Hopf Fibration

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Abstract

The Hopf Fibration is an amazing geometric map from 3-sphere to the 2-sphere and arises often in mathematics and physics. By considering rotations of the 2-sphere in $\mathbb{R}^3$ we obtain a concrete geometric realisation for the Hopf Fibration. This paper introduces some of the basic tools needed to generate the fibration as well as its geometric properties.

1 Introduction

The Hopf Fibration can be first analysed by looking at some simple definitions.

Definition 1.1 (Standard Unit n-sphere) The *standard unit n-sphere* S^n is the set of points $(x_0, x_1, \dots, x_n)$ in $\mathbb{R}^{n+1}$ that satisfy the condition,

$$\sum_{i=0}^{n} x_i^2 = x_0^2 + x_1^2 + \ldots + x_n^2 = 1$$

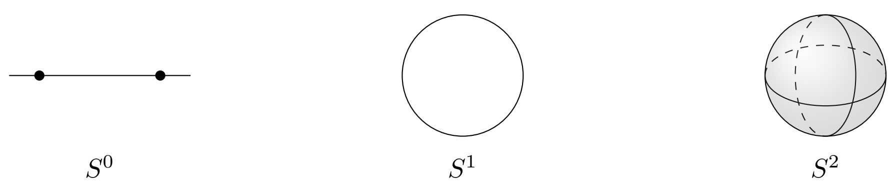


Figure 1: Examples of Spheres.

It's not uncommon to think of a sphere as a 3-dimensional object. Mathematically speaking however, spheres form the set of points with distance 1 from the origin. So S^0 is 2 points, S^1 is a unit circle on the plane and S^2 is what is typically considered a 'sphere'. [see Figure 1]

The term *fibration* refers to the decomposition of what is called the *Total Space* E, into two spaces, a *Base Space* B and a *Fiber* F. Formally we may define,

Definition 1.2 (Fibration) A *fibration* is a continuous surjection $\pi\colon E \to B$, where for points $b \in B$ the fibers are the preimages $\pi^{-1}(b)$. Furthermore it is required that for $b \in B$ there exists an open neighbourhood $U \subset B$ containing b, such that there is a *homeomorphism* $\varphi\colon \pi^{-1}(U) \to U \times F$. In this way the following diagram should commute,

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\text{proj}"] \\ U & \end{tikzcd}$$

Figure 2: The *homeomorphism* $\varphi\colon \pi^{-1}(U) \to U \times F$ should agree with the projection denoted as 'proj'.

The term *homeomorphism* vaguely means that a given space can be mapped to another without any 'gluing' or 'tearing'.

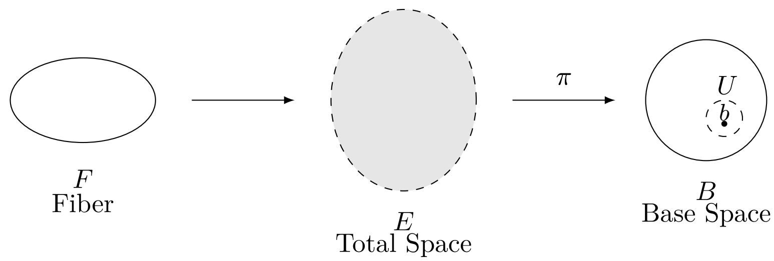


Figure 3: Diagrammatic representation of a fibration.

A fibration is typically denoted as follows:

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$

The horizontal and vertical arrows represent injective and surjective maps respectively. To familiarise the reader with this concept examples are given below.

Example 1.1. Consider a Total Space E formed by taking cross product of $[0,1] \times S^1$. This forms a cylinder. It is possible to project onto a base of S^1 with fiber is the interval $[0,1]$. The preimage $\pi^{-1}(U)$ is a section of our original cylinder.

Example 1.2. A Möbius Strip is a surface with only one boundary. Let this be the *Total Space* E. Along the center of the strip there is a circle. Consider a map which takes a line segment on Möbius Strip to a point on this circle. The resulting fibration has this circle as a *Base* B and line segment as *fiber* F. The preimage $\pi^{-1}(U)$ is a 'twisted' segment.

