Math 501 - Differential Geometry
Herman Gluck
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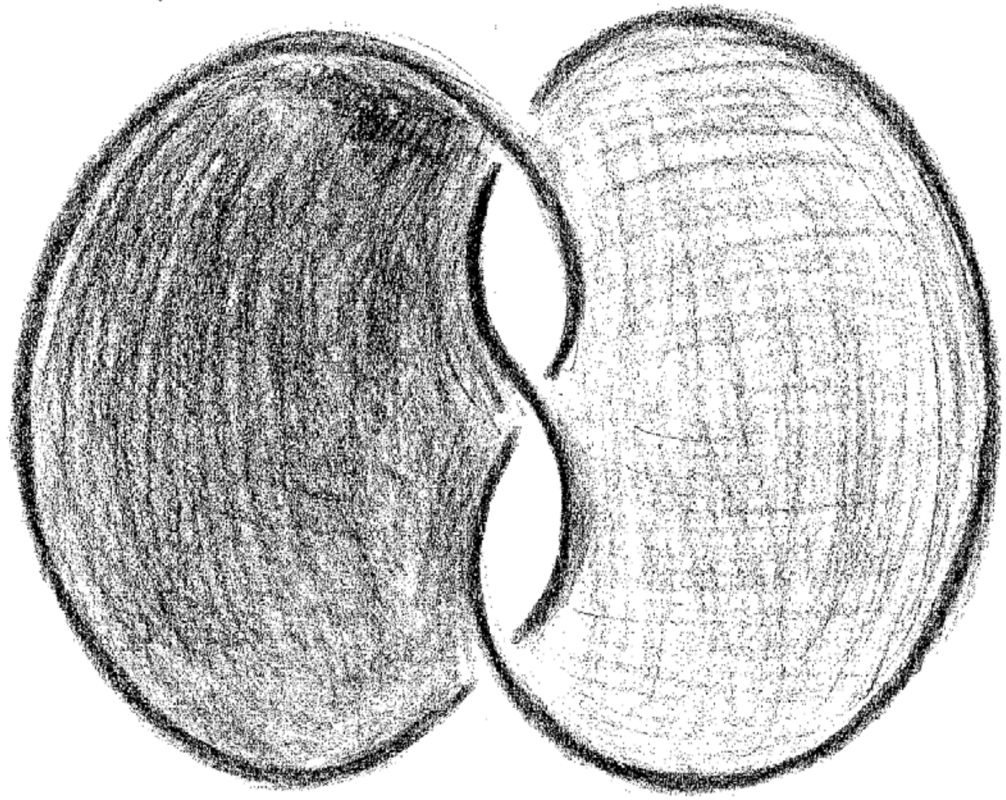
7. THE GAUSS-BONNET THEOREM

The Gauss-Bonnet Theorem is one of the most beautiful and one of the deepest results in the differential geometry of surfaces. It concerns a surface S with boundary ∂S in Euclidean 3-space, and expresses a relation between:

- the integral $\int_S K \, d(\text{area})$ of the Gaussian curvature over the surface,
- the integral $\int_{\partial S} \kappa_g \, ds$ of the geodesic curvature of the boundary of the surface, and
- the topology of the surface, as expressed by its Euler characteristic:

$$\chi(S) = \# \text{ Vertices} - \# \text{ Edges} + \# \text{ Faces} .$$



Do you recognize this surface?

We will approach this subject in the following way:

- First we'll build up some experience with examples in which we integrate Gaussian curvature over surfaces and integrate geodesic curvature over curves.

- Then we'll state and explain the Gauss-Bonnet Theorem and derive a number of consequences.

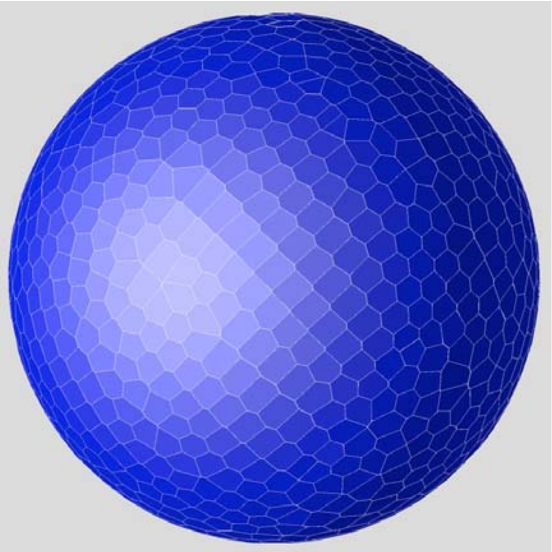
- Next we'll try to understand on intuitive grounds why the Gauss-Bonnet Theorem is true.

- Finally, we'll prove the Gauss-Bonnet Theorem.

Johann Carl Friedrich Gauss (1777 - 1855), age 26

Examples of the Gauss-Bonnet Theorem.

Round spheres of radius r.



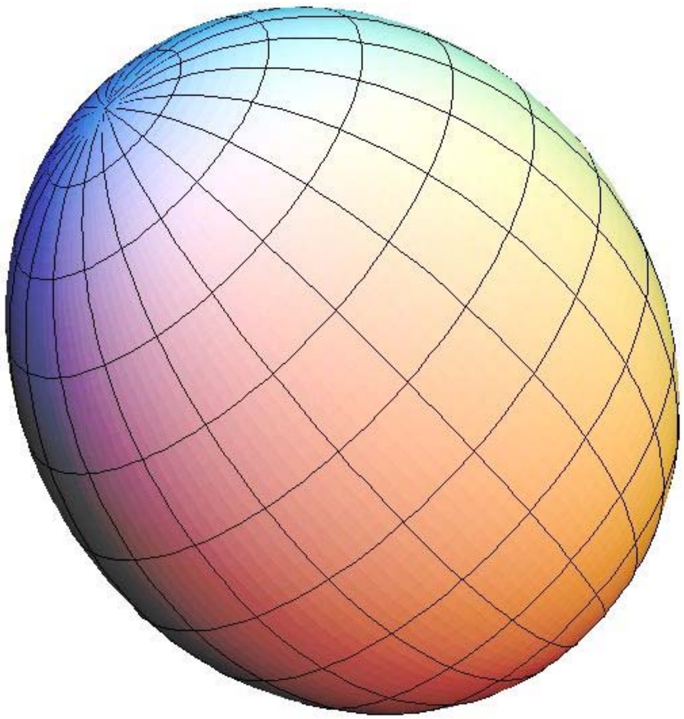
Gaussian curvature $K = 1/r^2$ Area $= 4\pi r^2$

$$\int_S K \, d(\text{area}) = K \times \text{Area} = (1/r^2)\, 4\pi r^2$$

$$= 4\pi = 2\pi \chi(S) .$$

Convex surfaces.

Let S be any convex surface in 3-space.



Let $N: S \to S^2$ be the Gauss map, with differential

$$dN_p: \; T_pS \to T_{N(p)}S^2 = T_pS \,.$$

The Gaussian curvature $K(p) = \det dN_p$ is the local explosion factor for areas under the Gauss map.

Hence by change of variables for multiple integrals,

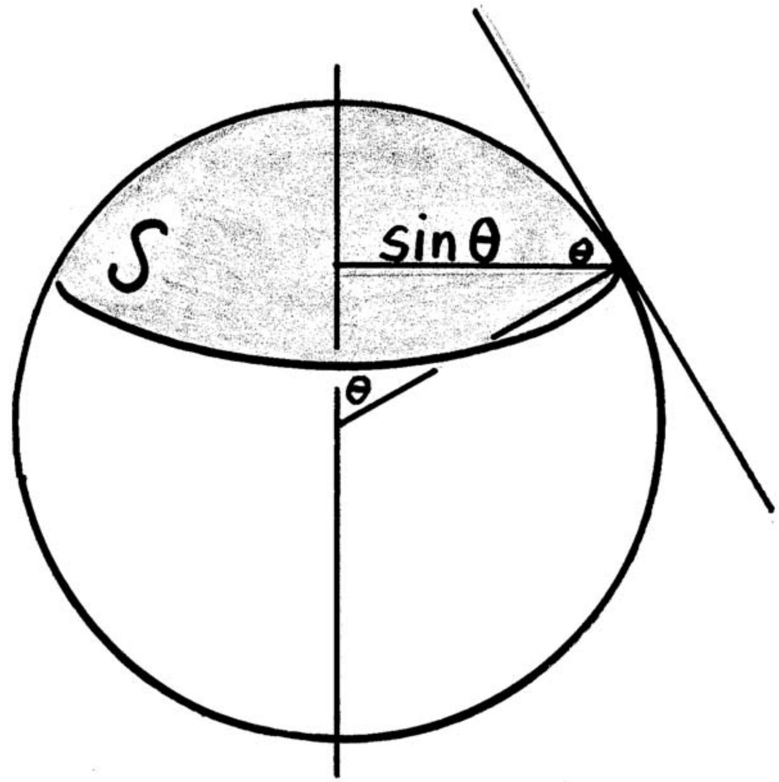
$$\int_S K \; d(\text{area}_S) = \int_S \det dN_p \; d(\text{area}_S) = \int_{S^2} d(\text{area}_{S^2})$$

$$= \text{Area}(S^2) = 4\pi = 2\pi\,\chi(S) \,.$$

Problem 1. How would you extend this argument to other closed, but not necessarily convex, surfaces in 3-space?

