AN ILLUSTRATED INTRODUCTION TO THE RICCI FLOW

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The Ricci flow is a central topic in differential geometry, and perhaps the most important example of a geometric evolution equation. In this paper, we will try to give some intuition for how this flow behaves and why it is so important in modern geometry.

The Ricci flow is most famous for its role in the proof of the Poincaré conjecture, so we start with a historical account of the Poincarè conjecture and some related problems to give context and motivation for studying the Ricci flow.

In the second half of the paper, we will provide an explanation for what the Ricci flow actually is. One very useful aphorism for the Ricci flow is that it is a "heat equation for curvature." This statement is not entirely correct, but is a useful starting point and our goal in this paper is to explain how the Ricci flow behaves similarly to (and differently from) a heat equation.

The goal throughout will be to provide an informal introduction which assumes minimal background knowledge. For readers who are already familiar which geometric analysis, there are additional details in the footnotes to fill in some of the imprecisions. However, these notes are not necessary to follow the main discussion, so can be ignored on an initial reading.

1. A short history of the Poincaré conjecture

In 1905, Henri Poincaré published *Analysis Situs* [Poi95], which laid the foundations for what is known today as *topology.*[1] Roughly speaking, topology studies "rubber-sheet geometry," in which spaces are allowed to bend and warp in a continuous way. One central focus of topology is to find properties of a space which do not change when they are deformed in this way.

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[1]Some topological ideas appeared in Leonard Euler's earlier work on the Bridges of Konigsburg, but Analysis Situs is a foundational work in this area.

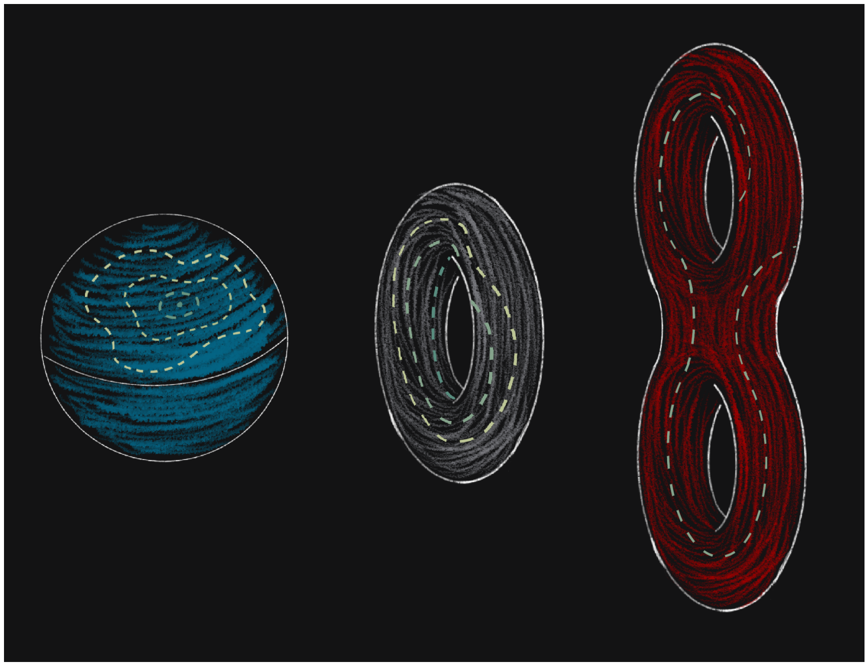

FIGURE 1. Three surfaces and some loops on them

While studying this topic, Poincaré conjectured that the three-sphere is the only simply-connected compact three-dimensional manifold.[2] To explain the meaning of this conjecture, it is helpful to first consider the corresponding result for two-dimensional surfaces, which is a special case of the *uniformization theorem.*

1.1. The uniformization theorem.

The uniformization theorem is a classification of two dimensional surfaces which was proven by Poincarè in 1907 [Poi08]. It states any surface can be deformed into one of three special types of geometry, depending on the number of holes that it has.[3] Surfaces without any holes can be deformed into a round sphere. A surface with one hole

[2]This is actually not Poincaré's original question, but a refinement after counterexamples were found to some earlier versions of the conjecture.

[3]The actual statement of the uniformization theorem is a bit stronger, and says that this deformation can be done in a conformal way.

(i.e., a donut) can be made into a flat space[4] and any surface with more than one hole can be given a hyperbolic geometry.[5]

When you first encounter this result, it might be somewhat hard to visualize a donut with flat geometry or a hyperbolic geometry on a surface with multiple holes. In fact, there is no way to realize these geometries if we think of these surfaces as sitting in three-dimensional Euclidean space.[6] [7] In modern geometry, one often considers the *intrinsic* geometry of a surface, without considering it as a surface lying in some ambient Euclidean space. This can be a major conceptual challenge. However, a deep theorem of John Nash shows that it is possible to realize any geometry on a surface if we consider it as living in 51-dimensional Euclidean space.[8]

1.2. **The Poincaré conjecture.** The uniformization conjecture has a straightforward, but very important, corollary about how the relationship between the topology of a surface and the behavior of loops on it. As shown in Figure 1, if we consider a surface with a positive number of holes and draw a loop around the hole, it is not possible to shrink that loop to a point without leaving the surface or cutting the loop.

As a result, the uniformization theorem implies that if we are have a surface where every loop drawn in it can be shrunk down continuously to a point, then the space is topologically equivalent to a sphere. Spaces which have this property are said to be *simply connected*.

The Poincarè conjecture is the three-dimensional version of this statement. In other words, if we have a closed three-dimensional space[9] which is bounded and has the property that every loop in it can be shrunk down

[4]Here, a surface is said to be *flat* if its curvature vanishes. In Section 3 we will discuss the notion of curvature in more detail.

[5]This provides a full classification of orientable surfaces. The uniformization theorem also gives a classification of non-orientable surfaces as well.

[6]Here, by sitting inside three-dimensional space, we mean smoothly immersed in $\mathbb{R}^3$.

[7]If you are familiar with the differential geometry of curves and surfaces, it is a good exercise to prove this. As a hint, suppose the surface contains the origin in its interior. What can you say about the curvature of the point furthest from the origin?

[8]In general, any n-dimensional Riemannian manifold can be embedding $\mathbb{R}^N$ where $N = n(n+1)(3n+11)/2$. The dimension 51 comes from applying this formula when $n = 2$. For compact surfaces, Mikhail Gromov proved that any compact surface can actually be isometrically embedded in $\mathbb{R}^5$, so for our purposes 46 of these 51 dimensions are redundant [Gro86].

[9]Throughout the rest of the paper, we will use the word "space" to mean "compact smooth manifold," so as to not introduce unnecessary terminology.

