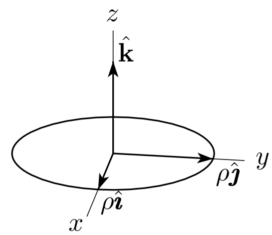
Parametrizing Circles

These notes discuss a simple strategy for parametrizing circles in three dimensions. We start with the circle in the xy–plane that has radius ρ and is centred on the origin. This is easy to parametrize:



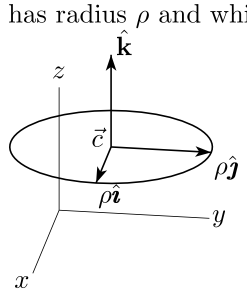
$$\vec{r}(t) = \rho\cos t\,\hat{\boldsymbol{\imath}} + \rho\sin t\,\hat{\boldsymbol{\jmath}} \qquad 0 \le t \le 2\pi$$

Note that we can check that $\vec{r}(t)$ lies on the desired circle by checking, firstly, that $\vec{r}(t)$ lies in the correct plane (in this case, the xy–plane) and, secondly, that the distance from $\vec{r}(t)$ to the centre of the circle is ρ:

$$\left|\vec{r}(t) - \vec{0}\right| = \left|\rho\cos t\,\hat{\boldsymbol{\imath}} + \rho\sin t\,\hat{\boldsymbol{\jmath}}\right| = \sqrt{(\rho\cos t)^2 + (\rho\sin t)^2} = \rho$$

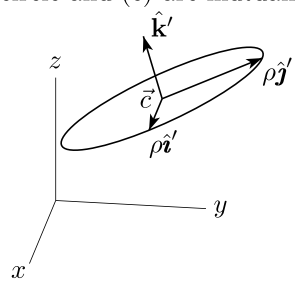
since $\sin^2 t + \cos^2 t = 1$.

Now let's move the circle so that its centre is at some general point $\vec{c}$. To parametrize this new circle, which still has radius ρ and which is still parallel to the xy–plane, we just translate by $\vec{c}$:



$$\vec{r}(t) = \vec{c} + \rho\cos t\,\hat{\boldsymbol{\imath}} + \rho\sin t\,\hat{\boldsymbol{\jmath}} \qquad 0 \le t \le 2\pi$$

Finally, let's consider a circle in general position. The secret to parametrizing a general circle is to replace $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ by two new vectors $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$ which (a) are unit vectors, (b) are parallel to the plane of the desired circle and (c) are mutually perpendicular.



$$\vec{r}(t) = \vec{c} + \rho\cos t\,\hat{\boldsymbol{\imath}}' + \rho\sin t\,\hat{\boldsymbol{\jmath}}' \qquad 0 \le t \le 2\pi$$

To check that this is correct, observe that

- $\vec{r}(t) - \vec{c}$ is parallel to the plane of the desired circle because $\vec{r}(t) - \vec{c} = \rho\cos t\,\hat{\boldsymbol{\imath}}' + \rho\sin t\,\hat{\boldsymbol{\jmath}}'$ and both $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$ are parallel to the plane of the desired circle
- $\vec{r}(t) - \vec{c}$ is of length ρ for all t because

$$\begin{aligned}
|\vec{r}(t) - \vec{c}\,|^2 &= (\vec{r}(t) - \vec{c}\,)\cdot(\vec{r}(t) - \vec{c}\,) \\
&= (\rho\cos t\,\hat{\boldsymbol{\imath}}' + \rho\sin t\,\hat{\boldsymbol{\jmath}}')\cdot(\rho\cos t\,\hat{\boldsymbol{\imath}}' + \rho\sin t\,\hat{\boldsymbol{\jmath}}') \\
&= \rho^2\cos^2 t\;\hat{\boldsymbol{\imath}}'\cdot\hat{\boldsymbol{\imath}}' + \rho^2\sin^2 t\;\hat{\boldsymbol{\jmath}}'\cdot\hat{\boldsymbol{\jmath}}' + 2\rho\cos t\sin t\;\hat{\boldsymbol{\imath}}'\cdot\hat{\boldsymbol{\jmath}}' \\
&= \rho^2(\cos^2 t + \sin^2 t) = \rho^2
\end{aligned}$$

since $\hat{\boldsymbol{\imath}}'\cdot\hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\jmath}}'\cdot\hat{\boldsymbol{\jmath}}' = 1$ ($\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$ are both unit vectors) and $\hat{\boldsymbol{\imath}}'\cdot\hat{\boldsymbol{\jmath}}' = 0$ ($\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$ are perpendicular).