Bilinear forms

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In this Chapter we study finite-dimensional vector spaces over an arbitrary field $\mathbb{F}$ with a bilinear form defined on the space. This is a generalisation of the notion of an inner product space over $\mathbb{R}$.

1 The notion of bilinear form. Matrix representation. Congruent matrices.

Let V be a vector space over $\mathbb{F}$.

Definition 1.1: *A bilinear form on V is a map $g : V \times V \to \mathbb{F}$ such that for any u, u', v, v' in V and scalar $a \in \mathbb{F}$ we have*

1. *(linearity in the first variable) $g(u+u', v) = g(u, v) + g(u', v)$ and $g(au, v) = ag(u, v)$;*
2. *(linearity in the second variable) $g(u, v+v') = g(u, v) + g(u, v')$ and $g(u, av) = ag(u, v)$.*

Remark 1.2: *Equivalently, $g : V \times V \to \mathbb{F}$ is a bilinear form if and only if for all $u \in V$ the map $l_u : V \to V$ defined by $l_u : v \mapsto g(u, v)$ is a linear form on V and for all $v \in V$ the map $r_v : V \to V$ defined by $r_v : u \mapsto g(u, v)$ is a linear form on V.*

Example 1.3:

1. Let $(V, \langle , \rangle)$ be an inner product space over $\mathbb{R}$. Then $g : V \times V \to \mathbb{R}$ defined by $g(u, v) = \langle u, v \rangle$ is a bilinear form. In particular, the standard dot product in $\mathbb{R}^n$ is a bilinear form. (Note, however, that this is not so in an inner product space over $\mathbb{C}$. The standard dot product in $\mathbb{C}^n$ is not a bilinear form!)
2. The zero form. $\mathbb{F}$ is an arbitrary field and $g : V \times V \to \mathbb{F}$ is defined by $g(u, v) = 0$ for all $u, v \in V$.
3. $V = \mathbb{F}^2_{col}$ and g is the determinant form:

$$g(u, v) = \det \begin{bmatrix} x^1 & y^1 \\ x^2 & y^2 \end{bmatrix} = x^1y^2 - x^2y^1$$

   for $u = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$, $v = \begin{bmatrix} y^1 \\ y^2 \end{bmatrix}$.

   (Since the determinant of a matrix is linear in each of its columns when the remaining $n-1$ columns are fixed, the example can be generalized to $V = \mathbb{F}^n_{col}$ for $n > 2$. Consider an $n \times n$ matrix with all but two columns fixed, then its determinant, considered as a function of the two remaining columns, is bilinear in its two arguments.)
4. $V = \mathbb{R}^4$ and $g(u, v) = x^1y^1 + x^2y^2 + x^3y^3 - x^4y^4$ for $u = (x^1, x^2, x^3, x^4)$ and $v = (y^1, y^2, y^3, y^4)$ (this form is called the Lorentz form, and $\mathbb{R}^4$ endowed with this form is called the Minkowski space – an important tool in the special relativity theory).
5. If g is a bilinear form on V and $f : V \to V$ is a linear operator, then $\tilde{g} : V \times V \to \mathbb{F}$ defined by $\tilde{g}(u, v) = g(f(u), v)$ is also bilinear.