2 Quaternions

Quaternions are a number system that extends the Complex Numbers. Thought of as a vector space, like $\mathbb{R}^4$ the quaternions are spanned by the basis $\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,1,0,0)\}$. These basis vectors are then denoted $1, i, j$ and k respectively. Typically we write a quaternion as a linear combination of these basis vectors, i.e in the form

$$a + bi + cj + dk \qquad (a,b,c,d \in \mathbb{R})$$

We call a the *real* part of a quaternion. Quaternions like Real and Complex Numbers can be multiplied. The basis elements are subject to the following rules of multiplication.

$$i^2 = j^2 = k^2 = ijk = -1,$$
$$ij = k, \qquad jk = i, \qquad ki = j.$$

An important aspect of quaternion multiplication is that multiplication of the basis elements is *anti-commutative* - That is to say reversing the order the product changes the sign,

$$ji = -k, \qquad kj = -i \qquad ik = -j$$

The set of all quaternions is often denoted $\mathbb{H}$.

Example 2.1. Quaternion Arithmetic follows obvious rules. As an illustration consider the following example,

$$\begin{aligned}(1+4j)(2+i+4k) &= 2+i+4k+8j+4ji+16jk && \text{(Distribution)}\\ &= 2+i+4k+8j-4k+16i && \text{(Quaternion Multiplicative Rules)}\\ &= 2+17i+8j && \text{(Collecting Terms)}\end{aligned}$$

Definition 2.1. (Quaternion Conjugate) The *conjugate* of a quaternion r is denoted $\overline{r}$ and given by,

$$\overline{r} = a - bi - cj - dk$$

This is analagous to the *Complex Conjugate*.

Definition 2.2. (Quaternion Norm) The *norm* or *length* denoted $\|r\|$ and is given as,

$$\|r\| = \sqrt{r\overline{r}} = \sqrt{a^2+b^2+c^2+d^2}$$

Definition 2.3. (Pure Quaternion) A quaternion is call *pure* if it is of the form,

$$xi + yj + zk \qquad (x, y, z \in \mathbb{R})$$

Property 2.1. Quaternion multiplication is associative. That is, for quaternions p, q and r

$$(pq)r = p(qr)$$

Property 2.2. Let $r, s \in \mathbb{H}$. Then,

$$\overline{rs} = \overline{s}\,\overline{r}$$

Proposition 2.1. Given a non-zero quatenrion r its corresponding multiplicative inverse is of the form,

$$r^{-1} = \frac{\overline{r}}{\|r\|^2}$$

Proof. Let $r \in \mathbb{H}$ be non-zero and let r^{-1} be its corresponding multiplicative inverse. Then,

$$rr^{-1} = r\frac{\overline{r}}{\|r\|^2} = \frac{r\overline{r}}{\|r\|^2} = \frac{\|r\|^2}{\|r\|^2} = 1$$

□

It follows from this that $r^{-1} = \overline{r}$ when $\|r\| = 1$.

Proposition 2.2. The norm of a quaternion product is equal to product of norms. That is, for any $r, s \in \mathbb{H}$, $\|rs\| = \|r\|\,\|s\|$

Proof. Let r, s be quaternions. Then,

$$\|rs\| = \sqrt{rs\overline{rs}} \qquad \text{(Definition 2.2.)}$$

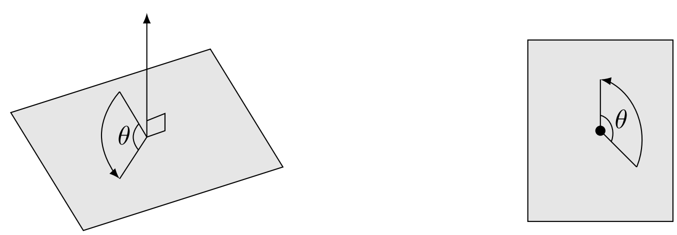
$$
\begin{aligned}
&= \sqrt{rs\bar{s}\,\bar{r}} && \text{(Property 2.2.)} \\
&= \sqrt{r(s\bar{s})\bar{r}} && \text{(Property 2.1.)} \\
&= \sqrt{r\,\|s\|^2\,\bar{r}} && \text{(Definition 2.2.)} \\
&= \|s\|\,\sqrt{r\bar{r}} \\
&= \|s\|\,\|r\| \\
&= \|r\|\,\|s\|
\end{aligned}
$$

□

This result means that product of two unit length quaternions will be another unit length quaternion.

3 Rotations in 3-space

In order to describe a rotation about the origin in $\mathbb{R}^3$ we need only an angle θ and a vector $\mathbf{v}$ specifying the axis of rotation (see Figure 7(i)). We define a rotation to be positive when in the counterclockwise direction as viewed from the tip of $\mathbf{v}$.(see Figure 7(ii))



(i) Side View (ii) Birds-Eye View

Figure 4: A rotation on 3-space can be described using a vector and an angle.

