Solve Cubic Equation

$$x^3 + ax^2 + bx + c = 0$$

Let $x = y - \frac{a}{3}$ Use substitute to remove x^2 term

$$(y-\frac{a}{3})^3 + a(y-\frac{a}{3})^2 + b(y-\frac{a}{3}) + c = 0 \qquad \text{Sub } x = y - \frac{a}{3} \text{ into the cubic eqa.}$$

$$\boxed{(y-\frac{a}{3})^2(y-\frac{a}{3})} + ay^2 + a(\frac{a}{3})^2 - a\frac{2a}{3}y + by - (\frac{ab}{3}) + c = 0$$

$$\boxed{(y^2+(\frac{a}{3})^2-\frac{2a}{3}y)(y-\frac{a}{3})} + ay^2 + a(\frac{a}{3})^2 - a\frac{2a}{3}y + by - (\frac{ab}{3}) + c = 0$$

$$\boxed{y^3+(\frac{a}{3})^2y-\frac{2a}{3}y^2-\frac{a}{3}y^2-(\frac{a}{3})^3+2(\frac{a}{3})^2y} + ay^2 + a(\frac{a}{3})^2 - a\frac{2a}{3}y + by - (\frac{ab}{3}) + c = 0$$

$$y^3 - \frac{2a}{3}y^2 - \frac{a}{3}y^2 + ay^2 + (\frac{a}{3})^2y + 2(\frac{a}{3})^2y - 2(\frac{a^2}{3})y + by - (\frac{a}{3})^3 + a(\frac{a}{3})^2 - (\frac{ab}{3}) + c = 0$$

$$y^3 + \left[3(\frac{a}{3})^2 - 6(\frac{a}{3})^2 + b\right]y + \left[-(\frac{a}{3})^3 - (\frac{ab}{3}) + a(\frac{a}{3})^2 + c\right] = 0$$

$$y^3 + \left[b - 3(\frac{a}{3})^2\right]y + \left[c - \frac{a^3}{27} - \frac{9ab}{27} + \frac{3a^3}{27}\right] = 0$$

$$y^3 + \left[b - 3(\frac{a}{3})^2\right]y + \left[c + \frac{2a^3}{27} - \frac{9ab}{27}\right] = 0$$

Where

$$p = b - 3(\frac{a}{3})^2$$

and

$$q = c + 2\frac{a^3}{27} - \frac{9ab}{27}$$

Substitute p, q into equation

$$\boxed{y^3 + py + q = 0}$$

Let $y = z - \frac{p}{3z}$ and substitute into the equation

$$(z-\frac{p}{3z})^3 + p(z-\frac{p}{3z}) + q = 0$$

Expand

$$\left[z^2 - \frac{2p}{3} + (\frac{p}{3z})^2\right](z-\frac{p}{3z}) + p(z-\frac{p}{3z}) + q = 0$$

$$\left[z^3 - \frac{2p}{3}z + (\frac{p}{3z})^2z - \frac{pz}{3} + \frac{2p^2}{9z} - (\frac{p}{3z})^3\right] + pz - \frac{p^2}{3z} + q = 0$$

$$\left[z^3 - \frac{2p}{3}z + \frac{p^2}{9z} - \frac{pz}{3} + \frac{2p^2}{9z} - \frac{p^3}{27z^3}\right] + pz - \frac{p^2}{3z} + q = 0$$

$$z^3 - \frac{p^3}{27z^3} + q = 0 \qquad \text{Simplify}$$

Multiply z^3 in both side

$$z^6 + qz^3 - \frac{p^3}{27} = 0$$

$$(z^3)^2 + qz^3 - \frac{p^3}{27} = 0 \qquad \text{Let } \omega = z^3$$

$$\omega^2 + q\omega - \frac{p^3}{27} = 0 \qquad ax^2 + bx^2 + c = 0 \qquad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad \begin{cases} a = 1 \\ b = q \\ c = -\frac{p^2}{27} \end{cases}$$

$$\omega = \frac{-q \pm \sqrt{q^2 + 4\frac{p^3}{27}}}{2} = -\frac{q}{2} \pm \sqrt{(\frac{p}{3})^3 + (\frac{q}{2})^2}$$

