Chapter 2

Projection Matrices

2.1 Definition

Definition 2.1 *Let $\boldsymbol{x} \in E^n = V \oplus W$. Then $\boldsymbol{x}$ can be uniquely decomposed into*

$$\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{x}_2 \ (\textit{where}\ \boldsymbol{x}_1 \in V \ \textit{and}\ \boldsymbol{x}_2 \in W).$$

The transformation that maps $\boldsymbol{x}$ into $\boldsymbol{x}_1$ is called the projection matrix (or simply projector) onto V along W and is denoted as ϕ. This is a linear transformation; that is,

$$\phi(a_1\boldsymbol{y}_1 + a_2\boldsymbol{y}_2) = a_1\phi(\boldsymbol{y}_1) + a_2\phi(\boldsymbol{y}_2) \tag{2.1}$$

for any $\boldsymbol{y}_1$, $\boldsymbol{y}_2 \in E^n$. This implies that it can be represented by a matrix. This matrix is called a projection matrix and is denoted by $\boldsymbol{P}_{V \cdot W}$. The vector transformed by $\boldsymbol{P}_{V \cdot W}$ (that is, $\boldsymbol{x}_1 = \boldsymbol{P}_{V \cdot W}\boldsymbol{x}$) is called the projection (or the projection vector) of $\boldsymbol{x}$ onto V along W.

Theorem 2.1 *The necessary and sufficient condition for a square matrix $\boldsymbol{P}$ of order n to be the projection matrix onto $V = \mathrm{Sp}(\boldsymbol{P})$ along $W = \mathrm{Ker}(\boldsymbol{P})$ is given by*

$$\boldsymbol{P}^2 = \boldsymbol{P}. \tag{2.2}$$

We need the following lemma to prove the theorem above.

Lemma 2.1 *Let $\boldsymbol{P}$ be a square matrix of order n, and assume that (2.2) holds. Then*

$$E^n = \mathrm{Sp}(\boldsymbol{P}) \oplus \mathrm{Ker}(\boldsymbol{P}) \tag{2.3}$$

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