4 Categorical Limits and Colimits

A comathematician is a device for turning cotheorems into ffee.

—Unknown (Please tell us if you made this joke up!)

Introduction. Categorical limits and colimits are one of the—and in some sense *the most*—efficient way to build a new mathematical object from old objects. The constructions introduced in chapter 1—subspaces, quotients, products, and coproducts—are examples in Top, though the discussion can occur in any category. In fact, there are a number of other important constructions—pushouts, pullbacks, direct limits, and so on—so it's valuable to learn the general notion.

In practice, limits are typically built by picking a subcollection according to some constraint, whereas colimits are typically built by "gluing" objects together. More formally, the defining property of a limit is characterized by morphisms whose *domain* is the limit. The defining property of a colimit is characterized by morphisms whose *codomain* is the colimit. Because of their generality, limits and colimits appear all across the mathematical landscape. A direct sum of abelian groups, the least upper bound of a poset, and a CW complex are all examples of limits or colimits, and we'll see more examples in the pages to come. Section 4.1 opens the chapter by answering the anticipated question, "A (co)limit of *what*?" The remaining two sections contain the formal definition of (co)limits followed by a showcase of examples.

4.1 Diagrams Are Functors

In topology, one asks for the limit of a sequence. In category theory, one asks for the (co)limit of a *diagram*. In what follows, it will be helpful to view a diagram as a functor. More specifically, a diagram in a category is a functor from the shape of the diagram to the category. For example, a commutative diagram like this

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & Z \arrow[ur, "h"'] & \end{tikzcd}$$

in a category C is a choice of three objects X, Y, and Z and some morphisms $f\colon X \to Y$, $g\colon X \to Z$, and $h\colon Z \to Y$, with $f = hg$. It can be viewed as the image of a functor from an *indexing category*; that is, from a picture like this:

$$\begin{tikzcd} \bullet \arrow[rr] \arrow[dr] & & \bullet \\ & \bullet \arrow[ur] & \end{tikzcd}$$

This is a small category—let's call it D—containing three objects pictured as bullets and three morphisms pictured as arrows. Though they must be in D, not shown are the identity morphisms and compositions. Here composition is determined by setting the composition of the two diagonal arrows to be the horizontal arrow. A functor $F\colon \mathsf{D} \to \mathsf{C}$ involves a choice of three objects and three morphisms and must respect composition. In summary,

$$\text{a diagram} \left(\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & Z \arrow[ur, "h"'] & \end{tikzcd} \in \mathsf{C} \right) \quad \text{is a functor} \left(\begin{tikzcd} \bullet \arrow[rr] \arrow[dr] & & \bullet \\ & \bullet \arrow[ur] & \end{tikzcd} \to \mathsf{C} \right)$$

The concept of identifying a map with its image is a familiar one. A sequence of real numbers, for instance, is a function $x\colon \mathbb{N} \to \mathbb{R}$, though one may write x_n for xn and think of the sequence as the collection $(x_1, x_2, \ldots)$. Likewise, a path in a topological space X is a continuous function $p\colon [0,1] \to X$, though one may often have the image $pI \subset X$ in mind. The idea that "a diagram is a functor" is no different.

Definition 4.1 Let D be a small category. A D-*shaped diagram* in a category C is a functor $\mathsf{D} \to \mathsf{C}$. If the categories C and D are understood, we'll simply say *diagram* instead of D-shaped diagram in C.

Because a diagram is a functor, it makes sense to ask for a morphism from one diagram to another—it's a natural transformation of functors. As we'll see below, a (co)limit of a diagram F involves a morphism between F and a diagram of a specific shape—a point. A point-shaped diagram is a functor that is constant at a given object of a category. Indeed we can view any object A of C as a D-shaped diagram for any category D and namely as the *constant functor*. It is defined by sending every object in D to A and every arrow in D to the identity at A in C.

$$\begin{tikzcd} \bullet \arrow[d] & & A \arrow[d, "\mathrm{id}_A"] \\ \bullet & & A \arrow[from=1-1, to=2-3, phantom, "\overset{A}{\longmapsto}"] \end{tikzcd}$$

Notice we're using the symbol A for both the object A and for the constant functor itself. In other words, we allow ourselves flexibility in viewing A as an object or as a functor. In this way, we introduce the phrase "a map from an object to a diagram" to mean a natural transformation from the constant functor to the diagram.