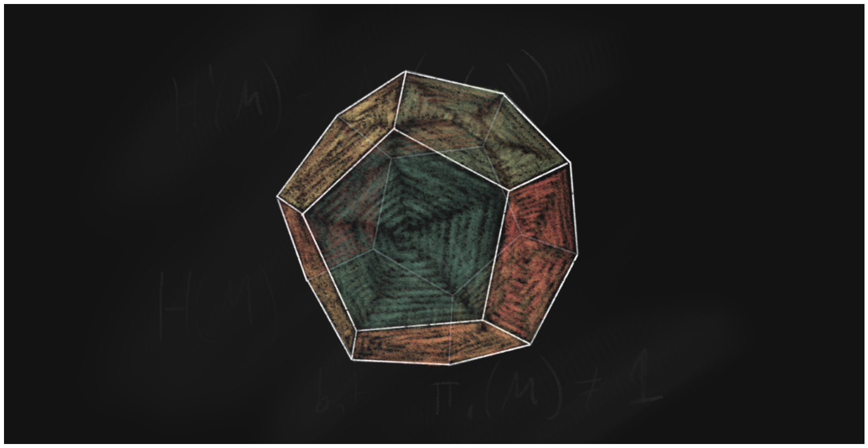

FIGURE 2. A counter-example to an earlier version of the Poincaré conjecture[10]

continuously to a point, then the space is topologically equivalent to a three-dimensional sphere.

This conjecture (and its higher dimensional analogues) motivated much of the early work in topology. The three-dimensional case in particular proved itself to be extremely subtle and intractable problem (see [Sta16] for some idea of why this is the case). It attracted the attention of many leading mathematicians who developed various tools in their attempts to solve it. As a result of all these efforts, by the 1980s the Poincaré conjecture was settled in all dimensions except for three.[11] However, Poincaré's original

[10]More precisely, if you take a solid dodecahedron and identify the opposite sides with a minimal twist, what results is a *homology sphere*, which is a manifold whose homology groups are the same as the sphere but which is topologically distinct from a sphere.

[11]There is an important subtlety here. It turns out that there is a both a topological and a smooth version of the Poincaré conjecture, which considers whether spaces are continuously equivalent to a sphere or smoothly equivalent to the round sphere. The former is what is traditionally known as the Poincaré conjecture. Stephen Smale proved the topological Poincaré conjecture in dimensions greater than four [Sma62] and Michael Freedman proved the four-dimensional case [Fre82]. On the other hand, work of Milnor showed that there are exotic smooth structures on the seven sphere, and thus the smooth Poincaré conjecture is false in dimension seven [Mil59], which started a long line of work study the possible smooth structures on spheres. All three were awarded Fields medals for their respective work. At this point, there is one major question remaining in this line of work, which is whether there exists exotic smooth structures on the four-dimensional sphere.

FIGURE 3. Thurston's Eight Geometries[12]

conjecture appeared to be outside the reach of the standard tools of low-dimensional topology, and it seemed that a new approach would be needed to solve it.

1.3. **The Geometrization Conjecture and the Ricci flow.** In the late 1970s and early 1980s, William Thurston studied the geometry of three-dimensional spaces and brought in many new ideas to understand their structure. He proposed a conjecture classifying the possible geometry of such spaces similar to how the uniformization theorem classifies the geometry of all two-dimensional surfaces. More precisely, he found eight natural geometries that three dimensional space can have and conjectured that any three-dimensional manifold can be obtained by gluing together pieces, each of which have one of the eight geometries. Thurston was able to prove this conjecture for a fairly broad class of three-dimensional spaces and was awarded the Fields Medal in 1982 for this work. His conjecture, known as Thurston's Geometrization conjecture, implied the Poincaré conjecture and gave a new avenue to attack Poincaré's question.

[12]In this illustration, the top row illustrates the five geometries which are either spaces of constant curvature or products thereof. The illustrations of the bottom row deserve further explanation.

(1) The depiction of Sol is meant to illustrate the metric
$$g = e^z dx^2 + e^{-z} dy^2 + dz^2,$$
which expands exponentially in the x-direction and contracts exponentially in the y-direction as the z-coordinate increases.

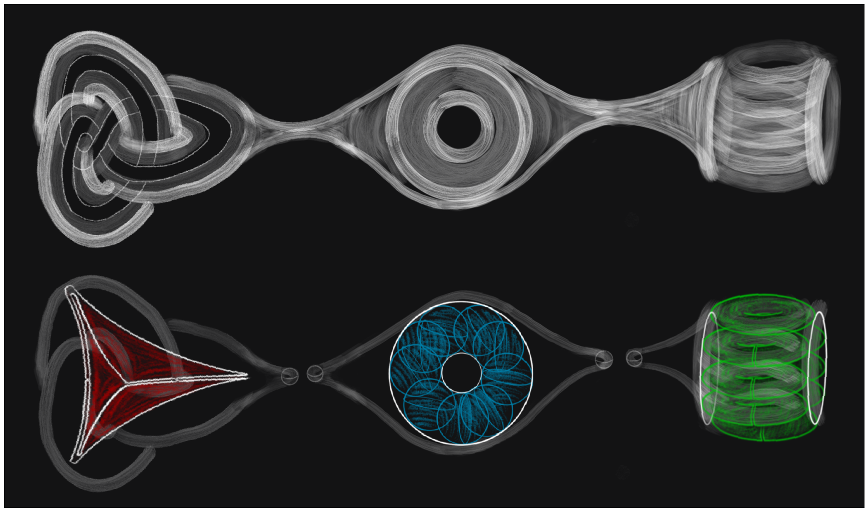

FIGURE 4. A three-dimensional space formed by connecting three different geometries

Later that year, Richard Hamilton proposed an ambitious program to attack both of these problems [Ham82]. Hamilton had also been studying work by Eells and Sampson [ES64], which used ideas about heat flows to understand harmonic maps. He thought that it might be possible to prove the Poincaré and Geometrization conjectures using a similar approach.[13] He defined an evolution equation, known as the Ricci flow, which would deform the shape of a space and hopefully allow its curvature to dissipate throughout the space. If the space satisfied the hypotheses of the Poincaré conjecture, he hoped the geometry would evolve to that of a round sphere, which would establish the result.

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(2) The fundamental example of a space which has $\widetilde{SL}(2,\mathbb{R})$-geometry is the complement of the trefoil knot in the three-sphere.
(3) Nil has the geometry of a torus bundle over a circle where the monodromy is given by a Dehn twist [CMST20]. Here, I chose to depict a vertical line as the universal cover of a circle so that the two tori would appear at different heights.

