4 Categorical Limits and Colimits

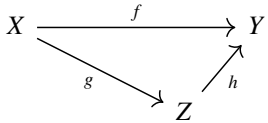
A comathematician is a device for turning cotheorems into ffee.
—Unknown (Please tell us if you made this joke up!)

Introduction. Categorical limits and colimits are one of the—and in some sense *the most*—efficient way to build a new mathematical object from old objects. The constructions introduced in chapter 1—subspaces, quotients, products, and coproducts—are examples in Top, though the discussion can occur in any category. In fact, there are a number of other important constructions—pushouts, pullbacks, direct limits, and so on—so it's valuable to learn the general notion.

In practice, limits are typically built by picking a subcollection according to some constraint, whereas colimits are typically built by "gluing" objects together. More formally, the defining property of a limit is characterized by morphisms whose *domain* is the limit. The defining property of a colimit is characterized by morphisms whose *codomain* is the colimit. Because of their generality, limits and colimits appear all across the mathematical landscape. A direct sum of abelian groups, the least upper bound of a poset, and a CW complex are all examples of limits or colimits, and we'll see more examples in the pages to come. Section 4.1 opens the chapter by answering the anticipated question, "A (co)limit of *what*?" The remaining two sections contain the formal definition of (co)limits followed by a showcase of examples.

4.1 Diagrams Are Functors

In topology, one asks for the limit of a sequence. In category theory, one asks for the (co)limit of a *diagram*. In what follows, it will be helpful to view a diagram as a functor. More specifically, a diagram in a category is a functor from the shape of the diagram to the category. For example, a commutative diagram like this



in a category C is a choice of three objects X, Y, and Z and some morphisms $f\colon X \to Y$, $g\colon X \to Z$, and $h\colon Z \to Y$, with $f = hg$. It can be viewed as the image of a functor from an