Extremely Advanced Calculus: Multivariable Analysis, Vectors, Forms, Metric

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Preface

The book

Here is a quick summary of the book. It is about differential and integral calculus in several variables. The first part on Analysis is mainly about calculus on an open subset of $\mathbf{R}^n$. The second part on Smooth Geometry is about calculus on a manifold modeled on an open subset of $\mathbf{R}^n$. Initially the manifold has no additional structure. This part has an emphasis on how to draw pictures. not only of vector fields, but also of differential forms and twisted differential forms. The third part on Metric Geometry continues with calculus on the same kind of manifold, but includes an additional structure: the metric tensor. Throughout the book the effort is to blend theoretical insight with practical calculation. The level of rigor should be adequate for a book at this level, but the ultimate goal is insight.

The title of the book is chosen to distinguish its content from real analysis: the Lebesgue theory of integration of measurable functions. The subject is calculus, that is, the theory of integration for smooth functions, for which the central result is the fundamental theorem of calculus and its generalizations. This is a sophisticated account, not the routine advanced calculus text aimed at presenting technique to disinterested students. The audience is readers who care to understand the subject, both its mathematical theory and its utility in applications.

Calculus with one variable is the workhorse of pure and applied mathematics. Calculus with several variables is a natural extension, but the geometry is more complicated. This leads to a richer theory where the role of differentials is much more explicit. Furthermore, the chain rule, the change of variable formula, and the fundamental theorem of calculus generalize in quite different ways. Here are examples. For the moment they are just landmarks, with details to be explained later.

- Vector fields are not at all like differential forms. Vector fields are represented by arrows; differential forms may often be represented by broken curves.

- The chain rule leads to pullback of differential forms and pushforward of vector fields.

- The change of variable in integrals becomes integration of differential forms.
- The fundamental theorem of calculus generalizes as Stokes' theorem.
- Some important operations are independent of metric ideas; the most striking example is the exterior derivative acting on differential forms.
- Other operations need a notion of volume; the standard example is divergence.
- Others require a metric; these include gradient and Laplacian, as well as the general covariant derivative.

The book originated in senior level courses in analysis at the University of Arizona. It took something more like the present form in lectures at NYU Shanghai (Fall 2014 and Fall 2018) for an advanced undergraduate course in multivariable analysis. Its purpose is to develop this subject in a framework that makes sense simultaneously for the purposes of both pure and applied mathematics. It presents intrinsic geometry (with pictures!) along with coordinate calculations, and it uses terminology that is appropriate to both contexts. While it could be used as a textbook, the intended audience is more general: ambitious undergraduate students, puzzled graduate students, curious mathematicians (pure and applied), and physicists—anyone who wants to make better sense of this sometimes confusing subject.

There are three main topics, Analysis, Smooth Geometry, and Metric Geometry. The Analysis part is about differentiation and integration for functions of several variables. It covers various topics relevant to the Riemann integral (including the dominated convergence theorem), but it does not attempt to treat the Lebesgue integral. The Smooth Geometry part treats vector fields and differential forms and their simplest interaction, that is, duality. It also describes the notion of volume element and its role in defining divergence. The Metric Geometry part deals with more complicated interactions between vector fields and differential forms needed to treat such objects as gradient and Laplacian. Advanced topics such as curvature get brief mention, and there are only hints of general tensor calculus. The Analysis and Geometry parts are conceptually related, but in spirit they are very different. To some extent they may be read independently.

The most important innovations are in the Smooth Geometry part. There are two possible approaches.

- The geometry of applied mathematics attempts to describe reality; it uses calculus with the coordinate system that is convenient for the application.
- The geometry of manifolds describes mathematical structures that depend on smoothness; it uses calculus in a version that either avoids coordinates or that works with every coordinate system.

An organizing principle of the book is that that these approaches are not only close to each other, but also that it is easy to translate back and forth between them. Throughout, the geometry is local geometry. This means that the focus is on small regions that may be described by a single coordinate system. Global geometry is beyond the scope of this account.

There are two prerequisites for this book. The first is a solid understanding of rigorous analysis (at least in one dimension) and of metric space ideas. The second is a background in linear algebra. The level of the book corresponds roughly to the the final undergraduate year in a university in the USA. It may also be useful for students beginning graduate study in mathematics or in various areas of applied mathematics. More generally, it should serve as a bridge between the world of geometric insight and the world of concrete calculations. The ultimate purpose is to give a useful picture of the landscape of this subject for any interested reader.

Analysis

Differentiation

The central object in this chapter is a smooth (that is, sufficiently differentiable) *numerical function* $\mathbf{f}$ defined on an open subset U of $\mathbf{R}^k$ with values in $\mathbf{R}^n$. Here k is the domain dimension and n is the target dimension. The function is called "numerical" to emphasize that both inputs and outputs involve numbers. (See below for other kinds of functions.) Sometimes a numerical function is denoted by an expression like $\mathbf{y} \mapsto \mathbf{f}(\mathbf{y})$. The choice of the variable name $\mathbf{y}$ is arbitrary.

When $k < n$ a numerical function $\mathbf{f}$ from an open subset U of $\mathbf{R}^k$ to $\mathbf{R}^n$ can give a *explicit* (parametric) representation of a k-dimensional surface in $\mathbf{R}^n$. Also, a numerical function $\mathbf{g}$ from an open subset W of $\mathbf{R}^n$ to $\mathbf{R}^{n-k}$ can give an *implicit* representation of a family of k-dimensional surfaces in W. The *implicit function theorem* gives conditions for when an implicit representation gives rise to an explicit representation.

When $k = n$ the function $\mathbf{f}$ may define a transformation from an open subset U of $\mathbf{R}^n$ to an open set V in $\mathbf{R}^n$. The *inverse function theorem* gives conditions that ensure that the transformation has an inverse transformation.

The multivariable *chain rule* says that the derivative of a composite function $(\mathbf{h} \circ \mathbf{g})(\mathbf{x}) = \mathbf{h}(\mathbf{g}(\mathbf{x}))$ is given by

$$(\mathbf{h} \circ \mathbf{g})'(\mathbf{x}) = \mathbf{h}'(\mathbf{g}(\mathbf{x}))\mathbf{g}'(\mathbf{x}). \tag{1}$$

These derivatives are matrices, and the product on the right is matrix multiplication.

Integration

This chapter is about the Riemann integral for functions of several variables. The *Fubini theorem* for Riemann integrals deals with iterated integrals. The

dominated convergence theorem for Riemann integrals is a result about pointwise convergence.

The *change of variables formula* states that

$$\int_T h(\mathbf{g}(\mathbf{x}))|\det \mathbf{g}'(\mathbf{x})|\,d\mathbf{x} = \int_{\mathbf{g}(T)} h(\mathbf{y})\,d\mathbf{y}. \tag{2}$$

Here $\mathbf{g}'(\mathbf{x})$ is a square matrix. The change of variables formula has an elegant proof via approximate delta functions and the dominated convergence theorem. This result is fundamental to the way integration is employed in the following chapters. Another related topic is fiber integration, which generalizes the Fubini theorem. This is illustrated by an application to probability.

The chapter includes formulas for *surface area*. A central idea is an amazing generalization of the theorem of Pythagoras. The classical theorem says that the length of a vector is the square root of the sum of squares of the lengths of its projections onto the coordinate axes. One case of the generalization says that the area of a parallelogram is the square root of the sum of the squares of the areas of its projections onto coordinate planes. The usual area formula works for an explicitly defined surface (that is, a parameterized surface). Another approach that works for a family of implicitly defined surfaces is based on the *coarea* formula.

Matrix theory

The material on matrix theory is a supplement to the analysis part.

Smooth Geometry

Vector Spaces and Modules

The material on vector spaces reviews fundamental linear algebra material. A brief section on modules introduces useful terminology. A module is a generalization of the vector space concept that applies in particular to vector fields and to differential forms.

Vector Fields

The chapter introduces a fundamental organizing principle of mathematical modeling that applies to geometry and to most applications of mathematics. In the approach of this book it is based on the elementary concept of an n-dimensional *manifold patch*. This term is used here for a differentiable manifold M modeled on some open subset of $\mathbf{R}^n$. In more detail, M is a set, and there are specified one-to-one functions from M onto open subsets of $\mathbf{R}^n$ called *coordinate systems*. Suppose $\mathbf{u} = (u_1, \ldots, u_n) : M \to U$ is a coordinate system, and suppose that $\mathbf{w} = (w_1, \ldots, w_n) : M \to W$ is another coordinate system, with $\mathbf{u} = \mathbf{g}(\mathbf{w})$. It is required that $\mathbf{g} : W \to U$ be a smooth numerical function with smooth inverse. There is no preferred coordinate system. The modeling process may employ several coordinate systems in the same discussion.

The general philosophy of modeling has two components:

- Variables are meaningful. In a problem involving pressure, volume, and temperature the variables should have meaningful names: p, V, T.

- They obey "variable democracy", that is, there is no preferred choice of independent variables.

Given two manifold patches K and M there is a concept of *manifold mapping* $\phi : K \to M$. The treatment in the book gives a practical notation that may be used in computations. Suppose K has coordinate system $\mathbf{t} = (t_1, \ldots, t_k)$ and M has coordinate system $\mathbf{u} = (u_1, \ldots, u_n)$. Then there is a smooth numerical function $\mathbf{f}$ such that the manifold map may be represented in the form

$$\phi = (\mathbf{u} \leftarrow \mathbf{f}(\mathbf{t})). \tag{3}$$

What this means is: take the input point in K, find its coordinate values using $\mathbf{t}$, apply the function $\mathbf{f}$ to these values to get new values, and finally define the output point in M to be the point that has these values as $\mathbf{u}$ coordinates.

When the two manifold patches are the same M, there is another important distinction. A *passive transformation* is a change of coordinates from $\mathbf{t}$ to $\mathbf{u} = \mathbf{f}(\mathbf{t})$. The manifold mapping $\mathbf{u} \leftarrow \mathbf{f}(\mathbf{t})$ is the identity mapping that sends every point in M to itself. It gives an alternative description of the same situation. An *active transformation* is a change in the object being modeled. For a given coordinate system $\mathbf{u}$ on M the transformation is $\mathbf{u} \leftarrow \mathbf{f}(\mathbf{u})$. The distinction between passive and active transformations is widely recognized but not always made explicit. (Some authors use terminology in which "passive" and "active" are replaced by the awkward terms "alias" and "alibi".)

A *scalar field* is a smooth function $s : M \to \mathbf{R}$ from the manifold patch to the reals. If $\mathbf{u} = (u_1, \ldots, u_n) : U \to \mathbf{R}^n$ is a coordinate system, then $s = f(\mathbf{u}) = f(u_1, \ldots, u_n)$ for some smooth numerical function $f : U \to \mathbf{R}$. A scalar field is usually pictured in terms of its contour curves (or contour surfaces).

A *vector field* is often pictured as a field of arrows. The arrow at a particular point describes how to differentiate a scalar field at that point. It is natural to think of a vector field as way of differentiating scalar fields to get new scalar fields. For instance, in one application a vector field is thought of as a fluid velocity. The corresponding linear differential operator is called the advective derivative.

A vector field is equivalent to an autonomous system of ordinary differential equations. (This means a system of ordinary differential equations expressing rate of change with respect to a time parameter via a rule that does not depend on time.) A point at which the vector field vanishes is a stationary solution for the system. The behavior of the solutions near such a point is influenced by (but not always determined by) the linearization of the vector field at the point.

Remark: The notation used for a vector field X relative to a particular

coordinate system $u_1, \ldots, u_n$ is

$$X = a_1 \frac{\partial}{\partial u_1} + \cdots + a_n \frac{\partial}{\partial u_n}. \tag{4}$$

The coefficents a_j are smooth functions representing the coefficients of the vector field, and the partial differential operators $\frac{\partial}{\partial u_j}$ are the basis vectors. These are the coordinate basis vectors. While this notation is unfamiliar to some, it has the remarkable advantage that changes of coordinates are instances of the chain rule. Furthermore, in the special case when it makes sense to talk of an orthogonal coordinate systems, the alternative expansion of the vector field in terms of normalized basis vectors follows immediately. A clear distinction between these two equivalent representations of vector fields is a major step in making sense of various formulas in the literature.

Differential Forms

The concept of scalar field also leads to the concept of *exact differential 1-form*. If s is a scalar field, the corresponding exact differential 1-form is written ds. A general *differential 1-form* is a linear combination of products of scalar fields with exact differential 1-forms. Differential forms are intrinsic expressions of change, and thus the basic mechanism of calculation with differential forms is equally simple for every possible choice of coordinate system. (For modeling a particular system one coordinate system may be more convenient than another.)

A physical example of an exact differential is a static electric field $E = E_1\,dx + E_2\,dy + E_3\,dz$, considered as a differential 1-form. There is an electric potential ϕ that is a scalar field, and $E = -d\phi$. If ϕ is measured in volts, then E is also measured in volts. The coefficients E_1, E_2, E_3 are measured in volts per meter. The integral of the electric field E over a path from one point to another is the voltage difference (the difference of the values of the scalar ϕ at the two points).

A vector field is a very different object from a differential 1-form. For instance, near every point for which the vector field is non-zero it is possible to find a new coordinate system with respect to which the vector field is constant. This is not true for differential forms; this is related to the fact that a differential form may or may not be exact. Vector fields are easy to picture with arrows. Differential forms are pictured in a quite different way. A 1-form determines a field of $(n-1)$-dimensional subspaces. In a discrete approximation they are stacked so that close spacing indicates a large form. If the form is a scalar field times an exact form, then these spaces mesh together to make $(n-1)$-dimensional surfaces, possibly with bounaries.

There is a general concept of *pullback* of a scalar field s or a differential form by a manifold mapping. In the case of a scalar field this is just composition. That is, if the scalar field

$$s = g(\mathbf{u}) : M \to \mathbf{R}$$

is composed with the manifold mapping $\phi = \mathbf{u} \leftarrow \mathbf{f}(\mathbf{t}) : K \to M$, then the result is

$$s \circ \phi = g(\mathbf{u})(\mathbf{u} \leftarrow \mathbf{f}(\mathbf{t})) = g(\mathbf{f}(\mathbf{t})) : K \to \mathbf{R}.$$

The notational device in the above equation is precise and convenient. It is also important as substantive mathematics: pullback together with change of variable is the key to integration of differential forms.

The pullback of a differential 1-form by a manifold mapping is another differential 1-form. There is also a concept of *pushforward* of a vector field by a manifold mapping. This is not a vector field, but it is a more general object, a *vector field along a mapping*. When the manifold mapping defines a parameterized surface, then a corresponding vector field along the mapping describes change tangent to the surface.

Differential forms provide a natural framework for optimization. If s a scalar field, then at a critical point $ds = 0$. More generally, suppose that s restricted to an implicitly given regular surface has a critical point. The *Lagrange multiplier* condition says that ds is a linear combination of the differentials of the scalar fields that define the surface. In other words, the only possible change in s is by relaxing the constraints that define the surface.

Remark: The notation for a differential 1-form

$$\omega = \sum_{k=1}^{n} p_k \, du_k \tag{5}$$

is parallel to the notation above for vector fields. The coefficients p_k are smooth functions representing the components of the form, and the du_k are the coordinate basis forms. In the special case of orthogonal coordinates it is also possible to expand in terms of normalized basis forms.

Multilinear Differential Forms

The central notion of this chapter is that of a *differential k-form* in an n dimensional space. The possible values of k range 0 to n. Such forms admit algebraic operations, such as the *exterior product*, and calculus operations, including the *differential*. The differential of a $(k-1)$-form ω is a k-form $d\omega$.

The main result is *Stokes' theorem* for a $(k-1)$-form ω. In one version it relates the integral of $d\omega$ over an k-dimensional oriented surface W to the integral of ω over its $(k-1)$-dimensional oriented boundary ∂W. It states that

$$\int_W d\omega = \int_{\partial W} \omega. \tag{6}$$

This is the ultimate expression of the fundamental theorem of calculus

The ordinary fundamental theorem of calculus (the case $n = 1$, $k = 1$) and its generalization to Stokes' theorem make no reference to notions of length and area; they simply describe the cumulative effect of small changes. Because of this intrinsic nature, the expression of Stoke's theorem is the same in every coordinate system.

There are various useful pictures associated with differential forms. One important special situation is that of a $(n-1)$-form ω. In a discrete approximation such a form is pictured by a family of (circulatory) *flux curves*. These curves have a transverse orientation, representing a kind of circulation around the curve. These curves may have (oriented) end points, and the cloud of such n-points represents the n-form $d\omega$. The *Gauss theorem (form version)* is a special case of the general Stokes' theorem. It says that the end points are the sources of the (circulatory) flux curves, so the total (oriented) source inside a region is equal to the (circulatory) flux within the boundary of the region.

Volume Element

There is also a related notion of *twisted differential k-form*. Such a form resembles a k-form, but has a $\pm$ sign determined by an orientation of space. This apparently small modification in the definition makes a considerable difference in the geometric picture and the associated physical interpretation. Twisted differential forms are almost as nice as differential forms—except for their awkward properties under pullback.

A twisted $(n-1)$-form is pictured by a family of (transport) flux curves. These curves have an orientation along the direction of the curve, representing the transport of some substance. The curves may have (signed) end points representing the twisted n-form obtained via the exterior derivative. The *Gauss theorem (twisted form version)* says that the end points are sources of the flux curves, so the total (signed) source inside a region is equal to the (transport) flux through the boundary of the region. This twisted $(n-1)$-form version of the Gauss theorem is a useful framework for *conservation laws*.

Electrostatics again gives a nice example. An electric field E may be considered as a 1-form. There is a closely related quantity D called the electric flux density. It is most naturally regarded as a twisted 2-form $D = D_1\,dy\,dz + D_2\,dz\,dx + D_3\,dx\,dy$. It is expressed in coulombs; the coefficients D_1, D_2, D_3 are in coulombs per square mater. The differential $dD = R$ is a twisted 3-form $R = \rho\,dx\,dy\,dz$. This is also measured in coulombs, while the charge density ρ is measured in coulombs per cubic meter. It follows from this divergence theorem that the charge inside a region is the electric flux through the boundary of the region.

In n dimensions a never-zero n-form is called a *volume form*. There is an associated never-zero twisted n-form $\mathrm{vol}|_+ \geq 0$ called a *volume element*. One can consider a situation where there is a given volume element as part of the mathematical structure. Then there is an associated *divergence* operation that sends vector fields to scalar fields, defined by

$$\mathrm{div}(X)\mathrm{vol}|_+ = d(X \lrcorner \mathrm{vol}|_+). \tag{7}$$

Here the (translport) flux $X \lrcorner \mathrm{vol}|_+$ is a twisted $(n-1)$-form resulting from the *interior product* of the vector field X with the volume element. The *divergence theorem* is a reformulation of the Gauss theorem (twisted form version) in terms

of vector fields. The twisted form picture is in some ways nicer, because it makes clear where the sources are located.

Metric Geometry

Affine and Euclidean Space

In an affine space there are translation vectors. These make it possible to differentiate scalar fields, vector fields, and differential forms in a straightforward manner. There is a covariant differential ∇ that accomplishes this in all cases. The problem is that in coordinate systems that are not Cartesian the basis vectors also need to be differentiated, and so the resulting formulas are complicated. One nice feature, however, is that the formulas carry over to the more general situation when there is the structure of a metric tensor.

A Euclidean space is not only an affine space, but there is also a notion of length of a vector. In this setting there is a conventional vector calculus that is a partial translation of the calculus of differential forms. This translation is explained in detail. There are some new features, such as the gradient operator. Again, the story carries over to the case when there is a metric tensor.

Metric Tensor

Some texts on multivariable analysis treat differential forms, but do not explain how they are related to notions of length, area, and volume. They make the connection only in Cartesian coordinates, or only in the three-dimensional situation. But such metric notions are unavoidable; the Laplace operator is a central object that makes full use of them. The treatment in this book gives an account that describes the situation in any number of dimensions.

The central object in the chapter is the *metric tensor*. The metric tensor on Euclidean space is rather simple, but it is possible to pull back a metric tensor to get another metric tensor, and the result can be quite complicated. One consequence of having a metric tensor is that there is an associated volume density. In particular, there is a well-defined operation of taking the divergence of a vector field.

The definition of *gradient* requires the use of the metric tensor. The *Laplace operator* is then the gradient followed by the divergence. With a metric tensor comes more elaborate formulas. Conventional vector calculus works with vector fields in two or three dimensional space. In this book the standard formulas for gradient and divergence and the Laplace operator are formulated so that they apply with arbitrary metric tensor and in arbitrary coordinates. There is also a systematic discussion of the curl and its variants in dimensions two and three.

In calculations there is always a choice of whether to use *coordinate bases* or *orthonormal bases*. The advantages of both appear when it is possible to use *orthogonal coordinates*. The vector field and differential form notations used in this book make the conversion between these two basis conventions essentially automatic.

Differential and Codifferential

Conventional vector algebra and calculus are very convenient and useful, but certain concepts, such as the cross product and the curl, are special to two or three dimensions. Furthermore, there is a proliferation of formulas that do not seem to fall into a meaningful pattern. There is a much more general theory of tensor algebra and tensor calculus that remedies these defects, but it is complicated enough to deserve a book of its own.

Fortunately, there is a fragment of tensor algebra and tensor calculus that helps to give a perspective on how things work in arbitrary dimensions and is also very useful. In the version used by mathematicians the fundamental objects are differential forms. This choice reflects the important role of Stokes' theorem.

In the general situation when there is no metric tensor the fundamental algebraic and calculus operations are the exterior product $\wedge$ and the differential d. In the presence of a metric tensor there is a new algebraic operation, the *metric interior product* $\rfloor$, and there is a new calculus operation, the *codifferential* δ.

The combination of the exterior product and the metric interior product gives an algebraic system that does in any number of dimensions what the conventional theory does in three dimensions. Furthermore, the operations make geometric sense. Finally, they rely on one fundamental calculational principle.

The combination of the differential and the codifferential give rise to a theory that generalizes all the operations of conventional vector calculus. In particular, there is a Laplace operator that acts on differential forms. In dimensions two and three there is as a systematic translation between the classical vector algebra and calculus and this more general theory. This translation is presented in considerable detail.

The differential and codifferential are fundamental operators on differential forms, and they are the natural setting for various calculations. Sometimes they are too special, and what is needed is a more general notion of derivative of a vector field or of a differential form. In the case of Euclidean space this can be accomplished by keeping track of how the basis vectors for a particular coordinate system turn as they move in space. In the case of a general metric tensor there is also such a notion, the *covariant differential* ∇. This chapter presents some of this theory, enough to give an idea of the mechanism that makes it work. The covariant differential acts on a vector field by the product rule:

$$\nabla X = \nabla \left(\sum_j a_j \frac{\partial}{\partial u_j} \right) = \sum_j \nabla a_j \frac{\partial}{\partial u_j} + \sum_j a_j \nabla \frac{\partial}{\partial u_j}, \tag{8}$$

where

$$\nabla \frac{\partial}{\partial u_j} = \sum_i \sum_k du_i \Gamma^k_{ij} \frac{\partial}{\partial u_k} \tag{9}$$

describes the differential change of the basis vector $\partial/\partial u_j$ in terms of other coordinate basis vectors $\partial/\partial u_k$. The coefficients Γ^k_{ij} are determined by the metric tensor in a straightforward way.

The chapter concludes with a discussion of the covariant differential ∇ acting on differential forms. There is a wonderful formula for the codifferential in terms of the metric interior product and the covariant derivative. The ∇ operator unifies everything in the subject. It may be used in two different ways to define two Laplace operators, each seeming quite natural. In curved space they are not the same. Their relationship is captured in the Weitzenböch formula, the last topic in the book.

General references

For mathematics at this level it is helpful to see multiple approaches. The course used Rudin [42] as an alternative text; it focused on the two chapters on Functions of Several Variables and on Integration of Differential Forms. The book by Spivak [45] gives a treatment at about the same level as Rudin. Flanders [14] presents differential forms along with many applications. The book of Jänich [22] has a straightforward account. The typewritten notes by Nickerson, Spencer, and Steenrod [37] are available again; it would be valuable to have an edition with proper typesetting. The books by Burke [6] and Frankel [15] are among the relatively few sources for information about twisted forms (pseudoforms).

For a more advanced version of the story, the reader may consult Morita [33]. The books by Barden and Thomas [4] and by Agricola and Friedrich [2] could also be useful. The subject matter considered here overlaps with tensor analysis. The book by Lovelock and Rund [25] has a relatively traditional approach with emphasis on tensors as indexed quantities that change with a change of coordinates. The notes by Nelson [34] presents a radically different approach: tensors are indexed quantities, but the indices are defined in such a way that the theory is global and independent of a coordinate representation.

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In these lectures the functions under consideration generally have a smoothness property, so the tools of differential calculus apply. This contrasts with the situation in real analysis and Lebesgue integration, where the functions are generally required only to be measurable. The differentiable situation deals with a greater variety of situations, and it is harder to produce a coherent exposition. It will be no surprise if the present version of these lectures is imperfect. The

author would be grateful to readers who signal typographical glitches, technical mistakes, conceptual errors, and expositional obscurities. There are few sources where this material is presented in a unified account, so it is worth trying to get it right.

Notation and terminology are always a problem; a few non-standard terms had to be introduced. In the index these are indicated by †.

Part I

Analysis

Chapter 1

Differentiation

1.1 Fixed point iteration (single variable)

1.1.1 The contraction mapping theorem

The foundational results in this chapter are the implicit function theorem and the inverse function theorem. Each of these depends on the fact that a certain equation has a unique solution. A technique for proving that an equation has a solution is fixed point iteration. That is the topic that begins these notes.

Let X be a metric space. Let $g : X \to X$ be a function from the space to itself. A *fixed point* is a point x^* in X with $g(x^*) = x^*$.

Take x_0 in X. Define the sequence of points by *iteration*. That is, define x_n in X by

$$x_{n+1} = g(x_n) \tag{1.1}$$

for $n \geq 0$. The set of iterates starting at x_0 is called the *orbit* of x_0.

Proposition 1.1 *Suppose that $g : X \to X$ is continuous. Let x_n be the sequence of iterates starting at x_0. Suppose that $x_n \to x^*$ as $n \to \infty$. Then x^* is a fixed point.*

Example: As an example, take X to be the real line with $g(x) = \cos(x)$. Start with an arbitrary real number. Then the iterates converge to a fixed point x^* that is equal to 0.739 in the first three decimal places. This is an experiment that is easy to do with a scientific calculator. ||

When X is the real number line there is a lovely way of picturing the function and the iteration process. The function g is given by its graph $y = g(x)$. The fixed points are where the graph crosses the diagonal $y = x$. The iterates are the sequence x_n defined by a given x_0 and by $x_{n+1} = g(x_n)$. These are plotted on the diagonal $y = x$ as pairs (x_n, x_n). The iteration is given by $(x_n, x_n) \mapsto (x_n, x_{n+1}) \mapsto (x_{n+1}, x_{n+1})$.

In this picture each iterate is represented by a point (x, x) on the diagonal. The next iterate is obtained by drawing the vertical line from (x, x) to $(x, g(x))$ and then the horizontal line from $(x, g(x))$ to $(g(x), g(x))$. The process is repeated as many times as is needed to show what is going on.

In the following we need stronger notions of continuity. A function $g : X \to X$ is *Lipschitz* if there is a constant c with $0 \leq c < +\infty$ such that for all x and y in X we have

$$d(g(x), g(y)) \leq cd(x, y). \tag{1.2}$$

A Lipschitz function is automatically continuous, in fact, it is even uniformly continuous. If the constant $c \leq 1$ then the function is called a *contraction*. If the constant $c < 1$, then the function is called a *strict contraction*.

Theorem 1.2 (Contraction mapping theorem) *Let X be a complete metric space. Let $g : X \to X$ be a function. Suppose that g is a strict contraction. Then g has a unique fixed point x^*. Furthermore, if x_0 is in X, then the corresponding sequence of iterates satisfies $x_n \to x^*$ as $n \to \infty$.*

Proof: There can only be one fixed point. Suppose $x = g(x)$ and $y = g(y)$ are fixed points. Then $d(x, y) \leq d(x, g(x)) + d(g(x), g(y)) + d(g(y), y) = d(g(x), g(y)) \leq cd(x, y)$. Since $c < 1$ we must have $d(x, y) = 0$, so $x = y$. This proves the uniqueness of the fixed point.

The existence of the fixed point is shown via iteration. Start with x_0 and define the corresponding iterates x_n. Then $d(x_{n+1}, x_n) \leq c^n d(x_1, x_0)$. Hence for $m > n$

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n) \leq [c^{m-1} + \cdots + c^n] d(x_1, x_0). \quad (1.3)$$

Hence

$$d(x_m, x_n) \leq [c^{m-n-1} + \cdots + 1] c^n d(x_1, x_0) \leq \frac{c^n}{1-c} d(x_1, x_0). \quad (1.4)$$

This shows that the x_n form a Cauchy sequence. Since the metric space is complete, this sequence must converge to some x^*. This is the desired fixed point. □

Example: Take again the example $g(x) = \cos(x)$. Let $1 \leq r < \pi/2$. Take the metric space to be the closed interval $X = [-r, r]$. Since $1 \leq r$ the function $g(x) = \cos(x)$ maps X into itself. Furthermore, since $r < \pi/2$ there is a $c < 1$ such that the derivative $g'(x) = -\sin(x)$ satisfied $|g'(x)| \leq c$. This is enough to show that g is a strict contraction. So the theorem guarantees the existence and uniqueness of the fixed point. Although there is no explicit formula for the solution of $\cos(x) = x$, the theorem defines this number with no ambiguity. ||

1.1.2 Derivative conditions

The reasoning in this example gives a way of rigorously proving the existence of fixed points. The hypothesis requires a bound on the derivative and the existence of an approximate fixed point p.

Proposition 1.3 *Let p be a real number, and consider the closed interval $[p - r, p + r]$ with $r > 0$. Suppose that $|g'(x)| \leq c < 1$ for x in this interval. Furthermore, suppose that $|g(p) - p| \leq (1 - c)r$. (This says that p is almost a fixed point.) Then g maps the interval into itself and is a strict contraction, so it has a unique fixed point in this interval. Furthermore, iteration starting in this interval converges to the fixed point.*

Proof: It is clear that $|g(x) - g(y)| \leq c|x - y|$. In order to show that g maps the interval into itself, suppose $|x - p| \leq r$. Then $|g(x) - p| \leq |g(x) - g(p)| + |g(p) - p| \leq c|x - p| + (1 - c)r \leq r$. □

Sometimes there is a fixed point that is *stable*, in the sense that fixed point iteration starting near the fixed point converges to it. The following proposition is a variant that captures this idea.

Proposition 1.4 *Let p be a fixed point of g. Suppose that g' is continuous and that $|g'(p)| < 1$. Then for c with $|g'(p)| < c < 1$ there is an $r > 0$ such that*

$|g'(x)| \leq c < 1$ for x in the closed interval $[p-r, p+r]$. Then g maps the interval into itself and is a strict contraction. Furthermore, iterates starting in this interval converge to the fixed point.

1.2 The implicit function theorem (single variable)

1.2.1 Solving equations

Fixed point iteration gives a rather general way of solving equations $f(x) = 0$. If a is an arbitrary non-zero constant, then the fixed points of

$$g(x) = x - \frac{1}{a} f(x) \tag{1.5}$$

are the solutions of the equation. The trick is to pick a such that g is a strict contraction. Since

$$g'(x) = 1 - \frac{1}{a} f'(x), \tag{1.6}$$

the strategy is to take a close to the values of $f'(x)$ at points where $f(x)$ is close to zero. This is illustrated in the following result, which is a reformulation of a previous result.

Proposition 1.5 *Let p be a real number, and consider the closed interval $[p-r, p+r]$ with $r > 0$. Suppose that for some c with $0 < c < 1$ we have $1 - c < \frac{1}{a} f'(x) < 1 + c$ for x in this interval. Furthermore, suppose that $|f(p)| \leq |a|(1-c)r$. Then the corresponding fixed point iteration starting in this interval converges to the solution of $f(x) = 0$ in this interval.*

Example: Suppose we want to solve the equation $x = 2\cos(x)$. Since $2\cos(\pi/3) = 1$, this is a fixed point equation with a solution near $\pi/3$. However, since $-2\sin(\pi/3) = -\sqrt{3}$ there is not much hope for stability. So we may write the equation as $f(x) = x - 2\cos(x) = 0$. The derivative is $f'(x) = 1 + 2\sin(x)$. The value of this derivative at $\pi/3$ is $1 + \sqrt{3}$ which is fairly close to 3. So we take $a = 3$, and the iteration function is $g(x) = x - \frac{1}{3}(x - 2\cos(x))$ with derivative $g'(x) = 1 - \frac{1}{3}(1 + 2\sin(x))$. This derivative decreases from $1 - \frac{1}{3}(1+1) = 1/3$ at $\pi/6$ to $1 - \frac{1}{3}(1+2) = 0$ at $\pi/2$. So we may take $c = 1/3$. Furthermore, $\pi/3 - g(\pi/3) = \frac{1}{3}(\pi/3 - 1)$. It is easy to see that this is bounded above by $(1-c)r = \frac{2}{3}\pi/6$. So $\pi/3$ is a sufficiently good approximation to the fixed point. Iteration shows that the fixed point is close to 1.03. ||

Example: Even with simple examples like the one above, it is convenient to use a computer program to do the calculations. Here, for example, is a program written in the computer language R.

```
f <- function (x) x - 2 * cos(x)
g <- function (x) x - f(x)/3
x <- 1
```

```
for (i in 1:10) x <- g(x)
x
[1] 1.029867  ||
```

Example: Fix z with $0 < z < 1$. Suppose we want to solve the equation $w = ze^w$ for w. We can plot $z = we^{-w}$ and see that there are always two positive solutions, one less than 1 and the other greater than 1. The smaller solution may be computed by fixed point iteration. On the other hand, the greater solution is an unstable fixed point. If we take the smaller solution, then we have a well-defined function $L(z)$ defined for $0 < z < 1$ with values $0 < L(z) < 1$ that satisfies $L(z) = ze^{L(z)}$. There is no obvious explicit formula for this function. ||

1.2.2 The implicit function theorem

The implicit function theorem (statement)

If one wants to solve an equation $f(x, y) = 0$ for x as a function of y (or for y as a function of x), there can be problems. Consider, for example, the equation $x - y^2 = 0$. There is no problem in solving for x as a function of y. On the other hand, near the origin there is a real issue of how to solve for y as a function of x. The obvious attempts are $y = \sqrt{x}$ for $x \geq 0$ and $y = -\sqrt{x}$ for $x \geq 0$. In either case the solution is non-differentiable at $x = 0$. The implicit function theorem gives insight into this sort of behavior. It is assumed here that the reader has some familiarity with partial derivatives and the chain rule.

Theorem 1.6 (Implicit function theorem) *Let $f(x, y)$ be continuous with continuous partial derivatives near some point $x = a, y = b$ with $f(a, b) = 0$. Suppose the goal is to solve $x = h(y)$, so x is to be the dependent variable. Suppose that at the point where x is a and y is b the partial derivative*

$$\frac{\partial f(x, y)}{\partial x}(x \leftarrow a, y \leftarrow b) \neq 0. \tag{1.7}$$

Then there is a continuous function $h(y)$ defined for y near b with $f(h(y), y) = 0$. In fact, the function $h(y)$ has a continuous derivative, and

$$0 = \frac{df(h(y), y)}{dy} = \frac{\partial f(x, y)}{\partial x}(x \leftarrow h(y))\frac{dh(y)}{dy} + \frac{\partial f(x, y)}{\partial y}(x \leftarrow h(y)). \tag{1.8}$$

In the following we shall sometimes write the partial derivatives in the form

$$\frac{\partial f(x, y)}{\partial x} = f'_{,1}(x, y) \tag{1.9}$$

and

$$\frac{\partial f(x, y)}{\partial y} = f'_{,2}(x, y). \tag{1.10}$$

The hypothesis is that $f'_{,1}(a, b) \neq 0$. This suggests that $f'_{,1}(x, y) \neq 0$ near this point, so the equation $f(x, y) = 0$ involves x in a non-trivial way. This should

allow x to be determined by y, so $x = h(y)$. The differentiated equation has the form

$$0 = f'_{,1}(h(y), y)h'(y) + f'_{,2}(h(y), y). \tag{1.11}$$

This may be solved to give

$$h'(y) = -\frac{f'_{,2}(h(y), y)}{f'_{,1}(h(y), y)}. \tag{1.12}$$

Example: As a very simple example, consider the equation $f(x, y) = x^2 + y^2 - 1 = 0$ of a circle. The partial derivatives are $\partial f(x, y)/\partial x = 2x$ and $\partial f(x, y)/\partial y = 2y$. Thus for every point on the circle where $x \neq 0$ it is possible to solve for x in terms of y near this point. However, if the point is on the right half of the circle, then $x = \sqrt{1 - y^2}$, while if the point is on the left half of the circle, then $x = -\sqrt{1 - y^2}$. In a completely symmetrical way, for every point on the circle where $y \neq 0$ it is possible to solve for y in terms of x near this point. If the point is on the upper half of the circle, then $y = \sqrt{1 - x^2}$, while if the point is on the lower half of the circle, then $y = -\sqrt{1 - x^2}$. ||

The idea of the proof presented below is to start with the solution $f(a, b) = 0$. For y near b one should solve $f(x, y) = 0$ for the unknown x. This is the same as finding a fixed point x for $g(x, y) = x - f(x, y)/A$. The issue is how to determine A so that this function defines a contraction mapping. One wants $g'_{,1}(x, y) = 1 - f'_{,1}(x, y)/A$ to be close to zero. A reasonable choice is to take $A = f'_{,1}(a, b)$.

Example: Take $f(x, y) = \cos(x) + \cos(xy) + \cos(y) - 2$. Then $f(\pi/2, 0) = 0$. In order to solve $x = h(y)$ near $y = 0$ we can use fixed point iteration. Say that, for instance, we want the value $h(1/10)$. Notice that $\partial f(x, y)/\partial y = -\sin(x) - y\sin(xy)$. At $x = \pi/2, y = 0$ this has the value -1. So we take the iteration function $g(x) = x + f(x, 1/10)$. There is a fixed point at $x = 1.553753$ which is near $\pi/2 = 1.570796$. This is the value of $h(1/10)$. ||

Example: Here is an R program to compute $h(1/10)$ from the previous example.

```
f <- function (x,y) cos(x) + cos(x * y) + cos(y) - 2
g <- function (x) x + f(x,1/10)
x <- pi/2
for (i in 1:30) x <- g(x)
x
[1] 1.553753  ||
```

The implicit function theorem (proof)*

Proof: Let $A = \partial f(x, y)/\partial x$ at the point where $x = a, y = b$, that is $A = f'_{,1}(a, b)$. Consider the iteration function

$$g(x, y) = x - \frac{1}{A}f(x, y). \tag{1.13}$$

This has partial derivative

$$\frac{\partial g(x,y)}{\partial x} = 1 - \frac{1}{A}\frac{\partial f(x,y)}{\partial x}. \tag{1.14}$$

At the point where $x = a, y = b$ this has the value 0. Pick some convenient value of c with $0 < c < 1$. Then for x, y sufficiently close to a, b this partial derivative has absolute value bounded by c. In particular, this is true in some box $|x - a| \leq r$, $|y - b| < s$. Fix $r > 0$. We know that $g(a, b) - a = 0$. Hence, if $s > 0$ is sufficiently small, $|g(a, y) - a| \leq (1 - c)r$. This shows that for all y with $|y - b| < s$ the contraction mapping result works for the function $x \mapsto g(x, y)$ on the interval of x with $|x - a| \leq r$. The resulting fixed point is given by some function $x = h(y)$.

It is not too difficult to show that $h(y)$ is a continuous function of y. Consider y' near y. Then $g(h(y), y) = h(y)$ and $g(h(y'), y') = h(y')$. So $h(y') - h(y) = g(h(y'), y') - g(h(y), y) = g(h(y'), y') - g(h(y), y') + g(h(y), y') - g(h(y), y)$. This gives $|h(y') - h(y)| \leq c|h(y') - h(y)| + |g(h(y), y') - g(h(y), y)|$. Write this as $(1 - c)|h(y') - h(y)| \leq |g(h(y), y') - g(h(y), y)|$. Then as $y' \to y$ we have $g(h(y), y') \to g(h(y), y)$, and hence $h(y') \to h(y)$.

It remains to show that h has a continuous derivative. The derivative of h at y is computed as the limit of the difference quotient $(h(y + k) - h(y))/k$ as $k \to 0$. In order to get a handle on this, compute

$$0 = 0 + 0 = f(h(y + k), y + k) - f(h(y), y). \tag{1.15}$$

Expand

$$\begin{aligned} 0 &= f(h(y + k), y + k) - f(h(y), y) \\ &= \int_0^1 \frac{d}{dt} f(th(y + k) + (1 - t)h(y), t(y + k) + (1 - t)y)\, dt. \end{aligned} \tag{1.16}$$

By the chain rule for partial derivatives

$$\begin{aligned} 0 &= f(h(y + k), y + k) - f(h(y), y) \\ &= \int_0^1 [f'_{,1}(th(y + k) + (1 - t)h(y), t(y + k) + (1 - t)y)(h(y + k) - h(y)) \\ &+ f'_{,2}(th(y + k) + (1 - t)h(y), t(y + k) + (1 - t)y)k]\, dt. \end{aligned} \tag{1.17}$$

This has solution

$$\frac{h(y + k) - h(y)}{k} = -\frac{\int_0^1 f'_{,2}(th(y + k) + (1 - t)h(y), t(y + k) + (1 - t)y)\, dt}{\int_0^1 f'_{,1}(th(y + k) + (1 - t)h(y), t(y + k) + (1 - t)y)\, dt}. \tag{1.18}$$

Now let $k \to 0$ in each integral on the right. For each fixed t between 0 and 1 the integrand converge to a limit as $k \to 0$. Furthermore, the integrands are bounded uniformly in k. In this circumstance the bounded convergence theorem (considered later in these notes) shows that the limit of the integral is

the integral of the limit. In fact, the integrands become independent of t, and we obtain the desired formula for $h'(y)$, that is,

$$h'(y) = -\frac{f'_{,2}(h(y), y)}{f'_{,1}(h(y), y)}. \tag{1.19}$$

The right hand side of this formula is continuous in y, so the left hand is also continuous in y. This shows that the function h has a continuous derivative. □

1.3 Differentiation (multivariable)

1.3.1 The derivative

The multivariable situation introduces new ideas. In particular, the size of a point $\mathbf{x}$ in $\mathbf{R}^n$ is measured by the *Euclidean norm*, the square root of the sum of squares of the components. The space $\mathbf{R}^n$ then has the geometric features of Euclidean space.

It is convenient to deal with a rather general class of functions, those defined on some domain $E \subseteq \mathbf{R}^n$ with values in $\mathbf{R}^m$. If $\mathbf{x}$ is in E, then there is a corresponding value

$$\mathbf{u} = \mathbf{f}(\mathbf{x}) \tag{1.20}$$

in $\mathbf{R}^m$. This may be written somewhat more explicitly as $u_i = f_i(\mathbf{x}) = f_i(x_1, \ldots, x_n)$.

It may help to see an example. Take $n = 3$ and $m = 2$. Then the equation says

$$\begin{aligned} u_1 &= f_1(x_1, x_2, x_3) \\ u_2 &= f_2(x_1, x_2, x_3). \end{aligned} \tag{1.21}$$

In many practical situations this is written without subscripts, for example

$$\begin{aligned} u &= f(x, y, z) \\ v &= g(x, y, z). \end{aligned} \tag{1.22}$$

Differential calculus has an important generalization to this situation. The starting point is the matrix of partial derivatives $\mathbf{f}'(\mathbf{x})$. We take E to be an open subset of $\mathbf{R}^n$. For each $\mathbf{x}$ in E we have the matrix of partial derivatives $\mathbf{f}'(\mathbf{x})$ with components

$$f'(\mathbf{x})_{i,j} = \frac{\partial u_i}{\partial x_j} = \frac{\partial f_i(\mathbf{x})}{\partial x_j}. \tag{1.23}$$

Each *partial derivative* is just an ordinary derivative in the situation when all variables but one are held constant during the limiting process. The matrix of partial derivatives is also called the *Jacobian matrix*.

Again it may help to look at an example. We have

$$\mathbf{f}'(\mathbf{x}) = \begin{bmatrix} f'_{1,1}(\mathbf{x}) & f'_{1,2}(\mathbf{x}) & f'_{1,3}(\mathbf{x}) \\ f'_{2,1}(\mathbf{x}) & f'_{2,2}(\mathbf{x}) & f'_{2,3}(\mathbf{x}) \end{bmatrix}. \tag{1.24}$$

An alternate notation using variables might be $(u_1, u_2) = \mathbf{f}(x_1, x_2, x_3)$ and

$$\mathbf{f}'(x_1, x_2, x_3) = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \end{bmatrix}. \tag{1.25}$$

If $u = f(x, y, z)$ and $v = g(x, y, z)$ we can also write

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}. \tag{1.26}$$

One way to interpret this equation is to think of x, y, z as functions of a parameter t. Then replace dx, dy, dz by $dx/dt, dy/dt, dz/dt$ and du, dv by $du/dt, dv/dt$.

Of course one can dispense with matrix notation altogether and write

$$\begin{aligned} du &= \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy + \frac{\partial u}{\partial z}\, dz \\ dv &= \frac{\partial v}{\partial x}\, dx + \frac{\partial v}{\partial y}\, dy + \frac{\partial v}{\partial z}\, dz. \end{aligned} \tag{1.27}$$

The existence of the matrix of partial derivatives does not fully capture the notion of differentiability. Here is the definition of the *derivative* $\mathbf{f}'(\mathbf{x})$. Write

$$\mathbf{f}(\mathbf{x} + \mathbf{h}) = \mathbf{f}(\mathbf{x}) + \mathbf{f}'(\mathbf{x})\mathbf{h} + \mathbf{r}(\mathbf{x}, \mathbf{h}). \tag{1.28}$$

Thus the function is written as the value at $\mathbf{x}$ plus a linear term (given by multiplying a matrix times a vector) plus a remainder term. The requirement on the remainder term is that

$$\frac{|\mathbf{r}(\mathbf{x}, \mathbf{h})|}{|\mathbf{h}|} \to 0 \tag{1.29}$$

as $\mathbf{h} \to 0$. Sometimes it is easier to deal with an equivalent definition that does not involve fractions:

$$|\mathbf{r}(\mathbf{x}, \mathbf{h})| \le \epsilon(\mathbf{x}, \mathbf{h})|\mathbf{h}|, \tag{1.30}$$

where $\epsilon(\mathbf{x}, \mathbf{h}) \to 0$ as $\mathbf{h} \to 0$. In other words, the remainder term has to be of higher order, in this precise sense of being bounded by the norm of $\mathbf{h}$ with a coefficient that goes to zero. From this definition one can conclude that if the derivative $\mathbf{f}'(\mathbf{x})$ exists in this sense, then the function is continuous, and the matrix of partial derivatives exists and coincides with $\mathbf{f}'(\mathbf{x})$. Since the derivative provides a linear transformation that approximates changes in the function, it is often called the *linearization* of the function.

Remark: There is persistent confusion in mathematical terminology regarding linear functions. In linear algebra we deal with a matrix A. A function that sends $\mathbf{h}$ to $A\mathbf{h}$ is called a *linear function* or linear transformation. A function that sends $\mathbf{h}$ to $A\mathbf{h} + \mathbf{b}$ is called an *affine function* or affine transformation. A function that sends $\mathbf{h}$ to $A(\mathbf{h} - \mathbf{k}) + \mathbf{b}$ is also an affine function, since $A(\mathbf{h} - \mathbf{k}) + \mathbf{b} = A\mathbf{h} + (\mathbf{b} - A\mathbf{k})$. The confusion is that in elementary mathematics an affine function is often called a linear function.

The derivative is defined by

$$\mathbf{g}(\mathbf{x} + \mathbf{h}) = \mathbf{g}(\mathbf{x}) + \mathbf{g}'(\mathbf{x})\mathbf{h} + \mathbf{r}(\mathbf{x}, \mathbf{h}), \tag{1.31}$$

where $\mathbf{r}(\mathbf{x}, \mathbf{h})/|\mathbf{h}| \to 0$ as $\mathbf{h} \to 0$. Thus for each $\mathbf{x}$ the function that sends $\mathbf{h}$ to $\mathbf{g}(\mathbf{x}) + \mathbf{g}'(\mathbf{x})\mathbf{h}$ is an affine function. On the other hand, for fixed $\mathbf{x}$ the function that sends $\mathbf{h}$ to $\mathbf{g}'(\mathbf{x})\mathbf{h}$ is a linear function. It make sense to call this the linearization of $\mathbf{g}$ at $\mathbf{x}$. So the *linearization* of $\mathbf{g}$ at $\mathbf{x}$ is just the derivative $\mathbf{g}'(\mathbf{x})$.

On the other hand, the *affine approximation* to $\mathbf{g}$ at $\mathbf{x}$ can mean several things. It can be the function that sends $\mathbf{h}$ to $\mathbf{g}(\mathbf{x}) + \mathbf{g}'(\mathbf{x})\mathbf{h}$. Since

$$\mathbf{g}(\mathbf{y}) = \mathbf{g}(\mathbf{x}) + \mathbf{g}'(\mathbf{x})(\mathbf{y} - \mathbf{x}) + \mathbf{r}(\mathbf{x}, \mathbf{y} - \mathbf{x}) \tag{1.32}$$

the function that sends $\mathbf{y}$ to $\mathbf{g}(\mathbf{x}) + \mathbf{g}'(\mathbf{x})(\mathbf{y} - \mathbf{x})$ can also be called the affine approximation to $\mathbf{g}$ at $\mathbf{x}$. Sometimes in elementary mathematics such an affine approximation may be called a linear approximation or linearization. However in advanced mathematics it is best to clearly distinguish linear from affine. ||

There is a theorem that says that if the matrix of partial derivatives exists and is continuous, then the derivative exists and is given by this matrix. This theorem is proved in a later section.

Example: There are examples where the matrix of partial derivatives exists and yet the derivative does not exist. Take $f(x, y) = xy/(x^2 + y^2)$ away from the origin, $f(0, 0) = 0$. Away from the origin it is easy to compute the partial derivatives $\partial f(x, y)/\partial x = y(y^2 - x^2)/(x^2 + y^2)^2$ and $\partial f(x, y)/\partial y = x(x^2 - y^2)/(x^2 + y^2)^2$. Since $f(x, 0) = 0$ and $f(0, y) = 0$, it follows that the two partial derivatives exist also at the origin. However the function is discontinuous at the origin. ||

A function defined on an open set is called a C^k *function* if all partial derivatives up to order k exist and are continuous. Thus a continuous function is a C^0 function, and a differentiable function with continuous derivative is a C^1 function. Often in theoretical work it is convenient to assume that one is dealing with a C^1 function, or even a C^2 function. So as not to worry too much about technicalities, it is quite common to assume that the function under consideration is a C^∞ *function*. This just means that it is in C^k for every $k = 0, 1, 2, 3, \ldots$. Often a C^∞ function is called a *smooth function*. In theoretical discussions one often has various things to worry about other than exactly how differentiable a function has to be. So it is common to limit the discussion to smooth functions. Sometimes one instead talks about "sufficiently smooth" function, which just means C^k with k large enough for the purpose at hand.

1.3.2 Surfaces and their tangent planes

The notion of differentiable function is very general, and it is revealing to see how various cases apply to important situations. First, consider the case when $n < m$. In that case it is common to think of the set of $\mathbf{u} = \mathbf{f}(\mathbf{x})$ for $\mathbf{x}$ in the domain to be a *parametrized surface* in $\mathbf{R}^m$ of dimension n. Suppose that the rank of $\mathbf{f}'(\mathbf{x})$ is n. (The rank is the dimension of the range, spanned by the columns.) For each $\mathbf{x}$ one can look at the *tangent space* to the surface at $\mathbf{u} = \mathbf{f}(\mathbf{x})$. The tangent space is n dimensional and has the parametric form $\mathbf{f}(\mathbf{x}) + \mathbf{f}'(\mathbf{x})\mathbf{h}$, where $\mathbf{h}$ is in $\mathbf{R}^n$ and parameterizes the tangent space.

In the special case when $n = 1$ the surface is actually just a curve. The derivative at a point is the tangent vector. The tangent space at a point on the curve is a line tangent to the point. For example, take the case when $m = 2$ and the curve is given by

$$\begin{aligned} u &= \cos(t) \\ v &= \sin(t). \end{aligned} \tag{1.33}$$

This is a parametric representation of a circle. The tangent to the circle at a particular point is given by

$$\begin{aligned} \bar{u} &= \cos(t) - \sin(t)k \\ \bar{v} &= \sin(t) + \cos(t)k, \end{aligned} \tag{1.34}$$

where k is an arbitrary real constant.

When $n = 2$ one gets a parametrized surface. Here the classic example with $m = 3$ is

$$\begin{aligned} u &= \sin(s)\cos(t) \\ v &= \sin(s)\sin(t) \\ w &= \cos(s) \end{aligned} \tag{1.35}$$

This describes a spherical surface. The angle s is the co-latitude and the angle t is the longitude. The tangent plane is given by two parameters h, k as

$$\begin{aligned} \bar{u} &= \sin(s)\cos(t) + \cos(s)\cos(t)h - \sin(s)\sin(t)k \\ \bar{v} &= \sin(s)\sin(t) + \cos(s)\sin(t)h + \sin(s)\cos(t)k \\ \bar{w} &= \cos(s) - \sin(s)h. \end{aligned} \tag{1.36}$$

Next, consider the case when $m < n$. In this case the set of $\mathbf{x}$ with $\mathbf{f}(\mathbf{x}) = \mathbf{c}$ can be the *implicit* definition of a surface of dimension $n - m$ in $\mathbf{R}^n$. Suppose that $\mathbf{f}'(\mathbf{x})$ has rank m. Then the tangent space to the surface at a point $\mathbf{x}$ should have dimension $n - m$. It should consist of the points $\bar{\mathbf{x}}$ such that $\mathbf{f}'(\mathbf{x})(\bar{\mathbf{x}} - \mathbf{x}) = 0$.

When $m = 1$ the surface has dimension $n - 1$ and is called a *hypersurface*. If $u = f(x_1, \ldots, x_n)$, then the derivative $\partial u/\partial x_i = f'_{,i}(x_1, \ldots, x_n)$ is a covector.

We often write this in the differential notation as

$$du = \frac{\partial u}{\partial x_1}\, dx_1 + \cdots + \frac{\partial u}{\partial x_n}\, dx_n. \tag{1.37}$$

A simple example is a sphere given by

$$x^2 + y^2 + z^2 = 1. \tag{1.38}$$

The differential of the left hand side is $2x\, dx + 2y\, dy + 2z\, dz$. The tangent plane at a point is found by solving the equation

$$2x(\bar{x} - x) + 2y(\bar{y} - y) + 2z(\bar{z} - z) = 0. \tag{1.39}$$

When $m < n$ it turns out that it is often possible to convert an implicit representation $\mathbf{f}(\mathbf{x}) = \mathbf{c}$ into a parametric representation. Information on how to do this is given by the implicit function theorem.

1.3.3 Active and passive interpretations

When $\mathbf{f}$ maps an open subset of $\mathbf{R}^n$ to $\mathbf{R}^m$, we often write this as a relation between variables $\mathbf{x}$ and $\mathbf{u}$, so $\mathbf{u} = \mathbf{f}(\mathbf{x})$. When $m \neq n$, then it is clear that these variables are describing different systems. The situation is more interesting when $m = n$. Then there is the possible that $\mathbf{u}$ and $\mathbf{x}$ are describing the same system.

Say for instance that $u = f(x, y)$ and $v = g(x, y)$. In the *passive* interpretation u, v is one list of coordinates for the system, and x, y is another list of coordinates for the same system. The functional relation describes how the coordinates are related. The coordinate systems are equivalent; they give alternative descriptions of the same object. In fact, one should be able to solve for x, y as a function of u, v. The inverse function theorem gives mathematical information that applies to this situation.

Example: A typical passive operation is the change from polar coordinates to Cartesian coordinates. This is just

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta). \end{aligned} \tag{1.40}$$

A given point has two descriptions. ||

Again consider a single system. In the *active* interpretation the functions describe how the state of the system is changed. Suppose for instance that the system is described by coordinates x, y at one stage. Then at the next stage it is described by $u = f(x, y), v = g(x, y)$. This process may be iterated. If the state is given at stage n by x_n, y_n, then at stage $n + 1$ it is given by $x_{n+1} = f(x_n, y_n), y_{n+1} = g(x_n, y_n)$. There is no special reason for the functions f and g to have inverses.

Example: Here is an example called the Hénon map. It occurred in a study of an extremely simplified model of atmospheric motion. It may be viewed as the composition of two simpler maps. The first is

$$\begin{aligned} p &= x \\ q &= 1 - ax^2 + y. \end{aligned} \tag{1.41}$$

Here $a > 0$ is a parameter. The derivative of this transformation preserves area. Combine this with the transformation

$$\begin{aligned} u &= q \\ v &= bp. \end{aligned} \tag{1.42}$$

Here b is a parameter with $0 < b < 1$, representing some kind of contraction or dissipation. It decreases area, but in a simple way. The combined transformation is the Hénon map

$$\begin{aligned} u &= 1 - ax^2 + y \\ v &= bx. \end{aligned} \tag{1.43}$$

The Hénon dynamics is a prediction of the future state of the system from the present state. Notice that this is an active operation; the state changes. It is possible to iterate this map many times, in an attempt to predict the state far into the future. This example has been the subject of much research. Reliable prediction far into the future is difficult. ||

1.4 The chain rule (multivariable)

1.4.1 The derivative of a composition

Consider a function $\mathbf{f}$ defined on an open subset of $\mathbf{R}^n$ with values in $\mathbf{R}^m$. Let $\mathbf{g}$ be another function defined on an open subset of $\mathbf{R}^m$ with values in $\mathbf{R}^p$. Supposed that the domain of $\mathbf{g}$ includes the range of $\mathbf{f}$. The *composition* of the two functions $\mathbf{g}$ and $\mathbf{f}$ is the function $(\mathbf{g} \circ \mathbf{f})$ defined by $(\mathbf{g} \circ \mathbf{f})(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x}))$. The *chain rule* describes the derivative of such a function.

Theorem 1.7 (Chain rule) *Suppose that the derivatives* $\mathbf{f}'(\mathbf{x})$ *and* $\mathbf{g}'(\mathbf{f}(\mathbf{x}))$ *exist. Then*

$$(\mathbf{g} \circ \mathbf{f})'(\mathbf{x}) = \mathbf{g}'(\mathbf{f}(\mathbf{x}))\mathbf{f}'(\mathbf{x}). \tag{1.44}$$

The left hand is the derivative of the composition of the two functions, while the right hand side is the matrix product representing the composition of their derivatives, evaluated at the appropriate points.

The chain rule may be written in components. Start with $\mathbf{g}(\mathbf{y})(\mathbf{y} \leftarrow \mathbf{f}(\mathbf{x})) = \mathbf{g}(\mathbf{f}(\mathbf{x}))$. The assertion is that

$$\frac{\partial g_i(\mathbf{f}(\mathbf{x}))}{\partial x_j} = \sum_{k=1}^{m} \frac{\partial g_i(\mathbf{y})}{\partial y_k}(\mathbf{y} \leftarrow \mathbf{f}(\mathbf{x}))\frac{\partial f_k(\mathbf{x})}{\partial x_j}. \tag{1.45}$$

Example: Start with $\mathbf{x}$ in $\mathbf{R}^3$. Let $\mathbf{u} = \mathbf{f}(\mathbf{x})$ in $\mathbf{R}^2$ and $\mathbf{p} = \mathbf{g}(\mathbf{u})$ also in $\mathbf{R}^2$, so the composition is $\mathbf{p} = \mathbf{g}(\mathbf{f}(\mathbf{x}))$. The chain rule says

$$(\mathbf{g} \circ \mathbf{f})'(\mathbf{x}) = \begin{bmatrix} g'_{1,1}(\mathbf{u}) & g'_{1,2}(\mathbf{u}) \\ g'_{2,1}(\mathbf{u}) & g'_{2,2}(\mathbf{u}) \end{bmatrix} \begin{bmatrix} f'_{1,1}(\mathbf{x}) & f'_{1,2}(\mathbf{x}) & f'_{1,3}(\mathbf{x}) \\ f'_{2,1}(\mathbf{x}) & f'_{2,2}(\mathbf{x}) & f'_{2,3}(\mathbf{x}) \end{bmatrix}. \tag{1.46}$$

We can also write

$$\begin{bmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} & \frac{\partial p}{\partial z} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} & \frac{\partial q}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial p}{\partial u} & \frac{\partial p}{\partial v} \\ \frac{\partial q}{\partial v} & \frac{\partial q}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}. \end{bmatrix}. \tag{1.47}$$

This matrix notation is just another way of writing six equations. ||

Proof: Suppose $\mathbf{f}(\mathbf{x}+\mathbf{h}) = \mathbf{f}(\mathbf{x})+\mathbf{f}'(\mathbf{x})\mathbf{h}+\mathbf{r}(\mathbf{x},\mathbf{h})$ with $|\mathbf{r}(\mathbf{x},\mathbf{h})| \leq \epsilon(\mathbf{x},\mathbf{h})|\mathbf{h}|$. Similarly, suppose $\mathbf{g}(\mathbf{u}+\mathbf{k}) = \mathbf{g}(\mathbf{u})+\mathbf{g}'(\mathbf{u})\mathbf{k}+\mathbf{s}(\mathbf{u},\mathbf{k})$ with $|\mathbf{s}(\mathbf{u},\mathbf{k})| \leq \eta(\mathbf{u},\mathbf{k})|\mathbf{k}|$. Take $\mathbf{u} = \mathbf{f}(\mathbf{x})$ and $\mathbf{k} = \mathbf{f}'(\mathbf{x})\mathbf{h}$. Then

$$\mathbf{g}(\mathbf{f}(\mathbf{x}+\mathbf{h})) = \mathbf{g}(\mathbf{u}+\mathbf{k}+\mathbf{r}(\mathbf{x},\mathbf{h})) = \mathbf{g}(\mathbf{u})+\mathbf{g}'(\mathbf{u})\mathbf{k}+\mathbf{g}'(\mathbf{u})\mathbf{r}(\mathbf{x},\mathbf{h})+\mathbf{s}(\mathbf{u},\mathbf{k}+\mathbf{r}(\mathbf{x},\mathbf{h})). \tag{1.48}$$

We need to show that the two remainder terms are appropriately small. First,

$$|\mathbf{g}'(\mathbf{u})\mathbf{r}(\mathbf{x},\mathbf{h})| \leq \|\mathbf{g}'(\mathbf{u})\|\epsilon(\mathbf{x},\mathbf{h})|\mathbf{h}|. \tag{1.49}$$

Second,

$$\begin{aligned} s(\mathbf{u},\mathbf{k}+\mathbf{r}(\mathbf{x},\mathbf{h})) &\leq \eta(\mathbf{u},\mathbf{k}+\mathbf{r}(\mathbf{x},\mathbf{h}))|\mathbf{k}+\mathbf{r}(\mathbf{x},\mathbf{h})| \\ &\leq \eta(\mathbf{u},\mathbf{f}'(\mathbf{x})\mathbf{h}+\mathbf{r}(\mathbf{x},\mathbf{h}))(\|\mathbf{f}'(\mathbf{x})\|+\epsilon(\mathbf{x},\mathbf{h}))|\mathbf{h}|. \end{aligned} \tag{1.50}$$

□

The chain rule has various important consequences. For instance, in the case when $m = n$ it is possible that $\mathbf{f}$ has an inverse function $\mathbf{g}$ such that $\mathbf{f}(\mathbf{g}(\mathbf{y})) = \mathbf{y}$. It follows from the chain rule that

$$\mathbf{g}'(\mathbf{y}) = \mathbf{f}'(\mathbf{g}(\mathbf{y}))^{-1}. \tag{1.51}$$

In other words, the derivative of an inverse function is obtained by taking the inverse of a matrix. In this case when the matrix is square, the Jacobian matrix has a determinant, called the *Jacobian determinant*. It follows that when the function has an inverse, the Jacobian determinant is non-zero.

1.4.2 Partial derivative with respect to a coordinate

In virtually every application of calculus, the variables that are employed have specific meanings, and it is natural to write the formulas in terms of these variables. However this leads to certain possible ambiguities. The first is that the notion of partial derivative with respect to a coordinate variable depends on the entire coordinate system. In other words, it is a partial derivative because the other variables are being held constant.