Polar caps on spheres.

Consider a polar cap S on a round sphere of radius 1 :



$$\text{Area } S = \int_0^\theta 2\pi \sin\theta \, d\theta = 2\pi(1 - \cos\theta),$$

hence $\quad \int_S K \, d(\text{area}) = 1 \times \text{Area } S = 2\pi(1 - \cos\theta)$.

The curvature κ of ∂S = 1/radius = $1/\sin\theta$.

The geodesic curvature κ_g of ∂S is $\cos\theta / \sin\theta$, hence

$$\int_{\partial S} \kappa_g \, ds = \kappa_g \times \text{length}(\partial S)$$

$$= (\cos\theta / \sin\theta)\, 2\pi \sin\theta = 2\pi \cos\theta .$$

Therefore

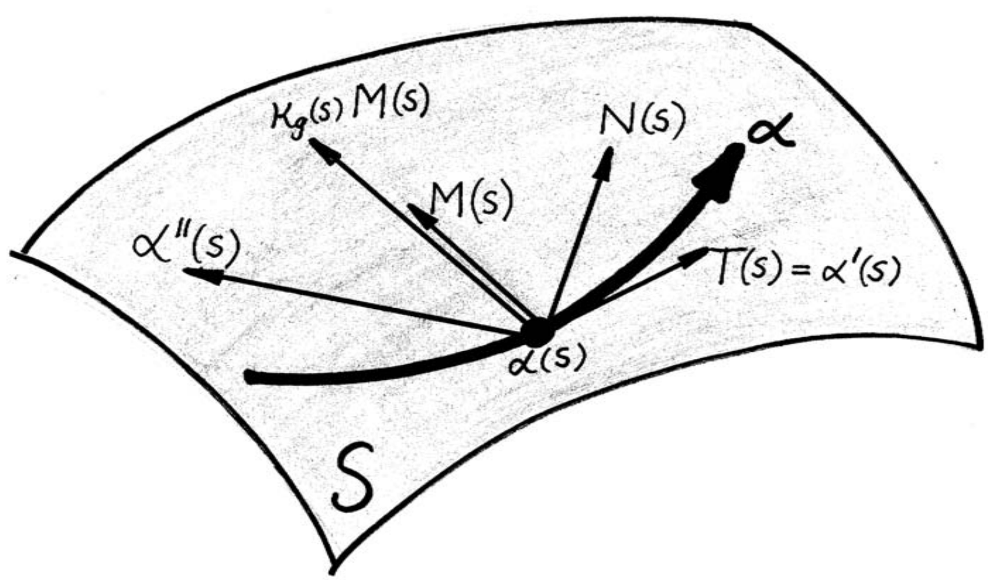
$$\int_S K \, d(\text{area}) + \int_{\partial S} \kappa_g \, ds = 2\pi(1 - \cos\theta) + 2\pi\cos\theta$$

$$= 2\pi = 2\pi\,\chi(S) .$$

Geodesic curvature as a rate of turning.

Let S be a regular surface in 3-space, and α: I $\rightarrow$ S a smooth curve on S parametrized by arc length.

Let T(s) , M(s) , N(s) be the right-handed O.N. frame at $\alpha(s)$ introduced earlier, where $T(s) = \alpha'(s)$ and N(s) is a unit normal to the surface at $\alpha(s)$.



The geodesic curvature of α at $\alpha(s)$ is

$$\kappa_g(s) = \langle \alpha''(s) , M(s) \rangle ,$$

and hence the covariant derivative

$$D\alpha'(s)/ds = \text{orthog proj of } \alpha''(s) \text{ on } S$$

$$= \kappa_g(s) M(s) .$$

Problem 2. Show that

$$\alpha''(s) = \kappa_g(s)\, M(s) + k_n(s)\, N(s) ,$$

where $k_n(s)$ denotes the normal curvature of S at the point $\alpha(s)$ in the direction of $T(s) = \alpha'(s)$.

Conclude that

$$\kappa^2 = \kappa_g{}^2 + k_n{}^2 ,$$

since $\kappa^2 = |\alpha''|^2$.

This conclusion was the content of a problem from an earlier chapter.

Problem 3. Let $\alpha: I \to S^2$ be a smooth curve parametrized by arc length on the unit 2-sphere S^2 in R^3. Show that

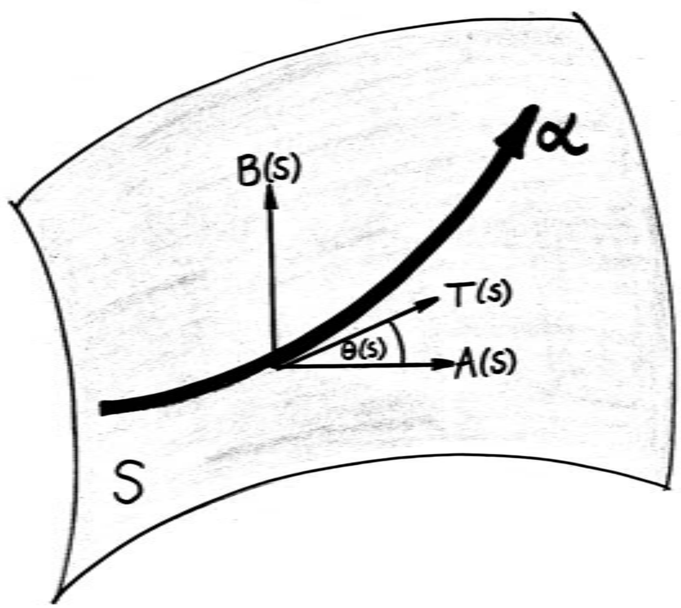
$$\kappa_g(s) = \alpha(s) \times \alpha'(s) \bullet \alpha''(s) .$$

Continuing with this example, let $A(s)$, $B(s)$ be an O.N. basis for the tangent space to S at $\alpha(s)$, chosen so that $A(s)$ and $B(s)$ are parallel vector fields along α.

Then we can write

$$T(s) \; = \; \cos\theta(s)\, A(s) \; + \; \sin\theta(s)\, B(s) \; ,$$

thus defining an ***angle of inclination*** of the unit tangent vector $T(s)$ with respect to the parallel frame $A(s)$, $B(s)$ along α .



Then

$$DT/ds \;=\; -\sin\theta(s)\,\theta'(s)\,A(s) \;+\; \cos\theta(s)\,\theta'(s)\,B(s)\,.$$

But

$$DT/ds \;=\; D\alpha'(s)/ds \;=\; \kappa_g(s)\,M(s)\,,$$

which tells us that

$$\kappa_g(s) \;=\; \theta'(s)\,.$$

In other words, we have shown

PROPOSITION. *The geodesic curvature of* α *is the rate of turning of the tangent line to* α *, reckoned with respect to a parallel frame along* α *.*

This generalizes the corresponding description of the curvature of a plane curve as the rate of turning of its tangent line.

Total geodesic curvature.

Let S be a regular surface in 3-space, and $\alpha: I \to S$ a smooth curve on S parametrized by arc length.

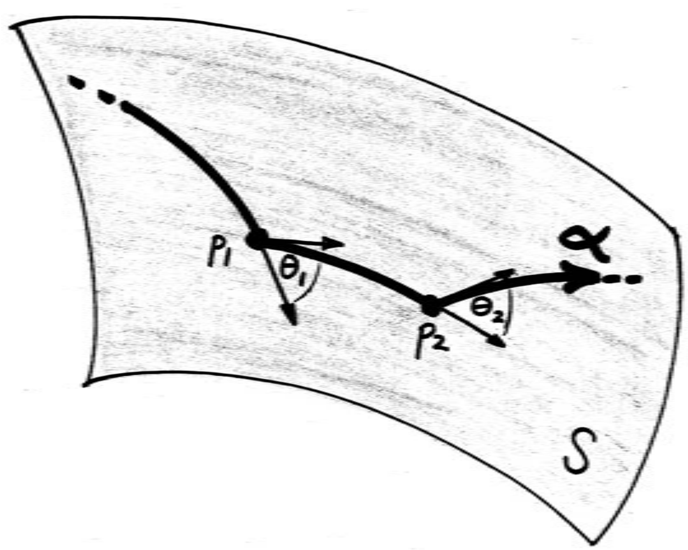
Let $A(s)$, $B(s)$ be an O.N. parallel frame along α .

