5 Adjunctions and the Compact-Open Topology

Birds fly high in the air and survey broad vistas of mathematics out to the far horizon. They delight in concepts that unify our thinking and bring together diverse problems from different parts of the landscape. Frogs live in the mud below and see only the flowers that grow nearby. They delight in the details of particular objects, and they solve problems one at a time.
—Freeman Dyson (2009)

Introduction. Early in chapter 0 it was noted that category theory is an appropriate setting in which to discuss the concept of "sameness" between mathematical objects. This concept is captured by an isomorphism: a morphism from one object to another that is both left and right invertible. The discussion becomes especially interesting when those objects are categories. Two categories are isomorphic if there exists a pair of functors—one in each direction—whose compositions equal the identities. But equality is a lot of ask for! Isomorphisms of categories are too strict to be of much use. Relaxing the situation yields something better: categories C and D are *equivalent* if there exists a pair of functors $L\colon \mathsf{C} \rightleftarrows \mathsf{D}\colon R$ and natural isomorphisms $\mathrm{id}_{\mathsf{C}} \to RL$ and $LR \to \mathrm{id}_{\mathsf{D}}$.

Relaxing this a step further yields another gem of category theory: adjunctions. A pair of functors $L\colon \mathsf{C} \rightleftarrows \mathsf{D}\colon R$ forms an *adjunction*, and L and R are called *adjoint functors*, if there are natural transformations (not necessarily isomorphisms) $\eta\colon \mathrm{id}_{\mathsf{C}} \to RL$ and $\epsilon\colon LR \to \mathrm{id}_{\mathsf{D}}$ that, in addition, interact compatibly in a sense that can be made precise. Here the categories may not be equivalent, but don't think that adjunctions are mere second (or third) best. Quite often, relaxing a notion of equivalence results in a trove of rich mathematics. That is indeed the case here.

In this chapter, then, we introduce adjoint functors and use them to highlight several constructions in topology. We'll present the formal definition in section 5.1 and give some examples—free constructions in algebra, a forgetful functor from Top, and the Stone-Čech compactification—in sections 5.2, 5.3, and 5.5, respectively. Then we'll use a particularly nice adjunction—the product-hom adjunction—as motivation for putting an appropriate topology on function spaces. In section 5.6, we'll take an in-depth look at this topology, called the compact-open topology. Quite a few pages are devoted to this endeavor and some of the difficulties involved. Finally, section 5.7 closes with a discussion on the category of compactly generated weakly Hausdorff spaces—a "convenient" category of topological spaces. So in the pages to come, we'll be both birds and frogs. A categorical point of view oftentimes highlights and elevates the important properties that characterize an object or construction but fails to establish that such objects exist. Existence can require getting down in the mud.