mathbf{P}$$

P is unitary matrix

$$\Rightarrow \mathbf{P}^* = \mathbf{P}^{-1}$$

$$\Rightarrow \mathbf{A}\mathbf{A}^* = \mathbf{P}^* \Delta \Delta^* \mathbf{P}$$
(4)

$$\Rightarrow \mathbf{A}^* \mathbf{A} = \mathbf{P}^* \Delta^* \mathbf{P} \mathbf{P}^* \Delta \mathbf{P}$$
$$\Rightarrow \mathbf{A}^* \mathbf{A} = \mathbf{P}^* \Delta^* \Delta \mathbf{P}$$
(5)

$$\Rightarrow \mathbf{A}\mathbf{A}^* - \mathbf{A}\mathbf{A}^* = \mathbf{P}^*(\Delta\Delta^* - \Delta^*\Delta)\mathbf{P} = \mathbf{0}$$

$$\Rightarrow \Delta\Delta^* - \Delta^*\Delta = \mathbf{0}$$

$$\Rightarrow \Delta\Delta^* = \Delta^*\Delta$$
(6)

 $\Rightarrow \Delta$ is normal matrix

From (6)

$$\Rightarrow \vec{e_i}^* \Delta \Delta^* \vec{e_i} = \vec{e_i}^* \Delta^* \Delta \vec{e_i}$$

$$\Rightarrow \langle \Delta^* \vec{e_i}, \Delta^* \vec{e_i} \rangle = \langle \Delta \vec{e_i}, \Delta \vec{e_i} \rangle$$

$$\Rightarrow \|\Delta^* \vec{e_i}\|^2 = \|\Delta \vec{e_i}\|^2$$

$$\Rightarrow \|\Delta^* \vec{e_i}\| = \|\Delta \vec{e_i}\|$$

 \Rightarrow The length of ith column and ith row in Δ are same (7)

 Δ is upper triangle matrix

Let i to be the first row with nonzero off-diagonal element

$$A_{n,n} = \begin{pmatrix} a_{1,1} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & a_{2,2} & \cdots & 0 & \cdots & 0 \\ 0 & 0 & a_{i,i} & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdot & \vdots & \vdots & a_{n,n} \end{pmatrix}$$

If Δ is not diagonal matrix, then the ith column is $|a_{i,i}| \neq |a_{i,i}| + |*|$

This contracts our previous (7), therefore Δ must be diagonal matrix