4 Quaternions and Rotations.

Quaternions can be used to determine a rotation in $\mathbb{R}^3$. This is done as follows. Let $r = a+bi+cj+dk$ be unit a quaternion and let $q = xi + yj + zk$ be associated with a point $(x, y, z) \in \mathbb{R}^3$. A standard computation will tell us that the quaternion product rpr^{-1} is also pure. A rotation in $\mathbb{R}^3$ is then given in the mapping,

$$R_r(q)\colon \mathbb{R}^3 \to \mathbb{R}^3, \qquad R_r(q) = rqr^{-1}$$

Allow us to analyse this map.

Proposition 4.1. The map R_r is norm preserving.

Proof. Let R_r be the map defined previously. It follows that,

$$\|R_r(q)\| = \left\|rqr^{-1}\right\|$$

$$
\begin{aligned}
&= \|r\| \, \|q\| \, \|r^{-1}\| && \text{(Proposition 2.2.)} \\
&= \|r\| \, \|r^{-1}\| \, \|q\| \\
&= \|rr^{-1}\| \, \|q\| && \text{(Proposition 2.2.)} \\
&= \|q\|
\end{aligned}
$$

□

Proposition 4.2. R_r is determines a rotation about the axis (b, c, d).

Proof. Let p and q be pure quaternions and take $a \in \mathbb{R}$. Then it can be seen that R_r is linear map since $R_r(ap+q) = aR_r(p)+R_r(q)$. Owing to this we can generate the matrix representation of the map R_r.

$$
\mathcal{M}(R_r) = \begin{pmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2bd+2ac \\ 2bc+2ad & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 2bd-2ac & 2cd+2ab & a^2-b^2-c^2+d^2 \end{pmatrix}
$$

A direct calculation will show that it has eigenvalue 1 and eigenvector (b, c, d). This directly implies that is a map rotation about the vector (b, c, d) □

Proposition 4.3. The map R_r determines a rotation of $\theta = 2\arccos(a) = 2\arcsin(\sqrt{b^2+c^2+d^2})$.

Proof. Given a 3×3 rotation matrix $\mathcal{A}$ it possible to extract the angle of rotation,

$$
Tr(\mathcal{A}) = 1 + 2\cos(\theta)
$$

where Tr denotes the trace of $\mathcal{A}$. Now $Tr(\mathcal{M}(R_r))$ can be calculated to be $3a^2-b^2-c^2-d^2$. So then we have,

$$
\begin{aligned}
3a^2-b^2-c^2-d^2 = 1+2\cos(\theta) &\implies 3a^2-b^2-c^2-d^2-1 = 2\cos(\theta) \\
&\implies 2a^2-2b^2-2c^2-2d^2 = 2\cos(\theta) \quad (a^2+b^2+c^2+d^2=1) \\
&\implies a^2-b^2-c^2-d^2 = \cos(\theta) \\
&\implies 2a^2-1 = \cos(\theta) \\
&\implies 2\arccos(a) = \theta
\end{aligned}
$$

The fact that $2\arccos(a) = 2\arcsin(\sqrt{b^2+c^2+d^2})$ follows from looking at a right angled triangle with hypotenuse length 1 and side lengths a and $\sqrt{b^2+c^2+d^2}$ □

5 Hopf Fibration

It is now possible to formulate the Hopf Map. The 3-sphere can be identified with the set of unit quaternions. Then fix the point $(1, 0, 0) \in S^2$ and identify it with the quaternion i. Then we can define,

Definition 5.1. (Hopf Fibration) Let $r = a+bi+cj+dk$ be a quaternion with unit norm. The *Hopf Fibration* or *Hopf's Map* is,

$$
r \mapsto R_r(i) = rir^{-1}
$$

Equivalently, expanding and simplifying a coordinate-wise map,

$$
h \colon S^3 \to S^2, \qquad h(a,b,c,d) = (a^2+b^2-c^2-d^2, 2(ad-bc), 2(bd-ac))
$$

where $a, b, c, d \in \mathbb{R}$. Here the base space is a 2-sphere and as we will see, the fibers are circles.

6 Rotational Geometry of the Hopf Fibration

As talked about in Section 4, the quaternion r induces a rotation in 3-space via map R_r. The Hopf Fibration takes this quaternion r then maps it to the image of $(1, 0, 0)$ under the rotation induced by R_r. [see Figure 5.]