Let $\theta(s)$ be the angle of inclination of the unit tangent vector $T(s) = \alpha'(s)$ with respect to this frame, as defined above.

Since the geodesic curvature $\kappa_g(s) = \theta'(s)$, it follows that the ***total geodesic curvature*** of α is given by

$$\int_I \kappa_g(s)\, ds = \theta(\text{end}) - \theta(\text{start}) .$$

If α: I $\rightarrow$ S is a piecewise smooth curve parametrized by arc length, then we need to deal with the ***exterior angles*** at the corners of α , as shown below.



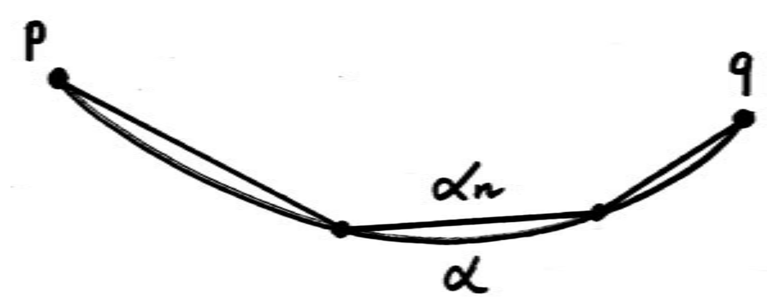
The signs of the exterior angles are determined by the orientation of S in the usual way.

If we repeat the preceding constructions for a piecewise smooth curve with corners at $p_1, \ldots, p_k$ and exterior angles $\theta_1, \ldots, \theta_k$ at these corners, then the total geodesic curvature of α is given by

$$\int_I \kappa_g(s)\ ds + \sum_{i=1}^k \theta_i = \theta(\text{end}) - \theta(\text{start})\ .$$

Problem 4. Let α_1 , α_2 , ... be a sequence of piecewise geodesic curves on the surface S , with a common starting point p and a common ending point q , which converges in some suitable sense to the smooth curve α from p to q .

Show that the sum of the exterior angles of the curve α_n converges to the total geodesic curvature of α as $n \to \infty$.



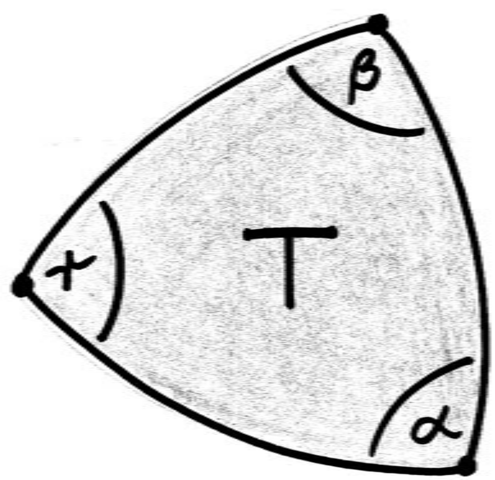
Note. Part of this problem is to decide in what sense the convergence takes place.

The area of a spherical triangle.

Given a triangle in the Euclidean plane with interior angles α , β , γ , we learned in high school that

$$\alpha + \beta + \gamma = \pi .$$

Consider a geodesic triangle on the unit 2-sphere S^2 with interior angles α , β , γ .



It turns out that

$$\alpha + \beta + \gamma > \pi ,$$

and that, furthermore, the excess is equal to the area T of the triangle:

$$\alpha + \beta + \gamma - \pi = T .$$

Proof that $\alpha + \beta + \gamma - \pi = T$.

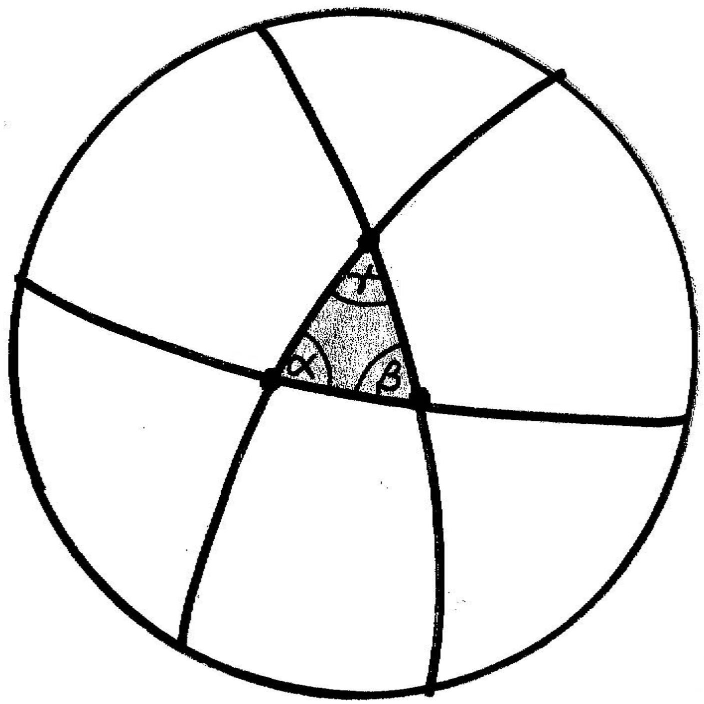
The sides of the triangle are geodesics, that is, arcs of great circles. Extend these arcs to full great circles, thus dividing the 2-sphere into six ***lunes***.

The area of the whole 2-sphere is 4π. A lune with vertex angle α represents a fraction $\alpha / 2\pi$ of the full sphere, and therefore has area

$$(\alpha / 2\pi)\, 4\pi = 2\alpha .$$

There are two such lunes in the picture below.

Two more lunes each have area 2β, and the final two lunes each have area 2γ.



Our geodesic triangle in covered three times by these lunes, and its antipodal image is also covered three times. The other regions are covered once by the union of the six lunes. Thus

$$2(2\alpha) + 2(2\beta) + 2(2\gamma) = 4\pi + 4\,T\,,$$

equivalently,

$$\alpha + \beta + \gamma = \pi + T\,,$$

or

$$\alpha + \beta + \gamma - \pi = T\,,$$

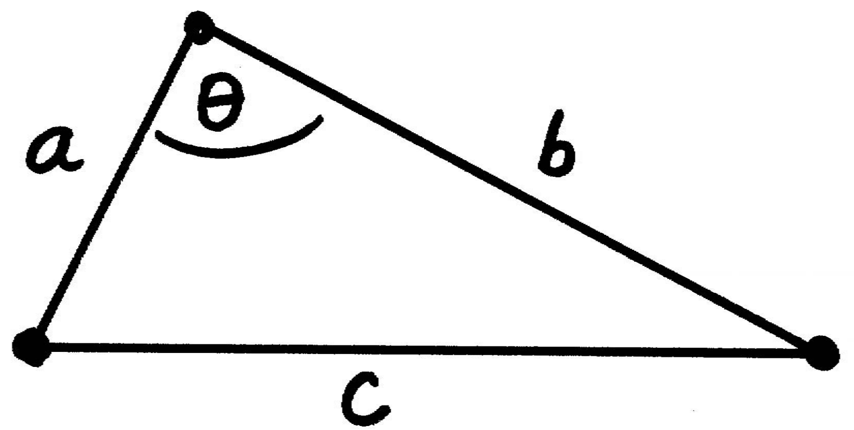
as claimed.

According to Marcel Berger in his book,

"A Panoramic View of Riemannian Geometry",

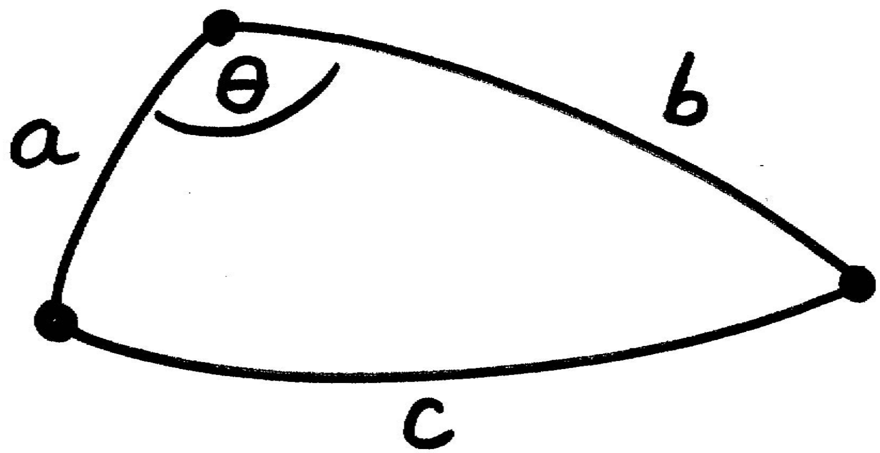
this formula for the area of a spherical triangle was discovered by Thomas Harriot (1560-1621) in 1603 and published (and perhaps rediscovered) by Albert Girard (1595-1632) in 1629.

