An Elementary Introduction to the Hopf Fibration

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Introduction

The Hopf fibration, named after Heinz Hopf who studied it in a 1931 paper [8], is an important object in mathematics and physics. It was a landmark discovery in topology and is a fundamental object in the theory of Lie groups. The Hopf fibration has a wide variety of physical applications including magnetic monopoles [13], rigid body mechanics [10] and quantum information theory [12].

Unfortunately, the Hopf fibration is little known in the undergraduate curriculum, in part because presentations usually assume background in abstract algebra or manifolds. However, this is not a necessary restriction. We present in this article an introduction to the Hopf fibration that requires only linear algebra and analytic geometry. In particular, no vector calculus, abstract algebra or topology is needed. Our approach uses the algebra of quaternions and illustrates some of the algebraic and geometric properties of the Hopf fibration. We explain the intimate connection of the Hopf fibration with rotations of 3-space that is the basis for its natural applications to physics.

We deliberately leave some of the development as exercises, called "Investigations," for the reader. The Investigations contain key ideas and are meant to be fun to think about. The reader may also take them as statements of facts that we wish to assume without interrupting the narrative.

Hopf's mapping

The *standard unit n-sphere* S^n is the set of points $(x_0, x_1, \ldots, x_n)$ in $\mathbb{R}^{n+1}$ that satisfy the equation

$$x_0^2 + x_1^2 + \cdots + x_n^2 = 1.$$