Math S21a: Multivariable calculus **Oliver Knill, Summer 2011**

Lecture 12: Gradient

The **gradient** of a function $f(x,y)$ is defined as

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle \, .$$

For functions of three dimensions, we define

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle \, .$$

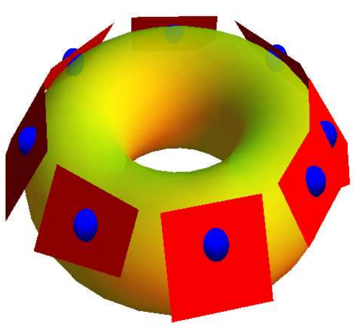
The symbol ∇ is spelled "Nabla" and named after an Egyptian harp. Here is a very important fact:

Gradients are orthogonal to level curves and level surfaces.

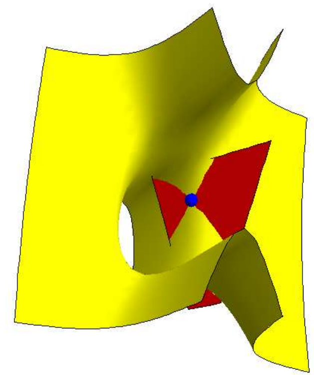
Proof. Every curve $\vec{r}(t)$ on the level curve or level surface satisfies $\frac{d}{dt} f(\vec{r}(t)) = 0$. By the chain rule, $\nabla f(\vec{r}(t))$ is perpendicular to the tangent vector $\vec{r}\,'(t)$.
Because $\vec{n} = \nabla f(p,q) = \langle a,b \rangle$ is perpendicular to the level curve $f(x,y) = c$ through (p,q), the equation for the tangent line is $ax + by = d$, $a = f_x(p,q)$, $b = f_y(p,q)$, $d = ap + bq$. Compactly written, this is

$$\nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = 0$$

and means that the gradient of f is perpendicular to any vector $(\vec{x} - \vec{x}_0)$ in the plane. It is one of the most important statements in multivariable calculus. since it provides a crucial link between calculus and geometry. The just mentioned gradient theorem is also useful. We can immediately compute tangent planes and tangent lines:



1 Compute the tangent plane to the surface $3x^2y + z^2 - 4 = 0$ at the point $(1,1,1)$. **Solution:** $\nabla f(x,y,z) = \langle 6xy, 3x^2, 2z \rangle$. And $\nabla f(1,1,1) = \langle 6,3,2 \rangle$. The plane is $6x + 3y + 2z = d$ where d is a constant. We can find the constant d by plugging in a point and get $6x + 3y + 2z = 11$.



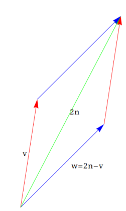
2 **Problem:** reflect the ray $\vec{r}(t) = \langle 1-t, -t, 1 \rangle$ at the surface

$$x^4 + y^2 + z^6 = 6 \, .$$

Solution: $\vec{r}(t)$ hits the surface at the time $t = 2$ in the point $(-1,-2,1)$. The velocity vector in that ray is $\vec{v} = \langle -1,-1,0 \rangle$ The normal vector at this point is $\nabla f(-1,-2,1) = \langle -4,4,6 \rangle = \vec{n}$. The reflected vector is

$$R(\vec{v} = 2\mathrm{Proj}_{\vec{n}}(\vec{v}) - \vec{v} \, .$$

We have $\mathrm{Proj}_{\vec{n}}(\vec{v}) = 8/68 \langle -4,-4,6 \rangle$. Therefore, the reflected ray is $\vec{w} = (4/17)\langle -4,-4,6 \rangle - \langle -1,-1,0 \rangle$.



If f is a function of several variables and $\vec{v}$ is a unit vector then $D_{\vec{v}} f = \nabla f \cdot \vec{v}$ is called the **directional derivative** of f in the direction $\vec{v}$.

The name directional derivative is related to the fact that every unit vector gives a direction. If $\vec{v}$ is a unit vector, then the chain rule tells us $\frac{d}{dt} D_{\vec{v}} f = \frac{d}{dt} f(x + t\vec{v})$.