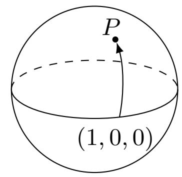


Figure 5: $(1, 0, 0) \in S^2$ is moved to another point $P \in S^2$ via R_r

This gives a concrete geometric realisation for the Hopf Fibration by considering rotations of the 2-sphere in $\mathbb{R}^3$. Given two points P and Q which are not antipodal we can consider two sets of rotations which take P to Q. To describe these sets first join P and Q with the arc of a great circle $\overline{PQ}$ and consider the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. The first set of rotations is obtained by rotating along the great circle adjoining P and Q. This is is a rotation about the vector $\overrightarrow{OP} \times \overrightarrow{OQ}$. Denote this rotation as R_1. The next set is a rotations is a rotation of π about the vector $\overrightarrow{OP} + \overrightarrow{OQ}$ which passes through the midpoint of the arc $\overline{PQ}$. Denote this rotation R_2

It is possible to derive explicit quaternions r_1 and r_2 which induce the two kinds of rotations, R_1 and R_2 and in particular induce the rotations R_{r_1} and R_{r_2} so that we have $R_1 = R_{r_1}$ and $R_2 = R_{r_2}$.

Proposition 6.1. Let $(p_1, p_2, p_3) \in S^2$. The quaternion r_1 is given by,

$$r_1 = \frac{1}{\sqrt{2(1+p_1)}}((1+p_1) - p_3 j + p_2 k)$$

Proof. Let $r_1 = a + bi + ck + dk$ be the unit quaternion which induces the rotation $R_1 = R_{r_1}$ which takes (1,0,0) to $(p_1, p_2, p_3) \in S^2$. Firstly the angle of rotation here can be found via,

$$\cos(\theta) = (1, 0, 0) \cdot (p_1, p_2, p_3) = p_1$$

Substitution of $\theta = 2\arccos(a)$ into LHS and gives us

$$\cos(\theta) = \cos(2\arccos(\theta)) = 2\cos^2(\arccos(a)) - 1 = 2a^2 - 1.$$

This means that $p_1 = 2a^2 - 1 \implies a = \sqrt{\frac{1+p_1}{2}}$. Then the axis of rotation is given by,

$$(1, 0, 0) \times (p_1, p_2, p_3) = \det \begin{pmatrix} i & j & k \\ 1 & 0 & 0 \\ p_1 & p_2 & p_3 \end{pmatrix} = (0, -p_3, p_2)$$

It is known from earlier that the eigenvector (b, c, d) determines the axis of rotation. In general however it can be shown that for any $n \in \mathbb{R}\backslash\{0\}$, $n(b, c, d)$ also forms an eigenvector which determines the same axis of rotation. Equating give us,

$$n(b, c, d) = (0, -p_3, p_2) \implies (b, c, d) = (0, -mp_3, mp_2),$$

where $m = \frac{1}{n}$. By hypothesis we have $a^2 + b^2 + c^2 + d^2 = 1$ From this we have,

$$1 = a^2 + b^2 + c^2 + d^2 \implies 1 = \frac{1 + p_1}{2} + m^2(p_2^2 + p_3^2)$$
$$\implies \frac{1 - p_1}{2} = m^2(p_2^2 + p_3^2).$$

Then since $(p_1, p_2, p_3) \in S^2$ we have $p_2^2 + p_3^2 = 1 - p_1^2 = (1 - p_1)(1 + p_1)$. It follows,

$$\frac{1 - p_1}{2} = m^2(1 - p_1)(1 + p_1) \implies \frac{1}{2(1 + p_1)} = m^2$$
$$\implies \frac{1}{\sqrt{2(1 + p_1)}} = m.$$

This gives us $c = \frac{-p_3}{\sqrt{2(1+p_1)}}$ and $d = \frac{p_2}{\sqrt{2(1+p_1)}}$ and thus the quaternion,

$$r_1 = \sqrt{\frac{1 + p_1}{2}} + \frac{-p_3}{\sqrt{2(1 + p_1)}}j + \frac{p_2}{\sqrt{2(1 + p_1)}}k = \frac{1}{\sqrt{2(1 + p_1)}}((1 + p_1) - p_3 j + p_2 k).$$

