COMPLEX DIFFERENTIABILITY

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1. The problem of extension

In complex analysis, we study a certain special class of functions of a complex variable, which has very strong analytical properties. This section introduces us to heuristic and historical reasons why we study this particular class.

Historically, complex functions arose from questions such as "What is e^z for a complex number z?" and "What is $\log i$?". The default understanding was that the values of e^z and $\log i$ are "out there", and we just need to "find" them. In our language, the question can be rephrased as follows.

Given a function $f(x)$ of a real variable, find an extension $F(z)$, that is in some sense natural.

A complex function $F(z)$ is an *extension* of $f(x)$ if $F(t) = f(t)$ for real t. Note that the essence of this problem is not that we start with a real function, but the question of how we classify complex functions into two classes: The first class consists of functions that we consider "natural" or "nice", and the second class consists of all the rest. Extensions of real functions will provide us with a large supply of complex functions, but we want to study complex functions on their own, regardless of whether or not they are extensions of real functions. Hence the classification problem just mentioned is the question we are really after.

In order to get some insight on the extension problem, let us consider a real polynomial

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n. \tag{1}$$

Then everybody would agree that the most natural extension of it to the complex setting is

$$P(z) = a_0 + a_1 z + \ldots + a_n z^n. \tag{2}$$

In particular, when we say that a complex number z is a root of $p(x)$, what we have in mind is the statement $P(z) = 0$. However, $P(z)$ is not the only possible extension of $p(x)$, as, for example,

$$Q(z) = a_0 + a_1 \mathrm{Re} z + \ldots + a_n (\mathrm{Re} z)^n, \tag{3}$$

and

$$R(z) = \begin{cases} p(z) & \text{if } \mathrm{Im}\, z = 0, \\ 0 & \text{if } \mathrm{Im}\, z \neq 0, \end{cases} \tag{4}$$

are both extensions of the polynomial $p(x)$.

Date: February 3, 2015.