Problem 5. Recall the ***law of cosines*** in Euclidean geometry:



$$\mathbf{c^2 = a^2 + b^2 - 2\,a\,b\cos\theta\ .}$$

Show that the law of cosines in spherical geometry is



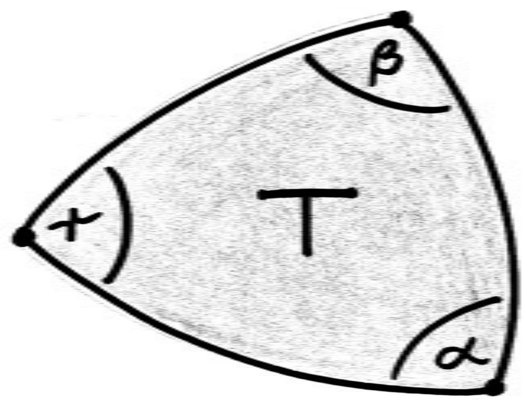
$$\cos c = \cos a \cos b + \sin a \sin b \cos \theta \, .$$

The Gauss-Bonnet Theorem for a spherical triangle with geodesic sides.

THEOREM. *Let* T *be a geodesic triangle on a round sphere of radius* R . *Then*

$$\int_T K \; d(\text{area}) \; + \; \sum \text{exterior angles} \; = \; 2\pi\, \chi(T) \; = \; 2\pi \, .$$

Proof. Let T also denote the area of the triangle T, and let α, β and γ denote the interior angles.



We showed that $\alpha + \beta + \gamma - \pi = T$ on a unit sphere, so on a sphere of radius R we have

$$\alpha + \beta + \gamma - \pi = T / R^2 .$$

Now $K = 1/R^2$, so $\int_T K \, d(\text{area}) = (1/R^2)\, T$.

The exterior angles are $\pi - \alpha$, $\pi - \beta$ and $\pi - \gamma$, so their sum is $3\pi - (\alpha + \beta + \gamma)$. Hence

$$\int_T K \; d(\text{area}) + \sum \text{exterior angles}$$

$$= T / R^2 + 3\pi - (\alpha + \beta + \gamma)$$

$$= (\alpha + \beta + \gamma - \pi) + 3\pi - (\alpha + \beta + \gamma)$$

$$= 2\pi ,$$

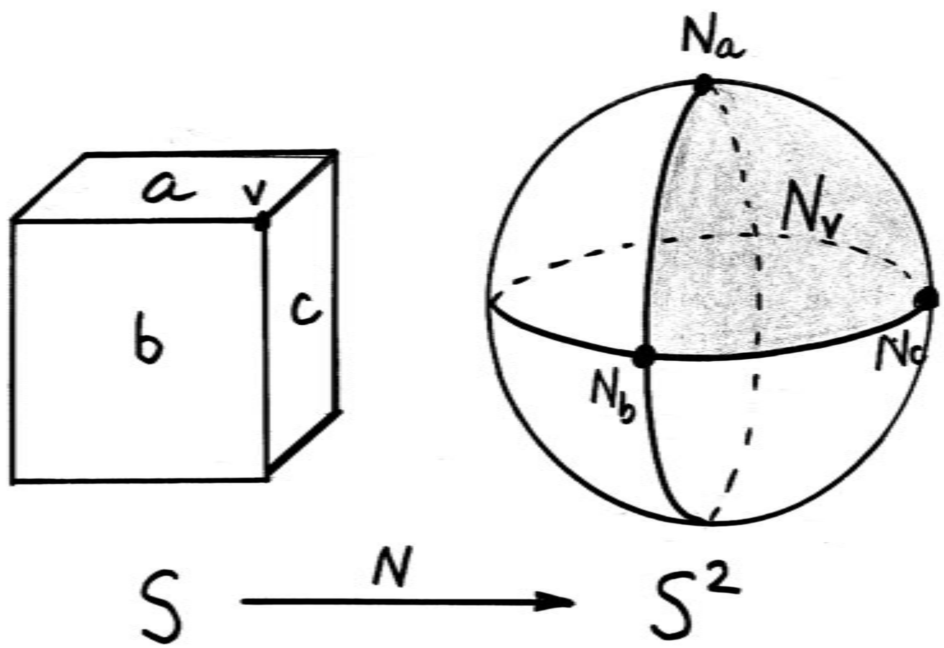
as claimed.

Problem 6. State and prove the Gauss-Bonnet Theorem for a spherical polygon with geodesic sides.

Gaussian curvature of polyhedral surfaces in 3-space.

To begin, suppose that S is the surface of a cube.

What will the Gauss "map" be?



No matter what point we choose in the top face a, the unit normal vector will point straight up. So the Gauss image N_a of the entire face a is the north pole of S^2.

Likewise, the Gauss image N_b of the entire front face b of the cube is the front pole of S^2, and the Gauss image N_c of the right face c is the east pole of S^2.

The Gauss image of the common edge shared by the faces a and b is not really defined, but common sense suggests that it should be the quarter circle connecting N_a and N_b. The points of this quarter circle represent the outer normals to the possible "tangent planes" to the cube along this edge.

The Gauss image of the vertex v, according to the same common sense, should be the geodesic triangle shaded in the figure, that is, the upper right front eighth of S^2. The points of this geodesic triangle represent the outer normals to the possible "tangent planes" to the cube at the vertex v.

Since the total Gaussian curvature of a region on a surface in 3-space is the area of its Gauss image on S^2, we see that the total curvature of the surface S of our cube is concentrated entirely at the eight vertices, with each contributing $\pi/2$ to the total Gaussian curvature of 4π.

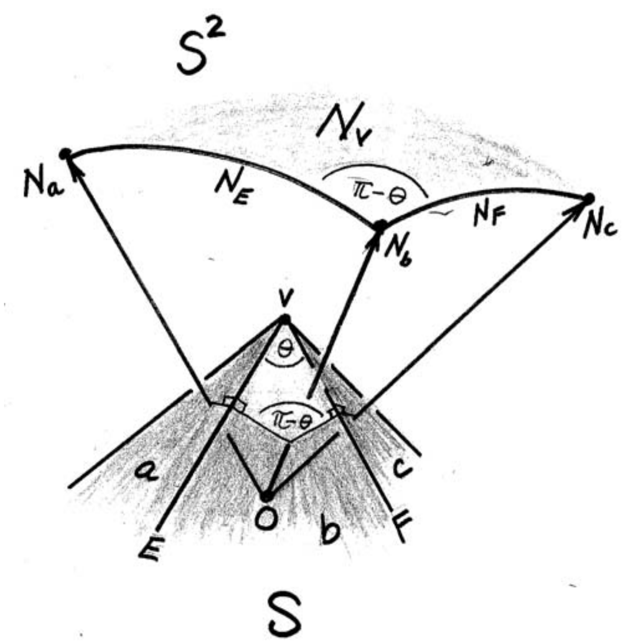
Conclusions.

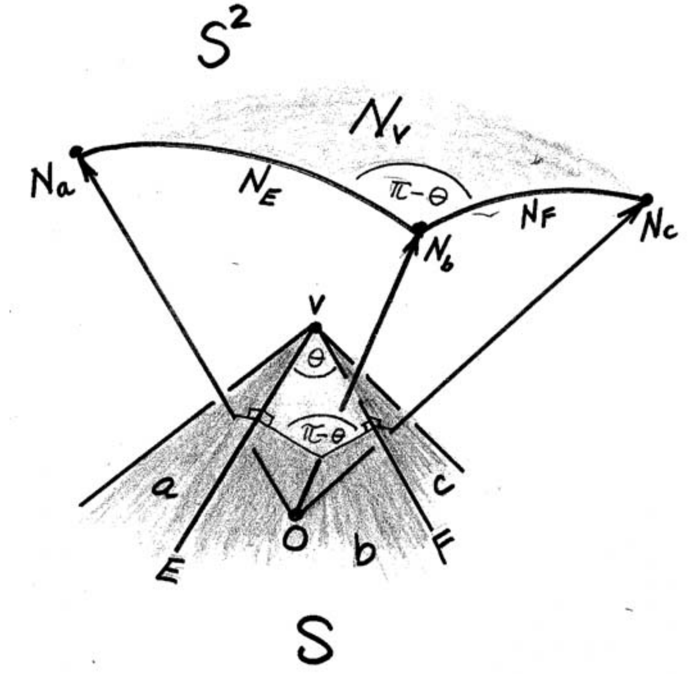
- The total Gaussian curvature of a polyhedral surface S in Euclidean 3-space is concentrated at its vertices.

- The Gauss image of a vertex is a region on S^2 bounded by a geodesic polygon.

Given a polyhedral surface in 3-space, how much Gaussian curvature is carried by each of its vertices?

The picture below represents the front portion of a neighborhood of a vertex v on a polyhedral surface S in 3-space. The faces a , b and c are shown, as are two edges E and F .