Definition 4.2 Given a functor $F\colon \mathsf{D} \to \mathsf{C}$, a map from an object A to F—that is, an element of $\mathsf{Nat}(A, F)$—is called a *cone from A to F*. Similarly, a *cone from F to A* is an element of $\mathsf{Nat}(F, A)$.

Unwinding the definition, a cone from A to $F\colon \mathsf{D} \to \mathsf{C}$ is a collection of maps

$$\{\, A \xrightarrow{\eta_\bullet} F\bullet \quad \text{where} \quad \bullet \quad \text{is an object in } \ \mathsf{D} \,\}$$

such that the diagrams:

$$\begin{tikzcd} & A \arrow[dl, "\eta_\bullet"'] \arrow[dr, "\eta_\circ"] & \\ F\bullet \arrow[rr, "F\varphi"] & & F\circ \end{tikzcd}$$

commute for every morphism $\bullet \xrightarrow{\varphi} \circ$ in D. For example, a cone from some object A to the functor F with which we opened the section consists of three maps η_X, η_Y, and η_Z fitting together in a commutative diagram:

$$\begin{tikzcd} & A \arrow[dl, "\eta_X"'] \arrow[dr, "\eta_Y"] \arrow[d, "\eta_Z"] & \\ X \arrow[rr, "f"] \arrow[r, "g"'] & Z \arrow[r, "h"'] & Y \end{tikzcd}$$

As we'll see in the next section, a limit of F is a special cone over F and a colimit of F is a special cone under F.

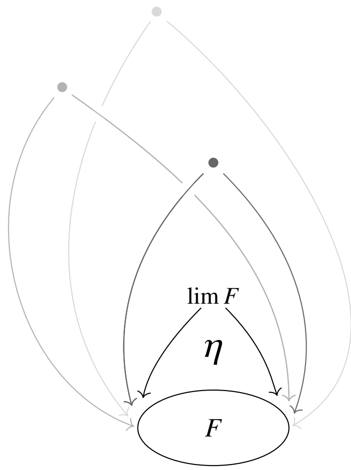
4.2 Limits and Colimits

Here are the formal definitions of limit and colimit.

Definition 4.3 A *limit* of the diagram $F\colon \mathsf{D} \to \mathsf{C}$ is a cone η from an object $\lim F$ to the diagram satisfying the universal property that for any other cone γ from an object B to the diagram, there is a unique morphism $h\colon B \to \lim F$ so that $\gamma_\bullet = \eta_\bullet h$ for all objects $\bullet$ in D.

$$\begin{tikzcd} & B \arrow[d, dashed, "h"] \arrow[ddl, bend right, "\gamma_\bullet"'] \\ & \lim F \arrow[dl, "\eta_\bullet"] \\ F\bullet & \end{tikzcd}$$

One may understand this colloquially in the following way. First, recognize that there may exist *many* cones over F—many objects with maps pointing down to the diagram. But only *one* of them can satisfy the property of limit, namely $\eta\colon \lim F \to F$. You might, however, come across another cone $\gamma\colon B \to F$ that behaves similarly. Perhaps γ also commutes with every arrow in the diagram F and thus seems to imitate η. But the similarity is no coincidence. The natural transformation γ behaves like η precisely because *it is built up from* η. That is, it factors through η as $\gamma = \eta \circ h$ for some unique morphism h. This is the universality of the limit cone. Informally, then, the limit of a diagram is the "shallowest" cone over the diagram. One might visualize all possible cones over the diagram as cascading down to the limit. It is the one that is as close to the diagram as possible:

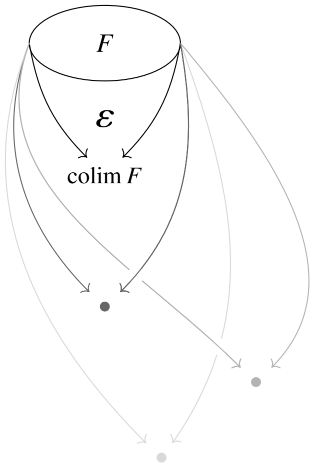


We can similarly ask for maps from a diagram to an object in the category. This leads to the following definition.