Consider for example $p = h(u, v)$, $u = f(x, y)$, $v = g(x, y)$. The system is described either by the coordinate system u, v or by the coordinate system x, y. One of the chain rule equations is

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial p}{\partial v}\frac{\partial v}{\partial x}. \tag{1.52}$$

The x partial derivatives are with respect to the x, y coordinate system, while the u and v partial derivatives are with respect to the u, v coordinate system. Some scientists have found it useful to use a more precise notation, where the coordinate system is explicitly indicated. In this case one would write

$$\frac{\partial p}{\partial x}|_y = \frac{\partial p}{\partial u}|_v \frac{\partial u}{\partial x}|_y + \frac{\partial p}{\partial v}|_u \frac{\partial v}{\partial x}|_y. \tag{1.53}$$

The variables that are held constant are part of the notation. This convention is typical, for instance, in thermodynamics.

This may seem more than is needed, but consider the rather common situation where

$$p = h(x, g(x, y)). \tag{1.54}$$

Then

$$\frac{\partial p}{\partial x} = h'_{,1}(x, g(x, y)) + h'_{,2}(x, g(x, y))g'_{,1}(x, y). \tag{1.55}$$

When the functional relationships are specified there is no ambiguity. However this could also be written with $p = h(x, v)$, $v = g(x, y)$ and hence $p = h(x, g(x, y))$. Then

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial v}\frac{\partial v}{\partial x}. \tag{1.56}$$

Now the problem is evident: the expression $\partial p/\partial x$ is ambiguous, at least until it is made clear what other variable or variables are held constant. The coordinate system could be x, v, or it could be x, y. If we indicate the variable that is held constant with a subscript, we get a more informative equation

$$\frac{\partial p}{\partial x}|_y = \frac{\partial p}{\partial x}|_v + \frac{\partial p}{\partial v}|_x \frac{\partial v}{\partial x}|_y. \tag{1.57}$$

Here is another possible ambiguity. In some applications there are variables u, v that form a coordinate system for some system, and other variables x, y that form a coordinate system for another system. One thinks of these as related by $u = f(x, y)$, $v = g(x, y)$. Suppose that $p = h(u, v)$. One often writes

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial p}{\partial v}\frac{\partial v}{\partial x}. \tag{1.58}$$

The expressions $\partial p/\partial x$ and $\partial p/\partial u$ seem to have a similar status. However in the first one u and v are first replaced by functions of x and y, and then the differentiation is performed. In the second one the differentiation is performed first, and then u and v are replaced by functions of x and y. This is usually not explicitly indicated, but it should be kept in mind.

1.4.3 The mean value theorem

Another nice consequence of the chain rule is the mean value theorem. There is a problem with the mean value theorem for vector-valued functions. One would like a theorem of the following form: There exists a t with $0 \le t \le 1$ and

$$\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) = \mathbf{f}'((1-t)\mathbf{x} + t\mathbf{y})\,(\mathbf{y} - \mathbf{x}). \tag{1.59}$$

This would say that there is a point on the segment between $\mathbf{x}$ and $\mathbf{y}$ where the derivative accurately predicts the change. But this can be false!

The following is a statement of a true version of the *mean value theorem*. The idea is to average over the segment. (The hypothesis of this particular version is that $\mathbf{f}'(\mathbf{x})$ not only exists but is continuous in $\mathbf{x}$. A version requiring differentiability but not continuous differentiability may be found in Rudin.)

Theorem 1.8 (Mean value theorem) *Let E be an open convex set in $\mathbf{R}^n$, and let $\mathbf{f}(\mathbf{x})$ be differentiable with derivative $\mathbf{f}'(\mathbf{x})$ continuous in $\mathbf{x}$ in E. Then the difference in function values is the average of the derivative over the segment:*

$$\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) = \int_0^1 \mathbf{f}'((1-t)\mathbf{x} + t\mathbf{y})\,dt\,(\mathbf{y} - \mathbf{x}). \tag{1.60}$$

Furthermore, suppose that for all $\mathbf{z}$ in E we have $\|\mathbf{f}'(\mathbf{z})\| \le M$. Then it follows that

$$|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})| \le M|\mathbf{y} - \mathbf{x}|. \tag{1.61}$$

Proof: Use the fundamental theorem of calculus and the chain rule to compute

$$\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) = \int_0^1 \frac{d}{dt}\mathbf{f}((1-t)\mathbf{x} + t\mathbf{y})\,dt = \int_0^1 \mathbf{f}'((1-t)\mathbf{x} + t\mathbf{y})(\mathbf{y} - \mathbf{x})\,dt. \tag{1.62}$$

The continuous differentiability assumption guarantees that the integrand is continuous in t. This gives the identity.

From the identity we get

$$|f(\mathbf{y}) - f(\mathbf{x})| \le \int_0^1 \|\mathbf{f}'((1-t)\mathbf{x} + t\mathbf{y})\||\mathbf{y} - \mathbf{x}|\,dt \le \int_0^1 M|\mathbf{y} - \mathbf{x}|\,dt. \tag{1.63}$$

But the integrand in the last integral does not depend on t. So this is just $M|\mathbf{y} - \mathbf{x}|$, as in the theorem. □

Corollary 1.9 (Error in affine approximation) *Let E be an open convex set in $\mathbf{R}^n$, and let $\mathbf{f}(\mathbf{x})$ be differentiable with derivative $\mathbf{f}'(\mathbf{x})$ continuous in $\mathbf{x}$ in E. Then the function may be represented as*

$$\mathbf{f}(\mathbf{a} + \mathbf{h}) = \mathbf{f}(\mathbf{a}) + \mathbf{f}'(\mathbf{a})\mathbf{h} + \mathbf{R}(\mathbf{a}, \mathbf{h})\mathbf{h}, \tag{1.64}$$

where

$$\mathbf{R}(\mathbf{a}, \mathbf{h}) = \int_0^1 [\mathbf{f}'(\mathbf{a} + t\mathbf{h}) - \mathbf{f}'(\mathbf{a})]\,dt \to 0 \tag{1.65}$$

as $\mathbf{h} \to 0$.

1.4.4 Continuous partial derivatives*

The mean value theorem idea also works to prove the result about continuous partial derivatives.

Theorem 1.10 *Consider a function* $\mathbf{f}(\mathbf{x})$ *that is defined and continuous on some open set. Then* $\mathbf{f}'(\mathbf{x})$ *exists and is continuous in* $\mathbf{x}$ *if and only if the partial derivatives exist and are continuous in* $\mathbf{x}$*.*

Proof: It is evident that if the derivative exists and is continuous, then the partial derivatives exist and are continuous. All the work is to go the other way.

The existence and continuity of the partial derivatives implies the following statement. Let $\mathbf{z}$ be in the open set on which the partial derivatives exist and are continuous. Let $\mathbf{h}$ be a vector in one of the coordinate directions. Then $d\mathbf{f}(\mathbf{z}+t\mathbf{h})/dt = \mathbf{f}'(\mathbf{z}+t\mathbf{h})\mathbf{h}$ exists for sufficiently small t, and the matrix of partial derivatives $\mathbf{f}'(\mathbf{z})$ is continuous in $\mathbf{z}$. From here on it is sufficient to work with this hypothesis.

We need to examine $\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x})$ when $\mathbf{h}$ is not in one of the coordinate directions. Denote the orthogonal projection of $\mathbf{h}$ on the ith coordinate direction by $\mathbf{h}_{(i)}$. Write $\mathbf{h}_{[i]} = \sum_{j=1}^{i} \mathbf{h}_{(j)}$. Then

$$\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) = \sum_{i=1}^{n} [\mathbf{f}(\mathbf{x}+\mathbf{h}_{[i-1]}+\mathbf{h}_{(i)}) - \mathbf{f}(\mathbf{x}+\mathbf{h}_{[i-1]})] \tag{1.66}$$

This represents the total change as the sum of changes resulting from incrementing one coordinate at a time. We can use the fundamental theorem of calculus to write this as

$$\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) = \sum_{i=1}^{n} \int_0^1 \mathbf{f}'(\mathbf{x}+\mathbf{h}_{[i-1]}+t\mathbf{h}_{(i)})\mathbf{h}_{(i)}\, dt. \tag{1.67}$$

Notice that each term in the sum only involves one coordinate direction. Furthermore each integrand is continuous in t. It follows that

$$\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) - \mathbf{f}'(\mathbf{x})\mathbf{h} = \sum_{i=1}^{n} \int_0^1 [\mathbf{f}'(\mathbf{x}+\mathbf{h}_{[i-1]}+t\mathbf{h}_{(i)}) - \mathbf{f}'(\mathbf{x})]\mathbf{h}_{(i)}\, dt. \tag{1.68}$$

Hence

$$|\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) - \mathbf{f}'(\mathbf{x})\mathbf{h}| \leq \sum_{i=1}^{n} \int_0^1 \|\mathbf{f}'(\mathbf{x}+\mathbf{h}_{[i-1]}+t\mathbf{h}_{(i)}) - \mathbf{f}'(\mathbf{x})\|\, dt |\mathbf{h}|. \tag{1.69}$$

Then it is not difficult to show that

$$\epsilon(\mathbf{x},\mathbf{h}) = \sum_{i=1}^{n} \int_0^1 \|\mathbf{f}'(\mathbf{x}+\mathbf{h}_{[i-1]}+t\mathbf{h}_{(i)}) - \mathbf{f}'(\mathbf{x})\|\, dt \to 0 \tag{1.70}$$

as $\mathbf{h} \to 0$. This is seen as follows. For each t between 0 and 1 we have the estimate $|\mathbf{h}_{[i-1]} + t\mathbf{h}_{(i)}| \leq |\mathbf{h}|$. If we take $|\mathbf{h}|$ small enough, then from the continuity assumption each integrand $\|\mathbf{f}'(\mathbf{x}+\mathbf{h}_{[i-1]}+t\mathbf{h}_{(i)}) - \mathbf{f}'(\mathbf{x})\|$ is small uniformly in t. So each integral is small. □

1.5 Fixed point iteration (multivariable)

1.5.1 The derivative condition

If A is a square matrix, then the spectral radius $\rho(A)$, Lipschitz norm $\|A\|$, and Euclidean norm $\|A\|_2$ satisfy

$$\rho(A) \leq \|A\| \leq \|A\|_2. \tag{1.71}$$

These quantities are described in the chapter on matrix theory; the most important for now is the Lipschitz norm, defined as the least number $\|A\|$ satisfying $|A\mathbf{x}| \leq \|A\||\mathbf{x}|$ for all $\mathbf{x}$.

Proposition 1.11 *Let $\mathbf{p}$ be in $\mathbf{R}^n$ and consider the closed ball of radius r about $\mathbf{p}$. Suppose $\mathbf{g}(\mathbf{x})$ and $\mathbf{g}'(\mathbf{x})$ are defined and continuous for $\mathbf{x}$ in some open set including this ball and have values in $\mathbf{R}^n$. Suppose that $\|\mathbf{g}'(\mathbf{x})\| \leq c < 1$ for $\mathbf{x}$ in this set. Furthermore, suppose that $|\mathbf{g}(\mathbf{p}) - \mathbf{p}| \leq (1-c)r$. Then $\mathbf{g}$ maps the ball into itself and is a strict contraction, so it has a unique fixed point in this ball. Furthermore, iteration starting in this ball converges to the fixed point.*

Proof: From the mean value theorem it follows that $|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})| \leq c|\mathbf{x} - \mathbf{y}|$. In order to show that g maps the ball into itself, suppose $|\mathbf{x} - \mathbf{p}| \leq r$. Then $|\mathbf{g}(\mathbf{x}) - \mathbf{p}| \leq |\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{p})| + |\mathbf{g}(\mathbf{p}) - \mathbf{p}| \leq c|\mathbf{x} - \mathbf{p}| + (1-c)r \leq r$. □

Sometimes it is helpful have a result where one knows there is a fixed point, but wants to show that it is stable, in the sense that fixed point iteration starting near the fixed point converges to it. The following proposition is a variant that captures this idea.

Proposition 1.12 *Let $\mathbf{p}$ be a fixed point of $\mathbf{g}$. Suppose that $\mathbf{g}'$ is continuous near $\mathbf{p}$ and that $\|\mathbf{g}'(\mathbf{p})\| < 1$. Then for c with $\|\mathbf{g}'(\mathbf{p})\| < c < 1$ there is an $r > 0$ such that $\|\mathbf{g}'(\mathbf{x})\| \leq c < 1$ for $\mathbf{x}$ satisfying $|\mathbf{x} - \mathbf{p}| < r$. Then $\mathbf{g}$ maps the ball into itself and is a strict contraction. Furthermore, iterates starting in this ball converge to the fixed point.*

In the multidimensional case this result need not give a particularly good account of stability, since the stability should be established by the spectral radius $\rho(\mathbf{g}'(\mathbf{p}))$, and the norm $\|\mathbf{g}'(\mathbf{p})\|$ can be much larger. So the following result is better.

Proposition 1.13 *Let $\mathbf{p}$ be a fixed point of $\mathbf{g}$. Suppose that $\mathbf{g}'$ is continuous near $\mathbf{p}$ and that the spectral radius $\rho(\mathbf{g}'(\mathbf{p})) < 1$. Then there is a closed ellipsoid centered on $\mathbf{p}$ such that $\mathbf{g}$ maps the ellipsoid into itself, and iterates starting in this ellipsoid converge to the fixed point.*

Proof: This proof relies on a fact from matrix theory: given a real square matrix A and $\epsilon > 0$, there is a new norm $|\mathbf{x}|_G$ on vectors with corresponding Lipschitz norm $\|A\|_G \leq \rho(A) + \epsilon$. This is a consequence the fact that the matrix is similar to its Jordan form.

This fact is applied to $A = \mathbf{g}'(\mathbf{p})$. For every $\epsilon > 0$ there is a new norm $|\mathbf{x}|_G$ so that $\|\mathbf{g}'(\mathbf{p}))\|_G \leq \rho(\mathbf{g}'(\mathbf{p})) + \epsilon$. Since $\rho(\mathbf{g}'(\mathbf{p})) < 1$, we can pick ϵ and the corresponding new norm so that $\|\mathbf{g}'(\mathbf{p}))\|_G < 1$. Then the continuity of $\mathbf{g}'(\mathbf{x})$ in $\mathbf{x}$ shows that for c with $\|\mathbf{g}'(\mathbf{p})\|_G < c < 1$ there is an $r > 0$ such that $|\mathbf{x} - \mathbf{p}|_G \leq r$ implies $\|\mathbf{g}'(\mathbf{x})\|_G \leq c < 1$. Then $\mathbf{g}$ maps the ball defined by $|\mathbf{x}-\mathbf{p}|_G \leq r$ into itself and is a strict contraction. Furthermore, iterates starting in this ball converge to the fixed point. On the other hand, with respect to the original Euclidean norm this ball is an ellipsoid. □

Remark: In general the various ways of measuring the size of a matrix give different answers. For instance, suppose

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \tag{1.72}$$

Then the spectral radius is $\rho(A) = 2$, the Lipschitz norm is $\|A\| = \sqrt{3 + \sqrt{5}} = \frac{1}{2}(\sqrt{2} + \sqrt{10})$, and the Euclidean norm is $\|A\|_2 = \sqrt{6}$. Since A is not symmetric, it is not surprising that $\rho(A) < \|A\| < \|A\|_2$. ||

When $n = 2$ there is another way of picturing the function and the iteration process. The function $\mathbf{g}$ is pictured by a vector field. Plot a grid of points $\mathbf{x}$, and at each point $\mathbf{x}$ in the grid draw an arrow from $\mathbf{x}$ to $\mathbf{g}(\mathbf{x})$. This gives at least a rough picture of the function. The fixed points are the points at which the corresponding arrow vanishes.

The iterates starting at $\mathbf{x}_0$ are obtained by the relation $\mathbf{x}_{n+1} = \mathbf{g}(\mathbf{x}_n)$. These may be plotted as points. An alternative is to plot arrows from $\mathbf{x}_n$ to $\mathbf{x}_{n+1}$. Such a broken line curve should roughly follow the vector field. If a number of initial points are used, then the resulting broken line curves give an alternative picture of the vector field.

Near a fixed point $\mathbf{p}$ the function $\mathbf{g}(\mathbf{x})$ is close to $\mathbf{g}(\mathbf{p}) + \mathbf{g}'(\mathbf{p})(\mathbf{x} - \mathbf{p}) = \mathbf{p} + \mathbf{g}'(\mathbf{p})(\mathbf{x} - \mathbf{p})$. Thus $\mathbf{g}(\mathbf{x}) - \mathbf{p}$ is close to $\mathbf{g}'(\mathbf{p})(\mathbf{x} - \mathbf{p})$. The picture resembles the picture near zero for the linear transformation given by multiplication by the matrix $\mathbf{g}'(\mathbf{p})$. In particular, near the fixed point $\mathbf{p}$ the eigenvalues of $\mathbf{g}'(p)$ strongly influence the behavior of the vector field.

Example: Define a function by

$$\begin{aligned} u &= f(x, y) = \frac{1}{2}(x^2 - y^2) + \frac{1}{2} \\ v &= g(x, y) = xy + \frac{1}{4} \end{aligned} \tag{1.73}$$

This has a fixed point where x and y are both equal to $1/2$. The linearization at the fixed point is

$$\begin{bmatrix} x & -y \\ y & x \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{1.74}$$

This has eigenvalues $\frac{1}{2} \pm \frac{1}{2}i$. The 2 norm is 1, which is at first alarming, but this is an overestimate. The Lipschitz norm is $\frac{\sqrt{2}}{2} < 1$, so the fixed point is certainly stable. In this special case the spectral radius is equal to the Lipschitz norm, so it gives no additional information. ||

Example: Here again is a simple computer program in R.

```
f <- function (x,y) (x^2 - y^2)/2 + 1/2
g <- function (x,y) x * y + 1/4
x <- 1
y <- 0
for (i in 1:20) {
u <- f(x,y)
v <- g(x,y)
x <- u
y <- v }
x
[1] 0.4998379
y
[1] 0.5015273
```

||

1.5.2 Solving systems of equations

Suppose $\mathbf{f}$ is a C^1 function from an open subset of $\mathbf{R}^n$ to $\mathbf{R}^n$. Fixed point iteration gives a rather general way of solving equations $\mathbf{f}(\mathbf{x}) = 0$. If A is an arbitrary non-singular matrix, then the fixed points of

$$\mathbf{g}(\mathbf{x}) = \mathbf{x} - A^{-1}\mathbf{f}(\mathbf{x}) \tag{1.75}$$

are the solutions of the equation. The trick is to pick A such that $\mathbf{g}$ is a strict contraction. However,

$$\mathbf{g}'(\mathbf{x}) = I - A^{-1}\mathbf{f}'(\mathbf{x}), \tag{1.76}$$

so the strategy is to take A close to values of $\mathbf{f}'(\mathbf{x})$ near a point where $\mathbf{f}(\mathbf{x})$ is close to zero. This is illustrated in the following result, which is a reformulation of a previous result.

Proposition 1.14 *Let $\mathbf{p}$ be a point, and consider the closed ball $|\mathbf{x} - \mathbf{p}| \leq r$ with $r > 0$. Suppose that for some c with $0 < c < 1$ and some invertible matrix A we have $\|I - A^{-1}\mathbf{f}'(\mathbf{x})\| \leq c$ for $\mathbf{x}$ in this ball. Furthermore, suppose that $|A^{-1}\mathbf{f}(\mathbf{p})| \leq (1 - c)r$. Then the corresponding fixed point iteration starting in this ball converges to the solution of $\mathbf{f}(\mathbf{x}) = 0$ in this ball.*

Remark: For this theorem to apply, it is necessary that $\mathbf{f}'(\mathbf{x})$ be an invertible matrix for $\mathbf{x}$ in the appropriate region. This is because $\|I - A^{-1}\mathbf{f}'(\mathbf{x})\| < 1$ implies that $A^{-1}\mathbf{f}'(\mathbf{x})$ is invertible. Since A is invertible, this implies that $\mathbf{f}'(\mathbf{x})$ is invertible. ||

1.6 The implicit function theorem (multivariable)

1.6.1 The implicit function theorem

The implicit function theorem (statement)

Theorem 1.15 (Implicit function theorem) *Let $m \leq n$ and let $\mathbf{f}(\mathbf{x}, \mathbf{y})$ be a function from an open subset of $\mathbf{R}^n$ to $\mathbf{R}^m$. Here $\mathbf{x}$ is in $\mathbf{R}^m$ and $\mathbf{y}$ is in $\mathbf{R}^{n-m}$. Suppose $\mathbf{f}$ is continuous with continuous derivative near some point $\mathbf{x} = \mathbf{a}, \mathbf{y} = \mathbf{b}$ belonging to the open subset. Let $I = \{1, \ldots, m\}$ and $J = \{m+1, \ldots, n\}$ be the indices corresponding to $\mathbf{x}$ and $\mathbf{y}$. Suppose that the m by m matrix $\mathbf{f}'_I(\mathbf{a}, \mathbf{b})$ has an inverse matrix. Then there is a continuous function $\mathbf{h}(\mathbf{y})$ defined for $\mathbf{y}$ near $\mathbf{b}$ with $\mathbf{f}(\mathbf{h}(\mathbf{y}), \mathbf{y}) = 0$. In fact, the function $\mathbf{h}(\mathbf{y})$ has a continuous derivative, and*

$$0 = \mathbf{f}'_I(\mathbf{h}(\mathbf{y}), \mathbf{y})\mathbf{h}'(\mathbf{y}) + \mathbf{f}'_J(\mathbf{h}(\mathbf{y}), \mathbf{y}). \tag{1.77}$$

The equation in the statement of the theorem may be solved to give

$$\mathbf{h}'(\mathbf{y}) = -\mathbf{f}'_I(\mathbf{h}(\mathbf{y}), \mathbf{y})^{-1}\mathbf{f}'_J(\mathbf{h}(\mathbf{y}), \mathbf{y}). \tag{1.78}$$

The proof of the theorem is given below. The idea is to use fixed point iteration, where for each fixed $\mathbf{y}$ near $\mathbf{b}$ the iteration function sends $\mathbf{x}$ to

$$\mathbf{g}(\mathbf{x}) = \mathbf{x} - \mathbf{f}'_I(\mathbf{a}, \mathbf{b})^{-1}\mathbf{f}(\mathbf{x}, \mathbf{y}). \tag{1.79}$$

The fixed point is the desired $\mathbf{h}(\mathbf{y})$.

The implicit function theorem has a geometric interpretation. Consider the case $m < n$ and a function $\mathbf{f}(\mathbf{x}, \mathbf{y})$ for $\mathbf{x}$ in $\mathbf{R}^m$ and $\mathbf{y}$ in $\mathbf{R}^{n-m}$, where the function values are in $\mathbf{R}^m$. A surface of dimension $n-m$ in $\mathbf{R}^n$ is given implicitly by $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{c}$, that is, $\mathbf{f}(\mathbf{x}, \mathbf{y}) - \mathbf{c} = 0$. The theorem says that we can write $\mathbf{x} = \mathbf{h}(\mathbf{y})$, where $\mathbf{h}$ is a function from $\mathbf{R}^{n-m}$ to $\mathbf{R}^m$, such that $\mathbf{f}(\mathbf{h}(\mathbf{y}), \mathbf{y}) = \mathbf{c}$. Thus $\mathbf{x} = \mathbf{h}(\mathbf{t}), \mathbf{y} = \mathbf{t}$ is a parametric representation of the surface.

Example: Consider the example from Rudin with

$$\begin{aligned} p &= f_1(x, y, u, v, w) = 2e^x + yu - 4v + 3 \\ q &= f_2(x, y, u, v, w) = y\cos(x) - 6x + 2u - w. \end{aligned} \tag{1.80}$$

The problem is to solve $f(x, y, u, v, w) = 0, g(x, y, u, v, w) = 0$ near (0,1,3,2,7) for x, y in terms of u, v, w. For this to be possible in principle, one needs to have x, y involved in the equation in a non-trivial way. It is sufficient to have the linearization

$$\begin{bmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} \end{bmatrix} = \begin{bmatrix} 2e^x & u \\ -y\sin(x) - 6 & \cos(x) \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix} \tag{1.81}$$

have an inverse. But the inverse is

$$\begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix}^{-1} = \frac{1}{20}\begin{bmatrix} 1 & -3 \\ 6 & 2 \end{bmatrix}. \tag{1.82}$$

So one can use the iteration function

$$r = g_1(x, y; u, v, w) = x - \frac{1}{20}(f(x, y, z, u, w) - 3g(x, y, z, u, w)) \tag{1.83}$$
$$s = g_2(x, y; u, v, w) = y - \frac{1}{20}(6f(x, y, z, u, w) + 2g(x, y, z, u, w)).$$

For each fixed value of u, v, w near $(3, 2, 7)$ this should have a fixed point near (0,1). ||

Example: Here is an R program to carry this out for the input $(u, v, w) = (3, 2, 6)$, which one hopes is near enough to $(u, v, w) = (3, 2, 7)$ where we know the solution.

```
f1 <- function (x,y,u,v,w) 2 * exp(x) + y * u - 4 * v + 3
f2 <- function (x,y,u,v,w) y * cos(x) - 6 * x + 2 * u - w
x <- 0
y <- 1
u <- 3
v <- 2
w <- 6
g1 <- function (x,y) x - ( f1(x,y,u,v,w) - 3 * f2(x,y,u,v,w) )/20
g2 <- function (x,y) y - ( 6 * f1(x,y,u,v,w) + 2 * f2(x,y,u,v,w) )/20
for (n in 1:40) {
r <- g1(x,y)
s <- g2(x,y)
x <- r
y <- s }
x
   [1] 0.1474038
y
   [1] 0.8941188
```

This result is for input (3,2,6), and it is reasonably close to the point (0,1) that one would get for input (3,2,7). Of course to get a better idea of the function this computation needs to be repeated for a variety of inputs near (3,2,7). Even then one only gets an idea of the function at inputs sufficiently near this point. At inputs further away fixed point iteration may fail, and the behavior of the function is harder to understand. ||

The implicit function theorem (proof)*

Proof: Let $A = \mathbf{f}'_I(\mathbf{a}, \mathbf{b})$. Consider the iteration function

$$\mathbf{g}(\mathbf{x}, \mathbf{y}) = \mathbf{x} - A^{-1}\mathbf{f}(\mathbf{x}, \mathbf{y}). \tag{1.84}$$

This has partial derivative

$$\mathbf{g}'_I(\mathbf{x}, \mathbf{y}) = I - A^{-1}\mathbf{f}'_I(\mathbf{x}, \mathbf{y}). \tag{1.85}$$

At $\mathbf{x} = \mathbf{a}, \mathbf{y} = \mathbf{b}$ this is the zero matrix. Pick some convenient value of c with $0 < c < 1$. Then for $\mathbf{x}, \mathbf{y}$ sufficiently close to $\mathbf{a}, \mathbf{b}$ this partial derivative has

absolute value bounded by c. In particular, this is true in some box $|\mathbf{x}-\mathbf{a}| \le r$, $|\mathbf{y}-\mathbf{b}| < s$. In this box $\|\mathbf{g}'_I(\mathbf{x},\mathbf{y})\| \le c$, so by the mean value theorem $|\mathbf{g}(\mathbf{x}'',\mathbf{y}) - g(\mathbf{x}',\mathbf{y})| \le c|\mathbf{x}''-\mathbf{x}'|$. Fix $r > 0$. We know that $\mathbf{g}(\mathbf{a},\mathbf{b}) - \mathbf{a} = 0$. Hence, if $s > 0$ is sufficiently small, $|\mathbf{g}(\mathbf{a},\mathbf{y}) - \mathbf{a}| \le (1-c)r$. We can put these results together to show that if $|\mathbf{x}-\mathbf{a}| \le r$, then

$$|\mathbf{g}(\mathbf{x},\mathbf{y}) - \mathbf{a}| \le |g(\mathbf{x},\mathbf{y}) - \mathbf{g}(\mathbf{a},\mathbf{y})| + |\mathbf{g}(\mathbf{a},\mathbf{y}) - \mathbf{a}| \le c|\mathbf{x}-\mathbf{a}| + (1-c)r \le r. \quad (1.86)$$

So the map $\mathbf{x} \to \mathbf{g}(\mathbf{x},\mathbf{y})$ is a contraction mapping that sends a complete metric space (the closed ball of $\mathbf{x}$ with $|\mathbf{x}-\mathbf{a}| \le r$) into itself. This shows that for all $\mathbf{y}$ with $|\mathbf{y}-\mathbf{b}| < s$ there is a fixed point $\mathbf{x} = \mathbf{h}(\mathbf{y})$. This proves that $\mathbf{g}(\mathbf{h}(\mathbf{y}),\mathbf{y}) = \mathbf{h}(\mathbf{y})$ and hence that $\mathbf{f}(\mathbf{h}(\mathbf{y}),\mathbf{y}) = 0$.

It is easy to see from the contraction mapping principle that the fixed point $\mathbf{h}(\mathbf{y})$ is a continuous function of the parameter $\mathbf{y}$. Consider $\mathbf{y}'$ near $\mathbf{y}$. Then $\mathbf{g}(\mathbf{h}(\mathbf{y}),\mathbf{y}) = \mathbf{h}(\mathbf{y})$ and $\mathbf{g}(\mathbf{h}(\mathbf{y}'),\mathbf{y}') = \mathbf{h}(\mathbf{y}')$. So $\mathbf{h}(\mathbf{y}') - \mathbf{h}(\mathbf{y}) = \mathbf{g}(\mathbf{h}(\mathbf{y}'),\mathbf{y}') - \mathbf{g}(\mathbf{h}(\mathbf{y}),\mathbf{y}) = \mathbf{g}(\mathbf{h}(\mathbf{y}'),\mathbf{y}') - \mathbf{g}(\mathbf{h}(\mathbf{y}),\mathbf{y}') + \mathbf{g}(\mathbf{h}(\mathbf{y}),\mathbf{y}') - \mathbf{g}(\mathbf{h}(\mathbf{y}),\mathbf{y})$. This gives $|\mathbf{h}(\mathbf{y}') - \mathbf{h}(\mathbf{y})| \le c|\mathbf{h}(\mathbf{y}') - \mathbf{h}(\mathbf{y})| + |\mathbf{g}(\mathbf{h}(\mathbf{y}),\mathbf{y}') - \mathbf{g}(\mathbf{h}(\mathbf{y}),\mathbf{y})|$. Write this as $(1-c)|\mathbf{h}(\mathbf{y}') - \mathbf{h}(\mathbf{y})| \le |\mathbf{g}(\mathbf{h}(\mathbf{y}),\mathbf{y}') - \mathbf{g}(\mathbf{h}(\mathbf{y}),\mathbf{y})|$. Then as $\mathbf{y}' \to \mathbf{y}$ we have $\mathbf{g}(\mathbf{h}(\mathbf{y}),\mathbf{y}') \to \mathbf{g}(\mathbf{h}(\mathbf{y}),\mathbf{y})$, and hence $\mathbf{h}(\mathbf{y}') \to \mathbf{h}(\mathbf{y})$.

It remains to show that $\mathbf{h}$ has a continuous derivative. Let $\mathbf{u}$ be a fixed vector. The directional derivative $\mathbf{h}'(\mathbf{y})\mathbf{u}$ is a vector computed as the limit of the difference quotient $(\mathbf{h}(\mathbf{y}+k\mathbf{u}) - \mathbf{h}(\mathbf{y}))/k$ as $k \to 0$. In order to get a handle on this, compute

$$0 = \mathbf{f}(\mathbf{h}(\mathbf{y}+k\mathbf{u}),\mathbf{y}+k\mathbf{u}) - \mathbf{f}(\mathbf{h}(\mathbf{y}),\mathbf{y}). \quad (1.87)$$

Each term on the right is zero. Then expand

$$0 = \mathbf{f}(\mathbf{h}(\mathbf{y}+k\mathbf{u}),\mathbf{y}+k\mathbf{u}) - \mathbf{f}(\mathbf{h}(\mathbf{y}),\mathbf{y}) = \quad (1.88)$$

$$\int_0^1 \frac{d}{dt}\mathbf{f}(t\mathbf{h}(\mathbf{y}+k\mathbf{u}) + (1-t)\mathbf{h}(\mathbf{y}), t(\mathbf{y}+k\mathbf{u}) + (1-t)\mathbf{y})\, dt. \quad (1.89)$$

By the chain rule for partial derivatives the right hand side is the sum of two terms, one from the I derivatives, and one from the J derivatives. The first term is given by $A(k)(\mathbf{h}(\mathbf{y}+k\mathbf{u}) - \mathbf{h}(\mathbf{y}))$, where

$$A(k) = \int_0^1 [\mathbf{f}'_I(t\mathbf{h}(\mathbf{y}+k\mathbf{u}) + (1-t)\mathbf{h}(\mathbf{y}), t(\mathbf{y}+k\mathbf{u}) + (1-t)\mathbf{y})]\, dt. \quad (1.90)$$

The second term is $B(k)k\mathbf{u}$, where

$$B(k) = \int_0^1 [\mathbf{f}'_J(t\mathbf{h}(\mathbf{y}+k\mathbf{u}) + (1-t)\mathbf{h}(\mathbf{y}), t(\mathbf{y}+k\mathbf{u}) + (1-t)\mathbf{y})]\, dt. \quad (1.91)$$

This gives the solution

$$\frac{\mathbf{h}(\mathbf{y}+k\mathbf{u}) - \mathbf{h}(\mathbf{y})}{k} = -A(k)^{-1}B(k)\mathbf{u}. \quad (1.92)$$

The quantities $A(k)$ and $B(k)$ are continuous in k, because of the continuity of $\mathbf{h}$. Now let $k \to 0$ and apply the dominated convergence theorem to justify taking the limit inside the integral. The integrands become independent of k, and we obtain the desired formula for this vector, namely

$$\mathbf{h}'(\mathbf{y})\mathbf{u} = -A(0)^{-1}B(0)\mathbf{u} = -\mathbf{f}'_I(\mathbf{h}(\mathbf{y}), \mathbf{y})^{-1}\mathbf{f}'_J(\mathbf{h}(\mathbf{y}), \mathbf{y})\mathbf{u}. \tag{1.93}$$

That is, the directional derivative is the matrix

$$\mathbf{h}'(\mathbf{y}) = -\mathbf{f}'_I(\mathbf{h}(\mathbf{y}), \mathbf{y})^{-1}\mathbf{f}'_J(\mathbf{h}(\mathbf{y}), \mathbf{y}). \tag{1.94}$$

The right hand side of this is continuous in $\mathbf{y}$. This shows that the left hand side is continuous in $\mathbf{y}$. As a consequence, $\mathbf{h}(\mathbf{y})$ as a function of $\mathbf{y}$ is differentiable, and the derivative $\mathbf{h}'(\mathbf{y})$ as a function of $\mathbf{y}$ is continuous. □

The last part of the above proof seems complicated but is actually a straightforward application of the technique of the mean value theorem. It follows unpublished notes of Joel Feldman.

1.6.2 The inverse function theorem

Theorem 1.16 (Inverse function theorem) *Let $\mathbf{y} = \mathbf{f}(\mathbf{x})$ be a function from an open subset of $\mathbf{R}^m$ to $\mathbf{R}^m$. Here $\mathbf{x}$ is in $\mathbf{R}^m$ and $\mathbf{y}$ is in $\mathbf{R}^m$. Suppose $\mathbf{f}$ is continuous with continuous derivative near some point $\mathbf{x} = \mathbf{a}$ in the open subset.. Suppose that the m by m matrix $\mathbf{f}'(\mathbf{a})$ has an inverse matrix. Then there is a continuous function $\mathbf{h}(\mathbf{y})$ defined for $\mathbf{y}$ near $\mathbf{b} = \mathbf{f}(\mathbf{a})$ with $\mathbf{f}(\mathbf{h}(\mathbf{y})) = \mathbf{y}$. In fact, the function $\mathbf{h}(\mathbf{y})$ has a continuous derivative, and*

$$\mathbf{h}'(\mathbf{y}) = \mathbf{f}'(\mathbf{h}(\mathbf{y}))^{-1}. \tag{1.95}$$

The inverse function theorem is the special case of the implicit function theorem when $\mathbf{f}(\mathbf{x}, \mathbf{y})$ is of the form $\mathbf{f}(\mathbf{x}) - \mathbf{y}$. So the iteration function for each fixed $\mathbf{y}$ is

$$\mathbf{g}(\mathbf{x}) = \mathbf{x} - \mathbf{f}'(\mathbf{a})^{-1}(\mathbf{f}(\mathbf{x}) - \mathbf{y}) \tag{1.96}$$

with derivative

$$\mathbf{g}'(\mathbf{x}) = I - \mathbf{f}'(\mathbf{a})^{-1}\mathbf{f}'(\mathbf{x}). \tag{1.97}$$

The fixed point of $\mathbf{g}$ is $\mathbf{h}(\mathbf{y})$.

This theorem is of great importance. For example, if a system is described by variables $x_1, \ldots, x_n$, and $u_i = f_i(x_1, \ldots, x_n)$ gives new variables, we might want to describe the system by $u_1, \ldots, u_n$. This would be a passive transformation. But can one recover the original variables? If the matrix $\partial u_i / \partial x_j$ is non-singular, then the inverse function theorem says that it should be possible. That is, we have $x_j = h_j(u_1, \ldots, u_n)$.

1.7 Second order partial derivatives

1.7.1 Equality of mixed partial derivatives

In this section we consider scalar functions of several variables. For instance, consider $u = f(\mathbf{x}) = f(x, y, z)$. The first order partial derivatives

$$[f'_{,1}(\mathbf{x}), f'_{,2}(\mathbf{x}), f'_{,3}(\mathbf{x})] = \left[\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right] \tag{1.98}$$

naturally form a covector. How about the second order partial derivatives? These can be arranged in a matrix

$$f''(\mathbf{x}) = f''(x, y, z) = \begin{bmatrix} f''_{,11}(\mathbf{x}) & f''_{,12}(\mathbf{x}) & f''_{,13}(\mathbf{x}) \\ f''_{,21}(\mathbf{x}) & f''_{,22}(\mathbf{x}) & f''_{,23}(\mathbf{x}) \\ f''_{,31}(\mathbf{x}) & f''_{,32}(\mathbf{x}) & f''_{,33}(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial x \partial z} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} & \frac{\partial^2 u}{\partial y \partial z} \\ \frac{\partial^2 u}{\partial z \partial x} & \frac{\partial^2 u}{\partial z \partial y} & \frac{\partial^2 u}{\partial z^2} \end{bmatrix}. \tag{1.99}$$

One ordinarily expects that this is a symmetric matrix, that is,

$$f''_{,12}(\mathbf{x}) = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = f''_{,21}(\mathbf{x}) \tag{1.100}$$

and similarly for the other three cases.

The symmetry of this matrix is due to the equality of mixed partial derivative. This fact is not completely obvious. Here is a strategy for showing that it is true. There is little loss of generality in considering the case of a function $f(x, y)$ of two variables. Define $\Delta_1(h)f(x, y) = f(x + h, y) - f(x, y)$ and $\Delta_2(k)f(x, y) = f(x, y + k) - f(x, y)$. Then $\Delta_1(h)f(x, y)/h \to f_{,1}(x, y)$ as $h \to 0$ and $\Delta_2(k)f(x, y)/k \to f_{,2}(x, y)$ as $k \to 0$.

The key to the mixed partial derivatives identity is the algebraic identity

$$\Delta_1(h)\Delta_2(k)f(x, y) = \Delta_2(k)\Delta_1(h)f(x, y). \tag{1.101}$$

Then the obvious attempt at a proof is to write

$$\lim_{h\to 0} \frac{\Delta_2(k)\Delta_1(h)f(x, y)}{hk} = \frac{\Delta_2(k)f'_{,1}(x, y)}{k} \tag{1.102}$$

and

$$\lim_{k\to 0} \frac{\Delta_2(k)f'_{,1}(x, y)}{k} = f''_{,21}(x, y). \tag{1.103}$$

Then

$$f''_{,12}(x, y) = \lim_{h\to 0} \lim_{k\to 0} \frac{\Delta_1(h)\Delta_2(k)f(x, y)}{hk} = \lim_{k\to 0} \lim_{h\to 0} \frac{\Delta_2(k)\Delta_1(h)f(x, y)}{hk} = f''_{,21}(x, y). \tag{1.104}$$

The trouble with this is that it is possible to find clever examples where the interchange of limits does not work. However if we make a continuity assumption, then this problem goes away.

Theorem 1.17 (Equality of mixed partial derivatives) *Assume that $f(x,y)$ and $f'_{,1}(x,y)$ and $f'_{,2}(x,y)$ and $f''_{,21}(x,y)$ are continuous on an open set. Then it follows that $f''_{,12}(x,y) = f''_{,21}(x,y)$.*

Proof: By the mean value theorem

$$\frac{\Delta_2(k)\Delta_1(h)f(x,y)}{hk} = \frac{\Delta_2(k)f'_{,1}(x+h^*,y)}{k} = f''_{,21}(x+h^*,y+k^*). \tag{1.105}$$

Here $x+h^*$ is between x and $x+h$ and $y+k^*$ is between y and $y+k$. Consider $\epsilon > 0$. Choose h and k so small that $f''_{,21}(x+h^*,y+k^*)$ is a distance less than ϵ from $f''_{,21}(x,y)$. Thus $\Delta_2(k)\Delta_1(h)f(x,y)/(hk)$ is a distance less than ϵ from $f''_{,21}(x,y)$. Now let $k \to 0$. We conclude that for h sufficiently small (depending on ϵ) we have that $\Delta_1(h)f'_{,2}(x,y)/h$ is a distance less than or equal to ϵ from $f''_{,21}(x,y)$. This proves that $f''_{,12}(x,y) = f''_{,21}(x,y)$. □

The conclusion is that, apart from technicalities, we may safely assume the equality of mixed partial derivatives. This has a number of important conclusions.

Theorem 1.18 *Consider a covector of functions $p_1 = h(x_1,\ldots,x_n),\ldots,p_n = h_n(x_1,\ldots,x_n)$ that have continuous derivatives. Suppose that this covector is integrable, in the sense that there is a function $u = f(x_1,\ldots,x_n)$ with continuous second partial derivatives satisfying*

$$du = p_1\,dx_1 + \cdots + p_n\,dx_n. \tag{1.106}$$

Then for each i,j the integrability condition

$$\frac{\partial p_i}{\partial x_j} = \frac{\partial p_j}{\partial x_i} \tag{1.107}$$

is satisfied.

Proof: The proof depends on the equality of mixed partial derivatives. For each k we have

$$p_k = \frac{\partial u}{\partial x_k}. \tag{1.108}$$

Therefore

$$\frac{\partial p_i}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}u = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u = \frac{\partial p_j}{\partial x_i}. \tag{1.109}$$

□

An expression $\omega = p_1\,dx_1 + \cdots p_n\,dx_n$ is called a differential form. When ω is integrable in the sense that $\omega = du$, then the form ω is said to be *exact*.

1.7.2 First and second derivative tests*

Another important application is to the first and second derivative tests. Suppose that the real function f is continuously differentiable in some open set. A point $\mathbf{a}$ where the covector $f'(\mathbf{a}) = 0$ is called a *critical point*. In particular, the partial derivatives $f'_{,j}(\mathbf{a}) = 0$ for each j. Each number $f(\mathbf{a})$ with $\mathbf{a}$ a critical point is called a *critical value*.

Theorem 1.19 (First derivative test) *Assume that* $\mathbf{f}$ *has continuous first partial derivatives on some open set. Suppose that there is a point* $\mathbf{a}$ *in the set where there is a local minimum or a local maximum. Then* $\mathbf{a}$ *must be a critical point.*

The first derivative test depends on the affine approximation. There is a second derivative test that depends on the quadratic approximation. Suppose that f has continuous second derivative and $f'(\mathbf{a}) = 0$. Let $f''(\mathbf{a})$ be the matrix of partial derivatives evaluated at the critical point. This is a symmetric square matrix called the *Hessian*. The entries of this matrix are $f''_{,ij}(\mathbf{a})$.

Theorem 1.20 (Quadratic approximation) *Let* E *be an open convex set in* $\mathbf{R}^n$*, and let* $f(\mathbf{x})$ *be differentiable with derivative* $f'(\mathbf{x})$ *and second derivative* $f''(\mathbf{x})$ *continuous in* $\mathbf{x}$ *in* E*. Then the function may be represented as*

$$f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + f'(\mathbf{a})\mathbf{h} + \frac{1}{2}\mathbf{h}^T f''(\mathbf{a})\mathbf{h} + \mathbf{h}^T\mathbf{R}(\mathbf{a},\mathbf{h})\mathbf{h}, \tag{1.110}$$

where the symmetric matrix

$$\mathbf{R}(\mathbf{a},\mathbf{h}) = \int_0^1 (1-t)[f''(\mathbf{a}+t\mathbf{h}) - f''(\mathbf{a})]\,dt \tag{1.111}$$

satisfies $\mathbf{R}(\mathbf{a},\mathbf{h}) \to 0$ *as* $\mathbf{h} \to 0$*.*

Proof: Here is the proof of the identity. It begins by noting that

$$\begin{aligned}
&\frac{d}{dt}((1-t)[f'(\mathbf{a}+t\mathbf{h})\mathbf{h} - \mathbf{h}^T f''(\mathbf{a})\mathbf{h}t]) = \\
&- f'(\mathbf{a}+t\mathbf{h})\mathbf{h} + \mathbf{h}^T f''(\mathbf{a})\mathbf{h}t \\
&+ (1-t)[\mathbf{h}^T f''(\mathbf{a}+t\mathbf{h})\mathbf{h} - \mathbf{h}^T f''(\mathbf{a})\mathbf{h}].
\end{aligned} \tag{1.112}$$

Integration gives

$$-f'(\mathbf{a})\mathbf{h} = -\int_0^1 f'(\mathbf{a}+t\mathbf{h})\mathbf{h}\,dt + \frac{1}{2}\mathbf{h}^T f''(\mathbf{a})\mathbf{h} + \mathbf{h}^T\mathbf{R}(\mathbf{a},\mathbf{h})\mathbf{h}. \tag{1.113}$$

This is

$$-f'(\mathbf{a})\mathbf{h} = -f(\mathbf{a}+\mathbf{h}) + f(\mathbf{a}) + \frac{1}{2}\mathbf{h}^T f''(\mathbf{a})\mathbf{h} + \mathbf{h}^T\mathbf{R}(\mathbf{a},\mathbf{h})\mathbf{h}. \tag{1.114}$$

□

Theorem 1.21 (Second derivative test) *Assume that $f(\mathbf{x})$ has continuous second partial derivatives on some open subset. Suppose that there is a critical point $\mathbf{a}$ in the subset where $f'(\mathbf{a}) = 0$. Consider the second derivative Hessian matrix $f''(\mathbf{a})$. This is a symmetric matrix defining a quadratic form. If this quadratic form is strictly positive definite, then the function has a local minimum. Similarly, if this quadratic form is strictly negative definite, then the function has a local maximum.*

Proof: Consider the case when $\mathbf{h}^T f''(\mathbf{a})\mathbf{h} \geq c|\mathbf{h}|^2$ for some $c > 0$. Take $\mathbf{h}$ so small that $\|R(\mathbf{a}, \mathbf{h})\| \leq \epsilon < \frac{1}{2}c$. Because $|\mathbf{h}^T R(\mathbf{a}, \mathbf{h})\mathbf{h}| \leq \epsilon|\mathbf{h}|^2$, it follows in particular that $\mathbf{h}^T R(\mathbf{a}, \mathbf{h})\mathbf{h} \geq -\epsilon|\mathbf{h}|^2$. Since $f'(\mathbf{a}) = 0$ we get

$$f(\mathbf{a} + \mathbf{h}) \geq f(\mathbf{a}) + (\frac{1}{2}c - \epsilon)|\mathbf{h}|^2. \tag{1.115}$$

This implies that f has a local minimum at $\mathbf{a}$. □

Problems

Fixed point iteration

1. Let $g(x) = \cos(x/2)$. It has a stable fixed point $r > 0$ with $g(r) = r$. Use fixed point iteration to find a numerical value for r. Also find $g'(r)$.

2. There is a theorem that says that if $|g'(x)| \leq c < 1$ in an interval of the form $[p - r, p + r]$, and if $|g(p) - p| \leq (1 - c)r$, then g has a fixed point in the interval. Use this to give a proof that $g(x) = \cos(x/2)$ has a fixed point in the interval $[\pi/6, \pi/2]$.

3. Say that $g : [a, b] \to [a, b]$ is a continuous function. Suppose that g is increasing, in the sense that $x \leq y$ implies $g(x) \leq g(y)$. Show that fixed point iteration starting with a always leads to a fixed point r.

4. In physics one calculates the spontaneous magnetization m by the fixed point equation

$$m = \tanh(Jm). \tag{1.116}$$

Here $J > 0$ is a fixed parameter. This equation may be solved by fixed point iteration using the function $g(x) = \tanh(Jx)$. Notice that $g'(x) = J\,\mathrm{sech}^2(Jx)$ satisfies $0 < g'(x) < J$ except at $x = 0$ where $g'(0) = J$. In particular it is increasing. Similarly, one can compute that $g''(x) < 0$ for $x > 0$ and $g''(x) > 0$ for $x < 0$. Also remember that $g(x) \to \pm 1$ as $x \to \pm\infty$.

(a) Suppose $J > 1$. Describe the fixed points. Discuss stability. Sketch orbits.

(b) For each of the fixed points, describe the set of x such that fixed point iteration starting at x converges to that fixed point.

5. Let $g(x) = (1/2)(x^2 + 2x^3 - x^4)$. This has four fixed points $r_1 < r_2 < r_3 < r_4$. Find them, and specify which ones are stable. Compute everything exactly.

6. In the preceding problem, prove that if $r_1 < x < r_3$, then fixed point iteration starting at x converges to r_2. Give a detailed discussion. Hint: It may help to carefully draw a graph and use the graphical analysis of fixed point iteration. Do not make assumptions about the graph that are not justified.

Fixed point iteration (continued)

1. Use fixed point iteration to numerically find the largest root r of $f(x) = x^3 - 5x^2 + 3x + 1 = 0$. Use $g(x) = x - f(x)/f'(s)$, where s is chosen close to the unknown root r. (Since $f(4) = -3$ is not very large, perhaps s could be near 4.) Start the iteration near the root.

2. Consider a smooth function $f(x)$ with a simple root r, that is, $f(r) = 0$ and $f'(r) \neq 0$. Let $g(x) = x - f(x)/f'(x)$. Find $g'(x)$. Find $g'(r)$.

3. Use the iteration function of the previous problem to numerically find the largest root for the example of the first problem.

4. Consider $g : [a, b] \to [a, b]$ an increasing function: $x \leq y$ implies $g(x) \leq g(y)$. Prove or disprove the following general assertion: For every such g there exists s in $[a, b]$ such that s is not a fixed point and iteration starting at s converges to a fixed point.

5. Consider $g : [a, b] \to [a, b]$ an increasing function: $x \leq y$ implies $g(x) \leq g(y)$. Prove or disprove the following general assertion: Every such function g has a fixed point.

The gradient descent method

1. Let F be a smooth function that has a local minimum at some point r and such that $F'(r) = 0$ and $F''(r) > 0$. The gradient descent method for finding r is to start near r and use an iteration function

$$g(x) = x - \epsilon F'(x). \tag{1.117}$$

   Here $\epsilon > 0$ with $\epsilon F''(r) < 1$. In this context gradient is another name for derivative, and the idea is to take a small step in the downward direction (opposite to the sign of the derivative). Show that this fixed point iteration is stable.

2. Take $F(x) = x^4 - x^2$. Apply the gradient method to numerically find the points where F has its minimum values. Check with the exact solution.

3. Suppose that the condition for the gradient descent method is replaced by $0 < \epsilon F''(r) < 2$. Would this extra liberty affect stability? What new behavior could result?

The derivative as a matrix

1. The problem is to solve $f(u, v) = 0$ for $u = h(v)$ as a function of v. Suppose $f(u_0, v_0) = 0$. If we want to find a solution with $u_0 = h(v_0)$, we can set $m = \partial f(u, v)/\partial u \neq 0$ evaluated at the point. Assume $m \neq 0$. Then we can set $g(u, v) = u - (1/m)f(u, v)$. Then $\partial g(u, v)/\partial u$ will be close to zero near the point. (Why?) So fixed point iteration $u \mapsto g(u, v)$ with v fixed near v_0 should work to produce a solution.

   Say that instead we want to solve $f(u, v) = 0$ for $v = k(u)$ as a function of u. Specify the assumption that is needed, and describe the procedure.

2. This continues the previous problem. Consider $f(u, v) = u^3 - v^2$. Describe all points u, v for which the above results give a solution for u as a function of v. Describe all points u, v for which the above results give a solution for v as a function of u. Graph the equation $f(u, v) = 0$ with some care. Indicate the functions $u = h(v)$ and $v = k(u)$ that you get. To what extent are they uniquely specified.

3. Say that $u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$. Find the 2 by 2 derivative matrix. What is the determinant of this matrix? What is its inverse?

4. This continues the previous problem. Say that $s = \sin(ue^v)$. Find $\partial s/\partial u$ and $\partial s/\partial v$. Find s as a function of x and y. Use the chain rule to evaluate the two entries of the derivative matrix (row covector) $\partial s/\partial x$ and $\partial s/\partial y$.

5. Let
$$u = f(x, y) = \frac{x^3}{x^2 + y^2} \tag{1.118}$$
   with $f(0, 0) = 0$ at the origin.

   a) Show that u is continuous at the origin by direct calculation using the definition of continuity.

   b) Evaluate $\partial u/\partial x = f'_{,1}(x, y)$ and $\partial u/\partial y = f'_{,2}(x, y)$ away from the origin. Evaluate $\partial u/\partial x$ and $\partial u/\partial y$ at the origin, using the definition of the (one-dimensional) derivative.

   c) Is $u = f(x, y)$ a C^1 function? That is, are the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ both continuous? Prove that your answer is correct by direct calculation.

   c)The condition for $u = f(x, y)$ to be differentiable at the origin is that
$$f(h, k) = f(0, 0) + f'_{,1}(0, 0)h + f'_{,2}(0, 0)k + r(0, 0; h, k) \tag{1.119}$$
   with $|r(0, 0; h, k)|/\sqrt{h^2 + k^2} \to 0$ as $\sqrt{h^2 + k^2} \to 0$. Is the function differentiable? Using only this definition, prove that your answer is correct.

The derivative as a matrix (continued)

1. Define

$$g_1(x, y) = xy - 2x - 2y + 6 \tag{1.120}$$
$$g_2(x, y) = xy - 2x + 1.$$

Find all fixed points by explicit computation. (Hint: Start by eliminating the xy term.) Find the derivative matrix at each fixed point. Discuss eigenvalues and stability.

2. The obvious general mean value theorem would say that if $\mathbf{f}(\mathbf{x})$ is a continuously differentiable function from a convex domain in $\mathbf{R}^n$ to $\mathbf{R}^m$, then there exists t with $0 \leq t \leq 1$ such that

$$\mathbf{f}(\mathbf{q}) - \mathbf{f}(\mathbf{p}) = \mathbf{f}'((1-t)\mathbf{q} + t\mathbf{p})(\mathbf{q} - \mathbf{p}). \tag{1.121}$$

But is this true? Hint: Take $n = 1$ and $m = 2$ and $\mathbf{f}(x) = [x^2, x^3]^T$.

3. Say that a function is C^k if it has k derivatives, all continuous.

a) For each k, find a function f from the real line to itself that is C^k but not C^{k+1}. Hint: It may help to take $f(x) = 0$ for $x \leq 0$.

b) A function is C^∞ if it is C^k for every k. Find a function with $f(x) = 0$ for $x \leq 0$ that is C^∞ and is not the zero function.

4. a) Let X and Y be metric spaces. Let $f : X \to Y$ be a function. Suppose that f is Lipschitz with Lipschitz constant c. This means that the distance from $f(x)$ to $f(y)$ is bounded by c times the distance from x to y. (The Lipschitz norm of f is the least such constant.) Prove that f is uniformly continuous.

b) Let $X = Y = [0, 1]$. Find a function $f : X \to Y$ that is uniformly continuous but not Lipschitz. Hint: Take $f(x) = \sqrt{x}$. Show that if $x < y$ and $y - x < \epsilon^2$, then $\sqrt{y} - \sqrt{x} < \epsilon$.

The implicit function theorem

1. Consider the surface in $\mathbf{R}^3$ given by

$$s = x^4 + x^2y^2 + y^4 + y^2z^2 + z^4 + z^2x^2 = 1. \tag{1.122}$$

(a) Calculate the differential of the function defining the surface. For which points on the surface does the differential vanish? (b) For which points on the surface does the implicit function theorem define at least one of the variables as a function of the other two near the point?

2. (a) In the preceding problem it should be possible to solve for y in terms of x, z near the point (0,1,0). Find a function $g(y; x, z)$ such that fixed point iteration $y \mapsto g(y; x, z)$ with this function (for fixed x, z) gives the

corresponding value y. Express this function in terms of $s-1 = f(x,y,z)$. (b) Answer the same question if y is to be expressed in terms of x, z near $(0,-1,0)$.

3. Consider the function given by

$$u = f_1(x,y) = x^4 - 6x^2y^2 + y^4 \tag{1.123}$$

$$v = f_2(x,y) = 4x^3y - 4y^3x. \tag{1.124}$$

   (a) Find the derivative matrix.

   (b) Find the determinant of the derivative matrix. Find all points at which this determinant vanishes.

   (c) Show that the four points $(x,y) = (\pm 1, 0)$, $(x,y) = (0, \pm 1)$ all map to $(u,v) = (1,0)$. Near which of these point is there an inverse function?

4. The function of the previous problem maps the point $(0,1)$ to $(1,0)$. There is an inverse function that sends points (u,v) near $(1,0)$ to points (x,y) near $(0,1)$. Find functions $g_1(x,y;u,v)$ and $g_2(x,y;u,v)$ such that fixed point iteration with these functions (for fixed (u,v)) give the corresponding inverse values x, y. Express these functions in terms of $f_1(x,y)$, $f_2(x,y)$ and u, v. (Use the algorithm involving the derivative matrix of $f_1(x,y)$, $f_2(x,y)$ evaluated at the point $(0,1)$.

5. Consider the iteration function $g_1(x,y;u,v)$, $g_2(x,y;u,v)$ found in the previous problem. Show that if $|x| < 1/100$ and $|y-1| < 1/100$, then the linearization at such a point has norm bounded by $1/2$. (Hint: Bound the 2-norm.)

6. (a) Consider an equation $f(p,v,t) = 0$. Consider a particular point in $\mathbf{R}^3$ that gives a solution of this equation. Describe the condition that guarantees that one can solve for v as a function of p, t near this point.

   (b) In physics the quantities p, v, t are related to pressure, volume, and temperature. An important equation relating these quantities is

$$f(p,v,t) = \left(p + \frac{3}{v^2}\right)\left(v - \frac{1}{3}\right) - \frac{8}{3}t = 0. \tag{1.125}$$

   Show that $p = 1$, $v = 1$, $t = 1$ is a solution.

   (c) Can the implicit function theorem be used to specify v as a function of p, t near this point where $p = 1, v = 1, t = 1$? Justify your answer.

Second derivatives

1. (a) Is $xy^4\,dx + 2x^2y^3\,dy$ exact? Justify your answer.

   (b) Is $3x^2y^2\,dx + 4x^3y\,dy$ exact? Justify your answer.

(c) Can $3x^2y^2\,dx + 4x^3y\,dy$ be multiplied by a power y^m to produce an exact differential? Justify your answer.

2. Let

$$u = f(x,y) = x^2y^2 - 2xy^2 + 5x^2 - 10xy + 7y^2 - 10x + 10y. \quad (1.126)$$

There is a point on the line $y = 0$ where $du = 0$. Find it, and find the corresponding value of u. Compute the Hessian matrix. Apply the second derivative test to establish whether it is a local minimum, local maximum, or saddle point (or something else).

3. The second derivative. Say that we want to minimize $w = h(u,v)$ by application of the first derivative test and the second derivative test. However we want to use coordinates x, y related to the u, v by $u = f(x,y)$, $v = g(x,y)$. The corresponding 2 by 2 derivative matrix is assumed to be invertible.

Show that at a point where $\partial w/\partial u = 0$ and $\partial w/\partial v = 0$ it is also true that $\partial w/\partial x = 0$ and $\partial w/\partial y = 0$.

Consider such a point where the first partial derivatives vanish. Suppose that at this point the quadratic form

$$\frac{\partial^2 w}{\partial u^2}p^2 + 2\frac{\partial^2 w}{\partial u \partial v}pq + \frac{\partial^2 w}{\partial v^2}q^2 > 0 \quad (1.127)$$

for all p, q other than $0, 0$. Show that $\partial^2 w/\partial x^2 > 0$.

4. Let $g(x,y) = (x^2 - y^2)/(x^2 + y^2)$ and let $g(0,0) = 0$. Then $g(x,y)$ is a bounded function that is C^2 away from (0,0). Show that $g(x,0) = 1$ for $x \neq 0$, $g(0,y) = -1$ for $y \neq 0$. (Thus g is not continuous at (0,0).) Show also that $x\partial g(x,y)/\partial x$ and $y\partial g(x,y)/\partial y$ are bounded functions in the region away from $(0,0)$. In the following use only these general properties of $g(x,y)$.

Let $f(x,y) = xyg(x,y)$. Calculate $\partial f(x,y)/\partial x$ and $\partial f(x,y)/\partial y$ at the origin. Show that $f(x,y)$ is C^1. Compute $\partial f(x,y)/\partial x$ along $y = 0$. Calculate $\partial f(x,y)/\partial y$ along $x = 0$. Show that your answers also work at the origin. Compute the mixed second partial derivatives at the origin.

The gradient descent method and the Hessian matrix

1. In this problem F is a smooth real function. The gradient ∇F is the column vector of first derivatives. The Hessian $H = \nabla\nabla F$ is the matrix of second derivatives, regarded here as a linear transformation.

Let F be a smooth function that has a local minimum at some point $\mathbf{r}$ and such that $\nabla F(\mathbf{r}) = 0$ and $H(\mathbf{r}) > 0$. The condition that $H(\mathbf{r}) > 0$ says that every eigenvalue λ of this symmetric Hesssian matrix is > 0.

The gradient method for finding $\mathbf{r}$ is to start near $\mathbf{r}$ and use an iteration function

$$\mathbf{g}(\mathbf{x}) = \mathbf{x} - \epsilon \nabla F(\mathbf{x}). \tag{1.128}$$

Here $\epsilon > 0$. It is also required that $0 < \epsilon H(\mathbf{r}) < 1$, in the sense that all the eigenvalues λ satisfy $0 < \epsilon\lambda < 1$. Show that this fixed point iteration is stable. Be sure to relate the eigenvalue condition to the norm.

2. Let $F(x, y) = 3x^2 + 2xy + y^2 - 10x - 6y$. Compute the gradient and the Hessian. Find the minimum point and the minimum value. Give numerical values of ϵ that are suitable for the gradient descent method.

Chapter 2

Integration

2.1 The Riemann integral

2.1.1 Infimum and supremum

In the following the integral is defined in terms of upper sums and lower sums. The notions of *supremum* (least upper bound) and *infimum* (greatest lower bound) play a role. For easy reference, here are the definitions.

Let S be a set of extended real numbers. (An extended real number is either a real number or is $+\infty$ or $-\infty$.) Then $\sup S$ is the least upper bound of S. For every b we have

$$\forall x \in S \;\; x \leq b \Leftrightarrow \sup S \leq b. \tag{2.1}$$

Equivalently,

$$b < \sup S \Leftrightarrow \exists x \in S \;\; b < x. \tag{2.2}$$

Similarly, $\inf S$ is the number with the property that is the greatest lower bound of S. For every b we have

$$\forall x \in S \;\; b \leq x \Leftrightarrow b \leq \inf S. \tag{2.3}$$

Equivalently,

$$\inf S < b \Leftrightarrow \exists x \in S \;\; x < b. \tag{2.4}$$

Suppose that f is a function with extended real values, and A is a subset of its domain. Then $\sup_A f$ and $\inf_A f$ are the supremum and infimum of the image of A under f.

2.1.2 The Riemann integral

An *interval* is a subset of $\mathbf{R}$ that is either empty or connected. An interval is *non-degenerate* if it has at least two points. (An interval is *degenerate* if it is empty or consists of only one point.)

An n-dimensional *cell* C is a subset of $\mathbf{R}^n$ that is a product of n intervals. An n dimensional cell is closed if and only if each of the n intervals is closed. An n-dimensional cell is bounded if and only if each of the n intervals is bounded. An n-dimensional cell is *non-degenerate* if and only if each of the n intervals is non-degenerate. (An n-dimensional cell is *degenerate* if at least one of the n intervals is degenerate.)

