7 Gaussian Elimination and LU Factorization

In this final section on matrix factorization methods for solving $A\boldsymbol{x} = \boldsymbol{b}$ we want to take a closer look at Gaussian elimination (probably the best known method for solving systems of linear equations).

The basic idea is to use left-multiplication of $A \in \mathbb{C}^{m\times m}$ by (elementary) lower triangular matrices, $L_1, L_2, \ldots, L_{m-1}$ to convert A to upper triangular form, i.e.,

$$\underbrace{L_{m-1}L_{m-2}\ldots L_2L_1}_{=\widetilde{L}} A = U.$$

Note that the product of lower triangular matrices is a lower triangular matrix, and the inverse of a lower triangular matrix is also lower triangular. Therefore,

$$\widetilde{L}A = U \quad \Longleftrightarrow \quad A = LU,$$

where $L = \widetilde{L}^{-1}$. This approach can be viewed as *triangular triangularization*.

7.1 Why Would We Want to Do This?

Consider the system $A\boldsymbol{x} = \boldsymbol{b}$ with LU factorization $A = LU$. Then we have

$$L\underbrace{U\boldsymbol{x}}_{=\boldsymbol{y}} = \boldsymbol{b}.$$

Therefore we can perform (a now familiar) 2-step solution procedure:

1. Solve the lower triangular system $L\boldsymbol{y} = \boldsymbol{b}$ for $\boldsymbol{y}$ by forward substitution.

2. Solve the upper triangular system $U\boldsymbol{x} = \boldsymbol{y}$ for $\boldsymbol{x}$ by back substitution.

Moreover, consider the problem $AX = B$ (i.e., many different right-hand sides that are associated with the same system matrix). In this case we need to compute the factorization $A = LU$ only once, and then

$$AX = B \quad \Longleftrightarrow \quad LUX = B,$$

and we proceed as before:

1. Solve $LY = B$ by many forward substitutions (in parallel).

2. Solve $UX = Y$ by many back substitutions (in parallel).

In order to appreciate the usefulness of this approach note that the operations count for the matrix factorization is $\mathcal{O}(\frac{2}{3}m^3)$, while that for forward and back substitution is $O(m^2)$.

Example Take the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix}$$