Let $R = (\frac{p}{3})^3 + (\frac{q}{2})^2$

$$\omega = -\frac{q}{2} \pm \sqrt{R}$$

$$z_1^3 \cdot z_2^3 = (-\frac{p}{3})^3 \qquad \because x^2 + bx + c = 0 \qquad x_1 \cdot x_2 = c$$

$$(-\frac{q}{2}+\sqrt{R})(-\frac{q}{2}-\sqrt{R}) = (-\frac{p}{3})^3$$

$$\left(-\frac{q}{2}+\sqrt{R}\right)^{\frac{1}{3}}\left(-\frac{q}{2}-\sqrt{R}\right)^{\frac{1}{3}} = -\frac{p}{3}$$

Let $z = (-\frac{q}{2}+\sqrt{R})^{\frac{1}{3}}$

$$\Rightarrow -\frac{p}{3z} = (-\frac{q}{2}-\sqrt{R})^{\frac{1}{3}} \qquad \because z(-\frac{p}{3z}) = -\frac{p}{3}$$

$$y = z - \frac{p}{3z}$$

$$\Rightarrow y = (-\frac{q}{2}+\sqrt{R})^{\frac{1}{3}} + (-\frac{q}{2}-\sqrt{R})^{\frac{1}{3}} \text{ is the cubic root for } \boxed{y^3 + py + q = 0}$$

$$z^3 = \frac{-q \pm \sqrt{q^2 + 4\frac{p^3}{27}}}{2}$$

$$\boxed{z = \left(\frac{-q \pm \sqrt{q^2 + 4\frac{p^3}{27}}}{2}\right)^{\frac{1}{3}} \quad \text{where} \quad \begin{cases} p = b - 3(\frac{a}{3})^2 \\ q = c + 2\frac{a^3}{27} - \frac{9ab}{27} \end{cases}}$$

Show

$$y = (-\frac{q}{2}+\sqrt{R})^{\frac{1}{3}} + (-\frac{q}{2}-\sqrt{R})^{\frac{1}{3}} \qquad \begin{cases} R &= (\frac{p}{3})^3 + (\frac{q}{2})^2 \\ p &= b - 3(\frac{a}{3})^2 \\ q &= c + 2\frac{a^3}{27} - \frac{9ab}{27} \end{cases}$$

is the root of cubic equation

$$y^3 + py + q = 0$$

$$y^3 + py + q \qquad \text{Sub } y = (-\frac{q}{2}+\sqrt{R})^{\frac{1}{3}} + (-\frac{q}{2}-\sqrt{R})^{\frac{1}{3}}$$

$$= \left[(-\frac{q}{2}+\sqrt{R})^{\frac{1}{3}} + (-\frac{q}{2}-\sqrt{R})^{\frac{1}{3}}\right]^3 + \boxed{p\left[(-\frac{q}{2}+\sqrt{R})^{\frac{1}{3}} + (-\frac{q}{2}-\sqrt{R})^{\frac{1}{3}}\right] + q}$$

$$=(-\frac{q}{2}+\sqrt{R}) + 3(-\frac{q}{2}+\sqrt{R})^{\frac{2}{3}}(-\frac{q}{2}-\sqrt{R})^{\frac{1}{3}} + 3(-\frac{q}{2}+\sqrt{R})^{\frac{1}{3}}(-\frac{q}{2}-\sqrt{R})^{\frac{2}{3}} + (-\frac{q}{2}-\sqrt{R}) + \boxed{p\left[(-\frac{q}{2}+\sqrt{R})^{\frac{1}{3}} + (-\frac{q}{2}-\sqrt{R})^{\frac{1}{3}}\right] + q}$$