We pick an origin O as shown, and draw through it the unique planes orthogonal to the edges E and F . Their intersection is a line orthogonal to the face b , and on that line we mark off the unit vector N_b , which is the Gauss image of the entire face b .

The dihedral angle between the planes orthogonal to the edges E and F is the same as the angle on S^2 between the geodesic arc connecting N_a to N_b and the geodesic arc connecting N_b to N_c .

This dihedral angle appears again on the white quadrilateral on the face b , and its value is seen to be $\pi - \theta$, where θ is the angle of the face b at the vertex v.

In other words, if θ_1 , θ_2 , ..., θ_p are the angles at v of the various faces of the polyhedral surface S , then $\pi - \theta_1$, $\pi - \theta_2$, ..., $\pi - \theta_p$ are the *interior* angles of the geodesic polygon N_v on S^2 which is the Gauss image of v .

The area of N_v on S^2 is the excess of the sum of its interior angles over the expected value $(p - 2)\,\pi$ from Euclidean geometry:

$$\mathrm{Area}(N_v) = (\pi - \theta_1) + \ldots + (\pi - \theta_p) - (p - 2)\,\pi$$

$$= 2\pi - (\theta_1 + \ldots + \theta_p)\,.$$

Note that $\theta_1, \ldots, \theta_p$ are the *exterior* angles at the vertices of the geodesic polygon N_v on S^2, so the above formula is just the Gauss-Bonnet formula for N_v.

If $K(v)$ denotes the total Gaussian curvature associated with the vertex v on the polyhedral surface S, then

$$K(v) = 2\pi - (\theta_1 + \ldots + \theta_p) .$$

If $\theta_1 + \ldots + \theta_p = 2\pi$, then a portion of S around v can be flattened out onto a plane, and the (total) Gaussian curvature at v is 0.

If $\theta_1 + \ldots + \theta_p < 2\pi$, then the Gaussian curvature at v is positive, and if $\theta_1 + \ldots + \theta_p > 2\pi$, then the Gaussian curvature at v is negative.

The Gauss-Bonnet Theorem for a closed polyhedral surface in Euclidean 3-space.

THEOREM. *Let* S *be a closed polyhedral surface in Euclidean 3-space. Then*

$$\textit{Total Gaussian curvature of } S = 2\pi\,\chi(S) .$$

We can write this total curvature as a sum over the vertices v of S ,

$$K(S) = \sum_v K(v) .$$

Proof. It will be convenient to assume that all faces of S are triangles. We can easily achieve this by adding some extra edges if necessary.

Having done this, let V, E and F denote the number of vertices, edges and faces of S.

Since the faces are all triangles, we have $3F = 2E$.

Then

$$K(S) = \sum_v K(v) = \sum_v 2\pi - (\text{angle sum at } v)$$

$$= 2\pi V - (\text{angle sum at all vertices})$$

$$= 2\pi V - (\text{angle sum of all triangles})$$

$$= 2\pi V - \pi F$$

$$= 2\pi V - 3\pi F + 2\pi F$$

$$= 2\pi V - 2\pi E + 2\pi F$$

$$= 2\pi (V - E + F)$$

$$= 2\pi \chi(S) ,$$

as claimed.

The Gauss-Bonnet Theorem for a compact polyhedral surface with boundary in Euclidean 3-space.

Let S be a compact polyhedral surface with boundary in Euclidean 3-space. The boundary consists of a finite number of polygonal simple closed curves.

If v is a vertex in the interior of S, then $K(v)$ denotes the (total) Gaussian curvature of S at v, defined above.

If v is a vertex on the boundary of S, let

$$\theta_{int}(v) = \text{angle sum at } v,$$

and

$$\theta_{ext}(v) = \pi - \theta_{int}(v) = \textbf{\textit{exterior angle at}} \ \mathbf{v}.$$

If we let $2S$ denote the (singular) surface obtained by superimposing two copies of S and sewing them together only along the boundary, then the exterior angle at a boundary vertex v of S is just half the Gaussian curvature of the surface $2S$ at v .

THEOREM. *Let* S *be a compact polyhedral surface with boundary* ∂S *in Euclidean 3-space. Then*

$$\sum_{v \in \mathrm{Int}(S)} K(v) + \sum_{v \in \partial S} \theta_{ext}(v) = 2\pi\, \chi(S) .$$

Proof. Let 2S denote the double of S , as described above. Note that

$$\chi(2S) = 2\,\chi(S) - \chi(\partial S) = 2\,\chi(S) ,$$

since ∂S is a disjoint union of simple closed curves, with Euler characteristic zero.

Note that

$$\textstyle\sum_{v \in 2S} K(v) \;=\; 2\left(\sum_{v \in \mathrm{Int}(S)} K(v) \;+\; \sum_{v \in \partial S} \theta_{ext}(v)\right) .$$

Then the Gauss-Bonnet Theorem for $2S$ implies the Gauss-Bonnet Theorem for S .

Problem 7. Show how the polyhedral version of the Gauss-Bonnet Theorem converges in the limit to the smooth version, first for smooth closed surfaces, and then for compact smooth surfaces with boundary.

Hint. There is an unexpected subtlety involved.

If we try to approximate a smooth surface by a sequence of inscribed polyhedral surfaces with smaller and smaller triangular faces, then the triangular faces may not get closer and closer to the tangent planes to the smooth surface.

Instead, they may begin to "pleat" like the folds of an accordian.

In such a case, the surface area of the approximating polyhedral surfaces need not converge to the surface area of the smooth surface. You can see an example at

http://mathworld.wolfram.com/SchwarzsPolyhedron.html

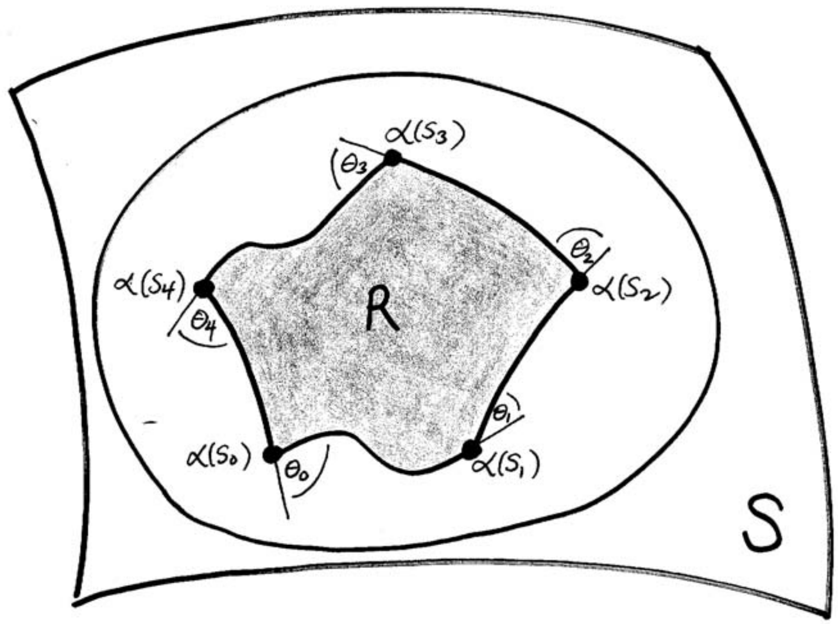
The antidote is to control the shapes of the triangles on the approximating polyhedra: all of their angles must be bounded away from zero.

The local Gauss-Bonnet Theorem.

LOCAL GAUSS-BONNET THEOREM. *Let* U *be an open set in* R^2 *and* $X: U \to S \subset R^3$ *a regular local parametrization of the surface* S . *Let* $R \subset X(U)$ *be a region homeomorphic to a disk. Let* $\alpha: I \to \partial R$ *be a piecewise smooth parametrization of* ∂R *by arc length, with vertices at* $\alpha(s_0)$, $\alpha(s_1)$, ..., $\alpha(s_k)$ *and with exterior angles* θ_0 , θ_1 , ..., θ_k *at these vertices. Then*

$$\int_R K \; d(\text{area}) + \int_{\partial R} \kappa_g(s) \; ds + \sum_{i=0}^{k} \theta_i = 2\pi .$$

$$\int_R K \; d(\text{area}) + \int_{\partial R} \kappa_g(s) \; ds + \sum_{i=0}^{k} \theta_i = 2\pi .$$



The middle term on the left is really a sum of integrals of geodesic curvature over the $k + 1$ smooth arcs on ∂R which, when added to the sum of the exterior angles, gives the total geodesic curvature of ∂R .

The local Gauss-Bonnet Theorem above says that

Total Gaussian curvature of **R**
+ ***total geodesic curvature of*** $\boldsymbol{\partial}$**R = 2$\boldsymbol{\pi}$.**

In the Euclidean plane, the first term on the left would be zero, and the second term 2π .