For a bounded cell C the n-dimensional *volume* (or *measure*) $m(C)$ is defined by

$$m(C) = \prod_{i=1}^{n} \Delta x_i, \tag{2.5}$$

where $\Delta x_i \geq 0$ is the length of side i. For $n = 1$ this is the length of an interval; for $n = 2$ it is the area inside a rectangle; for $n = 3$ it is the volume inside a box.

In mathematics a *set partition* of a set A is a collection of non-empty non-overlapping subsets whose union is A. The following notion of partition is different: the subsets are allowed to overlap, but only in lower dimension.

Let B be a subset of $\mathbf{R}^n$. Then $\mathcal{P}$ is a *finite cell partition* of B provided that $\mathcal{P}$ is a finite set of closed, bounded, non-degenerate cells, each pair of cells in $\mathcal{P}$ has an intersection that is a closed bounded degenerate cell, and B is the union of the cells in $\mathcal{P}$. Thus

$$B = \bigcup_{C \in \mathcal{P}} C. \tag{2.6}$$

A set B that has a finite cell partition is called a *bounded rectangular set*. A closed, bounded, non-degenerate cell is a bounded rectangular set.

A finite cell partition $\mathcal{P}$ of B is *finer* than a finite cell partition $\mathcal{Q}$ of B if for each C in $\mathcal{P}$ there is a D in $\mathcal{Q}$ with $C \subseteq D$. If $\mathcal{Q}$ and $\mathcal{R}$ are partitions, then there is always a partition $\mathcal{P}$ that is finer than $\mathcal{Q}$ and also finer than $\mathcal{R}$. This can be found by taking the cells of $\mathcal{P}$ to be of the form $C = D \cap E$, where D is a cell of $\mathcal{Q}$ and E is a cell of $\mathcal{R}$.

Let B be a bounded rectangular set. Let $f : B \to \mathbf{R}$ be a bounded real function. For each finite cell partition $\mathcal{P}$ of B define the *lower sum* by

$$L(f, \mathcal{P}) = \sum_{C \in \mathcal{P}} \inf_C f \; m(C) \tag{2.7}$$

and the *upper sum* by

$$U(f, \mathcal{P}) = \sum_{C \in \mathcal{P}} \sup_C f \; m(C). \tag{2.8}$$

Here each C is a closed bounded non-degenerate cell.

It is easy to see that when $\mathcal{P}$ is a refinement of $\mathcal{Q}$, then

$$L(f, \mathcal{Q}) \le L(f, \mathcal{P}) \le U(f, \mathcal{P}) \le U(f, \mathcal{Q}). \tag{2.9}$$

The *lower integral*

$$L(f) = \sup_{\mathcal{P}} L(f, \mathcal{P}) \tag{2.10}$$

and the *upper integral* is

$$U(f) = \inf_{\mathcal{P}} U(f, \mathcal{P}). \tag{2.11}$$

Here $\mathcal{P}$ ranges over all finite cell partitions of the bounded rectangular set B. It is not hard to show that $L(f) \le U(f)$.

If $L(f) = U(f)$, then we say that f is *Riemann integrable* with *integral* $I(f) = L(f) = U(f)$. Warning: There are bounded functions that are not Riemann integrable; for such functions $L(f) < U(f)$.

Example: A Riemann integrable function need not be continuous. In the case of dimension $n = 1$, if f is defined on $[a, b]$ and f is monotone increasing (or monotone decreasing), then f is Riemann integrable. ||

Example: Not every function is Riemann integrable. In the case $n = 1$, if f is defined on $[a, b]$ and $f(x) = 1$ for x rational, $f(x) = 0$ for x irrational, then $L(f) = 0$, while $U(f) = b - a$. ||

Proposition 2.1 *The lower integral is superadditive:*

$$L(f+g) \geq L(f) + L(g), \tag{2.12}$$

while the upper integral is subadditive:

$$U(f+g) \leq U(f) + U(g). \tag{2.13}$$

Proof: It is sufficient to prove this for the case of the lower integral. We have for each x in C the inequality

$$\inf_C f + \inf_C g \leq f(x) + g(x), \tag{2.14}$$

so the left hand side is an lower bound. Therefore the greatest lower bound satisfies

$$\inf_C f + \inf_C g \leq \inf_C (f+g). \tag{2.15}$$

Consider partitions $\mathcal{Q}$ and $\mathcal{R}$. There is a partition $\mathcal{P}$ that is finer than either of them. As a consequence, we have that

$$L(f, \mathcal{Q}) + L(g, \mathcal{R}) \leq L(f, \mathcal{P}) + L(g, \mathcal{P}) \leq L(f+g, \mathcal{P}) \leq L(f+g). \tag{2.16}$$

So we have an upper bound for the $L(f, \mathcal{Q})$. The least upper bound $L(f)$ must then satisfy

$$L(f) + L(g, \mathcal{R}) \leq L(f+g). \tag{2.17}$$

Similarly, we have an upper bound for the $L(f, \mathcal{R})$. The least upper bound $L(f)$ must then satisfy

$$L(f) + L(g) \leq L(f+g). \tag{2.18}$$

□

Theorem 2.2 *The Riemann integral is additive for Riemmann integrable functions:*

$$I(f+g) = I(f) + I(g). \tag{2.19}$$

The above theorem is the main reason for defining the Riemann integral as the common value of the lower and upper integrals.

Let A be an arbitrary bounded subset of $\mathbf{R}^n$. All these notions may be extended to the case of a bounded real function $f : A \to \mathbf{R}$. Choose a bounded rectangular set B (for example a closed, bounded, non-degenerate cell) such that $A \subseteq B$. Define $\bar{f}$ to be equal to f on A and to be equal to 0 on the complement of A in B. The lower integral is defined by $L(f) = L(\bar{f})$, while the upper integral is $U(f) = U(\bar{f})$. The integral of f is $I(f) = I(\bar{f})$, provided that $I(\bar{f})$ exists.

There are various notations that are used. Suppose f is a bounded real function defined on the bounded set $A \subseteq \mathbf{R}^n$. Then

$$L(f) = L(f(\mathbf{x}); \mathbf{x}) = \underline{\int} f(\mathbf{x})\, d\mathbf{x}. \tag{2.20}$$

$$U(f) = U(f(\mathbf{x}); \mathbf{x}) = \overline{\int} f(\mathbf{x})\, d\mathbf{x}. \tag{2.21}$$

If the integral exists, then

$$I(f) = I(f(\mathbf{x}); \mathbf{x}) = \int f(\mathbf{x})\, d\mathbf{x}. \tag{2.22}$$

Sometimes one wants to specify that a function f is to be integrated only over a subset E of the domain A of f. For a bounded set $E \subseteq A$ define the function 1_E to be the*indicator function* of E, equal to 1 on E and 0 on $A \setminus E$. (The indicator function is sometimes called the characteristic function, but this term has a conflicting use in probability theory.) If the product $1_E\, f$ is integrable, then the integral of f over E is defined to be

$$I_E(f) = I(1_E\, f). \tag{2.23}$$

This is more commonly written

$$\int_E f(\mathbf{x})\, d\mathbf{x} = \int 1_E(\mathbf{x})\, f(\mathbf{x})\, d\mathbf{x}. \tag{2.24}$$

2.1.3 Jordan content

Let E be a bounded set. If the indicator function 1_E is Riemann integrable, then E is said to be *Jordan measurable*. This means that E is a subset of a bounded rectangular set B, and 1_E is integrable as a function on B. The *Jordan content* (or *volume*) of E is defined to be

$$m(E) = I(1_E). \tag{2.25}$$

Jordan content is additive. This means that if $E = \bigcup_{i=1}^{k} E_i$ is a finite union of disjoint Jordan measurable sets E_i, then E is Jordan measurable, and $m(E) = \sum_{i=1}^{n} m(E_i)$.

Suppose that E is a bounded set that is Jordan measurable with Jordan content equal to zero. Then E is said to have *content zero* and is considered to be a set that is in some way very small.

Proposition 2.3 *Let E be a bounded set. Then E is Jordan measurable with Jordan content zero if and only if the upper integral $U(1_E) = 0$.*

2.1.4 Conditions for Riemann integrability

If S is a subset of a metric space, then the *diameter* of S is the supremum of the values $d(x, y)$ for x and y in S.

If f is a real function on $B \subseteq \mathbf{R}^n$, and $A \subseteq B$, define the *oscillation* of f on A by

$$\operatorname*{osc}_A(f) = \sup_A f - \inf_A f. \tag{2.26}$$

The oscillation is a special case of notion of *diameter* of a set. By definition of oscillation $\mathrm{osc}_A(f) = \sup_{\mathbf{z}\in A} f(\mathbf{z}) - \inf_{\mathbf{w}\in A} f(\mathbf{w})$. This is equivalent to $\mathrm{osc}_A(f) = \sup_{\mathbf{z},\mathbf{w}\in A} |f(\mathbf{z}) - f(\mathbf{w})|$. This shows that the oscillation is the diameter of the image of the restriction f_A of f to A.

There is a connection between oscillation and the Riemann integral. In fact

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) = \sum_{C\in\mathcal{P}} \operatorname*{osc}_C(f)\, m(C). \tag{2.27}$$

This implies that the integral exists if and only if the infimum over all these oscillation sums is zero.

Theorem 2.4 (Integrability of continuous functions) *If f is continuous on a bounded rectangular set B, then f is Riemann integrable on B.*

Proof: Since f is continuous on a compact set, it is bounded: there is a constant M such that for all $\mathbf{x}$ in B, $|\mathbf{f}(\mathbf{x})| \le M$. Also, since f is continuous on a compact subset of $\mathbf{R}^n$, it is also it is uniformly continuous: For every $\epsilon > 0$ there exists a $\delta > 0$ such that for every $\mathbf{x}, \mathbf{y}$ in B with $|\mathbf{x}-\mathbf{y}| < \delta$ it follows that $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \epsilon$.

Consider $\epsilon > 0$. By uniform continuity, there is a $\delta > 0$ such that for every $\mathbf{x}, \mathbf{y}$ in B with $|\mathbf{x} - \mathbf{y}| < \epsilon$ it follows that $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \epsilon/m(B)$. Choose a finite cell partition $\mathcal{P}$ such that each cell C in $\mathcal{P}$ has diameter less than δ. Then the oscillation $\mathrm{osc}_C(f) < \epsilon/m(B)$. It follows that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) \le \sum_{C\in\mathcal{P}} \frac{\epsilon}{m(B)} m(C) = \epsilon. \tag{2.28}$$

Since $\epsilon > 0$ is arbitrary, this proves that f is Riemann integrable on B. □

Theorem 2.5 (Integrability of certain discontinuous functions) *If f is bounded and is continuous on a bounded rectangular set B except on a set $D \subseteq B$ of content zero, then f is Riemann integrable on B.*

Proof: For every bounded non-degenerate cell C the restriction of 1_D to C has infimum 1 if $D \cap C \neq \emptyset$ and 0 otherwise. So for a finite cell partition $\mathcal{P}$ of B

$$U(1_D,\mathcal{P}) = \sum_{C\in\mathcal{P}, C\cap D\neq\emptyset} m(C). \tag{2.29}$$

Choose $\epsilon > 0$. Suppose that $|f|$ is bounded by M. Since D has content zero, there is a finite cell partition $\mathcal{P}$ of B such that this sum is less than $\epsilon/(4M)$. Let $\mathcal{P}'$ be the cells C such that $C \cap D \neq \emptyset$, and let B' be their union.

Consider the union B'' of the bounded non-degenerate cells C in $\mathcal{P}$ such that $C \cap D = \emptyset$. Then f is continuous on B''. By uniform continuity, there a finite cell partition $\mathcal{P}''$ of B'' such that $\mathrm{osc}_C(f) < \epsilon/(2m(B))$ for every C in $\mathcal{P}''$.

Now let $\mathcal{Q}$ be the cell partition of B consisting of $\mathcal{P}'$ together with $\mathcal{P}''$. It follows that

$$U(f,\mathcal{Q})-L(f,\mathcal{Q}) = \sum_{C\in\mathcal{P}'} \operatorname*{osc}_C(f)m(C)+\sum_{C\in\mathcal{P}''} \operatorname*{osc}_C(f)m(C) \leq 2M\frac{\epsilon}{4M}+\frac{\epsilon}{2m(B)}m(B'') \leq \epsilon. \tag{2.30}$$

Since ϵ is arbitrary, this establishes that f is Riemann integrable. □

Theorem 2.6 (Boundary of zero content) *Let B be a bounded rectangular region, and let $A \subseteq B$. If the boundary of A has Jordan content zero, then A is Jordan measurable.*

Theorem 2.7 *Suppose that f is Riemann integrable and that h is a Lipschitz function. Then the composition of $h \circ f$ is also Riemann integrable.*

Proof: To say that h is Lipschitz is to say that there is a constant M with $|h(y) - h(z)| \leq M|y - x|$. It follows that $\operatorname{osc}_I(h \circ f) \leq M \operatorname{osc}_I(f)$. So

$$U(h \circ f, \mathcal{P}) - L(h \circ f, \mathcal{P}) \leq M(U(f, \mathcal{P}) - L(f, \mathcal{P})). \tag{2.31}$$

If f is integrable, then so is $h \circ f$. □

Every C^1 function is Lipschitz on every bounded set. So the above result applies to functions such as $h(y) = y^2$.

Theorem 2.8 *Suppose that f and g are Riemann integrable. Then so is the product $f \cdot g$.*

Proof: This follows from the identity

$$f \cdot g = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right). \tag{2.32}$$

Since by the previous result the square of a Riemann integrable function is Riemann integrable, the right hand side defines a Riemann integrable function. □

2.1.5 Approximation by continuous functions

The following material is technical, but nevertheless useful. Riemann integrable functions are usually approximated by step functions, which are not continuous functions. However it is also possible to approximate Riemann integrable functions by continuous functions. In general these approximations are far from uniform, but this does not matter for the values of the integrals.

Given a real function f defined on $\mathbf{R}^n$, there is a subset of $\mathbf{x}$ where $f(\mathbf{x}) \neq 0$. The closure of this set is the *support* of f. To say that f has compact support is equivalent to saying that f vanishes outside of a bounded set.

Consider a function with compact support. For some purposes one wants the approximating continuous functions to also have compact support. This can also be accomplished.

Proposition 2.9 (Approximation from below) *Let K be a compact set. Consider arbitrary $\delta > 0$. Let f be a bounded real function with compact support inside K. For every $\epsilon > 0$ there exists a continuous function g with compact support in a δ neighborhood of K with $g \leq f$ and such that the lower integral satisfies*

$$L(f) - I(g) < \epsilon. \tag{2.33}$$

Proof: Choose a bounded rectangular set B and a bounded set partition $\mathcal{P}$ of B such that

$$L(f) - L(f, \mathcal{P}) = L(f) - \sum_{C \in \mathcal{P}} \inf f_C \, m(C) < \frac{\epsilon}{2}. \tag{2.34}$$

For each C in $\mathcal{P}$ define a continuous function g_C such that $g_C \leq \inf_C f \leq f$ on C and with $\inf_C f \, m(C) - I(g_C)$ very small. If $\inf_C f > 0$ the function $g_C \geq 0$ can be a continuous trapezoidal function that is equal to $\inf_C f$ on a large cell in the interior of C and has very steep sides, so that it is zero on the boundary of C and outside of C. If $\inf_C f < 0$ the function $g_C \leq 0$ can be a continuous trapezoidal function that is equal to $\inf_C f$ on C and has very steep sides, so that it is non-zero only very near C. In particular, it should be zero outside of the δ neighborhood of K. If $\inf_C f = 0$, then $g_C = 0$.

Define $g = \sum_{C \in \mathcal{P}} g_C$. Then $g \leq f$. Also, since there are only finitely many terms in this sum, g can be chosen so that $L(f, \mathcal{P}) - I(g) < \frac{\epsilon}{2}$. It follows that $L(f) - I(g) < \epsilon$. □

Proposition 2.10 (Approximation from above) *Let K be a compact set. Consider arbitrary $\delta > 0$. Let f be a bounded real function with compact support inside K. For every $\epsilon > 0$ there exists a continuous function h with compact support in a δ neighborhood of K with $f \leq h$ and such that the lower integral satisfies*

$$I(h) - L(f) < \epsilon. \tag{2.35}$$

Theorem 2.11 (Approximation from above and below) *Let K be a compact set. Consider arbitrary $\delta > 0$. Let f be a bounded real function with compact support inside K. Suppose that f is Riemann integrable. For every $\epsilon > 0$ there exists continuous functions g and h with compact support in a δ neighborhood of K with $g \leq f \leq h$ and such that the integrals satisfy $I(f) - I(g) < \epsilon$ and $I(h) - I(f) < \epsilon$.*

There is also a result in the other direction; approximation above and below by continuous functions implies the existence of the Riemann integral.

Theorem 2.12 (Criterion for Riemann integrability) *Let B be a bounded rectangular set. Let f be a bounded real function on B. Suppose that for every $\epsilon > 0$ there exists continuous functions g and h on B with $g \leq f \leq h$ on B and such that the integrals satisfy $I(h) - I(g) < 2\epsilon$. Then f is Riemann integrable.*

Proof: Let $\epsilon > 0$. Take $g \leq f \leq h$ as in the theorem. There is a finite cell partition $\mathcal{P}$ with $I(g) - L(g, \mathcal{P}) < \epsilon$ and $U(h, \mathcal{P}) - I(h) < \epsilon$. But $L(g, \mathcal{P}) \leq L(f, \mathcal{P})$ and $U(f, \mathcal{P}) \leq U(h, \mathcal{P})$. It follows that $I(g) - L(f, \mathcal{P}) < \epsilon$ and $U(f, \mathcal{P}) - I(h) < \epsilon$. Hence $U(f, \mathcal{P}) - L(f, \mathcal{P}) = I(h) - I(g) + 2\epsilon < 4\epsilon$. This is enough to establish Riemann integrability of f. □

2.1.6 The absolutely convergent Riemann integral

The usual definition of a Riemann integral is for a bounded real function defined on a bounded set. This section discusses an extension to certain cases where the function f is defined on an open subset $U \subseteq \mathbf{R}^n$. The set U might not be bounded, and the function f on U might not have bounded values.

It is required that the function be a locally Riemann integrable real function. This means that it is bounded and Riemann integrable on every closed, bounded, non-degenerate cell $C \subseteq U$. In addition, there is a requirement of absolute convergence, discussed in the following.

First consider the case of a function f restricted to a closed, bounded non-degenerate cell. The function f has a decomposition $f = f_+ - f_-$ into a positive part $f_+ \geq 0$ and a negative part $f_- \leq 0$. The absolute value of the function is $|f| = f_+ + f_-$. Recall that a Lipschitz function of a Riemann integrable function is also Riemann integrable. It follow that if f is Riemann integrable on C, then so are f_+ and f_- and $|f|$. Furthermore,

$$|I(f)| \leq I(|f|). \tag{2.36}$$

Next comes the extension to a locally Riemann integrable function. Let U be an open subset of $\mathbf{R}^n$. Define $\mathcal{P}$ to be an *infinite cell partition* of U provided that $\mathcal{P}$ is a countable set of closed, bounded, non-degenerate cells, each pair of cells in $\mathcal{P}$ has an intersection that is a closed bounded degenerate cell, and U is the union of the cells in $\mathcal{P}$. Thus

$$U = \bigcup_{C \in \mathcal{P}} C. \tag{2.37}$$

Lemma 2.13 *For each open subset $U \subseteq \mathbf{R}^n$ there is an infinite cell partition with union U.*

Proof: For each m let $\mathcal{P}_m$ be the infinite cell partition of $\mathbf{R}^n$ consisting of all cubical cells of side length $1/2^m$ all of whose sides are bounded by integer multiples of $1/2^m$. If $m' \geq m$, then $\mathcal{P}_{m'}$ is a refinement of $\mathcal{P}_m$. Let $\mathcal{P}'_1$ be the partition consisting of cells in $\mathcal{P}_1$ that are subsets of U. Then define P'_m inductively by requiring that $\mathcal{P}'_{m+1}$ consists of the cells in $\mathcal{P}'_m$ together with all cells in $\mathcal{P}_{m+1}$ that are subsets of U but are not subsets of cells in $\mathcal{P}'_m$. Let $\mathcal{P}'$ be the union of the $\mathcal{P}'_m$ for $m = 1, 2, 3, \ldots$.

Let V be the union of the cubes in $\mathcal{P}'$. Then $V \subseteq U$. On the other hand, $U \subseteq V$. The reason is this. Suppose that $\mathbf{x}$ is a point in U. Then it is a distance $\delta > 0$ from the complement of U in $\mathbf{R}^n$. Suppose that m satisfies $\sqrt{n}/2^m < \delta$. Then $\mathbf{x}$ is in the union of the cells in $\mathcal{P}'_m$. So $\mathbf{x}$ is in V. □

The locally Riemann integrable function f is said to be an absolutely convergent Riemann integrable if for each C in $\mathcal{P}$ the restriction f_C is integrable, and

$$\sum_{C \in \mathcal{P}} I(|f_C|) < +\infty. \tag{2.38}$$

In this case the absolutely convergent Riemann integral of f is defined by the absolutely convergent series

$$I(f) = \sum_{C \in \mathcal{P}} I(f_C). \tag{2.39}$$

Example: Here are some one-dimensional examples. In the first one the function is bounded, but the open set $(0, +\infty)$ is unbounded. The integral is

$$\int_0^\infty e^{-x^2} \, dx = \frac{1}{2}\sqrt{\pi}. \tag{2.40}$$

The cells can be intervals of length one. In the second one the function is unbounded. The integral is

$$\int_0^\infty e^{-t} \frac{1}{\sqrt{t}} \, dt = \sqrt{\pi}. \tag{2.41}$$

In the second example the cells between 1 and 0 may be taken to have lengths that decrease near 0 according to $1/2^n$. ||

In such a context, the term "improper Riemann integral" is often used. This sometimes may refer to integrals that are not absolutely convergent. The theory of integrals that are not absolutely convergent is much more delicate. When such an "integral" is given by a conditionally convergent sum that is not absolutely convergent, the result depends on the order in which the sum is taken.

Example: Here is a typical example of an integral to which many people would attach a value, but which is not absolutely convergent. The example is

$$\int_0^\infty \frac{\sin(x)}{x} \, dx = \frac{\pi}{2}. \tag{2.42}$$

It is true that

$$\lim_{r \to \infty} \int_0^r \frac{\sin(x)}{x} \, dx = \frac{\pi}{2}. \tag{2.43}$$

However if we write this as a sum over cells $C_k = [(k-1)\pi, k\pi]$, then the integral is an infinite sum that does not converge absolutely. Therefore its values depend on the order in which the summation is taken. In the natural order the answer is $\pi/2$, but this is a rather delicate fact. ||

Much of the theory of the Riemann integral carries over to this more general absolutely convergent Riemann integral. Moreover, the more general setting is needed for many practical problems.

On the other hand, the verification that this definition of absolutely convergent Riemann is well defined and leads to good properties can be tedious. It may not be worth spending too much effort on such details. The reason is that there is a more general definition of integral due to Lebesgue, and with this definition such questions are quickly resolved.

2.2 Fubini's theorem

2.2.1 Fubini's theorem

Fubini's theorem (statement in special case)

There is a famous theorem of integration theory called *Fubini's theorem* that has a version for the Riemann integral. It says that a multi-dimensional integral may be written as an *iterated integral*.

Computations of multidimensional integrals generally use Fubini's theorem to reduce the problem to one dimensional integrals. For one dimensional integrals there is then the powerful tool of the fundamental theorem of calculus.

Theorem 2.14 (Fubini) *Suppose that f is a bounded Riemann integrable function on a rectangle in $\mathbf{R}^{m+n}$. Think of a point in $\mathbf{R}^{m+n}$ as a pair $(\mathbf{x}, \mathbf{y})$ with $\mathbf{x}$ in $\mathbf{R}^m$ and $\mathbf{y}$ in $\mathbf{R}^n$. Suppose for each $\mathbf{y}$ the Riemann integral $\int f(\mathbf{x}, \mathbf{y})\, d\mathbf{x}$ exists. Then the function*

$$\mathbf{y} \mapsto \int f(\mathbf{x}, \mathbf{y})\, d\mathbf{x} \tag{2.44}$$

is Riemann integrable, and

$$\int f(\mathbf{x}, \mathbf{y})\, d\mathbf{x}\, d\mathbf{y} = \int \left[\int f(\mathbf{x}, \mathbf{y})\, d\mathbf{x} \right] d\mathbf{y}. \tag{2.45}$$

Similarly, suppose that for each $\mathbf{x}$ the Riemann integral $\int f(\mathbf{x}, \mathbf{y})\, d\mathbf{y}$ exists. Then

$$\mathbf{x} \mapsto \int f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \tag{2.46}$$

is Riemann integrable, and

$$\int f(\mathbf{x}, \mathbf{y})\, d\mathbf{x}\, d\mathbf{y} = \int \left[\int f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \right] d\mathbf{x}. \tag{2.47}$$

The rest of this section is devoted to a discussion and proof of this theorem and also of a more general version. We shall use a somewhat different notation. The reason for this is that we shall be comparing lower and upper sums, and the variant notation makes it easy to incorporate these notions. In particular, this version of the Fubini theorem is written

$$I(f(\mathbf{x}, \mathbf{y}); \mathbf{x}, \mathbf{y}) = I(I(f(\mathbf{x}, \mathbf{y}); \mathbf{x}); \mathbf{y}) \tag{2.48}$$

and

$$I(f(\mathbf{x}, \mathbf{y}); \mathbf{x}, \mathbf{y}) = I(I(f(\mathbf{x}, \mathbf{y}); \mathbf{y}); \mathbf{x}). \tag{2.49}$$

It is common to have an integral

$$\int_S f(x, y)\, dx\, dy \tag{2.50}$$

over a bounded region. Here is a strategy for how to apply Fubini's theorem. Write the iterated integrals as

$$\int_a^b \int_c^d 1((x,y) \in S) f(x,y)\, dx\, dy = \int_c^d \int_a^b 1((x,y) \in S) f(x,y)\, dy\, dx. \quad (2.51)$$

Here $1((x,y) \in S)$ is 1 for (x,y) in S and 0 otherwise. Sometimes one of these integrals is easier than the other.

Example: Conside

$$\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\, dy\, dx. \quad (2.52)$$

This looks impossible to calculate. But write it instead as

$$\int_0^6 \int_0^2 1(y > x/3) x\sqrt{y^3+1}\, dy\, dx = \int_0^2 \int_0^6 1(x < 3y) x\sqrt{y^3+1}\, dx\, dy. \quad (2.53)$$

But this is

$$\int_0^2 \int_0^{3y} x\sqrt{y^3+1}\, dx\, dy = \frac{9}{2}\int_0^2 y^2\sqrt{y^3+1} = \int_0^2 d(y^3+1)^{\frac{3}{2}} = 27 - 1 = 26. \quad (2.54)$$

One iterated integral may be much easier than the other iterated integral. And there is always a choice. ||

Fubini's theorem (statement in general case)

There are technical issues with more general versions of Fubini's theorem. For example, suppose that the Riemann integral $I(f(\mathbf{x},\mathbf{y});\mathbf{x},\mathbf{y})$ exists. Then it is not guaranteed that for fixed $\mathbf{y}$ the integral $I(f(\mathbf{x},\mathbf{y});\mathbf{x})$ exists. Nor is it guaranteed that for each fixed $\mathbf{x}$ that the integral $I(f(\mathbf{x},\mathbf{y});\mathbf{y})$ exists.

Example: Consider the following function defined on $[0,1]\times[0,1]$. Let $f(x,y) = 1$ when x is rational and $y = 1/2$, but $f(x,y) = 0$ elsewhere. This is Riemann integrable with integral zero. However for $y = 1/2$ the function $x \mapsto f(x,y)$ is not Riemann integrable. In fact, its lower integral is 0, while its upper integral is 1. ||

Theorem 2.15 (Fubini (general version)) *Suppose that f is a bounded Riemann integrable function on a rectangle in $\mathbf{R}^{m+n}$. Then the function*

$$\mathbf{y} \mapsto \overline{\int} f(\mathbf{x},\mathbf{y})\, d\mathbf{x} - \underline{\int} f(\mathbf{x},\mathbf{y})\, d\mathbf{x} \quad (2.55)$$

is Riemann integrable with integral zero. Furthermore,

$$\int f(\mathbf{x},\mathbf{y})\, d\mathbf{x}\, d\mathbf{y} = \int \left[\underline{\int} f(\mathbf{x},\mathbf{y})\, d\mathbf{x}\right] d\mathbf{y} = \int \left[\overline{\int} f(\mathbf{x},\mathbf{y})\, d\mathbf{x}\right] d\mathbf{y} \quad (2.56)$$

Similarly, the function

$$\mathbf{x} \mapsto \overline{\int} f(\mathbf{x},\mathbf{y})\,d\mathbf{y} - \underline{\int} f(\mathbf{x},\mathbf{y})\,d\mathbf{y} \tag{2.57}$$

is Riemann integrable with integral zero. Furthermore,

$$\int f(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} = \int \left[\underline{\int} f(\mathbf{x},\mathbf{y})\,d\mathbf{y}\right]\,d\mathbf{x} = \int \left[\overline{\int} f(\mathbf{x},\mathbf{y})\,d\mathbf{y}\right]\,d\mathbf{x} \tag{2.58}$$

Fubini's theorem (proof)*

Theorem 2.16 (Fubini's theorem for lower and upper integrals) *For lower integrals*

$$L(f(\mathbf{x},\mathbf{y});\mathbf{x},\mathbf{y}) \le L(L(f(\mathbf{x},\mathbf{y});\mathbf{x});\mathbf{y}), \tag{2.59}$$

while for upper integrals

$$U(U(f(\mathbf{x},\mathbf{y});\mathbf{x});\mathbf{y}) \le U(f). \tag{2.60}$$

There are similar results where the roles of $\mathbf{x}$ and $\mathbf{y}$ are reversed.

Proof: It is sufficient to prove the result for lower integrals. The result for upper integrals is proved in the same way, but with the inequalities reversed.

For the proof, it is useful to have the concept of a product partition. Let $C = C_1 \times C_2$ be the cell over which the integration takes place. The $\mathbf{x}$ variables range over C_1, while the $\mathbf{y}$ variables range over C_2. If $\mathcal{P}_1$ is a partition of C_1, and $\mathcal{P}_2$ is a partition of C_2, then the product partition $\mathcal{P}_1 \times \mathcal{P}_2$ is the partition of C consisting of all $I_1 \times I_2$ with I_1 from C_1 and I_2 from C_2. Given an arbitrary partition $\mathcal{P}$ of C, then there is a product partition that is finer than $\mathcal{P}$. So it is reasonable to first deal with product partitions.

First we need a simple lemma that only involves sums, not integrals. This states that

$$L(f, \mathcal{P}_1 \times \mathcal{P}_2) \le L(L(f(\mathbf{x},\mathbf{y});\mathbf{x},\mathcal{P}_1);\mathbf{y},\mathcal{P}_2). \tag{2.61}$$

The proof of the lemma uses $\inf_{(\mathbf{x},\mathbf{y})\in I_1\times I_2} f(\mathbf{x},\mathbf{y}) = \inf_{\mathbf{y}\in I_2} \inf_{\mathbf{x}\in I_1} f(\mathbf{x},\mathbf{y})$. The key ingredient is then the product property $m_{m+n}(I_1\times I_2) = m_m(I_1)m_n(I_2)$. We have

$$L(f, \mathcal{P}_1 \times \mathcal{P}_2) = \sum_{I_2\in\mathcal{P}_2} \sum_{I_1\in\mathcal{P}_1} \inf_{\mathbf{y}\in I_2} \inf_{\mathbf{x}\in I_1} f(\mathbf{x},\mathbf{y}) m_m(I_1) m_n(I_2). \tag{2.62}$$

From the general principle that $\sum_I \inf_{\mathbf{y}} h_I(\mathbf{y}) \le \inf_{\mathbf{y}} \sum_I h_I(\mathbf{y})$ we get

$$L(f, \mathcal{P}_1 \times \mathcal{P}_2) \le \sum_{I_2\in\mathcal{P}_2} \inf_{\mathbf{y}\in I_2} \sum_{I_1\in\mathcal{P}_1} \inf_{\mathbf{x}\in I_1} f(\mathbf{x},\mathbf{y}) m_m(I_1) m_n(I_2). \tag{2.63}$$

This translates to

$$L(f, \mathcal{P}_1 \times \mathcal{P}_2) \le \sum_{I_2\in\mathcal{P}_2} \inf_{\mathbf{y}\in I_2} L(f(\mathbf{x},\mathbf{y});\mathbf{x},\mathcal{P}_1) m_n(I_2). \tag{2.64}$$

This leads easily to the statement of the lemma. The proof of the lemma is complete.

Since lower sums are bounded by the lower integral, the lemma gives

$$L(f, \mathcal{P}_1 \times \mathcal{P}_2) \leq L(L(f(\mathbf{x}, \mathbf{y}); \mathbf{x}, \mathcal{P}_1); \mathbf{y}, \mathcal{P}_2) \leq L(L(f(\mathbf{x}, \mathbf{y}); \mathbf{x}), \mathbf{y}, \mathcal{P}_2). \quad (2.65)$$

Again for the same reason

$$L(f, \mathcal{P}_1 \times \mathcal{P}_2) \leq L(L(f(\mathbf{x}, \mathbf{y}); \mathbf{x}); \mathbf{y}). \quad (2.66)$$

Given an arbitrary partition $\mathcal{P}$, there is a finer product partition $\mathcal{P}_1 \times \mathcal{P}_2$, so we must also have

$$L(f, \mathcal{P}) \leq L(L(f(\mathbf{x}, \mathbf{y}); \mathbf{x}); \mathbf{y}). \quad (2.67)$$

That is, the iterated lower integral is an upper bound for the lower sums. Since $L(f)$ is the least of all such upper bounds, we have

$$L(f) \leq L(L(f(\mathbf{x}, \mathbf{y}); \mathbf{x}); \mathbf{y}) \quad (2.68)$$

as desired. □

Example: The theorem above gives a kind of Fubini theorem that works for the lower and upper integrals. Here is an example that shows that equality is not guaranteed. Consider the case of the upper integral of a function f defined on the cell $[0, 1] \times [0, 1]$. Suppose that there is a countable dense set D such that f is one on that set, zero on its complement. Then $U(f(x, y); x, y) = 1$. Now suppose that the set D has the property that for every horizontal line, there is at most one point on the line that is in D. Then for each y the function $x \mapsto f(x, y)$ has upper integral $U(f(x, y); x) = 0$. Thus $U(U(f(x, y), x), y) = 0$. So the iterated upper integral is smaller than the upper integral.

How can we find such a set D? First consider the set E of points in the plane with both coordinates rational. Consider all lines in the plane with fixed angle θ from the x axis, so that the slope is $m = \tan(\theta)$. Suppose that m is an irrational number. For instance, we could take lines at an angle $\theta = \pi/6$ from the x axis, with slope $m = 1/\sqrt{3}$. Every such line intersects E in at most one point. (Why?) Now rotate the picture by angle $-\theta$, so that we get a set D that consists of E rotated by this angle, and such that the lines become horizontal lines. ||

Theorem 2.17 (Fubini's theorem (general version)) *Suppose that the Riemann integral $I(f(\mathbf{x}, \mathbf{y}); \mathbf{x}, \mathbf{y})$ exists. Then for each fixed $\mathbf{y}$ the lower integral and upper integral are automatically defined and satisfy $L(f(\mathbf{x}, \mathbf{y}); \mathbf{x}) \leq U(f(\mathbf{x}, \mathbf{y}); \mathbf{x})$. Furthermore, as functions of $\mathbf{y}$ these each define Riemann integrable functions. Finally, we have both the formulas*

$$I(f(\mathbf{x}, \mathbf{y}); \mathbf{x}, \mathbf{y}) = I(L(f(\mathbf{x}, \mathbf{y}); \mathbf{x}); \mathbf{y}) \quad (2.69)$$

and

$$I(f(\mathbf{x}, \mathbf{y}); \mathbf{x}, \mathbf{y}) = I(U(f(\mathbf{x}, \mathbf{y}); \mathbf{x}); \mathbf{y}). \quad (2.70)$$

The result of course works in the other order. For the sake of completeness here is an explicit statement.

Theorem 2.18 (Fubini's theorem (general version)) *Suppose that the Riemann integral $I(f(\mathbf{x},\mathbf{y});\mathbf{x},\mathbf{y})$ exists. Then for each fixed $\mathbf{x}$ the lower integral and upper integral are automatically defined and satisfy $L(f(\mathbf{x},\mathbf{y});\mathbf{y}) \leq U(f(\mathbf{x},\mathbf{y});\mathbf{y})$. Furthermore, as functions of $\mathbf{x}$ these each define Riemann integrable functions. Finally, we have both the formulas*

$$I(f(\mathbf{x},\mathbf{y});\mathbf{x},\mathbf{y}) = I(L(f(\mathbf{x},\mathbf{y});\mathbf{y});\mathbf{x}) \tag{2.71}$$

and

$$I(f(\mathbf{x},\mathbf{y});\mathbf{x},\mathbf{y}) = I(U(f(\mathbf{x},\mathbf{y});\mathbf{y});\mathbf{x}). \tag{2.72}$$

Proof: The two preceding theorems are essentially the same; it is sufficient to prove the first one. The proof uses the results that relate lower integrals to iterated lower integrals and upper integrals to iterated upper integrals. Once we have these results, we are almost done. We have

$$L(f) \leq L(L(f(\mathbf{x},\mathbf{y});\mathbf{x});\mathbf{y}) \leq U(L(f(\mathbf{x},\mathbf{y});\mathbf{x});\mathbf{y}) \leq U(U(f(\mathbf{x},\mathbf{y});\mathbf{x});\mathbf{y}) \leq U(f). \tag{2.73}$$

If $L(f) = U(f)$, then $L(L(f(\mathbf{x},\mathbf{y});\mathbf{x});\mathbf{y}) = U(L(f(\mathbf{x},\mathbf{y});\mathbf{x});\mathbf{y})$. This proves the integrability of the function that sends $\mathbf{y}$ to $L(f(\mathbf{x},\mathbf{y});\mathbf{x})$.

Similarly, we have

$$L(f) \leq L(L(f(\mathbf{x},\mathbf{y});\mathbf{x});\mathbf{y}) \leq L(U(f(\mathbf{x},\mathbf{y});\mathbf{x});\mathbf{y}) \leq U(Uf(\mathbf{x},\mathbf{y});\mathbf{x});\mathbf{y}) \leq U(f). \tag{2.74}$$

If $L(f) = U(f)$, then $L(U(f(\mathbf{x},\mathbf{y});\mathbf{x});\mathbf{y}) = U(Uf(\mathbf{x},\mathbf{y});\mathbf{x});\mathbf{y})$. This proves the integrability of the function that sends $\mathbf{y}$ to $U(f(\mathbf{x},\mathbf{y});\mathbf{x})$. □

In the above proof it does not seem to matter whether one used the lower integral or the upper integral. This is clarified by the following remark. Define the difference function

$$h(\mathbf{y}) = U(f(\mathbf{x},\mathbf{y});\mathbf{x}) - L(f(\mathbf{x},\mathbf{y});\mathbf{x}). \tag{2.75}$$

Then $h \geq 0$. If f is Riemann integrable, then the analysis in the above proof shows that h is Riemann integrable and $I(h) = 0$. Thus h is negligible from the Riemann integral point of view. For instance, if h is continuous, then it is identically zero. In such a case the Fubini theorem is true in the form stated at the outset, that is, without any special consideration of upper and lower integrals.

Remark: Here is a technical comment about functions with Riemann integral zero. Suppose that $h \geq 0$ is a bounded function on a bounded set that is Riemann integrable. If the integral $I(h) = 0$, then for each real $k > 0$, the set where $h \geq k$ has Jordan measure zero. (This is because $1_{h\geq k} \leq (1/k)h$.) Conversely, if for each real $k > 0$ the set where $h \geq k$ has Jordan measure zero, then the integral $I(h) = 0$. (The reason in this case is that for each $k > 0$ there is a bound $h = h1_{h\geq k} + h1_{h<k} \leq M1_{h\geq k} + k$.)

There are situations where the properties of the Riemann integral are less pleasant. Suppose again that $h \geq 0$ is a bounded function on a bounded set that is Riemann integrable. The property that $I(h) = 0$ does not imply that the set where $h > 0$ has Jordan measure zero. In fact, while the sets where $h \geq 1/n$ increase to the set where $h > 0$, it does not follow that the set where $h > 0$ is Jordan measurable. (Here the relevant example is the function h defined on $(0, 1)$ by $h(x) = 1/q$ for $x = p/q$ rational and zero otherwise.) See [16] for a detailed discussion of such matters. ||

The Fubini theorem should also work for the more general absolutely convergent Riemann integrals. In the following it may be used for this case without further comment. There is a version of Fubini's theorem for the Lebesgue integral that is much more satisfactory, but that is a topic for another day.

The above exposition of Fubini's theorem for the Riemann integral follows unpublished notes on the Riemann integral by Mariusz Wodzicki [53].

2.2.2 Volume of a ball and area of a sphere

This section begins with the calculations of the n dimensional content (volume) of an n dimensional *ball* B_r of radius r in $\mathbf{R}^n$. This provides an excellent example of Fubini's theorem at work. The formula is of the form $v_n r^n$ with a coefficient v_n that depends on dimension n.

The formulas in low dimensions are familiar. When $n = 1, 2, 3$ the content is called length, area, or volume. The length of the interval is $v_1 r = 2r$. The area of the disk is $v_2 r^2 = \pi r^2$. The volume of the ball in three dimensions is $v_3 r^3 = \frac{4}{3}\pi r^3$.

The case $n = 0$ is harder to puzzle out, but it turns out that it is sensible to say that the ball of any radius has only one point, and $v_0 = 1$.

Theorem 2.19 (Recursion relation for volume of a ball) *The volume of the unit ball in n dimensions and in $n-2$ dimensions are related by*

$$v_n = \frac{2\pi}{n} v_{n-2}. \tag{2.76}$$

Proof: Write a point in $\mathbf{R}^n$ as $(\mathbf{x}, \mathbf{y})$, where $\mathbf{x}$ is in $\mathbf{R}^{n-2}$ and $\mathbf{y}$ is in $\mathbf{R}^2$. Then

$$v_n = m(B_1) = \int_{|\mathbf{y}| \leq 1} m(B_{\sqrt{1-|\mathbf{y}|^2}})\, d\mathbf{y} = \int_0^{2\pi} \int_0^1 v_{n-2} (1 - r^2)^{\frac{n-2}{2}} r\, dr\, d\theta. \tag{2.77}$$

The first equality is Fubini's theorem. The second equality is the change to polar coordinates in the plane. Make the change of variable $u = 1 - r^2$. This is

$$v_n = \pi v_{n-2} \int_0^1 u^{\frac{n-2}{2}}\, du = \frac{2\pi}{n} v_{n-2}. \tag{2.78}$$

It will then be no surprise that the final results for even dimension and for odd dimension have different forms. □

Corollary 2.20 (Even dimensions) *Suppose n is even. The volume of the unit ball in n dimensions is*

$$v_n = \frac{(2\pi)^{\frac{n}{2}}}{n \cdot (n-2) \cdots 4 \cdot 2}. \tag{2.79}$$

Corollary 2.21 (Odd dimensions) *Suppose n is odd. The volume of the unit ball in n dimensions is*

$$v_n = \frac{2(2\pi)^{\frac{n-1}{2}}}{n \cdot (n-2) \cdots 5 \cdot 3}. \tag{2.80}$$

It is also possible to write this result in terms of the Gamma function as

$$v_n = \frac{1}{n} \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{2.81}$$

From this it is easy to calculate the $n-1$ dimensional area of a $n-1$ dimensional *sphere* S_r of radius r in $\mathbf{R}^n$. The formula is of the form $a_{n-1} r^{n-1}$ with a coefficient a_{n-1} that depends on dimension n.

The formulas in low dimensions are familiar. When $n = 1, 2, 3$ the area counts two points, gives the length of the circle, or gives the area of the sphere. The count is $a_0 = 2$. The length is $a_1 = 2\pi r$. The area is $a_2 r^2 = 4\pi r^2$.

Theorem 2.22 (Area of a sphere) *The surface area of the unit sphere in n dimensions is*

$$a_{n-1} = n v_n. \tag{2.82}$$

Proof: The surface area is not a Jordan content, and in fact the surface area needs to be defined. In this proof we use the fact (proved later on) that the surface area may be computed in terms of the volume by the coarea formula. In the present case this takes the following form. Let $\Delta r > 0$ be a small number. Consider the shell consisting of the ball of radius $r + \Delta r$ minus the ball of radius r. This shell has volume $v_n (r + \Delta r)^n - v_n r^n$. To a good approximation this ball should have volume equal to the area $a_{n-1} r^{n-1}$ times the thickness Δr of the shell. As $\Delta r \to 0$ we see that the area $a_{n-1} r^{n-1}$ should be the derivative of $v_n r^n$. So it is $a_{n-1} r^{n-1} = n v_n r^{n-1}$. □

It is also possible to write this result in terms of the Gamma function as

$$a_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{2.83}$$

2.3 Convergence theorems

2.3.1 Uniform convergence

Say that f_n is a sequence of functions on a bounded set A, and f is another such function. We would like conditions that ensure the convergence of the integrals $I(f_n) \to I(f)$. The requirement of uniform convergence is certainly sufficient.

We review the basic definitions. Here is *pointwise convergence*. We say that f_n converges to f pointwise on A if for all $\mathbf{x}$ in A and for all $\epsilon > 0$ there is an N such that for $n \geq N$ we have that $|f_n(\mathbf{x}) - f(\mathbf{x})| < \epsilon$. In other words, the requirement is that for all $\mathbf{x}$ the limit as $n \to \infty$ of $f_n(\mathbf{x})$ is equal to $f(\mathbf{x})$. Sometimes this is written

$$\forall \mathbf{x} \, \forall \epsilon > 0 \, \exists N \, \forall n \geq N \, |f_n(\mathbf{x}) - f(\mathbf{x})| < \epsilon. \tag{2.84}$$

Notice that the $\forall \mathbf{x}$ may occur anywhere to the left of the $\exists N$. This says that the N may depend on $\mathbf{x}$.

Contrast that with the much stronger requirement of *uniform convergence*. We say that f_n converges to f uniformly on A if for all $\epsilon > 0$ there is an N such that for $n \geq N$ and for all $\mathbf{x}$ in A we have that $|f_n(\mathbf{x}) - f(\mathbf{x})| < \epsilon$. In other words, the requirement is that as n goes to infinity the function f_n approaches the function f in the sense that throughout the set A the deviation of f_n from f becomes arbitrary small. Sometimes this is written

$$\forall \epsilon > 0 \, \exists N \, \forall n \geq N \, \forall \mathbf{x} \, |f_n(\mathbf{x}) - f(\mathbf{x})| < \epsilon. \tag{2.85}$$

Notice that the $\forall \mathbf{x}$ may occur anywhere to the right of the $\exists N$ (but before the final inequality). Thus the N may not depend on $\mathbf{x}$. Clearly uniform convergence implies pointwise convergence.

The most famous result about uniform convergence is the following one. It is proved by a standard three ϵ argument.

Theorem 2.23 *Suppose that each f_n is continuous and that $f_n \to f$ uniformly. Then f is continuous.*

Sometimes it is necessary to find examples of functions that do not converge pointwise or do not converge uniformly. The condition for a function to not converge pointwise is

$$\exists \mathbf{x} \, \exists \epsilon > 0 \, \forall N \, \exists n \geq N \, |f_n(\mathbf{x}) - f(\mathbf{x})| \geq \epsilon. \tag{2.86}$$

The condition for a function to not converge uniformly is

$$\exists \epsilon > 0 \, \forall N \, \exists n \geq N \, \exists \mathbf{x} \, |f_n(\mathbf{x}) - f(\mathbf{x})| \geq \epsilon. \tag{2.87}$$

It is easier to prove that a function is not uniformly convergent, since the $\mathbf{x}$ is allowed to depend on N. In many cases it is possible to take $n = N$.

The following theorem involving uniform convergence and integration is elementary.

Theorem 2.24 *Suppose that all the f_n and f are Riemann integrable on the bounded set A. If f_n converges to f uniformly on A, then $I(f_n)$ converges to $I(f)$.*

One way to prove this is to first prove that $I(|f_n - f|)$ converges to zero.

There is a remarkable theorem of Dini that shows that under certain very special circumstances uniform convergence is automatic. The context is that of a sequence of functions that is decreasing. (There is an obvious variant with a sequence of functions that is increasing).

Theorem 2.25 (Dini's theorem) *Let A be a compact set. Suppose that f_n is a decreasing sequence of continuous functions on A. Suppose that $f_n \downarrow f$ converges pointwise. Finally, suppose that f is also continuous. Then f_n converges uniformly to f.*

Proof: Let $h_n = f_n - f$. Then $h_n \to 0$ is also decreasing, and it has pointwise limit zero. Furthermore, h_n is continuous. Let $\epsilon > 0$. By continuity, for each n the set U_n where $h_n < \epsilon$ is open. The open sets U_n are increasing. Now we use pointwise convergence. For each $\mathbf{x}$ in A, there are n sufficiently large (depending on $\mathbf{x}$) such that $h_n(x) < \epsilon$. Hence for such n we have $x \in U_n$. This shows that the union of the U_n for all n includes A. In other words, the U_n form an open cover of A. Since A is compact, there is a finite subcover. This implies that there is an N such that U_N includes A. Also the U_n for $n \geq N$ include A. In other words, for all $n \geq N$ the set where $h_n < \epsilon$ includes all of A. This is uniform convergence. □

2.3.2 The bounded convergence theorem

The bounded convergence theorem (statement)

It is rather amazing that uniform convergence is not necessary for convergence of the integrals. This section presents the *bounded convergence theorem* for the Riemann integral, first proved by Arzela in 1885.

Let A be a bounded set. Let f_n be a sequence of real Riemann integrable functions on A that are bounded by a fixed constant M. Thus there is an M such that $|f_n(x)| \leq M$ for all x in A and all n. Then the sequence of functions is said to be *bounded*. It is bounded in two ways: by the fixed bounded set A on which the functions are all defined, and by the fixed constant M by which their values are bounded.

Suppose that the sequence of functions f_n is bounded in this sense, and suppose that $f_n \to f$ pointwise as $n \to \infty$, where f is also Riemann integrable. This is called *bounded convergence* (for the Riemann integral). The theorem says that in this situation the integrals converge: $I(f_n) \to I(f)$ as $n \to \infty$.

Theorem 2.26 (Bounded convergence for the Riemann integral) *Let A be a bounded set. Let f_n be a sequence of real Riemann integrable functions on A that are bounded in absolute value by a fixed constant M. Thus there is an M such that*

$$|f_n(x)| \leq M < \infty \tag{2.88}$$

for all x in A and all n. Suppose that for all x in A that

$$f_n(x) \to f(x) \tag{2.89}$$

as $n \to \infty$, where f is also Riemann integrable. Then the integrals converge:

$$\int_A f_n(\mathbf{x})\, d\mathbf{x} \to \int_A f(\mathbf{x})\, d\mathbf{x} \tag{2.90}$$

as $n \to \infty$.

The bounded convergence theorem (proof)*

We begin with a lemma about approximation of lower integrals via continuous functions. This is a special case of a result proved earlier, but it is convenient to record it here.

Lemma 2.27 *Let A be a bounded set. Let $f \geq 0$ be a bounded real function on A. Then for every $\epsilon > 0$ there exists a continuous function g with $0 \leq g \leq f$ and $L(f) - I(g) < \epsilon$.*

Proof: From the definition of the lower integral it follows that there is a step function k with $0 \leq k \leq f$ and with $L(f) - I(k) < \epsilon/2$. However one can approximate each step by a continuous trapezoid with very steep sides, so that the resulting trapezoidal function g satisfies $0 \leq g \leq k$ and $I(k) - I(g) < \epsilon/2$. Then $I(f) - I(g) < I(f) - I(k) + I(k) - I(g) < \epsilon/2 + \epsilon/2 = \epsilon$. □

We now turn to an important result on monotone convergence; it will be used to prove the boundeded convergence theorem. In the monotone convergence result there is no assumption that the functions are Riemann integrable. However they have lower integrals, so the result is stated in terms of lower integrals.

Lemma 2.28 (Monotone convergence for the lower Riemann integral) *Let A be a bounded set. Let p_n be a sequence of real functions on A such that there is a constant M with $0 \leq p_n \leq M$. Suppose that $p_n \downarrow 0$ pointwise as $n \to \infty$. Then the lower integrals converge: $L(p_n) \downarrow 0$ as $n \to \infty$.*

Proof: The assumption is that p_n is a sequence of bounded functions with $p_n \downarrow 0$, that is, the functions are decreasing pointwise to zero. If Dini's theorem applied, then we would have $I(p_n) \downarrow 0$. However, there are obvious problems. The p_n need not be continuous, so Dini's theorem does not apply. Not only that, the p_n need not be Riemann integrable. In this case $I(p_n)$ is not even defined, and we must work with $L(p_n)$.

Ultimately the proof will reduce to Dini's theorem. Dini's theorem relies on the assumption that the functions are defined on a compact set, but that set can be taken to be a closed bounded non-degenerate cell C with $A \subseteq C$. (In fact this is how the Riemann integral is defined.) Uniform convergence is hiding somewhere in the world of pointwise convergence, but it is well hidden. We first use the fact that we can approximate by continuous functions. The essential idea is to approximate better and better as we go along the sequence.

Consider $\epsilon > 0$. Choose g_i continuous with $0 \leq g_i \leq p_i$ and

$$L(p_i) - I(g_i) \leq \epsilon \frac{1}{2^i}. \tag{2.91}$$

Unfortunately, there is no guarantee that the functions g_i are decreasing. To fix this, let

$$h_n = \min(g_1, \ldots, g_n). \tag{2.92}$$

Then the $h_n \downarrow 0$ are decreasing pointwise to zero, and each h_n is continuous. Hence by Dini's theorem $I(h_n) \downarrow 0$. This looks promising.

The problem is that in general the integral of a minimum can be much smaller than the integrals of the individual functions. So we need to use special features of the choices made above to ensure that $I(g_n) - I(h_n)$ is small. We have for each $j = 1, \ldots, n$ that

$$g_n - g_j \leq \max(g_j, g_n) - g_j \leq \sum_{i=1}^{n-1} (\max(g_i, g_n) - g_i) \tag{2.93}$$

since each $\max(g_i, g_n) - g_i \geq 0$. The sum on the right hand side does not depend on j, so it is an upper bound for all of the $g_n - g_j$. By definition of h_n we then have

$$g_n - h_n \leq \sum_{i=1}^{n-1} (\max(g_i, g_n) - g_i). \tag{2.94}$$

The above inequality only involves Riemann integrable functions. Hence we are allowed to use the additivity of the integral to write the integral of the right hand side as the sum of the integrals. This gives

$$I(g_n) - I(h_n) \leq \sum_{i=1}^{n-1} (I(\max(g_i, g_n)) - I(g_i)). \tag{2.95}$$

For $i = 1, \ldots, n$ we have $g_i \leq p_i$ and $g_n \leq p_n \leq p_i$, so $\max(g_i, g_n) \leq p_i$. Therefore we have

$$I(g_n) - I(h_n) \leq \sum_{i=1}^{n-1} (L(p_i) - I(g_i)). \tag{2.96}$$

Hence,

$$I(g_n) - I(h_n) \leq \sum_{i=1}^{n-1} \epsilon \frac{1}{2^i} = \epsilon \left(1 - \frac{1}{2^{n-1}}\right). \tag{2.97}$$

This is the result that is needed. We conclude by noting that

$$L(p_n) - I(h_n) = L(p_n) - I(g_n) + I(g_n) - I(h_n) \leq \epsilon \frac{1}{2^n} + \epsilon \left(1 - \frac{1}{2^{n-1}}\right). \tag{2.98}$$

This gives

$$L(p_n) \leq I(h_n) + \epsilon \left(1 - \frac{1}{2^n}\right) < I(h_n) + \epsilon. \tag{2.99}$$

So when n is so large that $I(h_n)$ is less than ϵ, then $L(p_n)$ is less than 2ϵ. $\square$

Remark: In the above proof it could be tempting to use $g_n - g_j \leq p_j - g_j$ for $j \leq n$ to prove $g_n - h_n \leq \sum_{i=1}^{n}(p_i - g_i)$. The problem would be that the right hand side only has a lower integral. Furthermore, the lower integral is only known to be superadditive, so the lower integral of the sum could be much larger than the sum of the lower integrals. This was avoided in the proof by using $\max(g_i, g_n)$ in place of p_i. ||

Proof: The next step is to use the monotone convergence theorem for the lower Riemann integral to prove the bounded convergence theorem for the Riemann integral. It actually suffices to prove an apparently weaker version of the theorem. This weaker version says that if $\bar{f}_n \geq 0$ and $\bar{f}_n \leq 2M$ and $\bar{f}_n \to 0$ pointwise, then $I(\bar{f}_n) \to 0$. If we set $\bar{f}_n = |f_n - f|$, we see that $I(|f_n - f|) \to 0$. Hence $|I(f_n) - I(f)| = |I(f_n - f)| \leq I(|f_n - f|) \to 0$.

Thus suppose that $\bar{f}_n \geq 0$ with $\bar{f}_n \leq 2M$ and $\bar{f}_n \to 0$ pointwise. It is sufficient to show that $I(\bar{f}_n) \to 0$. Here is the strategy. For each n let $p_n = \sup_{k \geq n} \bar{f}_k$ be the pointwise supremum of the $\bar{f}_k$ for all $k \geq n$. This is an infinite supremum! Then $0 \leq \bar{f}_n \leq p_n$. Furthermore, $p_n \downarrow 0$ is decreasing pointwise to zero. (Why?) There is an apparent problem: p_n need not be Riemann integrable. In this case $I(p_n)$ is not defined. However the lemma on monotone convergence proves that the lower integrals $L(p_n) \to 0$. But then $I(\bar{f}_n) = L(\bar{f}_n) \leq L(p_n) \to 0$, so we have the desired result. □

This bounded convergence theorem for the Riemann integral was first proved by Arzela in 1885. It states that if one has a suitably bounded sequence of Riemann integrable functions that converges pointwise to a Riemann integrable function, then the limit of the integrals is the integral of the limit. This was long before the introduction of the Lebesgue integral; Lebesgue's thesis was published in 1902. The Lebesgue theory gives a much more powerful result. It says that if one has a dominated sequence of Lebesgue integrable functions that converge pointwise, then the limit is automatically Lebesgue integable, and the limit of the integrals is the integral of the limit.

This proof given above is elegant but somewhat strange. The theorem is a result about a sequence of Riemann integrable functions, but the proof uses a sequence of functions that need not in general be Riemann integrable. In fact, the proof uses ideas that are perhaps more natural in the context of the Lebesgue integral. It is taken from a paper by Luxemburg [26]. In this paper Luxemburg says that his proof is essentially the same as Hausdorff's proof published in 1927. However at one point Hausdorff gives an incorrect argument. This needs to be replaced by other reasoning; this is supplied by Luxemburg.

2.3.3 Bounded convergence with a continuous parameter*

In rough summary, the boundeded convergence theorem says that if A is bounded and the functions $f_n(\mathbf{x})$ are uniformly bounded ($|f_n(\mathbf{x})| \leq M$), then the condition that $\lim_{n\to\infty} f_n(\mathbf{x}) = f(\mathbf{x})$ pointwise in $\mathbf{x}$ implies that

$$\lim_{n\to\infty} \int_A f_n(\mathbf{x})\, d\mathbf{x} = \int_A f(\mathbf{x})\, d\mathbf{x}. \tag{2.100}$$

There is a variant form of the boundeded convergence theorem in which the functions depend on a continuous parameter.

Theorem 2.29 (Continuous parameter bounded convergence) *If A is bounded and the functions $f(\mathbf{x}, \mathbf{y})$ are uniformly bounded ($|f(\mathbf{x}, \mathbf{y})| \leq M$), then the condition that $\lim_{\mathbf{y}\to\mathbf{y}_0} f(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}, \mathbf{y}_0)$ pointwise in $\mathbf{x}$ implies that*

$$\lim_{\mathbf{y}\to\mathbf{y}_0} \int_A f(\mathbf{x}, \mathbf{y})\, d\mathbf{x} = \int_A f(\mathbf{x}, \mathbf{y}_0)\, d\mathbf{x}. \tag{2.101}$$

This variant form is a consequence of the original theorem and of a general fact about continuity in the metric space setting. The general fact is that if for every sequence $\mathbf{a}_n$ with $\lim_n \mathbf{a}_n = \mathbf{y}_0$ we have $\lim_n g(\mathbf{a}_n) = g(\mathbf{y}_0)$, then $g(\mathbf{y})$ is continuous at $\mathbf{y}_0$. We can then apply this to $g(\mathbf{y}) = \int_A f(\mathbf{x}, \mathbf{y})\, d\mathbf{x}$.

Theorem 2.30 (Differentiating inside an integral) *Let A be a bounded set that is Jordan measurable, and U be an open set. Suppose that $f(\mathbf{x}, \mathbf{y})$ is C^1, and write $f'_{,2}(\mathbf{x}, \mathbf{y})$ for the covector of partial derivatives with respect to the $\mathbf{y}$ variables. Suppose also that there is a constant M with $|f'_{,2}(\mathbf{x}, \mathbf{y})| \leq M$. Let*

$$g(\mathbf{y}) = \int_A f(\mathbf{x}, \mathbf{y})\, d\mathbf{x}. \tag{2.102}$$

Then

$$g'(\mathbf{y})\mathbf{h} = \int_A f'_{,2}(\mathbf{x}, \mathbf{y})\mathbf{h}\, d\mathbf{x}. \tag{2.103}$$

Proof: Write

$$g(\mathbf{y} + \mathbf{h}) - g(\mathbf{y}) - \int_A f'_{,2}(\mathbf{x}, \mathbf{y})\mathbf{h}\, d\mathbf{x} = \int_A (f(\mathbf{x}, \mathbf{y} + \mathbf{h}) - f(\mathbf{x}, \mathbf{y}) - f'_{,2}(\mathbf{x}, \mathbf{y})\mathbf{h})\, d\mathbf{x}. \tag{2.104}$$

This can also be written

$$g(\mathbf{y} + \mathbf{h}) - g(\mathbf{y}) - \int_A f'_{,2}(\mathbf{x}, \mathbf{y})\mathbf{h}\, d\mathbf{x} = \int_A \int_0^1 (f'_{,2}(\mathbf{x}, \mathbf{y} + t\mathbf{h}) - f'_{,2}(\mathbf{x}, \mathbf{y}))\mathbf{h}\, dt\, d\mathbf{x}. \tag{2.105}$$

This has absolute value bounded by

$$\epsilon(\mathbf{y}, \mathbf{h}) = \int_A \int_0^1 |f'_{,2}(\mathbf{x}, \mathbf{y} + t\mathbf{h}) - f'_{,2}(\mathbf{x}, \mathbf{y})|\, dt\, d\mathbf{x}. \tag{2.106}$$

times $|\mathbf{h}|$. All that remains is to show that $\epsilon(\mathbf{y}, \mathbf{h}) \to 0$ as $\mathbf{h} \to 0$. For fixed $\mathbf{x}$ and t the integrand approaches zero as $\mathbf{h} \to 0$. The conclusion follows from the bounded convergence theorem. □

This theorem gives a practical condition for differentiating an integral depending on a parameter with respect to the parameter. For the theorem to apply it is important that the bound on $f'_{,2}(\mathbf{x}, \mathbf{y})$ be independent of $\mathbf{x}$ and of $\mathbf{y}$.

2.3.4 The dominated convergence theorem

The *dominated convergence theorem* applies to the absolutely convergent Riemann integral. This integral is defined by a series expansion. The dominated convergence theorem for a series is the following result.

Proposition 2.31 *For each n consider a sequence $a_{n1}, a_{n2}, a_{n3}, \ldots$. Suppose that the the sequences are dominated by some series $b_1, b_2, b_3, \ldots$ in the sense that*

$$|a_{nk}| \leq b_k, \tag{2.107}$$

where

$$\sum_k b_k < +\infty. \tag{2.108}$$

Suppose that for each k we have

$$a_{nk} \to a_k \tag{2.109}$$

as $n \to \infty$. Then

$$\sum_k a_{nk} \to \sum_k a_k \tag{2.110}$$

as $n \to \infty$.

The proof of this proposition is left for the reader.

The same idea gives the following important theorem for the absolutely convergent Riemann integral. Say that a real function on an open subset U of $\mathbf{R}^n$ is *locally Riemann integrable* if it is Riemann integrable on each bounded, closed, non-degenerate cell that is a subset of U.

