1 Complex Number 1

Addition

$$\begin{aligned}\text{let } z_1 &= x_1 + y_1 i \\ \text{let } z_2 &= x_2 + y_2 i \\ z &= z_1 + z_2 \\ z &= (x_1 + x_2) + (y_1 + y_2)i\end{aligned}$$

Multiplication

$$\begin{aligned}z &= z_1 z_2 \\ z &= (x_1 + y_1 i)(x_2 + y_2 i) \\ z &= (x_1 y_2 + x_2 y_1) + (x_1 x_2 - y_1 y_2)i\end{aligned}$$

Division

$$\begin{aligned}z &= \frac{z_1}{z_2} \\ z &= \frac{x_1 + y_1 i}{x_2 + y_2 i} \\ z &= \frac{(x_1 + y_1 i)(x_2 - y_2 i)}{(x_2 + y_2 i)(x_2 - y_2 i)} \\ z &= \frac{(x_1 x_2 + y_1 y_2) + (-x_1 y_2 + x_2 y_1)i}{x_2^2 + y_2^2} \\ z &= \frac{(x_1 x_2 + y_1 y_2)}{x_2^2 + y_2^2} + \frac{(-x_1 y_2 + x_2 y_1)}{x_2^2 + y_2^2} i\end{aligned}$$

Conjugation

$$\begin{aligned}\overline{z} &= x_1 - y_1 i \\ z\overline{z} &= (x_1 + y_1 i)(x_1 - y_1 i) \\ z\overline{z} &= x_1^2 + y_1^2 \\ z\overline{z} &= |z|^2\end{aligned}$$

Modulo

$$r = \sqrt{x^2 + y^2}$$

Polar form

$$(r, \beta) = (\sqrt{x^2 + y^2}, \cos^{-1} \frac{x}{r})$$

Matrix represents Complex Number 1

$$
\begin{aligned}
\text{let } z_1 &= x_1 + y_1 i \\
\text{let } z_2 &= x_2 + y_2 i \\
z_1 z_2 &= (x_1 + y_1 i)(x_2 + y_2 i) \\
&= (x_1 x_2 - y_1 y_2) + (y_1 x_2 + x_1 y_2) i \\
(x_1 x_2 - y_1 y_2) &= \left[\begin{array}{cc} x_1 & -y_1 \end{array}\right] \left[\begin{array}{c} x_2 \\ y_2 \end{array}\right] \\
(y_1 x_2 + x_1 y_2) &= \left[\begin{array}{cc} y_1 & x_1 \end{array}\right] \left[\begin{array}{c} x_2 \\ y_2 \end{array}\right]
\end{aligned}
$$

Combine two dot products, we have

$$
\begin{aligned}
\mathcal{M} &= \begin{bmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{bmatrix} \\
\vec{V} &= \mathcal{M} \left[\begin{array}{c} x_2 \\ y_2 \end{array}\right] \\
(x_1 x_2 - y_1 y_2) + (y_1 x_2 + x_1 y_2) i &= \left[\begin{array}{cc} 1 & i \end{array}\right] \vec{V}
\end{aligned}
$$

Identity, $1 \in \mathbb{C}$ can be represented as identity matrix

$$
\begin{aligned}
x_1 + y_1 i &= 1, \quad \text{where } x_1 = 1, y_1 = 0 \\
1 = \mathcal{M} &= \begin{bmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

Imaginary unit, $i \in \mathbb{C}$ can be represented as matrix

$$
\begin{aligned}
x_1 + y_1 i &= i \quad \text{where } x_1 = 0, y_1 = 1 \\
i = \mathcal{M} &= \begin{bmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\end{aligned}
$$

Show $i^2 = -1$

$$
\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

Conjugate $\overline{z}$ corresponds to transpose of the matrix

$$\begin{aligned}
z = x + yi &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} \\
z = x + yi &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
z = x + yi &= \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \\
\overline{z} = x - yi &= \begin{bmatrix} x & -y \\ y & x \end{bmatrix}^T \\
x - yi &= \begin{bmatrix} x & y \\ -y & x \end{bmatrix}
\end{aligned}$$

