WHAT THE FUNCTOR?: CATEGORY THEORY AND THE CONCEPT OF ADJOINTNESS

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ABSTRACT. Category theory provides a more abstract and thus more general setting for considering the structure of mathematical objects. In this paper we will define basic concepts related to category theory and discuss examples, such as groups and sets as categories and forgetful and free functors, following Eugenia Cheng's notes on category theory [1]. Our ultimate goal will be to examine the concept of adjointness through the example of free and forgetful functors.

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1. SOME DEFINITIONS AND EXAMPLES

First we define some basic concepts related to categories and see some helpful examples.

Definition 1.1. A *category* $\mathcal{C}$ consists of:

- A collection of objects, $\mathrm{ob}\,\mathcal{C}$
- For every pair of objects $X, Y \in \mathrm{ob}\,\mathcal{C}$, a collection $\mathcal{C}(X, Y)$ of morphisms (also called maps) $f : X \to Y$.

These morphisms are equipped with:

- For each object $X \in \mathrm{ob}\,\mathcal{C}$ an identity map $1_X \in \mathrm{ob}\,\mathcal{C}$
- For each $X, Y, Z \in \mathrm{ob}\,\mathcal{C}$ a composition map $m_{XYZ} : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$ that sends (g, f) to $g \circ f = gf$

Composition of morphisms satisfies the following:

- Unit laws: if $f : X \to Y$ then $1_Y \circ f = f = f \circ 1_X$.
- Associativity: if $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$, then $h(gf) = (hg)f$.

Examples 1.2. Categories:

(1) The category **Set**, where objects are sets and morphisms are functions.
(2) The category **Grp**, where objects are groups and morphisms are group homomorphisms.