[13]Strictly speaking, the Ricci flow is not a typical heat flow. In particular, its diffeomorphism-invariance induces zero terms in its symbol, so it is not parabolic. As such, establishing the existence of the flow was a major accomplishment. Hamilton was able to do so via a technical argument [Ham82] but a few years later Dennis DeTurck found a much simpler argument by conjugating the Ricci flow to obtain a strictly parabolic flow [DeT81].

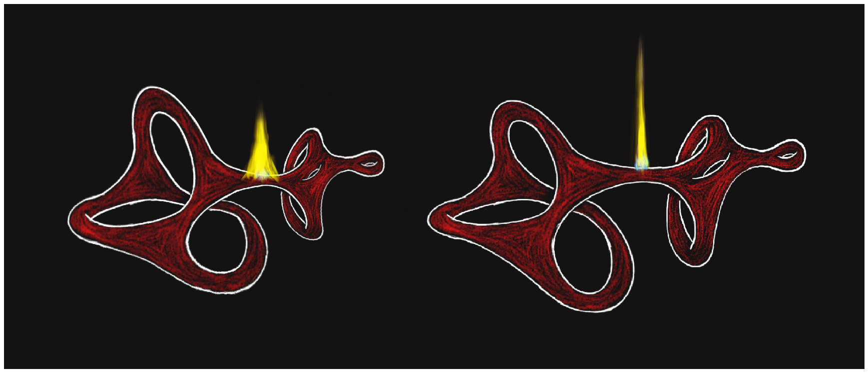

FIGURE 5. An evolving surface with a backwards heat equation

Hamilton and others were able to make partial progress towards this goal. In particular, Hamilton showed that the Poincaré conjecture was true under the additional assumption that the Ricci curvature of the space is positive. Furthermore, he was nearly[14] able to use the Ricci flow to prove the uniformization theorem of Riemann surfaces. These results gave a proof of concept that the Ricci flow was a promising approach to the problem. However, in three dimensions the flow would sometimes violently rip a space apart, and this presented a fundamental obstacle to finishing the proof.

1.4. **Perelman's breakthrough.** In 2000, the Clay Millenium institute listed seven major open problems in mathematics and provided a $1,000,000 prize for a solution to any of them. The Poincaré conjecture was one of the problems and, much like the other problems, no one expected there to be a solution in the near future.

However, just two years later, Grigori Perelman posted a terse preprint to the arXiv containing several major breakthroughs in the Ricci flow [Per02]. In particular, he showed that by combining the Ricci flow with a backwards heat equation, there was a non-decreasing quantity (which corresponds to the entropy of the heat distribution) which could be used to control the behavior of the flow.

Over the next year, two more preprints [Per03b, Per03a] followed the first, which used the ideas from the first paper as well as a geometric process

[14]There were some missing steps dealing with genus zero metrics of mixed curvature which were established by Bennett Chow [Cho91] and Xiuxiong Chen, Peng Lu, and Gang Tian [CLT06].

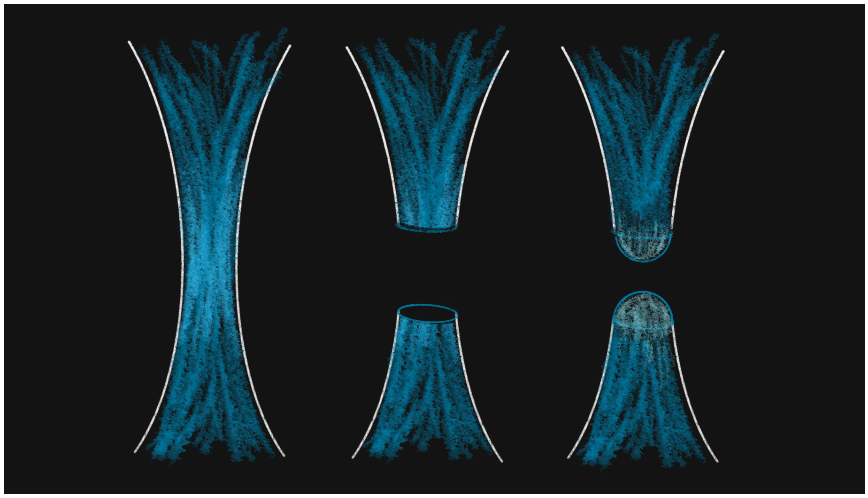

Figure 6. Using surgery to tame singularities

known as surgery[15] to excise regions of space in the process of being ripped apart and replace them with pieces with better behavior. By combining the Ricci flow with surgery, Perelman was able to show that any simply-connected three-dimensional space would converge to a round sphere (or possibly a connected sum of several round spheres), and thus is topologically equivalent to the standard sphere. For more general three-dimensional spaces, he proved that after a large amount of time, Ricci flow with surgery would decompose the space into several pieces whose geometric structure was well understood.[16] This established both the Poincaré conjecture as well as the Geometrization Conjecture.

Perelman had toiled in relative isolation for nearly a decade crafting his proof. It took several years and an authorship scandal for the mathematical community to accept the work as correct (and to appropriately credit it to Perelman). In 2006, Perelman was awarded a Fields medal for his efforts.

[15]Ricci flow with surgery was invented by Hamilton in 1993 [Ham93, Ham97], but he was unable to use it to handle the singularities in the three dimensional case.

[16]Ricci flow with surgery does not necessarily converge to one of Thurston's geometries on each connected component. Indeed, several of the geometries collapse under the flow, so are not even fixed points. Instead, what occurs is that if we perform the surgeries in an effective way, there are only finitely many surgeries [Bam18], and after the last one the space decomposes into several pieces. Each of these pieces is one of four types (a hyperbolic manifold, a round sphere, a Haken manifold, or a graph manifold), all of which were previously known to satisfy the Geometrization conjecture[Cal20].

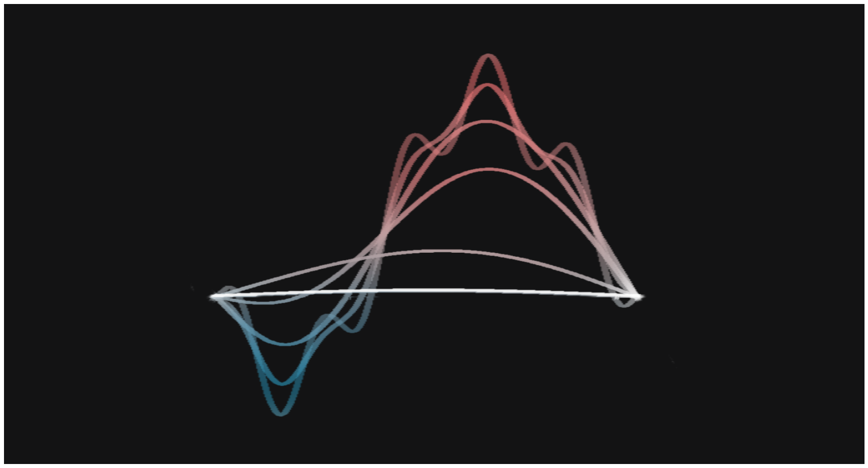

FIGURE 7. A heat distribution flowing to equilibrium[17]

He declined the award and later turned down the Millenium Prize, as well. He has since withdrawn from mathematics.