The square of absolute values of z corresponds to the determinant of the matrix

$$|z|^2 = \det \begin{bmatrix} x & -y \\ y & x \end{bmatrix} = x^2 + y^2$$

Polar form in Complex Number which has matrix form

$$\begin{aligned}
e^{i\theta} &= \cos(\theta) + i\sin(\theta) \\
e^{-i\theta} &= \cos(\theta) + i\sin(-\theta) \\
e^{i\theta} + e^{-i\theta} &= 2\cos(\theta) \\
\frac{e^{i\theta} + e^{-i\theta}}{2} &= \cos(\theta)
\end{aligned}$$

Polar form in Complex Number which has matrix form

$$\cos(\beta) + \sin(\beta)i \implies \begin{bmatrix} \sin(\beta) & -\cos(\beta) \\ \cos(\beta) & \sin(\beta) \end{bmatrix}$$

Proof. $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

From Euler's formular

$$\cos\theta + i\sin\theta = e^{i\theta} \tag{1}$$
$$\cos\theta - i\sin\theta = e^{-i\theta} \tag{2}$$

Adding and substracting (1) and (2)

$$\begin{aligned}(\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) &= e^{i\theta} + e^{i\theta} \\ (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta) &= e^{i\theta} - e^{i\theta} \\ \sin\theta &= \frac{e^{i\theta} - e^{i\theta}}{2i} \\ \cos\theta &= \frac{e^{i\theta} + e^{i\theta}}{2}\end{aligned}$$

□

Euler's formula can be represented in matrix form

$$\begin{aligned}e^{i\theta} &= \cos(\theta) + i\sin(\theta) \\ \exp(\theta \begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}) &= \begin{bmatrix}\sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta)\end{bmatrix}\end{aligned}$$

Above matrix is one way to represent Complex number, other matrix such as

$$\mathcal{J} = \begin{bmatrix}p & q \\ r & -p\end{bmatrix}, \quad p^2 + rq + 1 = 0$$

has the properties that its square is the negative of the identity matrix: $\mathcal{J}^2 = -I$

Proof. $\mathcal{J}^2 = -I$

$$\begin{aligned}\mathcal{J}^2 &= \begin{bmatrix}p & q \\ r & -p\end{bmatrix}\begin{bmatrix}p & q \\ r & -p\end{bmatrix} \\ &= \begin{bmatrix}p^2 + rq & pq - pq \\ rp - rp & p^2 + rq\end{bmatrix} \\ &= -I\end{aligned}$$

□

2 Quaternion

2.1 Addition and Multiplication

Add two quaternions acts component-wise, let two quaternions **q** and **p**

$$\begin{aligned}\mathbf{q} &= q_0 + q_1 i + q_2 j + q_3 k = q_0 + \vec{q} \\ \mathbf{p} &= p_0 + p_1 i + p_2 j + p_3 k = p_0 + \vec{p}\end{aligned}$$

We have

$$\begin{aligned}\mathbf{q}+\mathbf{p} &= (q_0+q_1 i+q_2 j+q_3 k)+(p_0+p_1 i+p_2 j+p_3 k)\\ &= (q_0+p_0)+(q_1+p_1)i+(q_2+p_2)j+(q_3+p_3 k)\end{aligned}$$

The product of two quaternions satisfies these fundamental rules introducted by Hamilton:

$$\begin{aligned}&\hat{i}^2=\hat{j}^2=\hat{k}^2=\hat{i}\hat{j}\hat{k}=-1\\ &\hat{i}\hat{j}=\hat{k},\quad \hat{j}\hat{k}=\hat{i},\quad \hat{k}\hat{i}=\hat{j}\\ \mathbf{qp} &= (q_0+q_1\hat{i}+q_2\hat{j}+q_3\hat{k})(p_0+p_1\hat{i}+p_2\hat{j}+p_3\hat{k})\\ &= [q_0+(q_1\hat{i}+q_2\hat{j}+q_3\hat{k})][p_0+(p_1\hat{i}+p_2 j+p_3\hat{k})]\\ &= q_0p_0+q_0(p_1\hat{i}+p_2\hat{j}+p_3\hat{k})+p_0(q_1\hat{i}+q_2\hat{j}+q_3\hat{k})+\\ &\quad (q_1\hat{i}+q_2\hat{j}+q_3\hat{k})(p_1\hat{i}+p_2\hat{j}+p_3\hat{k})\\ &= (q_0+\vec{q})(p_0+\vec{p})\quad \text{where}\quad \vec{q}=q_1\hat{i}+q_2\hat{j}+q_3\hat{k}\quad \vec{p}=p_1\hat{i}+p_2\hat{j}+p_3\hat{k}\\ &= q_0p_0+q_0\vec{p}+p_0\vec{q}-\vec{q}\cdot\vec{p}+\vec{q}\times\vec{p}\end{aligned}$$