Theorem 2.32 (Dominated convergence theorem) *Let f_n be a real function on an open subset U of $\mathbf{R}^n$ that is locally Riemann integrable. Suppose that f_n is dominated by another locally Riemann integrable function g in the sense that*

$$|f_n(x))| \leq g(x), \tag{2.111}$$

where

$$\int_U g(\mathbf{x})\, d\mathbf{x} < +\infty. \tag{2.112}$$

Suppose that there is a locally Riemann integrable function f such that for each $\mathbf{x}$ we have

$$f_n(\mathbf{x}) \to f(\mathbf{x}) \tag{2.113}$$

as $n \to \infty$. Then

$$\int_U f_n(\mathbf{x})\, d\mathbf{x} \to \int_U f(\mathbf{x})\, d\mathbf{x} \tag{2.114}$$

as $n \to \infty$.

2.4 Affine change of variable and approximate delta functions

2.4.1 Affine change of variable

Affine change of variable (statement)

If A is a square matrix, then there is an associated number $\det(A)$, the *determinant* of A. The determinant of a product of matrices is the product of the determinants. The determinant of the identity matrix I is $\det(I) = 1$. A square matrix A has an inverse matrix A^{-1} if and only if $\det(A) \neq 0$.

An affine transformation is given by $\mathbf{y} = A\mathbf{x} + \mathbf{b}$. It is invertible if A has an inverse matrix.

Theorem 2.33 (Affine change of variable) *Under an invertible affine transformation* $\mathbf{y} = A\mathbf{x} + \mathbf{b}$ *an integral transforms by*

$$\int f(\mathbf{y})\, d\mathbf{y} = \int f(A\mathbf{x} + \mathbf{b})\, d\mathbf{x}\, |\det(A)|. \tag{2.115}$$

Affine change of variable) (proof)*

Various linear algebra operations may be expressed in terms of *elementary matrices*. These are square matrices E. For each such matrix there is a corresponding linear transformation $\mathbf{x} \mapsto E\mathbf{x}$. There are three kinds of elementary matrix:

1. A *scaling* matrix that is obtained by multiplying row i of the identity matrix by a constant $a \neq 0$. The corresponding linear transformation sends x_i to ax_i and leaves the other coordinates unchanged. The determinant is a.

2. An *interchange* matrix is obtained by interchanging rows i and j of the identity matrix. The corresponding linear transformation sends x_i to x_j and x_j to x_i and leaves the other coordinates unchanged. The determinant is -1.

3. A *shear* matrix is obtained by modifying row i of the identity matrix by adding a constant c times row j. The corresponding linear transformation sends x_i to $x_i + cx_j$ and leaves the other coordinates unchanged. The determinant is 1.

Every invertible matrix may be written as a product of elementary matrices. This gives a way of computing the determinant. The significance of the determinant is that its absolute value gives a factor by which volumes are multiplied. This is reflected in the ways that integrals are computed.

1. For a scaling the integral transforms by

$$|a| \int f(ax_i)\, dx_i = \int f(y_i)\, dy_i. \tag{2.116}$$

   Volumes are multiplied by the absolute value $|a|$ of the scale factor, which is the absolute value of the determinant.

2. For an interchange the integral transforms by

$$\int f(x_i, x_j)\, dx_i\, dx_j = \int f(y_j, y_i)\, dy_i\, dy_j. \tag{2.117}$$

   The absolute value of the determinant is 1. Volumes are left unchanged.

3. For an shear use Fubini's theorem to integrate with respect to x_i with x_j fixed.

$$\int f(x_i + cx_j, x_j)\, dx_i\, dx_j = \int f(y_i, x_j)\, dy_i\, dx_j = \int f(y_i, y_j)\, dy_i\, dy_j. \tag{2.118}$$

   The determinant is 1. Volumes are left unchanged

These considerations lead to a proof of the change of variables theorem for an affine transformation.

Proof: Write $A = E_1 \cdots E_k$ as a product of elementary matrices. Then repeated use of the integral identities above gives

$$\int f(\mathbf{y})\, d\mathbf{y} = |\det(E_1)| \cdots |\det(E_k)| \int f(E_1 \cdots E_k \mathbf{x})\, d\mathbf{x}. \tag{2.119}$$

By the multiplicative properties of absolute value and determinant we have

$$|\det(E_1)| \cdots |\det(E_k)| = |\det(E_1) \cdots \det(E_k)| = |\det(E_1 \cdots E_k)|. \tag{2.120}$$

This gives the result. □

2.4.2 Approximate delta functions

Consider a function $\delta_1(\mathbf{x}) \geq 0$ that is positive and has integral one. Often we shall suppose for convenience that it is continuous and has compact support. For instance, it might vanish outside the closed ball of radius $c > 0$. However there are important examples that do not satisfy these extra properties.

Construct a family of functions $\delta_\epsilon(\mathbf{x}) \geq 0$ defined for each $\epsilon > 0$ by scaling according to the rule

$$\delta_\epsilon(\mathbf{x}) = \delta_1(\frac{\mathbf{x}}{\epsilon})\frac{1}{\epsilon^n}. \tag{2.121}$$

Then it is easy to see that for each $\epsilon > 0$ the n-dimensional integral

$$\int \delta_\epsilon(\mathbf{x})\, d\mathbf{x} = 1. \tag{2.122}$$

This family of functions (considered for $\epsilon > 0$ small) will be called a family of *approximate delta functions*. In the case of compact support δ_ϵ vanishes outside the closed ball of radius ϵc.

Theorem 2.34 *Let f be a bounded continuous function, and let δ_ϵ be a family of approximate delta functions. Then*

$$\lim_{\epsilon \to 0} \int f(\mathbf{x}) \delta_\epsilon(\mathbf{x})\, d\mathbf{x} = f(0). \tag{2.123}$$

Proof: Write

$$\int f(\mathbf{x}) \delta_\epsilon(\mathbf{x})\, d\mathbf{x} = \int f(\mathbf{x}) \delta_1(\frac{\mathbf{x}}{\epsilon}) \frac{1}{\epsilon^n}\, d\mathbf{x}. \tag{2.124}$$

Make the change of variable $\mathbf{x} = \epsilon \mathbf{u}$. This gives

$$\int f(\mathbf{x}) \delta_\epsilon(\mathbf{x})\, d\mathbf{x} = \int f(\epsilon \mathbf{u}) \delta_1(\mathbf{u})\, d\mathbf{u}. \tag{2.125}$$

The integrand converges pointwise in $\mathbf{u}$ to $f(0)\delta_1(\mathbf{u})$ on the closed ball of radius c and is bounded by a constant independent of ϵ. By the dominated convergence theorem the integral converges to $f(0)$. $\square$

Remark: The amazing thing about this theorem is that the right hand side is independent of the particular choice of approximate delta function. For this reason, it is customary to write it in the form

$$\int f(\mathbf{x}) \delta(\mathbf{x})\, d\mathbf{x} = f(0). \tag{2.126}$$

Of course, there is no such *delta function* $\delta(\mathbf{x})$ with this property, but it is still convenient to describe its properties. While the left hand side does not have a literal meaning, it gives an easy way to remember the result. Furthermore, it allows one to summarize various useful facts, such as

$$\int \delta(\mathbf{x})\, d\mathbf{x} = 1 \tag{2.127}$$

and

$$f(\mathbf{x})\delta(\mathbf{x}) = f(0)\delta(\mathbf{x}). \tag{2.128}$$

Also, the delta function is even

$$\delta(-\mathbf{x}) = \delta(\mathbf{x}) \tag{2.129}$$

and transforms under a change of scale $a \neq 0$ by

$$|a|^n \delta(a\mathbf{x}) = \delta(\mathbf{x}). \tag{2.130}$$

More generally, we have

$$|a|^n \delta(a\mathbf{x} - \mathbf{y}) = \delta(\mathbf{x} - \frac{1}{a}\mathbf{y}). \tag{2.131}$$

The reader may check that each of these suggests a meaningful statement about approximate delta functions. ||

The integrals involving approximate delta functions are often of the form

$$\int h(\mathbf{y})\delta_\epsilon(\mathbf{z}-\mathbf{y})\,d\mathbf{y} = \int h(\mathbf{z}-\mathbf{x})\delta_\epsilon(\mathbf{x})\,d\mathbf{x}. \tag{2.132}$$

The two integrals expressions are equivalent after the change of variable $\mathbf{y} = \mathbf{z} - \mathbf{x}$. The new feature is that we look at the result as a function of $\mathbf{z}$. If $|\mathbf{h}(\mathbf{y})| \leq M$, then each integral above as a function of $\mathbf{z}$ has magnitude bounded by M. Furthermore, it is a continuous function of $\mathbf{z}$. Also, suppose that $\mathbf{h}$ has compact support K. Then this integral as a function of $\mathbf{z}$ has compact support in the set $K_{\epsilon c}$ of points a distance at most ϵc from K. The result may be stated in this context as follows.

Theorem 2.35 *Let h be a bounded continuous function, and let δ_ϵ be a family of approximate delta functions. Then*

$$\lim_{\epsilon\to 0}\int h(\mathbf{y})\delta_\epsilon(\mathbf{z}-\mathbf{y})\,d\mathbf{y} = \lim_{\epsilon\to 0}\int h(\mathbf{z}-\mathbf{x})\delta_\epsilon(\mathbf{x})\,d\mathbf{x} = h(\mathbf{z}). \tag{2.133}$$

Example: There are many examples of approximate delta functions, some of which have particularly nice algebraic properties. In one dimension one of the most famous is the Gaussian

$$\delta_\epsilon(x) = \frac{1}{\sqrt{2\pi\epsilon^2}}\exp(-\frac{1}{2\epsilon^2}x^2). \tag{2.134}$$

This has the advantage that for each $\epsilon > 0$ it is an entire function of x. The Gaussian example has an obvious generalization to n dimensions. Anther important example in one dimension is the Poisson kernel

$$\delta_\epsilon(x) = \frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2}. \tag{2.135}$$

This has the advantage that it is a rational function. An obvious example is

$$\delta_\epsilon(x) = \frac{1}{2\epsilon}1_{(-\epsilon,\epsilon)}(x). \tag{2.136}$$

This is uniformly distributed in an interval, but is not given by a continuous function. A final example is

$$\delta_\epsilon(x) = \frac{1}{\epsilon^2}(\epsilon - |x|) \vee 0. \tag{2.137}$$

This is also concentrated in an interval, but is also continuous. ||

2.4.3 Change of variable for approximate delta functions

Change of variable for approximate delta functions (statement)

In general a change of variable $\mathbf{y} = \mathbf{g}(\mathbf{x})$ is not an affine function. However near a point $\mathbf{a}$ it is near to an affine function $\mathbf{y} = \mathbf{g}(\mathbf{a}) + \mathbf{g}'(\mathbf{a})(\mathbf{y} - \mathbf{a})$. For the case of an approximate delta function this formula becomes exact. The following discussion makes this idea precise.

- The approximate delta function in the following is taken so that each δ_ϵ is continuous and has support in a ball of radius c about zero.
- The change of variable function $\mathbf{g} : U \to V$ is one-to-one from the open set U onto the open set V and is C^1. Furthermore, $\det \mathbf{g}'$ is never zero.
- The geometrical setting is the following. Let K be a compact subset of V. Write $K_{c\epsilon}$ for the compact subset of points within distance $c\epsilon$ of K. Choose $\epsilon_1 > 0$ so that $\mathbf{g}$ maps a compact set $L \subseteq U$ onto $K_{c\epsilon_1} \subseteq V$.

Proposition 2.36 *For $0 < \epsilon \leq \epsilon_1$ the functions*

$$\mathbf{y} \mapsto \int_L h(\mathbf{x})\delta_\epsilon(\mathbf{g}(\mathbf{x}) - \mathbf{y})\, d\mathbf{x} \tag{2.138}$$

are continuous on K, and these functions are bounded independently of ϵ.

Theorem 2.37 *Let h be a bounded continuous function on U. For every $\mathbf{y}$ in K*

$$\lim_{\epsilon \to 0} \int_L h(\mathbf{x})\delta_\epsilon(\mathbf{g}(\mathbf{x}) - \mathbf{y})\, d\mathbf{x}\, |\det \mathbf{g}'(\mathbf{g}^{-1}(\mathbf{y}))| = h(\mathbf{g}^{-1}(\mathbf{y})). \tag{2.139}$$

This formula can also be written as

$$\lim_{\epsilon \to 0} \int_L h(\mathbf{x})\delta_\epsilon(\mathbf{g}(\mathbf{x}) - \mathbf{y})\, d\mathbf{x}\, |\det \mathbf{g}'(\mathbf{g}^{-1}(\mathbf{y}))| = \lim_{\epsilon \to 0} \int_L \mathbf{h}(\mathbf{x})\delta_\epsilon(\mathbf{x} - \mathbf{g}^{-1}(\mathbf{y}))\, d\mathbf{x}. \tag{2.140}$$

Remark: Again there is a common abbreviation for this kind of result. One could write

$$\int_L h(\mathbf{x})\delta(\mathbf{g}(\mathbf{x}) - \mathbf{y})\, d\mathbf{x}\, |\det \mathbf{g}'(\mathbf{g}^{-1}(\mathbf{y}))| = h(\mathbf{g}^{-1}(\mathbf{y})). \tag{2.141}$$

Even more radically, one could write

$$\delta(\mathbf{g}(\mathbf{x}) - \mathbf{y})\, |\det \mathbf{g}'(\mathbf{g}^{-1}(\mathbf{y}))| = \delta(\mathbf{x} - \mathbf{g}^{-1}(\mathbf{y})). \tag{2.142}$$

Even if there is no such thing as a delta function, such identities involving delta functions make perfectly good sense. ||

Change of variable for approximate delta functions (proof)*

Proof: First it is helpful to take some care about the region of integration. The integral is over the set of $\mathbf{x}$ with $|g(\mathbf{x}) - \mathbf{y}| \leq \epsilon c$. Consider the set $K_{\epsilon c}$ consisting of all points with distance no greater than ϵc from K. There is an ϵ_1 such that $K_{\epsilon_1 c}$ is a subset of $V = \mathbf{g}(U)$. Since this is a compact set, the function $\|(\mathbf{g}^{-1})'\|$ is bounded there by some constant λ. Now suppose that $\mathbf{y}$ is in K and $|\mathbf{y}' - \mathbf{y}| \leq \epsilon c$ for some $\epsilon \leq \epsilon_1$. Then $|\mathbf{g}^{-1}(\mathbf{y}') - \mathbf{g}^{-1}(\mathbf{y})| \leq \lambda |\mathbf{y}' - \mathbf{y}| \leq \lambda c \epsilon$. In particular for $\mathbf{x}$ in the region of integration $|\mathbf{x} - \mathbf{g}^{-1}(\mathbf{y})| \leq \lambda |\mathbf{g}(\mathbf{x}) - \mathbf{y}| \leq \lambda \epsilon c$.

Make the change of variable $\mathbf{x} = \mathbf{g}^{-1}(\mathbf{y}) + \epsilon \mathbf{t}$. The integration region is now $|\mathbf{t}| \leq \lambda c$. This is a fixed bounded set, independent of ϵ. The integral on the left hand side is

$$\int h(\mathbf{x}) \delta_1 \left(\frac{\mathbf{g}(\mathbf{x}) - \mathbf{y}}{\epsilon} \right) \frac{1}{\epsilon^n} \, d\mathbf{x} = \int h(\mathbf{g}^{-1}(\mathbf{y}) + \epsilon \mathbf{t}) \delta_1 \left(\frac{\mathbf{g}(\mathbf{g}^{-1}(\mathbf{y}) + \epsilon \mathbf{t}) - \mathbf{g}(\mathbf{g}^{-1}(\mathbf{y}))}{\epsilon} \right) \, d\mathbf{t}. \tag{2.143}$$

The integrand is bounded, independent of ϵ. By the dominated convergence theorem the limit as $\epsilon \to 0$ of this is

$$\int h(\mathbf{g}^{-1}(\mathbf{y})) \delta_1(\mathbf{g}'(\mathbf{g}^{-1}(\mathbf{y}))\mathbf{t}) \, d\mathbf{t} = \int h(\mathbf{g}^{-1}(\mathbf{y})) \frac{1}{|\det(\mathbf{g}'(\mathbf{g}^{-1}(\mathbf{y})))|} \delta_1(\mathbf{u}) \, d\mathbf{u}. \tag{2.144}$$

The last step is the change of variables $\mathbf{u} = \mathbf{g}'(\mathbf{g}^{-1}(\mathbf{y}))\mathbf{t}$. This involves a matrix $\mathbf{g}'(\mathbf{g}^{-1}(\mathbf{y}))$ that only depends on the parameter $\mathbf{y}$ and so may be regarded as constant. Performing the $\mathbf{u}$ integral gives the result on the right hand side. □

Remark: For later use we note that the integral $\int_V h(\mathbf{x}) \delta_\epsilon(\mathbf{g}(\mathbf{x}) - \mathbf{y}) \, d\mathbf{x}$ bas a function of $\mathbf{y}$ is uniformly bounded on K, independent of ϵ. Furthermore, for each ϵ it is continuous in $\mathbf{y}$. (The dominated convergence also applies to integrals that depend on a real parameter such as $\mathbf{y}$.) ||

2.5 Change of variable for the Riemann integral

2.5.1 Change of variables

Change of variables for the Riemann integral (statement)

Theorem 2.38 (Change of variables for the Riemann integral) *Consider a one-to-one C^1 function $\mathbf{g}$ from open set U onto open set V. Furthermore, suppose that $\det \mathbf{g}'(\mathbf{x}) \neq 0$. Let f be a Riemann integrable function on V with compact support. Suppose that $T \subseteq U$ is a compact set such that the support of f is a subset of $\mathbf{g}(T) \subseteq V$. Then*

$$\int_{\mathbf{g}(T)} f(\mathbf{y}) \, d\mathbf{y} = \int_T f(g(\mathbf{x})) |\det \mathbf{g}'(\mathbf{x})| \, d\mathbf{x}. \tag{2.145}$$

Sometime the function $f(\mathbf{g}(\mathbf{x}))$ is called the *pullback* of the function $f(\mathbf{y})$. Also, the integrand $f(g(\mathbf{x}))|\det \mathbf{g}'(\mathbf{x})| \, d\mathbf{x}$ is called the pullback of the integrand $f(\mathbf{y}) \, d\mathbf{y}$. The change of variable formula is about the pullback of an integrand.

Th theorem could also be written in a compact notation

$$I_{\mathbf{g}(T)}(f) = I_T(f \circ \mathbf{g} \,|\det \mathbf{g}'|). \tag{2.146}$$

Example: The classical example is the case of polar coordinates, where x and y are written as functions of r and θ by $x = r\cos(\theta)$ and $y = r\sin(\theta)$ The derivative is

$$\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}. \tag{2.147}$$

The determinant is r. This proves that

$$\int f(x,y)\,dx\,dy = \int f(r\cos(\theta), r\sin(\theta))\, r\,dr\,d\theta. \tag{2.148}$$

The integral on the right hand side may be taken with $0 < r < +\infty$ and $0 < \theta < 2\pi$. This maps into an open subset of the plane that is sufficiently large for integration purposes. ||

Change of variable for the Riemann integral (proof)*

Proof: First we give the proof for the case when f is a continuous function with compact support K. The function f is equal to zero on $V \setminus \mathbf{g}(T)$. Similarly, the function $f \circ \mathbf{g}$ is equal to zero on $U \setminus T$.

The idea is to use an approximate δ function such that each δ_ϵ is continuous and has support in a closed ball of radius $c\epsilon$ about zero. Let $K_{c\epsilon}$ be the compact set consisting of all points within $c\epsilon$ of K. Choose $\epsilon_1 > 0$ so that $K_{c\epsilon_1} \subseteq V$ and $L = \mathbf{g}^{-1}(K_{c\epsilon_1}) \subseteq U$. Thus $T \subseteq L$ and $\mathbf{g}(T) = K \subseteq K_{c\epsilon_1}$.

The proof depends on the fact that Fubini's theorem shows that two double integrals give the same result. The integral of $\delta_\epsilon(\mathbf{g}(\mathbf{x}) - \mathbf{y})$ may be done either with respect to the non-linear variable $\mathbf{x}$ or the linear variable $\mathbf{y}$. When the non-linear variable is integrated first we get

$$N(\epsilon) = \int_K \int_L f(\mathbf{y})|\det \mathbf{g}'(\mathbf{g}^{-1}(\mathbf{y}))|\delta_\epsilon(\mathbf{g}(\mathbf{x}) - \mathbf{y})\,d\mathbf{x}\,d\mathbf{y}. \tag{2.149}$$

When the linear variable is integrated first we get

$$L(\epsilon) = \int_L \int_K f(\mathbf{y})|\det \mathbf{g}'(\mathbf{g}^{-1}(\mathbf{y}))|\delta_\epsilon(\mathbf{g}(\mathbf{x}) - \mathbf{y})\,d\mathbf{y}\,d\mathbf{x}. \tag{2.150}$$

By Fubini's theorem $N(\epsilon) = L(\epsilon)$.

The plan is to take $\epsilon \to 0$. The limit of $L(\epsilon)$ must then be equal to the limit of $R(\epsilon)$. This will give the result for the case when f is continuous with support K.

Here is the part dealing with $L(\epsilon)$. Let K be the support of f. Integrating a function of $\mathbf{y}$ times $\delta_\epsilon(\mathbf{g}(\mathbf{x}) - \mathbf{y})$ and taking the limit in the inner integral gives

$$\int_K f(\mathbf{y})|\det \mathbf{g}'(\mathbf{g}^{-1}(\mathbf{y}))|\delta_\epsilon(\mathbf{g}(\mathbf{x}) - \mathbf{y})\,d\mathbf{y} \to f(g(\mathbf{x}))|\det \mathbf{g}'(\mathbf{x})| \tag{2.151}$$

as $\epsilon \to 0$. This result uses that f is continuous with compact support in K. Thus it is actually unnecessary to restrict the integral to K.

The integral in the above formula vanishes unless $\mathbf{g}(\mathbf{x})$ is in $K_{\epsilon c}$. Choose ϵ_1 so that $K_{\epsilon_1 c}$ is in $\mathbf{g}(V)$. So for $\epsilon \leq \epsilon_1$ we may integrate $\mathbf{x}$ over the compact set $L = \mathbf{g}^{-1}(K_{\epsilon_1 c})$. Furthermore, these integrals are uniformly bounded, independent of ϵ. By the bounded convergence theorem we get the limit of double integrals

$$L(\epsilon) = \int_L \int_K f(\mathbf{y}) |\det \mathbf{g}'(\mathbf{g}^{-1}(\mathbf{y}))| \delta_\epsilon(\mathbf{g}(\mathbf{x}) - \mathbf{y})\, d\mathbf{y}\, d\mathbf{x} \to \int_L f(g(\mathbf{x})) |\det \mathbf{g}'(\mathbf{x})|\, d\mathbf{x} \tag{2.152}$$

as $\epsilon \to 0$.

Here is the part dealing with the $N(\epsilon)$. This is the non-trivial step, and it uses the change of variable formula for the $\mathbf{x}$ variable in $\delta_\epsilon(\mathbf{g}(\mathbf{x}) - \mathbf{y})$. This gives a limit for the inner integral

$$\int_L f(\mathbf{y}) |\det \mathbf{g}'(\mathbf{g}^{-1}(\mathbf{y}))| \delta_\epsilon(\mathbf{g}(\mathbf{x}) - \mathbf{y})\, d\mathbf{x} \to f(\mathbf{y}) \tag{2.153}$$

as $\epsilon \to 0$. The restriction to L is no restriction at all, since the integrand is continuous with compact support inside L.

As a function of $\mathbf{y}$ the above integral is bounded by a constant independent of ϵ and is supported on K. The bounded convergence theorem gives a limit of double integrals

$$N(\epsilon) = \int_K \int_L f(\mathbf{y}) |\det \mathbf{g}'(\mathbf{g}^{-1}(\mathbf{y}))| \delta_\epsilon(\mathbf{g}(\mathbf{x}) - \mathbf{y})\, d\mathbf{x}\, d\mathbf{y} \to \int_K f(\mathbf{y})\, d\mathbf{y} \tag{2.154}$$

as $\epsilon \to 0$.

The proof for general Riemann integrable functions requires additional comment. For convenience, change notation so that the change of variable function is called ϕ, and ϕ maps the compact set $T \subset U$ to the compact set $K \subseteq V$. The function f is zero outside $K = \phi(T)$, and the function $f \circ \phi$ is zero outside T. So we may as well write $I_f = I(f \circ \phi \,|\det \phi'|)$.

Consider $\epsilon > 0$. Choose continuous real functions g, h with compact support with $g \leq f \leq h$ and $I(h) - I(g) < 2\epsilon$. Then $I(g) = I(g \circ \phi)|\det \phi'|)$ and $I(h) = I(h \circ \phi)|\det \phi'|)$. Furthermore, $(g \circ \phi)|\det \phi'| \leq (f \circ \phi)|\det \phi'| (f \circ \phi)|\det \phi'|$. Finally, $I(h \circ \phi)|\det \phi'|) - I(g \circ \phi)|\det \phi'|) < 2\epsilon$. This is enough to prove that $(f \circ \phi)|\det \phi'|$ is Riemann integrable with value $I(f)$. $\square$

This beautiful proof of the change of variable formula (using approximate delta functions) is from a recent paper by Ivan Netuka [36]. (A few details of the proof have been changed.) That formulation in this paper is for the Lebesgue integral, but it also works for the Riemann integral if one recognizes that the Riemann integral also has a bounded convergence theorem.

Remark: A physicist or engineer who is familiar with delta functions might summarize the entire proof by recalling that

$$|\det \mathbf{g}'(\mathbf{g}^{-1}(\mathbf{y}))| \delta_\epsilon(\mathbf{g}(\mathbf{x}) - \mathbf{y}) = \delta(\mathbf{x} - \mathbf{g}^{-1}(\mathbf{y})). \tag{2.155}$$

So

$$\int f(g(\mathbf{x}))|\det \mathbf{g}'(\mathbf{x})|\,d\mathbf{x} = \int\int f(\mathbf{y})|\det \mathbf{g}'(\mathbf{g}^{-1}(\mathbf{y}))|\delta(\mathbf{g}(\mathbf{x})-\mathbf{y})\,d\mathbf{y}\,d\mathbf{x} \quad (2.156)$$

which in turn is equal to

$$\int\int f(\mathbf{y})\delta(\mathbf{x}-\mathbf{g}^{-1}(\mathbf{y}))\,d\mathbf{x}\,d\mathbf{y} = \int f(\mathbf{y})\,d\mathbf{y} \quad (2.157)$$

Thus the proof is immediately memorable. ||

The change of variable formula also works for the absolutely convergent Riemann integral. In this case the integral is over an open subset U of $\mathbf{R}^n$, not necessarily bounded. It says that if $\mathbf{g} : U \to \mathbf{R}^n$ is a one-to-one C^1 function, and if $\mathbf{g}'(\mathbf{x})$ is invertible for every $\mathbf{x}$ in U, then for an absolutely convergent Riemann integrable function f defined on $\mathbf{g}(U)$ we have

$$\int_{\mathbf{g}(U)} f(\mathbf{y})\,d\mathbf{y} = \int_U f(g(\mathbf{x}))|\det \mathbf{g}'(\mathbf{x})|\,d\mathbf{x}. \quad (2.158)$$

Such a result for unbounded regions is usually proved in the context of the Lebesgue integral.

Here is another form for the change of variable formula. Take the function f to have the special form $f(\mathbf{y}) = h(\mathbf{g}^{-1}(\mathbf{y}))/|\det \mathbf{g}'(\mathbf{g}^{-1}(\mathbf{y}))|$. Then

$$\int_{\mathbf{g}(U)} h(\mathbf{g}^{-1}(\mathbf{y}))\frac{1}{|\det \mathbf{g}'(\mathbf{g}^{-1}(\mathbf{y}))|} = \int_U h(\mathbf{x})\,d\mathbf{x}. \quad (2.159)$$

It gives an expression for the integral of h over region U. This may also be thought of as the original change of variable formula with $\mathbf{g}^{-1}$ in place of $\mathbf{g}$ and with $\mathbf{g}(U)$ in place of U.

2.5.2 The oriented integral in one dimension

For calculus in one dimension the most common convention is that an integral is oriented, so

$$\int_b^a f(y)\,dy = -\int_a^b f(y)\,dy. \quad (2.160)$$

For oriented integrals the change of variable theorem has the form

$$\int_{g(a)}^{g(b)} f(y)\,dy = \int_a^b f(g(x))g'(x)\,dx. \quad (2.161)$$

This works for an arbitrary C^1 function g. There is no requirement that it be one-to-one. Also, there is no absolute value in the formula.

It is clear that the natural setting for calculus in one dimension is that of oriented integrals. The same is true in n dimensions; this will be the subject of the chapter on Multilinear Differential Forms.

It is possible to do calculus with unoriented integrals. It is even possible when g is not one-to-one. But in this case the change of variable formula would be

$$\int_{g([a,b])} f(y) N_g(y)\, dy = \int_{[a,b]} f(g(x)) |g'(x)|\, dx. \tag{2.162}$$

Here $N_g(y)$ is the number of x in $[a, b]$ such that $g(x) = y$. A more general version of this type of formula is presented in the next section.

It is even more perverse to recover the oriented integral from the unoriented integral. The change of variable formula would then be

$$\int_{g([a,b])} f(y) D_g(y)\, dy = \int_{[a,b]} f(g(x)) g'(x)\, dx. \tag{2.163}$$

Here the *degree* $D_g(y)$ is the sum over x in $[a, b]$ with $g(x) = y$ and $g'(x) \neq 0$ of the sign of $g'(x)$.

2.5.3 Fiber summation*

There are generalizations of the change of variable formula to more complicated situations. This section deals with the case of a change of variable function $\mathbf{g}$ that is not one-to-one. Suppose that $\mathbf{g}$ is a C^1 function from an open subset of $\mathbf{R}^n$ to an open subset $\mathbf{g}(W)$ of $\mathbf{R}^n$. Suppose that W is the disjoint union of open sets W_j, where $\mathbf{g}$ is one-to-one on each W_j. Then $\mathbf{g}(W)$ is the union of the sets $\mathbf{g}(W_j)$, but these sets may overlap.

Theorem 2.39 *Consider a function $\mathbf{g}$ that is a C^1 map of an open subset W in $\mathbf{R}^n$ to the open subset $\mathbf{g}(W)$ in $\mathbf{R}^n$. Suppose that $\mathbf{g}$ satisfies the conditions of the preceding paragraph. Assume that for $\mathbf{x}$ in W the determinant $\det \mathbf{g}'(\mathbf{x}) \neq 0$. Then*

$$\int_W h(\mathbf{x})\, d\mathbf{x} = \int_{\mathbf{g}(W)} \left(\sum_{\{\mathbf{x} | \mathbf{g}(\mathbf{x}) = \mathbf{y}\}} h(\mathbf{x}) \frac{1}{|\det \mathbf{g}'(\mathbf{x})|} \right) d\mathbf{y}. \tag{2.164}$$

The set $\{\mathbf{x} \mid \mathbf{g}(\mathbf{x}) = \mathbf{y}\}$ is called the *fiber* over y. The corresponding *fiber sum*

$$\mathbf{h}_*(\mathbf{y}) = \sum_{\{\mathbf{x} | \mathbf{g}(\mathbf{x}) = \mathbf{y}\}} h(\mathbf{x}) \frac{1}{|\det \mathbf{g}'(\mathbf{x})|} \tag{2.165}$$

defines the *pushforward* function of the original $\mathbf{h}(\mathbf{x})$. This is the function that goes in the integrand.

The reasoning behind the formula makes clear the conditions for which it is true. Write W as the union of sets W_j such that $\mathbf{g}$ one to one from W_j to $\mathbf{g}(W_j)$. It is supposed that the restricted $\mathbf{g}$ has an inverse function $\mathbf{f}_j$ from $\mathbf{g}(W_j)$ to W_j. The usual pullback formula says that

$$\int_{W_j} h(\mathbf{x})\, d\mathbf{x} = \int_{\mathbf{g}(W_j)} h(\mathbf{f}_j(\mathbf{y})) |\det \mathbf{f}_j'(\mathbf{y})|\, d\mathbf{y}. \tag{2.166}$$

Sum over j to get

$$\int_W h(\mathbf{x})\,d\mathbf{x} = \sum_j \int 1_{\mathbf{g}(W_j)}(\mathbf{y}) h(\mathbf{f}_j(\mathbf{y}))|\det \mathbf{f}_j'(\mathbf{y})|\,d\mathbf{y}. \tag{2.167}$$

This is the same as

$$\int_W h(\mathbf{x})\,d\mathbf{x} = \int \sum_j 1_{\mathbf{g}(W_j)}(\mathbf{y}) h(\mathbf{f}_j(\mathbf{y})) \frac{1}{|\det \mathbf{g}'(\mathbf{f}_j(\mathbf{y}))|}\,d\mathbf{y}. \tag{2.168}$$

However $\mathbf{y} \in \mathbf{g}(W_j)$ and $\mathbf{f}_j(\mathbf{y}) = \mathbf{x}$ is equivalent to $\mathbf{x} \in W_j$ and $\mathbf{g}(\mathbf{x}) = \mathbf{y}$. So this is

$$\int_W h(\mathbf{x})\,d\mathbf{x} = \int \sum_j 1_{\{\mathbf{x}\in W_j | \mathbf{g}(\mathbf{x})=\mathbf{y}\}} h(\mathbf{x}) \frac{1}{|\det \mathbf{g}'(\mathbf{x})|}\,d\mathbf{y}. \tag{2.169}$$

2.5.4 Fiber integration*

This section deals with a change of variable theorem involving a function $\mathbf{g}$ from an n-dimensional space to an $n-k$ dimensional space. For each $\mathbf{y}$ there is a k-dimensional surface consisting of those $\mathbf{x}$ where where $\mathbf{g}(\mathbf{x}) = \mathbf{y}$. The idea is to first integrate over each such surface and then integrate over the possible values of $\mathbf{y}$.

Let W be an open subset of $\mathbf{R}^n$. Let $1 < k < n$. Let $\mathbf{g} : W \to \mathbf{R}^{n-k}$ be a C^1 function that maps onto the open subset $\mathbf{g}(W)$. It is natural to suppose that for each $\mathbf{x}$ in W the derivative $\mathbf{g}'(\mathbf{x})$ has rank $n-k$. The function $\mathbf{g}$ defines a family of implicitly defined surfaces. For each $\mathbf{y}$ the corresponding surface is the inverse image $\mathbf{g}^{-1}(\mathbf{y})$.

The next assumption is that this family of surfaces is also given explicitly. Here $\mathbf{F}$ is a one to one C^1 function from some open subset Z of $\mathbf{R}^k \times \mathbf{g}(W)$ onto W. It is required that

$$\mathbf{g}(\mathbf{F}(\mathbf{u}, \mathbf{y})) = \mathbf{y}. \tag{2.170}$$

The idea is that for each fixed $\mathbf{y}$ the mapping $\mathbf{u} \mapsto \mathbf{F}(\mathbf{u}, \mathbf{y})$ is a parametric representation of $g^{-1}(\mathbf{y})$. In fact, it is assumed that $\mathbf{F} : Z \to W$ is a C^1 function with C^1 inverse. We speak of this as a *parametrized family of surfaces*.

The function $\mathbf{F}'_{,1}(\mathbf{u}, \mathbf{y}_0)$ should have rank $n-k$. Furthermore, according to the chain rule $\mathbf{g}'(\mathbf{F}(\mathbf{u}, \mathbf{y}))\mathbf{F}'_{,1}(\mathbf{u}, \mathbf{y}) = 0$. This says that the tangent space to the surface at a particular point $\mathbf{x} = \mathbf{F}(\mathbf{u}, \mathbf{y})$ may be given either as the nullspace of $\mathbf{g}'(\mathbf{x})$ or as the range of $\mathbf{F}'_{,1}(\mathbf{u}, \mathbf{y})$.

The surface $\mathbf{g}^{-1}(\mathbf{y})$ is called the *fiber* over $\mathbf{y}$ in V. For each such $\mathbf{y}$ the function that sends $\mathbf{u}$ to $\mathbf{F}(\mathbf{u}, \mathbf{y})$ sends a parameter region $Z_{\mathbf{y}}$ in $\mathbf{R}^k$ to the corresponding k dimensional surface that is the fiber over $\mathbf{y}$.

The function $\mathbf{F} : W \to Z$ has an inverse $\mathbf{G} : W \to Z$. This has two components: For $\mathbf{x} = \mathbf{F}(\mathbf{u}, \mathbf{v})$ the inverse relation is

$$\begin{aligned} \mathbf{u} &= \mathbf{G}_1(\mathbf{x}) \\ \mathbf{y} &= \mathbf{G}_2(\mathbf{x}) = \mathbf{g}(\mathbf{x}). \end{aligned} \tag{2.171}$$

Theorem 2.40 (Fiber integral) *Consider such a parametrized family of surfaces. Then*

$$\int_W h(\mathbf{x})\,d\mathbf{x} = \int_{\mathbf{g}(W)} \left(\int_{Z_{\mathbf{y}}} h(\mathbf{F}(\mathbf{u},\mathbf{y}))|\det \mathbf{F}'(\mathbf{u},\mathbf{y})|\,d\mathbf{u} \right) d\mathbf{y}. \tag{2.172}$$

In other words, the integral over the region is equal to an integral over each surface corresponding to a value of $\mathbf{y}$*, followed by an integral over the possible values of* $\mathbf{y}$*.*

Proof: By the change of variable formula

$$\int_W h(\mathbf{x})\,d\mathbf{x} = \int_Z h(F(\mathbf{u},\mathbf{y}))|\det \mathbf{F}'(\mathbf{u},\mathbf{y})|\,d\mathbf{u}\,d\mathbf{y}. \tag{2.173}$$

The result then follows from Fubini's theorem. □

Define the pushforward of h by

$$h_*(\mathbf{y}) = \int_{Z_{\mathbf{y}}} h(\mathbf{F}(\mathbf{u},\mathbf{y}))|\det \mathbf{F}'(\mathbf{u},\mathbf{y})|\,d\mathbf{u}. \tag{2.174}$$

Then the result says that the integral is equal to the integral over the pushforward. This formula is for fixed $\mathbf{y}$, but the derivative $\mathbf{F}'(\mathbf{u},\mathbf{y})$ is for $\mathbf{F}(\mathbf{u},\mathbf{y})$ as a function of both variables. Thus the determinant is that of a square matrix.

Proposition 2.41 *The pushforward in the fiber integral does not depend on the choice of parameterization.*

Proof: Say that we have a new parameterization given by $\bar{\mathbf{F}}(\mathbf{w},\mathbf{y}) = \mathbf{F}(\phi(\mathbf{w},\mathbf{y}),\mathbf{y})$. Calculate

$$\bar{h}_*(\mathbf{y}) = \int_{\bar{Z}_{\mathbf{y}}} h(\bar{\mathbf{F}}(\mathbf{w},\mathbf{y}))|\det \bar{\mathbf{F}}'(\mathbf{w},\mathbf{y})|\,d\mathbf{u}. \tag{2.175}$$

Write $\bar{F}(\mathbf{w},\mathbf{y}) = \mathbf{F}(\mathbf{K}(\mathbf{w},\mathbf{y}))$ with $\mathbf{K}(\mathbf{w},\mathbf{y}) = (\phi(\mathbf{w},\mathbf{y}),\mathbf{y})$. The derivative of $\mathbf{K}$ has the form

$$\mathbf{K}'(\mathbf{w},\mathbf{y}) = \begin{bmatrix} \phi'_{,1}(\mathbf{w},\mathbf{y}) & \phi'_{,2}(\mathbf{w},\mathbf{y}) \\ 0 & I \end{bmatrix}. \tag{2.176}$$

The determinant of this is $\det \phi'_{,1}(\mathbf{w},\mathbf{y})$. Compute

$$\det \bar{\mathbf{F}}'(\mathbf{w},\mathbf{y})| = \det \mathbf{F}'(\mathbf{K}(\mathbf{w},\mathbf{y}))\det \mathbf{K}'(\mathbf{w},\mathbf{y}) = \det \mathbf{F}'(\phi(\mathbf{w},\mathbf{y}),\mathbf{y}))\det \phi'_{,1}(\mathbf{w},\mathbf{y}). \tag{2.177}$$

Hence

$$\bar{h}_*(\mathbf{y}) = \int_{\bar{Z}_y} h(F(\phi(\mathbf{w},\mathbf{y}),\mathbf{y}))|\det \mathbf{F}'(\phi(\mathbf{w},\mathbf{y}),\mathbf{y})||\det \phi'_{,1}(\mathbf{w},\mathbf{y})|\,d\mathbf{w}. \tag{2.178}$$

With a change of variables this is the same $h_*(\mathbf{y})$. □

It is convenient to define quantities $h^{(\mathbf{y})} = h(\mathbf{F}(\mathbf{u}, \mathbf{y}))$ and the *fiber form*

$$\text{fiber}^{(\mathbf{y})} = |\det \mathbf{F}'(\mathbf{u}, \mathbf{y})|\, d\mathbf{u} \tag{2.179}$$

that suppress the details of the particular parametrization. This leads to a simple statement of the main result. The idea is that one imagines that the integrals defining the pushforward are over the fibers $\mathbf{g}^{-1}(\mathbf{y})$.

Theorem 2.42 (Fiber integral) *Consider such a parametrized family of k-dimensional surfaces $\mathbf{g}(\mathbf{x}) = \mathbf{y}$ in open set W. Then*

$$\int_W h(\mathbf{x})\, d\mathbf{x} = \int_{\mathbf{g}(W)} \left(\int_{\mathbf{g}^{-1}(\mathbf{y})} h^{(\mathbf{y})} \text{fiber}^{(\mathbf{y})} \right) d\mathbf{y}. \tag{2.180}$$

The fiber integral

$$h_*(\mathbf{y}) = \int_{\mathbf{g}^{-1}(\mathbf{y})} h^{(\mathbf{y})} \text{fiber}^{(\mathbf{y})} \tag{2.181}$$

is the *pushforward* of $h(\mathbf{x})$ via $\mathbf{g}$.

Another way of thinking of the fiber integral is in terms of delta functions. The relation is

$$\int \delta(\mathbf{g}(\mathbf{x}) - \mathbf{y}) h(\mathbf{x})\, d\mathbf{x} = \int_{\mathbf{g}^{-1}(\mathbf{y})} h^{(\mathbf{y})} \text{fiber}^{(\mathbf{y})}. \tag{2.182}$$

The delta function on the left enforces $n-k$ equations, leaving an k dimensional integral on the right.

Example: The classic example of a fiber integral is polar coordinates. Here are the details in three dimensions. The sphere of radius r has the implicit equation $g(x, y, z) = \sqrt{x^2 + y^2 + z^2} = r$. For each r this is parametrized by spherical polar coordinates r, ϕ, θ, where ϕ is longitude and θ is co-latitude. The function that sends ϕ, θ, r to x, y, z has derivative matrix with determinant $-r^2 \sin(\theta)$. So the fiber form is $r^2 \sin(\theta)\, d\phi\, d\theta$. The polar coordinate formula is

$$\int_{\mathbf{R}^3} h(x, y, z)\, dx\, dy\, dz = \int_0^\infty \left(\int_0^\pi \int_0^{2\pi} h^{(r)}(\phi, \theta)\, r^2 \sin(\theta)\, d\phi\, d\theta \right) dr. \tag{2.183}$$

The fibers are spheres. Note that this calculation makes no mention of surface area. However the function $g(x, y, z)$ has a derivative matrix $[x/r, y/r, z/r]$ with length 1 at every point. In this special case (and only in this special case) the fiber form is the same as the area form. This is a consequence of the coarea formula presented in a later section. ||

The discussion in this section has avoided various technical issues. The reader may consult the fascinating article by Ponomarev [39] for further information on this subject.

2.5.5 Probability*

The central notion of probability is that of *expectation* of a function of a vector *random variable* $\mathbf{x}$. A common case is when the expectation is given by a *probability density* $\rho(\mathbf{x})$. This is a positive function with integral one. Say that $\mathbf{y} = \mathbf{g}(\mathbf{x})$ is a random variable that is a function of $\mathbf{x}$. Then it may or may not be the case that the expectation of $f(\mathbf{y}) = f(\mathbf{g}(\mathbf{x}))$ is given by a pushed forward probability density $\rho_*(\mathbf{y})$. When this is the case, we should have

$$\int f(\mathbf{y})\rho_*(\mathbf{y})\, d\mathbf{y} = \int f(\mathbf{g}(\mathbf{x}))\rho(\mathbf{x})\, d\mathbf{x}. \tag{2.184}$$

First consider the case when n random variables are mapped to n random variables. There $\rho(\mathbf{x})$ is a joint probability density for random variables $\mathbf{x}$, $\mathbf{g}(\mathbf{x})$ is a vector of n random variables, and $f(\mathbf{g}(\mathbf{x}))$ is a function of these n random variables. The right hand side is the expectation. If one wants to write this expectation in terms of the random variables $\mathbf{y} = \mathbf{g}(\mathbf{x})$, then one has to push forward the density. The change of variables formula suggests that the new density is

$$\rho_*(\mathbf{y}) = \sum_{\mathbf{g}(\mathbf{x})=\mathbf{y}} \frac{1}{|\det \mathbf{g}'(\mathbf{x})|}\rho(\mathbf{x}). \tag{2.185}$$

This only works when the regions where $\det \mathbf{g}'(\mathbf{x}) = 0$ can be neglected, and this is not always the case. If, for instance, there is a region C of non-zero volume with $\mathbf{g}(\mathbf{x}) = \mathbf{y}^*$ for $\mathbf{x}$ in C, then the extra contribution $f(\mathbf{y}^*)\int_C \rho(\mathbf{x})\, d\mathbf{x}$ must be added to the left hand side for the identity to be valid. (Technical note: Sard's theorem does nothing to help, since there is no longer a factor that vanishes on the set of critical points. Even though the set of critical values has measure zero, there can be a lot of probability on a set of measure zero.)

Example: An example is when $n = 1$ and the density is $\rho(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2)$. This is the density for a standard normal (Gaussian) distribution. Let $y = x^2$. Then $\rho_*(y) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} \exp(-\frac{1}{2}y)$ for $y > 0$. This density is that of a chi-squared distribution (with one degree of freedom). ||

Next consider the case when n random variables are mapped to m random variables with $m < n$ by $\mathbf{y} = \mathbf{g}(\mathbf{x})$. The pushforward formula suggests that

$$\rho_*(\mathbf{y}) = \int_{\mathbf{g}^{-1}(\mathbf{y})} \rho^{(\mathbf{y})}\text{fiber}^{(\mathbf{y})}. \tag{2.186}$$

Example: Here is an example when $n = 2$ and $m = 1$. Consider the joint normal (Gaussian) distribution with density $\rho(x_1, x_2) = \frac{1}{2\pi}\exp(-\frac{1}{2}(x_1^2 + x_2^2))$. Let $y = x_2/x_1$ be the quotient. The fiber over y is the line $x_2 = yx_1$. One choice of parameter is $u = x_1$. Then we have $x_1 = u, x_2 = yu$. The Jacobian determinant is u, so the fiber form is $\text{fiber}^{(y)} = |u|\, du$. Then $\rho^{(y)}\text{fiber}^{(y)} = \frac{1}{2\pi}\exp(-\frac{1}{2}(1 + y^2)u^2)|u|\, du$. The integral of this form over the fiber is $\rho(y) = \frac{1}{\pi}\frac{1}{1+y^2}$. This is the density for the Cauchy distribution. ||

Perhaps the moral of the story is that one should calculate with the original density $\rho(\mathbf{x})$. In probability theory expectations (or measures) push forward in

a routine way. When you try to express them in terms of densities, then the expressions are less pleasant. Densities are functions. Functions pull back with ease, but push forward with considerable difficulty.

2.6 Area

2.6.1 The area formula

This section records formulas for the k-dimensional area of a surface in n-dimensional space. This is for the case when the surface is given parametrically.

These formulas are justified by corresponding formulas for the area of the parallelepiped spanned by k vectors. If $\mathbf{a}_1, \ldots, \mathbf{a}_k$ are vectors, then the matrix of inner products $\mathbf{a}_i \cdot \mathbf{a}_j = \mathbf{a}_i^T \mathbf{a}_j$ may be written in the elegant form $A^T A$, where A is the matrix with these vectors as columns. The area is the square root of the determinant of this matrix. See the Supplement on Matrix Theory for a more detailed discussion.

Remark: In the case of two vectors $\mathbf{a}_1, \mathbf{a}_2$ this is an elementary formula. The determinant is $(\mathbf{a}_1 \cdot \mathbf{a}_1)(\mathbf{a}_2 \cdot \mathbf{a}_2) - (\mathbf{a}_1 \cdot \mathbf{a}_2)^2$. If we write this in terms of the lengths of the vectors, this is $|\mathbf{a}_1|^2 |\mathbf{a}_2|^2 - |\mathbf{a}_1|^2 |\mathbf{a}_2|^2 \cos^2(\theta)$. This is the same as $|\mathbf{a}_1|^2 |\mathbf{a}_2|^2 \sin^2(\theta)$. The square root is $|\mathbf{a}_1||\mathbf{a}_2| \sin(\theta)$. This is a familiar formula for the area of a parallelogram. ||

Remark: In the case of two vectors in three dimensions there is another familiar formula. The cross product of the two vectors is a third vector that is orthogonal to both of them. The area of the parallelogram is the length of this cross product vector. As shown in the Supplement on Matrix Theory, this is a very special case of the Cauchy-Binet identity. ||

Say that $k < n$ and $\mathbf{f}$ is a C^1 function from a bounded open subset U of $\mathbf{R}^k$ to $\mathbf{R}^n$. Let h be a function defined on the range of $\mathbf{f}$. The area integral of h over the surface is defined to be

$$A(h) = \int_U h(\mathbf{f}(\mathbf{u})) \sqrt{\det(\mathbf{f}'(\mathbf{u})^T \mathbf{f}'(\mathbf{u}))}\, d\mathbf{u}. \tag{2.187}$$

Thus the integrand involves the determinant of a k by k square matrix. Area is the special case when h is the indicator function of some set.

The intuitive idea behind this formula is that the columns of $\mathbf{f}'(\mathbf{u})$ for a fixed $\mathbf{u}$ are k vectors in $\mathbf{R}^n$ that are each tangent to the surface at the point $\mathbf{f}(\mathbf{u})$. The $\sqrt{\det(\mathbf{f}'(\mathbf{u})^T \mathbf{f}'(\mathbf{u}))}$ is a k-dimensional area associated with these vectors. (In the case $k = 1$ the vector is usually interpreted as a velocity vector, and the quantity with the square root is the speed.)

The vectors and the area that they determine all depend on the parameterization. However there is a sense in which $\sqrt{\det(\mathbf{f}'(\mathbf{u})^T \mathbf{f}'(\mathbf{u}))}\, d\mathbf{u}$ is independent of the parameterization, as shown in the following proposition.

Proposition 2.43 *The area integral is independent of the parameterization.*

Proof: Suppose that there is another bounded open set V such that $\mathbf{g}$ is a smooth one-to-one function from V onto U. Then $\mathbf{f} \circ \mathbf{g}$ is another parameterization of the surface. With this choice the area is

$$A(h) = \int_V h(\mathbf{f}(\mathbf{g}(\mathbf{w})))\sqrt{\det((\mathbf{f}'(\mathbf{g}(\mathbf{w}))\mathbf{g}'(\mathbf{w}))^T\mathbf{f}'(\mathbf{g}(\mathbf{w}))\mathbf{g}'(\mathbf{w}))}\, d\mathbf{w}. \quad (2.188)$$

But this is

$$A(h) = \int_V h(\mathbf{f}(\mathbf{g}(\mathbf{w})))\sqrt{\det(\mathbf{f}'(\mathbf{g}(\mathbf{w}))^T\mathbf{f}'(\mathbf{g}(\mathbf{w})))|}\,|\det(\mathbf{g}'(\mathbf{w}))|\, d\mathbf{w}. \quad (2.189)$$

By the change of variable formula this is the same area as before. □

Since many of the details of the parametrization are irrelevant, it is convenient to introduce an area form

$$\text{area} = \sqrt{\det(\mathbf{f}'(\mathbf{u})^T\mathbf{f}'(\mathbf{u}))}\, d\mathbf{u}. \quad (2.190)$$

Let $h^*(\mathbf{u}) = h(\mathbf{f}(\mathbf{u}))$. Then

$$A(h) = \int h^* \,\text{area}. \quad (2.191)$$

The integral is over the appropriate parameter region.

When $k = n - 1$ there is an other expression for the area integral. This is a special case of the Cauchy-Binet identity. In this case $\mathbf{f}'(\mathbf{u})$ is a n by $n - 1$ matrix. Let $\mathbf{f}'(\mathbf{u})_{\setminus j}$ be the $n - 1$ by $n - 1$ square matrix obtained by deleting the row j. The identity is

$$\sqrt{\det(\mathbf{f}'(\mathbf{u})^T\mathbf{f}'(\mathbf{u}))} = \sqrt{\sum_{j=1}^{n}(\det \mathbf{f}'(\mathbf{u})_{\setminus j})^2}. \quad (2.192)$$

This identity may be interpreted in terms of a row vector ν with components

$$\nu_j(\mathbf{u}) = (-1)^{j-1} \det \mathbf{f}'(\mathbf{u})_{\setminus j} \quad (2.193)$$

The vector $\nu(\mathbf{u})$ is orthogonal to the vectors in the columns of $\mathbf{f}'(\mathbf{u})$. In fact, the equation $\nu(\mathbf{u})\mathbf{f}'(\mathbf{u}) = 0$ says that $\nu(\mathbf{u})$ is orthogonal to the surface. By the Cauchy-Binet theorem says that the area factor is just the length $|\nu(\mathbf{u})|$ of $\nu(\mathbf{u})$.

The conclusion is that the area may also be given by the formula

$$A(h) = \int_U h^* \,|\nu(\mathbf{u})|\, d\mathbf{u}. \quad (2.194)$$

Thus $\nu(\mathbf{u})$ represents a vector orthogonal to the surface at $\mathbf{f}(\mathbf{u})$ with length that determines the element of area at $\mathbf{u}$.

The classic situation is when $k = 2$ and $n = 3$. Then it is conventional to write the matrix

$$\mathbf{f}'(\mathbf{u})^T\mathbf{f}'(\mathbf{u}) = \begin{bmatrix} E & F \\ F & G \end{bmatrix}. \quad (2.195)$$

where E, F, G are functions of $\mathbf{u}$. Then

$$A(h) = \int_U h^* \sqrt{EG - F^2}\, du\, dv. \tag{2.196}$$

Alternately,

$$A(h) = \int_U h^* \sqrt{\nu_1^2 + \nu_2^2 + \nu_3^2}\, du\, dv. \tag{2.197}$$

Example: Take the surface given in Cartesian coordinates by $z = x^2 + y^2$ with $z \leq 1$. Write $x = u$, $y = v$ and $z = u^2 + v^2$. Then $E = 1 + 4u^2$, $F = 4uv$, and $G = 1 + 4v^2$. So with these parameters the area element is $\sqrt{EG - F^2}\, du\, dv = \sqrt{1 + 4u^2 + 4v^2}\, du\, dv$. This is integrated over the region $u^2 + v^2 \leq 1$.

With the alternative calculation $\nu_x = -2u$ and $\nu_y = -2v$ and $\nu_z = 1$. So the area element is $\sqrt{\nu_x^2 + \nu_y^2 + \nu_z^2}\, dx\ dy = \sqrt{4u^2 + 4v^2 + 1}\, du\, dv$, exactly the same.

Other parameterizations are possible. Take, for instance, $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = r^2$. Then $E = 1 + 4r^2$, $F = 0$, and $G = r^2$. So with these parameters the area element is $\sqrt{EG - F^2}\, dr\, d\theta = r\sqrt{1 + 4r^2}\, dr\, d\theta$.

We may as well go ahead and compute the area. It is 2π times the integral from 0 to 1 of $r\sqrt{1 + 4r^2}\, dr$. The area is thus $(\pi/6)(5^{\frac{3}{2}} - 1)$. ||

The formulas in this example can be generalized.

Proposition 2.44 (Area of graph of a function) *Say that $\mathbf{f}(\mathbf{u}) = (\mathbf{u}, f(\mathbf{u}))$ maps from $W \subseteq \mathbf{R}^{n-1}$ to $\mathbf{R}^n$. Consider the row vector $f'(\mathbf{u})$, which represents the slope of the function. Then the area is obtained by*

$$A = \int_W \sqrt{1 + f'(\mathbf{u}) f'(\mathbf{u})^T}\, d\mathbf{u}. \tag{2.198}$$

Proof: This proposition is remarkable, since there is no more determinant. Start with $\mathbf{f}'(\mathbf{u})^T = [I, f'(\mathbf{u})]^T$. Then

$$\mathbf{f}'(\mathbf{u})^T \mathbf{f}'(\mathbf{u}) = I + f'(\mathbf{u})^T f'(\mathbf{u}). \tag{2.199}$$

Sylvester's determinant identity (below) shows that

$$\det(\mathbf{f}'(\mathbf{u})^T \mathbf{f}'(\mathbf{u})) = 1 + f'(\mathbf{u}) f'(\mathbf{u})^T. \tag{2.200}$$

□

Proposition 2.45 (Sylvester) *If Z is an m by k matrix and W is a k by m matrix, then ZW is m by m and WZ is k by k. The identity says that*

$$\det(I_m + ZW) = \det(I_k + WZ). \tag{2.201}$$

This identity is a consequence of the block matrix formulas in the Supplement on Matrix Theory. Here is a quick proof using these ideas. It uses the fact that

the determinant of a block triangular matrix is given by the usual formula. (See the Supplement on Matrix Theory for this.) We have

$$\begin{bmatrix} I & -Z \\ W & I \end{bmatrix} = \begin{bmatrix} I & -Z \\ 0 & I \end{bmatrix} \begin{bmatrix} I + ZW & 0 \\ W & I \end{bmatrix} \tag{2.202}$$

Also

$$\begin{bmatrix} I & -Z \\ W & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ W & I \end{bmatrix} \begin{bmatrix} I & -Z \\ 0 & I + WZ \end{bmatrix} \tag{2.203}$$

The identity follows by taking determinants.

It would also be nice to compute the area for a function that is defined implicitly. This is the subject of the coarea formula of the next section. Here is a special case when the surface is the graph of a function.

Proposition 2.46 (Coarea calculation of area) *Say that $\mathbf{f}(\mathbf{u}) = (\mathbf{u}, f(\mathbf{u}))$ maps from $W \subseteq \mathbf{R}^{n-1}$ to $W \times \mathbf{R}_+$. Suppose also that $g(\mathbf{f}(\mathbf{u})) = g(\mathbf{u}, f(\mathbf{u})) = 0$, so that g describes the surface implicitly. Finally, assume that $g'_{,n}(\mathbf{f}(\mathbf{u})) \neq 0$. Let $g'(\mathbf{x})$ be the row vector that is the derivative of $g(\mathbf{x})$. Then the area integral is*

$$A = \int_W |g'(\mathbf{f}(\mathbf{u}))| \frac{1}{|g'_{,n}(\mathbf{f}(\mathbf{u})|} \, d\mathbf{u}. \tag{2.204}$$

Proof: This depends on the calculation of the derivative

$$0 = (g \circ \mathbf{f})'_{,k}(\mathbf{u}) = g'_{,k}(\mathbf{f}(\mathbf{u})) + g'_{,n}(\mathbf{f}(\mathbf{u})) f'_k(\mathbf{u}). \tag{2.205}$$

This says that the row vector

$$g'(\mathbf{f}(\mathbf{u})) = g'_{,n}(\mathbf{f}(\mathbf{u})) \, [-f'(\mathbf{u}), 1]. \tag{2.206}$$

The two vectors $g'(\mathbf{f}(\mathbf{u}))$ and $[-f'(\mathbf{u}), 1]$ are both zero on the tangent vectors (the columns of $\mathbf{f}'(\mathbf{u})$), and in fact they are proportional. Compute the length of the vector on each side of this equation. The result is then seen to conincide with the preceding result. □

The preceding result looks promising, since it involves the length $|g'(\mathbf{x})|$ of the row vector associated with the equation $g(\mathbf{x}) = 0$ giving the implicit definition. But the extra factor is ugly. Here is a version that is truly beautiful.

Proposition 2.47 (Coarea calculation with delta function) *Suppose that g is a function defined on $W \times \mathbf{R}^+$ such that $g'_{,n} \neq 0$. Suppose that g has an inverse function in the last variable. That is, suppose that there is a function $h(x_1, \ldots, x_n, c) \geq 0$ such that for every $x_1, \ldots, x_{n-1}$ we have*

$$g(x_1, \ldots, x_{n-1}, h(x_1, \ldots, x_{n-1}, c)) = c. \tag{2.207}$$

Then (under reasonable assumptions about interchanging order of integration) the $n-1$ dimensional area of the surface defined by $g(\mathbf{x}) = 0$ has the expression

$$A = \lim_{\epsilon \to 0} \int_{W \times \mathbf{R}^+} \delta_\epsilon(g(\mathbf{x})) |g'(\mathbf{x})| \, d\mathbf{x}. \tag{2.208}$$

Proof: Let $f(x_1, \ldots, x_{n-1}) = h(x_1, \ldots, x_{n-1}, 0)$ be the inverse function of g in the last variable evaluated at 0. Then $f(x_1, \ldots, x_n) \geq 0$, and also $g(x_1, \ldots, x_{n-1}, f(x_1, \ldots, x_{n-1})) = 0$.

The integral can be written as an integral over x_n over $\mathbf{R}_+ = [0, \infty)$ followed by an integral over $x_1, \ldots, x_{n-1}$ over W. It is supposed that the limit as $\epsilon \to 0$ may be taken inside the outer integral. The inner integral is then

$$\int_0^\infty \delta_\epsilon(g(x_1, \ldots, x_n))|g'(x_1, , \ldots, x_n)|\, dx_n. \tag{2.209}$$

In the limit this becomes

$$|g'(x_1, , \ldots, x_{n-1}, f(x_1, \ldots, x_{n-1}))| \frac{1}{|g'_{,n}(x_1, \ldots, x_{n-1}, f(x_1, \ldots, x_{n-1}))|}. \tag{2.210}$$

When this is integrated over the other variables, the result coincides with the result of the previous proposition. □

Would it be possible, at least in some cases, to define surface area by the formula

$$A = \lim_{\epsilon \to 0} \int_{\mathbf{R}^n} \delta_\epsilon(g(\mathbf{x}))|g'(\mathbf{x})|\, d\mathbf{x}? \tag{2.211}$$

This would be both beautiful and intuitive.

2.6.2 The coarea formula*

The co-area formula arises in the context of surface area for a surface that is defined implicitly. It relates fiber integration with area. It is possible that this is not always a good idea, since fiber integration is often easy to carry out, while calculations involving area are usually a nightmare (because of the square roots).

Suppose that $\mathbf{g}(\mathbf{x}) = \mathbf{y}$ is a parametrized family of surfaces with parameterization $\mathbf{F}(\mathbf{u}, \mathbf{y})$ satisfying

$$\mathbf{g}(\mathbf{F}(\mathbf{u}, \mathbf{y})) = \mathbf{y}. \tag{2.212}$$

Let $\mathbf{F}'_1(\mathbf{u}, \mathbf{y})$ be the n by k matrix of partial derivatives with respect to the $\mathbf{u}$ variables. The columns of this matrix represent a basis of tangent vectors to the surface. From

$$\mathbf{g}(\mathbf{F}(\mathbf{u}, \mathbf{y})) = \mathbf{y} \tag{2.213}$$

we see that

$$\mathbf{g}'(\mathbf{x})\mathbf{F}'_1(\mathbf{u}, \mathbf{y}) = 0. \tag{2.214}$$

This expresses in matrix form the fact that the $n-k$ independent row covectors in $\mathbf{g}'(\mathbf{x})$ are zero on the k independent column vectors in $\mathbf{F}'_1(\mathbf{u}, \mathbf{y})$.

Define the *co-area factor* $C(\mathbf{x})$ by

$$C(\mathbf{x}) = \sqrt{\det \mathbf{g}'(\mathbf{x})\mathbf{g}'(\mathbf{x})^T}. \tag{2.215}$$

This is the determinant of an $n-k$ square Gram matrix. Define the *area factor* by

$$A(\mathbf{u},\mathbf{y}) = \sqrt{\det \mathbf{F}_1'^T(\mathbf{u},\mathbf{y})\mathbf{F}_1'(\mathbf{u},\mathbf{y})}. \tag{2.216}$$

The determinant is that of a k square Gram matrix.