Definition 4.4 A *colimit* of the diagram $F\colon \mathsf{D} \to \mathsf{C}$ is a cone ϵ from the diagram to an object $\operatorname{colim} F$ satisfying the universal property that for any other cone γ from the diagram to an object B, there is a unique map $h\colon \operatorname{colim} F \to B$ so that $\gamma_\bullet = h\epsilon_\bullet$ for all objects $\bullet$ in D.

$$\begin{tikzcd}
F\bullet \arrow[dr, "\epsilon_\bullet"] \arrow[ddr, bend right, "\gamma_\bullet"'] & \\
& \operatorname{colim} F \arrow[d, dashed, "h"] \\
& B
\end{tikzcd}$$

Informally, a colimit of a diagram F is the "shallowest" cone under F. There many exist *many* cones—many objects with maps pointing away from the diagram—under F, but a colimit is the cone that is closest to the diagram. Again, one makes sense of the informal words "shallow" and "close" via the universal property in the definition:



Be aware that the (co)limit of a diagram may not exist, but if it does, then—as the reader should verify—it is unique up to a unique isomorphism. We will therefore refer to *the* (co)limit of a diagram.

4.3 Examples

Depending on the shape of the indexing category, the (co)limit of a diagram may be given a familiar name: intersection, union, Cartesian product, kernel, direct sum, quotient, fibered product, and so on. The following examples illustrate this idea. In each case, recall that the data of a (co)limit is an object together with maps to or from that object, satisfying a universal property.

4.3.1 Terminal and Initial Objects

If the indexing category D is empty—no objects and no morphisms—then a functor $\mathsf{D} \to \mathsf{C}$ is an empty diagram. The limit of an empty diagram is called a *terminal object*. It is an object T in C such that for every object X in C there is a unique morphism $X \to T$. In other words, all objects in the category terminate at T. In Set the terminal object is the one-point set; in Top it's the one-point space; in Grp it's the trivial group; in FVect, it's the zero vector space; in a poset, its the greatest element, if it exists. This highlights an important point: not every category has a terminal object. For example, $\mathbb{R}$ with the usual ordering is a poset without a greatest element—it is a category without a terminal object.

Dually, the colimit of an empty diagram is called an *initial object*. It is an object I in C such that for every object X in C there is a unique morphism $I \to X$. In other words, all objects in the category initialize from I. In Set the initial object is the empty set; in Top

it's the empty space; in Grp it's the trivial group; in FVect, it's the zero vector space; in a poset, it's the least element, if it exists. Again, not every category has an initial object.

Many times, when one is interested in the limit of a diagram, the colimit of the diagram will be trivial, or vice versa. For example, the colimit of a diagram that has a terminal object Y is just the object Y, together with the morphisms in the diagram. For example, the object Y is terminal in this diagram:

$$\begin{tikzcd} & Z \arrow[dr, "h"] & \\ X \arrow[ur, "g"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

and indeed the colimit of this diagram is just Y with the map of the diagram given by $f\colon X \to Y$, $h\colon Z \to Y$, and $\mathrm{id}_Y\colon Y \to Y$. It has the universal property since for any object S, a map from the diagram to S includes a map from Y to S making everything commute. That map is the one satisfied by the universal property of Y being the colimit.

$$\begin{tikzcd} & Z \arrow[dr, "h"] \arrow[dd] & \\ X \arrow[ur, "g"] \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl, dashed] \\ & S & \end{tikzcd}$$

Similarly, the limit of a diagram with an initial object X is just the initial object X, together with the morphisms in the diagram.

4.3.2 Products and Coproducts

If the indexing category D has no nonidentity morphisms—that is, if D is a *discrete category*—then a diagram $\mathsf{D} \to \mathsf{C}$ is just a collection of objects parametrized by D. In this case, the limit of the diagram is called the *product* and the colimit is called the *coproduct*. When C = Set, you can verify the universal properties of the product and coproduct given in chapter 0 and show that they are indeed the limit and colimit of discrete diagrams. When the category is Top, the limit is the product of the spaces in the diagram, equipped with the product topology, together with projection maps, down to each of the factors. Likewise, the coproduct is the disjoint union of spaces, equipped with the coproduct topology, together with inclusion maps from each space.