Conjugation

$$\mathbf{q}^*=q_0-q_1 i-q_2 j-q_3 k$$

Unit Quaternion

$$\mathbf{q}=q_0+q_1 i+q_2 j+q_3 k=q_0^2+q_1^2+q_2^2+q_3^2=1$$

Inverse Quaternion

$$\mathbf{q}^{-1}=\frac{\mathbf{q}^*}{|\mathbf{q}|}$$

$$\text{If } |\mathbf{q}|=1 \text{ then } \mathbf{q}^*=\mathbf{q}^{-1}\quad \mathbf{qq}^*=1$$

Proof. $\mathbf{qq}^*=1$

$$\begin{aligned}|\mathbf{q}|=1 &\implies |q_0|^2+|\vec{q}|^2=1\\ \mathbf{qq}^* &= (q_0+\vec{q})(q_0-\vec{q})\\ &= q_0q_0-2q_0\vec{q}+\vec{q}\cdot\vec{q}+\vec{q}\times\vec{q}\\ &= |q_0|^2+|\vec{q}|^2\\ &= 1\end{aligned}$$

□

Norm of Quaternion

$$\mathbf{q}\mathbf{q}^* = |\mathbf{q}|^2$$

Multiplication

$$\begin{aligned}
\mathbf{qp} &= (q_0 + q_1 i + q_2 j + q_3 k)(p_0 + p_1 i + p_2 j + p_3 k) \\
&= [q_0 + (q_1 i + q_2 j + q_3 k)][p_0 + (p_1 i + p_2 j + p_3 k)] \\
&= q_0 p_0 + q_0(p_1 i + p_2 j + p_3 k) + p_0(q_1 i + q_2 j + q_3 k) + (q_1 i + q_2 j + q_3 k)(p_1 i + p_2 j + p_3 k) \\
&= (q_0 + \vec{q})(p_0 + \vec{p}) \\
&= q_0 p_0 + q_0 \vec{p} + p_0 \vec{q} - \vec{q} \cdot \vec{p} + \vec{q} \times \vec{p}
\end{aligned}$$

Multiplication table for $(1, i, j, k)$

	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

Multiplication table for (p_0, p_1, p_2, p_3) (q_0, q_1, q_2, q_3)

	p_0	p_1i	p_2j	p_3k
q_0	$q_0 p_0$	$q_0 p_1$i	$q_0 p_2$j	$q_0 p_3$k
q_1i	$p_0 q_1$i	$-q_1 p_1$	$q_1 p_2$k	$-q_1 p_3$j
q_2j	$p_0 q_2$j	$-q_2 p_1$k	$-q_2 p_2$	$q_2 p_3$i
q_3k	$p_0 q_3$k	$q_3 p_1$j	$-q_3 p_2$i	$-q_3 p_3$

$$\Longrightarrow p_0 q_0 + p_0 \vec{q} + q_0 \vec{p} - \vec{q}\vec{p} + \vec{q} \times \vec{p}$$

Multiplication table contains $p_0 q_0$ and $p_0 \vec{q}$, $q_0 \vec{p}$

	p_0	p_1i	p_2j	p_3k
q_0	$q_0 p_0$	$q_0 p_1$i	$q_0 p_2$j	$q_0 p_3$k
q_1i	$p_0 q_1$i			
q_2j	$p_0 q_2$j			
q_3k	$p_0 q_3$k			

$$\Longrightarrow p_0 q_0 + p_0 \vec{q} + q_0 \vec{p}$$

Multiplication table contains inner product of $\vec{q}$ and $\vec{p}$

	1	i	j	k
1				
i		$-q_1 p_1$		
j			$-q_2 p_2$	
k				$-q_3 p_3$

$$\Longrightarrow -\vec{p} \cdot \vec{q}$$

Multiplication table contains outer product of $\vec{p}$ and $\vec{q}$

	1	i	j	k
1				
i			$q_1 p_2 \mathrm{k}$	$-q_1 p_3 \mathrm{j}$
j		$-q_2 p_1 \mathrm{k}$		$q_2 p_3 \mathrm{i}$
k		$q_3 p_1 \mathrm{j}$	$-q_3 p_2 \mathrm{i}$	

