Proof whether there is matrix T such as $TA = A^T$, $\quad T$ is *transpose* matrix

First Try. Assume there is matrix T such as TA = A^T. then T T = T^T $\implies$ T T T = T

$\implies (TT)T = T$

$\implies T^2 = I$, That Does not mean T is Identity matrix $\because T$ can be matrix swapping two rows or two columns

If $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ then $TT = I$

It seems to me we can not go further. $\square$

Second try. let assume there is 2 by 2 matrix T such as $TA = A^T$

T = $\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

T A = A^T

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\begin{bmatrix} x_{11}a + x_{12}c & x_{11}b + x_{12}d \\ x_{21}a + x_{22}c & x_{21}b + x_{22}d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\begin{aligned} x_{11}a + x_{12}c &= a \\ x_{21}a + x_{22}c &= b \\ x_{11}b + x_{12}d &= c \\ x_{21}b + x_{22}d &= d \end{aligned} \implies \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} a & a & b & b \\ c & c & d & d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Sub **??** into

$$x_{11}a + x_{12}c = a$$
$$\implies a = \frac{x_{12}c}{1 - x_{11}}$$
$$x_{21}a + x_{22}c = b$$
$$x_{21}\left(\frac{x_{12}c}{1 - x_{11}}\right) + x_{22}c = b$$
$$x_{21}b + x_{22}d = d$$
$$d = \frac{x_{21}b}{1 - x_{22}}$$
$$x_{11}b + x_{12}d = c$$
$$x_{11}b + x_{12}\left(\frac{x_{21}b}{1 - x_{22}}\right) = c$$

$$\boxed{x_{21}\left(\frac{x_{12}c}{1 - x_{11}}\right) + x_{22}c = b}$$
$$c\left[x_{21}\left(\frac{x_{12}}{1 - x_{11}}\right) + x_{22}\right] = b$$
$$\boxed{x_{11}b + x_{12}\left(\frac{x_{21}b}{1 - x_{22}}\right) = c}$$
$$b\left[x_{11} + x_{12}\left(\frac{x_{21}}{1 - x_{22}}\right)\right] = c$$
$$c\left[x_{22} + x_{21}\left(\frac{x_{12}}{1 - x_{11}}\right)\right] = b$$
$$\text{If } b \neq 0$$
$$x_{11} + x_{12}\left(\frac{x_{21}}{1 - x_{22}}\right) = x_{22} + x_{21}\left(\frac{x_{12}}{1 - x_{11}}\right)$$
$$x_{11} - x_{22} = x_{12}x_{21}\left(\frac{1}{1 - x_{11}} - \frac{1}{1 - x_{22}}\right)$$
$$x_{11} - x_{22} = x_{12}x_{21}\left(\frac{1 - x_{22}}{(1 - x_{11})(1 - x_{22})} - \frac{1 - x_{11}}{(1 - x_{11})(1 - x_{22})}\right)$$
$$x_{11} - x_{22} = x_{12}x_{21}\left(\frac{x_{11} - x_{22}}{(1 - x_{11})(1 - x_{22})}\right)$$
$$\text{If } x_{11} \neq x_{22}$$
$$1 = \frac{x_{12}x_{21}}{(1 - x_{11})(1 - x_{22})}$$
$$x_{12}x_{21} = (1 - x_{11})(1 - x_{22})$$

$$\mathrm{R}(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

$$R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix}$$

$$R(\tfrac{\pi}{2}) = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$R(\pi) = \begin{bmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

$TA = A^T$

$T(TA) = A$

$(TT)A = A$

$T^2 = I$