Quite often, one is only interested in a subset (or subspace) of the product. For instance, given sets or spaces X and Y, it's often interesting to consider only those pairs $(x, y) \in X \times Y$ where x and y relate to each other according to some equation. In a dual sense, one may be interested in identifying parts of sets (or spaces) rather than considering their full coproduct. The next few categorical constructions provide different ways of accomplishing these tasks.

4.3.3 Pullbacks and Pushouts

A functor from $\bullet \to \bullet \leftarrow \bullet$ is a diagram

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

and its limit is called the *pullback* of X and Y along the morphisms f and g. In Set the pullback is realized by the set consisting of all pairs (x, y) satisfying $fx = gy$, along with projection maps onto each factor X and Y. The set is denoted by $X \times_Z Y$. Diagrammatically, there is a special notation to describe pullbacks. A square diagram decorated with a caret "$\lrcorner$" in the upper left corner denotes a pullback diagram. For example, this diagram

$$\begin{tikzcd} \circ \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

should be read as saying, "the square commutes and the object $\circ$ is the pullback." As a concrete example, suppose $X = *$ is the one-point set so that a function $f: * \to Z$ picks out an element $z \in Z$. Then the pullback consists of the set of points $y \in Y$ such that $gy = z$. In other words, the pullback is the preimage $g^{-1}z \subset Y$.

In Top, the pullback has $X \times_Z Y$ as its underlying set and it becomes a topological space when viewed as a subspace of the product $X \times Y$. It satisfies the universal property described by this diagram:

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

Explicitly, the pullback topology (first characterization) is the finest topology for which the projection maps $\pi_x: X \times_Z Y \to X$ and $\pi_Y: X \times_Z Y \to Y$ are continuous. Alternatively, the pullback topology (second characterization) is determined by specifying that maps into the pullback from any space W are continuous if and only if the maps $W \to X$ and $W \to Y$ obtained by postcomposing with f and g are continuous. Said the other way around: maps to the pullback from a space W are specified by maps $a: W \to X$ and $b: W \to Y$ with $fa = gb$.

At some point you might encounter a statement such as "the map $p: Y \to X$ is the *pullback of* $\pi: E \to B$ *along the map* $f: X \to B$." This means that p fits into a pullback square with f and π as pictured

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

and that Y together with $p\colon Y \to X$ and the unnamed map is the pullback of the rest of the diagram. The unnamed map $Y \to E$ exists and is part of the pullback but might not be explicitly mentioned.

Dually, a functor from $\bullet \leftarrow \bullet \to \bullet$ is a diagram,

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & X \\ Y & \end{tikzcd}$$

and its colimit is called the *pushout* of X and Y along the morphisms f and g. In Set the pushout of maps $f\colon Z \to X$ and $g\colon Z \to Y$ is realized by the quotient set $X \sqcup_Z Y := X \coprod Y/\sim$ where $\sim$ is the equivalence relation generated by $fz \sim gz$ for all $z \in Z$, together with maps from X and Y into the quotient. By analogy with pullbacks, we will use a caret to denote a pushout square. That is, a diagram such as

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \circ \end{tikzcd}$$

means that the square commutes and that $\circ$ is the pushout. As a concrete example of a pushout in Set, note that given any two sets A and B, the intersection $A \cap B$ is a subset of A and B. So we have the diagram of inclusions

$$\begin{tikzcd} A \cap B \arrow[r, hook] \arrow[d, hook] & A \\ B & \end{tikzcd}$$

The pushout of this diagram is the union $A \cup B$. Said explicitly, the union fits into the diagram, and for any other set S that fits into the diagram, there's a unique function from $A \cup B \to S$ as pictured by the dashed line below:

$$\begin{tikzcd} A \cap B \arrow[r, hook] \arrow[d, hook] & A \arrow[d] \arrow[ddr, bend left] & \\ B \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] \arrow[drr, bend right] & A \cup B \arrow[dr, dashed] & \\ & & S \end{tikzcd}$$

The universal property of the pushout in this example says that a function $A \cup B \to S$ is the same as a pair of functions $A \to S$ and $B \to S$ that agree on $A \cap B$.