It is hard to think of praise too high for Perelman's work. It is one of the greatest mathematical works of all time and a marvel of the power of hard analysis. It is also one of the main reasons why the Ricci flow has received so much attention in the past twenty years.

2. The Heat Equation

In the rest of the paper, our goal will be to explain what the Ricci flow actually is and give some intuition. In order to do so, we start our journey by trying to understand heat flows.

2.1. Intuition.

The heat equation models the behavior of a heat distribution as time goes by. To give a concrete example, suppose that we have a warm cup of coffee outside in a cold day. Left alone, the heat from the coffee will dissipate into the air and the temperature throughout will converge to that of the outside environment. More generally, the temperature of a system tends to flow from an ordered distribution to an equilibrium state of

[17]In this diagram, we have imposed Dirichlet boundary conditions on the solution. The time slices were chosen to show qualitative behavior of the solution, and were not taken uniformly.

constant temperature.[18] The hottest regions will cool down and the coldest ones will warm up.

The heat equation[19] is given by

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t). \tag{1}$$

In this equation, $u(x,t)$ is a heat distribution which depends both on the position x and the time t. This equation was introduced by Joseph Fourier in 1822, and was foundational in the development of Fourier analysis. We will not provide a physical derivation for this equation, but instead try to understand what its solutions look like. Before doing so, let us first consider the Laplacian term on the right-hand side.

2.2. **The Laplacian.** Equation 1 involves the Laplacian Δu. You might remember this from vector calculus as being

$$\Delta u = \mathrm{div}(\,\mathrm{grad}\ u).$$

However, there are several other perspectives on the Laplacian that will tie in more naturally to Ricci flow. One interpretation of the Laplacian which is helpful is to consider the Hessian matrix (in $\mathbb{R}^3$)

$$\mathrm{Hess}(u) = \begin{bmatrix} \frac{\partial^2}{\partial x^2}u & \frac{\partial}{\partial x}\frac{\partial}{\partial y}u & \frac{\partial}{\partial x}\frac{\partial}{\partial z}u \\ \frac{\partial}{\partial y}\frac{\partial}{\partial x}u & \frac{\partial^2}{\partial y^2}u & \frac{\partial}{\partial y}\frac{\partial}{\partial z}u \\ \frac{\partial}{\partial z}\frac{\partial}{\partial x}u & \frac{\partial}{\partial z}\frac{\partial}{\partial y}u & \frac{\partial^2}{\partial z^2}u \end{bmatrix}$$

and note that the Laplacian is the trace of this matrix:

$$\Delta u = \frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u + \frac{\partial^2}{\partial z^2}u.$$

Later on, we will see that the Ricci curvature is the trace of the Riemann curvature tensor, so this identity makes the connection between the Ricci curvature and the Laplacian fairly natural.

The goal of this paper is to define the Ricci flow with the minimal amount of Riemannian geometry necessary. For this, there is a third perspective that is even more useful. We consider a point x and a small ball of radius ε around

[18] Using the notion of entropy, it is possible to formalize the statement that heat distributions flow from more orderly states to less orderly ones.

[19] We are ignoring the boundary conditions here. In the context of the Ricci flow, we will be dealing with compact manifolds without boundary so this is not an issue.

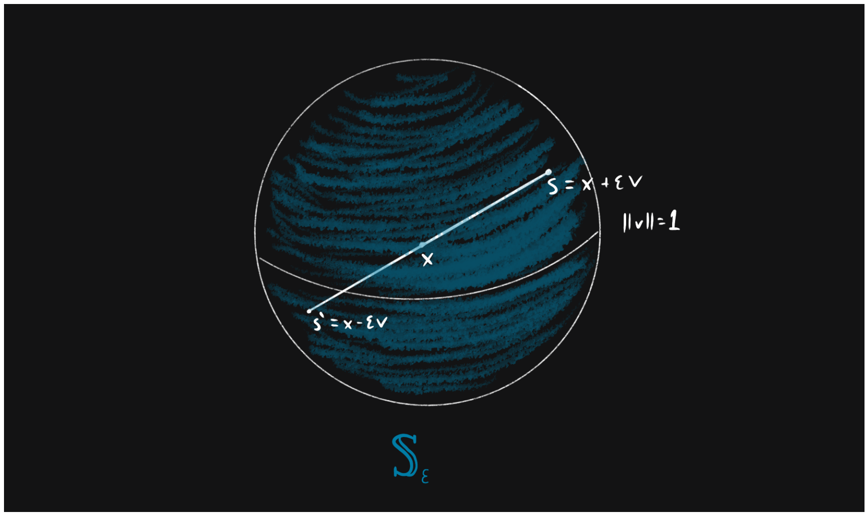

FIGURE 8. The Laplacian as an integral

x, which we denote $\mathbb{S}_\varepsilon$. It turns out that the Laplacian is proportional to the difference between $f(x)$ and the average value of f on $\mathbb{S}_\varepsilon$:

$$\Delta f(x) = \lim_{\varepsilon \to 0+} \frac{2n}{\varepsilon^2} \frac{1}{\omega(\mathbb{S}_\varepsilon)} \int_{\mathbb{S}_\varepsilon} (f(s) - f(x))\, \mathrm{d}\omega(s). \tag{2}$$

Here, $\omega(\mathbb{S}_\varepsilon)$ is the surface area of a sphere of radius ε in Euclidean n-space, which is given explicitly by

$$\omega(\mathbb{S}_\varepsilon) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \varepsilon^{n-1}.$$

From this we can see that Δu is the average of second derivatives, which is a perspective that will help us understand Ricci curvature. It's worth convincing yourself that the right-hand side of (2) really does have something to do with second derivatives. Figure 8 is meant to serve as a hint for why this is the case.[20]

2.3. **The maximum principle.** Now that we have defined the Laplacian, we consider the heat equation (1) and try to understand its behavior. Recall

