3 Limits of Sequences and Filters

The Axiom of Choice is obviously true, the well-ordering theorem is obviously false; and who can tell about Zorn's Lemma?
—Jerry Bona (Schechter, 1996)

Introduction. Chapter 2 featured various properties of topological spaces and explored their interactions with a few categorical constructions. In this chapter we'll again discuss some topological properties, this time with an eye toward more fine-grained ideas. As introduced early in a study of analysis, properties of nice topological spaces X can be detected by sequences of points in X. We'll be interested in some of these properties and the extent to which sequences suffice to detect them. But take note of the adjective "nice" here. What if X is any topological space, not just a nice one? Unfortunately, sequences are not well suited for characterizing properties in arbitrary spaces. But all is not lost. A sequence can be replaced with a more general construction—*a filter*—which is much better suited for the task. In this chapter we introduce filters and highlight some of their strengths.

Our goal is to spend a little time inside of spaces to discuss ideas that may be familiar from analysis. For this reason, this chapter contains less category theory than others. On the other hand, we'll see in section 3.3 that filters are a bit like functors and hence like generalizations of points. This perspective thus gives us a coarse-grained approach to investigating fine-grained ideas. We'll go through some of these basic ideas—closure, limit points, sequences, and more—rather quickly in sections 3.1 and 3.2. Later in section 3.2 we'll see exactly why sequences don't suffice to detect certain properties in all spaces. We'll also discover those "nice" spaces for which they do. Section 3.3 introduces filters with some examples and results about them. These results include the claim that filters, unlike sequences, do suffice to characterize certain properties. Finally, in section 3.4 we'll use filters to share a delightfully short proof of Tychonoff's theorem.

3.1 Closure and Interior

Here are a few basic definitions, which may be familiar from analysis. Given any subset B of a space X, two topological constructions suggest themselves. There is the *closure* $\overline{B}$ which is the smallest closed set containing B, and there is the *interior* B° which is the largest open set contained in B. When $\overline{B} = X$, we say B is *dense* in X. If $\left(\overline{B}\right)^\circ = \varnothing$, we say B is *nowhere dense*.