In Top, the pullback has the quotient $X \sqcup_Z Y$ as its underlying set. It becomes a topological space as a quotient space of the coproduct. This pushout satisfies the universal property described by this diagram; that is, by making use of the caret notation for pushouts:

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "i_X"] \\ Y \arrow[r, "i_Y"] & X \sqcup_Z Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The pushout topology (first characterization) is the coarsest topology for which the maps $i_X\colon X \to X \sqcup_Z Y$ and $i_Y\colon Y \to X \sqcup_Z Y$, which send an element to its equivalence class, are continuous. Alternatively, the pushout topology (second characterization) is determined by specifying that maps from the pushout to any space W are continuous if and only if the maps $X \to W$ and $Y \to W$ obtained by precomposing with f and g are continuous. Said the other way around: maps from the pushout to a space W are specified by maps $a\colon X \to W$ and $b\colon Y \to W$ with $af = bg$.

Pushout diagrams like this are commonly used to describe the space obtained by attaching a disc D^n to a space X along a map $f\colon S^{n-1} \to X$.

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "i_X"] \\ D^n \arrow[r, "i_Y"] & X \sqcup_{S^{n-1}} D^n \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

In this case, the map $S^{n-1} \hookrightarrow D^n$ is usually understood as the inclusion, and one describes the pushout succinctly by saying "the disc D^n is attached to X via the attaching map f" and writes $X \sqcup_f D^n$. Some authors (Brown, 2006) call these pushouts "adjunction spaces," but we will not. We reserve "adjunction" for another purpose (see chapter 5).

To summarize, pullbacks provide one way to obtain a limit from the product while pushouts provide one way to obtain a quotient of the coproduct. If we change the shape of the indexing categories, then the (co)limit of the resulting diagrams provide additional constructions: inverse and directed limits.

4.3.4 Inverse and Direct Limits

The limit of a diagram of the shape $\bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \cdots$, such as

$$X_1 \overset{f_1}{\leftarrow} X_2 \overset{f_2}{\leftarrow} X_3 \leftarrow \cdots$$

is sometimes called the *inverse limit* of the objects $\{X_i\}$. As in the case of pullbacks, the inverse limit in Set is a certain subset of the product of the objects. Explicitly, the inverse limit is realized by the set of sequences $(x_1, x_2, \ldots) \in \prod_i X_i$ satisfying $f_i x_{i+1} = x_i$ for all i, together with projection maps from the product down to each factor. It is denoted by $\varprojlim X_i$ and can be thought of as the smallest object that projects down to the factors. The inverse limit in Top has this set of sequences as its underlying set. It becomes a topological space when endowed with the subspace topology of the product. For example, the limit of the diagram of spaces

$$\mathbb{R} \leftarrow \mathbb{R}^2 \leftarrow \mathbb{R}^3 \leftarrow \cdots$$

where the maps $\mathbb{R}^{n+1} \to \mathbb{R}^n$ are given by $(x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n)$ is the product $X = \prod_{n\in\mathbb{N}} \mathbb{R}$, the set of all sequences $(x_1, x_2, x_3, \ldots)$ with the product topology. The projections $X \to \mathbb{R}^n$ defined by $(x_1, x_2, \ldots) \mapsto (x_1, \ldots, x_n)$ define the map from X to the diagram, and the topology on X is the coarsest topology making the maps from X to the diagram continuous. Here, the limit X of the diagram $\mathbb{R} \leftarrow \mathbb{R}^2 \leftarrow \mathbb{R}^3 \cdots$ agrees in both Top and $\mathsf{Vect}_{\mathbf{k}}$.

Dually, the colimit of a diagram of the shape $\bullet \to \bullet \to \bullet \to \cdots$, such as

$$X_1 \to X_2 \to X_3 \to \cdots$$

is sometimes called the *directed limit* of the $\{X_i\}$. It is denoted by $\varinjlim X_i$ and consists of an object X, together with maps $i_k \colon X_k \to X$ that assemble to be a map from the diagram. In a concrete category in which the objects are sets with some additional structure and the $X_k \to X_{k+1}$ are injections, one can think of the diagram as an increasing sequence of objects. The colimit, if it exists, may be thought of as the union of the objects.