[20]As an additional hint, given a unit vector V, what is

$$\lim_{\varepsilon \to 0} \frac{f(x+\varepsilon V) - f(x) + f(x-\varepsilon V) - f(x)}{\varepsilon^2}?$$

that this equation is the partial differential equation

$$\frac{\partial u}{\partial t} = \Delta u.$$

Here, the derivatives on the right-hand side are space derivatives (with respect to x), whereas the derivative on the left-hand side indicates the change with respect to time (i.e., t). Given an initial distribution of heat (once again, disregarding issues about the boundary), a solution to this equation will give the temperature of points at time t. We call such a solution a "heat flow." To check our intuition that the hottest points should cool down, suppose at some time t_0 the point (x_0, y_0, z_0) in the interior was the hottest point. Then, since $u(x_0, y_0, z_0)$ is at a maximum, $\mathrm{Hess}(u)$ is non-positive definite. This implies that $\mathrm{tr}(\mathrm{Hess}(u))$ is non-positive and thus that

$$\frac{u(x_0, y_0, z_0)}{\partial t} \leq 0.$$

This type of argument is known as a *maximum principle,* and plays an essential role in the analysis of heat-type equations. In order to show that the temperature of the hottest point is strictly decreasing, we would need a stronger version of the maximum principle. The maximum principle alone also doesn't show that the heat converges to a constant as t goes to infinity, but for reasonable initial conditions, this is indeed the case.[21]

3. Curvature

In order to discuss the Ricci flow, it is first necessary to discuss the notion of curvature. Unfortunately, rigorously defining curvature requires some background knowledge in Riemannian geometry and a more in-depth discussion of extrinsic versus intrinsic geometry, which are both outside the scope of this introduction. As a result, in this section we will try to give an intuitive notion of curvature without worrying about being too precise.

3.1. **Sectional curvature.** Even if you have no idea what curvature is, it should be intuitive that Euclidean space is flat, and thus has no curvature. With this idea, one way to define curvature is to understand how the geometry of a given space differs from that of Euclidean space. In other words, Euclidean space has no bumps or valleys, and we want to find a way to measure this mathematically.

[21] For instance, using Fourier analysis it is possible to show that the heat will converge to a constant if the initial data is $W^2([0,1])$ (or even less regular) with periodic boundary conditions.

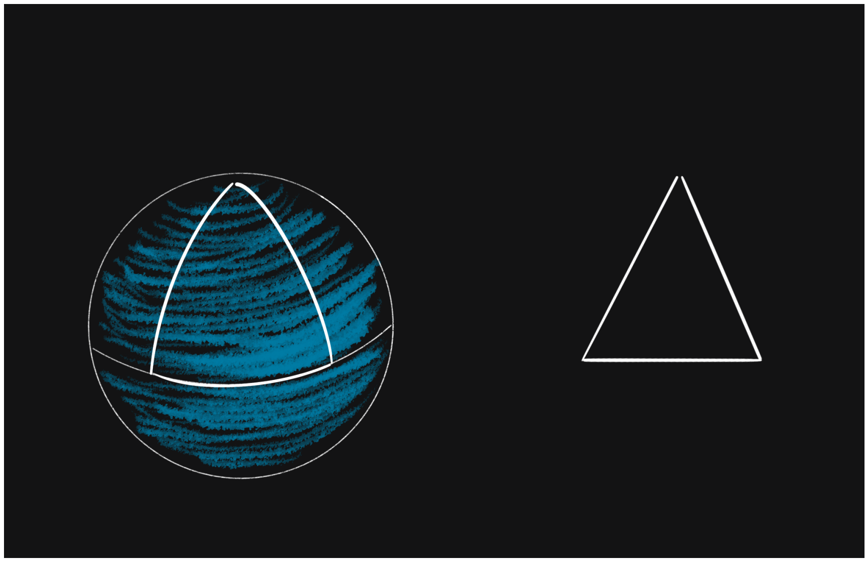

FIGURE 9. A spherical and flat triangle

It turns out that the simplest way to characterize curvature is to compare triangles in Euclidean space and the curved space. To make sense of this, let us consider a sphere, which is a surface of positive curvature.

If we look at triangles on a sphere,[22] we can see that they tend to appear "fatter" than triangles in flat space. For example, take a look at the two triangles above. The left is a spherical triangle whereas the right is a Euclidean triangle.

The sum of the angles of the spherical triangle is strictly greater than π, and actually depends on the area of the triangle.[23] Apart from the angles, we can also see from the picture that the sides of a spherical triangle seem to be bowed away from the other sides. This bowing occurs in any space of positive curvature, and is one of the characteristic features of positive sectional curvature. It's a bit counter-intuitive, but the key observation is that the geodesics are actually turning towards each other, which results in

[22]More precisely, we are considering geodesic triangles, where each side is length-minimizing.

[23]If you have studied spherical trigonometry, this will be a familiar fact. On a two dimensional sphere, it is actually a consequence of the Gauss-Bonnet theorem and for those who are familiar with the differential geometry of curves and surfaces, it's a good exercise to compute the exact formula for the sum of the angles.

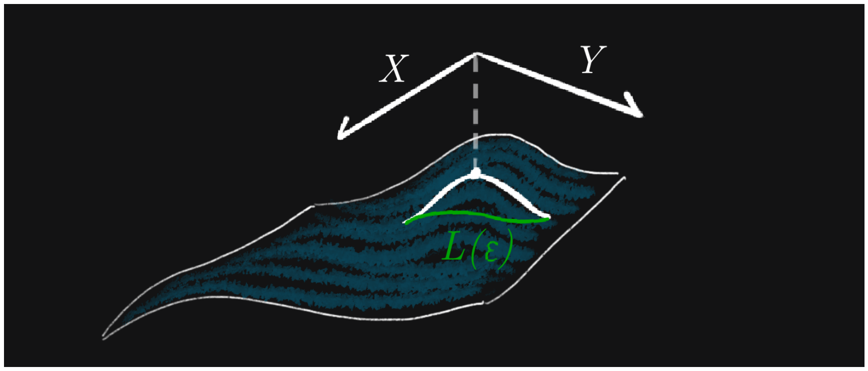

FIGURE 10. A heuristic diagram for $L(\varepsilon)$

the sides of the triangle appearing to bow outward. Here, the word "turning" is informal and somewhat imprecise, but hopefully the meaning is somewhat apparent from the pictures.

However, before we can use this idea to discuss curvature, it is worthwhile to note that round spheres are very special spaces in that every point looks like every other point.[24] However, the earth is not exactly a sphere; the geometry of Mount Everest looks very little like that of the Grand Canyon. This will be true for most of the spaces we are interested in, so we want some way to define curvature on spaces that are not symmetric. To do so, instead of using a giant triangle like on our sphere, we use very small ones.

