Families of Commuting Normal Matrices

Definition M.1 (Notation)

i) $\mathbb{C}^n \;=\; \big\{\ \mathbf{v} = (v_1, \cdots, v_n)\ \big|\ v_i \in \mathbb{C} \text{ for all } 1 \le i \le n\ \big\}$

ii) If $\lambda \in \mathbb{C}$ and $\mathbf{v} = (v_1, \cdots, v_n),\ \mathbf{w} = (w_1, \cdots, w_n) \in \mathbb{C}^n$, then

$$\lambda\mathbf{v} = (\lambda v_1,\ \cdots,\ \lambda v_n) \in \mathbb{C}^n$$
$$\mathbf{v} + \mathbf{w} = (v_1 + w_1,\ \cdots,\ v_n + w_n) \in \mathbb{C}^n$$
$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{j=1}^{n} \bar{v}_j w_j \in \mathbb{C}$$

The $^-$ means complex conjugate.

iii) Two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$ are said to be orthogonal (or perpendicular, denoted $\mathbf{v} \perp \mathbf{w}$) if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.

iv) If $\mathbf{v} \in \mathbb{C}^n$ and A is the $m \times n$ matrix whose (i,j) matrix element is $A_{i,j}$, then $A\mathbf{v}$ is the vector in $\mathbb{C}^m$ with

$$(A\mathbf{v})_i = \sum_{j=1}^{n} A_{i,j} v_j \qquad \text{for all } 1 \le i \le m$$

v) A linear subspace V of $\mathbb{C}^n$ is a subset of $\mathbb{C}^n$ that is closed under addition and scalar multiplication. That is, if $\lambda \in \mathbb{C}$ and $\mathbf{v}, \mathbf{w} \in V$, then $\lambda\mathbf{v},\ \mathbf{v} + \mathbf{w} \in V$.

vi) If V is a subset of $\mathbb{C}^n$, then its orthogonal complement is

$$V^\perp = \big\{\ \mathbf{v} \in \mathbb{C}^n\ \big|\ \mathbf{v} \perp \mathbf{w} \text{ for all } \mathbf{w} \in V\ \big\}$$

Problem M.1 Let $V \subset \mathbb{C}^n$. Prove that $V^\perp$ is a linear subspace of $\mathbb{C}^n$.

Lemma M.2 *Let V be a linear subspace of $\mathbb{C}^n$ of dimension at least one. Let A be an $n \times n$ matrix that maps V into V. Then A has an eigenvector in V.*

Proof: Let $\mathbf{e}_1,\ \cdots,\ \mathbf{e}_d$ be a basis for V. As A maps V into itself, there exist numbers $a_{i,j},\ 1 \le i, j \le d$ such that

$$A\mathbf{e}_j = \sum_{i=1}^{d} a_{i,j} \mathbf{e}_i \qquad \text{for all } 1 \le j \le d$$