We will assume that the parametrization $X: U \rightarrow S$ is *orthogonal*, i.e., that $F = \langle X_u , X_v \rangle = 0$, and will see later, when we prove the global Gauss-Bonnet Theorem, why there is no loss of generality.

In the following sections we state and prove two lemmas needed for the proof of the local Gauss-Bonnet Theorem, and then follow with the proof.

Lemma 1.

Let $W(t)$ be a smooth field of unit vectors along a smooth curve $\alpha: I \to S$ on an oriented surface S .

Since $W(t)$ is a unit vector field, both the ordinary derivative dW/dt and the covariant derivative DW/dt are orthogonal to $W(t)$. If N is the normal to S , then DW/dt is some multiple of $N \times W(t)$,

$$DW/dt = [DW/dt]\, N \times W(t) .$$

The multiplier $[DW/dt]$ is called by do Carmo the ***algebraic value*** of the covariant derivative.

We can write $[DW/dt] = < DW/dt , N \times W(t) >$

$$= < dW/dt , N \times W(t) >$$

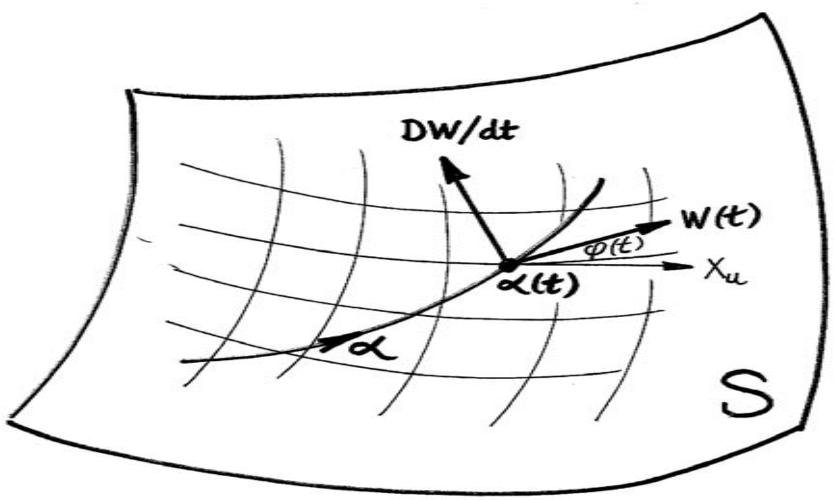
For example, if $\alpha(s)$ is parametrized by arc length, then the geodesic curvature of α is the algebraic value of the covariant derivative of the unit tangent vector $\alpha'(s)$,

$$\kappa_g(s) = [D\alpha'/ds] .$$

LEMMA 1. *Let* X(u, v) *be an orthogonal parametrization of a neighborhood on an oriented surface* S *in* R^3 , *and let* W(t) *be a smooth field of unit vectors along the curve* $\alpha(t) = X(u(t), v(t))$ *on* S . *Then*

$$[DW/dt] = \tfrac{1}{2}(EG)^{-1/2} \{G_u\, dv/dt - E_v\, du/dt\} + d\varphi/dt ,$$

where $\varphi(t)$ *is the angle from* X_u *to* W(t) *in the given orientation.*



Example 1.

Let S be the xy-plane.

Let $W(t) = (\cos \varphi(t), \sin \varphi(t))$ be a unit vector field along any curve $(x(t), y(t))$.

Then $N \times W(t) = (-\sin \varphi(t), \cos \varphi(t))$ and

$DW/dt = dW/dt = (-\sin \varphi(t)\, d\varphi/dt, \cos \varphi(t)\, d\varphi/dt)$.

Hence $[DW/dt] = \langle DW/dt, N \times W(t) \rangle = d\varphi/dt$.

$E = 1, \ F = 0, \ G = 1$

So $G_u = 0 = E_v$.

Lemma 1 then says that

$$[DW/dt] = d\varphi/dt ,$$

which is correct.

Example 2 (Polar coordinates in the punctured plane).

Let $X(u, v) = (v \cos u , v \sin u)$.

$X_u = (- v \sin u , v \cos u)$ $\quad\quad$ $X_v = (\cos u , \sin u)$.

$E = < X_u , X_u > = v^2$, $F = 0$, $G = < X_v , X_v > = 1$

$E_v = 2v$ $\quad\quad$ $G_u = 0$

Now let $u = t$ and $v = v_0 =$ constant , so that our curve $X(u(t), v(t)) = (v_0 \cos t , v_0 \sin t)$ is a circle of radius v_0 .

Then for any vector field $W(t)$ along this curve, we have

$$[DW/dt] = \tfrac{1}{2}(EG)^{-1/2} \{G_u \, dv/dt - E_v \, du/dt\} + d\varphi/dt$$

$$= \tfrac{1}{2} (v^2)^{-1/2} \{ -2v \, du/dt \} + d\varphi/dt$$

$$= - du/dt + d\varphi/dt \; .$$

Now, to be specific, let $W(t) = (-\sin t , \cos t)$ be the unit tangent vector field along our circle of radius v_0 .

Then the angle φ between $W(t)$ and the coordinate vector X_u is always 0 . So $d\varphi/dt = 0$.

And since $u = t$, we have $du/dt = 1$.

So the above formula gives $[DW/dt] = -1$.

On the other hand, $DW/dt = dW/dt = (-\cos t, -\sin t)$.

And $N \times W(t) = (\cos t, \sin t)$.

So we get $[DW/dt] = \langle DW/dt, N \times W(t) \rangle = -1$,

agreeing with the result above.

Proof of Lemma 1.

Instead of focusing on the unit vector field $W(t)$ along our curve $\alpha(t) = X(u(t), v(t))$, let's focus first on the unit vector field $V(t) = X_u / |X_u| = X_u / \sqrt{E}$, which points along the u-coordinate curves.

By definition,

$$[DV/dt] = < DV/dt , N \times V > = = < dV/dt , N \times V >$$

$$= < dV/dt , X_v / |X_v| > = < dV/dt , X_v / \sqrt{G} > .$$

Now

$$dV/dt = (d/dt)(X_u / \sqrt{E}) = E^{-1/2} \, dX_u/dt + dE^{-1/2}/dt \, X_u .$$

Since we are going to take the inner product of this with X_v , and since our coordinate system is orthogonal, we need only pay attention to the term $E^{-1/2}\, dX_u/dt$ above. Thus

$$[DV/dt] \;=\; < dV/dt\ ,\ G^{-1/2}\, X_v >$$

$$= \;< E^{-1/2}\, dX_u/dt\ ,\ G^{-1/2}\, X_v >$$

$$= \;(EG)^{-1/2} < dX_u/dt\ ,\ X_v >$$

$$= \;(EG^{-1/2} < X_{uu}\, du/dt \;+\; X_{uv}\, dv/dt\ ,\ X_v >$$

$$= \;(EG)^{-1/2}\, \{<X_{uu}\ ,\ X_v>\, du/dt \;+\; <X_{uv}\ ,\ X_v>\, dv/dt\}.$$

Recall from Chapter 4. Intrinsic Geometry of Surfaces, Problem 7, that

$$\langle X_{uu}, X_v \rangle = F_u - \tfrac{1}{2} E_v \quad \text{and} \quad \langle X_{uv}, X_v \rangle = \tfrac{1}{2} G_u .$$

In the present case, $F = 0$, so we insert the above values and get

$$[DV/dt] = (EG)^{-1/2} \{- \tfrac{1}{2} E_v \, du/dt + \tfrac{1}{2} G_u \, dv/dt\}$$

$$= \tfrac{1}{2}(EG)^{-1/2} \{G_u \, dv/dt - E_v \, du/dt\} ,$$

which agrees with the Lemma, since in this case $\varphi \equiv 0$.

Problem 8. If $V(t)$ and $W(t)$ are unit vector fields along the curve $\alpha(t)$, and $\varphi(t)$ is the angle from $V(t)$ to $W(t)$, show that

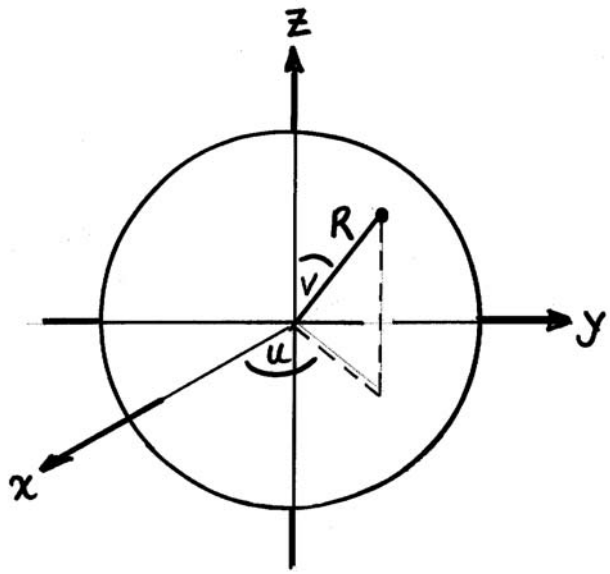
$$[DW/dt] = [DV/dt] + d\varphi/dt .$$

This problem finishes the proof of Lemma 1.