To do this, we consider a point p in our space and two unit directions X and Y (more precisely, X and Y are unit tangent vectors). Then we consider the triangle which has a vertex at p and has sides εX and εY, where ε is a very small number. We have not said what it means to make a small triangle where the side is a direction, and this takes some work to make precise. However, the intuitive idea is that we consider a segment which "heads in the direction" of X.[25] With the intuition that the triangle will turn in on itself in a space which is positively curved, we then consider the length of the third edge of the triangle, which we denote $L(\varepsilon)$.

If $L(\varepsilon)$ is smaller than it would be in Euclidean space, the curvature is positive. Conversely, if $L(\varepsilon)$ is greater than a corresponding triangle in Euclidean space, the curvature is negative. To make this idea a bit more precise, we consider the Taylor polynomial of $L(\varepsilon)$ in terms of ε. Doing so,

[24] More precisely, round spheres are symmetric spaces.

[25] More precisely, we take a unit tangent vector X and consider the geodesic $\gamma(t) = \exp_p(Xt)$. However, I didn't want to define exponential maps.

we find the following:

$$L(\varepsilon) = \varepsilon\|X - Y\| \left(1 - \frac{1}{12}K(X,Y)(1 + \langle X, Y\rangle)\varepsilon^2\right) + O\left(\varepsilon^4\right) \tag{3}$$

In this expression, $K(X,Y)$ is the sectional curvature of the tangent plane spanned by X and Y. Note that in flat space, $K(X,Y) \equiv 0$ so this matches with our intuition. On the other hand, the unit sphere has constant positive sectional curvature (i.e., $K(X,Y) \equiv 1$). Since $K(X,Y)$ is positive, the third side of the triangle will be shorter than the corresponding side in flat space. Conversely, whenever the sectional curvature is negative, the third side of the triangle will be longer than that of the corresponding Euclidean triangle.

3.2. **Ricci curvature.** Now that we have defined the sectional curvature, we can use it to define the Ricci curvature.[26] To do so, we consider a unit vector X and say that the Ricci curvature $\mathrm{Ric}(X,X)$ is $(n-1)$ times the average of all of the sectional curvatures of tangent planes containing X. In other words, the Ricci curvature satisfies the identity[27]

$$\mathrm{Ric}(X,X) = \frac{1}{2}(n-1) \oint_{\|Y\|=1 \text{ and } X \perp Y = 0} K(X,Y)\,\mathrm{d}\mathbb{S}^{n-2}(Y), \tag{4}$$

where $\mathrm{d}\mathbb{S}^{n-2}$ is the unit measure on the $(n-2)$-dimensional sphere. This definition is concise, but doesn't give any intuition for what the Ricci curvature actually is. To do so, it is possible to draw a picture for the Ricci curvature which is similar to the one for sectional curvature but uses geodesic cones rather than geodesic triangles.[28]

[26]The standard definition of the Ricci curvature is that it is the contraction of the Riemann curvature tensor along second and last indices. In other words, given an orthonormal frame $\{e_i\}_{i=1}^n$ and two vector fields X and Z,

$$\mathrm{Ric}(X,Z) = \sum_{i=1}^{n} \langle R(X,e_i)Z, e_i\rangle$$

Here, $R(\cdot,\cdot)\cdot$ is the Riemannian curvature tensor, which is defined as

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]}Z.$$

Here, ∇_X denotes covariant derivative in the X direction with respect to the Levi-Civita connection. I used Equation (4) to avoid having to define the notions of Riemannian curvature and covariant derivatives.

[27]Note that the pairs (X,Y) and $(X,-Y)$ span the same plane, which is why there is a factor of $\frac{1}{2}$ in Equation 4.

[28]The idea of Ricci curvature as the distortion of narrow geodesic cones is taken from Chapter 14 of Villani's text [Vil09].

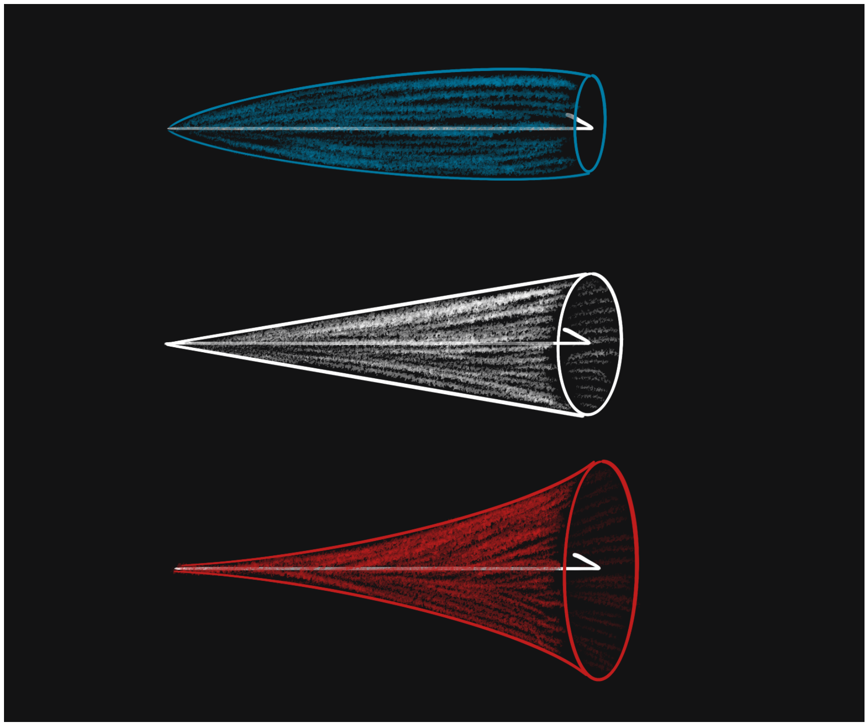

FIGURE 11. Cones in positively curved space, flat space and negatively curved space

We consider a point p and a unit direction X at p. We then take a short segment[29] of length ε in the X direction and put a narrow circular cone whose vertex is p around the segment (here, the cone being narrow means that the cone angle θ satisfies $\tan(\theta) = \varepsilon$). We then consider the area of the base of the cone, which we denote $A(\varepsilon)$.

Again, the intuition is that in positively curved space, the cone will close in on itself whereas in negatively curved space, the cone will turn outward (as shown in Figure 11). In this spirit, when we compute the Taylor expansion of $A(\varepsilon)$, we find the following:

$$A(\varepsilon) = \varepsilon^{2(n-1)} \left(D_{n-1} - C_n \operatorname{Ric}(X, X)\varepsilon^2\right) + O(\varepsilon^{2n+1}). \tag{5}$$

Here D_{n-1} is the volume of a unit disk in Euclidean $(n-1)$-space and C_n is a complicated, but positive, constant which depends on the dimension. The factor of $\varepsilon^{2(n-1)}$ on the right-hand comes from the fact that the cone is both short and narrow, both of which contribute to the area of the base being small.