□

Proposition 6.2. For $(p_1, p_2, p_3) \in S^2$. The quaternion r_2 has the form,

$$r_2 = \frac{1}{\sqrt{2(1 + p_1)}}((1 + p_1)i + p_2 j + p_3 k).$$

Proof. Let $r_2 = a + bi + cj + dk$ be the unit quaternion which induces the rotation$R_2 = R_{r_1}$ which takes $(1, 0, 0)$ to (p_1, p_2, p_3). The angle of rotation about this axis is π. Immediately we have $a = \cos(\frac{\pi}{2}) = 0$. Then the axis of rotation is given by,

$$(1, 0, 0) + (p_1, p_2, p_3) = (1 + p_1, p_2, p_3)$$

Equating to our generalised eigenvector gives us

$$(b, c, d) = m(1 + p_1, p_2, p_3)$$

By hypothesis $b^2 + c^2 + d^2 = 1$. This gives us,

$$b^2 + c^2 + d^2 = 1 \implies m^2\big((1 + p_1)^2 + p_2^2 + p_3^2\big) = 1$$
$$\implies m^2(1 + 2p_1 + p_1^2 + 1 - p_1^2) = 1$$
$$\implies m^2(2 + 2p_1) = 1$$
$$\implies m^2 = \frac{1}{2 + 2p_1}$$
$$\implies m = \frac{1}{\sqrt{2(1 + p_1)}}$$

This gives us,

$$r_2 = \frac{1}{\sqrt{2(1 + p_1)}}((1 + p_1)i + p_2 j + p_3 k).$$

□

7 Fibers of the Hopf Fibration

Let $e^{it} = \cos(t) + i\sin(t)$. In $\mathbb{R}^4$ this is a rotation on the xy-plane. As t varies this sweeps out a circle. The fiber $h^{-1}(P)$ for a point $P = (p_1, p_2, p_3) \in S^2$ can be given parametrically as,

$$h^{-1}(P) = \left\{r_1 e^{it}\right\}_{0\le t\le 2\pi},$$
$$h^{-1}(P) = \left\{r_2 e^{it}\right\}_{0\le t\le 2\pi}.$$

Respectively expanding these gives,

$$h^{-1}(P) = \frac{1}{\sqrt{2(1+p_1)}}\Big((1+p_1)\cos(t), (1+p_1)\sin(t), p_2\sin(t) - p_3\cos(t), p_3\sin(t) + p_2\cos(t)\Big),$$
$$h^{-1}(P) = \frac{1}{\sqrt{2(1+p_1)}}\Big(-(1+p_1)\sin(t), (1+p_1)\cos(t), p_2\cos(t) + p_3\sin(t), p_2\cos(t) - p_3\sin(t)\Big).$$

If $P = (-1, 0, 0)$ then the preimage is,

$$h^{-1}((-1,0,0)) = \{ke^{it}\}_{0\le t\le 2\pi}.$$

Expanded, this is,

$$h^{-1}((-1,0,0)) = (0, 0, \sin(t), \cos(t)).$$

From this we can see one of the more interesting geometric properties of the Hopf Fibration. The fibers of the Hopf Fibration are *great circles* on S^3 since they have unit norm.

8 Stereographic Projection

Stereographic Projection is a powerful tool that allows us to see the fibers of the Hopf Fibration. We begin using the example of Stereographic Projection of S^2 onto $\mathbb{R}^2$.

Example 8.1. Imagine S^2 sitting on top of $\mathbb{R}^2$. Stereographic Projection of S^2 to $\mathbb{R}^2$ can be described as follows: consider a ray of light beginning at $N = (1, 0, 0)$ that passes through some point P on S^2 and carries this point to a point $P' \in \mathbb{R}^2$. (see Figure 6.)

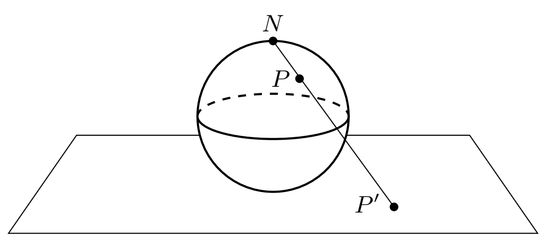


Figure 6: Stereographic Projection

From this it is important to notice that N has no sensible projection. Thus is the reason we restrict this mappng to $S^2\backslash\{N\} \to \mathbb{R}^2$. We can give an formula for stereographic projection as follows. Let $(x, y, z) \in S^2$ then,

$$(x, y, z) \mapsto \left(\frac{y}{1-z}, \frac{z}{1-z}\right).$$

Proof. Let S^2 sit on top of $\mathbb{R}^2$, then let $(x, y, z) \in S^2$ and the north pole $N = (1, 0, 0)$. The projection which takes (x, y, z) to $\mathbb{R}^2$ is parametrised by $r(t) = (1-t, 0, 0) + t(x, y, z)$ for $t \in [0, 1]$. When the last coordinate is equal 0, i.e when on the plane we find

$$\begin{aligned} 1 - t + tx = 0 &\iff 1 = t - tx \\ &\iff 1 = t(1-x) \\ &\iff \frac{1}{1-x} = t \end{aligned}$$