As a closing remark, notice that the limit of the diagram $X_1 \to X_2 \to X_3 \to \cdots$ or the colimit of the diagram $X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots$ are both just the object X_1.

Example 4.1 In linear algebra, the colimit of $\mathbb{N}$ copies of $\mathbb{R}$ is the set of sequences of real numbers for which all but finitely many are zero and is denoted $\oplus_{n\in\mathbb{N}}\mathbb{R}$. This is not the same as the colimit of $\mathbb{N}$ copies of $\mathbb{R}$ in Top, which is $\coprod_{n\in\mathbb{N}} \mathbb{R}$. To make the vector space $\oplus_{n\in\mathbb{N}}\mathbb{R}$ into a topological space, we need to view it in a different way.

Specifically, $X = \oplus_{n\in\mathbb{N}}\mathbb{R}$ is the colimit of the diagram of (vector and topological) spaces

$$\mathbb{R} \to \mathbb{R}^2 \to \mathbb{R}^3 \to \cdots$$

where the map $\mathbb{R}^n \to \mathbb{R}^{n+1}$ is given by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$. Think of the diagram as an increasing union: $\mathbb{R}$ sits inside $\mathbb{R}^2$ as the x axis, then $\mathbb{R}^2$ sits inside $\mathbb{R}^3$ as the xy-plane, and so on. The colimit X of the diagram is an infinite dimensional space in which all these finite dimensional spaces sit inside and is the *smallest* such space, meaning that if Y is any other space that has maps $X_i \to Y$, these maps factor through a map $X \to Y$. The space X is realized as the set of sequences $(x_1, x_2, \ldots)$ for which all but finitely many x_i are nonzero, together with the maps $\mathbb{R}^n \to X$ defined by

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, 0, \ldots)$$

which identify the $\mathbb{R}^n$s with increasing subsets of X. The vector space structure is addition and scalar multiplication of sequences. The topology on X coming from the colimit can be described explicitly by saying that a set U of sequences is open if and only if the intersection $U \cap \mathbb{R}^n$ is open for all $n \in \mathbb{N}$. This is the finest topology that makes the inclusions $R^n \hookrightarrow X$ continuous.

4.3.5 Equalizers and Coequalizers

The limit of a diagram of the shape $\bullet \rightrightarrows \bullet$, such as

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

is called the *equalizer* of f and g. In Set the equalizer is realized as the set $E = \{x \in X \mid fx = gx\}$, together with the inclusion map $E \to X$. It's the largest subset of the domain X on which the two maps agree. In Top, the equalizer has E as its underlying set and becomes a space when endowed with the subspace topology. The universal property is captured in this diagram:

$$\begin{tikzcd} S \arrow[r, dashed] \arrow[rr, bend left] & E \arrow[r] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

In algebraic categories, such as Grp, $\mathsf{Vect}_{\mathbf{k}}$, $R\mathsf{Mod}$, the equalizer of $f\colon G \to H$ and the unique map from the initial object $0\colon G \to H$ is called the *kernel* of f.

Dually, the colimit of the same diagram

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

is called the *coequalizer* of f and g. In Set and Top, the coequalizer is realized as the quotient $Y/\sim$ where $\sim$ is the equivalence relation generated by $fx \sim gx$ for each $x \in X$, endowed with the quotient topology in the case of Top. It's the quotient of the codomain Y by the smallest relation that makes the maps agree. The universal property is captured in this diagram:

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r] \arrow[rr, bend left] & C \arrow[r, dashed] & S \end{tikzcd}$$

In algebraic categories, such as Grp, $\mathsf{Vect}_{\mathbf{k}}$, $R\mathsf{Mod}$, the coequalizer of $f\colon G \to H$ and the unique map from the initial object $0\colon G \to H$ is the called the *cokernel* of f.

After reading through the examples in this chapter, you might suspect that a limit is always, in some sense, either a product or a construction obtained from a product. This suspicion is indeed correct and provides a prescription for constructing limits in general. In fact, it is a theorem: if a category has all products and all equalizers, then it has all limits. Likewise, the feeling that a colimit may be regarded as either a coproduct or a quotient of a coproduct is also a theorem: if a category has all coproducts and all coequalizers, then it has all colimits. We'll close with these results in the next section.