Theorem 2.48 (Co-area identity) *The co-area and area factors are related by*

$$C(\mathbf{F}(\mathbf{u},\mathbf{y}))|\det \mathbf{F}'(\mathbf{u},\mathbf{y})| = A(\mathbf{u},\mathbf{y}). \tag{2.217}$$

Proof: Let $\mathbf{F}'(\mathbf{u},\mathbf{y})$ be the n by n derivative matrix of $F(\mathbf{u},\mathbf{y})$. Let $\mathbf{F}_1'(\mathbf{u},\mathbf{y})$ be the n by k derivative matrix of $\mathbf{F}(\mathbf{u},\mathbf{v})$ with respect to $\mathbf{u}$, and let $\mathbf{F}_2'(\mathbf{u},\mathbf{y})$ be the n by $n-k$ derivative matrix with respect to $\mathbf{y}$. Then

$$\mathbf{F}'(\mathbf{u},\mathbf{v}) = [\mathbf{F}_1(\mathbf{u},\mathbf{v}), \mathbf{F}_2'(\mathbf{u},\mathbf{y})]. \tag{2.218}$$

Let $\mathbf{G}(\mathbf{x})$ be the inverse function to $\mathbf{F}(\mathbf{u},\mathbf{v})$. Thus

$$\mathbf{G}'(\mathbf{x}) = \begin{bmatrix} \mathbf{G}_1'(\mathbf{x}) \\ \mathbf{G}_2'(\mathbf{x}) \end{bmatrix}. \tag{2.219}$$

Here $\mathbf{G}_1(\mathbf{x})$ and $\mathbf{G}_2(\mathbf{x})$ have derivatives $\mathbf{G}_1'(\mathbf{x})$ and $\mathbf{G}_2'(\mathbf{x})$ that are k by n and $n-k$ by n matrices.

By the chain rule we have $\mathbf{G}'(\mathbf{F}(\mathbf{u},\mathbf{y}))\mathbf{F}'(\mathbf{u},\mathbf{y}) = I$. In the following we write this in abbreviated form as

$$\mathbf{G}'\mathbf{F}' = I. \tag{2.220}$$

This says that

$$\begin{bmatrix} \mathbf{G}_1' \\ \mathbf{G}_2' \end{bmatrix} \begin{bmatrix} \mathbf{F}_1' & \mathbf{F}_2' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{2.221}$$

The expression $\det \mathbf{G}_2'\mathbf{G}_2'^T = \det \mathbf{g}'\mathbf{g}'^T$ is the square of the coarea factor. The expression $\det \mathbf{F}_1'^T\mathbf{F}_1'$ is the square of the area expression. The algebraic fact proved in the Supplement on Matrix Theory is

$$\det \mathbf{G}_2'\mathbf{G}_2'^T \det \mathbf{F}'^T\mathbf{F}' = \det \mathbf{F}_1'^T\mathbf{F}_1'. \tag{2.222}$$

This can be written

$$\det \mathbf{g}'\mathbf{g}'^T (\det \mathbf{F}')^2 = \det \mathbf{F}_1'^T\mathbf{F}_1'. \tag{2.223}$$

This gives the required identity. □

In the language of differential forms, this formula says that

$$C^{(\mathbf{y})}\text{fiber}^{(\mathbf{y})} = \text{area}^{(\mathbf{y})}. \tag{2.224}$$

Here $C^{(\mathbf{y})} = C(\mathbf{F}(\mathbf{u},\mathbf{y}))$, while $\text{fiber}^{(\mathbf{y})} = |\det \mathbf{F}'(\mathbf{u},\mathbf{y})|\, d\mathbf{u}$ and $\text{area}^{(\mathbf{y})} = A(\mathbf{u},\mathbf{y})\, d\mathbf{u}$.

Theorem 2.49 (Co-area formula as fiber integral) *Consider a parametrized family of surfaces* $\mathbf{g}(\mathbf{x}) = \mathbf{y}$ *for* $\mathbf{x}$ *in* W*. Then*

$$\int_W h(\mathbf{x})\, d\mathbf{x} = \int_{\mathbf{g}(W)} \left(\int_{\mathbf{g}^{-1}(\mathbf{y})} h^{(\mathbf{y})} \frac{1}{C^{(\mathbf{y}}} \mathrm{area}^{(\mathbf{y})} \right) d\mathbf{y}. \tag{2.225}$$

This formula gives an expression for the fiber form in terms of rather concrete objects: the coarea factor and the area form. The negative aspect is that it introduces complicated metric quantities involving square roots in numerator and denominator. The general fiber form expression shows that these always cancel out!

This coarea formula may also be thought of as a formula for fiber integrals in terms of delta functions. Thus

$$\int_W h(\mathbf{x}) \delta(\mathbf{g}(\mathbf{x}) - \mathbf{y})\, d^n\mathbf{x} = \int_{\mathbf{g}^{-1}(y)} \frac{1}{C^{(\mathbf{y})}} h^{(y)} \mathrm{area}^{(y)}. \tag{2.226}$$

Theorem 2.50 (Co-area formula as area representation) *Consider a parametrized family of surfaces* $\mathbf{g}(\mathbf{x}) = \mathbf{y}$ *for* $\mathbf{x}$ *in* W*. Then*

$$\int_W h(\mathbf{x}) C(\mathbf{x})\, d\mathbf{x} = \int_{\mathbf{g}(W)} \left(\int_{\mathbf{g}^{-1}(\mathbf{y})} h^{(\mathbf{y})} \mathrm{area}^{(\mathbf{y})} \right) d\mathbf{y}. \tag{2.227}$$

This coarea formula may also be thought of as a formula for area integrals in terms of delta functions. Thus

$$\int_W h(\mathbf{x}) \delta(\mathbf{g}(\mathbf{x}) - \mathbf{y}) C(\mathbf{x})\, d^n\mathbf{x} = \int_{\mathbf{g}^{-1}(\mathbf{y})} h^{(\mathbf{y})} \mathrm{area}^{(\mathbf{y})}. \tag{2.228}$$

Example: The co-area formula may seem unfamiliar, but there is a case when it becomes quite transparent. Consider a function $g : W \to \mathbf{R}$ defined on some bounded open set W. Then the fibers $g^{-1}(y)$ are hypersurfaces that belong to W. In the integral form above replace $h(\mathbf{x})$ by $h(\mathbf{x}) 1_{\{g(\mathbf{x}) \leq s\}}$. The result is

$$\int_{\{g(\mathbf{x}) \leq s\}} h(\mathbf{x}) |g'(\mathbf{x})|\, d\mathbf{x} = \int_{-\infty}^{s} \int_{g^{-1}(y)} h^{(y)} \mathrm{area}^{(y)}\, dy. \tag{2.229}$$

It follows that

$$\frac{d}{ds} \int_{\{g(\mathbf{x}) \leq s\}} h(\mathbf{x}) |g'(\mathbf{x})|\, d\mathbf{x} = \int_{g^{-1}(s)} h^{(s)} \mathrm{area}^{(s)}. \tag{2.230}$$

In particular, this gives an area representation

$$\mathrm{area}(g^{-1}(s)) = \frac{d}{ds} \int_{\{g(\mathbf{x}) \leq s\}} |g'(\mathbf{x})|\, d\mathbf{x} \tag{2.231}$$

This is a formula for the (n-1) dimensional area of a surface $g^{-1}(s)$ given implicitly by a function $g : W \to \mathbf{R}$. It says that this area is obtained by taking

an n-dimensional integral over a region in W where g has values bounded by s and then taking the derivative. The integrand is the co-area factor, which in this case is just the magnitude of the derivative $g'(\mathbf{x})$. ||

There are versions of the co-area formula that apply in much greater generality. An account of these matters may be found in the book by Lin and Yang [24].

Problems

Convergence of integrals

1. (a) Give an example of functions f and g on $[0,1]$ such that their lower integrals satisfy $L(f+g) > L(f) + L(g)$.

   (b) Give an example of functions f and g on $[0,1]$ such that their upper integrals satisfy $U(f+g) < U(f) + U(g)$.

2. Consider functions $f_n(x)$ with $f_n(x) = n^2 x$ for $0 \leq x \leq 1/n$ and $f_n(x) = n - n^2(x - 1/n)$ for $1/n \leq x \leq 2/n$, zero elsewhere. Find the integral of the pointwise limit. Find the limit of the integrals.

3. Consider functions defined on $[a,b]$. Say that $f_n \downarrow f$ if for each $m \geq n$ we have $f_m(x) \leq f_n(x)$, and if $f(x)$ is the infimum of the $f_n(x)$. (a) Show that the limit of the $f_n(x)$ as $n \to \infty$ is $f(x)$. (b) Suppose each f_n is continuous. Prove or disprove: The convergence must be uniform.

4. Let f_n be a sequence of Riemann integrable functions on $[a,b]$ with each $f_n(x) \geq 0$, uniformly bounded. We would like to have a sequence p_n of functions such that for each n we have $f_n(x) \leq p_n(x)$ and such that $p_n(x) \downarrow p(x)$. An obvious device is to take $p_n(x) = \sup_{k \geq n} f_k(x)$. (a) Prove that the sequence $p_n(x)$ is decreasing. (b) Must each p_n be Riemann integrable? Give a careful discussion and proof.

5. Consider the functions $f_n(x) = x^n$ defined for $0 \leq x \leq 1$. Find the integral of the limit and the limit of the integrals. (a) Does the bounded convergence theorem apply? Explain in detail. (b) Does Dini's theorem apply? Explain in detail. (c) Is there uniform convergence? Justify your answer directly from the definition of uniform convergence.

6. For each $t > 0$ define the function

$$\delta_t(x) = \max(\frac{1}{\sqrt{t}} - \frac{1}{t}|x - \sqrt{t}|, 0) \tag{2.232}$$

   Define $\delta_0(x) = 0$ and $\delta_{-t}(x) = \delta_t(x)$.

   Also, for each t define

$$\phi(x,t) = t\delta_t(x). \tag{2.233}$$

(a) Compute the x integral of $\delta_t(x)$ for each $t \neq 0$. Let $t \to 0$. Compute the x integral of the pointwise limit and the limit of the x integrals.

(b) Compute the x integral of $\phi(x,t)$. Compute the t derivative of the x integral of $\phi(x,t)$.

(c) For fixed x, compute the t derivative of $\phi(x,t)$ at $t=0$. Compute the x integral of this t partial derivative.

(d) What does this say about differentiating under the integral sign?

Fubini's theorem

1. Evaluate
$$\int_0^2 \int_0^3 y e^{-xy}\, dy\, dx \tag{2.234}$$
without using integration by parts.

2. (a) Consider the function f defined on the unit square with $f(x,y)=1$ if x is rational and $y=\frac{1}{2}$, zero otherwise. Is it Riemann integrable? Prove that your answer is correct.

(b) Consider the function f defined on the unit square with $f(x,y)=1$ if x is rational and y is rational, zero otherwise. Is it Riemann integrable? Prove that your answer is correct.

(c) Consider the function f defined on the unit square with $f(x,y)=1$ if x is rational and y is rational and $x=y$, zero otherwise. Is it Riemann integrable? Prove that your answer is correct.

(d) Consider the function f defined on the unit square with $f(x,y)=1$ if x is rational and y is rational and $x \neq y$, zero otherwise. Is it Riemann integrable? Prove that your answer is correct.

(e) Consider the function f defined on the unit square with $f(x,y)=1$ if x is rational or y is rational, zero otherwise. Is it Riemann integrable? Prove that your answer is correct.

3. It is possible that the iterated integrals exist, while the double integral does not exist. Give a detailed proof of this via the following example.

Let B be the set of all numbers (x,y) in the interior of the unit square of the form $(x,y)=(\frac{k}{2^n},\frac{\ell}{2^n})$ where k and ℓ are both odd. Let f be the indicator function of B.

4. Evaluate
$$\int_0^1 \int_{3y}^3 e^{-x^2}\, dx\, dy. \tag{2.235}$$

Change of variables

1. Let $u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$. Consider the region D of x, y in the first quadrant such that the corresponding u, v satisfy $|u| < 3$ and $|v - 2| < 1$. Evaluate
$$\int_D (x^2 + y^2)^2 \, dx \, dy. \tag{2.236}$$

2. Show that
$$\int_0^\infty \int_0^\infty f(\sqrt{t^2 + s^2}) \, t^{n-3} \, dt \, s \, ds = \frac{1}{n-2} \int_0^\infty f(u) u^{n-1} \, du. \tag{2.237}$$

3. Define $a_0 = 2$, $a_1 = 2\pi$ and $a_{n+1} = (2\pi/n)a_{n-1}$. Prove by induction that
$$\int_{\mathbf{R}^n} f(|\mathbf{x}|) \, d\mathbf{x} = a_{n-1} \int_0^\infty f(r) r^{n-1} \, dr. \tag{2.238}$$

   Hint: Start by proving that
$$\int_{\mathbf{R}^n} f(|\mathbf{x}|) \, d\mathbf{x} = \int_0^{2\pi} \int_0^\infty \int_{\mathbf{R}^{n-2}} f(\sqrt{\mathbf{y}^2 + s^2}) \, d\mathbf{y} \, s \, ds \, d\theta. \tag{2.239}$$

4. Evaluate
$$\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)} \, dx \, dy. \tag{2.240}$$

5. Do the integral
$$\int_{-\infty}^\infty e^{-x^2} \, dx \tag{2.241}$$
   by applying Fubini's theorem to the previous problem.

6. Show using Fubini that
$$\int_{\mathbf{R}^n} e^{-\mathbf{x}^2} \, d\mathbf{x} = \pi^{\frac{n}{2}}. \tag{2.242}$$

7. Prove that
$$a_{n-1} \int_0^\infty e^{-r^2} r^{n-1} \, dr = \pi^{\frac{n}{2}}. \tag{2.243}$$
   Remark: This proves the identity $a_{n-1} \frac{1}{2} \Gamma(\frac{n}{2}) = \pi^{\frac{n}{2}}$.

8. For $\epsilon > 0$ define
$$\delta_\epsilon(\mathbf{x}) = \frac{1}{\pi^{\frac{n}{2}}} \frac{1}{\epsilon^n} e^{-\frac{\mathbf{x}^2}{\epsilon^2}}. \tag{2.244}$$
   a) Find the integral over $\mathbf{R}^n$ of δ_ϵ.
   b) Find the pointwise limit of $\delta_\epsilon(\mathbf{x})$ as $\epsilon \to +\infty$.

9. For $\epsilon > 0$ let
$$d_\epsilon(\mathbf{x}) = \frac{2}{n}\frac{1}{\pi^{\frac{n}{2}}}\frac{1}{\epsilon^{n+2}}\mathbf{x}^2 e^{-\frac{\mathbf{x}^2}{\epsilon^2}}. \tag{2.245}$$

   a) Find the integral over $\mathbf{R}^n$ of d_ϵ.

   b) Find the pointwise limit of $d_\epsilon(\mathbf{x})$ as $\epsilon \to +\infty$.

   c) Find the pointwise limit of $d_\epsilon(\mathbf{x})$ as $\epsilon \to 0$.

Change of variables for delta functions

1. (a) For each $k = 0, 1, 2, 3, \ldots$ find a C^k approximate delta function that vanishes outside of a bounded set? (b) Is there a C^∞ approximate δ function that vanishes outside of a bounded set? Prove or disprove.

2. (a) Evaluate
$$\int_{-\infty}^{\infty} h(x)\,dx \tag{2.246}$$
   via the change of variable $t = x^2$.

   (b) Let $a > 0$. Compute $\delta(x^2 - a^2)$ as a linear combination of delta functions of the form $\delta(x \pm a)$. Hint: A formula involving delta functions is defined by multiplying by $f(x)$ and integrating. Thus use the result of part (a) to evaluate
$$\int_{-\infty}^{\infty} \delta(x^2 - a^2) f(x)\,dx. \tag{2.247}$$

3. According to the co-area formula, the area of a surface $g(x, y, z) = c$ is
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(g(x, y, z) - c)|g'(x, y, z)|\,dx\,dy\,dz. \tag{2.248}$$
   Use this to find the area of a sphere of radius a. Thus evaluate
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x^2 + y^2 + z^2 - c) 2\sqrt{x^2 + y^2 + z^2}\,dx\,dy\,dz. \tag{2.249}$$
   Hint: Use the result of the preceding problem to reduce this to an integral over y and z. This integral is easy in polar coordinates.

Surface area

1. Consider an $n - 1$ surface given implicitly by $w = g(x_1, \ldots, x_n) = C$ and explicitly by $x_i = f_i(u_1, \ldots, u_{n-1})$. Suppose the differential
$$dw = \sum_j \frac{\partial w}{\partial x_j}\,dx_j \tag{2.250}$$

has components that form a non-zero row vector. Suppose also that there are $n-1$ linearly independent tangent vectors X_α to the surface, each with components $\partial x_i/\partial u_\alpha$ forming a column vector.

Prove that dw on the surface is zero on the tangent space. That is, prove that for each β we have

$$\sum_{j=1}^{n} \frac{\partial w}{\partial x_j} \frac{\partial x_j}{\partial u_\beta} = 0. \tag{2.251}$$

In other words, the row vector consisting of the components of dw annihilates each of $n-1$ column vectors formed by the components of X_β.

2. Define a row vector with components ν_j given by $(-1)^{j-1}$ times the determinant of $\partial x_i/\partial u_\alpha$ with row j deleted. This defines a form $\nu = \sum_j \nu_j \, dx_j$. Show that ν is also zero on the tangent space. That is, show that for each β

$$\sum_{j=1}^{n} \nu_j \frac{\partial x_j}{\partial u_\beta} = 0. \tag{2.252}$$

Since it is also zero on the tangent space, it must be a multiple of the differential dw on the surface. Hint: Consider the square matrix with first column $\partial x_i/\partial u_\beta$ and remaining columns $\partial x_i/\partial u_\alpha$ for $\alpha = 1, \ldots, n-1$. Here β is an arbitrary choice of one of the α indices.

3. Consider the surface given parametrically by $x = uv, y = u + v, z = u - v$. Find the two tangent vectors X_1 and X_2, regarded as column vectors.

4. Define the two by two matrix with entries $E = X_1^T X_1$ and $F = X_1^T X_2 = X_2^T X_1$ and $G = X_2^T X_2$. Write the quadratic form $E\,du^2 + 2F\,du\,dv + G\,dv^2$ for this surface.

5. a) Show that the same surface is given implicitly $w = 4x - y^2 + z^2 = 0$. Find the form dw.

   b) Express the coefficients of dw in terms of u and v and form a row vector. Show that this row vector is zero on both column vectors X_1 and X_2.

6. Find the form ν for this surface. Express its coefficients as a row vector with entries in terms of u and v. Verify that the two forms are multiples of each other.

7. For the same surface, calculate the area for parameter region $u^2 + v^2 \leq 1$ by integrating $\sqrt{EG - F^2}\,du\,dv$ over the region.

8. For the same surface, calculate the area for parameter region $u^2 + v^2 \leq 1$ by integrating $\sqrt{(\nu_1^2 + \nu_2^2 + \nu_3^2)}\,du\,dv$ over the region.

Chapter 3

Supplement: Matrix Theory

3.1 Matrices and linear transformations

3.1.1 The algebra of matrices

Basic matrix operations

A m by n matrix A is an array a_{ij} of numbers arranged as entries in m rows and n columns. The jth column has the entries a_{ij} as i ranges from 1 to m. The ith row has the entries a_{ij} as j ranges from 1 to n. There is a special m by n zero matrix 0 that has all entries equal to zero. This matrix could be denoted more explicitly by 0_{mn}. In this treatment the usual assumption is that the entries in matrices are real numbers. Occasionally it will be useful to allow complex numbers.

A sum $A + B$ of m by n matrices is given by adding the components of the matrices. More generally, a linear combination $sA + tB$ of m by n entries is obtained by multiplying the entries of the individual matrices by numbers and then adding.

Multiplication is more interesting. If B is a p by m matrix, and A is a m by n matrix, then the matrix product $C = BA$ is a p by n matrix. Its entries are given by

$$c_{kj} = \sum_{i=1}^{m} b_{ki} a_{ij}. \tag{3.1}$$

The entry for C in row k and column j is obtained from row k of B combined with column j of A.

There are two special cases. A n by 1 matrix is a single column. This is a *vector* or *column vector* $\mathbf{x}$ with n components x_j. The zero vector is denoted $\mathbf{0}$. The product $A\mathbf{x}$ is a vector $\mathbf{y}$ with m components

$$y_i = \sum_{j=1}^{n} a_{ij} x_j. \tag{3.2}$$

A 1 by m matrix is a single row. This is a *covector* or *row vector* or *dual vector* or *linear form* $\mathbf{p}$ with m components $\mathbf{p}_i$. The product $\mathbf{p}A$ is another row vector $\mathbf{q}$ with n components

$$q_j = \sum_{i=1}^{m} p_i a_{ij}. \tag{3.3}$$

A n component row vector $\mathbf{p}$ on the left with a n component column vector $\mathbf{x}$ on the right gives a number

$$\mathbf{px} = \sum_{j=1}^{n} p_j x_j. \tag{3.4}$$

The most common interpretation of a m by n matrix is as a linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$. If A is an m by n matrix, and $\mathbf{x}$ is an n component

column vector, then $A\mathbf{x}$ is an m component column vector. This is indicated by writing $A : \mathbf{R}^n \to \mathbf{R}^m$. If B is a p by m matrix, then similarly $B : \mathbf{R}^m \to \mathbf{R}^p$. The matrix product BA is a p by n matrix, and this defines a transformation $BA : \mathbf{R}^n \to \mathbf{R}^p$. The product of BA with $\mathbf{x}$ is the same as the product of B with $A\mathbf{x}$. Thus matrix multiplication coincides with the composition of the functions.

The defining property of a linear transformation is that is preserves linear combinations. This is expressed by the identity $A(s\mathbf{x} + t\mathbf{y}) = sA\mathbf{x} + tA\mathbf{y}$. A space of column vectors is a *vector subspace* if it is non-empty and closed under taking linear combinations. There are two specially important vector subspaces associated with A.

Suppose $A : \mathbf{R}^n \to \mathbf{R}^m$. The *nullspace* is the vector subspace consisting of all $\mathbf{x}$ in $\mathbf{R}^n$ such that $A\mathbf{x} = \mathbf{0}$. The transformation is one-to-one if and only if its nullspace consists only of the zero vector. The equation $A\mathbf{x} = \mathbf{0}$ is a *implicit description* of the $\mathbf{x}$ that belong to this vector subspace.

Again suppose $A : \mathbf{R}^n \to \mathbf{R}^m$. The *range* or *column space* of the transformation is the subspace of $\mathbf{R}^m$ consisting of all vectors $\mathbf{y} = A\mathbf{x}$ where $\mathbf{x}$ is in $\mathbf{R}^n$. The transformation is onto $\mathbf{R}^m$ if and only if its range is $\mathbf{R}^m$. The equation $\mathbf{y} = A\mathbf{x}$ is a *parametric representation* of the $\mathbf{y}$ in the range. The vector $\mathbf{x}$ is the parameter vector.

Another interpretation of a m by k matrix Y is as a list of k column vectors in $\mathbf{R}^m$. The column vectors of A are *linearly independent* when the nullspace consists only of the zero vector. They *span* a vector subspace W when the range is W. When the vectors are both linearly independent and span W, then they are a *basis* for W. The number of vectors in a basis for W is the *dimension* of W.

Suppose $A : \mathbf{R}^n \to \mathbf{R}^m$. The *rank* $\text{rank}(A)$ of the m by n matrix A is the dimension of the range of A. Also, $\text{null}(A)$ denotes the dimension of the nullspace of A. There is an important relation

$$\text{rank}(A) + \text{null}(A) = n. \tag{3.5}$$

It is obvious that $\text{rank}(A) \leq m$, and from the relation it is clear that $\text{rank}(A) \leq n$.

The m by n matrix A has *full rank* if $r = \text{rank}(A)$ satisfies $r = n$ or $r = m$. There are two cases.

- If $n \leq m$, then A has full rank if and only if it defines a one-to-one linear transformation.
- If $m \leq n$, then A has full rank if and only if it defines a transformation that is onto $\mathbf{R}^m$.

Operations on square matrices

Square matrices have special properties. If A is an n by n square matrix, then $A : \mathbf{R}^n \to \mathbf{R}^n$ sends a space to itself. There is an n by n *identity matrix*, usually

denoted I, with entries 1 on the main diagonal and all other entries 0. This may also be denoted more explicitly by I_n. This sends every vector in $\mathbf{R}^n$ to itself.

An n by n square matrix Q has an *inverse* R if $QR = I$ and $RQ = I$. The inverse of Q is denoted Q^{-1}. It is well-known linear algebra fact that if the inverse exists on one side, for instance $QR = I$, then also $RQ = I$, so the inverse exists on both sides. A square matrix has an inverse if and only if it has full rank. A square matrix with an inverse is often called *invertible* or *non-singular*. It is said to be *singular* if it does not have an inverse.

The inverse of a 2 by 2 matrix is particularly simple:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{3.6}$$

For square n by n matrices there is a natural algebra involving addition and subtraction and multiplication. Multiplication is not necessarily commutative. In particular, there are two possible notions of division: PQ^{-1} is in general different from $Q^{-1}P$. One basic property of the inverse is that $A^{-1-1} = A$. The general relation between multiplication and inverse is that when A^{-1} and B^{-1} exist, then

$$(AB)^{-1} = B^{-1}A^{-1}. \tag{3.7}$$

The relation to addition is more complicated; when A^{-1} and B^{-1} exists, then

$$B^{-1} - A^{-1} = A^{-1}(A - B)B^{-1}. \tag{3.8}$$

If A is an n by n square matrix, the *trace* of A is the number $\mathrm{tr}(A)$ obtained by taking the sum of the diagonal entries. If A and B are n by n matrices, then

$$\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B) \tag{3.9}$$

and

$$\mathrm{tr}(AB) = \mathrm{tr}(BA). \tag{3.10}$$

The most interesting number associated to a square matrix A is its *determinant* $\det(A)$. It has a relatively complicated definition, to be treated in a later sections. The most important property is that if A and B are n by n matrices, then then

$$\det(AB) = \det(A)\det(B). \tag{3.11}$$

Also $\det(I) = 1$. A matrix A has an inverse A^{-1} if and only if $\det(A) \neq 0$. In that case $\det(A^{-1}) = 1/\det(A)$.

One interpretation of the determinant of A is that it measures the extent to which n-dimensional volumes are changed by the corresponding linear transformation. There is also a $\pm$ sign that captures change in orientation. This interpretation is obvious in certain examples.

- If the matrix D has zero entries except on the main diagonal, then it is a *diagonal matrix*, and $\det A$ is the product of the entries on the main diagonal.

- If the matrix S is an upper triangular (or lower triangular) matrix with 1s on the main diagonal, then it is a *shear matrix* and has $\det(S) = 1$.
- If the matrix P is obtained by permuting rows (or columns) of the identity matrix, then it is a *permutation matrix* with $\det(P) = \pm 1$, with sign depending on whether it is an even or odd permutation.

It is a standard result of linear algebra that every matrix may be represented as a product of matrices, each of which is of one of these three types. The corresponding volume change factors are multiplied, so the matrix gives the resulting volume change.

The determinant of a 2 by 2 matrix is particularly simple:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc. \tag{3.12}$$

If, for example, $a \neq 0$, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & \frac{ad-bc}{a} \end{bmatrix} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix} = ad - bc. \tag{3.13}$$

The two shear transformations do not change volumes: the diagonal transformation multiplies volumes (up to sign) by $ad - bc$.

Similarity of square matrices

A square matrix A is *similar* to a square matrix B if there is an invertible matrix P with $AP = PB$. This can also be written $A = PBP^{-1}$ or $B = P^{-1}AP$. For two similar matrices the trace and the determinant are the same.

The similarity relation satisfies three basic properties. First, A is similar to A. Second, if A is similar to B, then B is similar to A. Third, if A is similar to B and B is similar to C, then A is similar to C. The last of these is proved by noting that if $AP = PB$ and $BQ = QC$, then $APQ = PQC$.

Remark: The space $\mathbf{R}^n$ is the space of n component column vectors. This serves as a standard vector space of dimension n. It is natural to take $\mathbf{R}^0$ to be a vector space of dimension zero, whose only vector is the zero vector $\mathbf{0}$.

If W is an n dimensional subspace of $\mathbf{R}^m$, then a basis of W is given by the columns of a m by n matrix of full rank. The linear combinations of the columns span W. In particular, if $n = 0$, the basis is given by the columns of a m by 0 matrix. This is an empty list of columns. The linear combinations of these columns span W, that is, the empty linear combination gives the zero vector in W. As a linear transformation from $\mathbf{R}^0$ to $\mathbf{R}^m$ this matrix sends the zero vector in $\mathbf{R}^0$ to the zero vector in $\mathbf{R}^m$. The natural notation for this transformation is 0_{m0}.

This definition is not just theoretical. Consider a computer algorithm to find a basis for the nullspace of a matrix. It has to return a value for the case when the nullspace consists only of the zero vector. So it must be able to return the empty basis.

Summary: It makes sense to speak of a m by n matrix even when $m = 0$ or $n = 0$. Such a matrix has no entries. As a linear transformation it sends the zero vector in $\mathbf{R}^n$ to the zero vector in $\mathbf{R}^m$. Since every space of m by n matrices has a special matrix 0_{mn}, this notation is reasonable when $m = 0$ or $n = 0$ to denote the unique matrix in the space.

It also seems reasonable to have a 0 by 0 identity matrix I_0. This is the same matrix as 0_{00}. The usual convention is that an empty sum is zero and an empty product is one. Hence $\mathrm{tr}(I_0) = 0$ and $\det(I_0) = 1$. This is compatible with the general relations $\mathrm{tr}(I_n) = n$ and $\det(I_n) = 1$. ||

3.1.2 Jordan form

Two by two examples

This section is devoted to a deeper analysis of the structure of a square matrix A with real entries. A common problem is to work with powers A^n. Also, in the study of systems of ordinary differential equations it is important to deal with the $\exp(tA)$ for real t. This satisfies the differential equation $dX/dt = AX$ with initial condition $X = I$ at $t = 0$. It will help to begin with some 2 by 2 examples. It turns out that every 2 by 2 matrix is similar to one of these.

A first special case is a diagonal matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{3.14}$$

The power is also a diagonal matrix

$$\Lambda^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \tag{3.15}$$

and so it its exponential

$$\exp(t\Lambda) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}. \tag{3.16}$$

The next example is the dilation-rotation

$$K = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}. \tag{3.17}$$

The case of main interest is when $b \neq 0$. The power is another dilation-rotation

$$K^n = r^n \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix} \tag{3.18}$$

and the exponential also has this form

$$\exp(tK) = e^{at} \begin{bmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix}. \tag{3.19}$$

There is a final example where the diagonal entry is repeated and the matrix is upper triangular.

$$L = \begin{bmatrix} \lambda & \epsilon \\ 0 & \lambda \end{bmatrix} \tag{3.20}$$

with $\epsilon \neq 0$. The power is another matrix of the same form

$$L^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1}\epsilon \\ 0 & \lambda^n \end{bmatrix} \tag{3.21}$$

and so is the exponential

$$\exp(tL) = e^{\lambda t} \begin{bmatrix} 1 & \epsilon t \\ 0 & 1 \end{bmatrix}. \tag{3.22}$$

In the case when $\lambda \neq 0$ the final example may be written in an alternative form

$$L = \lambda \begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix} \tag{3.23}$$

with $\delta \neq 0$. This can be recognized as a multiple of a shear matrix.

It turns out that every 2 by 2 real matrix is similar to one of these examples. This allows the power and exponential to be computed for all such matrices. Suppose it is possible to easily compute B^n and $\exp(tB)$. Suppose also $A = PBP^{-1}$. Then $A^n = PB^nP^{-1}$ and $\exp(tA) = P\exp(tB)P^{-1}$.

Eigenvalues

Consider a complex square matrix A. A complex number λ is an *eigenvalue* of A if and only if there is a non-zero complex vector $\mathbf{x}$ with

$$A\mathbf{x} = \lambda\mathbf{x}. \tag{3.24}$$

Such a vector $\mathbf{x}$ is called an *eigenvector* for λ. The existence of such a vector is equivalent to the property that $A - \lambda I$ is a singular matrix. Similar matrices have the same eigenvalues. The set of eigenvalues of A is sometimes called the *spectrum* of A. The *spectral radius* $\rho(A)$ is defined to be the largest of the absolute values $|\lambda|$, where λ is a complex eigenvalue.

The eigenvalues are the solutions of the nth degree polynomial equation $\det(A - \lambda I) = 0$. This polynomial $\det(A - \lambda I)$ is called the *characteristic polynomial*. For a real matrix A the characteristic polynomial has only real coefficients, and the roots are either real or occur in complex conjugate pairs.

In the 2 by 2 case it is not difficult to compute the eigenvalues. The sum of the eigenvalues is $\mathrm{tr}(A)$ and the product of the eigenvalues is $\det(A)$. From this it is easy to see that the eigenvalues are the solutions of the quadratic equation $\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$. In general finding eigenvalues can be a challenging task.

Theorem 3.1 (Diagonal form) *Suppose that A is a n by n complex matrix with a basis of complex eigenvectors. Then there is a complex diagonal matrix Λ and a complex invertible matrix P with*

$$AP = P\Lambda. \tag{3.25}$$

In other words, A is similar to a diagonal matrix. The diagonal entries of Λ are the eigenvalues, and the columns of P are the eigenvectors.

It may be shown that if A has n distinct eigenvalues, then A has a basis of eigenvectors. So the situation described in the theorem is the typical case.
Example: If A is a real matrix, then it can happen that it has n distinct real eigenvalues with corresponding real eigenvectors. Consider the matrix

$$A = \begin{bmatrix} 11 & -18 \\ 6 & -10 \end{bmatrix}. \tag{3.26}$$

The trace is 1 and the determinant is -2. It follows that the eigenvalues are 2 and -1. This determines Λ. The matrix P is found by solving $AP = P\Lambda$. The result is the representation

$$A = P\Lambda P^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}. \tag{3.27}$$

That is, A is similar to a real diagonal matrix. ||

Suppose that A is a real matrix with a complex eigenvalue that is not real. Then $A(\mathbf{u} + i\mathbf{v}) = (a - ib)(\mathbf{u} + i\mathbf{v})$ with $b \neq 0$. This can be written in the real form $A\mathbf{u} = a\mathbf{u} + b\mathbf{v}$, $A\mathbf{v} = -b\mathbf{u} + a\mathbf{v}$. Consider the n by 2 matrix $[\mathbf{u}\ \mathbf{v}]$. Then

$$A[\mathbf{u}\ \mathbf{v}] = [a\mathbf{u} + b\mathbf{v}\ \ -b\mathbf{u} + a\mathbf{v}] = [\mathbf{u}\ \mathbf{v}] \begin{bmatrix} a & -b \\ b & a \end{bmatrix}. \tag{3.28}$$

The 2 by 2 matrix on the right is a dilation-rotation matrix K.

Corollary 3.2 (Diagonal form (real version)) *Suppose that A is a n by n real matrix with a basis of complex eigenvectors. Then there is a real matrix Λ that has entries that are either real numbers on the diagonal or 2 by 2 dilation-rotation blocks K centered on the diagonal, and there is a real invertible matrix P such that*

$$AP = P\Lambda. \tag{3.29}$$

Example: Consider the matrix

$$A = \begin{bmatrix} 10 & -13 \\ 5 & -6 \end{bmatrix}. \tag{3.30}$$

The trace is 4 and the determinant is 5. It follows that the eigenvalues are $2 \pm i$. This determines Λ. The matrix P is found by solving $AP = P\Lambda$. The result is the representation

$$A = P\Lambda P^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}. \tag{3.31}$$

That is, A is similar to a dilation rotation matrix. The dilation factor is $\sqrt{5}$. ||

Jordan form

There are matrices that are not diagonalizable. The simplest examples are truncated shift matrices. For instance, the linear transformation

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.32}$$

shifts a vector by one and truncates. The matrix

$$T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.33}$$

shifts a vector by two and truncates. In each of these example the characteristic polynomial has a repeated root, and there is only one eigenvector direction.

When there are repeated roots of the characteristic polynomial, it is usually necessary to introduce new vectors that are not eigenvectors. For instance, one can have a eigenvector $\mathbf{u}$ with $(A - \lambda I)\mathbf{u} = \mathbf{0}$ together with another vector $\mathbf{v}$ with $(A - \Lambda I)\mathbf{v} = \epsilon\mathbf{u}$. Here $\epsilon \neq 0$ is a parameter that may be chosen arbitrarily. In that case we have $(A - \lambda I)[\mathbf{u}\ \mathbf{v}] = [\mathbf{0}\ \epsilon\mathbf{u}]$. This says that $A[\mathbf{u}\ \mathbf{v}] = [\mathbf{u}\ \mathbf{v}]B_2$, where

$$B_2 = \begin{bmatrix} \lambda & \epsilon \\ 0 & \lambda \end{bmatrix}. \tag{3.34}$$

A more complicated variant is to have an eigenvector $\mathbf{u}$ with $(A - \lambda I)\mathbf{u} = \mathbf{0}$ together with vectors $\mathbf{v}$ and $\mathbf{w}$ with $(A - \Lambda I)\mathbf{v} = \epsilon\mathbf{u}$ and $(A - \lambda I)\mathbf{w} = \epsilon\mathbf{v}$. In that case we have $(A - \lambda I)[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = [\mathbf{0}\ \epsilon\mathbf{u}\ \epsilon\mathbf{v}]$. This says that $A[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = [\mathbf{u}\ \mathbf{v}\ \mathbf{w}]B_3$, where

$$B_3 = \begin{bmatrix} \lambda & \epsilon & 0 \\ 0 & \lambda & \epsilon \\ 0 & 0 & \lambda \end{bmatrix}. \tag{3.35}$$

There is an obvious generalization to even larger blocks. Call each of these a *repeated eigenvalue Jordan block*.

In general a *complex Jordan matrix* is made of *complex Jordan blocks* centered on the main diagonal. Each complex Jordan block is either a single eigenvalue on the diagonal or a repeated eigenvalue Jordan block.

Theorem 3.3 (Jordan form) *Suppose that A is a n by n complex matrix. Fix $\epsilon \neq 0$. Then there is a complex Jordan matrix J with parameter ϵ and a complex invertible matrix P (depending on ϵ) with*

$$AP = PJ. \tag{3.36}$$

For the case of a real matrix with repeated complex eigenvalues it is possible to write the Jordan blocks in a real form. There is a 4 by 4 matrix

$$C_2 = \begin{bmatrix} K & \epsilon I \\ 0 & K \end{bmatrix}. \tag{3.37}$$

with K a 2 by 2 dilation-rotation matrix. Similarly, there is a 6 by 6 matrix

$$C_3 = \begin{bmatrix} K & \epsilon I & 0 \\ 0 & K & \epsilon I \\ 0 & 0 & KR \end{bmatrix}. \tag{3.38}$$

The generalization to larger blocks goes the same way. In these formulas K is a 2 by 2 dilation-rotation matrix. The 0 and I stand for 2 by 2 zero matrices or 2 by 2 identity matrices. The parameter ϵ is real and non-zero. Each of these is a *repeated dilation-rotation Jordan block*.

A *real Jordan matrix* is constructed from *real Jordan blocks* arranged along the main diagonal. Each real Jordan block is either a real eigenvalue block, dilation-rotation block, repeated real eigenvalue Jordan block, or repeated dilation-rotation Jordan block.

Theorem 3.4 (Jordan form (real version)) *Let A be an n by n matrix with real entries. Fix real $\epsilon \neq 0$. Then there exists a real Jordan matrix J with parameter ϵ and a real invertible matrix P (depending on ϵ) such that*

$$AP = PJ. \tag{3.39}$$

This theorem says that A has the representation $A = PJP^{-1}$. The matrix $P^{-1}x$ introduces a distortion, the matrix J has a simple description, and the matrix P undoes the distortion. The Jordan form gives useful formulas for A^n and for $\exp(tA)$. In fact, $A^n = PJ^nP^{-1}$ and $\exp(tA) = P\exp(tJ)P^{-1}$. The powers of Jordan matrices are relatively easy to compute.

Example: Consider the matrix

$$A = \begin{bmatrix} -5 & 12 \\ -3 & 7 \end{bmatrix}. \tag{3.40}$$

The trace is 2 and the determinant is 1. It follows that the only eigenvalue is 2. This is non-similar to a diagonal matrix, so it must be one of the exceptional Jordan blocks. The parameter is arbitrary except that it must not be zero. Take $\epsilon = 3$. The matrix P is found by solving $AP = PJ$. The result is the representation

$$A = PJP^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}. \tag{3.41}$$

That is, A is similar to a shear matrix. ||

Here is another way to think of the Jordan form. A square matrix N is said to be *nilpotent* if some power $N^p = 0$ for $p \geq 1$. Such a matrix is small in the

sense that every eigenvalue of N must be zero. In other words, the spectral radius $\rho(N) = 0$. The classic example of a nilpotent matrix is a matrix that is strictly upper (or strictly lower) triangular (so it has only zeros on the diagonal).

Theorem 3.5 (Jordan decomposition) *If A is a real square matrix, then*

$$A = B + N, \tag{3.42}$$

where B is a real matrix with a basis of complex eigenvectors, and where N is a real nilpotent matrix. The matrices B and N commute.

Proof: The matrix J may be written in the form

$$J = \Lambda + \Delta, \tag{3.43}$$

where Λ has real diagonal entries or dilation rotation blocks, and Δ consists of blocks that are truncated shift matrices multiplied by ϵ. Since Δ is strictly upper triangular, it is nilpotent. It is easy to see that Λ and Δ commute. Then $A = PJP^{-1} = P\Lambda P^{-1} + P\Delta P^{-1} = B + N$. Since N is similar to Δ, it follows that N is also nilpotent. Similarly, B and N commute. □

In the situation described in the corollary $\exp(tA) = \exp(B)\exp(tN)$. The $\exp(tB) = P\exp(t\Lambda)P^{-1}$ part is expressed in terms of functions of t that are exponential or trigonometric functions. Since N is nilpotent, the $\exp(tN)$ has an expansion in powers of tN that eventually terminates. So $\exp(tN)$ is a polynomial in t with matrix coefficients.

3.2 Matrices and quadratic forms

3.2.1 The transpose of a matrix

Transpose, inner product, norm

Let A be an m by n matrix. Then its *transpose* A^T is the n by m matrix with the columns and rows reversed. Thus A^T has entries $a^T_{ij} = a_{ji}$. The most basic identities involving the transpose are $A^{TT} = A$ and $(A+B)^T = A^T + B^T$. With multiplication the key property is

$$(BA)^T = A^T B^T. \tag{3.44}$$

Another useful property is that the rank of A^T is equal to the rank of A.

The transpose of an n by n square matrix A is another n by n square matrix A^T. The trace and determinant satisfy $\mathrm{tr}(A) = \mathrm{tr}(A^T)$ and $\det(A) = \det(A^T)$. An n by n real matrix A is *symmetric* if $A = A^T$.

The *Euclidean inner product* of two real column vectors is

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y} = \sum_{i=1}^{n} x_i y_i. \tag{3.45}$$

The *Euclidean norm* of a column vector is

$$|\mathbf{x}| = \sqrt{\mathbf{x}^T\mathbf{x}} = \sqrt{\sum_{i=1}^{n} x_i^2}. \tag{3.46}$$

The famous Schwarz inequality says that $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$. One important consequence is the triangle inequality $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$.

Two vectors are *orthogonal* if $\mathbf{x} \cdot \mathbf{y} = 0$. The *Theorem of Pythagoras* says that for orthogonal vectors $|\mathbf{x} + \mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2$.

Proposition 3.6 (Projection theorem) *Consider a vector* $\mathbf{y}$ *in* $\mathbf{R}^n$ *and a vector subspace* W *of* $\mathbf{R}^n$*. There is a unique vector* $\hat{\mathbf{y}}$ *in* W *such that* $\mathbf{y} - \hat{\mathbf{y}}$ *is orthogonal to every vector* $\mathbf{w}$ *in* W*. Furthermore,* $|\mathbf{y} - \mathbf{w}|^2 = |\mathbf{y} - \hat{\mathbf{y}}|^2 + |\hat{\mathbf{y}} - \mathbf{w}|^2$ *for every* $\mathbf{w}$ *in* W*. Thus* $\hat{\mathbf{y}}$ *is the vector in* W *closest to* $\mathbf{y}$*. It also follows that* $|\mathbf{y}|^2 = |\mathbf{y} - \hat{\mathbf{y}}|^2 + |\hat{\mathbf{y}}|^2$

The vector given by the theorem is called the *orthogonal projection vector*. A square matrix P is an *orthogonal projection matrix* if $P^2 = P$ and $P^T = P$. It is the orthogonal projection onto the range of P. This means that if $\mathbf{y}$ is in $\mathbf{R}^n$, then $P\mathbf{y}$ is in the range of P, and the difference $\mathbf{y} - P\mathbf{y}$ is orthogonal to every vector in the range of P.

Gram matrices

An m by n matrix A^TA or AA^T is sometimes called a *Gram matrix*. . The rank of the n by n matrix A^TA is the same as the rank of the m by m matrix AA^T, and these are both equal to the rank of A. As a consequence

- If $n \leq m$, then the transformation A is one-to-one if and only A^TA is invertible.
- If $m \leq n$, then the transformation A is onto $\mathbf{R}^m$ if and only if AA^T is invertible.

These matrices can give left and right inverses to matrices that are not square.

- If $n \leq m$ and A^TA is invertible, then $(A^TA)^{-1}A^T$ is a left inverse to A.
- If $m \leq n$ and AA^T is invertible, then $A^T(AA^T)^{-1}$ is a right inverse to A.
- If $n \leq m$ and A^TA is invertible, then $A(A^TA)^{-1}A^T$ is the orthogonal projection onto the range of A.
- If $m \leq n$ and AA^T is invertible, then $A^T(AA^T)^{-1}A$ is the orthogonal projection onto the space of vectors orthogonal to the nullspace of A

Example: Fitting parameters to data. Consider the case $n \leq m$ with a matrix A of rank n. The range of A is a vector subspace of dimension $n \leq m$. Suppose $\mathbf{y}$ is a given data vector in $\mathbf{R}^m$. There may be no parameter vector $\mathbf{x}$ with $A\mathbf{x}$

equal to $\mathbf{y}$. However there is a parameter vector $\mathbf{x}$ such that $\hat{\mathbf{y}} = A\mathbf{x}$ is closest to $\mathbf{y}$. It is given by the left inverse via $\mathbf{x} = (A^T A)^{-1} A^T \mathbf{y}$. The corresponding estimate vector $\hat{\mathbf{y}} = A(A^T A)^{-1} A^T \mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto the range of A. ||

3.2.2 Symmetric matrices as quadratic forms

Quadratic forms

An n by n symmetric square real matrix G, so

$$G = G^T. \tag{3.47}$$

It defines a linear transformation from column vectors to row vectors. The value of this transformation on $\mathbf{x}$ is

$$(G\mathbf{x})^T = \mathbf{x}^T G. \tag{3.48}$$

It also defines a symmetric *bilinear form*

$$\mathbf{x}^T G \mathbf{y} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i g_{ij} y_j. \tag{3.49}$$

This in turn gives a *quadratic form*

$$\mathbf{x}^T G \mathbf{x} = \sum_{i=1}^{n} x_i g_{ij} x_j. \tag{3.50}$$

Congruence

A symmetric matrix G is *congruent* to a symmetric matrix H if there is an invertible matrix P with $P^T G P = H$.

The congruence relation satisfies three basic properties. First, GA is similar to G. Second, if G is similar to H, then H is similar to G. Third, if G is congruent to H, and H is congruent to K, then G is congruent to K. Thus if $P^T G P = H$ and $Q^T H Q = K$, then $Q^T P^T G P Q = K$, and hence $(PQ)^T G (PQ) = K$.

Theorem 3.7 *Let $G = G^T$ be a symmetric matrix. Then there exists an invertible matrix P such that that $P^T G P = J$, where J is diagonal with entries that are each either 1 or 0 or -1. In other words, G is congruent to a matrix J of this type.*

A quadratic form G is *positive definite* if $\mathbf{x}^T G \mathbf{x} \geq 0$ for all vectors $\mathbf{x}$. If G is positive definite, then the corresponding J has diagonal entries 1 or 0.

A quadratic form G is *strictly positive definite* if $\mathbf{x}^T G \mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$. If G is strictly positive definite, then J has diagonal entries 1, that is, J is the

identity matrix I. One useful consequence of the theorem is that G is a strictly positive definite form if and only if $G = Q^T Q$ for some invertible matrix Q.

When G is strictly positive definite, then $\mathbf{x}^T G\mathbf{y}$ is called the G *inner product*. There is a corresponding G *norm* defined by $|\mathbf{x}|_G = \sqrt{\mathbf{x}^T G\mathbf{x}}$. Much of linear algebra carries over to these more general inner products and norms.

Remark: The theorem in this section classifies symmetric real matrices under the congruence relation in a simple and useful way. There is a similar theorem for anti-symmetric real matrices under congruence that is even simpler. How about a real square matrix that is not assumed to be either symmetric or anti-symmetric? (This would do for congruence what the Jordan form does for similarity.) It turns out that there is a result that classifies such matrices under the congruence relation, but it is significantly more complicated. See [20] or [49] for a discussion. ||

3.2.3 Symmetric matrices as linear transformations

The spectral theorem

Let $A = A^T$ be a symmetric n by n real matrix. The plan is to consider A both as a linear transformation and as a symmetric quadratic form. Suppose that B is another symmetric matrix, and that A is both similar to B and congruent to B via the same transformation P. Then $P^{-1}AP = B$ and $P^T AP = B$, with $P^{-1} = P$.

An n by n real matrix P is *orthogonal* if $P^{-1} = P^T$. If P is orthogonal, then so are P^{-1} and P^T, and in fact they are the same. If P is orthogonal, then $(\det P)(\det P) = \det(P)\det(P^T) = \det(PP^T) = \det(I) = 1$, so $\det(P) = \pm 1$.

Theorem 3.8 (Spectral theorem) *Suppose A is real and symmetric. Then there is a real orthogonal matrix P and a real diagonal matrix Λ such that*

$$AP = P\Lambda. \tag{3.51}$$

In particular, A is both similar and congruent to a real diagonal matrix via P.

The diagonal entries λ_j in Λ are the eigenvalues of A. The columns $\mathbf{p}_j$ of P form an orthonormal basis, and these are the eigenvectors of A. This gives the *spectral representation* of the matrix:

$$A = P\Lambda P^T = \sum_j \lambda_j \mathbf{p}_j \mathbf{p}_j^T. \tag{3.52}$$

Example: Here is a simple example where one can do the computation. The symmetric matrix is

$$A = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}. \tag{3.53}$$

It is easy to see that this matrix has dependent rows, so the determinant is zero. As a consequence at least one eigenvalue is zero. In this situation it is

easy to find the other two eigenvalues λ_1, λ_2. Use $\lambda_1 + \lambda_2 = \text{tr}(A) = 34$ and $\lambda_1^2 + \lambda_2^2 = \text{tr}(A^2) = 706$. All that remains is to solve a quadratic equation to get the non-zero eigenvalues 25, 9. The corresponding eigenvectors are found by solving linear systems. The eigenvectors form the column matrix

$$R = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & -2 \\ 0 & 4 & -1 \end{bmatrix}. \tag{3.54}$$

Since the eigenvalues are distinct, the eigenvectors are automatically orthogonal. This says that $R^T R$ is a diagonal matrix. If we normalize each column to be a vector of length one, then we get a new matrix P such that $P^T P$ is the identity matrix. In particular, we get the representation

$$A = 25\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} + 9\frac{1}{18}\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}\begin{bmatrix} 1 & -1 & 4 \end{bmatrix}. \tag{3.55}$$

||

Singular values

The spectral theorem applies to Gram matrices.

Theorem 3.9 (Singular values) *Suppose A be a real matrix, not necessarily square. Then the matrix*

$$H = A^T A \tag{3.56}$$

is symmetric with all eigenvalues ≥ 0. In particular, there is a real orthogonal matrix P and a real diagonal matrix $\Sigma^2 \geq 0$ such that

$$HP = P\Sigma^2. \tag{3.57}$$

The numbers σ_j^2 on the diagonal of Σ^2 are the eigenvalues of $H = A^T A$. Their square roots $\sigma_i \geq 0$ are the *singular values* of A.

Corollary 3.10 *Every Gram matrix $A^T A$ with eigenvalues σ_i^2 has a square root $\sqrt{A^T A}$ which is also self-adjoint and has eigenvalues $\sigma_i \geq 0$.*

Proof: The matrix $A^T A = P\Sigma^2 P^{-1}$ where Σ is diagonal with diagonal entries σ_i^2. Define the corresponding diagonal matrix Σ with diagonal entries $\sigma_i \geq 0$. The matrix $P\Sigma P^{-1}$ is the square root. □

Theorem 3.11 (Polar decomposition) *Consider an m by n matrix A with $n \leq m$. Suppose that A has full rank. Then there exists another m by n matrix U with $U^T U = I$ such that*

$$A = U\sqrt{A^T A}. \tag{3.58}$$

Proof: Let $U = A(\sqrt{A^T A})^{-1}$. $\square$

The condition that $U^T U = I$ says that the columns of U form an orthonormal basis. One consequence of the polar decomposition is that the orthogonal projection onto the range of A is

$$A(A^T A)^{-1} A^T = UU^T. \tag{3.59}$$

The polar decomposition together with the spectral theorem gives the *singular value decomposition*

$$A = UP\Sigma P^T. \tag{3.60}$$

All the distortion is isolated in the diagonal matrix Σ.

3.2.4 Matrix norms

The Lipschitz norm

If A is a real m by n matrix, the norm of the matrix A is the least number $\|A\|$ such that for all real n component column vectors we have

$$|A\mathbf{x}| \leq \|A\||\mathbf{x}|. \tag{3.61}$$

It follows that

$$|A\mathbf{x} - A\mathbf{y}| = |A(\mathbf{x} - \mathbf{y})| \leq \|A\||\mathbf{x} - \mathbf{y}|, \tag{3.62}$$

so $\|A\|$ is the best Lipschitz constant for the linear transformation A. In these lectures it will be called the *Lipschitz norm* of A.

The Lipschitz norm behaves reasonably under sum and product. In fact, $\|A + B\| \leq \|A\| + \|B\|$ and $\|AB\| \leq \|A\|\|B\|$. The norm also preserves the transpose: $\|A^T\| = \|A\|$.

The basic existence result for the inverse is the following.

Proposition 3.12 *Suppose that $\|C\| < 1$. Then $(1 - C)^{-1}$ exists.*

Proof: Consider the equation $(I - C)X = I$, that is, $I + CX = X$. The space of n by n matrices with the Lipschitz norm is a complete metric space. Define the map $g(X) = I + CX$. This is a contraction mapping. Hence it has a fixed point. $\square$.

Proposition 3.13 *Suppose that A^{-1} exists and that $\|A^{-1}(B - A)\| < 1$. Then B^{-1} exists, and*

$$B^{-1} = A^{-1} - A^{-1}(B - A)B^{-1}. \tag{3.63}$$

Proof: By the previous proposition $I + A^{-1}(B - A)$ has an inverse. Then one can verify that

$$B^{-1} = (I + A^{-1}(B - A))^{-1})A^{-1}. \tag{3.64}$$

Multiply on the left by $I + A^{-1}(B - A)$ to get the identity. $\square$

This norm $\|A\|$ is quite difficult to compute. Use the spectral theorem for $A^T A = P\Sigma^2 P^T$ to compute

$$|A\mathbf{x}|^2 = \mathbf{x}^T A^T A\mathbf{x} = (P^T\mathbf{x})^T \Sigma^2 P^T\mathbf{x} \leq \sigma_{\max}^2 |P^T\mathbf{x}|^2 = \sigma_{\max}^2 |\mathbf{x}|^2. \tag{3.65}$$

Thus $|A\mathbf{x}| \leq \sigma_{\max}|\mathbf{x}|x$, and this is the least such bound. This is summarized in the following result.

Theorem 3.14 *The Lipschitz norm of A has the explicit expression*

$$\|A\| = \sigma_{\max}, \tag{3.66}$$

where $\sigma_{\max}$ is the largest singular value of A.

For the real symmetric matrix $A^T A$ the norm is the largest eigenvalue $\sigma_{\max}^2$. So another consequence of this reasoning is

$$\|A\| = \sqrt{\|A^T A\|}. \tag{3.67}$$

In other words, to compute the norm of A one computes the norm of the symmetric matrix $A^T A$ and takes the square root. Unfortunately, this requires finding an eigenvalue.

The Lipschitz norm $\|A\|$ and the spectral radius $\rho(A)$ are related by $\rho(A) \leq \|A\|$. In general they are not equal. However for a Jordan matrix they are very close to being equal.

Proposition 3.15 *Let J be a Jordan matrix whose blocks are real diagonal elements, dilation-rotation blocks, Jordan blocks with repeated real diagonal elements, or Jordan blocks with repeated dilation-rotation blocks. Suppose that in the last two cases the entries of the shift and truncated parts are multiplied by the parameter $\epsilon > 0$. Then $\|J\| \leq \rho(J) + \epsilon$.*

Proof: Write $J = \Lambda + \Delta$. The first term corresponds to the real diagonal elements and dilation-rotation blocks. Furthermore, $\rho(J) = \rho(\Lambda)$. The second term has the ϵ factors times shift and truncate parts. The shift and truncation parts can only decrease norm. Then $|J\mathbf{x}| \leq |\Lambda\mathbf{x}| + |\Delta\mathbf{x}| \leq \rho(J)|\mathbf{x}| + \epsilon|\mathbf{x}|$. □

Let G be a positive definite quadratic form. It is possible to define a new G norm $|\mathbf{x}|_G$ on $\mathbf{R}^n$ such that $|\mathbf{x}|_G^2 = \mathrm{tr}(\mathbf{x}^T G\mathbf{x})$. For square matrices A there is also a corresponding G Lipschitz norm $\|A\|_G$ on square matrices such that $|A\mathbf{x}|_G \leq \|A\|_G|\mathbf{x}|_G$.

Suppose G is factored as $G = Q^T Q$. Then $|\mathbf{x}|_G = |Q\mathbf{x}|$. This implies that $|\mathbf{x}|_G \leq \|Q\||\mathbf{x}|_G$ and $|\mathbf{x}| = |Q^{-1}Q\mathbf{x}| \leq \|Q^{-1}\||\mathbf{x}|_G$. In particular, convergence in the usual vector norm is equivalent to convergence in the G norm. Similar reasoning shows that $\|A\|_G = \|QAQ^{-1}\|$.

Theorem 3.16 *Suppose that A is a real square matrix with spectral radius $\rho(A)$. Choose $\epsilon > 0$. Then there is a strictly positive definite quadratic form G (depending on ϵ) such that the G Lipschitz norm $\|A\|_G \leq \rho(A) + \epsilon$.*

Proof: By the Jordan form theorem there is a Q with $J = QAQ^{-1}$ a Jordan matrix with parameter ϵ. Hence $\|A\|_G = \|J\| \leq \rho(J) + \epsilon = \rho(A) + \epsilon$. □

The Euclidean norm

There is another norm of A that is easier to compute. This is the *Euclidean norm*

$$\|A\|_2 = \sqrt{\mathrm{tr}\, A^T A}. \tag{3.68}$$

It has the happy feature that it may be obtained from the matrix entries. The kj entry of $A^T A$ is $\sum_i a_{ik} a_{ij}$. Thus the diagonal jj entry is $\sum_i a_{ij} a_{ij} = \sum_i a_{ij}^2$ Thus

$$\|A\|_2^2 = \mathrm{tr}(A^T A) = \sum_j \sum_i a_{ij}^2. \tag{3.69}$$

(The Euclidean norm can also be called the Hilbert-Schmidt norm. Another name that is sometimes used is Frobenius norm.)

Even better news is that the Euclidean norm is an upper bound for the Lipschitz norm. In fact,

$$|A\mathbf{x}|^2 = \sum_i (\sum_j a_{ij} x_j)^2 \le \sum_i \sum_j a_{ij}^2 \sum_k x_k^2 = \|A\|_2^2 |x|^2. \tag{3.70}$$

This is summarized in the following proposition.

Proposition 3.17 *The Lipshitz norm of a matrix is related to the Euclidean norm of the matrix by*

$$\|A\| \le \|A\|_2. \tag{3.71}$$

The other direction is not so precise. There is a bound, but it depends on dimension.

Proposition 3.18 *The Euclidean norm of a matrix $A : \mathbf{R}^n \to \mathbf{R}^m$ is related to the Lipschitz norm of the matrix by*

$$\|A\|_2 \le \sqrt{n} \|A\|. \tag{3.72}$$

The proof is easy. Let δ_j be the column vector in $\mathbf{R}^n$ whose jth entry is 1 and whose other entries are zero. Then $a_{ij} = (A\delta_j)_i$. Furthermore, the norm $|\delta_j| = 1$. So

$$\|A\|_2^2 = \sum_{j=1}^n \sum_{i=1}^m a_{ij}^2 = \sum_{j=1}^n \sum_{i=1}^m (A\delta_j)_i^2 = \sum_{j=1}^n |A\delta_j|^2 \le \sum_{j=1}^n \|A\|^2 = n\|A\|^2. \tag{3.73}$$

Example: Consider the matrix

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}. \tag{3.74}$$

It is easy to see that $\|A\|_2 = \sqrt{34}$. On the other hand, to compute $\|A\|$ takes some work. But the matrix $A^T A$ is the matrix H of the previous example, which has eigenvalues 25, 9, 0. So $\|A\| = \sqrt{25} = 5$.

There is a considerably easier way to do this computation, namely compute $\|A^T\|$, which is square root of the largest eigenvalue of AA^T. The pleasure of this is left to the reader. ||

3.3 Area and Coarea

3.3.1 Area

Determinants

The *Levi-Civita permutation symbol* $\epsilon_{j_1\cdots j_n}$ is equal to 0 if there are repeated indices, equal to 1 if the indices form an even permutation of $1,\ldots,n$, and equal to -1 if the indices form an odd permutation of $1,\ldots,n$. The advantage of the Levi-Civita symbol is that the sums are over all values of the indices; the symbol itself enforces the permutation condition. This is particularly useful in dealing with determinants. One elementary property of this symbol is

$$\sum_{j_1,\ldots,j_n} \epsilon_{j_1\cdots j_n}\epsilon_{j_1\cdots j_n} = n!. \tag{3.75}$$

A general definition of determinant is

$$\det(A) = \sum_{j_1,\ldots,j_n} \epsilon_{j_1\cdots j_n} a_{1j_1}\cdots a_{nj_n}. \tag{3.76}$$

This says that the determinant is a sum of products, each product having coefficient 0, 1, or -1. There are n^n such products, many of them equal to zero. Each product with a non-zero coefficient corresponds to picking a distinct element from each row and multiplying them together. The number of products with possibly non-zero coefficient is $n!$, which is still a very large number for computational purposes.

The determinant formula above depends on the fact that the rows are taken in order $1,\ldots,n$. If we instead take them in the order $i_1,\ldots,i_n$ we get

$$\sum_{j_1,\ldots,j_n} \epsilon_{j_1\cdots j_n} a_{i_1j_1}\cdots a_{i_nj_n} = \epsilon_{i_1\cdots i_n}\det(A). \tag{3.77}$$

There is an even more complicated but considerably more symmetric formula for the determinant:

$$\det(A) = \frac{1}{n!}\sum_{i_1,\ldots,i_n}\sum_{j_1,\ldots,j_n} \epsilon_{i_1\cdots i_n}\epsilon_{j_1\cdots j_n} a_{i_1j_1}\cdots a_{i_nj_n}. \tag{3.78}$$

This formula immediately leads to the fact that $\det(A) = \det(A^T)$.

Theorem 3.19 (Multiplicative property of determinants)

$$\det(AB) = \det(A)\det(B). \tag{3.79}$$

The proof follows immediately from the symmetric expression for the determinant applied to the matrix AB. Thus

$$\det(AB) = \frac{1}{n!}\sum_{i_1,\ldots,i_n}\sum_{j_1,\ldots,j_n}\sum_{k_1,\ldots,k_n} \epsilon_{i_1\cdots i_n}\epsilon_{j_1\cdots j_n} a_{i_1k_1}b_{kj_1}\cdots a_{i_nk_n}b_{k_nj_n}. \tag{3.80}$$

This simplifies to

$$\det(AB) = \frac{1}{n!} \sum_{k_1,\ldots,k_n} \epsilon_{k_1\cdots k_n}\epsilon_{k_1\cdots k_n} \det(A)\det(B) = \det(A)\det(B). \quad (3.81)$$

Area of a parallelepiped

First we need a definition of the area of a parallelepiped. Take $k \leq n$ and let A be an n by k matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_k$. If A has rank less than k, then the corresponding k-dimensional area of the parallelepiped is zero. Otherwise, let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be an orthonormal basis for the range of A, and write the corresponding matrix as U. Then $A = UK$, where K is an invertible matrix. The area of the parallelepiped is then defined by $\text{area}(\mathbf{a}_1, \ldots, \mathbf{a}_k) = |\det K|$.