LEMMA 2. *In an orthogonal parametrization, the Gaussian curvature* K *is given by*

$$K = -\tfrac{1}{2}\,(EG)^{-1/2}\,\{\,(E_v\,(EG)^{-1/2})_v + (G_u\,(EG)^{-1/2})_u\,\}\,.$$

Example. Let's check out this formula for a round sphere of radius R .



$$X(u, v) = (R \sin v \cos u , R \sin v \sin u , R \cos v)$$

$$X_u = (-R \sin v \sin u , R \sin v \cos u , 0)$$

$$X_v = (R \cos v \cos u , R \cos v \sin u , -R \sin v)$$

$$E = \langle X_u , X_u \rangle = R^2 \sin^2 v$$

$$F = \langle X_u , X_v \rangle = 0$$

$$G = \langle X_v , X_v \rangle = R^2$$

$$EG = R^4 \sin^2 v \qquad (EG)^{1/2} = R^2 \sin v$$

$$E_v = 2 R^2 \sin v \cos v \qquad G_u = 0$$

$$K = -\tfrac{1}{2} (EG)^{-1/2} \{ (E_v (EG)^{-1/2})_v + (G_u (EG)^{-1/2})_u \}$$

$$= -\tfrac{1}{2} R^{-2}(\sin v)^{-1}\{ (2 R^2 \sin v \cos v (R^2 \sin v)^{-1})_v\}$$

$$= -\tfrac{1}{2} R^{-2}(\sin v)^{-1} \{-2 \sin v\}$$

$$= R^{-2} ,$$

as expected.

Proof of Lemma 2.

We must show that in an orthogonal parametrization, the Gaussian curvature K is given by

$$K = -\tfrac{1}{2} (EG)^{-1/2} \{ (E_v (EG)^{-1/2})_v + (G_u (EG)^{-1/2})_u \} .$$

To start, we go to Chapter 4. Intrinsic Geometry of Surfaces, formula (4) :

$$\begin{aligned} -E K = \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12,u} + \Gamma^2_{12}\Gamma^2_{12} \\ - \Gamma^1_{11}\Gamma^2_{12} - \Gamma^2_{11,v} - \Gamma^2_{11}\Gamma^2_{22} . \end{aligned}$$

This was the formula used to prove Gauss's Theorema Egregium, that the Gaussian curvature of a surface depends only on its intrinsic geometry.

In that same chapter, we saw how the Christoffel symbols can be determined from the coefficients of the first fundamental form and their first partial derivatives:

$$\tfrac{1}{2} E_u = \Gamma^1_{11} E + \Gamma^2_{11} F$$
$$F_u - \tfrac{1}{2} E_v = \Gamma^1_{11} F + \Gamma^2_{11} G$$

$$\tfrac{1}{2} E_v = \Gamma^1_{12} E + \Gamma^2_{12} F$$
$$\tfrac{1}{2} G_u = \Gamma^1_{12} F + \Gamma^2_{12} G$$

$$F_v - \tfrac{1}{2} G_u = \Gamma^1_{22} E + \Gamma^2_{22} F$$
$$\tfrac{1}{2} G_v = \Gamma^1_{22} F + \Gamma^2_{22} G$$

In the present case, we have an orthogonal parametrization of our surface, so $F \equiv 0$. So we can write simple explicit formulas for the Christoffel symbols, as follows:

$$\Gamma^1{}_{11} = \tfrac{1}{2} E_u / E \qquad \Gamma^2{}_{11} = -\tfrac{1}{2} E_v / G$$

$$\Gamma^1{}_{12} = \tfrac{1}{2} E_v / E \qquad \Gamma^2{}_{12} = \tfrac{1}{2} G_u / G$$

$$\Gamma^1{}_{22} = -\tfrac{1}{2} G_u / E \qquad \Gamma^2{}_{22} = \tfrac{1}{2} G_v / G \,.$$

If we insert these values of the Christoffel symbols into formula (4) from Chapter 4, which we copied above, we get

$$K = \tfrac{1}{4} E_v{}^2 / E^2G - \tfrac{1}{2} G_{uu} / EG + \tfrac{1}{4} G_u{}^2 / EG^2$$

$$+ \tfrac{1}{4} E_uG_u / E^2G - \tfrac{1}{2} E_{vv} / EG + \tfrac{1}{4} E_vG_v / EG^2 .$$

If we take the formula proposed in Lemma 2,

$$K = -\tfrac{1}{2} (EG)^{-1/2} \{ (E_v (EG)^{-1/2})_v + (G_u (EG)^{-1/2})_u \} ,$$

and carry out the indicated differentiations, we get the same result. This completes the proof of Lemma 2.

Proof of the local Gauss-Bonnet Theorem.

We state the theorem again for easy reference.

LOCAL GAUSS-BONNET THEOREM. *Let* U *be an open set in* R^2 *and* $X: U \to S \subset R^3$ *a regular local parametrization of* S . *Let* $R \subset X(U)$ *be a region homeomorphic to a disk. Let* $\alpha: I \to \partial R$ *be a piecewise smooth parametrization of* ∂R *by arc length, with vertices at* $\alpha(s_0)$, $\alpha(s_1)$, ..., $\alpha(s_k)$ *and with exterior angles* θ_0 , θ_1 , ..., θ_k *at these vertices. Then*

$$\int_R K \, d(\text{area}) + \int_{\partial R} \kappa_g(s) \, ds + \sum_{i=0}^{k} \theta_i = 2\pi .$$

Proof. Consider the unit vector field $W(s) = \alpha'(s)$ along the curve $\alpha: I \to \partial R$. This is well-defined along each of the smooth arcs $\alpha([s_i , s_{i+1}])$, but has two possible values at each vertex $\alpha(s_i)$, with the exterior angle θ_i representing the angle between them.

Along each of these arcs, the geodesic curvature $\kappa_g(s)$ is given by the formula from Lemma 1,

$$\kappa_g(s) = [D\alpha'/ds]$$

$$= \tfrac{1}{2}(EG)^{-1/2} \{G_u \, dv/ds - E_v \, du/ds\} + d\varphi/ds .$$

Consider now two of the three terms from the left-hand side of the local Gauss-Bonnet Theorem:

$$\int_{\partial R} \kappa_g(s)\ ds + \sum_{i=0}^{k} \theta_i$$

$$= \int_{\partial R} \tfrac{1}{2}(EG)^{-1/2} \{G_u\ dv/ds - E_v\ du/ds\}\ ds$$

$$+ \int_{\partial R} d\varphi/ds\ ds + \sum_{i=0}^{k} \theta_i\ .$$

Since $\varphi(s)$ is the angle from $X_u(s)$ to $\alpha'(s)$, the two terms

$$\int_{\partial R} d\varphi/ds\ ds + \sum_{i=0}^{k} \theta_i$$

represent the total change in angle of inclination of the unit tangent vector $\alpha'(s)$ with respect to a fixed coordinate system as we travel once counterclockwise around ∂R .
This total change is 2π .

Now consider the other integral,

$$\int_{\partial R} \tfrac{1}{2}(EG)^{-1/2} \{G_u \, dv/ds \; - \; E_v \, du/ds\} \, ds \, .$$

We will use Greens' Theorem to change this to a surface integral over the region R, so that it may be compared with the remaining integral $\int_R K \; d(\text{area})$ on the left hand side of the local Gauss-Bonnet Theorem.