4.4 Completeness and Cocompleteness

A category is called *complete* if it contains the limits of small diagrams and is called *cocomplete* if it contains the colimits of all small diagrams.[1] The categories Set and Top are both complete and cocomplete. In Set, one can construct the colimit of any diagram by taking the disjoint union of every set in the diagram and then quotienting by the relations required for the diagram to map into the resulting set. In Top, this set gets the quotient topology of the disjoint union. This topology is the finest topology for which all the maps involved in the map from the diagram are continuous.

Dually, to construct the limit of any diagram of sets, first take the product of all the sets that appear in the diagram. The product then maps to all objects in the diagram. The limit of the diagram is simply the subset of the product so that the projection maps to the objects assemble to be a map to the diagram. In Top, this set gets the subspace topology of the product. This topology is the coarsest topology for which all the maps involved in the map to the diagram are continuous.

Seeing how to define an arbitrary colimit of sets as a quotient of the disjoint union or how to define an arbitrary limit as a subspace of the product gives the idea of how to prove the following theorem: small (co)limits are all "generated by" a set's worth of (co)products and (co)equalizers.

Theorem 4.1 If a category has products and equalizers, then it is complete. If it has coproducts and coequalizers, then it is cocomplete.

Proof. Here's how to construct the colimit of a diagram in a category with coproducts and coequalizers. Proceed in two steps. First, take the coproduct Y of all the objects X_α in the diagram that have morphisms $X_\alpha \to X_\beta$ from them (there may be multiple copies of X_αs) and take the coproduct Z of all the objects X_β that appear in the diagram (just one copy each):

$$Y := \coprod_{X_\alpha \to X_\beta} X_\alpha \qquad \Bigg| \qquad Z := \coprod_{\beta} X_\beta$$

There are two maps $Y \to Z$. One map, call it f, is defined by taking the coproduct of the morphisms in the diagram, and one map is defined simply by identities. The coequalizer of these two maps

$$Y \underset{\mathrm{id}}{\overset{f}{\rightrightarrows}} Z$$

[1] A category is *small* if both the collection of objects and the collection of morphisms are sets.

is the colimit of the diagram. The idea for limits is similar. $\square$

A summary of the ideas discussed in this chapter can be organized as seen in table 4.1. We end this chapter with a definition and an example.

Table 4.1 Common categorical limits and colimits.

(index)	$\xrightarrow{\text{functor}}$	(diagram)	its limit	its colimit
	$\longmapsto$		terminal object	initial object
$\bullet \quad \bullet \quad \bullet$	$\longmapsto$	$A \quad B \quad C$	product	coproduct
$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$	$\longmapsto$	$\begin{tikzcd} & B \arrow[d] \\ A \arrow[r] & C \end{tikzcd}$	pullback	—
$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$	$\longmapsto$	$\begin{tikzcd} C \arrow[r] \arrow[d] & B \\ A & \end{tikzcd}$	—	pushout
$\bullet \leftarrow \bullet \leftarrow \cdots$	$\longmapsto$	$A_1 \leftarrow A_2 \leftarrow \cdots$	inverse limit	—
$\bullet \rightarrow \bullet \rightarrow \cdots$	$\longmapsto$	$A_1 \rightarrow A_2 \rightarrow \cdots$	—	direct limit
$\bullet \rightrightarrows \bullet$	$\longmapsto$	$A \rightrightarrows B$	equalizer	coequalizer

Definition 4.5 A functor is *continuous* if and only if it takes limits to limits. It is *cocontinuous* if it takes colimits to colimits.

Example 4.2 Given a object X in a category C, the hom functor $\mathsf{C}(X,-)\colon \mathsf{C} \to \mathsf{Set}$ is continuous. One may wish for a dual statement, "$\mathsf{C}(-,X)\colon \mathsf{C}^{op} \to \mathsf{Set}$ is cocontinuous," but this is not the case. The contravariance of $\mathsf{C}(-,X)$ does imply that colimits are sent to limits. In fact, we've seen a special case of these results in the discussion of the universal property of products and coproducts in Set in section 0.3.4. An example of a cocontinuous functor may be found in exercise 4.5 at the end of the chapter.