Substituting in this value of $t = \frac{1}{1-x}$ into $r(t)$ yields,

$$\begin{aligned} r\left(\frac{1}{1-x}\right) &= \left(1 - \frac{1}{1-x}, 0, 0\right) + \left(\frac{1}{1-x}\right)(x, y, x) \\ &= \left(\frac{-x}{1-x}, 0, 0\right) + \left(\frac{x}{1-x}, \frac{y}{1-x}, \frac{x}{1-x}\right) \\ &= \left(0, \frac{y}{1-x}, \frac{z}{1-x}\right) \end{aligned}$$

This agrees with mapping shown previously. □

Remark 8.1. The Inverse map for Stereographic Projection are as follows. Let (X, Y) be points on the plane then the Inverse Stereographic Projection is,

$$(X, Y) \mapsto \left(\frac{2X}{X^2+Y^2+1}, \frac{2Y}{X^2+Y^2+1}, \frac{X^2+Y^2-1}{X^2+Y^2+1}\right)$$

Remark 8.2. Stereographic Projection generalizes to all dimensions. That is if we let $(x_0, x_1, \ldots, x_n) \in S^n$, and take $N = (1, 0, \ldots, 0)$ to be the North Pole, *Stereographic Projection* is then the mapping $s\colon S^n \setminus \{N\} \to \mathbb{R}^n$

$$s\colon (x_0, x_1, \ldots, x_n) \mapsto \left(\frac{x_1}{1-x_0}, \frac{x_2}{1-x_0}, \ldots, \frac{x_n}{1-x_0}\right)$$

The inverse also generalises. That is for $(X_0, X_1 \ldots X_n) \in \mathbb{R}^{n+1}$ Inverse Stereographic Projection is the mapping,

$$(X_0, X_1 \ldots X_n) \mapsto \left(\frac{2X_0}{\sum_{i=0}^n X_i^2 + 1}, \frac{2X_1}{\sum_{i=0}^n X_i^2 + 1}, \ldots, \frac{2X_n}{\sum_{i=0}^n X_i^2 + 1}, \frac{\sum_{i=0}^n X_i^2 - 1}{\sum_{i=0}^n X_i^2 + 1}\right)$$

where $i \in \{0, 1, 2, \ldots, n\}$.

A nice property about Stereographic Projection is that it preservers circles which do not contain the point of projection N. This property is easily seen in the example of Stereographic Projection of S^2 to $\mathbb{R}^2$. More specifcally, circles on S^n which do not contain the point of projection are mapped to circles in $\mathbb{R}^n$ and circles that do contain this point are mapped to lines in $\mathbb{R}^n$. (see Figure 7). For a proof of this see [6], Chapter 18. This fact allows to allows us to 'see' fibers of the Hopf Fibration.

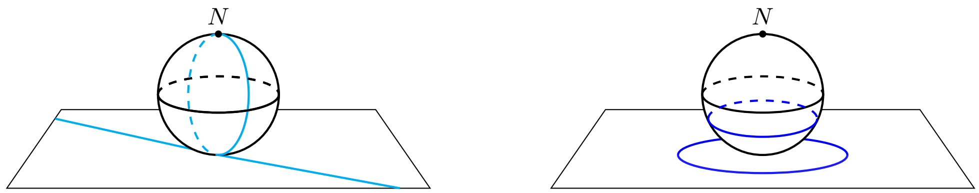


Figure 7: Under Stereographic Projection circles which do not pass through the point of projection are projected to circles.

9 Linked Hopf Fibers

Let us continue on with a discussion of the fibers. Another interesting property they have is that they are *Hopf Linked*.

Definition 9.1. (Hopf Link) A *Hopf Link* consists of two circles which are linked together exactly once.

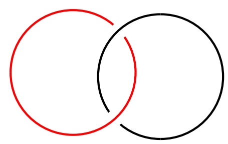


Figure 8: A Hopf Link.