The motivation for this definition is the following. The unit vectors that form the columns of U span a cubical region of area one. Since $A = UK$, the columns of U are mapped by K onto the the columns of A. In fact, $a_{ij} = \sum_i u_{ip}K_{pj}$ gives $\mathbf{a}_j = \sum_i \mathbf{u}_p K_{pj}$. The determinant of K then determines (up to sign) how the area is transformed as we pass from the cubical region to the parallelepiped.

The next result shows that this notion of area is well-defined, and it even gives an explicit formula for the area in terms of A.

Theorem 3.20 (Area formula) *Take $k \leq n$, and consider k vectors $\mathbf{a}_1, \ldots, \mathbf{a}_k$ in $\mathbf{R}^n$. Let A be the n by k matrix whose columns are these vectors. Let A^TA be the corresponding k by k Gram matrix. Then*

$$\text{area}(\mathbf{a}_1, \ldots, \mathbf{a}_k) = \sqrt{\det(A^TA)}. \quad (3.82)$$

Proof: The subspace W spanned by the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_k$ is the range of A. If W has dimension less than k, then the area should be zero. So we may as wall assume that W has dimension k, so A has full rank. Write $A = UK$, where the columns of U form an orthonormal basis for W. This orthonormal basis condition is equivalent to the equation $U^TU = I$. The area is defined as $|\det K|$. It follows that $A^TA = K^TU^TUK = K^TK$, so $\det(A^TA) = \det(K^TK) = \det(K^T)\det(K) = \det(K)^2$. The result follows. □

Corollary 3.21 (Area formula for top dimension) *Consider k vectors $\mathbf{a}_1, \ldots, \mathbf{a}_k$ in $\mathbf{R}^k$. Let A be the k by k matrix whose columns are these vectors. Then*

$$\text{area}(\mathbf{a}_1, \ldots, \mathbf{a}_k) = |\det(A)|. \quad (3.83)$$

The Cauchy-Binet formula

Theorem 3.22 (Cauchy-Binet for Gram determinants) *Take $k \leq n$, and let A be n by k matrix. Let A^TA be the corresponding k by k Gram matrix. For each k-element subset K of $\{1, \ldots, n\}$, let A_K be the corresponding square submatrix of A. Then*

$$\det(A^TA) = \sum_K \det(A_K^TA_K) = \sum_K \det(A_K)^2. \quad (3.84)$$

Proof: Let A be an n by k matrix. For each sequence $i_1, \ldots, i_k$ of rows of X, let

$$J_{i_1 \cdots i_k} = \sum_{\alpha_1 \cdots \alpha_k} \epsilon_{\alpha_1 \cdots \alpha_k} A_{i_1 \alpha_1} \cdots A_{i_k \alpha_k}. \tag{3.85}$$

represent the determinant of the corresponding k by k minor obtained by retaining only those rows.

We have

$$\det(A^T A) = \frac{1}{k!} \sum_{\alpha_1 \cdots \alpha_k} \sum_{\beta_1 \cdots \beta_k} \epsilon_{\alpha_1 \cdots \alpha_k} \epsilon_{\beta_1 \cdots \beta_k} (A^T A)_{\alpha_1\, \beta_1} \cdots (A^T A)_{\alpha_k\, \beta_k}. \tag{3.86}$$

However

$$(A^T A)_{\alpha\, \beta} = \sum_i A_{i\alpha} A_{i\beta}. \tag{3.87}$$

So

$$\det(A^T A) = \frac{1}{k!} \sum_{i_1 \ldots i_k} \sum_{\alpha_1 \cdots \alpha_k} \sum_{\beta_1 \cdots \beta_k} \epsilon_{\alpha_1 \cdots \alpha_k} \epsilon_{\beta_1 \cdots \beta_k} A_{i_1 \alpha_1} \cdots A_{i_k \alpha_k} A_{i_1 \beta_1} \cdots A_{i_k \beta_k}. \tag{3.88}$$

From the definition of J we get

$$\det(A^T A) = \frac{1}{k!} \sum_{i_1 \ldots i_k} J^2_{i_1 \cdots i_k}. \tag{3.89}$$

This is equivalent to the statement of the theorem. □

Theorem 3.23 (Cauchy-Binet for areas) *Take $k \leq n$, and let $\mathbf{a}_1, \ldots, \mathbf{a}_k$ be vectors in $\mathbf{R}^n$. Each k-element subset K of $\{1, \ldots, n\}$ defines a k-dimensional coordinate plane in $\mathbf{R}^n$. Let $\mathbf{a}_{1K}, \ldots, \mathbf{a}_{kK}$ be the projection of the vectors onto the coordinate plane defined by K. Then*

$$\text{area}(\mathbf{a}_1, \ldots, \mathbf{a}_k)^2 = \sum_K \text{area}(\mathbf{a}_{1K}, \ldots, \mathbf{a}_{kK})^2. \tag{3.90}$$

The following special case shows that a parallelepiped of dimension $n-1$ may be represented by an orthogonal vector. The area of the parallelepiped is the length of the vector.

Theorem 3.24 (Cauchy-Binet for hyperplane areas) *Let A be n by $n-1$ matrix. Let $A^T A$ be the corresponding $n-1$ by $n-1$ Gram matrix. For each j in $\{1, \ldots, n\}$, let $A_{\setminus j}$ be the $n-1$ by $n-1$ square submatrix obtained from A by omitting row j. Define the vector*

$$\nu_j = (-1)^{j-1} \det(A_{\setminus j}) \tag{3.91}$$

Then ν is orthogonal to the column space of A, and the area is equal to the length of the vector:

$$\det A^T A = |\nu|^2. \tag{3.92}$$

Proof: Since $\nu_k^2 = \det(A_{\setminus j})^2$, the identity follows. It remains to show that ν is orthogonal to the $\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}$. Fix i with $1 \le i \le n-1$. It is sufficient to show that ν is orthogonal to $\mathbf{a}_i$. Consider the n by n matrix obtained by augmenting this list by $\mathbf{a}_i$. Its determinant is given by the cofactor expansion

$$\det[\mathbf{a}_i, \mathbf{a}_1, \ldots, \mathbf{a}_{n-1}] = \sum_{j=1}^{n} (-1)^{j-1} a_{ij} \det(A_{\setminus j}). \tag{3.93}$$

But this says that

$$0 = \sum_{j=1}^{n} a_{ij} \nu_j. \tag{3.94}$$

This completes the argument. □

3.3.2 Co-area*

The co-area formula

Let A_1 be an n by k matrix with $k < m$. Recall that $A_1^T A_1$ is a k by k matrix. The area associated with A_1 is $\sqrt{\det(A_1^T A_1)}$. Now let A_2 be an n by $n-k$ matrix. Place these together to form a n by n matrix

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \tag{3.95}$$

The columns of A_2 are put in to make this a square matrix.

The matrix A has an inverse

$$B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}. \tag{3.96}$$

The matrix B_1 is k by n and the matrix B_2 is $n-k$ by n. Thus B is n by n.

The fact that A and B are inverses is expressed by

$$BA = I \tag{3.97}$$

or

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \end{bmatrix} = \begin{bmatrix} I_k & 0 \\ 0 & I_{n-k} \end{bmatrix} \tag{3.98}$$

In particular $B_2 A_1 = 0$. This says that the rows of B_2 applied to the columns of A_1 always give zero. In other words, B_2 gives an implicit definition of the space spanned by the columns of A_1. The matrix $B_2 B_2^T$ is an $n-k$ by $n-k$ matrix. The co-area associated with B_2 is defined to be $\sqrt{\det(B_2 B_2^T)}$.

Of course B_2 is not the only matrix that gives an implicit definition of the space spanned by the columns of A_1. However there is also the equation $B_2 A_2 = I$. The role of A_2 is that it gives extra information that characterizes B_2.

The goal is to related the co-area to the area. This is accomplished via the volume ratio $\sqrt{\det(A^T A)} = |\det(A)|$. The volume ratio is the area divided by the co-area.

Theorem 3.25 (Co-area formula) *The co-area times the volume ratio is the area:*

$$\sqrt{\det(B_2 B_2^T)}\sqrt{\det(A^T A)} = \sqrt{\det(A_1^T A_1)} \tag{3.99}$$

Proof: Since $BA = I$, it follows that $AB = I$, $B^T A^T = I$, $BB^T A^T A = BIA = BA = I$. The conclusion is that

$$BB^T AA^T = I. \tag{3.100}$$

More explicitly, we could write this as

$$\begin{bmatrix} B_1 {B_1}^T & B_1 {B_2}^T \\ B_2 {B_1}^T & B_2 {B_2}^T \end{bmatrix} \begin{bmatrix} {A_1}^T A_1 & {A_1}^T A_2 \\ {A_2}^T A_1 & {A_2}^T A_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{3.101}$$

There is a theorem in linear algebra that gives a relation between the determinants of submatrices of matrices that are inverse to each other. The statement and proof are in the next section. In this case it says that

$$\det(B_2 {B_2}^T)\det(A^T A) = \det({A_1}^T A_1). \tag{3.102}$$

This gives the required identity. □

Block matrices

This section presents a theorem on block matrices that is useful for the co-area theorem. The block matrices considered have four blocks, so they resemble in some respects 2 by 2 matrices with four entries. Some of the formulas for 2 by 2 matrices carry over to this situation, at least after appropriate modifications. We begin with a lemma about the determinant of a matrix for which one of the blocks is the zero matrix. The main theorem relates determinants of blocks of a matrix and of its inverse matrix.

Lemma 3.26 *Consider a block triangular block matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}. \tag{3.103}$$

Then $\det A = \det A_{11} \det A_{22}$.

Proof: It is sufficient to consider the case when A_{11} is non-singular. Decompose

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1} A_{12} \\ 0 & I \end{bmatrix}. \tag{3.104}$$

It is easy to work out each of the determinants on the right. So $\det A = \det A_{11} \cdot \det A_{22} \cdot 1$. □

Theorem 3.27 *Consider a block matrix A with inverse B, so that the product $AB = I$ has the form*

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{3.105}$$

Then $\det A \det B = 1$. *Furthermore,*

$$\det A_{22} \det B = \det B_{11} \tag{3.106}$$

and

$$\det A_{11} \det B = \det B_{22}. \tag{3.107}$$

Proof: The inverse of A is given by the block Cramer's rule

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -(A_{22}A_{12}^{-1}A_{11} - A_{21})^{-1} \\ -(A_{11}A_{21}^{-1}A_{22} - A_{12})^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{bmatrix}. \tag{3.108}$$

There is a triangular factorization

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ A_{22}^{-1}A_{21} & I \end{bmatrix}. \tag{3.109}$$

By the lemma above this gives $\det A = \det A_{22} \det B_{11}^{-1}$, which leads to the first result. We also have the triangular factorization

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & I \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}. \tag{3.110}$$

This gives $\det A = \det A_{11} \det B_{22}^{-1}$, which is equivalent to the second result. □

Problems

Eigenvalues

1. Describe all 2 by 2 matrices with only one eigenvalue that are not diagonalizable.

2. Consider the matrix
$$A = \begin{bmatrix} -3 & 16 \\ -1 & 5 \end{bmatrix}. \tag{3.111}$$
Find the eigenvalue. Let $\delta \neq 0$. Find an invertible matrix P with $AP = PJ$, where
$$J = \begin{bmatrix} \lambda & \delta \\ 0 & \lambda \end{bmatrix}. \tag{3.112}$$
The matrix P will depend on δ.

3. Consider $0 \leq r \leq s$. Must there be a 2 by 2 matrix with spectral radius r and norm s? Prove that your answer is correct.

4. Let
$$R = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}. \tag{3.113}$$
This is the same as
$$R = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \end{bmatrix} - \begin{bmatrix} \sin(\theta) \\ -\cos(\theta) \end{bmatrix} \begin{bmatrix} \sin(\theta) & -\cos(\theta) \end{bmatrix}. \tag{3.114}$$
Check this identity. Find the eigenvalues and eigenvectors. Find R^2.

5. Consider the matrix
$$A = \begin{bmatrix} 1 & \frac{1}{10} \\ \frac{1}{10} & 1 \end{bmatrix} \tag{3.115}$$
For which $\mathbf{x}$ does $A^n\mathbf{x} \to 0$ as $n \to \infty$? For which $\mathbf{x}$ does $A^n\mathbf{x}$ run off to infinity as $n \to \infty$. In what direction does it run off to infinity?

Norms

1. Let H be the real symmetric matrix
$$H = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}. \tag{3.116}$$
This matrix has determinant zero, so one eigenvalue is zero. Find all eigenvalues. Find the corresponding eigenvectors, as column vectors. (Are they orthogonal?) Produce a matrix P with the normalized eigenvectors as columns. Show that $P^TP = I$. Show by explicit computation that $HP = P\Lambda$, where Λ is diagonal. Find the spectral representation of H.

2. Let A be the real matrix
$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}. \tag{3.117}$$
Find the Lipschitz norm of A (the square root of the largest eigenvalue of A^TA). Find the 2 norm of A (the square root of sum of squares of entries, or, equivalently, the square root of the trace of A^TA). Compare them.

3. This problem deals with the Lipschitz norm. Say that A is a real square matrix. The claim is that it is always true that $\|A^2\| = \|A\|^2$. Prove or disprove.

4. Let R be the real symmetric matrix

$$R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \tag{3.118}$$

   Find the Lipschitz norm $\|R\|$.

5. Find all real square matrices A such that $\|A\| = \|A\|_2$. If you need a hint, see below.

   Hint: Consider a vector $\mathbf{x}$ that is not the zero vector, and another vector $\mathbf{a}$. The Schwarz inequality says that the inner product $\mathbf{a}\cdot\mathbf{x}$ satisfies $|\mathbf{a}\cdot\mathbf{x}| \leq |\mathbf{a}||\mathbf{x}|$ with equality only when $\mathbf{a} = c\mathbf{x}$. (Since $\mathbf{a}\cdot\mathbf{x} = |\mathbf{a}||\mathbf{x}|\cos(\theta)$, this is when $\cos(\theta) = \pm 1$, so the vectors are either pointing in the same or opposite direction.)

   Use the Schwarz inequality for each i to prove

$$|A\mathbf{x}|^2 = \sum_i (\sum_j a_{ij}x_j)^2 \leq \sum_i (\sum_j a_{ij}^2 \sum_k x_k^2) = \|A\|_2^2 |\mathbf{x}|^2. \tag{3.119}$$

   When is this an equality? (Consider the situation for each fixed i.) Once you have the form of the matrix you can calculate $A^T A$ and evaluate the norms.

Part II

Smooth Geometry

Chapter 4

Prelude: Vector Spaces and Modules

4.1 Vector spaces

4.1.1 Vector spaces and linear transformations

This section has linear algebra preliminaries. This begins with a brief introduction to geometrical vector spaces. In a geometric vector space a vector is thought of as a geometric object, an arrow, rather than as something numerical.

There is a standard axiomatic definition of such vector spaces. A *vector space* consists of a set V that is a commutative group under the operation $+$ of addition. There is also a field F of scalars that acts on V. (Here the field F of scalars is just the real number field.) Thus if a is in F and $\mathbf{u}$ is in V, then $a\mathbf{u}$ is in V. The action is required to satisfy the following axioms:

- $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
- $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$
- $(ab)\mathbf{u} = a(b\mathbf{u})$
- $1\mathbf{u} = \mathbf{u}$.

The first axiom is about addition in V; the other axioms are about operations in F.

The fundamental definition in the theory is that of *linear transformation*. A linear transformation $T : V \to W$ from one vector space to another must satisfy:

- $T(\mathbf{u} + \mathbf{v}) = T\mathbf{u} + T\mathbf{v}$.
- $T(a\mathbf{u}) = a\,T\mathbf{u}$.

A standard example of a finite dimensional vector spce is $\mathbf{R}^n$. The elements of $\mathbf{R}^n$ are often written as column vectors. A linear transformation $A : \mathbf{R}^n \to \mathbf{R}^m$ is then represented by left multiplication by a m by n matrix.

Suppose that $L : \mathbf{R}^k \to V$ is a linear transformation. Let δ_i be the column vector with 1 in the i component and zero elsewhere. Then the vectors $\mathbf{u}_i = L\delta_i$ for $i = 1, \ldots, k$ form a list of vectors in V. It follows that L is given by the formula $L\mathbf{a} = \sum_{i=1}^{k} a_i \mathbf{u}_i$. The *span* of the vectors in this list is the range of the transformation L. These vectors are *linearly independent* if and only if the transformation L is one-to-one.

Suppose that there is a linear transformation $B : \mathbf{R}^n \to V$ that is one-to-one and onto V. In that case the list of vectors $\mathbf{b}_i = B\delta_i$ are linearly independent with span V. In that case the vector space V is said to be n-dimensional, and B defines a *basis* for V. The inverse transformation $B^{-1} : V \to \mathbf{R}^n$ is called a *coordinate mapping*.

Suppose that V is n-dimensional with basis B, and W is m dimensional with basis C. For each linear transformation $T : V \to W$ there is a corresponding m by n matrix $A = C^{-1}TB$.

Figure 4.1: Pictures of vector $\mathbf{v}$, form α, and value $\langle \alpha \mid \mathbf{v} \rangle$

The conclusion of this discussion is that, for finite-dimensional vector spaces, vectors may be represented by column vectors, and linear transformations may be represented by matrices. However, these representations may be dependent on a choice of basis. Sometimes a particular choice of basis has nice properties.

Suppose that V is n-dimensional. For each linear transformation $T : V \to V$ and basis B there is a corresponding n by n matrix $A = B^{-1}TB$. The Jordan form theorem says that B can be chosen so that A has real Jordan form.

4.1.2 The dual space of a vector space

Let V be an n-dimensional real vector space. Then there is another n-dimensional real vector space V^* called the *dual space* of V. It consists of all linear transformations $\alpha : V \to \mathbf{R}$. The value of a linear form α on a vector $\mathbf{v}$ could be denoted in the usual way $\alpha(\mathbf{v})$. Instead the practice here is to use a special notation $\langle \alpha \mid \mathbf{v} \rangle$. This notation will turn out to be useful for writing complicated expressions in a clear way.

An element of the dual space is called a *linear form*, but this is also commonly abbreviated to *form*. Another name for such an object is *covector*.

The dual space is a very important concept, because the intuition for a dual space is very different from the intuition for the original vector space. The original vector space may be thought of as a collection of arrows starting at the origin. An element α of the dual space is a real function, and so should be expressed in terms of the subsets where it has fixed values. For instance, one can choose a small number $\delta > 0$. For each integer j look at the subset of V where α has the value $k\delta$. Typically this is a line or plane of dimension $(n-1)$. So one should picture an element of the dual space as a collection of closely spaced lines or planes. It is also helpful to specify the δ and the direction of increase. Figure 4.1 shows the pictures of a vector, a linear form, and the number that results from their pairing.

If $T : V \to W$ is a linear transformation, then $T^* : W^* \to V^*$ is a linear transformation defined by

$$\langle T^*\alpha \mid \mathbf{u} \rangle = \langle \alpha \mid T\mathbf{u} \rangle. \tag{4.1}$$

This is called the *dual transformation*

Say that $B : \mathbf{R}^n \to V$ is a basis for V. The natural dual space for $\mathbf{R}^n$ is the space of row vectors $\mathbf{R}^{nT}$. Then $B^* : V^* \to \mathbf{R}^{nT}$ is a coordinate mapping for

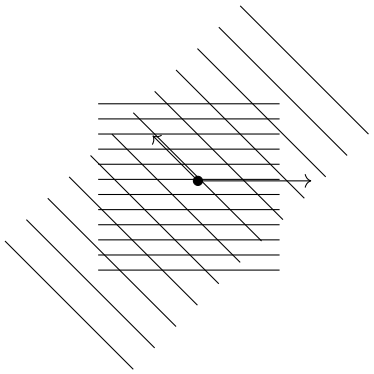

Figure 4.2: Picture of basis and dual basis

V^*. It follows that $B^{*-1} : \mathbf{R}^{nT} \to V^*$ is a basis for V^*. This is called the *dual basis* for V^*.

Say that the basis vectors are $\mathbf{b}_k = B\delta_k$. Then the dual basis vectors are $\beta_j = B^{*-1}\delta_j^T$. Their relation is $\langle \beta_j \mid \mathbf{b}_k \rangle = \langle B^{*-1}\delta_j^T \mid B\delta_k \rangle = \langle \delta_j^T \mid \delta_k \rangle$. In other words, the dual basis and basis vectors are related by

$$\langle \beta_j \mid \mathbf{b}_k \rangle = \delta_{jk}. \tag{4.2}$$

Figure 4.2 shows a picture of a basis of vectors and a dual basis of linear forms.

Define a *bilinear form* as a linear transformation $\mathbf{Q} : V \to V^*$. The reason for this terminology is that the expression $\langle \mathbf{Qu}, \mathbf{v} \rangle$ is a real quantity that depends linearly on $\mathbf{u}$ and also on $\mathbf{v}$.

A bilinear form $Q : V \to V^*$ is *symmetric* if it satisfies the identity $\langle Q\mathbf{u} \mid \mathbf{v} \rangle = \langle Q\mathbf{v} \mid \mathbf{u} \rangle$. A bilinear form is *antisymmetric* if it satisfies the identity $\langle Q\mathbf{u} \mid \mathbf{v} \rangle = -\langle Q\mathbf{v} \mid \mathbf{u} \rangle$.

A symmetric bilinear form is usually studied via its associated *quadratic form*, a function that sends $\mathbf{u}$ in V to the numerical value $\langle Q\mathbf{u} \mid \mathbf{u} \rangle$. Since this is a numerical function, it may be pictured by the surfaces where it has constant values. These are quadratic surfaces.

The quadratic form is *positive definite* if it satisfies $\langle Q\mathbf{u} \mid \mathbf{u} \rangle \geq 0$ for all $\mathbf{u}$. It is *strictly positive definite* if it satisfies $\langle Q\mathbf{u} \mid \mathbf{u} \rangle > 0$ for every vector $\mathbf{u}$ other than the zero vector.

An antisymmetric bilinear form is harder to visualize in this way, since it always satisfies $\langle Q\mathbf{u} \mid \mathbf{u} \rangle = 0$. What this says is that the vector $\mathbf{u}$ always belongs to the $(n-1)$-dimensional subspace where the form $Q\mathbf{u}$ is zero.

There are results that say that symmetric bilinear forms and anti-symmetric bilinear forms have properties that are captured by matrices that are diagonal or near diagonal.

The result for a symmetric bilinear form $\mathbf{Q} : V \to V^*$ is that there is a basis $\mathbf{B}$ for V and a diagonal matrix D with $+1, 0, -1$ entries on the diagonal such

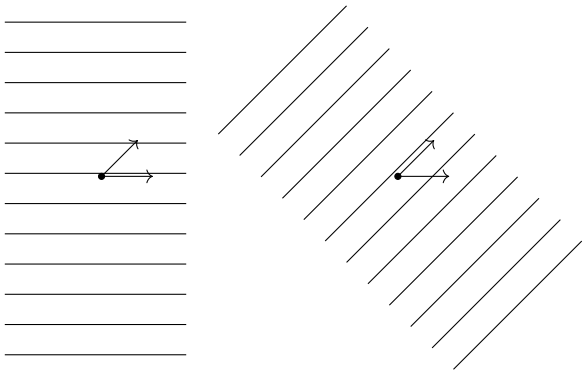

Figure 4.3: An anti-symmetric form $\mathbf{Q}$: Pictures of $\langle \mathbf{Qu} \mid \mathbf{v} \rangle$ and of $\langle \mathbf{Qv}, \mathbf{u} \rangle$

that for all column vectors $\mathbf{x}, \mathbf{y}$ in $\mathbf{R}^n$

$$\langle \mathbf{QBx} \mid \mathbf{By} \rangle = \mathbf{x}^T D \mathbf{y}. \tag{4.3}$$

This result may also be stated in the form

$$\langle Q\mathbf{b}_i \mid \mathbf{b}_j \rangle = D_{ij} \tag{4.4}$$

In this basis the symmetric bilinear form is given by a diagonal matrix.

The result for a antisymmetric bilinear form $\mathbf{Q} : V \to V^*$ involves matrices F that are not quite diagonal. The matrix F has the special antisymmetric form that has all entries zero except for blocks of the form

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{4.5}$$

with the two zeros along the diagonal of F.

The result for an antisymmetric bilinear form $\mathbf{Q} : V \to V^*$ says that there is special antisymmetric matrix F such that for all column vectors $\mathbf{x}, \mathbf{y}$ in $\mathbf{R}^n$ it follows that

$$\langle \mathbf{QBx} \mid \mathbf{By} \rangle = \mathbf{x}^T F \mathbf{y}. \tag{4.6}$$

Anothe way of stating this is by

$$\langle \mathbf{Qb}_i \mid \mathbf{b}_j \rangle = F_{ij}. \tag{4.7}$$

It is not particularly easy to picture an antisymmetric bilinear form $\mathbf{Q}$, but it is worth making the attempt. If $\mathbf{u}$ is a vector, then $\langle \mathbf{Qu} \mid \mathbf{u} \rangle = 0$, so the vector $\mathbf{u}$ belongs to the subspace where the linear form $\mathbf{Qu}$ is zero. Keeping this in mind, Figure 4.3 illustrates the meaning of the antisymmetry property $\langle \mathbf{Qu} \mid \mathbf{v} \rangle = -\langle \mathbf{Qv} \mid \mathbf{u} \rangle$.

4.1.3 Orientation

A vector space of dimension k has two *orientations*. In one dimension an orientation is just a choice of direction. In two dimensions the two possible orientations are often called clockwise and counter clockwise. These are not absolute terms; they depend on an external view of the vector space. In three dimensions the orientations are called right handed and left handed. Again these are not absolute notions.

The usual way of specifying an orientation of a vector space V of dimension n is by giving a sequence $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ of n vectors that form a basis for the space. (The ordering of the vectors is important!) Each basis uniquely determines one of the two orientations. This orientation will be denoted $[(\mathbf{v}_1, \ldots, \mathbf{v}_k)]$. Two bases determine the same orientation $\mathcal{O}$ if the matrix that relates one to the other has determinant > 0. Two bases determine opposite orientations if the matrix that relates them has determinant < 0. If $\mathcal{O}$ denotes an orientation, then $-\mathcal{O}$ denotes the opposite orientation.

This idea may also be expressed in matrix language. If the columns of two bases are assembled to define basis transformations $\mathbf{V} : \mathbf{R}^n \to V$ and and $\mathbf{W} : \mathbf{R}^n \to V$, then the change of basis matrix is $C = \mathbf{W}^{-1}\mathbf{V} : \mathbf{R}^n \to \mathbf{R}^n$. This matrix is invertible, so $\det C \neq 0$. The condition for having the same orientation is $\det C > 0$.

For the case $k = 0$ the vector space has only the zero element. The convention is that there are two orientations, $+$ and $-$. In contrast to the case of higher dimensions, these are absolute notions of positive and negative orientations.

An orientation is a rather abstract construction, but there is a number associated with a pair of orientations. This is the *relative orientation* defined by

$$\begin{aligned} \operatorname{sign}(\mathcal{O}, \mathcal{O}) &= 1 \\ \operatorname{sign}(\mathcal{O}, -\mathcal{O}) &= -1. \end{aligned} \tag{4.8}$$

There is an important construction that relates different orientations. Let $W \subseteq V$ be a vector subspace of the vector space V. Then there is a *quotient space* V/W that consists of vectors in V, but with two vectors regarded as equivalent if they differ by a vector in W. If V has dimension n and W has dimension k, then V/W has dimension $n - k$. An orientation of V/W is called a *transverse orientation* or *external orientation*.

Consider a list of vectors $(\mathbf{u}_1, \ldots, \mathbf{u}_{n-k}, \mathbf{w}_1, \ldots, \mathbf{w}_k)$ that is a basis for V and such that $(\mathbf{w}_1, \ldots, \mathbf{w}_k)$ is a basis for W. Then the $n - k$ vectors $(\mathbf{u}_1, \ldots, \mathbf{u}_{n-k})$ determine a basis for V/W. This gives a more concrete way of specifying a transverse orientation.

Consider two such pairs of lists $(\mathbf{u}, \mathbf{w})$ and $(\mathbf{u}', \mathbf{w}')$. They are related by

$$\begin{aligned} \mathbf{u}' &= A\mathbf{u} + B\mathbf{w} \\ \mathbf{w}' &= D\mathbf{w}, \end{aligned}$$

where A and D are invertible matrices. The two lists $\mathbf{u}$ and $\mathbf{u}'$ determine the same transverse (external) orientation of W if $\det(A) > 0$; they determine

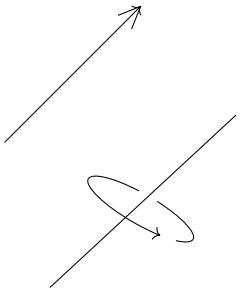

Figure 4.4: 3d: vector; twisted vector

opposite transverse (external) orientations if $\det(A) < 0$. Also, the two lists $\mathbf{w}$ and $\mathbf{w}'$ determine the same (internal) orientation of W if $\det(D) > 0$. Finally, the two bases for V have the same orientation if $\det(A)\det(D) > 0$.

Proposition 4.1 *Consider vector spaces $W \subseteq V$. A transverse (external) orientation $\mathcal{O}_{V/W}$ followed by an (internal) orientation $\mathcal{O}_W$ of W determine an orientation $\mathcal{O}_V$ of V. In general, any two of these orientations determines the third orientation.*

In applications a vector in a vector space V of dimension n may represent a displacement, a velocity, an acceleration, a force, and so on. For a non-zero vector $\mathbf{w}$ the associated one-dimensional vector space W has an orientation. The vector $\mathbf{w}$ is represented as a segment, together with an arrow indicating its one-dimensional (internal) orientation.

There is a closely related concept that is used for angular velocity, angular acceleration, torque and such concepts. A non-zero *twisted vector* (or *pseudovector*) in a vector space V is a segment with an $(n-1)$-dimensional transverse (external) orientation. See Figure 4.4 for pictures of a vector and of a twisted vector in the case of dimension $n = 3$.

Example: Here is the example of angular velocity. A wheel spinning on an axis is represented by a segment along the axis representing how fast it is spinning, together with an 2-dimensional transverse orientation that represents the sense of rotation. ||

There is an equivalent way to define a twisted vector. It is an odd function from orientations of V to vectors. If the function sends the orientation $\mathcal{O}$ of V to the vector $\mathbf{w}$, then it sends the opposite orientation $-\mathcal{O}$ to $-\mathbf{w}$.

The equivalence of these two definitions follows from the preceding proposition. A non-zero vector $\mathbf{w}$ determines a vector subspace W and an orientation of that space. Giving the orientation $\mathcal{O}$ of V and the orientation of W determined by $\mathbf{w}$ determines the transverse orientation. The same transverse orientation is obtained from $-\mathcal{O}$ and $-\mathbf{w}$.

One can go the other way, starting with the transverse orientation. The given transverse orientation followed by the vector orientation given by $\mathbf{w}$ is

the orientation $\mathcal{O}$. Or the given transverse orientation followed by the vector orientation given by $-\mathbf{w}$ is the orientation $-\mathcal{O}$

In a three dimensional vector space with inner product there is a common construction that produces twisted vectors. The product combines two vectors $\mathbf{u}, \mathbf{v}$. They have a *vector product* (or *cross product*) defined as follows. Start with an an orientation $\mathcal{O}$. The corresponding vector $\mathbf{w}$ is perpendicular both to $\mathbf{u}$ and to $\mathbf{v}$. It has a length that is equal to the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$. If $\mathbf{u}, \mathbf{v}$ are linearly dependent, then $\mathbf{w}$ is the zero vector. Otherwise, the vector $\mathbf{w}$ is determined by the property that the ordered triple $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ defines the orientation $\mathcal{O}$. If $\mathcal{O}$ is replaced by $-\mathcal{O}$, then $\mathbf{w}$ is replaced by $-\mathbf{w}$. This somewhat strange construction will be studied and put in its proper context in the course of the chapters to follow.

In following chapters it will also be useful to have the concept of *boundary orientation*. Consider a list of linearly independent vectors $(\mathbf{v}_1, \ldots, \mathbf{v}_k)$. Define the corresponding *oriented cell* as the set C of all vectors of the form $\sum_j c_j \mathbf{v}_j$ with $0 \leq c_j \leq 1$. The list of vectors may be regarded as giving an orientation of this cell. This cell has a boundary cells. The boundary cell C_i^+ consists of the points in the cell that satisfy $c_i = 1$, while the boundary cell C_i^- consists of the points that satisfy $c_i = 0$. There are $2k$ such boundary cells. Each one of them may be thought of as an *oriented boundary cell*. The orientation of a boundary cell is obtained by requiring that the outward pointing vector followed by vectors giving the orientation of the boundary cell gives the orientation of the original cell. For C_i^+ the outward pointing vector is $\mathbf{v}_i$, while for C_i^- the outward pointing vector is $-\mathbf{v}_i$. It follows that the orientations C_i^+ must be given by $((-1)^{j-1}\mathbf{v}_1, \cdots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_k)$. Similarly the orientation of C_i^- must be given by $(-(-1)^{j-1}\mathbf{v}_1, \cdots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_k)$.

Example: Consider the parallelogram defined by vectors $(\mathbf{v}_1, \mathbf{v}_2)$. As we go around the boundary the four orientations are $\mathbf{v}_2$, then $-\mathbf{v}_1$, then $-\mathbf{v}_2$, and then $\mathbf{v}_1$. ||

Example: Consider the parallelepiped defined by vectors $(\mathbf{v}_1, \mathbf{v}_1, \mathbf{v}_3)$. Then the six boundary faces may be oriented in terms of the vectors in this list (without minus signs). The boundary faces have orientations given by $(\mathbf{v}_2, \mathbf{v}_3)$ and $(\mathbf{v}_3, \mathbf{v}_1)$ and $(\mathbf{v}_1, \mathbf{v}_2)$ for the + faces and $(\mathbf{v}_3, \mathbf{v}_2)$ and $(\mathbf{v}_1, \mathbf{v}_3)$ and $(\mathbf{v}_2, \mathbf{v}_1)$ for the − faces. ||

Example: Consider the segment defined by a vector $\mathbf{v}$. The two boundary points have orientations + and −. ||

4.2 Modules

A *module* is a generalization of a vector space. In a module $\mathcal{E}$ there are vector-like objects that can be added via an operation +. With this operation they form a commutative group. There is also a set $\mathcal{R}$ of scalars that act on the vector-like objects. The scalars are required to form a *ring* $\mathcal{R}$. In our situation this will be a commutative ring. This means that all the usual laws of algebra involving addition and multiplication are satisfied, except there is no requirement about

multiplicative inverses.

The elements of $\mathcal{R}$ that have multiplicative inverses form a commutative group under multiplication. The ring is a field if and only if this group consists of all elements $\neq 0$.

Example: The set $\mathcal{R} = \mathbf{R}^m$ forms a ring under pointwise addition and multiplication. The elements that have multiplicative inverses are those for which each component is $\neq 0$. ||

Example: The set of smooth functions $f(x)$ forms a ring under pointwise addition and mulitplication. The elements that have multiplicative inverses are those for which $f(x)$ is never zero. ||

In a module it makes sense to talk about linear combinations, where the vector-like objects are combined by multiplying them by scalars and then adding them. This leads to the usual notion of linear dependence and span. A basis is a list of vector-like objects that are linearly independent and span the module. A module is said to be a *free module* if it has a basis. In the case considered here, when the ring of scalars is commutative, the number of elements in a basis is always the same. This is the *rank* of the free module. In the typical situation that we encounter in this exposition the module will be free and have finite rank. Much of the usual reasoning about a vector space of dimension n carries over to a free module of rank n.

Example: Consider first order differential operations in the plane of the form

$$X = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}. \tag{4.9}$$

Here $f(x,y)$ and $g(x,y)$ are smooth functions of x and y. As a vector space over $\mathbf{R}$ these operators form an infinite dimension vector space. The alternative viewpoint is to take the ring $\mathcal{R}$ to consist of all smooth functions $h(x,y)$. This makes the operators into a free module of rank two. ||

A mapping of modules is defined similarly to a a linear transformation of vector spaces, except that the scalars are taking from the ring. If $\mathcal{E}$ is a module with ring $\mathcal{R}$, then the *dual module* $\mathcal{E}^*$ consists of the module mappings from $\mathcal{E}$ to $\mathcal{R}$. The value of α in $\mathcal{E}^*$ on $\mathbf{v}$ in $\mathcal{E}$ will be denoted $\langle \alpha \mid \mathbf{v} \rangle$. A finite-rank free module $\mathcal{E}$ has a dual module $\mathcal{E}^*$ that is also a finite-rank free module, with the same rank. If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis for the module $\mathcal{E}$, then the dual basis for $\mathcal{E}^*$ is $\alpha_1, \ldots, \alpha_n$ satisfying $\langle \alpha_j \mid \mathbf{v}_k \rangle = \delta_{jk}$.

In the following the module $\mathcal{E}$ will often consist of smooth vector fields defined on a connected open subset of Euclidean space. In that case there are only two kinds of invertible elements in the ring $\mathcal{R}$ of smooth function: functions that are everywhere strictly positive, and their negatives. This fact underlies an algebraic formulation of orientation for such a module.

This brief description of the module concept is here for a dual purpose: to introduce terminology, and to signal that much of the material of this and following chapters can be formulated in a completely algebraic context. The book by Nelson [34] gives a sophisticated and original account of this point of view.

Chapter 5

Vector Fields

5.1 Manifold geometry

5.1.1 Mathematical models

The construction of mathematical models follows a general plan.

- Identify the objects to be described and a question about those objects.
- Assign meaningful names for numbers associated to these objects.
- Identify which names represent variables and which represent constants.
- Write equations relating these quantities. Identify the dimension of the system.
- Use mathematics to answer the question.

Example: Find the shape of a cylindrical can holding a fixed amount of liquid that uses the least amount of material. Here are the stages.

- Think of tall narrow cans and short fat cans. The question is to find an intermediate shape that is most economical. The space of possibilities is the set M_V of all cylindrical cans with fixed volume V.
- Call the radius r and the height h and the area A and the volume V. These are numbers attached to the cylindrical cans. They may be thought of as functions from M_V to $\mathbf{R}$.
- The r, h, A are variable, while the volume V is a constant.
- The equations are $A = 2\pi r^2 + 2\pi rh$ and $V = \pi r^2 h$. Since V is fixed, there are two equations relating the three variables r, h, A. So the system is 1-dimensional.
- The question is to find the shape of the can for which A is maximal. We can calculate the effect of a small change in the shape. The equations are

$$\begin{aligned} dA &= (4\pi r + 2\pi h)\, dr + 2\pi r\, dh \\ dV &= 2\pi rh\, dr + \pi r^2\, dh. \end{aligned} \tag{5.1}$$

  Since V is constant, $dV = 0$. This says $2h\, dr + r\, dh = 0$. For the optimum shape A should be making the transition from decreasing to increasing, that is, $dA = 0$. This implies $(2r + h)\, dr + r\, dh = 0$. Subtracting these gives $(h - 2r)\, dr = 0$. The conclusion is that the optimum can shape is where $h = 2r$.

There are several things to note about this example. One is that all quantities are restricted to $r > 0, h > 0, A > 0, V > 0$. Also V constant gives further restrictions. So this is not a problem about Euclidean space. The underlying space is the space M_V of cylinders of fixed volume. These cylinders are not numbers, and so they can only be described indirectly. For example, the optimal

cylinder may be be prescribed as that shape for which $h = 2r$, or, alternatively, as that shape for which $2\pi r^3 = V$.

Another thing to observe is that r, h, A may be thought of as functions whose domain is the set M_V of cylinders of fixed volume. These are not functions that take numbers as input. It is possible to express h as a function of r or r as a function of h. These are functions with numerical input and numerical output, but neither of these functions gets any special preference. This concept is sometimes called *variable democracy*.

Another remark is that the differentials dA, dr, dh are meaningful expressions of small change. If calculus is to be a useful tool in modeling, such differentials must play a role. This will be the subject of the next chapter on differential forms. ||

Example: The above cylindrical can example may also serve as an example of a differential equation in a modeling context. Compute

$$\frac{dA}{dr} = 4\pi r - \frac{2V}{r^2}. \tag{5.2}$$

Consider the differential equation

$$\frac{dr}{dt} = -\frac{dA}{dr} = \frac{2V}{r^2} - 4\pi r. \tag{5.3}$$

If r is very small, then it increases, while if r is very large, then it decreases. The point at which r is stationary is when the right hand side is zero, that is, where $2\pi r^3 = V$. Giving the differential equation describes a vector field that is pushing the cylindrical cans toward their optimal shape. The vector fields in this example should be thought of as velocity vector fields. (There is no concept of displacement vector.) The vector field has value zero only at the optimal shape. At this shape the velocity is zero, so the shape remains constant in time. It is a pleasant exercise to solve the differential equation for r as a function of t. This represents the flow of the vector field.

The differential equation was chosen so that the area is always decreasing. This follows from

$$\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt} = -\left(\frac{dA}{dr}\right)^2. \tag{5.4}$$

An alternative is to consider the equation $dh/dt = -dA/dh$. This is not quite the same; in the r variable it is equivalent to $dr/dt = -(dA/dr)(dh/dr)^2$. Since $(dh/dr)^2 > 0$ this also makes the area decrease. ||

Example: There is another way to think of the can problem. Let M be the 2-dimensional space of all cans, with arbitrary values of $r > 0$ and $h > 0$. The area $A = 2\pi r^2 + 2\pi rh$ and volume $V = \pi r^2 h$ are both functions on M. The problem is to minimize the restriction of A to the one-dimensional curve M_V where the volume is constant.

The idea of Lagrange is to realize that for the particular minimal area can shape the only way that A can change is by violating the condition that the

volume is constant. In other words, for this shape the change in A must be proportional to the change in volume:

$$dA = \lambda\, dV. \tag{5.5}$$

This equation says that

$$(4\pi r + 2\pi h)\, dr + 2\pi r\, dh = \lambda(2\pi r h\, dr + \pi r^2\, dh). \tag{5.6}$$

Equating the coefficients of dh gives $2 = \lambda r$. Then equating the coefficients of dr gives $2r + h = \lambda r h = 2h$. So the optimum shape is $2r = h$. ||

5.1.2 Coordinates on a manifold patch

In the following a function is said to be *smooth* if it is C^∞. Two open subsets U, V of $\mathbf{R}^n$ are said to be *diffeomorphic* if there is a one-to-one smooth function $\mathbf{f} : U \to V$ with smooth inverse function $\mathbf{f}^{-1} : V \to U$.

This notion of smoothness is convenient; there is no need to worry about whether derivatives exist. Furthermore, there is a version of the inverse function theorem that says that if a smooth function has derivative that is invertible at a particular point, then near the image of that point there is a smooth inverse function. There is a similar version of the implicit function theorem.

An n-dimensional *manifold patch* is a set M together with a collection of functions defined on M. Each such function is one-to-one from M onto some open subset of $\mathbf{R}^n$. Such a function is called a *coordinate system*. There are two requirements on the coordinate system functions.

- If $\mathbf{x} = (x_1, \ldots, x_n) : M \to U$ is a coordinate system onto the open set U, and if $\mathbf{f} : U \to V$ is a diffeomorphism from the open set U to the open set V, then $\mathbf{f}(\mathbf{x}) : M \to V$ is a coordinate system onto V.

- If $\mathbf{x} = (x_1, \ldots, x_n) : M \to U$ is a coordinate system mapping M onto U, and if $\mathbf{u} = (u_1, \ldots, u_n) : M \to V$ is a coordinate system mapping M onto V, then there is a diffeomorphism $f : U \to V$ such that $\mathbf{u} = \mathbf{f}(\mathbf{x})$.

The purpose of this definition is to is to capture the idea of a set with many different coordinate systems, each of which is defined on the same set.

For $n = 1$ a typical manifold patch is something like a featureless curve without end points. It has no particular shape, but it cannot cross itself or close on itself. Since no notion of distance along the curve is assumed, it may help to think of it as a broken rubber band. For $n = 2$ a typical manifold patch is like a patch of cloth, possibly with holes. It has no border. In this case it may help to think of it as a rubber sheet. The typical $n = 0$ case is a point; a coordinate system attaches the number 0 to this point.

The coordinate functions attach numbers to the points of M. This can be done in many way, for instance by $\mathbf{x}$ or by $\mathbf{u} = \mathbf{f}(\mathbf{x})$. The philosophy is that no one coordinate system is superior to all the others, at least not without further consideration. The points in M do not have names, at least not in a direct

sense. To specify a point μ in M, one gives the value $\mathbf{x}(\mu)$ or the value $\mathbf{u}(\mu)$. These are typically not the same number. Say, for instance, that $\mathbf{x}(\mu) = \mathbf{c}$. Then $\mathbf{u}(\mu) = \mathbf{f}(\mathbf{c})$.

Here is a useful indirect way of talking about a point in M. The point μ where $\mathbf{x}(\mu) = \mathbf{c}$ is denoted $\mathbf{x} \leftarrow \mathbf{c}$. Thus the value of $\mathbf{x}$ at this point is $\mathbf{x}(\mathbf{x} \leftarrow \mathbf{c}) = \mathbf{c}$. The value of $\mathbf{u}$ at the same point is $\mathbf{u}(\mathbf{x} \leftarrow \mathbf{c}) = \mathbf{f}(\mathbf{x})(\mathbf{x} \leftarrow \mathbf{c}) = \mathbf{f}(\mathbf{c})$.

Example: Here is an example of the relation of two coordinate systems. Let $x + iy = \frac{1}{2}(u + iv)^2$, with $y > 0$. The function relating these variables maps the upper half plane to the plane with a line removed. In terms of real coordinates this is $x = \frac{1}{2}(u^2 - v^2)$ and $y = uv$. The u, v are parabolic coordinates related to Cartesian coordinates x, y. ||

The concept of manifold patch is natural in geometry and in many applications of mathematics. A manifold patch M is (at least at the outset) assumed to be featureless, except for the fact that it can have various coordinate systems. These coordinate systems are supposed to be on a democratic status; one is as good as another. Since the notions of open set, closed set, compact set, continuous function, smooth function, smooth curve, and so on are independent of coordinate system, they make sense for a manifold patch. On the other hand, notions of distance, angle, congruence, and so on are not defined (although they may be introduced later). It is amazing how much useful mathematics can be done in this general context.

The manifold patch concept is particularly suited to the *local* notion of geometry, that is, it gives a description of what happens near a point. This is because a manifold patch is modeled on an open subset of $\mathbf{R}^n$. There is a more general concept of *manifold* that consists of many manifold patches joined together. This is a *global* or big picture notion of geometry, and it is a fascinating topic in advanced mathematics. Here we focus on the local story, with only brief mention of possible global issues.

Starting with a manifold patch M of dimension n, it is possible to construct other manifold patches. In fact, an open subset of M may be regarded as a (smaller) manifold patch of dimension n. Also, there is a construction that gives manifold patches of lower dimension. Suppose that M has coordinates $u_1, \ldots, u_n$. Let $c_{k+1}, \ldots, c_n$ be constants. Then the set S of all points μ with $u_{k+1}(\mu) = c_{k+1}, \ldots, u_n(\mu) = c_n$ is a manifold patch of dimension k. The restrictions of $u_1, \ldots, u_k$ to S form a coordinate system for S. Such a manifold patch is an example of a *regular surface* of dimension k in M.

Example: Here is a typical example of a manifold patch in applied mathematics. Consider a box of gas with pressure p and volume V and temperature T. These are related by an equation of state $f(p, V, T) = 0$. In nice cases this equation may be solved for one variable in terms of the others via the implicit function theorem. Finding the equation of state is a major task of physics. But even after this is determined, it is not clear which coordinates will be most convenient to use. In this case the manifold patch M is the set of possible states of the gas. One possible coordinate system is p, V. Another is p, T. Yet another is V, T. Physicists and chemists and geologists use whichever coordinate system is appropriate to the problem under consideration. ||

Here is our convention on the use of letters of the alphabet to indicate coordinates.

- When the underlying space is a manifold patch M but not a Euclidean space, the coordinates may be written as x, y, z or $x_1, \ldots, x_n$ or $\mathbf{x}$ or u, v, w or $u_1, \ldots, u_k$ or $\mathbf{u}$. Any other convenient symbols may be used. In an application the symbols are chosen to reflect the meaning of the objects being described. (The use of the symbols x, y, z or $x_1, \ldots, x_n$ or $\mathbf{x}$ do not necessarily imply that the setting is Euclidean space.)

- When the underlying space is Euclidean space E, then the Cartesian coordinates are always written as x, y, z or $x_1, \ldots, x_n$ or $\mathbf{x}$. Other coordinates may be used. For example three dimensional Euclidean space may be described by Cartesian coordinates x, y, z or by spherical polar coordinates r, θ, ϕ.

5.2 Scalar fields and vector fields

5.2.1 Scalar fields

There are three notions that are important at the outset, that of scalar field, vector field, and differential 1-form. We look at each of them in turn. In general, we shall require that these be defined on a manifold patch M.

The term *constant* may refer to a real number. As we know, the real number system $\mathbf{R}$ has natural operations of addition, subtraction, multiplication, and division (except that division by zero is prohibited). (An algebraic system of this sort is called a *field*.) A constant may also refer to a real function on a manifold patch M whose values are always the same real number. Such constant real functions also satisfy these algebraic laws.

A *scalar field* is a function $s = h(\mathbf{u}) = h(u_1, \ldots, u_n)$ from M to $\mathbf{R}$. Here $\mathbf{u} = (u_1, \ldots, u_n)$ is some coordinate system mapping M onto the open set V, and h is a smooth function on V. The set of all scalar fields on M is an algebraic system. It is a vector space (with operations of addition, subtraction, and multiplication by constants). It also has a product (multiplication of scalars) that satisfies the commutative and distributive laws. A scalar field may or may not have an inverse under multiplication. Such an inverse exists only when the scalar field never has the value zero. (An algebraic system of this sort, where an element not identically zero may lack a multiplicative inverse, is called a *ring*.)

Usually we picture a scalar field by drawing *level sets* of s on M. These are sets where s has constant value. To specify the scalar field it is also necessary to give an indication of the values of s on various level sets. The sketch may indicate maxima and minima and saddle points of s. While we may express $s = h(\mathbf{u})$, we may also express $s = h(\mathbf{f}(\mathbf{x}))$ in terms of some other coordinate system. The notion of scalar field is independent of coordinate system. Other names for level set are *contour curve* (for $n = 2$) or *contour surface*.

In the following the term constant may also be used for an element $\mathbf{c}$ in $\mathbf{R}^k$ for some k. (However for $k > 1$ the system $\mathbf{R}^k$ with pointwise addition and multiplication is not a field, only a ring.) This gives a convenient way of denoting a point M: The point where the coordinates $\mathbf{u}$ have the value $\mathbf{c}$ is called $\mathbf{u} \leftarrow \mathbf{c}$.

If s is a scalar field, then the value of s at a point μ in M is a constant $s(\mu)$. This constant value may also be written s_μ. If for some coordinate system $\mathbf{u}$ the scalar field is of the form $s = h(\mathbf{u})$, and if $\mu = (\mathbf{u} \leftarrow \mathbf{c})$, then $s(\mu) = h(\mathbf{u})(\mathbf{u} \leftarrow \mathbf{c}) = h(\mathbf{c})$.

Example: Here is a simple example that illustrates the notion of manifold patch in an elementary application. Consider the problem of making a box with a given amount A of material to contain the maximum volume. The box will have five sides: a base and four vertical sides. It is open at the top. In this case the manifold patch is the set M_A of possible boxes made with this material.

Say that the side lengths of the base are u, v and the height is w. The amount of material available is A. Thus $uv+2uw+2vw = A$. There are various possible coordinate systems to describe the boxes. One possibility is u, v. Another is u, w. Yet another is v, w. Each coordinate system may be expressed in terms of each other coordinate system. For instance, we may express u, v as in terms of u, w by noting that $v = (A - 2uw)/(u + 2w)$ for $0 < 2uw < A$.

The volume $V = uvw$ is a function on M_A that is a scalar field. For every constant $c > 0$ the set of points in M_A (box shapes) where $V = c$ is either a curve (the set of boxes with volume c), a single point (the box where V assumes its maximum value), or empty. In the next chapter we shall solve the problem of finding the box shape for which V assumes its maximum value. The answer is that at this particular shape the coordinates satisfy $u = 2w$ and $v = 2w$. That is, the open top box with maximal volume has a square base and a height half the length of each side of the base. ||

5.2.2 Vector fields

There is an intuitive picture of a vector field X on a manifold M in terms of a field of arrows. It is a function that assigns to each point in the manifold an arrow based at that point that represents a velocity. At each point μ in M the corresponding arrow is $X_{[\mu]}$. (This is a special case of the notation for a restricted vector field, to be introduced later.)

Suppose that s is a scalar field on M. The rate of change of s at μ should be the derivative of s along $X_{[\mu]}$. This can be written as $X_{[\mu]}\, s$. In general, the rate of change of s at an arbitrary point on the manifold is given by $X\, s$, which is another scalar field. The value of this scalar field at μ is the constant $(X\, s)_\mu = X_{[\mu]}\, s$. This suggests the following definition of vector field, which makes no reference to arrows.

A *vector field* is a *derivation* of the algebra of scalar fields. This means that if X is a vector field, and s is a scalar field, then $X\, s$ is another scalar field.

Furthermore, for every constant c and scalar field s

$$\begin{aligned} X\,c &= 0 \\ X\,(r+s) &= X\,r + X\,s \\ X\,(rs) &= (X\,r)s + r(X\,s). \end{aligned} \tag{5.7}$$

Both sides of each of these equations are scalar fields. The last two are the sum rule and the product rule for differentiation.

Theorem 5.1 (Coordinate representation of a vector field) *Suppose $s = h(\mathbf{x})$ is a scalar field, and X is a vector field. Then*

$$X\,s = \sum_{k=1}^{n} a_k h'_{,k}(\mathbf{x}), \tag{5.8}$$

Here each a_k is a smooth real function on M. In other words,

$$X = \sum_{k=1}^{n} a_k \frac{\partial}{\partial x_k}. \tag{5.9}$$

This theorem is proved in various sources [4]. An indication of the argument may be found in a later subsection. For the moment it may be thought of as an alternative, more concrete, definition of a vector field. We take it as our working definition.

Theorem 5.2 (Coordinate invariance of vector field) *The vector field X is independent of the coordinate system. Suppose that $\mathbf{u} = \mathbf{g}(\mathbf{x})$ is a change of coordinates. Then*

$$X = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial u_i}{\partial x_j} a_j \right) \frac{\partial}{\partial u_i}. \tag{5.10}$$

The coefficients change by taking the product of the matrix

$$\frac{\partial u_i}{\partial x_j} = g'_{i,j}(\mathbf{u}) \tag{5.11}$$

with the column vector with components a_j.

Example: Consider parabolic coordinates given by $x = (1/2)(u^2 - v^2)$, $y = uv$. It is slightly awkward to express u, v in terms of x, y, but it can be done. The Cartesian coordinate basis vectors are given in terms of parabolic coordinate basis vectors by the matrix

$$\begin{aligned} \frac{\partial}{\partial u} &= u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \\ \frac{\partial}{\partial v} &= -v\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}. \end{aligned} \tag{5.12}$$

The other direction is given by

$$\frac{\partial}{\partial x} = \frac{1}{u^2+v^2}\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right)$$
$$\frac{\partial}{\partial y} = \frac{1}{u^2+v^2}\left(v\frac{\partial}{\partial u} + u\frac{\partial}{\partial u}\right). \tag{5.13}$$

This involves the inverse matrix. ||

The addition of vector fields has a natural definition: $(X+Y)\,s = X\,s + Y\,s$. Also, there is a natural operation of multiplication of scalar field and vector field: If s is a scalar field and X is a vector field, then sX is another vector field. The multiplication satisfies $s(X+Y) = sX + sY$ and $(r+s)X = rX + sX$ and $(rs)X = r(sX)$. The space of $\mathcal{E}$ of all vector fields on the manifold patch resembles a vector space in many respects. The main difference is that the scalar fields are not constants, and so they do not have a very good notion of division. (This kind of algebraic system, for which the scalar multiples come from a ring, is called a *module*. If the ring is a field, then the module is a vector space.)

There is another important operation on the module $\mathcal{E}$ of vector fields. This is the *Lie product* defined by

$$[X, Y] = XY - YX. \tag{5.14}$$

More explicitly, $[X, Y]\,s = X\,(Y\,s) - Y\,(X\,s)$. The result is another vector field. The interpretation of this is that X determines a change in the space, and $[X, Y]$ describes the corresponding change in Y. The coefficients of $[X, Y]$ involve derivatives of the coefficients of both X and Y.

Remark: The expression $X\,Y$ can of course be defined by $(XY)\,s = X\,(Y\,s)$. This involves second derivatives, so it is not a vector field. In later developments it will be occasionally useful to define $X\,Y$ in a completely different way, so that $X\,Y$ is the covariant derivative of Y along X, which is another vector field. Because this notation runs the risk of ambiguity, it may be better to denote this covariant derivative by $X \lrcorner \nabla Y$. All this will be explained in due course. ||

Here is a fluid flow picture of a vector field $X = \sum_j a_j \frac{\partial}{\partial x_j}$. The space in which the flow takes place is M. The vector field is the fluid velocity field. At each point of M there is an arrow. This arrow is not to be thought of as a displacement in M, but as a rate of change at this point of M. More precisely, the components a_k of the vector field are velocities, measured in meters per second. The vector field X (regarded as a differential operator) describes how scalar quantities change with time, so it is measured in inverse seconds.

In fluid dynamics the differential operator X represents the effect of the fluid flow given by the velocity vector field. It is the derivative following the motion of the particle. It is so important that it has many names: *advective derivative*, *particle derivative*, *material derivative*, *substantial derivative*, *Lagrangian derivative*, and so on. Later on there will be other notations for the advective derivative of a scalar field s, such as $X \lrcorner \nabla s$ in place of $X\ s$. Even more important is the advective derivative (covariant derivative) of a vector field Y,

denoted $X \lrcorner \nabla Y$. This does not make sense on a manifold patch unless it has additional structure. The discussion is postponed to later chapters that treat the covariant differential ∇.

Remark: This is a remark intended to indicate certain issues that will be treated in more detail later on. The module $\mathcal{E}$ of all vector fields on a given manifold patch M does not form a vector space, but the structure is remarkably similar. The main difference is that the coefficients are smooth functions on M instead of constants. Nevertheless, many vector space ideas carry over, and in particular there is a notion of basis.

One kind of basis is obtained from a coordinate system $u_1, \ldots, u_n$. The basis vectors are $\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_n}$. This is called a *coordinate basis*. Every vector field has the form

$$Y = \sum_{j=1}^{n} b_j \frac{\partial}{\partial u_j}, \tag{5.15}$$

where the coefficients are smooth functions on M. Thinking of vector fields as differential operators is initially confusing, but the reward is great. For one thing, the transformation properties of coefficients under change of basis are given by the chain rule. Many other properties of vector fields become transparent with this point of view.

In many applications of mathematics there are notions of length of a vector and of orthogonal vectors. In this special situation another kind of basis vector is available. A common situation is that the coordinates $u_1, \ldots, u_n$ are what is known as *orthogonal coordinates*. In this special situation the coordinate basis vectors $\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_n}$ form an orthogonal basis. This basis may not be an orthonormal basis, because the vectors are not all of length one. In fact, their lengths are given by quantities $h_1, \ldots, h_n$ that are smooth functions on M, strictly positive at each point. So the appropriate orthonormal basis is $\frac{1}{h_1}\frac{\partial}{\partial u_1}, \ldots, \frac{1}{h_n}\frac{\partial}{\partial u_n}$. This can be called a *normalized basis*. The vector Y above may then be written

$$Y = \sum_{j=1}^{n} a_j \frac{1}{h_j} \frac{\partial}{\partial u_j}. \tag{5.16}$$

It is the same vector field, but the coefficients are different, because a different basis is used.

The simplest non-trivial example of these concepts is polar coordinates. The coordinate basis vectors are $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$. The corresponding normalized basis vectors are $\frac{\partial}{\partial r}$ and $\frac{1}{r}\frac{\partial}{\partial \theta}$. The polar coodinate example will be described in the following chapter in greater detail and with pictures.

Both kinds of basis are in common use. In tensor analysis the practice is to use coordinate bases. In vector analysis the practice is to try to find orthogonal coordinates and then use normalized bases. One advantage of the present notation is that the relation between these various kinds of bases is easy to compute. This will be seen in later chapters. ||

Remark: It is also possible to abbreviate

$$\begin{aligned} \frac{\partial}{\partial x} s &= s_{,x} \\ \frac{\partial}{\partial y} s &= s_{,y}. \end{aligned} \tag{5.17}$$

A variant of this notation is common in connection with partial differential equations that arise in applied mathematics. In this setting the perhaps risky convention is to write the partial derivatives simply as s_x and s_y.

As an example, consider a velocity vector field

$$V = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y}. \tag{5.18}$$

The components of velocity are written as v_x and v_y. The $\partial/\partial x$ and $\partial/\partial y$ are basis vectors, but they can alson be used to compute rate of change. The rate of change of a scalar field s is

$$V\ s = v_x s_{,x} + v_y s_{,y}. \tag{5.19}$$

The present work will not make further use of this abbreviated notation for partial derivatives. ||

5.2.3 Vectors at a point

The definition of *vector at a point* is as a derivation of the algebra of scalar fields whose values are constants. If Y is a vector field at point μ in M, and s is a scalar field, then $Y\,s$ is a constant. Furthermore, for every constant c and scalar field s

$$\begin{aligned} Y\,c &= 0 \\ Y\,(r+s) &= Y\,r + Y\,s \\ Y\,(rs) &= (Y\,r)s(\mu) + r(\mu)(Y\,s). \end{aligned} \tag{5.20}$$

Theorem 5.3 (Coordinate form of vector at a point) *Let μ be a point in M with $\mathbf{x}$ coordinate value $\mathbf{c} = \mathbf{x}(\mu)$, so $\mathbf{c}$ is constant. A vector Y at μ is given in coordinate system $\mathbf{x}$ by constants a_j*

$$Y\,h(\mathbf{x}) = \sum_{j=1}^{n} a_j h'_{,j}(\mathbf{c}). \tag{5.21}$$

Given a vector field X on M and a point μ in M, there is a vector $X_{[\mu]}$ at μ that is called the *restriction* of the vector field to μ. It is defined by

$$X_{[\mu]}\,s = (X\,s)_\mu. \tag{5.22}$$

Proposition 5.4 (Coordinate form of restriction of a vector to a point) *Let μ be a point in M with $\mathbf{x}$ coordinate value $\mathbf{c} = \mathbf{x}(\mu)$, so $\mathbf{c}$ is a constant vector. Suppose that*

$$X = \sum_j g_j(\mathbf{x}) \frac{\partial}{\partial x_j}. \tag{5.23}$$

The restriction $X_{[\mu]}$ to μ is given in coordinate system $\mathbf{x}$ by

$$X_{[\mu]} = \sum_{j=1}^{n} g_j(\mathbf{c}) \frac{\partial}{\partial x_j}{}_{[\mu]}. \tag{5.24}$$

The vectors at a particular point in M form a vector space of dimension n. This is called the *tangent space* to M at this point. The reason for this terminology will become clear during the discussion of surfaces. A vector in the tangent space at μ may be thought of as an arrow describing change at this particular point.

Consider a vector field X on M. At each point μ of M the vector field defines a tangent vector $X_{[\mu]}$ at that point. This in turn defines a function $\mu \mapsto X_{[\mu]}$ whose value at each point in M is a vector at that point.

Proposition 5.5 *For a vector field X define the corresponding function $\mu \mapsto X_{[\mu]}$. Then the vector field may be recovered from this function by the formula*

$$X\, s = (\mu \mapsto X_{[\mu]} s) \tag{5.25}$$

Remark: It seems reasonable on a first account to define a vector field as a differential operator that takes different forms in different coordinates systems. However suppose that one wants to define a vector field in a way that makes no mention of a coordinate system. All that is required is the sum rule and the product rule. There is then the following result.

Proposition 5.6 (Coordinate representation of a vector) *Suppose $s = h(\mathbf{x})$ is a scalar, and X is a vector field. Let μ be the point in M where $\mathbf{x}$ has the value $\mathbf{c}$. Then*

$$(X\, s)(\mu) = \sum_{k=1}^{n} v_k h'_{,k}(\mathbf{c}), \tag{5.26}$$

Here each $v_k = (X\,(x_k - c_k))(\mu)$ is a constant. The action of the vector field is to take the partial derivatives in the coordinate directions and weight them by these constants.