$\Longrightarrow \vec{p} \times \vec{q}$

$$(q_0 + q_1 i + q_2 j + q_3 k)(p_0 + p_1 i + p_2 j + p_3 k) = [1, i, j, k] \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$M_j = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$M_k = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$(\pm M_1, \pm M_i, \pm M_j, \pm M_k)(\pm 1, \pm i, \pm j, \pm k)$ forms a group

3 Derive Tayor Series

Q: What is the problem that we try to solve?
A: We try to come up an equation to approximate $f(x) = e^x$
Q: What kind of equation that we can think of?
A: we can think about a polynomial e.g. $g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$
Q: Approximate $f(x) = e^x$ with $g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$? Sure:0
Q: What I should do next? come up a table and see what is on the table?

x	0	1	2	3
$f(x)$	e^0	e^1	e^2	e^3
$g(x)$	a_0	$a_0 + a_1 + a_2 + a_3$	$a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 + a_3 \cdot 2^3$	$a_0 + a_1 3 + a_2 \cdot 3^2 + a_3 \cdot 3^3$

We have four equations and four unknows and $a_0 = e^0 = 1$. Therefore, only a_1, a_2, a_3 need to be solved.

$$
\begin{aligned}
e^0 &= a_0 \\
e^1 &= a_0 + a_1 + a_2 + a_3 \\
e^2 &= a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 + a_3 \cdot 2^3 \\
e^3 &= a_0 + a_1 \cdot 3 + a_2 \cdot 3^2 + a_3 \cdot 3^3
\end{aligned}
$$

$$
\begin{aligned}
e^1 - a_0 &= a_1 + a_2 + a_3 \\
e^2 - a_0 &= a_1 \cdot 2 + a_2 \cdot 2^2 + a_3 \cdot 2^3 \\
e^3 - a_0 &= a_1 \cdot 3 + a_2 \cdot 3^2 + a_3 \cdot 3^3
\end{aligned}
$$

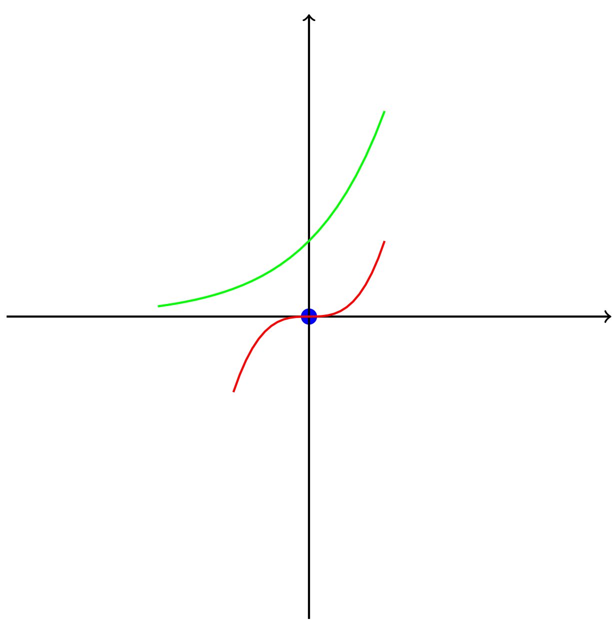
$$
\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2^2 & 2^3 \\ 3 & 3^2 & 3^3 \end{bmatrix}
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}
=
\begin{bmatrix} e^1 - a_0 \\ e^2 - a_0 \\ e^3 - a_0 \end{bmatrix}
$$

try to solve for a_0, a_1, a_2. However, polynomial $f(x)$ passes four points:

$$(0, e^0) \quad (1, e^1) \quad (2, e^2) \quad (3, e^3)$$

Assume we solve the matrix and find a_0, a_1, a_2, a_3

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$



$$(x_1 + y_1 i)(x_2 + y_2 i) = x_1 x_2 + x_1 y_2 i + x_2 y_1 i - y_1 y_2 = x_1 x_2 - y_1 y_2 + (x_1 y_2 + x_2 y_1) i$$