A compact formula for the derivative of a 3-D rotation in exponential coordinates

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***Abstract*—We present a compact formula for the derivative of a 3-D rotation matrix with respect to its exponential coordinates. A geometric interpretation of the resulting expression is provided, as well as its agreement with other less-compact but better-known formulas. To the best of our knowledge, this simpler formula does not appear anywhere in the literature. We hope by providing this more compact expression to alleviate the common pressure to reluctantly resort to alternative representations in various computational applications simply as a means to avoid the complexity of differential analysis in exponential coordinates.**

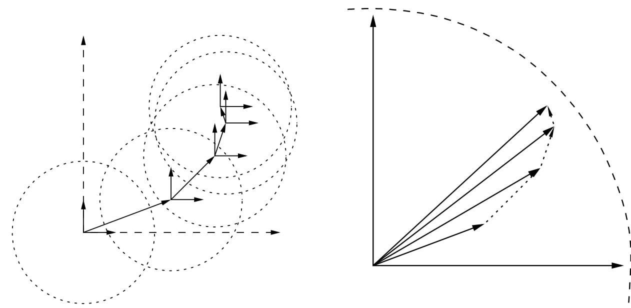
***Index Terms*—Rotation, rotation group, derivative of rotation, exponential map, cross-product matrix, rotation vector, Rodrigues parameters, optimization.**

I. INTRODUCTION

THREE-DIMENSIONAL rotations have numerous applications in many scientific areas, from quantum mechanics to stellar and planetary rotation, including the kinematics of rigid bodies. In particular, they are widespread in computer vision and robotics to describe the orientation of cameras and objects in the scene, as well as to describe the kinematics of wrists and other parts of a robot or a mobile computing device with accelerometers.

Space rotations have three degrees of freedom, and admit several ways to represent and operate with them. Each representation has advantages and disadvantages. Among the most common representations of rotations are Euler angles, axis-angle representation, exponential coordinates, unit quaternions, and rotation matrices. Euler angles [14, p. 31], axis-angle and exponential coordinates [14, p. 30] are very easy to visualize because they are directly related to world models; they are also compact representations, requiring 3-4 real numbers to represent rotations. These representations are used as parametrizations of 3×3 rotation matrices [14, p. 23], which are easier to work with but require nine real numbers. Unit quaternions (also known as Euler-Rodrigues parameters) [2], [14, p. 33] are a less intuitive representation, but nevertheless more compact (4 real numbers) than 3×3 matrices, and are also easy to work with. Historical notes as well as additional references on the representations of rotations can be found in [13, p. 43], [3].

In many applications, it is not only necessary to know how to represent rotations and carry out simple group operations but also to be able to perform some differential analysis. This often requires the calculation of derivatives of the rotation matrix, for example, to find optimal rotations that control some process or that minimize some cost function (in cases where a closed form solution does not exist) [7], [10]. Such is the case for the optimal pose estimation problem long studied within the computer vision and photogrammetry communities [15], [9], [6], as well as for other related problems [19], [5], [12], [16].



(a) Optimization by means of a sequence of local parametrizations (i.e., local charts, represented by circles centered at the corresponding chart origin and axes).

(b) Usual finite-dimensional optimization paradigm that uses a global chart to reach all valid locations of parameter space (interior of fictitious circle).

Figure I.1: Two optimization paradigms: local vs. global parametrizations of the search space.

The space of rotations, i.e., the Lie group $SO(3)$ or rotation group, has the structure of both a group and a manifold, and it is not isomorphic to $\mathbb{R}^3$ [17]. The usual approach to numerical optimization in the rotation group consists of constructing a sequence of local parametrizations (*charts* in the language of differential geometry), as illustrated in Fig. I.1a, rather than relying on a single global parametrization [18], [11], such as Euler angles, to avoid problems caused by singularities in the latter case [17]. This procedure of calculating incremental steps in the tangent space to the manifold relies on the fact that the rotation group has a natural parametrization based on the exponential operator associated with the Lie group.

The previous approach differs from most numerical finite-dimensional optimization paradigms, where the unknown parameters are assumed to lie in some vector space isomorphic to $\mathbb{R}^n$ and a global parametrization is defined, as depicted in Fig. I.1b. The exponential coordinates, however, can be used in this setting since it is a parametrization that covers the whole space of rotations, thus avoiding local charts, with an isolated and removable singularity at the origin.

Here we study rotations parametrized by exponential coor-

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