Examples of Manifolds

A manifold is a generalization of a surface. Roughly speaking, a d–dimensional manifold is a set that looks locally like $\mathbb{R}^d$. It is a union of subsets each of which may be equipped with a coordinate system with coordinates running over an open subset of $\mathbb{R}^d$. Here is a precise definition.

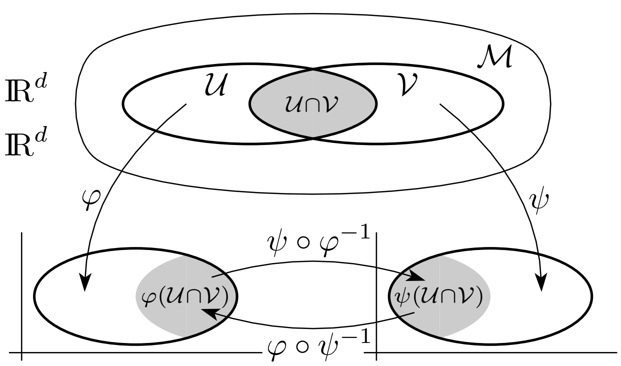
Definition 1 We now define what is meant by the statement that $\mathcal{M}$ is a d–dimensional manifold of class C^k (with $1 \le k \le \infty$ — we shall deal almost exclusively with manifolds of class C^∞).

(a) Let $\mathcal{M}$ be a Hausdorff topological space[(1)]. A *coordinate system* (or *chart* or *coordinate patch*) on $\mathcal{M}$ is a pair $(\mathcal{U}, \varphi)$ with $\mathcal{U}$ a connected open subset of $\mathcal{M}$ and φ a homeomorphism (a 1–1, onto, continuous function with continuous inverse) from $\mathcal{U}$ onto an open subset of $\mathbb{R}^d$. Think of φ as assigning coordinates to each point of $\mathcal{U}$. A coordinate system $(\mathcal{U}, \varphi)$ is called a cubic coordinate system if $\varphi(U)$ is an open cube about the origin in $\mathbb{R}^d$. (That is, if there are numbers $a_1, \cdots, a_d, b_1, \cdots, b_d > 0$ such that $\varphi(U) = \left\{ x \in \mathbb{R}^d \,\middle|\, -a_i < x_i < b_i, \ \text{ for all } 1 \le i \le d \right\}$.) If $m \in U$ and $\varphi(m) = 0$, then the coordinate system is said to be centred at m.

(b) A *locally Euclidean space of dimension* d, is a Hausdorff topological space for which every point has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^d$.

(c) Two charts $(\mathcal{U}, \varphi)$ and $(\mathcal{V}, \psi)$ are said to be *compatible* of class C^k if the transition functions

$$\psi \circ \varphi^{-1} \colon \ \varphi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^d \to \psi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^d$$
$$\varphi \circ \psi^{-1} \colon \ \psi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^d \to \varphi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^d$$



are C^k. That is, all partial derivatives up to order k (for C^∞, all partial derivatives of all orders) of $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ exist and are continuous.

(d) An *atlas* of class C^k for a locally Euclidean space $\mathcal{M}$ is a family $\mathcal{A} = \left\{ (\mathcal{U}_i, \varphi_i) \,\middle|\, i \in \mathcal{I} \right\}$ of coordinate systems on $\mathcal{M}$ such that $\bigcup_{i \in \mathcal{I}} \mathcal{U}_i = \mathcal{M}$ and such that every pair of charts

(1) If you don't know what this means, substitute "metric space" for "Hausdorff topological space" and read the notes "A Little Point Set Topology".