Recall the statement of Green's Theorem:

$$\int_{\partial R} P \, du + Q \, dv \;=\; \int_R (\partial Q/\partial u \; - \; \partial P/\partial v) \, du \, dv \, .$$

To apply this in the present situation, let

$$P = -\tfrac{1}{2}(EG)^{-1/2} E_v \quad \text{and} \quad Q = \tfrac{1}{2}(EG)^{-1/2} G_u \ .$$

Then

$$\partial Q/\partial u - \partial P/\partial v = (\tfrac{1}{2}(EG)^{-1/2} G_u)_u + (\tfrac{1}{2}(EG)^{-1/2} E_v)_v \ ,$$

so

$$\int_{\partial R} \tfrac{1}{2}(EG)^{-1/2} \{G_u \, dv/ds - E_v \, du/ds\} \, ds$$

$$= \int_R (\tfrac{1}{2}(EG)^{-1/2} G_u)_u + (\tfrac{1}{2}(EG)^{-1/2} E_v)_v \ du \, dv \ .$$

On the other hand, by Lemma 2, we have

$$K = -\tfrac{1}{2} (EG)^{-1/2} \{ (E_v (EG)^{-1/2})_v + (G_u (EG)^{-1/2})_u \} ,$$

and therefore

$$\int_R K \ d(\text{area}) = \int_R K \ (EG)^{1/2} \ du\, dv$$

$$= \int_R -\tfrac{1}{2} \{ (E_v (EG)^{-1/2})_v + (G_u (EG)^{-1/2})_u \} \ du\, dv .$$

$$= - \int_R (\tfrac{1}{2}(EG)^{-1/2} G_u)_u + (\tfrac{1}{2}(EG)^{-1/2} E_v)_v \ du\, dv .$$

Thus

$$\int_R K \, d(\text{area}) + \int_{\partial R} \kappa_g(s) \, ds + \sum_{i=0}^{k} \theta_i$$

$$= - \int_R (\tfrac{1}{2}(EG)^{-1/2} G_u)_u + (\tfrac{1}{2}(EG)^{-1/2} E_v)_v \, du \, dv$$

$$+ \int_R (\tfrac{1}{2}(EG)^{-1/2} G_u)_u + (\tfrac{1}{2}(EG)^{-1/2} E_v)_v \, du \, dv$$

$$+ 2\pi$$

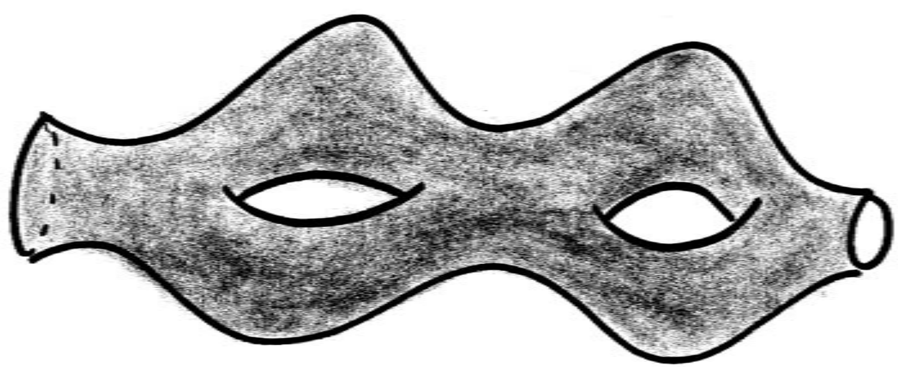
$$= 2\pi ,$$

completing the proof of the local Gauss-Bonnet Theorem.

The global Gauss-Bonnet Theorem.

GLOBAL GAUSS-BONNET THEOREM.
Let $S \subset R^3$ *be a regular surface, and* R *a compact region on* S *whose boundary* ∂R *consists of the piecewise smooth simple closed curves* $C_1, \ldots, C_n$. *Let* $\theta_1, \ldots, \theta_p$ *be the set of all exterior angles on these curves. Then*

$$\int_R K \; d(\text{area}) + \int_{\partial R} \kappa_g(s)\, ds + \sum_{i=1}^{p} \theta_i = 2\pi\, \chi(R) .$$



A surface with Euler characteristic $\chi = -4$

Comments.

(1) We first orient the region R, and then orient $\partial R = C_1 \cup \ldots \cup C_n$ accordingly.

(2) As in the case of the local Gauss-Bonnet Theorem, the term $\int_{\partial R} \kappa_g(s)\, ds$ is really a sum of integrals of geodesic curvature over the smooth arcs on ∂R which, when added to the sum $\sum_{i=1}^{p} \theta_i$ of the exterior angles, gives the total geodesic curvature of ∂R.

Proof of the global Gauss-Bonnet Theorem.

We begin by subdividing the given region R into many small triangles, each of which fits inside the image of an orthogonal local parametrization.

We intend to apply the local Gauss-Bonnet Theorem to each of these small triangles, add up the results and, with a little bit of management, get the global Gauss-Bonnet Theorem.

To fix notation, say that our triangulation of R has

$V = V_{int} + V_{bdry}$ vertices

$E = E_{int} + E_{bdry}$ edges

F faces .

We note that

$$V_{bdry} = E_{bdry} \quad \text{and} \quad 3F = 2E_{int} + E_{bdry} .$$

The local Gauss-Bonnet Theorem for each triangle T in R can be written as

$$\int_T K \ d(\text{area}) + \int_{\partial T} \kappa_g(s) \ ds + \sum \text{ext angles} = 2\pi .$$

We'll rewrite this as

$$\int_T K \ d(\text{area}) + \int_{\partial T} \kappa_g(s) \ ds + \sum (\pi - \text{int angles}) = 2\pi .$$

We intend to add up these equations, one for each triangular face in R .

The integrals of the Gaussian curvature K over the triangles add up to the integral of K over all of R .

The integrals of the geodesic curvature κ_g along the interior edges will cancel, since each of these edges appears twice, with opposite orientations, and hence with opposite signs for κ_g . So these terms will add up to the integral of κ_g along ∂R .

We will get

$$\int_R K \, d(\text{area}) + \int_{\partial R} \kappa_g(s) \, ds + 3\pi F - \sum \text{int angles} = 2\pi F \, .$$

When we add up the interior angles, we will get 2π at each interior vertex, and some unknown angle between 0 and 2π at each boundary vertex. Because two edges of a triangle can be tangent at a vertex, we can actually get interior angle sums of 0 and 2π at the boundary vertices.

At each boundary vertex v, we have

$$\sum \text{int angles at } v = \pi - \text{ext angle at } v .$$

Thus

$$\sum \text{int angles} = 2\pi V_{\text{int}} + \pi V_{\text{bdry}} - \sum \text{ext angles} .$$

Inserting this value, we so far have the equation

$$\int_R K \, d(\text{area}) + \int_{\partial R} \kappa_g(s)\, ds + 3\pi F$$

$$- (2\pi V_{int} + \pi V_{bdry} - \sum \text{ext angles}) = 2\pi F ,$$

or

$$\int_R K \, d(\text{area}) + \int_{\partial R} \kappa_g(s)\, ds + \sum \text{ext angles}$$

$$= 2\pi V_{int} + \pi V_{bdry} - 3\pi F + 2\pi F .$$

Our job is to recognize the right hand side of this equation,

$$2\pi V_{int} + \pi V_{bdry} - 3\pi F + 2\pi F ,$$

as the Euler characteristic

$$\chi(R) = V - E + F .$$

Recall that

$$V_{bdry} = E_{bdry} \quad \text{and} \quad 3F = 2E_{int} + E_{bdry} .$$

Hence

$$2\pi V_{int} + \pi V_{bdry} - 3\pi F + 2\pi F$$

$$= 2\pi V_{int} + \pi V_{bdry} - 2\pi E_{int} - \pi E_{bdry} + 2\pi F .$$

Since $V_{bdry} = E_{bdry}$, we can add πV_{bdry} to the right hand side above, and compensate by subtracting πE_{bdry}, giving

$$2\pi V_{int} + 2\pi V_{bdry} - 2\pi E_{int} - 2\pi E_{bdry} + 2\pi F$$

$$= 2\pi V - 2\pi E + 2\pi F = 2\pi \chi(R) .$$

We have shown that

$$\int_R K\, d(\text{area}) + \int_{\partial R} \kappa_g(s)\, ds + \sum \text{ext angles}$$

$$= 2\pi V_{int} + \pi V_{bdry} - 3\pi F + 2\pi F .$$

$$= 2\pi V - 2\pi E + 2\pi F$$

$$= 2\pi \chi(R) ,$$

completing the proof of the global Gauss-Bonnet Theorem.

Problem 9. Let $S \subset R^3$ be a smooth closed surface (automatically orientable) which is not homeomorphic to S^2. Show that there are points on S where the Gaussian curvature is positive, zero and negative.

Problem 10. Let S be a torus of revolution in R^3. Visualize the image of the Gauss map, and see directly (without using the Gauss-Bonnet Theorem) that

$$\int_S K \; d(\text{area}) = 0 .$$

Problem 11. Let $S \subset R^3$ be a smooth surface homeomorphic to S^2 . Suppose $\Gamma \subset S$ is a simple closed geodesic, and let A and B be the two regions on S which have Γ as boundary. Let $N: S \rightarrow S^2$ be the Gauss map. Prove that $N(A)$ and $N(B)$ have the same area on S^2 .

Problem 12. Let $S \subset R^3$ be a surface with Gaussian curvature $K \leq 0$. Show that two geodesics γ_1 and γ_2 on S which start at a point p can not meet again at a point q in such a way that together they bound a region S' on S which is homeomorphic to a disk.

Problem 13. Let $S \subset R^3$ be a surface homeomorphic to a cylinder and with Gaussian curvature $K < 0$. Show that S has at most one simple closed geodesic.

Hint. Use the result of the preceding problem.

Problem 14. Let $S \subset R^3$ be a smooth closed surface of positive curvature, and thus homeomorphic to S^2. Show that if Γ_1 and Γ_2 are two simple closed geodesics on S, then they must intersect one another.