# A Linearity Property of Determinants

On. p. 173 is a property of determinants that I didn't mention in lecture, assuming you'd pick up on it in reading Section 3.2. That property is useful for at least one WebWork problem that a couple of people have asked about.

Suppose $A = [\boldsymbol{a_1} \ldots\ \boldsymbol{a_j}\ \ldots\ \boldsymbol{a_n}]$ is an $n \times n$ matrix with columns $\boldsymbol{a_1}, \ldots\ \boldsymbol{a_j}, \ldots\ \boldsymbol{a_n}$

Supppose <u>one column</u>, say column $\boldsymbol{a_j}$, is replaced by a variable column $\boldsymbol{x}$ from $\mathbb{R}^n$.

Let $T(\boldsymbol{x}) = \det\,[\boldsymbol{a_1} \ldots\ \boldsymbol{x}\ \ldots\ \boldsymbol{a_n}]$. $T(\boldsymbol{x})$ is a number so $\boldsymbol{x} \mapsto T(\boldsymbol{x})$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^1$.

<u>$T$ is a linear transformation.</u> this means that

i) $T(c\boldsymbol{x}) = cT(\boldsymbol{x})$ for all $\boldsymbol{x}$ in $\mathbb{R}^n$ and all scalars $c$, and
ii) $T(\boldsymbol{x} + \boldsymbol{y}) = T(\boldsymbol{x}) + T(\boldsymbol{y})$ for all $\boldsymbol{x}$, $\boldsymbol{y}$ in $\mathbb{R}^n$

i) is true becuase of a property we already know about how factoring a number out of a column affects the determinant:

$$T(c\boldsymbol{x}) = \det\,[\boldsymbol{a_1} \ldots\ c\boldsymbol{x}\ \ldots\ \boldsymbol{a_n}] = c\det\,[\boldsymbol{a_1} \ldots\ \boldsymbol{x}\ \ldots\ \boldsymbol{a_n}] = cT(\boldsymbol{x})$$

ii) is checked by expanding the determinants for $T(\boldsymbol{x} + \boldsymbol{y})$, $T(\boldsymbol{x})$, and $T(\boldsymbol{y})$ down the $j$th column (see the $3 \times 3$ example in Exercise 43 of Section 3.2.

To illustrate with a $3 \times 3$ example.

Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, and replace (say) column 2 with a veriable vector $\boldsymbol{x}$ from $\mathbb{R}^3$:

$$T(\boldsymbol{x}) = \det \begin{bmatrix} 1 & x_1 & 3 \\ 4 & x_2 & 6 \\ 7 & x_3 & 9 \end{bmatrix}.$$

Since $T$ is linear, $T(\boldsymbol{x} + \boldsymbol{y}) = \det \begin{bmatrix} 1 & x_1 + y_1 & 3 \\ 4 & x_2 + y_2 & 6 \\ 7 & x_3 + y_3 & 9 \end{bmatrix}$

$$= \det \begin{bmatrix} 1 & x_1 & 3 \\ 4 & x_2 & 6 \\ 7 & x_3 & 9 \end{bmatrix} + \det \begin{bmatrix} 1 & y_1 & 3 \\ 4 & y_2 & 6 \\ 7 & y_3 & 9 \end{bmatrix} = T(\boldsymbol{x}) + T(\boldsymbol{y})$$

Read in reverse order, this says that if we have two matrices that are identical except for 1 column, then in computing the determinants

$$\det\begin{bmatrix} 1 & x_1 & 3 \\ 4 & x_2 & 6 \\ 7 & x_3 & 9 \end{bmatrix} + \det\begin{bmatrix} 1 & y_1 & 3 \\ 4 & y_2 & 6 \\ 7 & y_3 & 9 \end{bmatrix} = \det\begin{bmatrix} 1 & x_1 + y_1 & 3 \\ 4 & x_2 + y_2 & 6 \\ 7 & x_3 + y_3 & 9 \end{bmatrix}$$

two columns the same

we can add together the two different columns

note: we are not adding column 1's or column 3's together