$$=-q + 3(-\frac{q}{2}+\sqrt{R})^{\frac{1}{3}}(-\frac{q}{2}-\sqrt{R})^{\frac{1}{3}}\left[(-\frac{q}{2}+\sqrt{R})^{\frac{1}{3}} + (-\frac{q}{2}-\sqrt{R})^{\frac{1}{3}}\right] + p\left[(-\frac{q}{2}+\sqrt{R})^{\frac{1}{3}} + (-\frac{q}{2}-\sqrt{R})^{\frac{1}{3}}\right] + q$$

$$= \left[(-\frac{q}{2}+\sqrt{R})^{\frac{1}{3}} + (-\frac{q}{2}-\sqrt{R}^{\frac{1}{3}})\right]\left[3(-\frac{q}{2}+\sqrt{R})^{\frac{1}{3}}(-\frac{q}{2}-\sqrt{R})^{\frac{1}{3}} + p\right]$$

$$= \left[(-\frac{q}{2}+\sqrt{R})^{\frac{1}{3}} + (-\frac{q}{2}-\sqrt{R}^{\frac{1}{3}})\right]\left[3\left((\frac{q}{2})^2 - R\right)^{\frac{1}{3}} + p\right] \quad \text{where} \quad R = (\frac{p}{3})^3 + (\frac{q}{2})^2$$

$$= \left[(-\frac{q}{2}+\sqrt{R})^{\frac{1}{3}} + (-\frac{q}{2}-\sqrt{R}^{\frac{1}{3}})\right]\left[3\left((\frac{q}{2})^2 - (\frac{q}{2})^2 - (\frac{p}{3})^3\right)^{\frac{1}{3}} + p\right] \quad \text{where} \quad R = (\frac{p}{3})^3 + (\frac{q}{2})^2$$

$$= \left[(-\frac{q}{2}+\sqrt{R})^{\frac{1}{3}} + (-\frac{q}{2}-\sqrt{R}^{\frac{1}{3}})\right]\left(3(-\frac{p}{3}) + p\right)$$

$$= \left[(-\frac{q}{2}+\sqrt{R})^{\frac{1}{3}} + (-\frac{q}{2}-\sqrt{R}^{\frac{1}{3}})\right]0$$

$$=0$$

Solve the following cubic equation

$$(x-1)(x-2)(x-3) = 0 \tag{1}$$

$$(x-1)(x-2)(x-3) = 0$$

$$(x^2-3x+2)(x-3) = 0$$

$$x^3 - 3x^2 + 2x - 3x^2 + 9x - 6 = 0$$

$$x^3 - 6x^2 + 11x - 6 = 0 \qquad \begin{cases} a = -6 \\ b = 11 \\ c = -6 \end{cases}$$

Let $y = x - \frac{a}{3}$ Sub into eqa. (1)

$$y^3 + \left[b - 3(\frac{a}{3})^2\right]y + \left[c + \frac{2a^3}{27} - \frac{9ab}{27}\right] = 0 \qquad \begin{cases} p &= b - 3(\frac{a}{3})^2 = 11 - 12 = -1 \\ q &= c + \frac{2a^3}{3^3} - \frac{9ab}{3^3} = -6 - \frac{2\cdot 6\cdot 36}{3^3} - \frac{9\cdot(-6)11}{3^3} = -6 - 16 + 22 = 0 \end{cases}$$

$$R = (\frac{p}{3})^3 + (\frac{q}{2})^2$$

$$R = (\frac{-1}{3})^3$$

$$y = z - \frac{p}{3z}$$

$$y = (-\frac{q}{2}+\sqrt{R})^{\frac{1}{3}} + (-\frac{q}{2}-\sqrt{R})^{\frac{1}{3}}$$

$$y = (-\frac{0}{2}+\sqrt{R})^{\frac{1}{3}} + (-\frac{0}{2}-\sqrt{R})^{\frac{1}{3}}$$

$$y = \sqrt{\frac{-1}{3}} - \sqrt{\frac{-1}{3}} = 0$$

$$x = y - \frac{a}{3}$$

$$x = 0 - \frac{-6}{3}$$

$$\Rightarrow x = 2 \qquad \text{is the root of equation}$$