We will prove that the fibers are linked using their images under Stereographic Projection. For our proof we suppose that the fibers are were originally given by $h^{-1}(P) = \{r_1 e^{it}\}_{0 \le t \le 2\pi}$. The result that the fibers are linked holds the same if $h^{-1}(P) = \{r_2 e^{it}\}_{0 \le t \le 2\pi}$. (The proof given was originally outlined in [1] Investigation L,M). Our proof will rely on the following remark.

Remark 9.1. A continuous bijection preserves Hopf Links.

Proof. Denote now s as the stereographic projection $s \colon S^3 \backslash (1,0,0,0) \to \mathbb{R}^3$. Consider the fiber $h^{-1}(-1,0,0) = \sin(t)j + \cos(t)k$. We have, $s \circ h^{-1}((-1,0,0)) = (0, \cos(t), \sin(t))$, that is too say that this fiber is projected on the unit circle on the yz-plane. Next $s \circ h^{-1}(1,0,0) = \left(\frac{\sin t}{1-\cos(t)}, 0, 0\right)$, which is the x-axis. If can add a point at infinity to the x-axis we can consider this to be a circle Now consider any other generic projected fiber $s \circ h^{-1}(P)$. It has the following form,

$$s \circ h^{-1}(P) = \frac{1}{\sqrt{2(1+p_1)}} \left(\frac{(1+p_1)\sin(t)}{1 - d\cos(t)}, \frac{p_2 \sin(t) - p_3 \cos(t)}{1 - d\cos(t)}, \frac{p_2 \cos(t) + p_3 \sin(t)}{1 - d\cos(t)} \right)$$

This projected fiber intersects the yz-plane at two distinct points, when $t = 0$ or $t = \pi$, in other words when,

$$s \circ h^{-1}(P) = \left(0, \frac{-p_3}{1+d}, \frac{p_2}{1+d}\right)$$

when $t = 0$ and

$$s \circ h^{-1}(P) = \left(0, \frac{p_3}{1-d}, \frac{-p_2}{1-d}\right)$$

for $t = \pi$. We wish to show that one of these interesections lies outside the unit circle and the other inside. In order to do this it suffices to show that as vectors on the yz-plane one has norm greater than one and the other less than one.

Now consider the vector $\left(\frac{-p_3}{1+d}, \frac{p_2}{1+d}\right)$ on the yz-plane. Then taking the norm gives us,

$$\left\| \left(\frac{-p_3}{1+d}, \frac{p_2}{1+d} \right) \right\| = \frac{1}{1+d}\sqrt{p_3^2 + p_2^2}$$
$$= \frac{1}{1+d}\sqrt{1 - p_1^2}$$

Denote this function of p_1 as f. Now since $p_1 \in S^2$ it must be that $p_1 \in [0, 1]$. From this we may find that $f(p_1) \in [0, 2 - \sqrt{2}]$, and thus intersects the inside of the unit circle on the yz-plane.

Next consider the vector $\left(\frac{p_3}{1-d}, \frac{-p_2}{1-d}\right)$and applying the same process gives,

$$\left\| \left(\frac{p_3}{1-d}, \frac{-p_2}{1-d} \right) \right\| = \frac{1}{1-d}\sqrt{1 - p_1^2}$$

Denoting this function g. Then it can be shown that $g(p_1) \in [2 + \sqrt{2}, \infty)$ meaning this intersects the yz-plane outside the unit circle. This proves that any projected fiber is linked with unit circle in the yz-plane.

Proving that any two fibers are linked can be done as follows. Let C_1 and C_2 be projected fibers. Let $\zeta\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous bijective map which takes C_1 to the unit circle in the yz-plane and C_2 to another projected fiber. Explicitly ζ is the following. Let a point $P \in C$ and take $r = s^{-1}(P)$. We have $\zeta = s \circ f \circ s^{-1}$ where $f(x) = kr^{-1}x$. This is quaternion multiplication.

$\zeta(C_2)$ will be linked with the unit circle in yz-plane. So it follows that C_1 and C_2 are also linked. Finally stereographic projection is a continuous bijection so doesn't unlink the circles. Hence we can say that the fibers were originally linked. $\square$

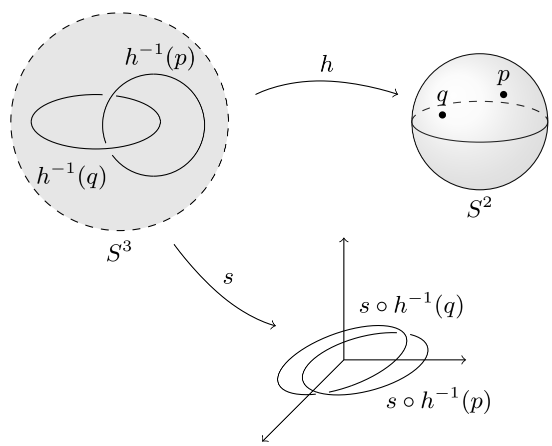


Figure 9: The map h denotes the *Hopf Map* and s denotes *Stereographic Projection*. The linked circles in S^3 are projected to linked circles in $\mathbb{R}^3$.