[29]More precisely, we consider a geodesic γ of length ε with $\dot{\gamma}(0) = X$.

3.3. **The Ricci curvature as a geometric Laplacian.** Heuristically, the sectional curvature can be understood as a geometric second derivative. In particular, if you ignore the ε at the front of Equation (3) (which comes from the fact that the associated triangle has very short sides), the sectional curvature appears in the second-order position. With this intuition, comparing Equations (2) and (4) suggests that the Ricci curvature is a sort of geometric Laplacian. Making this heuristic precise requires a bit of Riemannian geometry, because the sectional curvature $K(X, Y)$ is non-linear in terms of X and Y and it is not clear how to relate these two formulas.

We won't go into too much detail about how to make this idea rigorous. However, the basic idea is that you can use the sectional curvature to construct the *Riemann curvature tensor* (denoted R), which is comparable to a geometric Hessian.[30] The Ricci curvature turns out to be the trace of the Riemann curvature tensor, which gives further credence to its interpretation as a geometric Laplacian.[31]

There is one other important analogy between the usual Laplacian and the Ricci curvature, which is that the Laplacian measures how much the volume of an infinitesimal cube changes under gradient flow. In a similar way, the Ricci curvature has the nice property that it fully determines how the volumes on a curved space differ from volumes in flat space.[32]

[30]It's possible to make this reasonably precise. In particular, in any set of geodesic normal coordinates, the Riemannian metric satisfies

$$g_{ij} = \delta_{ij} - \sum_{k,l=1}^{n} \frac{1}{3} R_{ikjl} x^k x^l + O\left(|x|^3\right).$$

The key intuition of this formula is that it strongly resembles a second-order Taylor polynomial for a multivariate function, and the terms $-\frac{1}{3}R_{ijkl}$ correspond to the second-order terms in the expansion.

[31]There are other ways to make this precise. For instance, in harmonic coordinates,

$$\mathrm{Ric}_{ij} = -\frac{1}{2}\Delta g_{ij} + \text{lower order terms...}$$

[32]There is another curvature tensor, known as the Weyl tensor, which is orthogonal to the Ricci curvature and measures the "tidal forces" of the curvature tensor. In other words, the Weyl tensor determines how the shape of small objects deform when they move along short geodesics.

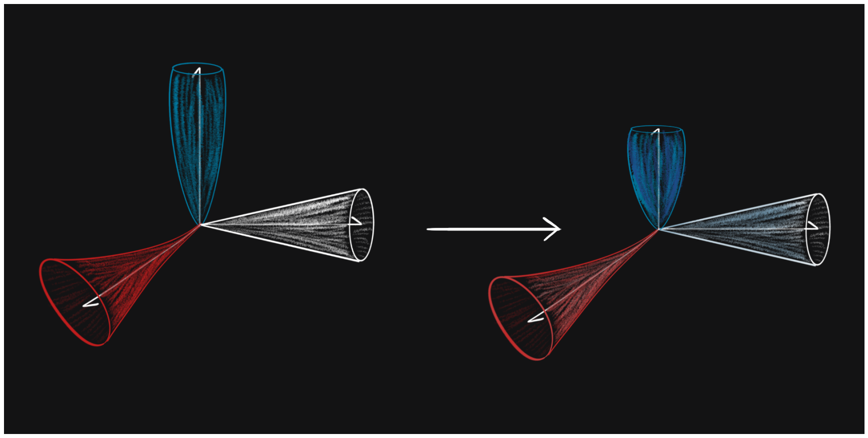

Figure 12. Ricci flow deforming a basis which is initially orthonormal[33]

4. The Ricci Flow

Now that we have discussed both the heat equation as well as the notion of Ricci curvature, we can finally talk about the Ricci flow. To get started, we will provide an informal geometric explanation.

Definition. *The Ricci flow changes the shape of a space proportional to -2 times the Ricci curvature.*

In other words, directions which have negative Ricci curvature get longer whereas directions with positive Ricci curvature get shorter (as depicted in Figure 12). However, there is a major roadblock in defining the Ricci flow more precisely. We have discussed what "curvature" is, but have not given a formal definition for what the "shape" of a space is. In order to do so, we would need to define the notion of Riemannian metric, which are a generalization of the dot product in Euclidean space. However, the technical definition of Riemannian metrics and why they correspond to the "shape" of a space requires some background, so we will not try to give a precise definition.[34]

Instead, we're just going to write out the Riemannian metric as a symbol g and say that it is some geometric object that encodes the "shape" of our

[33]Under Ricci flow, the curvature evolves in a complicated way, which is why the colors of the cones are also changing.

[34]More precisely, given a smooth manifold, a Riemannian metric g is smoothly-varying collection of positive-definite inner products on the tangent space of each point.

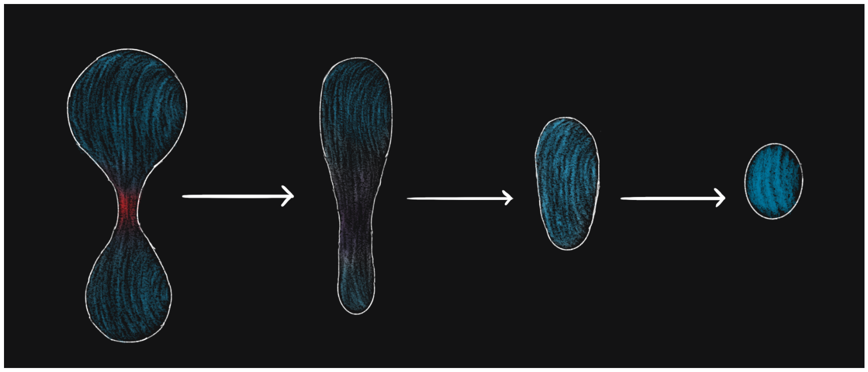

FIGURE 13. The Ricci flow converging to a round sphere[35]

space. Using the metric, the equation for the Ricci flow becomes

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Ric}(g). \tag{6}$$

The first objection you might raise is that our definition of Ricci curvature required that we input a vector (which we wrote, somewhat bizarrely, as two separate entries). In fact, the metric g also requires that we input two vectors, so this is actually part of what makes the Ricci flow work.

The second objection to considering this equation as a heat equation is that this formula has a -2 whereas the heat equation $\frac{\partial u}{\partial t} = \Delta u$ does not. The factor of -2 appears because analysts and geometers cannot decide on the definition of the Laplacian and their conventions differ by a sign.

