GROUP ACTIONS

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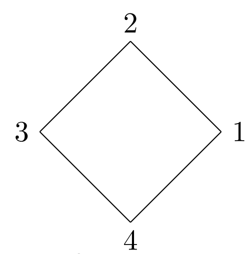
1. Introduction

The groups S_n, A_n, and (for $n \geq 3$) D_n behave, by their definitions, as permutations on certain sets. The groups S_n and A_n both permute the set $\{1, 2, \ldots, n\}$ and D_n can be considered as a group of permutations of a regular n-gon, or even just of its n vertices, since rigid motions of the vertices determine where the rest of the n-gon goes. If we label the vertices of the n-gon in a definite manner by the numbers from 1 to n then we can view D_n as a subgroup of S_n. For instance, the labeling of the square below lets us regard the 90 degree counterclockwise rotation r in D_4 as (1234) and the reflection s across the horizontal line bisecting the square as (24). The rest of the elements of D_4, as permutations of the vertices, are in the table below the square.



1	r	r^2	r^3	s	rs	r^2s	r^3s
(1)	(1234)	(13)(24)	(1432)	(24)	(12)(34)	(13)	(14)(23)

If we label the vertices in a different way (*e.g.*, swap the labels 1 and 2), we turn the elements of D_4 into a different subgroup of S_4.

More abstractly, if we are given a set X (not necessarily the set of vertices of a square), then the set $\mathrm{Sym}(X)$ of all permutations of X is a group under composition, and the subgroup $\mathrm{Alt}(X)$ of even permutations of X is a group under composition. If we list the elements of X in a definite order, say as $X = \{x_1, \ldots, x_n\}$, then we can think about $\mathrm{Sym}(X)$ as S_n and $\mathrm{Alt}(X)$ as A_n, but a listing in a different order leads to different identifications of $\mathrm{Sym}(X)$ with S_n and $\mathrm{Alt}(X)$ with A_n.[1]

The "abstract" symmetric groups $\mathrm{Sym}(X)$ really do arise naturally:

Theorem 1.1 (Cayley)**.** *Every finite group G can be embedded in a symmetric group.*

Proof. To each $g \in G$, define the left multiplication function $\ell_g \colon G \to G$, where $\ell_g(x) = gx$ for $x \in G$. Each ℓ_g is a permutation of G as a set, with inverse $\ell_{g^{-1}}$. So ℓ_g belongs to $\mathrm{Sym}(G)$. Since $\ell_{g_1} \circ \ell_{g_2} = \ell_{g_1 g_2}$ (that is, $g_1(g_2 x) = (g_1 g_2)x$ for all $x \in G$), associating to g the mapping ℓ_g gives a homomorphism of groups, $G \to \mathrm{Sym}(G)$. This homomorphism is one-to-one since ℓ_g determines g (after all, $\ell_g(e) = g$). Therefore the correspondence $g \mapsto \ell_g$ is an embedding of G as a subgroup of $\mathrm{Sym}(G)$. $\square$

[1]When $X = \emptyset$, consider $\mathrm{Sym}(X)$ and $\mathrm{Alt}(X)$ to be trivial groups. The number of permutations of a set of size 0 is $0! = 1$.