Proof: Start with the mean value theorem identity

$$h(\mathbf{x}) - \mathbf{h}(\mathbf{c}) = \int_0^1 \frac{dh(\mathbf{c} + t(\mathbf{x} - \mathbf{c}))}{dt}\, dt = \sum_{k=1}^{n} \int_0^1 (x_k - c_k) h'_{,k}(\mathbf{c} + t(\mathbf{x} - \mathbf{c}))\, dt. \tag{5.27}$$

This can be written

$$h(\mathbf{x}) = \mathbf{h}(\mathbf{c}) + \sum_{k=1}^{n} (x_k - c_k) g_{\mathbf{c},k}(\mathbf{x}), \tag{5.28}$$

where the function on the right satisfies $g_{\mathbf{c},k}(\mathbf{c}) = h'_k(\mathbf{c})$. Apply X to this equation. By the sum rule and the product rule $X\, h(\mathbf{x})$ has the representation

$$X\, \mathbf{h}(\mathbf{c}) + \sum_{k=1}^{n} X((x_k - c_k) g_{\mathbf{c},k}(\mathbf{x})) = 0 + \sum_{k=1}^{n} (X\,(x_k - c_k) g_{\mathbf{c},k}(\mathbf{x}) + (x_k - c_k) X\, g_{\mathbf{c},k}(\mathbf{x})). \tag{5.29}$$

Evaluation at μ means replacing $\mathbf{x}$ by $\mathbf{c}$. This gives the result. □

The technique used in this proposition is related to a result called the *Hadamard lemma*. ||

5.2.4 Zeros of a vector field

Getting numbers out of a vector field might seem elusive, since the components of a vector field depend on the coordinate system. However, at each point at which the vector field is zero there is a linearization, and the eigenvalues of this linearization are numbers that do not depend on the coordinate system.

Theorem 5.7 *Suppose that*

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} = \sum_{i=1}^{n} g_i(\mathbf{x}) \frac{\partial}{\partial x_i} \tag{5.30}$$

is a vector field. Consider a point where each $g_i(\mathbf{x}) = 0$. Then the eigenvalues (and Jordan form) of the matrix

$$\frac{\partial a_i}{\partial x_k} = \frac{\partial g_i(\mathbf{x})}{\partial x_k} = g'_{j,k}(\mathbf{x}) \tag{5.31}$$

do not depend on the coordinate system.

These eigenvalues can help sketching the vector field. If they are negative, then the arrows tend to point in the general direction of the zero. If they are positive, the the arrows tend to point in the general direction away from the zero. If they have both signs, then the arrows do something more complicated.

Proof: Here is a proof in a style making heavy use of coordinates. We know that

$$X = \sum_{j=1}^{n} b_j \frac{\partial}{\partial u_j} \tag{5.32}$$

with

$$b_j = \sum_{i=1}^{n} \frac{\partial u_j}{\partial x_i} a_i. \tag{5.33}$$

Therefore

$$\frac{\partial b_j}{\partial u_k} = \sum_{i=1}^{n} \frac{\partial u_j}{\partial x_i} \frac{\partial a_i}{\partial u_k} + \sum_{i=1}^{n} \frac{\partial^2 u_j}{\partial x_i \partial u_k} a_i. \tag{5.34}$$

Since we are at a zero of the vector field, this becomes

$$\frac{\partial b_j}{\partial u_k} = \sum_{i=1}^{n} \frac{\partial u_j}{\partial x_i} \frac{\partial a_i}{\partial u_k}. \tag{5.35}$$

Then by the chain rule we get

$$\frac{\partial b_j}{\partial u_k} = \sum_{i=1}^{n} \sum_{\ell=1}^{n} \frac{\partial u_j}{\partial x_i} \frac{\partial a_i}{\partial x_\ell} \frac{\partial x_\ell}{\partial u_k}. \tag{5.36}$$

This says that the two matrices are similar. □

Proof: Here is the proof of the theorem in matrix language. Let $\mathbf{u} = \mathbf{f}(\mathbf{x})$ be the change of coordinates. Let $\mathbf{g}(\mathbf{x})$ be the column vector of components in the $\mathbf{x}$ basis, and let $\mathbf{j}(\mathbf{u})$ be the column vector of components in the $\mathbf{u}$ basis.

Let $s = h(\mathbf{u}) = k(\mathbf{x})$ be a scalar field. Then

$$X\, s = h'(\mathbf{u})\mathbf{j}(\mathbf{u}) = k'(\mathbf{x})\mathbf{g}(\mathbf{x}). \tag{5.37}$$

In particular,

$$h'(\mathbf{u})\mathbf{j}(\mathbf{u}) = (h \circ \mathbf{f})'(\mathbf{x})\mathbf{g}(\mathbf{x}) = h'(\mathbf{u})\mathbf{f}'(\mathbf{x})\mathbf{g}(\mathbf{x}). \tag{5.38}$$

Hence

$$\mathbf{j}(\mathbf{u}) = \mathbf{f}'(\mathbf{x})\mathbf{g}(\mathbf{x}). \tag{5.39}$$

This says that the coefficients change according to the derivative of the change of coordinates function.

Write this as

$$\mathbf{j}(\mathbf{u}) = \mathbf{f}'(\mathbf{f}^{-1}(\mathbf{u}))\mathbf{g}(\mathbf{f}^{-1}(\mathbf{u})). \tag{5.40}$$

This is a product, but because we are at a zero we get only one term from the product rule. Furthermore, we can use the fact that the derivative of $\mathbf{f}^{-1}$ is $(\mathbf{f}' \circ \mathbf{f}^{-1})^{-1}$. So

$$\mathbf{j}'(\mathbf{u}) = \mathbf{f}'(\mathbf{f}^{-1}(\mathbf{u}))\mathbf{g}'(\mathbf{f}^{-1}(\mathbf{u}))\mathbf{f}'(\mathbf{f}^{-1}(\mathbf{u}))^{-1}. \tag{5.41}$$

That is,

$$\mathbf{j}'(\mathbf{u}) = \mathbf{f}'(\mathbf{x})\mathbf{g}'(\mathbf{x})\mathbf{f}'(\mathbf{x})^{-1}. \tag{5.42}$$

This says that the matrices are similar. □

Proof: It would be nice to have a proof that defined the linear transformation directly. This may be done considering two vector fields X and Y. Their relative motion is described by a third vector field, the Lie product $[Y, X] = YX - XY$. At a point μ where $X = 0$ the mapping $Y \mapsto [Y, X]$ sends vectors Y at μ to vectors $[Y, X]$ at μ. This is a linear transformation of this n-dimensional vector space, so it has eigenvalues and eigenvectors.

The way this works can be seen by choosing a coordinate representation. If

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \tag{5.43}$$

and

$$Y = \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}, \tag{5.44}$$

then

$$[Y, X] = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(b_j \frac{\partial a_i}{\partial x_j} - a_j \frac{\partial b_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}. \tag{5.45}$$

This is valid at every point in an arbitrary coordinate system. At a point μ where X vanishes (so the $a_j = 0$) all that matters is the value of Y at that point:

$$[Y, X] = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial a_i}{\partial x_j} b_j \right) \frac{\partial}{\partial x_i}. \tag{5.46}$$

In this expression all coefficients and coordinate basis vectors are considered to be located at the point μ. This is the linear transformation expressed in this coordinate basis. □

5.3 Integral curves of a vector field

5.3.1 Parameterized curves

Say that M is an n-dimensional manifold patch. Let P be a one-dimensional connected manifold patch. A *parameterized curve* ϕ is a smooth mapping from P to M.

Consider some coordinate system $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ on M. Let t be a coordinate system on P. Then the parameterized curve ϕ may be expressed by

$$\phi = (\mathbf{x} \leftarrow \mathbf{f}(t)). \tag{5.47}$$

This means take the point in P, compute its t value, apply the function $\mathbf{f} : \mathbf{R} \to \mathbf{R}^n$, and finally find the point in M where $\mathbf{x}$ has the value $\mathbf{f}(t)$.

One effect of ϕ is to make a replacement in a scalar field. Suppose that $s = h(\mathbf{x})$ is a scalar field on M. Then $s \circ \phi$ is a scalar field on the parameter space P given by

$$s \circ \phi = h(\mathbf{x})(\mathbf{x} \leftarrow \mathbf{f}(t)) = h(\mathbf{f}(t)). \tag{5.48}$$

As an important special case, take $s = x_i$ to be a coordinate. Then

$$x_i \circ \phi = f_i(t). \tag{5.49}$$

A common interpretation is that t is a time coordinate. Then $\mathbf{x} \circ \phi = \mathbf{f}(t)$ is the location of a particle as a function of time. The corresponding velocity along the path of the particle has components given by

$$\frac{d(x_i \circ \phi)}{dt} = f_i'(t). \tag{5.50}$$

5.3.2 Autonomous systems of differential equations

A vector field is equivalent to an *autonomous system* of first order ordinary differential equations. (The word autonomous means that the vector field does not change with time.)

Theorem 5.8 (Existence and uniqueness of integral curve) *Suppose M is a manifold patch, X is a vector field on M. For every point μ in M there exists a number a with $0 < a \leq +\infty$ with the following properties. There is a one-dimensional parameter manifold patch P with coordinate t having values $-a < t < a$. There is a unique manifold mapping $\phi : P \to M$ with $\phi(t \leftarrow 0) = \mu$ and such that for every scalar field s on M we have*

$$\frac{d\,(s \circ \phi)}{dt} = (X\,s) \circ \phi. \tag{5.51}$$

This theorem is proved in texts on differential equations. There is an excellent treatment in [1].

Theorem 5.9 (Coordinate form of differential equation) *Suppose that the vector field is*

$$X = \sum_j g_j(\mathbf{x}) \frac{\partial}{\partial x_j}. \tag{5.52}$$

Suppose $\mathbf{x}(\mu) = \mathbf{c}$. The integral curve is $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{c}, t)$. The equation then says that

$$\frac{df_j(\mathbf{c}, t)}{dt} = g_j(\mathbf{f}(\mathbf{c}, t)). \tag{5.53}$$

Often the differential equation is written

$$\frac{dx_j}{dt} = g_j(\mathbf{x}). \tag{5.54}$$

This is just another way of describing the vector field. The expression becomes meaningful only after inserting the solution $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{c}, t)$.
Example: Consider the one-dimensional vector field $X = x^2 \frac{\partial}{\partial x}$. The equation $dx/dt = x^2$ has solution $x \leftarrow f(x_0, t) = x_0/(1 - x_0 t)$. This example shows that the solution may exist only for small value of t. ||
Example: Consider the two-dimensional vector field $X = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$. The system $dx/dt = x^2$, $dy/dt = xy$ has solution $(x, y) \leftarrow \mathbf{f}(x_0, y_0, t) = (x_0(1 - x_0 t)^{-1}, y_0(1 - x_0 t)^{-1})$. ||

5.3.3 The image curve of a solution

A solution corresponding to vector field X is a mapping from the time parameter space to the manifold patch M. One can ask about its image. This should ordinarily be a curve in M having nothing to do with time.

Suppose that the vector field is never zero. Then each image curve should be a one-dimensional curve inside M. It should be possible to describe this curve by a system of first order ordinary differential equations. This system says that for each $i \neq j$ we have

$$\frac{dx_i}{g_i(\mathbf{x})} = \frac{dx_j}{g_j(\mathbf{x})} \tag{5.55}$$

This looks like many equations, but there are only $n-1$ independent equations.
Example: Consider again the two-dimensional vector field $X = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$. In two dimensions there is only a single equation, which is $dx/x^2 = dy/(xy)$. This is equivalent to $dx/x = dy/y$ which has solution $x = Cy$. Indeed this is satisfied in the previous example. ||
Example: Consider the system

$$\begin{aligned} \frac{dx}{dt} &= -y \\ \frac{dy}{dt} &= x \\ \frac{dz}{dt} &= 1. \end{aligned} \tag{5.56}$$

This has solution

$$\begin{aligned} x &\leftarrow x_0 \cos(t) + y_0 \sin(t) \\ y &\leftarrow x_0 \sin(t) - y_0 \cos(t) \\ z &\leftarrow z_0 + t. \end{aligned} \tag{5.57}$$

The corresponding equations for the image are

$$\begin{aligned} -\frac{dx}{y} &= \frac{dy}{x} \\ \frac{dy}{x} &= dz \\ -\frac{dx}{y} &= dz. \end{aligned} \tag{5.58}$$

The first equation $x\,dx + y\,dy = 0$ has the solution $x^2 + y^2 = c^2$. The other two are satisfied by $y = a\sin(z-b)$ and $x = a\cos(z-b)$. ||

5.3.4 Structure of a vector field

Locally, a vector field is a fairly boring object, with one exception. This is at a point μ in the manifold patch M where the vector field X vanishes, that is,

where $X_{[\mu]} = 0$. This is a *stationary point* or *equilibrium point*. The solution ϕ passing through this point has the constant value μ. If the coordinate value corresponding to μ is $\mathbf{c}$, then the solution $\mathbf{f}(\mathbf{c}, t) = \mathbf{c}$ for all t.

Away from such points the local behavior of the vector field is no more interesting than uniform motion. The following theorem will be proved in a later section.

Theorem 5.10 (Straightening out theorem) . *If*

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \neq 0 \tag{5.59}$$

is a vector field that is non-zero near some point, then near that point there is another coordinate system $u_1, \ldots, u_n$ and a value of j such that

$$X = \frac{\partial}{\partial u_j}. \tag{5.60}$$

Example: Suppose that $X = u\frac{\partial}{\partial v}$. It is non-zero wherever $u \neq 0$. We can make the change of variables

$$\begin{aligned} w &= \frac{v}{u} \\ z &= u \end{aligned} \tag{5.61}$$

Then

$$X = u\frac{\partial}{\partial v} = u\left(\frac{\partial w}{\partial v}\frac{\partial}{\partial w} + \frac{\partial z}{\partial v}\frac{\partial}{\partial z}\right) = \frac{\partial}{\partial w}. \tag{5.62}$$

In the new coordinates X is constant. ||

Say that a vector field defining a system of ordinary differential equations has an isolated zero. Thus the equation $d\mathbf{x}/dt = \mathbf{g}(\mathbf{x})$ has a point $\mathbf{x}^*$ where $\mathbf{g}(\mathbf{x}^*) = 0$. Near this point there is an approximation $\mathbf{g}(\mathbf{x}) \approx \mathbf{g}'(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}*)$. This suggests considering the approximate equation $d\mathbf{x}/dt \approx \mathbf{g}'(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}*)$. By changing to new coordinates $\bar{\mathbf{x}} = \mathbf{x} - \mathbf{x}^*$ this becomes $d\bar{\mathbf{x}}/dt \approx \mathbf{g}'(\mathbf{x}^*)\bar{\mathbf{x}}$.

The *linearization* of the differential equation $d\mathbf{x}/dt = \mathbf{g}(\mathbf{x})$ at a point $\mathbf{x}^*$ with $\mathbf{g}(\mathbf{x}^*) = 0$ is defined to be the differential equation

$$\frac{d\bar{\mathbf{x}}}{dt} = \mathbf{g}'(\mathbf{x}^*)\bar{\mathbf{x}}. \tag{5.63}$$

The behavior of the linearization is studied by finding the eigenvalues of the matrix $\mathbf{g}'(\mathbf{x}^*)$. Thus if $\mathbf{g}'(\mathbf{x}^*)\bar{\mathbf{x}}_0 = \lambda\bar{\mathbf{x}}_0$, then a solution is given by $\bar{\mathbf{x}} \leftarrow \bar{\mathbf{x}}_0 e^{\lambda t}$. In the case when there are no repeated eigenvalues, every solution is a linear combination of solutions of this form. Some of the eigenvalues may be complex.

This story is already of interest in the case $n = 2$. Suppose that the differential equation is

$$\begin{aligned} \frac{dx}{dt} &= g_1(x, y) \\ \frac{dy}{dt} &= g_2(x, y). \end{aligned} \tag{5.64}$$

Suppose that x^* and y^* are constants such that $g_1(x^*, y^*) = 0$ and $g_2(x^*, y^*) = 0$. Then the linearization is obtained by setting $\bar{\mathbf{x}} = \mathbf{x} - \mathbf{x}^*$ and $\bar{\mathbf{y}} = \mathbf{y} - \mathbf{y}^*$. The equation is

$$\begin{aligned} \frac{d\bar{x}}{dt} &= g'_{1,1}(x^*, y^*)\bar{x} + g'_{1,2}(x^*, y^*)\bar{y} \\ \frac{d\bar{y}}{dt} &= g'_{2,1}(x^*, y^*)\bar{x} + g'_{2,2}(x^*, y^*)\bar{y}. \end{aligned} \tag{5.65}$$

The special solution that are used to build general solutions are of the form $(\bar{x}, \bar{y}) \leftarrow e^{\lambda t}(\bar{x}_0, \bar{y}_0)$. There are three kinds of Jordan form.

1. There is the case of real eigenvalues. In this case write the equation as

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}. \tag{5.66}$$

Let $P = [\mathbf{p}_1, \mathbf{p}_2]$ be the matrix whose columns are the eigenvectors of A. Make the change of variables $\mathbf{x} = P\mathbf{z}$ and use $AP = \Lambda A$. This gives

$$\frac{dz_i}{dt} = \lambda_i z_i \tag{5.67}$$

with solution

$$z_i \leftarrow c_i e^{\lambda_i t}. \tag{5.68}$$

Then the solution is

$$\mathbf{x} \leftarrow c_1 e^{\lambda_1 t}\mathbf{p}_1 + c_2 e^{\lambda_2 t}\mathbf{p}_2. \tag{5.69}$$

This includes the following situations.

Stable node Real eigenvalues with $\lambda_1 < 0, \lambda_2 < 0$.

Unstable node Real eigenvalues with $\lambda_1 > 0, \lambda_2 > 0$.

Hyperbolic fixed point (saddle) Real eigenvalues with $\lambda_1 < 0 < \lambda_2$.

2. There is also the case of conjugate nonreal eigenvalues. In this case the eigenvalues of A are $\mu \pm i\omega$ with eigenvectors $\mathbf{p} \mp i\mathbf{q}$. Here $\omega \neq 0$. Write $P = [\mathbf{p}, \mathbf{q}]$. Then $AP = PJ$, where

$$J = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix}. \tag{5.70}$$

This is the real Jordan form. If $d\mathbf{x}/dt = A\mathbf{x}$, we can set $\mathbf{x} = P\mathbf{z}$ and get

$$\frac{d\mathbf{z}}{dt} = J\mathbf{z}. \tag{5.71}$$

This has solution

$$\begin{aligned} z_1 &\leftarrow r_0 e^{\mu t}\cos(\omega t + \theta_0) \\ z_2 &\leftarrow r_0 e^{\mu t}\sin(\omega t + \theta_0). \end{aligned} \tag{5.72}$$

In the original coordinates this is

$$\mathbf{x} \leftarrow r_0 e^{\mu t}(\cos(\omega t + \theta_0)\mathbf{p} + \sin(\omega t + \theta_0)\mathbf{q}). \tag{5.73}$$

This includes the following situations.

Stable spiral Nonreal eigenvalues with $\lambda = \mu \pm i\omega$, $\mu < 0$.

Unstable spiral Nonreal eigenvalues with $\lambda = \mu \pm i\omega$, $\mu > 0$.

Elliptic fixed point (center) Nonreal eigenvalues $\lambda = \pm i\omega$.

3. The third case is that of the repeated real eigenvalue with the exceptional Jordan form. That can also arise, but it is not typical.

Example: Here is a representative analysis of a vector field via fixed points and linearizations. Consider the system

$$\begin{aligned} \frac{du}{dt} &= u(v-1) \\ \frac{dv}{dt} &= 4 - u^2 - v^2. \end{aligned} \tag{5.74}$$

The analysis proceeds in stages.

- Find the zeros of the vector field (that is, the fixed points for the differential equation). For this problem there are fixed points in the u, v plane at $(0, 2)$ and $(0, -2)$ and $(\sqrt{3}, 1)$ and $(-\sqrt{3}, 1)$.

- Compute the linearizations at each of these fixed points, and find the general behavior of the each of these linear differential equations.

  The first two are hyperbolic fixed points with vertical and horizontal eigenspaces. The eigenvalues at $(0, -2)$ are -3 in the horizontal direction and 4 in the vertical direction. The eigenvalues at $(0, 2)$ are 1 in the horizontal direction and -4 in the vertical direction.

  The eigenvalues at $(\sqrt{3}, 1)$ and $(-\sqrt{3}, 1)$ are both stable spirals with eigenvalues $-1 \pm \sqrt{5}i$. The first one spirals in clockwise, while the other one spirals in counter-clockwise.

- Go back to the original differential equation and connect the information from the fixed points to build up an overall picture.

  By setting $u = 0$ in the original equation one can see that v moves along the vertical axis $u = 0$ from the fixed point at $(0, -2)$ to the fixed point at $(0, 2)$. Thus the stable direction for the fixed point at $(0, 2)$ is the line $u = 0$.

  For further analysis one might as well look at $u > 0$, since the picture is symmetric. The region is divided into the interior of the circle $u^2 + v^2 = 4$ where $dv/dt > 0$ and the exterior where $dv/dt < 0$. Furthermore, there is

the region $v > 2$ where $du/dt > 0$ and the region $v < 2$ where $du/dt < 0$. Thus one can sketch the vectors in these four regions.

There is a very special orbit that approaches $(0, -2)$ from the right in the stable direction. This orbit comes from large positive values of v. Points to the right of this orbit sweep past everything to large negative values of v. Points to the left of this orbit get attracted to the fixed point at $(\sqrt{3}, 1)$ and spiral into this point.

Here is a remark that may make clear the nature of linearization. Consider this example, and suppose that the interest is the behavior of the solution near the fixed point at $(0, -2)$. Write the nonlinear equation as

$$\frac{du}{dt} = -3u + u(v+2) \tag{5.75}$$
$$\frac{dv}{dt} = 4(v+2) - u^2 - (v+2)^2.$$

This is just writing it with the linearization and the higher order terms given explicitly. From this one can see various things. For example, starting with $u > 0$ and $v = -2$ does not give a solution that approaches the fixed point, since the linearization does not tell the whole story. In the nonlinear version the value of dv/dt at such a point is negative, so such a solution instead gets swept away to large negative values of v.

Suppose that the same system were viewed in another coordinate system. Many details would differ. However there would still be four fixed points, and the eigenvalues of the linearizations would be the same. So the qualitative picture would be much the same, but perhaps less symmetric. ||

Example: A classic example from physics is the nonlinear pendulum. This is a standard example in physics textbooks that treat mechanics; these may be consulted for background. The equations for displacement q and momentum p are

$$\frac{dq}{dt} = \frac{1}{m}p \tag{5.76}$$
$$\frac{dp}{dt} = -mg\sin(\frac{1}{a}q). \tag{5.77}$$

Here $q = a\theta$ represents displacement, and p represents momentum. The nonlinear term is $\sin(\theta)$. The zeros are at $\theta = n\pi$. When n is even this is the pendulum at rest in a stable position; when n is odd this is the pendulum at rest upside down, in a very unstable position. The linearization at a zero is

$$\frac{dq}{dt} = \frac{1}{m}p \tag{5.78}$$
$$\frac{dp}{dt} = -\frac{mg}{a}(-1)^n q. \tag{5.79}$$

In matrix form this is

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} \\ -(-1)^n \frac{mg}{a} & 0 \end{pmatrix} \begin{pmatrix} \tilde{q} \\ \tilde{p} \end{pmatrix}. \tag{5.80}$$

The eigenvalues λ are given by $\lambda^2 = -(-1)^n \frac{g}{a}$. When n is even we get an elliptic fixed point, while when n is odd we get a hyperbolic fixed point.

The equation for the image in phase space is

$$\frac{1}{m} p\, dp + mg \sin(\frac{1}{a} q)\, dq = 0. \tag{5.81}$$

This has solution

$$\frac{1}{2m} p^2 - mga \cos(\frac{1}{a} q) = H. \tag{5.82}$$

Each value of the energy H defines the image of a solution in phase space. While the energy does not describe the time dependence of the solutions, it gives considerable insight into what they look like. ||

Remark: The following question is natural. Suppose that a vector field has an isolated zero. At that zero it has a linearization. Is it possible to choose coordinates nearby so that the vector field is given in those new coordinates by its linearization? It turns out that this can often be done. The answer to the question is negative in general. See Nelson [35] for a discussion of this delicate matter.

Here are some facts relevant to this topic. There is a theorem of Hartman-Grobman that says that if the linearization of the vector field at a zero has no pure imaginary eigenvalues, then there is a continuous transformation to a linear form near the point. Thus in this case the rough form of the solution is clear. However it does not establish that there is a smooth change of coordinates that brings the vector field to a linear form.

Consider integer multiples of the eigenvalues of the form $m_1\lambda_1 + m_2\lambda_2 + \cdots + m_n\lambda_n$. Here each $m_i \geq 0$. Suppose this combination is non-trivial, in the sense that $m_1 + m_2 + \cdots + m_n \geq 2$. Call any such combination *resonant*. The result of Sternberg (discussed in the Nelson reference) is that if no eigenvalue λ_i is resonant, then there is a smooth change of coordinates that linearizes the vector field near the zero.

Notice that if λ and $-\lambda$ are both eigenvalues, then λ is resonant. In fact, then $\lambda = 2\lambda + 1(-\lambda)$. In particular, a pure imaginary eigenvalue is always resonant. ||

5.3.5 Picturing vector fields

There are two common ways of picturing a vector field.

- The first is to do a grid plot. That is, one makes a grid of closely spaced points, and at each point one draws an arrow representing the vector at that point. The best way to think of this vector is as a sort of velocity. In the most typical interpretation it does not represent a displacement in space (that is, in the manifold patch.) Because the vectors are velocities, the length of a vector compared to a distance in the space is of no particular importance. So it may be useful to choose the scale for the length of the vectors to make the picture give a reasonably detailed description without

being too cluttered with arrows. In any case, one particularly important part of a picture of a vector field is around the points where it is zero. It may be good practice to try to include all the relevant zeros in the picture. There are many vector field plotting programs.

In the case of Euclidean space (or affine space) it is possible to recover a displacement vector in space by multiplying the velocity vector by a time value, thus obtaining a displacement vector. However this is a special situation. In general the solutions of the differential equation associated with the vector field gives a motion that is not naturally described by adding a vector.

- The other way to picture a vector field is by drawing solution curves obtained by solving the differential equations. One draws enough solution curves that the resulting picture gives a reasonable overall picture that is more or less equivalent to the vector field picture. The arrows of the vector field are tangent to the curves. The motion along a solution curve is in space. As the curve twists and turns, it is telling a story about the cumulative effect of the velocity vectors as they are integrated in time. A zero of the vector field corresponds to a solution that stays at a particular point.

  One weakness of the solution curve picture it that it does not by itself give an indication of time dependence, that is, of the velocities. It is as if one knows the directions of the arrows but not how long they are. It may be a good idea to give an indication of the direction of motion along the solution curves. The speed could also be indicated by appropriate coding. One method would be to pick an appropriate unit of time and place marks on the curve indicating points spaced according to this time interval. At points where the flow is slow the marks would be very close together.

  Producing solution curves is mainly a job for the computer. But with available software it should be almost as easy as drawing the vector field. See Figures 5.1 and 5.2 for examples of the grid plot and solution curve methods drawing a vector field. See Figure 5.3 for an idea of the supplemental information that should be included in the solution curve to indicate how fast the solution is moving along the curve. The marks indicate one second intervals. (The arrow is to indicate the direction of time.) It would be valuable to have this extra information incorporated in the plotting software.

Problems

Mathematical modeling

A cylindrical can has area $A = 2\pi r^2 + 2\pi rh$ and volume $V = \pi r^2 h$.

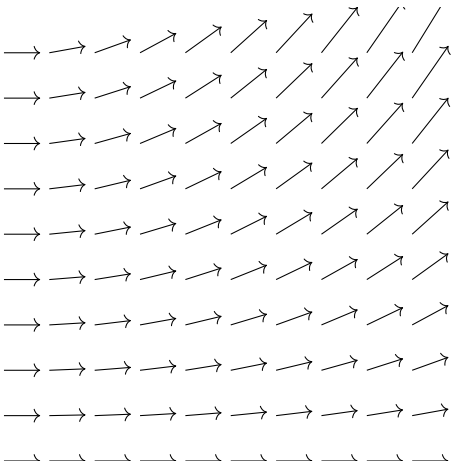

Figure 5.1: A vector field in 2-dimensions: grid plot

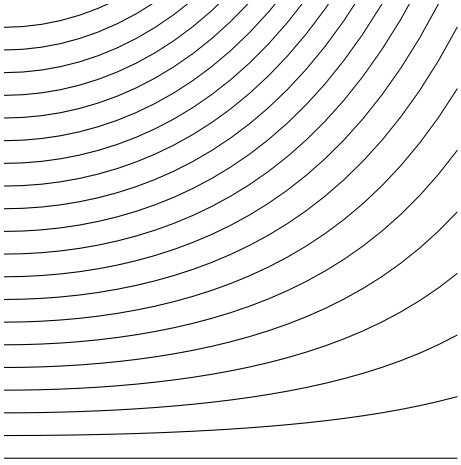

Figure 5.2: A vector field in 2 dimensions: solution plot

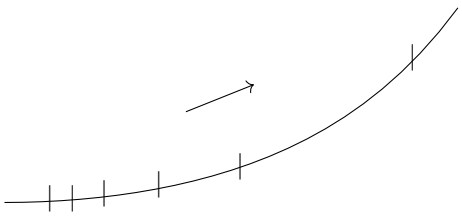

Figure 5.3: Indicating velocity with one second intervals

1. Solve the cylindrical can fixed volume least area problem by expressing A as a function of r.

2. Solve the cylindrical can fixed volume least area problem by expressing A as a function of h.

3. In the cylindrical can problem, the rate $dr/dt = -dA/dr$ is expressed in terms of r. Solve this equation for r as a function of t. Start with the initial condition $r = r_0$ at $t = 0$. Check that the solution converges to the optimum shape as $t \to \infty$.

Vector fields

1. Straightening out. A vector field that is non-zero at a point can be transformed into a constant vector field near that point by a change of coordinate system. Pick a point away from the origin, and find coordinates u, v so that
$$-\frac{y}{x^2+y^2}\frac{\partial}{\partial x} + \frac{x}{x^2+y^2}\frac{\partial}{\partial y} = \frac{\partial}{\partial u}. \tag{5.83}$$

2. Linearization. Consider the vector field
$$X = x(4-x-y)\frac{\partial}{\partial x} + (x-2)y\frac{\partial}{\partial y}. \tag{5.84}$$
Find its zeros. At each zero, find its linearization. For each linearization, find its eigenvalues.

3. Use this and other information to sketch the vector field of the preceding problem.

4. Nonlinearity. Consider the vector field

$$Y = (1 + x^2 + y^2)y\frac{\partial}{\partial x} - (1 + x^2 + y^2)x\frac{\partial}{\partial y}. \tag{5.85}$$

(a) Find its linearization at zero. Solve the linear system. Find the period of a solution.

(b) Solve the original nonlinear system. Find the period of a solution. Hint: Polar coordinates.

5. Consider the vector field of the previous problem. Show that there is no coordinate system near 0 in which the vector field is expressed by its linearization.

6. Nonlinear instability. Here is an example of a fixed point where the linear stability analysis gives an elliptic fixed point, but changing to polar coordinates shows the unstable nature of the fixed point:

$$\frac{dx}{dt} = -y + x(x^2 + y^2) \tag{5.86}$$
$$\frac{dy}{dt} = x + y(x^2 + y^2). \tag{5.87}$$

Change the vector field to the polar coordinate representation, and solve the corresponding system of ordinary differential equations.

7. A predator-prey system. Fix $\alpha > 0$. In the region with $0 < u$ and $0 < v$ consider the system

$$\frac{du}{dt} = u(1 - v) \tag{5.88}$$
$$\frac{dv}{dt} = \alpha v(u - 1).$$

The u variable represents prey; the v variable represents predators.

Find the zero. What kind of linearization is there at this zero? Sketch this vector field.

8. (a) Show that each solution satisfies

$$\alpha v(u - 1)\, du + u(v - 1)\, dv = 0. \tag{5.89}$$

Show that $1/(uv)$ is an integrating factor for the differential form.

(b) Integrate this form to find an equation for the solution curves in the u, v plane. Show that these are compatible with the sketch. What value of the constant of integration corresponds to the fixed point?

Chapter 6

Differential Forms

6.1 Differential 1-forms

6.1.1 Differential forms

This chapter deals with differential 1-forms. They provide the natural setting for the fundamental theorem of calculus in higher dimensions, and they have many other uses. The context is a manifold patch M and vector fields X on M.

A *differential 1-form* ω assigns to each vector field X a corresponding scalar field $\langle \omega \mid X \rangle$. This is linear with respect to scalar fields, in the sense that $\langle \omega, (X + Y) \rangle = \langle \omega \mid X \rangle + \langle \omega \mid Y \rangle$ and $\langle \omega, sX \rangle = s \langle \omega \mid X \rangle$. A differential 1-form is also called a *covector field*.

The module $\mathcal{E}^*$ of all differential 1-forms is not a vector space, but in many ways it resembles one. In fact, it is a module that is dual to the module $\mathcal{E}$ of vector fields. The addition satisfies $\langle \omega + \sigma \mid X \rangle = \langle \omega \mid X \rangle + \langle \sigma \mid X \rangle$. There is also natural definition of the product of a scalar field with a 1-form: $s\omega$ is defined by $\langle s\omega \mid X \rangle = s \langle \omega \mid X \rangle$. This satisfies the usual algebraic properties: $s(\omega + \sigma) = s\omega + s\sigma$, $(r + s)\omega = r\omega + s\omega$, $(rs)\omega = r(s\omega)$.

If μ is a point in M, then there is an n-dimensional vector space of tangent vectors at this point. This leads to a picture of what the differential form ω looks like at a point. It is a linear real function on the space of tangent vectors, and such a function may be pictured in terms of its level sets. If $n = 2$ these are contour lines, if $n = 3$ these are contour planes, in general they are contour hyperplanes.

Proposition 6.1 (Coordinate expression of a 1-form) *In a coordinate system a differential 1-form takes the form*

$$\omega = \sum_{j=1}^{n} p_j \, dx_j. \tag{6.1}$$

Here each $p_j = h_j(\mathbf{x})$ is a smooth function on the manifold patch, and

$$\left\langle dx_j \mid \frac{\partial}{\partial x_i} \right\rangle = \delta_{ij}. \tag{6.2}$$

The coordinate basis forms dx_i are sometimes called the *dual basis* of the coordinate basis vector fields $\partial/\partial x_j$.

Proposition 6.2 (Coordinate invariance of a 1-form) *Suppose that $\mathbf{x} = \mathbf{f}(\mathbf{u})$ is a change of coordinates. The differential 1-form ω is independent of the coordinate system. Thus*

$$\omega = \sum_{k=1}^{n} p_k \, dx_k = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} p_k \frac{\partial x_k}{\partial u_i} \right) du_i \tag{6.3}$$

The coefficients change by taking the product of the covector with components p_k with the matrix

$$\frac{\partial x_k}{\partial u_i} = f'_{k,i}(\mathbf{u}). \tag{6.4}$$

Proposition 6.3 (Coordinate invariance of pairing) *The general 1-form may be written in the form*

$$\omega = \sum_{j=1}^{n} p_j \, dx_j. \tag{6.5}$$

Here the p_j are smooth real functions on M. Its value on the vector field

$$X = \sum_{k=1}^{n} a_k \frac{\partial}{\partial x_k} \tag{6.6}$$

is the scalar field

$$\langle \omega \mid X \rangle = \sum_{j=1}^{n} p_j a_j. \tag{6.7}$$

The scalar field is the product of a row vector on the left with a column vector on the right. It is independent of the coordinate system.

Remark: There are other notations for the value of a differential 1-form ω on a vector field X. The form is a linear function, so the function notation $\omega(X)$ makes sense. Other possibilities are

$$\langle \omega \mid X \rangle = \omega \mid X = \omega \, X. \tag{6.8}$$

Sometimes it is convenient to write the vector first, in which case a suitable notation is

$$X \lrcorner \omega = \langle \omega \mid X \rangle. \tag{6.9}$$

The expression on the left hand side is read X *interior product* ω. The interior product of a vector field with a differential 1-form is a scalar field. ||

The usual way to picture a vector field is through a grid plot. That is, one picks a grid of points and sketches a velocity arrow at each point in the grid. See Figure 6.1 for an idea of how this goes. The integral curves of such a vector field have these velocity vectors as tangent vectors.

For a differential 1-form a grid plot also gives a reasonable picture. At each point of the grid there is a real linear function defined for the velocity vectors. Each such function may be pictured by contour hyperplanes (contour lines in two dimensions, actual planes in three dimensions). A larger form has more closely spaced hyperplanes. The picture is in Figure 6.2. One extra complication is that it is necessary to indicate which direction is the upward direction. There is no need to introduce a special symbol; this can be done in words. (For instance, in this example it could be generally to the right.)

The pairing of vector field and differential 1-form is a scalar field. This is pictured in Figure 6.3. At each point the tip of the vector is on a certain contour of the linear function; the value of the scalar is value of the form at that contour.
Remark: Again this is a brief look toward later developments. As in the case of vector fields there is a concept of basis for the differential forms on a manifold

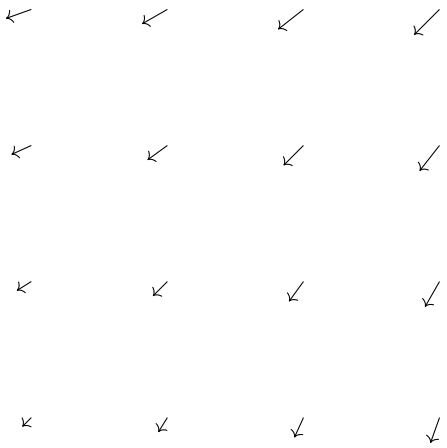

Figure 6.1: A vector field: grid plot

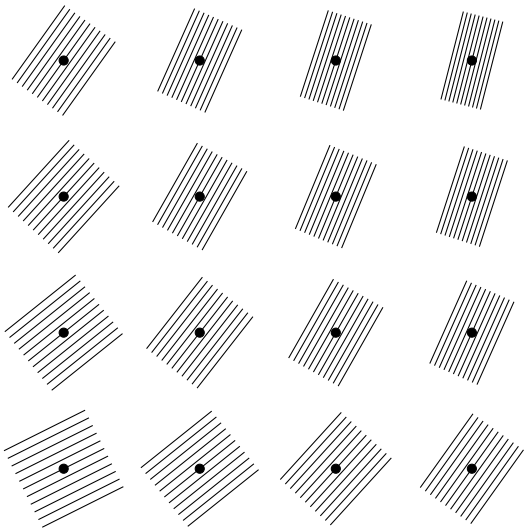

Figure 6.2: A differential form: grid plot

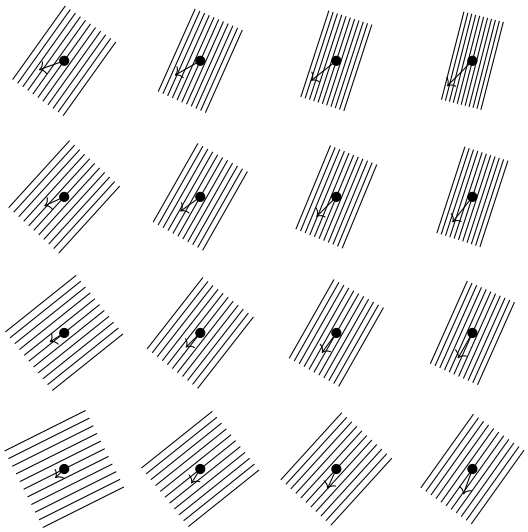

Figure 6.3: Differential 1-form paired with vector field gives scalar field

patch. If the coordinate system is $u_1, \ldots, u_n$, then the *coordinate basis forms* are $du_1, \ldots, du_n$. The general 1-form may be written

$$\omega = \sum_{k=1}^{n} q_k \, du_k. \tag{6.10}$$

Again in the special case of orthogonal coordinates there are *normalized basis forms*. They are of the form $h_1 \, du_1, \ldots, h_n \, du_n$. So the same 1-form is

$$\omega = \sum_{k=1}^{n} p_k \, h_k \, du_k. \tag{6.11}$$

The normalized basis forms are typically used to measure distances in the coordinate directions. There is more discussion of these issues in following chapters. ||

6.1.2 Differential of a scalar field

There is a very important way of constructing a differential 1-form from a scalar field s. This is called the *differential* ds of the scalar field. The definition is

$$\langle ds \mid X \rangle = X \, s. \tag{6.12}$$

This is perhaps the fundamental definition in the entire subject. It is easy to check that

$$\begin{aligned} dc &= 0 \\ d(r+s) &= dr + ds \\ d(rs) &= s\,dr + r\,ds. \end{aligned} \tag{6.13}$$

Here c is a constant. These equations are equations for differential 1-forms. The last two are the sum rule and product rule for differentials. The product identity comes from $X\,(rs) = (X\,r)s + r(X\,s)$ and the definitions.

Remark: Since $X \lrcorner \alpha = \langle \alpha \mid X \rangle$, another notation for how a vector field differentiates a scalar field is

$$X \lrcorner ds = \langle ds \mid X \rangle. \tag{6.14}$$

This notation (or something close to it) is more common in many contexts than the $X\ s$ notation. In fact, the latter chapters of this book abandon the use of $X\ s$ in favor of the $X \lrcorner ds$ that more clearly indicates that a differentiation is being performed. ||

A differential 1-form of the form ds is said to be *exact*. There is a picture of ds that comes from the picture of s. The scalar field s is pictured in terms of its level sets, which are its contour hypersurfaces. For $n = 2$ these are contour lines; for $n = 3$ these are ordinary surfaces. These are the hypersurfaces on which s has constant values. Giving these values is part of the specification of s. Close to a particular point μ in M, these contour surfaces resemble contour hyperplanes passing through the point. These hyperplanes are the level sets for a real linear function on the tangent space at μ. The idea is that the action of the differential form ds on tangent vectors at μ is a limiting case in which the contour hypersurfaces become the contour hyperplanes of an affine function. When picturing ds in terms of contour hypersurfaces only the differences of the values on different hypersurfaces are important.

There is an important difference between the grid plot picture picture of an exact form and the contour curve picture of an exact form. In the grid plot the contour lines or planes represent a certain specified change in velocity. However in the picture in terms of contour curves or surfaces the change is in space. As we shall see, this is accomplished by integration.

Take

$$X = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}. \tag{6.15}$$

If we write out the definition explicitly, we get

$$\langle ds \mid X \rangle = \sum_{j=1}^{n} a_j \frac{\partial s}{\partial x_j}. \tag{6.16}$$

If we apply this to the scalar $s = x_i$ that is one of the coordinates, then we get

$$\langle dx_i \mid X \rangle = a_i. \tag{6.17}$$

It follows that

$$\langle ds \mid X\rangle = \sum_{j=1}^{n} \frac{\partial s}{\partial x_j}\langle dx_j \mid X\rangle = \left\langle \sum_{j=1}^{n} \frac{\partial s}{\partial x_j}\, dx_j \mid X \right\rangle. \tag{6.18}$$

Proposition 6.4 (Coordinate expression of differential) *The differential ds is given by*

$$ds = \sum_{j=1}^{n} \frac{\partial s}{\partial x_j}\, dx_j. \tag{6.19}$$

This applies in every coordinate system.

Remark: There is a dramatic difference between vector field coordinate bases and 1-form coordinate bases. The notation $\partial/\partial z$ does not make sense unless z is a variable that belongs to a given coordinate system. For instance, if the coordinate system is q, z, s, then $\partial/\partial z$ means to differentiate with respect to z holding q, s both constant. On the other hand, a differential dy makes sense for an arbitrary scalar field y, whether or not it belongs to a coordinate system. ||
Example: Consider parabolic coordinates given by $x = (1/2)(u^2 - v^2)$, $y = uv$. The coordinate basis 1-forms are given by

$$\begin{aligned} dx &= u\, du - v\, dv \\ dy &= v\, du + u\, dv. \end{aligned} \tag{6.20}$$

The inverse relation is

$$\begin{aligned} du &= \frac{1}{u^2+v^2}\,(u\, dx + v\, dy) \\ dv &= \frac{1}{u^2+v^2}\,(-v\, dx + u\, dy)\,. \end{aligned} \tag{6.21}$$

These parabolic coordinates are just another complicated way of describing the Euclidean plane. ||
Example: Consider spherical polar coordinates r, θ, ϕ given by

$$\begin{aligned} x &= r\cos(\phi)\sin(\theta) \\ y &= r\sin(\phi)\sin(\theta) \\ z &= r\cos(\theta) \end{aligned} \tag{6.22}$$

This is another description of the three dimensional Euclidean space. The convention is the one that is most common in physics and engineering. (In mathematics it is common to reverse the roles of θ and ϕ.) The angle ϕ is longitude with $0 < \phi < 2\pi$. The angle θ is co-latitude, with $0 < \theta < \pi$. (The latitude is $\pi - \theta$.) The differentials are

$$\begin{aligned} dx &= \cos(\phi)\sin(\theta)\, dr + r\cos(\phi)\cos(\theta)\, d\theta - r\sin(\phi)\sin(\theta)\, d\phi \\ dy &= \sin(\phi)\sin(\theta)\, dr + r\sin(\phi)\cos(\theta)\, d\theta + r\cos(\phi)\sin(\theta)\, d\phi \\ dz &= \cos(\theta)\, dr - r\sin(\theta)\, d\theta \end{aligned} \tag{6.23}$$

The normalized basis forms in spherical polar coordinates are dr, $r\,d\theta$, and $r\sin(\theta)\,d\phi$. ||

Example: Consider a two-dimensional sphere S of radius a with coordinates θ, ϕ. Write x_S, y_S, z_S for the restrictions of x, y, z to S. Then

$$\begin{aligned} x_S &= a\cos(\phi)\sin(\theta) \\ y_S &= a\sin(\phi)\sin(\theta) \\ z_S &= a\cos(\theta) \end{aligned} \tag{6.24}$$

The sphere S is not Euclidean space. However there is no problem with differential form calculations. The differentials are

$$\begin{aligned} dx_S &= a\cos(\phi)\cos(\theta)\,d\theta - a\sin(\phi)\sin(\theta)\,d\phi \\ dy_S &= a\sin(\phi)\cos(\theta)\,d\theta + a\cos(\phi)\sin(\theta)\,d\phi \\ dz_S &= -a\sin(\theta)\,d\theta \end{aligned} \tag{6.25}$$

The normalized basis forms are $a\,d\theta$, and $a\sin(\theta)\,d\phi$. For people living on the surface of a planet, this is an important example. ||

Example: Here is an application. Consider the problem of making a box with a given amount A of material to contain the maximum volume. The box will have five sides: a base and four vertical sides. It is open at the top. In this case the manifold patch is the set M_A of possible boxes made with this material.

Say that the side lengths of the base are u, v and the height is w. The amount of material available is a fixed number A. Thus $uv + 2uw + 2vw = A$. Since A is a constant, we have

$$(v + 2w)\,du + (u + 2w)\,dv + 2(u + v)\,dw = 0. \tag{6.26}$$

This relation is valid on all of M_A. We are interested in the point of M_A (that is, in the particular shape of box) with the property that the volume $V = uvw$ is maximized. At this point we have

$$dV = vw\,du + uw\,dv + uv\,dw = 0. \tag{6.27}$$

Now it is time to choose a coordinate system to work with, and it is convenient to choose u, v. Thus we eliminate dw from the system. Multiplication by $2(u + v)/(uv)$ gives

$$(2\frac{vw}{u} + 2w)\,du + (2\frac{uw}{v} + 2w)\,dv + 2(u + v)\,dw = 0 \tag{6.28}$$

at the point. Subtracting the equations gives

$$v(1 - 2\frac{w}{u})\,du + u(1 - 2\frac{w}{v})\,dv = 0 \tag{6.29}$$

at the point. Since u, v is a coordinate system, the coefficients must be zero at the point. This gives $w = u/2$ and $w = v/2$ as the dimensions of the box. The

box with maximal volume has a square base and a height half the length of each side of the base. ||

Example: Here is a more elegant solution of the box problem using the method of Lagrange. Let M be the 3-dimensional manifold patch of all open boxes. The problem is to maximized the volume V with the area A fixed. At the optimum box shape the only way to change V is to relax the constraint that A is fixed. Thus

$$dV = \lambda\, dA. \tag{6.30}$$

This says that

$$vw\, du + uw\, dv + uv\, dw = \lambda((v + 2w)\, du + (u + 2w)\, dv + (2u + 2v)\, dw). \tag{6.31}$$

This gives three expressions for $1/\lambda$. Solving gives $2w = u = v$. ||

6.1.3 Vectors and 1-forms in Euclidean space

In Euclidean space there is a mapping $\mathbf{g}$ from vector fields to 1-forms given in Cartesian coordinates by

$$\mathbf{g}\left(a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n}\right) = a_1\, dx_1 + \cdots + a_n\, dx_n. \tag{6.32}$$

It might seem that there is no need to distinguish vectors from 1-forms. This is not true for two reasons. First: the manifold patch may not be Euclidean space. Second: Even if the manifold patch is Euclidean space, it may be natural to use coordinates that are not Cartesian. In that case the expression is more complicated.

Example: For the case of parabolic coordinates the calculation is easy. The result is that $\mathbf{g}(\frac{\partial}{\partial u}) = (u^2 + v^2)\, du$ and $\mathbf{g}(\frac{\partial}{\partial v}) = (u^2 + v^2)\, dv$. This may be seen by converting to Cartesian coordinates, applying the simple recipe above, and then converting back to parabolic coordinates. There is an extra factor. Even though parabolic coordinates are coordinates on Euclidean space, they are not Cartesian coordinates. ||

For Euclidean space there are several important operations. The *gradient* of a scalar field is defined for Cartesian coordinates by

$$\operatorname{grad} s = \mathbf{g}^{-1}\, ds = \frac{\partial s}{\partial x_1}\frac{\partial}{\partial x_1} + \cdots + \frac{\partial s}{\partial x_n}\frac{\partial}{\partial x_n}. \tag{6.33}$$

Thus a scalar field on Euclidean space defines a vector field. A solution curve corresponding to this vector field has the property that the scalar increases with time along this curve. (This can be seen by applying grad s to s itself. The result is a sum of squares.) Unfortunately, the expression for the gradient field becomes more complicated in coordinates that are not Cartesian. Some of these formulas will appear in later chapters.

Another important operation for Euclidean space is the *divergence*. This is defined for Cartesian coordinates by

$$\operatorname{div}\left(a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n}\right) = \frac{\partial a_1}{\partial x_1} + \cdots + \frac{\partial a_n}{\partial x_n}. \tag{6.34}$$

In other coordinates it is not so simple. This operation will also be studied in more detail in later chapters.

A final operation of great importance is the Laplacian

$$\nabla^2 s = \text{div grad } s = \frac{\partial^2 s}{\partial x_1^2} + \cdots + \frac{\partial^2 s}{\partial x_n^2}. \tag{6.35}$$

6.1.4 Matrix notation*

It is not very common to combine these ideas with matrix notation, but it may be helpful to make an attempt. The basic quantities are the row vector

$$\frac{\partial}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \end{bmatrix} \tag{6.36}$$

and the column vector

$$d\mathbf{x} = \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}. \tag{6.37}$$

If

$$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}. \tag{6.38}$$

is a column vector of scalar functions, then

$$\frac{\partial}{\partial \mathbf{x}} \triangleright \mathbf{a} = a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n} \tag{6.39}$$

is the corresponding vector field. The special symbol $\triangleright$ is supposed to be a reminder that the components of the column vector appear before the differential operators. (This is not a standard notation.) It may help to realize that when this vector field is applied to the scalar field s, the result of row times column is

$$\frac{\partial s}{\partial \mathbf{x}} \mathbf{a} = a_1 \frac{\partial s}{\partial x_1} + \cdots + a_n \frac{\partial s}{\partial x_n}. \tag{6.40}$$

Similarly, if

$$\mathbf{p} = \begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix} \tag{6.41}$$

is a row vector of scalar functions, then $\mathbf{p}\, d\mathbf{x}$ is a differential form. If s is a scalar field, then ds is $(\frac{\partial}{\partial \mathbf{x}} s)\, d\mathbf{x}$ with components in the row vector $\frac{\partial}{\partial \mathbf{x}} s$. In summary, components of vectors go in columns, while components of covectors go in rows.

There is also a matrix $\frac{\partial}{\partial \mathbf{x}}^T \mathbf{p}$. If the differential form is exact, then this matrix is symmetric. In particular the Hessian matrix $\frac{\partial}{\partial \mathbf{x}}^T \frac{\partial}{\partial \mathbf{x}} s$ is symmetric.

If the coordinates are Cartesian, then grad s is $\frac{\partial}{\partial \mathbf{x}} \triangleright (\frac{\partial}{\partial \mathbf{x}} s)^T$ with components in the column vector $(\frac{\partial}{\partial \mathbf{x}} s)^T$. Furthermore, suppose the coordinates are Cartesian and $X = \frac{\partial}{\partial \mathbf{x}} \triangleright \mathbf{a}$. Then the divergence of X is $\frac{\partial}{\partial \mathbf{x}} \mathbf{a}$. Finally, in Cartesian coordinates the Laplacian of s is $\frac{\partial}{\partial \mathbf{x}} (\frac{\partial}{\partial \mathbf{x}} s)^T$.

This matrix notation need not be taken very seriously, but does make the connection with earlier material. In particular, the components of vector fields are column vectors, and the components of differential 1-forms are row vectors.
Remark: There is another closely related notation, in which an operator $\overrightarrow{\nabla}$ (called nabla or del) is defined by $\overrightarrow{\nabla} = \frac{\partial}{\partial \mathbf{x}}^T$. Then $\overrightarrow{\nabla} s$ gives the components of the gradient vector. In particular, $\overrightarrow{\nabla}^T \overrightarrow{\nabla} s = \frac{\partial}{\partial \mathbf{x}} \frac{\partial}{\partial \mathbf{x}}^T s$ is the Laplacian of s. Similarly, $\overrightarrow{\nabla} \overrightarrow{\nabla}^T s = \frac{\partial}{\partial \mathbf{x}}^T \frac{\partial}{\partial \mathbf{x}} s$ is the Hessian of s. This notation may give comfort to those who like their partial derivatives to appear in a column vector. But there is not always a good reason to want this. ||

6.1.5 Closed and exact 1-forms

The 1-form ω is said to be *exact* in an open set if $\omega = ds$ for some scalar field s defined on that open set. It is also sometimes called an *exact differential*. The 1-form $\omega = \sum_j p_j \, dx_j$ is said to be *closed* if for each $j \neq k$ we have

$$\frac{\partial p_j}{\partial x_k} = \frac{\partial p_k}{\partial x_j}. \tag{6.42}$$

These are $\binom{n}{2}$ conditions. They are preserved under change of coordinates. The following theorem is both trivial and supremely important.

Theorem 6.5 *If ω is an exact 1-form in some open set, then ω is also a closed form.*

In dimension $n = 2$ it is possible to picture a differential 1-form that is not exact. The idea is to draw many fragments of contour curves that are each allowed to end in a point. The cloud of end points is located in the part of the region where the form is not exact.
Example: Take an example when $n = 2$. A simple example of a differential 1-form that is not exact is $y \, dx$. It vanishes on lines where x is constant. The fragmented contour lines are all vertical. The form indicates increase to the right in the upper half plane and increase to the left in the lower half plane. As the x axis is approached the density of these fragmented contour lines must diminish at constant rate. Some of the lines have end points at their lower ends (in the upper half plane) or at their upper ends (in the lower half plane). ||

The general rule for finding the contour curves representing $p \, dx + q \, dy$ is to solve the nonlinear ordinary differential equation

$$\frac{dy}{dx} = -\frac{p}{q}. \tag{6.43}$$

The uphill directions across the curves is given by vectors $a\partial/\partial x + b\partial/\partial y$ with $pa + qb > 0$. When the form is exact the relevant curves are the entire family of solution curves. When the form is not exact, endpoints are encountered where $\partial q/\partial x - \partial p/\partial y \neq 0$

Example: Consider the differential form $-2v\,du + u\,dv$. The contour curves are where

$$\frac{dv}{v} = 2\frac{du}{u}, \tag{6.44}$$

The solutions are $v = Cu^2$. These are parabolas.

To see which arrows $a\partial/\partial u + b\partial/\partial v$ point uphill, try taking a arbitrary and $b = 0$. Then the condition is $-2av < 0$, or $av < 0$, which means that these arrows are in the direction of decreasing u where $v > 0$ and in the direction of increasing u where $v < 0$. (One can also try $a = 0$, resulting in the condition $bu > 0$.) A look at the picture shows that one can go around the origin in a closed curve continuously going uphill.

The value of the difference of partial derivatives is $\partial u/\partial u + 2\partial v/\partial v = 3$, indicating that these curves have end points throughout. One can think of each curve as starting at some arbitrary point and continuing outward along a parabola. In some sense the density of the parabolas is increasing as one moves further and further from the origin. This indicates that the total amount of uphill travel around a closed curve surrounding the origin will increase as one moves away from the origin. The explanation for this is that a larger curve encloses more end points. ||

6.1.6 Differential 1-forms at a point

A *differential 1-form at a point* is a real linear function defined on the n-dimensional vector space of vectors at the point. Thus if η is a differential form at point μ, and Y is a vector at μ, then $\langle \eta \mid Y \rangle$ is a constant. The differential 1-forms at a particular point form an n-dimensional vector space. This the *cotangent space*.

If ω is a differential form defined on M, then there is a *restriction to a point* $\omega_{[\mu]}$. If $\omega = ds$, and Y is a vector at μ, then this is $\langle ds_{[\mu]} \mid Y \rangle = Y\,s$. If $\omega = p\,ds$, then this is $p_\mu ds_{[\mu]}$. If ω is a sum of forms, then the restriction of the sum is the sum of the restrictions.

Proposition 6.6 (Coordinate expression for restriction of 1-form to a point) *Let μ be a point in M with $\mathbf{x}$ coordinate value $\mathbf{c} = \mathbf{x}(\mu)$, so $\mathbf{c}$ is a constant. Suppose that*

$$\omega = \sum_{j=1}^{n} h_i(\mathbf{x})\,dx_i. \tag{6.45}$$

The restriction $\omega_{[\mu]}$ to μ is given in coordinate system $\mathbf{x}$ by

$$\omega_{[\mu]} = \sum_{j=1}^{n} h_j(\mathbf{c})\,dx_{i\,[\mu]}. \tag{6.46}$$

In other words, the restriction to a point μ amounts to making the coefficients have the appropriate constant values and measuring coordinate changes only for vectors at μ.

In many calculations an exact differential such as ds is thought of as a small change in s. This can be made precise. Consider a point μ and a vector field X and a particular time parameter t. Let ϕ be the path determined by X that passes through μ at time zero. Let Δs be the difference in values of s along the path ϕ between time zero and time t. Then

$$\Delta s = \langle ds \mid tX \rangle + r(t)t. \tag{6.47}$$

Here $r(t) \to 0$ as $t \to 0$. This means that calculations involving ds give corresponding results for changes Δs, up to a small error.

6.1.7 Picturing differential 1-forms

The focus will be on pictures of 1-forms in two dimensions. The higher dimensional story has extra complications.

There are two basic classes of 1-forms. An exact 1-form ω is given by $\omega = ds$ for some scalar field. We shall see that the fundamental theorem of calculus implies that the integral of such a form around a closed curve is always zero.

The other kind of form is less pleasant but more common. Green's theorem (which will be studied in a later chapter) says that the integral around a closed curve is

$$\int_C p\,dx + q\,dy = \int_R \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) dx\,dy. \tag{6.48}$$

Here C is the oriented boundary of the region R. This result is valid in an arbitrary coordinate system. The integrand on the right hand side of Green's theorem vanishes when ω is an exact form. However when the form is not exact it can give a non-zero result.

One can picture a form either through a grid plot or via curves. We have already seen the grid plot picture. The form attaches numbers to velocity vectors. The contour lines for a given form are parallel straight lines. The other picture is in terms of curves obtained by integrating the ordinary differential equation $p\,dx + q\,dy = 0$. These are curves in space. Again it is necessary to give some indication (perhaps in words) of which direction is upward. This is in effect assigning a transverse orientation to the curves (that is, directions across the curves).

For an exact 1-form the curves look like the contour curves of a scalar field. They are not quite the contour curves of a scalar field, since there are no numbers attached to the contours. On the other hand, for two contours there is a number that represents the change going from one to the other. In the picture one needs to indicate the direction of increase. This can be done in words. For instance, in Figure 6.4 the upward direction could be generally upward and to the left on the page. For an exact form the integral around a closed curve is always zero. This is because the contribution from places where the curve crosses contours

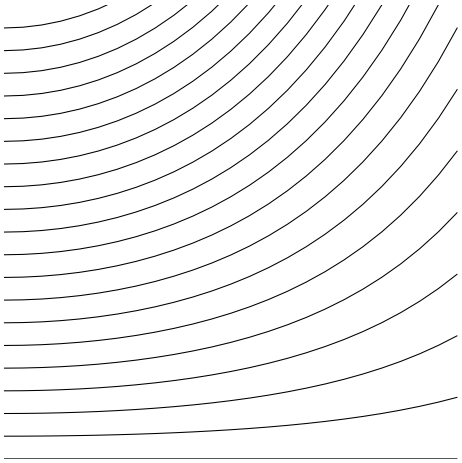

Figure 6.4: An exact 1-form in 2 dimensions

in an upward direction cancels that from places where the curve crosses the contour in a downward direction.

For a form that is not exact the picture is somewhat different: the curves must have end points. The density of these end points is specified by planar integrals of the kind appearing in Green's theorem. Figure 6.5 presents a curve plot for a form that is not exact. The picture needs to be supplemented by an indication of what is the upward direction. For this case it might be taken to be generally upward and to the left on the page. Thus each curve has a transverse orientation.

Also, each end point has an associated orientation (clockwise or counterclockwise). Since the orientation of a point is orinarily just a ± 1 sign, this two-dimensional orientation of an end point should actually be regarded as a transverse orientation of the point. The general rule is that the transverse orientation of an end point (clockwise or counterclockwise) is the inward orientation of the point followed by the transverse orientation of the curve. The end point is a hinge on which the curve wants to rotate in the upward direction. See Figure 6.6 for a typical curve with end points. For the 1-form illustrated in Figure 6.5 (transverse orientation to the upper left) the end points will then have a counterclockwise orientation.

The integral of 1-form around a closed curve is generally not equal to zero. There is more travel up than down (or the other way), and so the contributions do not cancel. In fact, the lack of cancellation is explained by the density of points in the interior of the curve, together with their (clockwise or counterclockwise) orientations. This is the intuitive content of Green's theorem.