However, the key observation here is that there is a parallel between this formula and the heat equation $\frac{\partial u}{\partial t} = \Delta u$. We have tried to justify why it is possible to think of the Ricci curvature tensor as being analogous to Δ (it is more like $-\frac{1}{2}\Delta$, but the particular constant is not important), which suggests that the Ricci flow is a heat flow of "shape". As a demonstration of this fact, it's worthwhile to see an example of the Ricci flow in action, which is depicted in Figure 13.

There are also some animations of the flow, which are helpful to gain an intuition for how it behaves. Notice how the surface becomes more and more spherical as time goes on. [Pro] In the same way that the heat equation spreads the heat evenly throughout the space, the Ricci flow spreads the curvature evenly throughout the space.

[35]To be more precise, this flow is converging to a round point, which means that it is shrinking to a point while asymptotically converging to a sphere.

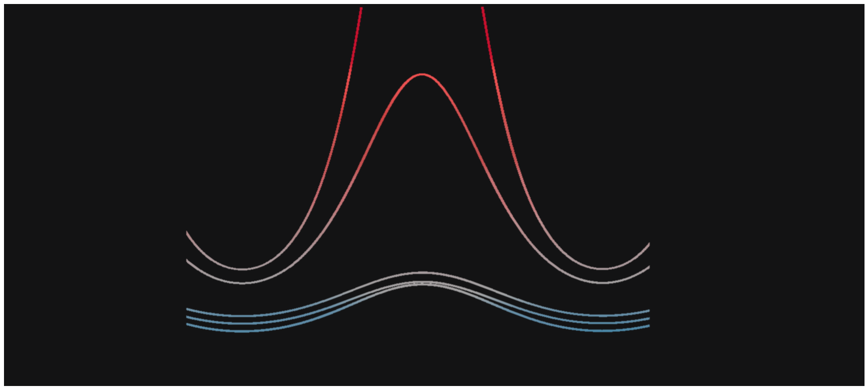

FIGURE 14. A reaction-diffusion equation going to infinity[37]

4.1. **Some caveats.** Going back to our intuition about how heat behaves, we expect that the Ricci flow should smooth out our space and make it more uniform. In reality, the Ricci flow is more complicated than a heat flow. First, it is non-linear (the linear combination of two solutions is no longer a solution) because curvature depends in a highly non-linear way on the geometry of the space. Second, it actually behaves more similarly to the reaction-diffusion equation

$$\frac{\partial}{\partial t}u = \Delta u + u^2. \tag{7}$$

The first term on the right-hand side behaves as a diffusion term that disperses heat throughout the space whereas the second acts a reaction term that concentrates heat at a point. Reaction-diffusion equations can be thought as a tug-of-war between the diffusion process and the reaction process. If diffusion wins, the solution will smooth itself out much like the normal heat equation. If the reaction term wins out, the heat will become more and more intense and can sometimes even become become infinite in a finite amount of time.[36]

With the Ricci flow, a key computation in understanding its behavior shows that the curvature evolves according to the equation

$$\frac{\partial}{\partial t}R = \Delta R + R^2 + R^\sharp. \tag{8}$$

[36]This occurs with the equation $\frac{\partial}{\partial t}u = \Delta u + u^2$ if you start off with a constant positive function $u \equiv C$.

[37]Unlike with the standard heat equation, this equation does not have a closed form solution. As such, I used the Fourier transform to numerically approximate a solution.

It takes some background in Lie algebra to define the terms on the right-hand side of this formula [Wil13], but the intuitive idea is that there is a notable similarity between this equation and Equation (7).

In particular, sometimes the reaction term wins out and the curvature becomes larger and larger until the shape tears itself apart (or shrinks to a point).[38] This is part of what made analysis of the flow so difficult. It is not obvious when singularities occur or what they look like.[39]

5. Conclusion

The Ricci flow remains an active area of research, both as a tool to prove geometric and topological results but also as a topic of interest in its own right. It would be impossible to give a complete overview of the current state of research, but let us mention a few broad areas of interest.

5.1. **The long-term behavior and singularity formation of the Ricci flow.** The singularity formation and convergence of the Ricci flow is a fascinating and difficult topic. There are many open questions about when singularities occur and what their possible geometries are. The most famous open problem in this direction is to determine whether the scalar curvature (i.e., the trace of the Ricci curvature) necessarily becomes infinite at a singularity.[40]

There has also been research into the structure of Ricci flows which are allowed to become singular and the singularities that form under "generic" initial conditions. For details on this line of work, a good reference is the recent survey article by Richard Bamler [Bam21].

5.2. **Geometric classification results.** Apart from the Poincaré conjecture, the Ricci flow has been used to establish other geometric classification results. Prior to Perelman's work, Hamilton had already used Ricci flow to understand the geometry of three-dimensional spaces with positive Ricci

[38] Once the short-time existence of the Ricci flow has been established, it is not hard to show the flow will exist until the sectional curvature goes to plus or minus infinity. As such, a singularity is generally defined to be a point in space-time where the norm of the sectional curvature tensor goes to infinity.

[39] In three dimensions, one of Perelman's key contributions was to classify all the possible structures of singularities and to show that a singularity known as the "cigar soliton" would not develop.

[40] Nataša Šešum showed that at any singularity, the Ricci curvature must become infinite [Šeš05].

curvature [Ham82] and four-dimensional spaces with positive curvature operator [Ham97]. In 2007, Simon Brendle and Richard Schoen proved the Differentiable Sphere Theorem using the Ricci flow [Bre10] and Brendle recently used it to establish a classification of higher-dimensional spaces with positive isotropic curvature [Bre19].

5.3. **Other geometric flows.** The idea of using a heat-type flow to take an arbitrary object and deform it to some canonical configuration predates the Ricci flow, but this approach has exploded in popularity in the four decades since the Ricci flow was first studied. At present, there are many geometric flows studied in the literature: mean curvature flow, harmonic map heat flow, Kähler-Ricci flow, Chern-Ricci flow, pluriclosed flow, anomaly flow, Calabi flow, Yamabe flow...

These flows play a significant role in differential geometry and have many applications, both within pure mathematics and more broadly. There is still much to be said about the behavior of geometric flows, and the field is an active and vibrant area of research.

6. Acknowledgements

This project started as lecture notes for a talk at the "What is...?" seminar at Ohio State University. The seminar was aimed at advanced high school students at the Ross program, who had taken multivariate calculus and linear algebra, but were not expected to know any differential geometry. I would like to thank the organizers for letting me give a talk about a subject well outside the normal purview of the seminar.

Discussing the Ricci flow without assuming any background in differential geometry was a challenge, and I relied on the help of several people to find explanations which avoided discussing Riemannian geometry or PDEs in detail. In particular, thanks to Kori Khan for her helpful suggestions and to Mizan Khan for his help editing the paper. I would also like to thank Frank Nielsen for some corrections.

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