Chapter 16

Metric tensor

• A **metric** on a vector space V is a function $g : V \times V \to \mathbb{R}$ which is

$i)$ bilinear:

$$\begin{aligned} g(av_1 + v_2, w) &= ag(v_1, w) + g(v_2, w) \\ g(v, w_1 + aw_2) &= g(v, w_1) + ag(v, w_2)\,, \end{aligned} \tag{16.1}$$

i.e., g is a (0,2) tensor;

$ii)$ symmetric:

$$g(v, w) = g(w, v); \tag{16.2}$$

$iii)$ non-degenerate:

$$g(v, w) = 0 \qquad \forall w \qquad \Rightarrow v = 0\,. \tag{16.3}$$

□

• If for some $v, w \neq 0$, we find that $g(v, w) = 0$, we say that v, w are **orthogonal**. □

• Given a metric g on V, we can always find an **orthonormal basis** $\{e_\mu\}$ such that $g(e_\mu, e_\nu) = 0$ if $\mu \neq \nu$ and ± 1 if $\mu = \nu$. □

• If the number of $(+1)$'s is p and the number of (-1)'s is q, we say that the metric has **signature** (p, q).

We have defined a metric for a vector space. We can generalize this definition to a manifold $\mathcal{M}$ by the following.

• A **metric** g on a manifold $\mathcal{M}$ is a (0, 2) tensor field such that if (v, w) are smooth vector fields, $g(v, w)$ is a smooth function on $\mathcal{M}$, and has the properties (16.1), (16.2) and (16.3) mentioned earlier. □