CHAPTER 4

FOURIER SERIES AND INTEGRALS

4.1 FOURIER SERIES FOR PERIODIC FUNCTIONS

This section explains three Fourier series: **sines, cosines, and exponentials** $\boldsymbol{e^{ikx}}$. Square waves (1 or 0 or -1) are great examples, with delta functions in the derivative. We look at a spike, a step function, and a ramp—and smoother functions too.

Start with $\sin x$. It has period 2π since $\sin(x+2\pi) = \sin x$. It is an odd function since $\sin(-x) = -\sin x$, and it vanishes at $x = 0$ and $x = \pi$. Every function $\sin nx$ has those three properties, and Fourier looked at *infinite combinations of the sines*:

Fourier sine series $$S(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots = \sum_{n=1}^{\infty} b_n \sin nx \quad (1)$$

If the numbers $b_1, b_2, \ldots$ drop off quickly enough (we are foreshadowing the importance of the decay rate) then the sum $S(x)$ will inherit all three properties:

Periodic $S(x+2\pi) = S(x)$ **Odd** $S(-x) = -S(x)$ $S(0) = S(\pi) = 0$

200 years ago, Fourier startled the mathematicians in France by suggesting that *any function* $S(x)$ with those properties could be expressed as an infinite series of sines. This idea started an enormous development of Fourier series. Our first step is to compute from $S(x)$ the number b_k that multiplies $\sin kx$.

Suppose $S(x) = \sum b_n \sin nx$. *Multiply both sides by* $\sin kx$. *Integrate from* 0 *to* π:

$$\int_{\mathbf{0}}^{\boldsymbol{\pi}} S(x) \sin kx \, dx = \int_{\mathbf{0}}^{\boldsymbol{\pi}} b_1 \sin x \sin kx \, dx + \cdots + \int_{\mathbf{0}}^{\boldsymbol{\pi}} \boldsymbol{b_k \sin kx \ \sin kx \, dx} + \cdots \quad (2)$$

On the right side, all integrals are zero except the highlighted one with $n = k$. This property of "**orthogonality**" will dominate the whole chapter. The sines make 90° angles in function space, when their inner products are integrals from 0 to π:

Orthogonality $$\int_0^{\pi} \sin nx \ \sin kx \, dx = 0 \quad \text{if} \quad n \neq k \, . \quad (3)$$