Exercises

1. Let $f\colon Y \to X$ be an embedding of a space Y into a space X. Construct a diagram for which Y (and the map f) is a limit. Hint: exercise 1.12 at the end of chapter 1 shows that quotients are coequalizers and hence colimits.

2. Define the infinite dimensional sphere S^∞ to be the colimit of the diagram

$$S^0 \hookrightarrow S^1 \hookrightarrow S^2 \hookrightarrow S^3 \hookrightarrow \cdots$$

Prove that S^∞ is contractible.

3. From a diagram

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{h} Z$$

construct a commutative square

$$\begin{tikzcd} X \coprod X \arrow[r, "(f,g)"] \arrow[d] & Y \arrow[d, "h"] \\ X \arrow[r] & Z \end{tikzcd}$$

Prove that the first diagram is a coequalizer precisely if the second is a pushout. Now, from a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "p"] \\ X \arrow[r, "q"'] & Z \end{tikzcd}$$

construct a diagram

$$X \underset{g}{\overset{f}{\rightrightarrows}} X \coprod Y \xrightarrow{(p,q)} Z$$

Prove that the first diagram is a pushout if and only if the second is a coequalizer.

Conclude that a category that has pushouts and coproducts has all colimits. Give a similar argument to prove that a category that has pullbacks and products is closed (Mac Lane, 2013, p. 72, exercise 9).

4. In any category, prove that $f\colon X \to Y$ is an epimorphism if and only if the following square is a pushout:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, "\mathrm{id}_Y"] \\ Y \arrow[r, "\mathrm{id}_Y"] & Y \end{tikzcd}$$

(Mac Lane, 2013, p. 72, exercise 4)

5. For any set X, show that the functor $X \times - : \mathsf{Set} \to \mathsf{Set}$ is cocontinuous (Mac Lane, 2013, p. 118, exercise 4).

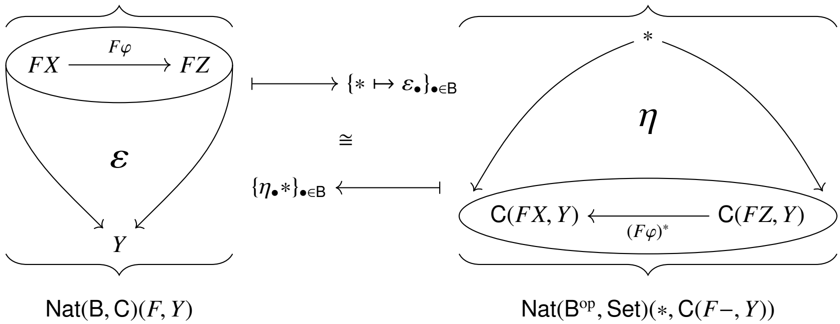
6. In a poset, what is the limit, if it exists, of any nonempty diagram? What is the colimit, if it exists?

7. Explain the blanks in table 4.1.

8. Using the image below, prove that a functor $F\colon \mathsf{B} \to \mathsf{C}$ has a colimit if and only if for all objects $Y \in \mathsf{C}$ there is a natural isomorphism:

$$\mathsf{C}(\operatorname{colim} F, Y) \quad \cong \quad \lim \mathsf{C}(F-, Y)$$

The following provides a guide for the proof. Below on the left, observe that by the universal property of the colimit of F, elements of $\mathsf{C}(\operatorname{colim} F, Y)$ correspond to natural transformations from the diagram F to (the constant functor at) Y. Below on the right, oberve that by the universal property of the limit of the functor $\mathsf{C}(F-, Y)\colon \mathsf{B}^{\mathrm{op}} \to \mathsf{Set}$, the set of all natural transformations from (the constant functor at) the terminal, one-point set $*$ to $\mathsf{C}(F-, Y)$ is the limit, $\lim \mathsf{C}(F-, Y)$. Lastly, in the center we claim the indicated maps form a natural isomorphism:



In short, to exchange colimits in the first argument of homs with limits of homs, one need only send maps to precompositions.