Chapter 4

Differentiation of vectors

4.1 Vector-valued functions

In the previous chapters we have considered real functions of several (usually two) variables $f\colon D \to \mathbb{R}$, where D is a subset of $\mathbb{R}^n$, where n is the number of variables. These are *scalar-valued* functions in the sense that the result of applying such a function is a real number, which is a scalar quantity. We now wish to consider *vector-valued* functions $\mathbf{f}\colon D \to \mathbb{R}^m$. In principal, m can be any positive integer, but we will only consider the cases where $m = 2$ or 3, and the results of applying the function is either a 2D or 3D vector.

4.2 Parametric equations of curves

The simplest type of vector-valued function has the form $\mathbf{f}\colon I \to \mathbb{R}^2$, where $I \subset \mathbb{R}$. Such a function returns a 2D vector $\mathbf{f}(t)$ for each $t \in I$, which may be regarded as the position vector of some point on the plane.

For example, recall the Section Formula from Level 1. This states that the position vector of any point P on the line through points A and B is

$$\mathbf{p} = \frac{\alpha\mathbf{a} + \beta\mathbf{b}}{\alpha + \beta},$$

for any scalars α, β. If we define $t = \beta/(\alpha + \beta)$, then this may be rewritten as

$$\mathbf{p}(t) = (1 - t)\mathbf{a} + t\mathbf{b}.$$

As t changes, we get different points on the line through A and B and in particular, $\mathbf{p}(0) = \mathbf{a}$ and $\mathbf{p}(1) = \mathbf{b}$. We may think of $\mathbf{p}$ as a vector-valued function

$$\mathbf{p}\colon \mathbb{R} \to \mathbb{R}^3,$$

the image of which is the whole of the line, or

$$\mathbf{p}\colon [0, 1] \to \mathbb{R}^3,$$

the image of which is the line segment from A to B.

In general, a curve, in 2D or 3D space, can be represented as the image of a vector-valued function on an interval I; the position vector of a point on the curve is

$$\mathbf{r} = \mathbf{f}(t), \quad t \in I.$$

This is called a *parametric* description of the curve and t is called a *parameter*. This may also be written in component form; if $\mathbf{r} = (x, y, z)$ and $\mathbf{f} = (f_1, f_2, f_3)$ then

$$x = f_1(t),\ y = f_2(t),\ z = f_3(t), \quad t \in I.$$