A PRIMER ON SESQUILINEAR FORMS

BRIAN OSSERMAN

This is an alternative presentation of most of the material from §8.1, 8.2, 8.3, 8.4, 8.5 and 8.8 of Artin's book. Any terminology (such as sesquilinear form or complementary subspace) which is discussed here but not in Artin is optional for the course, and will not appear on homework or exams.

1. Sesquilinear forms

The dot product is an important tool for calculations in $\mathbb{R}^n$. For instance, we can use it to measure distance and angles. However, it doesn't come from just the vector space structure on $\mathbb{R}^n$ – to define it implicitly involves a choice of basis. In Math 67, you may have studied inner products on real vector spaces and learned that this is the general context in which the dot product arises. Now, we will make a more general study of the extra structure implicit in dot products and more generally inner products. This is the subject of bilinear forms. However, not all forms of interest are bilinear. When working with complex vector spaces, one often works with Hermitian forms, which toss in an extra complex conjugation. In order to handle both of these cases at once, we'll work in the context of sesquilinear forms.

For convenience, we'll assume throughout that our vector spaces are finite dimensional.

We first set up the background on field automorphisms.

Definition 1.1. Let F be a field. An **automorphism** of F is a bijection from F to itself which preserves the operations of addition and multiplication. An automorphism $\varphi : F \to F$ is an **involution** if $F \circ F$ is the identity map.

Example 1.2. Every field has at least one involution: the identity automorphism!

Example 1.3. Since for $z, z' \in C$, we have $\overline{z + z'} = \bar{z} + \bar{z}'$ and $\overline{zz'} = \bar{z}\bar{z}'$, complex conjugation is an automorphism. Since $\bar{\bar{z}} = z$, it is an involution.

Example 1.4. Consider the field $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. Then the map $a + b\sqrt{2} \mapsto a - b\sqrt{2}$ is an involution.

We will assume throughout that we are in the following

Situation 1.5. Let V be a vector space over a field F, and suppose we have an involution on F which we denote by $c \mapsto \bar{c}$ for all $c \in F$.

Definition 1.6. A **sesquilinear form** on a vector space V over a field F is a map

$$\langle , \rangle : V \times V \to F$$

which is linear on the right side, and almost linear on the left: that is,

$$\langle v_1, cw_1 \rangle = c \langle v_1, w_1 \rangle, \quad \text{and} \quad \langle v_1, w_1 + w_2 \rangle = \langle v_1, w_1 \rangle + \langle v_1, w_2 \rangle$$
$$\langle cv_1, w_1 \rangle = \bar{c} \langle v_1, w_1 \rangle, \quad \text{and} \quad \langle v_1 + v_2, w_1 \rangle = \langle v_1, w_1 \rangle + \langle v_2, w_1 \rangle$$

for all $c \in F$ and $v_1, v_2, w_1, w_2 \in V$.

A special case of sesquilinear forms that works for any field arises when the involution is the identity map.