10 Other Hopf Fibrations

In this paper we have discussed the *Classical* Hopf Fibration. The term *Hopf Fibration* more generally refers to class of fibrations in which the spaces E, B and F are all spheres. There exist exactly four *Hopf Fibrations* which are,

$$\begin{array}{cccc} S^0 \longrightarrow S^1 & S^1 \longrightarrow S^3 & S^3 \longrightarrow S^7 & S^7 \longrightarrow S^{15} \\ \quad\quad\downarrow & \quad\quad\downarrow & \quad\quad\downarrow & \quad\quad\downarrow \\ \quad\quad S^1 & \quad\quad S^2 & \quad\quad S^4 & \quad\quad S^8 \end{array}$$

The lowest dimensional Hopf Fibration maps pairs of antipodal points on a circle to a single point on a new circle. There are some interesting ways to visualise the this. The first is to imagine S^1 sitting next to a line. Consider the mapping which takes pairs of anti-podal points to this this line. [see Figure 10]

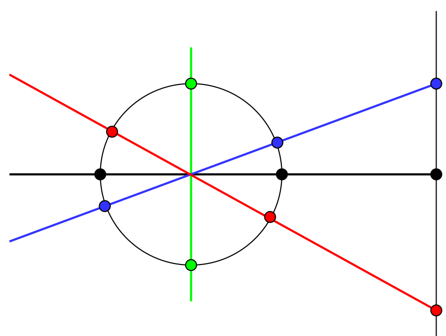


Figure 10: Pairs of anti-podal points can be identified as 0-spheres.

Here the pair of anti-podal points at top and bottom S^1 (shown in green) will seemingly not be mapped to the line. In order to remedy this, add a point to this line at 'infinity' and send this pair anti-podal points there. The line with infinity added forms a circle, completing the fibration.

Another way to visualise this fibration is to think of a circle that 'twisted' then 'folded' over itself. How does work? First consider a circle with pairs of anti-podal points identified. [see Figure 11]

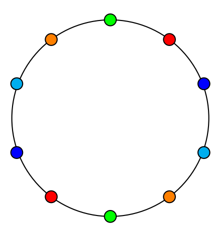


Figure 11: A circle with pairs of anti-podal points identified.

To the circle apply a 'twist'. [see Figure 12]

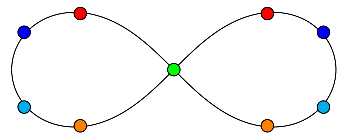


Figure 12: 'Twisting' of a circle to form 'figure-8' or 'infinity' symbol. Anti-podal points from our original circle are still identified.

From here 'folding' along the vertical axis passing through the center point (shown in green) produces a circle. [see Figure 13]

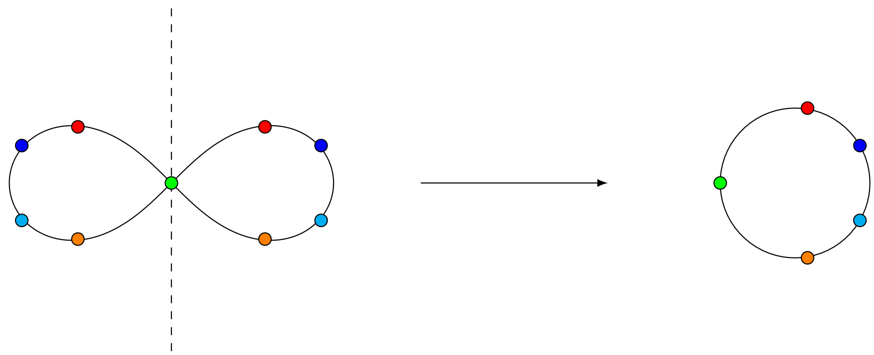


Figure 13: The anti-podal points on our original circle here are mapped to the same point on the new circle.

These two actions may be likened to 'twisting' and 'folding' of a rubber band or hairtie. Again this completes the fibration.

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