Example: Consider the exact form

$$x\,dx + y\,dy = \frac{1}{2}\,d(x^2 + y^2) = \frac{1}{2}dr^2 = r\,dr. \tag{6.49}$$

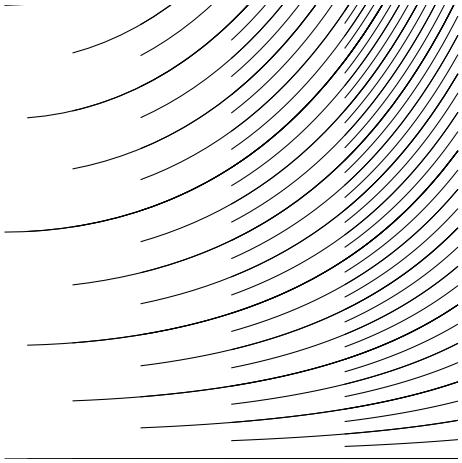

Figure 6.5: A 1-form in 2 dimensions with non-zero circulation

The contour curves are circles. ||
Example: Consider the exact form

$$x\,dx - y\,dy = \frac{1}{2}(x^2 - y^2). \tag{6.50}$$

The contour curves are hyperbolas. ||
Example: Point source. Consider the form

$$\omega = \frac{-y\,dx + x\,dy}{x^2 + y^2} = d\theta. \tag{6.51}$$

The contour curves are lines from the origin. This form is exact in any region for which the angle can be consistently defined. The integral over every closed oriented curve that does not surround the origin is zero. The origin acts as a (counterclockwise) point source that makes this form not exact in a global sense. An integral over a closed oriented curve that surrounds the origin can have a non-zero value. We can say that the form is locally exact but not globally exact. ||
Example: Extended source everywhere. Consider the form

$$-y\,dx + x\,dy = r^2 d\theta. \tag{6.52}$$

The contour curves are lines whose end point are a scattering of points throughout the plane. This form is very far from being exact. One can think of the entire plane as an extended source of non-exactness. ||
Example: Extended source in a disk. Consider the form

$$\alpha = \min(r^2, 1)\,d\theta. \tag{6.53}$$

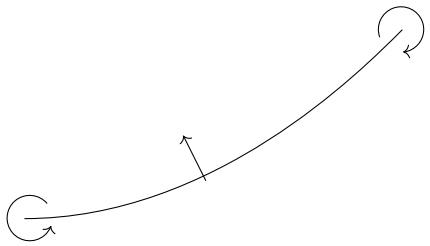

Figure 6.6: 2d: 1-form flux density curve orientation and end-point orientations

The contour curves are lines whose end points are a scattering of points throughout the disk of radius 1 about the origin. This is also an example of an extended source of non-exactness, but the source is confined to this disk. Outside this disk the form is locally exact. ||

Example: Opposite point sources. Let $a > 0$. Consider the form

$$\omega_- = \frac{-y\,dx + (x+a)\,dy}{(x+a)^2 + y^2} - \frac{-y\,dx + (x-a)\,dy}{(x-a)^2 + y^2}. \tag{6.54}$$

This is locally exact except at two points. The contour curves are complicated. There are two point sources, one at $-a$ and one at a, with opposite transverse orientations. This means that there are curves connecting one counterclockwise oriented point on the left to the other clockwise oriented point on the right. Between them the increase in upward. In fact, on the line where $x = 0$, the form ω_- restricts to $2a\,dy/(a^2 + y^2)$. ||

Example: Equal point sources. Let $a > 0$. Consider the form

$$\omega_+ = \frac{-y\,dx + (x+a)\,dy}{(x+a)^2 + y^2} + \frac{-y\,dx + (x-a)\,dy}{(x-a)^2 + y^2}. \tag{6.55}$$

This is locally exact except at two points. There are two point sources, one at $-a$ and one at a, with the same counterclockwise orientation. Each curve from either source remains in its own half plane. On the line where $x = 0$, the form ω_+ restricts to $-2y\,dx/(a^2 + y^2)$. Thus this line is one of the curves; it separates the two regions. At great distances this form looks like $2\,d\theta$. ||

6.1.8 Polar coordinates

Polar coordinates provide a useful example of these ideas. Consider the manifold patch M that is the Euclidean plane with Cartesian coordinates x and y. These are expressed in polar coordinates by

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta). \end{aligned} \tag{6.56}$$

The basic calculations of differentials are

$$\begin{aligned} dx &= \cos(\theta)\, dr - \sin(\theta) r\, d\theta \\ dy &= \sin(\theta)\, dr + \cos(\theta) r\, d\theta. \end{aligned} \tag{6.57}$$

The differentials are naturally expressed in terms of normalized basis forms dr and $r\, d\theta$. The inverse relation is

$$\begin{aligned} dr &= \frac{x}{r}\, dx + \frac{y}{r}\, dy \\ r\, d\theta &= -\frac{y}{r}\, dx + \frac{x}{r}\, dy. \end{aligned} \tag{6.58}$$

The corresponding equations for vector fields are

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} \\ \frac{1}{r}\frac{\partial}{\partial \theta} &= -\frac{y}{r}\frac{\partial}{\partial x} + \frac{x}{r}\frac{\partial}{\partial y} \end{aligned} \tag{6.59}$$

with inverse relation

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos(\theta)\frac{\partial}{\partial r} - \sin(\theta)\frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \sin(\theta)\frac{\partial}{\partial r} + \cos(\theta)\frac{1}{r}\frac{\partial}{\partial \theta}. \end{aligned} \tag{6.60}$$

In what sense are these relations valid? Consider the manifold patch M that is the Euclidean plane. The points in the planes are not numbers; they are geometrical objects. Let $M^\bullet$ be M with a point removed. This is the *punctured plane*. The coordinates x, y are chosen to vanish at this point. Let $M^\dagger$ be $M^\bullet$ after removal of a half-line that starts at the missing point. This is the *cut plane*. These relations are certainly valid in the cut plane $M^\dagger$. The values of r range in $0 < r$. The values of θ jump by 2π across the half-line.

Suppose that the line where $x \le 0$ and $y = 0$ is the half-line in the definition of $M^\dagger$. Define open subsets U and V of $\mathbf{R}^2$ by taking $U = \{(a, b) \mid a > 0, -\pi < b < \pi\}$ and $V = \mathbf{R}^2 \setminus \{(p, q) \mid p \le 0, q = 0\}$. Then the coordinate system (x, y) maps $M^\dagger$ to V, and the coordinate system (r, θ) maps $M^\dagger$ to U. The change of coordinates is a smooth one-to-one function $\mathbf{f}$ from U to V. The result is that $(x, y) = \mathbf{f}(r, \theta)$ as functions from $M^\dagger$ to V. The two coordinate systems provide two numerical descriptions of the same object $M^\dagger$.

Define the differential form

$$\omega^\dagger = \frac{x\, dy - y\, dx}{x^2 + y^2} = d\theta \tag{6.61}$$

on the cut plane $M^\dagger$. This is an exact form.

The form on the cut plane is not a particularly natural object. Instead, define the *angle form*

$$\omega^\bullet = \frac{x\, dy - y\, dx}{x^2 + y^2} \tag{6.62}$$

the punctured plane $M^\bullet$. This is a closed form on $M^\bullet$. The price to pay is that it is not an exact form on the punctured plane $M^\bullet$. The interpretation of this form is that near every point in $M^\bullet$ it represents angular change. In fact, locally we can always define an angle variable such that $\omega^\bullet = d\chi$. But there is no such angle variable defined on all of $M^\bullet$.

Consider the vector fields

$$\begin{aligned} E &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\ R &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}. \end{aligned} \tag{6.63}$$

These are both defined on the plane M. The vector field E is sometimes called the *Euler vector field*; it plays a role in the study of homogeneous functions. It can also be called the *radial vector field* or *position vector field*. The vector field R is the *rotation vector field*. It describes rotations of constant angular frequency. The origin is a zero of the vector field R. Rotations of the origin leave it fixed.

If we restrict the vector fields E and R to the punctured plane $E^\bullet$, then they can be locally straightened out. In fact this is true for arbitrary non-zero vector fields. For differential forms the condition for local exactness is that the form be a closed form. There is no such restriction for vector fields.

For the Euler operator we can take $r = e^t$ and get $E = r\partial/\partial r = \partial/\partial t$ in all of $E^\bullet$. For the rotation operator the straightening out is only local. For every point in $E^\bullet$ we can choose an angular variable χ that is defined near the point, and then $R = \partial/\partial\chi$ near the point.

The differential equation corresponding to E is

$$\begin{aligned} \frac{dx}{dt} &= x \\ \frac{dy}{dt} &= y \end{aligned} \tag{6.64}$$

with solution given by the substitution $(x, y) \leftarrow (x_0 e^t, y_0 e^t)$. In polar coordinates the equation is

$$\begin{aligned} \frac{dr}{dt} &= r \\ \frac{d\chi}{dt} &= 0. \end{aligned} \tag{6.65}$$

The solution is $(r, \theta) \leftarrow (r_0 e^t, \chi_0)$. These are the same solution.

The differential equation corresponding to R is

$$\begin{aligned} \frac{dx}{dt} &= -y \\ \frac{dy}{dt} &= x \end{aligned} \tag{6.66}$$

with solution given by the substitution $(x, y) \leftarrow (x_0 \cos(t) - y_0 \sin(t), x_0 \sin(t) + y_0 \cos(t))$. In polar coordinates the equation is

$$\begin{aligned} \frac{dr}{dt} &= 0 \\ \frac{d\chi}{dt} &= 1. \end{aligned} \tag{6.67}$$

The solution is $(r, \chi) \leftarrow (r_0, \chi_0 + t)$.

This discussion shows that it is useful to think of Cartesian coordinates and polar coordinates as scalars defined on a manifold patch. Then the equations above are identities, either for scalars or for differential forms or for vector fields. The closed differential form $\omega^\bullet$ defined on the punctured plane is a fundamental mathematical object; for instance it underlies many of the calculations in complex variable theory. The rotation operator R has an associated system of differential equations. These are the equations for a linear oscillator, a basic system that occurs throughout applied mathematics. These examples also illustrate that differential forms and vector fields are quite different objects.

The example in this section is not typical in one respect: there are natural notions of length and angle. Thus in Cartesian coordinates dx and dy are orthogonal unit forms. In polar coordinates dr and $r\,d\theta$ are orthogonal unit forms. (Note: The form $r\,d\theta$ is not a closed form.) There is a similar story for vector fields. In Cartesian coordinates $\partial/\partial x$ and $\partial/\partial y$ are orthogonal unit vectors. In polar coordinates $\partial/\partial r$ and $1/r\,\partial/\partial\theta$ are orthogonal unit vectors. The reason for this is that the underlying manifold M is the Euclidean plane, which has natural notions of length and angle. Such special structure need not be present in other manifolds.

Figure 6.7 gives a picture of the coordinate basis radial vector field $\partial/\partial r$. This only varies in the angular direction. Figure 6.8 give a picture of the coordinate basis angular vector field $\partial/\partial\theta$. Figure 6.9 gives the corresponding normalized basis angular vector field $(1/r)\partial/\partial\theta$. This normalized vector field has the advantage that it only varies in the angular direction, and its variation is purely radial.

These statements about the variation of vector fields in various directions make sense because the underlying space is Euclidean. In this case there is a notion of *covariant derivative*. The covariant derivatives of the normalzed basis vector fields are given by

$$\begin{aligned} \frac{\partial}{\partial r}\,\frac{\partial}{\partial r} &= 0 \\ \frac{\partial}{\partial \theta}\,\frac{\partial}{\partial r} &= \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial r}\,\frac{1}{r}\frac{\partial}{\partial \theta} &= 0 \\ \frac{\partial}{\partial \theta}\,\frac{1}{r}\frac{\partial}{\partial \theta} &= -\frac{\partial}{\partial r}. \end{aligned} \tag{6.68}$$

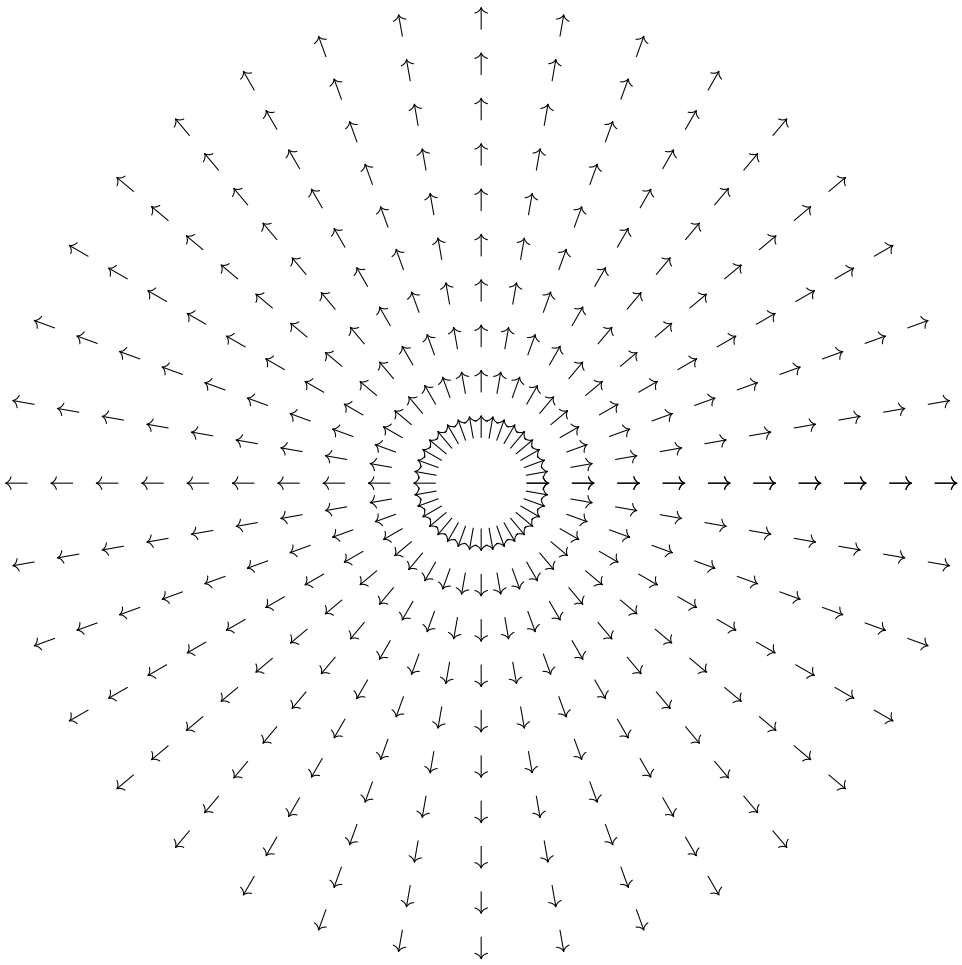

Figure 6.7: The vector field $\frac{\partial}{\partial r}$.

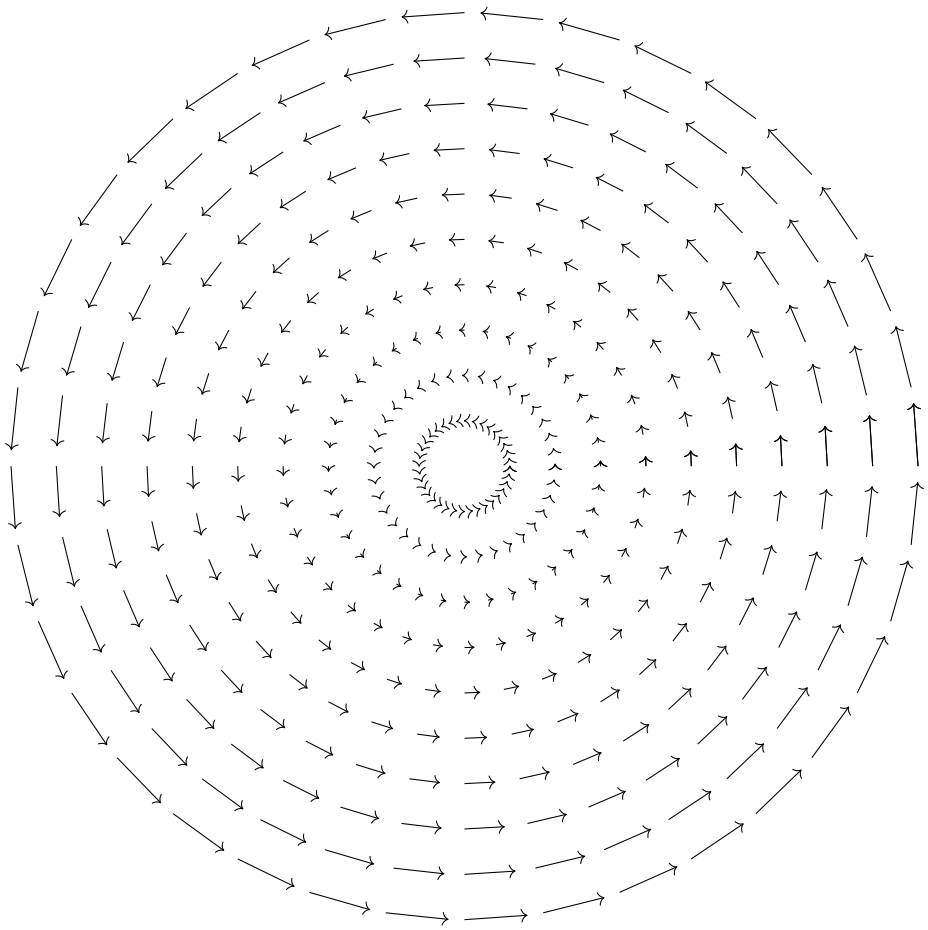

Figure 6.8: The vector field $\frac{\partial}{\partial \theta}$.

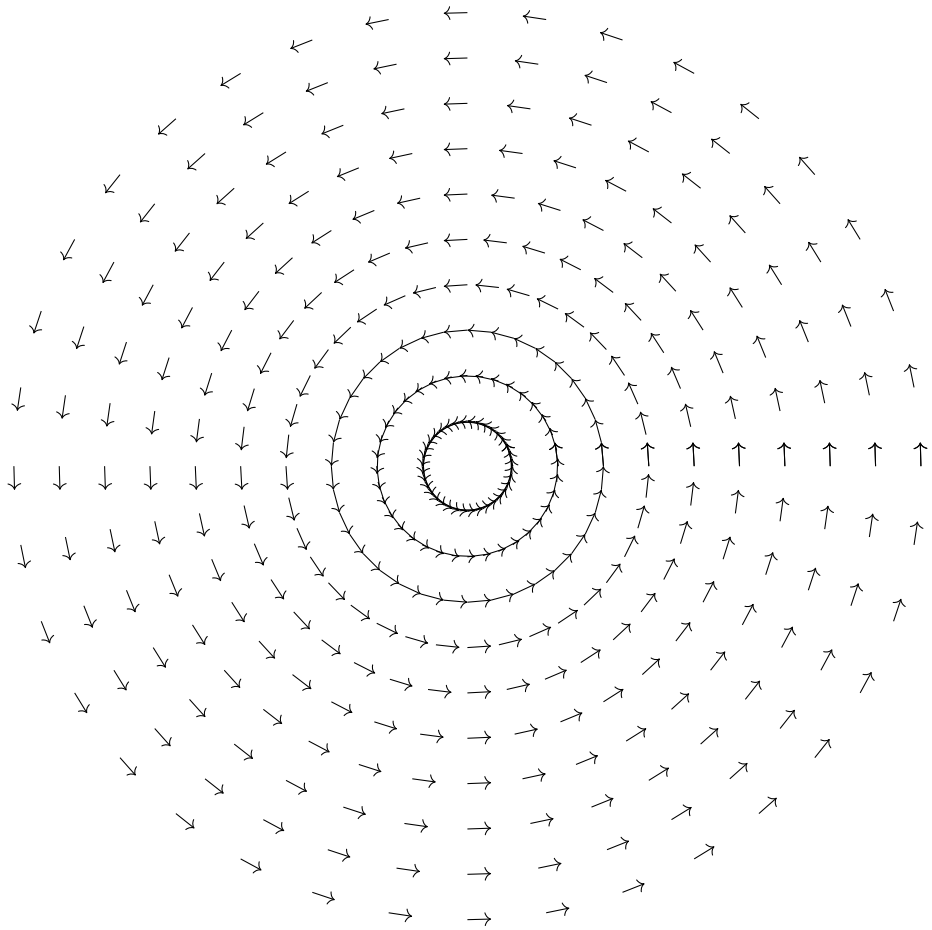

Figure 6.9: The vector field $\frac{1}{r}\frac{\partial}{\partial\theta}$.

Similarly, Figure 6.10 gives a picture of the coordinate basis form dr. It varies only in the angular direction, and its variation is purely angular. Figure 6.11 gives a picture of the coordinate basis form $d\theta$. Figure 6.12 gives the corresponding normalized form $r\,d\theta$. Notice that it is not exact. However it has the advantage that it varies only in the angular direction, in the sense that the spacing between contours (if properly drawn) should not vary with the radius. Its variation is purely radial. In summary, the covariant derivatives of the normalized basis vectors are given by

$$\begin{aligned}
\frac{\partial}{\partial r}\, dr &= 0 \\
\frac{\partial}{\partial \theta}\, dr &= r\, d\theta \\
\frac{\partial}{\partial r} r\, d\theta &= 0 \\
\frac{\partial}{\partial \theta} r\, d\theta &= -dr.
\end{aligned} \tag{6.69}$$

6.1.9 Differential equations and integrating factors*

Everything in this subsection will be in two-dimensions and local. An ordinary differential equation may be written in the form

$$f(x,y)\frac{dx}{du} + g(x,y)\frac{dy}{du} = 0. \tag{6.70}$$

The variable u is often taken to be x. In some instances it may be appropriate to take u to be y. Such a differential equation would seem to be related to differential forms. This section explores the connection. In particular it explains how solving such an equation is equivalent to finding an integrating factor for the form.

Proposition 6.7 (Straightening out for exact 1-forms) *Let M be a two-dimensional manifold patch. Suppose that at some point in M the 1-form ds is not zero. Then locally there is a coodinate system u, v such that $v = s$. Thus $ds = dv$ has constant coefficients 0,1 in this coordinate system.*

Proof: Write $ds = \frac{\partial s}{\partial x}\, dx + \frac{\partial s}{\partial y}\, dy$. One of the coefficients must be non-zero. Suppose it is $\frac{\partial s}{\partial y}$. Define $u = x$ and $v = s = g(x, y)$. This defines u, v in terms of x, y. Then the Jacobian matrix of this change of variables has determinant $\partial s/\partial y \neq 0$. So by the inverse function theorem x, y may be expressed in terms of u, v. □

Suppose that M is a 2-dimensional manifold patch. A coordinate system u, v always defines a *family of regular curves*. These are the curves where v has a constant value. The value of v defines the curve, while the value of u describes points within a given curve.

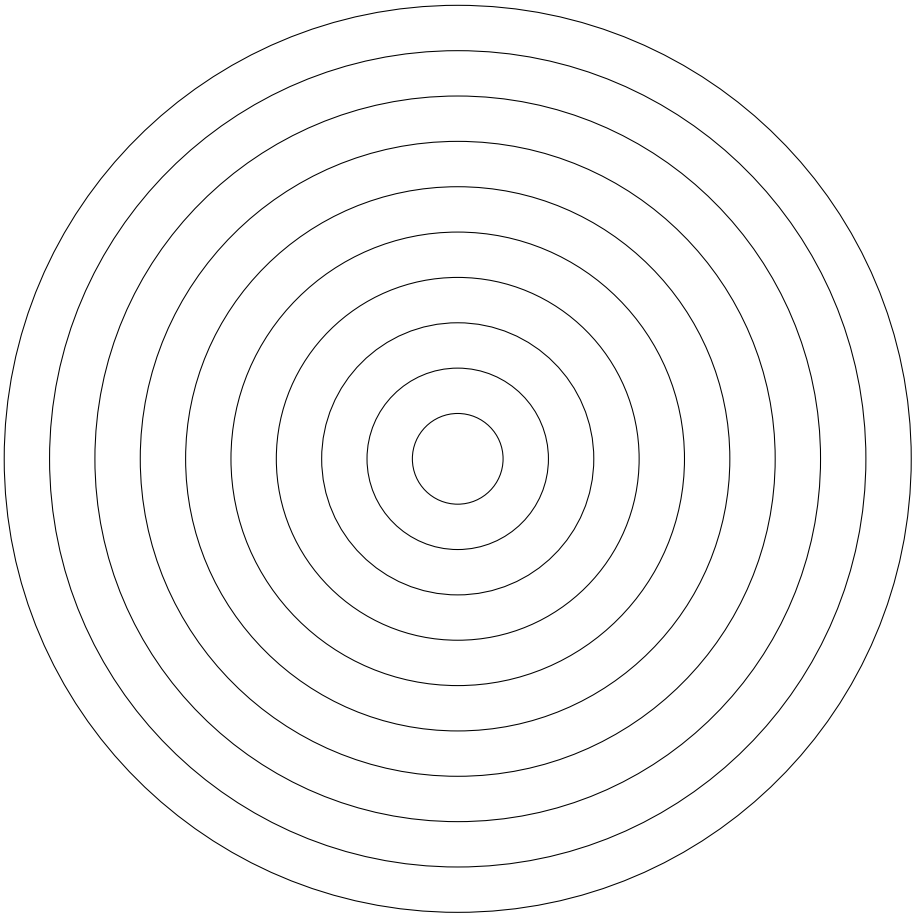

Figure 6.10: The 1-form dr.

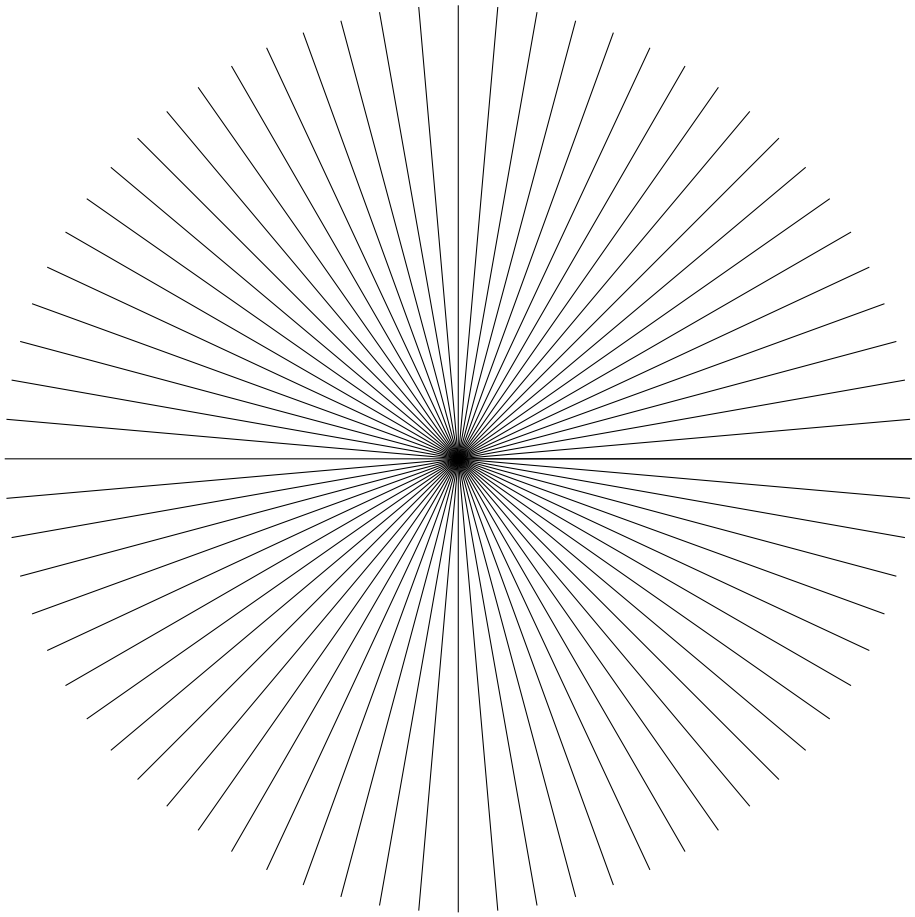

Figure 6.11: The 1-form $d\theta$.

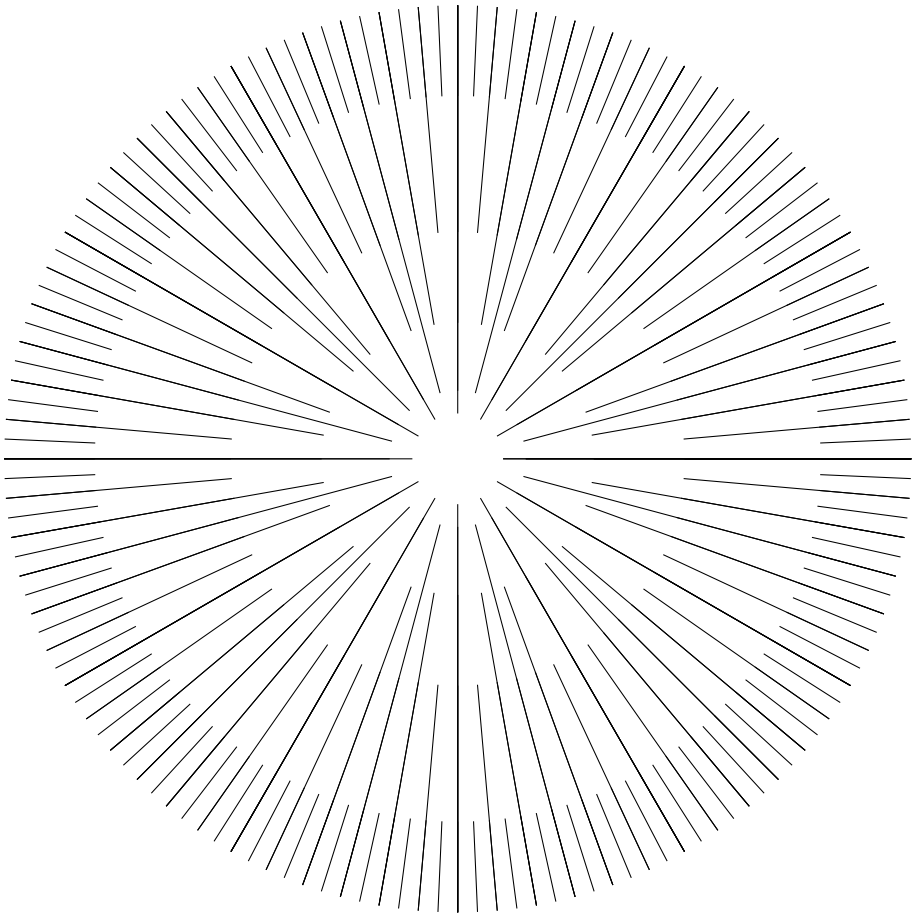

Figure 6.12: The 1-form $r\,d\theta$.

A *differential equation* on M is specified by giving a non-zero differential form α. The equation is written

$$\alpha = p\,dx + q\,dy = 0. \tag{6.71}$$

Here $p = f(x, y)$ and $q = g(x, y)$ are functions of x, y. A *general solution* of such an equation is a family of regular curves such that

$$p\frac{\partial x}{\partial u} + q\frac{\partial y}{\partial u} = 0. \tag{6.72}$$

An individual solution corresponds to a fixed value of v. The variable u then parametrizes the soluton. The reason this is written with partial derivatives is to emphasis that the constant of integration v is held constant.

A non-zero differential form α is said to have an *integrating factor* if there are non-zero scalar fields r and s such that $r\alpha = ds$.

Theorem 6.8 (Solutions and integrating factors) *Suppose that $\alpha = p\,dx + q\,dy$ is a non-zero 1-form and that there is a family of regular curves that satisfies the equation $\alpha = 0$. Let u, v be a coordinate system for which the curves are given by constant values of v. Then there is a non-zero scalar field r such that*

$$r\alpha = dv. \tag{6.73}$$

Proof: The form α may be written as

$$\alpha = \left(p\frac{\partial x}{\partial u} + q\frac{\partial y}{\partial u}\right)\,du + \left(p\frac{\partial x}{\partial v} + q\frac{\partial y}{\partial v}\right)\,dv. \tag{6.74}$$

If u, v provide a general solution, then

$$\alpha = \left(p\frac{\partial x}{\partial v} + q\frac{\partial y}{\partial v}\right)\,dv = w\,dv. \tag{6.75}$$

It suffices to take $r = 1/w$. □

Theorem 6.9 (Existence of solutions) *Suppose that $\alpha = p\,dx + q\,dy$ is a differential form in two dimensions that is non-zero near some point. Then there is a family of regular curves near the point that satisfies the differential equation. It follows that α has a non-zero integrating factor r near the point, so $r\alpha = dv$ for some scalar field v.*

Proof: This is where it is important that the dimension is two. In that case there is a a non-zero vector field $X = q\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}$ with $\langle \alpha \mid X \rangle = 0$. By the straightening out theorem, there is a new coordinate system u, v such that $X = \frac{\partial}{\partial u}$. The other coordinate v then defines the family of regular curves.

The integrating factor may be computed as follows. The change of coordinates implies that that $\frac{\partial x}{\partial u} = q$ and $\frac{\partial y}{\partial u} = -p$. It is easy to check that the differential equation is satisfied. As a consequence

$$\alpha = p\,dx + q\,dy = \left(p\frac{\partial x}{\partial v} + q\frac{\partial y}{\partial v}\right)\,dv = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\,dv = w\,dv, \tag{6.76}$$

where w is a non-zero scalar. This computation even shows that this scalar w is given by a determinant associated with the change of coordinates. We can take $r = 1/w$. $\square$

Here is the procedure for solving ordinary differential equations in the plane.

- The equation is determined by the differential form $\alpha = p\,dx + q\,dy$, but two different forms may determine equivalent equations. For example, if $r = h(x, y)$ is a non-zero scalar, then the form $r\alpha$ is a quite different form, but it determines an equivalent differential equation.

- If $\alpha = p\,dx + q\,dy$ is exact, then $p\,dx + q\,dy = ds$, for some scalar s depending on x and y. Each solution of the differential equation is then given implicitly by $s = c$, where c is the constant of integration.

- If $\alpha = p\,dx + q\,dy$ is not exact, then one looks for an integrating factor r such that
$$r\alpha = r(p\,dx + q\,dy) = ds \tag{6.77}$$
is exact. Once this is done, again the general solution of the differential equation is then given implicitly by $s = c$, where c is the constant of integration.

Finding an explicit integrating factor may be no easy matter. However, there is a strategy that may be helpful. Recall that if a differential form is exact, then it is closed. So if r is an integrating factor, then

$$\frac{\partial rp}{\partial y} - \frac{\partial rq}{\partial x} = 0. \tag{6.78}$$

This condition may be written in the form

$$p\frac{\partial r}{\partial y} - q\frac{\partial r}{\partial x} + \left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right) r = 0. \tag{6.79}$$

Say that by good fortune there is an integrating factor r that depends only on x. Then this gives a linear ordinary differential equation for r that may be solved by integration.

Example: Consider the standard problem of solving the linear differential equation

$$\frac{dy}{dx} = -ay + b, \tag{6.80}$$

where a, b are functions of x. Consider the differential form $(ay-b)\,dx+dy$. Look for an integrating factor r that depends only on x. The differential equation for r is $-dr/dx = ar$. This has solution $r = e^A$, where A is a function of x with $dA/dx = a$. Thus

$$e^A(ay - b)\,dx + e^A\,dy = d(e^A y - S), \tag{6.81}$$

where S is a function of x with $dS/dx = e^A b$. So the solution of the equation is $y = e^{-A}(S + c)$. $||$

Example: There are many other special tricks for finding integrating factors. These work in various special solutions. For example, suppose that the differential form is $\omega = p\,dx + q\,dy$ where p/q is a function of y/x. (This is the case for a homogeneous differential form.) Make the change of coordinates $x = x$ and $y = vx$, so p/q is a function of v. Then the form is

$$\omega = (p + qv)\,dx + qx\,dv = x(p + qv)(\frac{1}{x}\,dx + \frac{1}{\frac{p}{q} + v}\,dv). \tag{6.82}$$

This displays an integrating factor for the form.

As an example, take

$$\omega = (4xy^2 - y^3)\,dx + (x^3 + 8x^2y)\,dy. \tag{6.83}$$

All the terms are homogeneous of third order. So the ratio is zero order, and it is easy to write it as a function of y/x. The calculation from then on is straightforward, up to the point where there are some unpleasant integrals.

The same differential form may be written in more than one coordinate system. This example illustrates how useful this can be. ||

6.1.10 Canonical forms*

The theory of differential forms is extraordinarily different from the theory of vector fields. A nonzero vector field may always be straightened out locally. For differential forms this is only possible if the form is closed (and hence locally exact).

Theorem 6.10 *Consider a differential form $\alpha = p\,dx + q\,dy$ in two dimensions. Suppose that near some point α is not zero. Then*

- *If α is closed near this point, then there is a scalar field z with $\alpha = dz$.*
- *If α is not closed, then there there is a new coordinate system w, v with $\alpha = w\,dv$.*

Proof: Since $\alpha = p\,dx + q\,dy$ is not zero, there is a new coordinate system u, v in which it has the form $\alpha = w\,dv$. In this coordinate system the condition that α is a closed form is that $\frac{\partial w}{\partial u} = 0$.

If α is closed, then w is a function of v. Thus $\alpha = w\,dv$ has an integral z that is a function of v.

If α is not closed, then the matrix that expresses the partial derivatives of w, v in terms of u, v is non-singular. By the inverse function theorem w, u is also a coordinate system. □

A theorem of Darboux gives a list of standard representations of non-zero 1-forms in higher dimensions. The differential equations book by Ince [21] treats the three dimensional situation. The treatise by Şuhubi [48] gives a full discussion for n dimensions. There is another nice account in the book by Olver [38]. Yet another treatment is in the work by Bryant and others [5].

Here is what can happen with a non-zero 1-form in dimensions greater than two. In three dimensions coordinates can be found so that the form is either $\omega = dx$ or $\omega = x\,dy$ or $\omega = x\,dy + dz$. In the third case there is no integrating factor. In four dimensions it can get even worse: there can be forms like $x\,dy + z\,dw$. In cases where there is no integrating factor it can be difficult to get a clear geometrical picture of what the form looks like. Differential 1-forms are more complicated than vector fields. What happens near a zero is even more complicated; the book by Olver [38] gives references. Among these are [29] and [54]. There also is a particulary nice survey by Golubitsky and Tischler [17]. See a later section of the present work for the case of a form in two dimensions vanishing at a point.

6.2 The integral of a 1-form along a path

6.2.1 Integral along a path

Let M be an n-dimensional manifold patch. Let P be a one-dimensional manifold patch with a coordinate t. (This coordinate maps to an interval (a, b) of real numbers, where $= \infty \leq a < b \leq +\infty$.) A *parametrized curve* ϕ is a smooth function from P to M. Suppose that $\mathbf{x}$ is a coordinate system for M. Then a parametrized curve may be denoted $\mathbf{x} \leftarrow \mathbf{h}(t)$. This means take the point in P, find its t value, apply the function $\mathbf{h}$ to get a new value $\mathbf{h}(t)$, and then return the point in M where $\mathbf{x}$ has the value $\mathbf{h}(\mathbf{t})$.

The curve is generally thought of as being inserted in M. It could be a simple curve, like those made by the letter C or the letter S. In general the curve can intersect itself, as in the numeral 8. There is also the possiblity of almost touching, as shown by the numeral 6, where the one end of the curve has a completely different part of the curve as a limit.

Suppose that ω is a 1-form on M. Consider a parametrized curve. Then there is another 1-form $\phi^*\omega$ on P called the *pullback* of ω by ϕ. If $\omega = \sum_j g_j(\mathbf{x})\,dx_j$ is a 1-form on M, and the parametrized curve ϕ is given by $\mathbf{x} \leftarrow \mathbf{h}(t)$, then the pullback $\phi^*\omega$ is defined by

$$\phi^*\omega = \sum_j g_j(\mathbf{x} \circ \phi) \frac{d(x_j \circ \phi)}{dt}\,dt. \tag{6.84}$$

This can be written more explicitly as

$$\phi^*\omega = \sum_j g_j(\mathbf{h}(t)) \frac{dh_j(t)}{dt}\,dt = \sum_j g_j(\mathbf{h}(t)) h_j'(t)\,dt. \tag{6.85}$$

Let I be a compact interval in P. Then there is a start point in I with the smallest value of t and an end point with the largest value of t. A *path* is a parametrized curve restricted to such an interval. The image of the start point (end point) in I is the start point (end point) of the path in M.

Example: Here are some pullback examples. Suppose we have two variables x, y. The parametrized curve is ϕ equal to $(x, y) \leftarrow (t^2, t^3)$. The pullback of the exact form $d(xy^2) = y^2\,dx + 2xy\,dy$ using the right hand side is $t^6\,dt^2 + 2t^5\,dt^3 = 2t^7\,dt + 6t^7\,dt = 8t^7\,dt$. Using the left hand side we also get $dt^8 = 8t^7\,dt$.

For a form like $\omega = y\,dx + 2xy\,dy$ that is not exact, pulling it back by ϕ gives the result $2t^4\,dt + 6t^7\,dt = (2t^4 + 6t^7)\,dt = d(\frac{2}{5}t^5 + \frac{3}{4}t^8)$. Thus $\phi^*\omega$ is exact, though ω is not exact. The pullback is a non-trivial operation on differential forms. ||

The integral of ω along the curve ϕ is defined by

$$\int_\phi \omega = \int_I \phi^*\omega = \int_I \sum_j g_j(\mathbf{h}(t))h_j'(t)\,dt. \tag{6.86}$$

Proposition 6.11 *The integral of a differential 1-form ω over a path is the same for every parametrization with the same orientation.*

Proof: Suppose that $t = f(u)$, where $f'(u) > 0$. The t interval I corresponds to a u interval J. By the change of variable formula, the integral over I with the $\mathbf{x} \leftarrow \mathbf{h}(t)$ parametrization is equal to

$$\int_J \sum_j g_j(\mathbf{h}(f(u))h_j'(f(u))f'(u)\,du = \int_J \sum_j g_j(\mathbf{h}(f(u))\frac{dh_j(f(u))}{du}\,du. \tag{6.87}$$

The final expression on the right is the integral over J with the $\mathbf{x} \leftarrow \mathbf{h}(f(u))$ parametrization. So the two parameterizations give the same answer. □

Example: Consider the oriented curve C along $y = x^2$, running from the origin to the point where x and y are both 1. Then

$$\int_C y\,dx + 2xy\,dy = \int_0^1 t^2\,dt + 2t^3 d(t^2) = \frac{1}{3} + \frac{4}{5}. \tag{6.88}$$

The parametrization reduces this to a one-dimensional integral. ||

6.2.2 The fundamental theorem of calculus

The equivalent of integration for a scalar field is a special kind of summation. Consider a finite set of points μ in M, together with a non-zero integer k_μ for each point. If K is such a *signed point set*, the sum for scalar field s is then $\sum_K s = \sum_\mu k_\mu s(\mu)$.

A *path chain* is a finite sequence of paths. If C is a path chain, then the *boundary* ∂C is determined by the final points with plus signs together with the initial points with minus signs. This signed point set is called the *oriented boundary* of C.

Theorem 6.12 (Fundamental theorem of calculus) *Suppose that s is a scalar field on M with differential ds. Consider a path chain C in M with oriented boundary ∂C. Then the integral of ds over C depends only on ∂C, and*

$$\int_C ds = \sum_{\partial C} s. \tag{6.89}$$

Example: Consider the oriented curve C along $y = x^2$, running from the origin to the point where x and y are both 1. Then

$$\int_C y^2\,dx + 2xy\,dy = \int d(xy^2) = 1 - 0 = 1. \tag{6.90}$$

Since the form is exact this can be computed in the original variables; there is no need to introduce a parameter. ||

Example: Let $\omega = x\,dy$. Integrate from $(x, y) \leftarrow (0, 0)$ to $(x, y) \leftarrow (1, 1)$. The first path is up to $(x, y) \leftarrow (0, 1)$ and then over. The answer is 0. The second path is over to $(x, y) \leftarrow (1, 0)$ and then up. The answer is 1. The third path is the diagonal line segment. The integrand is $t\,dt$ and the answer is $1/2$. ||

Example: Let

$$\omega = \frac{x\,dx + y\,dy}{x^2 + y^2}. \tag{6.91}$$

This is exact in the region $M^\bullet$, a plane with one point removed. Let $r = \sqrt{x^2 + y^2}$. If $s = \log(r)$, then

$$ds = d\log(r) = \frac{dr}{r} = \frac{1}{2}\frac{dr^2}{r^2} = \omega. \tag{6.92}$$

It follows that for every path in $M^\bullet$

$$\int_\phi \omega = \log(r_+) - \log(r_-). \tag{6.93}$$

This is the nicest case. ||

Example: Let

$$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2}. \tag{6.94}$$

Restrict to the region $M^\dagger$ that is the complement of the half line where $x \leq 0$. Take θ between $-\pi$ and π. Then $x\,dy - y\,dx = r^2\,d\theta$, so

$$d\theta = \omega. \tag{6.95}$$

It follows that for every path in $M^\dagger$

$$\int_\phi \omega = \theta_+ - \theta_-. \tag{6.96}$$

The important complication here is that the form is not exact in the larger region that only excludes the origin. This is the fundamental fact behind most computations in complex variable, for example in residue theory. ||

There is a duality between the notions of integrating a vector field and integrating a differential 1-form. For a vector field X the idea is to find a path ϕ such that for every scalar field s we have $d(s \circ \phi)/dt = (X\,s) \circ \phi$. For a differential 1-form ω the idea is to find a scalar field s such that for every path

ϕ the integral of ω along ϕ only depends on the difference of the values of s at the two end points. This only works if $\omega = ds$ is exact.

In the vector field case the solution is a path $\phi : I \to M$ whose tangent vectors are given by the vector field. The constant of integration is the starting point of the path. In the differential 1-form case the solution is a scalar field $s : M \to \mathbf{R}$ whose contour surfaces determine the differential form. The constant of integration is the value of s at the initial point of the integration path.

One common interpretation of a differential form $\omega = p\,dx + q\,dy + r\,dz$ is in terms of *work*. The coefficients p, q, r represent *force*, and the dx, dy, dz are increments of distance. So ω is the increment of work. The integral along an oriented curve C represents the work done by the force from the beginning point to the end point.

If the form ω is exact, then $\omega = -ds$, where s is the *potential energy*. The minus sign in this definition is standard in physics. In that case the work integral along the oriented curve C from point 1 to point 2

$$\int_C \omega = s_1 - s_2 \tag{6.97}$$

is the difference of the potentials at the end points. The work is that performed by the force.

Example: A force acts in the x direction with work form $F\,dx$, where F is constant. The potential is $s = -Fx$. An object under the influence of this force is accelerated in the x direction. As its potential energy decreases, its kinetic energy increases. ||

Example: A fixed massive object at a point in three-dimensional space has gravitational potential energy proportional to $s = -1/r$. Here r is the distance from the object. The corresponding work form is

$$-ds = d\frac{1}{r} = -\frac{1}{r^2}\,dr. \tag{6.98}$$

So the force is $F = -1/r^2$. If another object is subject to this force, it loses potential energy as it is accelerated toward the fixed object. ||

6.2.3 Forms in one dimension

Consider a connected 1-dimensional manifold patch. Suppose that x is a scalar field (not necessarily a coordinate). Then $f(x)\,dx$ is a differential 1-form. Define

$$\int_{x\leftarrow c}^{x\leftarrow d} f(x)\,dx = \int_C f(x)\,dx, \tag{6.99}$$

where C is any oriented curve running from any point where x has the value c to any other point where x has the value d. Since $f(x)\,dx = dF(x)$ is exact, this is

$$\int_{x\leftarrow c}^{x\leftarrow d} f(x)\,dx = F(d) - F(c). \tag{6.100}$$

The reader will recongnize this as something very close to the classical fundamental theorem of calculus. This should be compared with the more general version discussed in the previous section.

Now suppose that $x = g(u)$, where u is another scalar. Then

$$f(x)\,dx = f(g(u))g'(u)\,du. \tag{6.101}$$

Furthermore, if $g(a) = c$ and $g(b) = d$, then

$$\int_{x\leftarrow c}^{x\leftarrow d} f(x)\,dx = \int_{u\leftarrow a}^{u\leftarrow b} f(g(u))g'(y)\,du. \tag{6.102}$$

This is because at every point where u has the value a or b the corresponding x has the value $c = g(a)$ or $d = g(b)$.

Conclusion: One way to think of change of variable in one dimension is passive. Nothing has changed; the same differential form is written in more than one way. This point of view is particularly appealing in applications that describe the same situation with different variables.

6.3 The first and second differential*

6.3.1 The first differential*

This section deals with the critical points of a scalar function. Topics include the coordinate invariant versions of the first derivative test and the second derivative test. The first derivative test involves first partial derivatives, that is, the differential. The second derivative test involves the Hessian matrix of second partial derivatives, which at a critical point gives a second differential.

Proposition 6.13 (Coordinate invariance of first derivative test) *Suppose that M is a manifold patch and z is a scalar field on M. If z has a local maximum or local minimum at a certain point, then at that point*

$$dz_{[\mu]} = \sum_{i=1}^{n} \frac{\partial z}{\partial x_i}(\mu)\,dx_{i\,[\mu]} = 0. \tag{6.103}$$

This condition may be expressed in any coordinate system.

6.3.2 The second differential*

For every scalar field s the first order differential ds is well-defined. In general the second differential is only defined when there is extra structure. However if ds is zero at a particular point, then d^2s is well-defined at that point. It is computed in a coordinate system as follows. Say that $s = h(\mathbf{x})$. Then $ds = \sum_j h'_{,j}(\mathbf{x})\,dx_j$. At the point where this vanishes,

$$d^2s = \sum_j \sum_k h''_{,jk}(\mathbf{x})\,dx_k\,dx_j. \tag{6.104}$$

In other words, the calculation is done by the product rule using $d^2x_j = 0$. The products $dx_k\,dx_j$ are taken to be symmetric, that is, $dx_j\,dx_k = dx_k\,dx_j$.

If X and Y are vectors at the point where ds is zero, then at that point d^2s defines a symmetric bilinear form $\langle X \mid d^2s \mid Y\rangle$ and a quadratic form $\langle X \mid d^2s \mid X\rangle$. The quadratic form defines a second order change in s. That is, if Δs is a change along the path defined by X starting from the point where ds is zero and running for time t, then

$$\Delta s = \frac{1}{2}\langle tX \mid d^2s \mid tX\rangle = r(t)t^2, \tag{6.105}$$

where $r(t) \to 0$ as $t \to 0$.

Remark: This use of the expression d^2s is not the same as the d^2s in the exterior differential calculus, in which $d^2s = 0$. The former is a description exhibiting symmetry, while that latter is a calculation method based on anti-symmetry. ||

Proposition 6.14 (Coordinate invariance of second derivative test) *Suppose that M is a manifold patch and z is a scalar field on M. Consider a point μ where $dz_{[\mu]} = 0$. Consider tangent vectors*

$$Y = \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i}{}_{[\mu]} \tag{6.106}$$

at μ. The symmetric Hessian matrix of second partial derivatives defines a corresponding quantities

$$\langle Y \mid d^2z_{[\mu]} \mid Y\rangle = \sum_{i=1}^{n}\sum_{\ell=1}^{n} \frac{\partial^2 z}{\partial x_i \partial x_\ell}(\mu) c_i c_j \tag{6.107}$$

This quadratic form may be denoted

$$d^2z_{[\mu]} = \sum_{i=1}^{n}\sum_{\ell=1}^{n} \frac{\partial^2 z}{\partial x_i \partial x_\ell}(\mu)\, dx_{i\,[\mu]}\, dx_{j\,[\mu]}. \tag{6.108}$$

If the quadratic form is strictly positive definite (strictly negative definite), then the function z has a local minimum (local maximum).

The computation that underlies these results begins with

$$\frac{\partial z}{\partial y_i} = \sum_{j=1}^{n} \frac{\partial z}{\partial x_j}\frac{\partial x_j}{\partial y_i}. \tag{6.109}$$

If we differentiate again, we get

$$\frac{\partial^2 z}{\partial y_i \partial y_k} = \sum_{j=1}^{n}\sum_{\ell=1}^{n} \frac{\partial^2 z}{\partial x_j \partial x_\ell}\frac{\partial x_j}{\partial y_i}\frac{\partial x_\ell}{\partial y_k} + \sum_{j=1}^{n} \frac{\partial z}{\partial x_j}\frac{\partial^2 x_j}{\partial y_i \partial y_k}. \tag{6.110}$$

The second derivative in the second term on the right is a rather complicated factor. But if the first derivatives $\partial z/\partial x_j = 0$ for $j = 1, \ldots, n$ at a certain point, then we are left with the Hessian matrix at this point transformed by the coordinate transformations on left and right. This is a matrix congruence, so it preserves the positive definite or negative definite property.

In the case of a function of two variables, there is a simple criterion for application of the second derivative test. Suppose that $z = h(x, y)$ is a smooth function. Consider a point where the first derivative test applies, that is, the differential $dz = d\,h(x, y)$ is zero. Consider the case when the Hessian is non-degenerate, that is, has determinant not equal to zero. Suppose first that the determinant of the Hessian matrix is strictly positive. Then the function has either a local minimum or a local maximum, depending on whether the trace is strictly positive or strictly negative. Alternatively, suppose that the determinant of the Hessian matrix is strictly negative. Then the function has a saddle point.

The case of n dimensions is more complicated. The Hessian matrix may be transformed by matrix congruence transformations to a diagonal matrix with entries ϵ_j that are $+1$, -1, or 0. In the non-degenerate case the entries are ± 1. If they are all $+1$ then we have a local minimum, while if they are all -1 we have a local maximum. Otherwise we have a saddle.

Example: Here is a simple practice problem. Suppose that $z = x^2 + 3y^2 + 2xy - 4x - 4y$. Then $dz = (2x + 2y - 4)\,dx + (6y + 2x - 4)\,dy$. This vanishes at the point $(x, y) \leftarrow (2, 0)$. The second differential at this point is $d^2z = 2(\,dx^2 + 2\,dx\,dy + 3\,dy^2)$. Its value on $X = a\partial/\partial x + b\partial/\partial y + c\partial/\partial z$ is $2(a^2 + 2ab + 3b^2)$. Since $a^2 + 2ab + 3b^2 = (a + b)^2 + 2b^2$, this is positive definite. This is a minimum point, and the value of z at this point is -4. ||

Example: Here is an example to show that the technique is independent of the coordinate system. Let $s = x^2 + y^2 - 6y$. Then $ds = 2x\,dx + (2y = 6)\,dy$ vanishes at $(x, y) \leftarrow (0, 3)$. The second differential at this point is $2(\,dx^2 + dy^2)$, indicating a minimum.

In polar coordinates $s = r^2 - 6r\sin(\theta)$. Then $ds = (2r - 6\sin(\theta))\,dr - 6r\cos(\theta)\,d\theta$. This vanishes at $(r, \theta) \leftarrow (3, \frac{\pi}{2})$. The second differential is $d^2s = 2\,dr^2 - 12\cos(\theta)\,dr\,d\theta + 6r\sin(\theta)\,d\theta^2$ which is $d^2s = 2(\,dr^2 + 9\,d\theta^2)$ at this point. This is the same result, again indicating the minimum. ||

There is a more powerful insight into these results that comes from changing to a new coordinate system. The first result states that away from a critical point nothing interesting happens.

Theorem 6.15 *Let $z = f(x_1, \ldots, x_n)$ be a smooth function on an n-dimensional manifold patch such that at a certain point $dz \neq 0$. Then there is a new coordinate system $u_1, \ldots, u_n$ near the point such that $z = u_1$.*

Proof: We may assume without loss of generality that $\partial z/\partial x_1 \neq 0$. Let $u_1 = z$ and let $u_j = x_j$ for $j = 2, \ldots, n$. Then the matrix of partial derivatives $\partial u_i/\partial x_j$ is non-singular. So by the inverse function theorem the x_j may be expressed in terms of the u_j. □

The next result says that even when the first derivative vanishes, there are common circumstances when there is nothing interesting going on with the second derivative. See Milnor [30] for a proof.

Theorem 6.16 (Morse lemma) *Let z be a smooth function on an n-dimensional manifold such that dz vanishes at a certain point. Let z_0 be the value of the function at that point. Suppose that the Hessian is non-degenerate at this point. Then there is a coordinate system $u_1, \ldots, u_n$ near the point with*

$$z = z_0 + \sum_{i=1}^{n} \epsilon_i u_i^2, \tag{6.111}$$

where ϵ_i are constants that each have the value ± 1.

6.4 Zeros of a differential form*

6.4.1 Invariants of a bilinear form*

A vector field $X = a\partial/\partial x + b\partial/\partial y$ generates a flow. The flow is particularly interesting at points where $a = b = 0$. Then there is a linearization with interesting invariants, including eigenvalues. Similarly, a differential form $\alpha = p\,dx + q\,dy$ will have invariants at a point where $p = q = 0$. The following discussion describes what heppens in this situation.

Consider a differential form $\alpha = p\,dx + q\,dy$ and a point where both p and q are zero. At that point there is a constant matrix

$$H = \begin{bmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} \end{bmatrix}. \tag{6.112}$$

A change of coordinates produces a new matrix, but this matrix is congruent to the original matrix. This is not the matrix of a linear transformation, as in the vector field case. It is the matrix of a bilinear form. This raises the question: Are there numerical invariants of such a matrix, perhaps analogous to or somehow related to eigenvalues?

It would first seem that the answer is no. Consider the case when $\alpha = ds$ is exact. In that case the matrix H is symmetric and is known as the Hessian matrix of s. The classification of symmetric matrices is well-known. The canonical forms are diagonal with entries $0, 0$ or $0, 1$ or $0, -1$ or $-1, -1$ or $-1, 1$ or $1, 1$. In the last three cases the matrix is non-singular, and the Hessian matrix provides the well-known test for whether s has a local minimum, a local maximum, or a saddle point.

In the general case congruence for 2 by 2 matrices is still a rather simple relation: the matrix H is replaced by $P^T H P$, where P is invertible. The following are the most convenient congruences.

- Multiply the matrix by a strictly positive real number $\epsilon > 0$

- Multiply the diagonal elements by reciprocal factors of ϵ and $1/\epsilon$, with $\epsilon > 0$.
- Change the signs of both off-diagonal elements.
- Interchange the diagonal elements and simultaneously exchange the off-diagonal elements.

The first three are accomplished by diagonal matrices P; the last one works if P is an interchange matrix.

Turn to the more interesting case where the matrix H is not symmetric. In this case there are interesting invariants under the congruence relation. The paper of Horn and Sergeichuk [20] gives a complete classification. See also [49] for the history of such results.

Since congruence is invariant under multiplication by a strictly positive real number, the representative matrices have to be normalized in some way. The result in the paper of Horn and Sergeichuk [20] is that each non-symmetric H is congruent to one of the following three matrices. The first case is

$$H_1 = \begin{bmatrix} 0 & 1 \\ a & 0, \end{bmatrix}, \tag{6.113}$$

where the real number $a \neq 1$ to avoid symmetry, and where for $a \neq 0$ the two values a and $1/a$ represent the same congruence class. In this case $-H_1$ is congruent to H_1. The second is

$$H_2 = \begin{bmatrix} c & -1 \\ 1 & c \end{bmatrix}, \tag{6.114}$$

with real $c \neq 0$. (The case $c = 0$ would give a matrix congruent to the first case with $a = -1$.) In this case $-H_2$ is congruent to H_2 with c replaced by $-c$. The third is exceptional with no parameters:

$$H_3 = \begin{bmatrix} 0 & -1 \\ 1 & \pm 1 \end{bmatrix}. \tag{6.115}$$

In this case $-H_3$ is congruent to H_3 with the opposite sign. The matrices H_1, H_2, H_3 are all invertible, except for the first case with $a = 0$.

One way to make sense of bilinear forms is to compare them to linear transformations. The linear transform matrices are converted to bilinear form matrices by interchanging rows and then multiplying the top row by -1. More explicitly, the conversion from linear transformations to bilinear forms is accomplished by left multiplication by the matrix

$$F = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{6.116}$$

Proposition 6.17 *Suppose that K and A are 2 by 2 real matrices with $FA = K$. Then*

$$P^T K P = \det(P) F P^{-1} A P. \tag{6.117}$$

Thus if two linear transformation matrices are similar via a transformation P with $\det(P) > 0$, then the corresponding bilinear form matrices are congruent.

Proof: If P is a non-singular 2 by 2 matrix, then $\det(P)P^{-1} = P^\sharp$, where $P^\sharp$ is the adjugate of P. Also $FP^\sharp = P^T F$. Thus $P^T KP = P^T FAP = FP^\sharp AP$. The result follows. □

If

$$H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}, \tag{6.118}$$

define its *non-symmetry* by $\mathrm{ns}(H) = h_{21} - h_{12}$.

Proposition 6.18 *If $FA = K$, then $\mathrm{tr}\, A = \mathrm{ns}(K)$.*

Corollary 6.19 $\mathrm{ns}(P^T KP) = \det(P)\mathrm{ns}(K)$.

Proof: Let $FA = K$. Then $\mathrm{ns}(P^T KP) = \det(P)F\,\mathrm{tr}(P^{-1}AP) = \mathrm{tr}(A) = \mathrm{ns}(K)$. □

Corollary 6.20 *Let*

$$\frac{1}{\mathrm{ns}(H)}H = FA. \tag{6.119}$$

Then

$$\frac{1}{\mathrm{ns}(P^T HP)}P^T HP = FP^{-1}AP. \tag{6.120}$$

Corollary 6.21 *The map*

$$H \mapsto F^{-1}\frac{1}{\mathrm{ns}(H)}H \tag{6.121}$$

sends bilinear form matrices to linear transformation matrices with trace 1. It sends bilinear form matrices H and $-H$ to the same linear transformation matrix. Furthermore, it sends sends congruence classes to similarity classes.

The invariants of a real matrix A under similarity are classified by Jordan form. In the case of 2 by 2 real matrices these are of three kinds. These are recorded here in the version that will be most useful for the comparison. The first case is that of real eigenvalues, where

$$A_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \tag{6.122}$$

where λ_1 and λ_2 are real. The eigenvalues may be interchanged by a similarity transformation with determinant having either sign (rotation or reflection). The second is that of complex eigenvalues $\lambda \pm i\mu$, where

$$A_2 = \begin{bmatrix} \lambda & -\mu \\ \mu & \lambda \end{bmatrix}, \tag{6.123}$$

with λ real and $\mu \neq 0$. The matrices with μ and $-\mu$ are similar; the matrix implementing the symmetry has negative determinant. The third is the exceptional Jordan form

$$A_3 = \begin{bmatrix} \lambda & \delta \\ 0 & \lambda \end{bmatrix}, \tag{6.124}$$

with λ real and $\delta \neq 0$. In this third case there is only one important number, since δ may be given any non-zero value by a change of coordinates. (The traditional choice is $\delta = 1$.) The matrices with δ and $-\delta$ are similar; the matrix implementing the symmetry has negative determinant.

For the purpose of classifying non-symmetric bilinear forms we take these matrices to have trace 1.

One can apply the transformation F to the Jordan form matrices. Two Jordan form linear transformation matrices that are similar by a negative determinent matrix need not map to congruent bilinar form matrices. The bilinear form matrices that are obtained are the following. First

$$K_1 = \begin{bmatrix} 0 & -\lambda_2 \\ \lambda_1 & 0 \end{bmatrix}. \tag{6.125}$$

Second,

$$K_2 = \begin{bmatrix} -\mu & -\lambda \\ \lambda & -\mu \end{bmatrix}. \tag{6.126}$$

with $\mu \neq 0$. As a bilinear form matrix the cases μ and $-\mu$ are distinct. Third,

$$K_3 = \begin{bmatrix} 0 & -\lambda \\ \lambda & \delta \end{bmatrix}, \tag{6.127}$$

with $\delta \neq 0$. As a bilinear form matrix the cases δ and $-\delta$ are distinct.

Theorem 6.22 *The above matrices K_1, K_2, K_3 normalized to have non-symmetry parameter 1 are canonical forms for non-symmetric biiinear forms .*

Remark: The matrices H_1, H_2, H_3 are the matrices K_1, K_2, K_3 normalized by division to have 1 in a corner entry. ||

6.4.2 Invariants of a differential form at a zero*

Suppose $\alpha = p\,dx + q\,dy$ is a differential form. If $\alpha = ds$ is exact, then the Morse lemma describes what happens at a point where $\alpha = 0$.

Otherwise consider the situation when α is not closed in some region but vanishes at a point, say the origin. Then its linearization is

$$\omega = p\,dx + q\,dy = (h_{11}x + h_{12}y)\,dx + (h_{21}x + h_{22}y)\,dy. \tag{6.128}$$

The corresponding vector field is

$$Y = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \tag{6.129}$$

The relation is that $au = q$, $bu = -p$, where $u = (\partial q/\partial x - \partial p/\partial y) \neq 0$. The condition of trace 1 is $\partial a/\partial x + \partial b/\partial y = 1$.

For the three cases above the canonical form may be taken to be

$$\omega_1 = y\,dx + ax\,dy \tag{6.130}$$

with $a \neq 1$ and

$$\omega_2 = (cx - y)\,dx + (x + cy)\,dy \tag{6.131}$$

with $c \neq 0$ and

$$\omega_3 = -y\,dx + (x \pm y)\,dy. \tag{6.132}$$

None of these forms are exact.

Example: These forms determine the solution curves for the corresponding vector field (but without the time dependence). For instance, in the first case the form has an integrating factor $1/(xy)$, so the equation for a curve is equivalent to

$$\frac{dx}{x} + a\frac{dy}{y} = 0. \tag{6.133}$$

This says that $xy^a = C$. ||

Remark: There is an excellent survey article by Golubitsky and Tischler [17] that puts such results in a broader context. In particular the results for 1-forms in 2-dimensions have analogs for $(n-1)$-forms in n dimensions. ||

6.5 Manifold mappings

6.5.1 Manifold mappings and parametrized surfaces

Now suppose that N is a manifold patch of dimension k and M is a manifold patch of dimension n. Suppose $\phi : N \to M$ is a smooth function. We shall call such a ϕ a *manifold mapping*. Sometimes we may just say *mapping*.

The notation $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u})$ is a way of defining the manifold mapping ϕ in terms of coordinate systems. Thus ϕ takes a point in N, reads the numbers $\mathbf{u}$, computes the numbers $\mathbf{f}(\mathbf{u})$, and then finds the point in M where $\mathbf{x}$ has this value. It can be read $\mathbf{x}$ "becomes" $\mathbf{f}(\mathbf{u})$. The left arrow may be called *replacement*. One can also say $\mathbf{x}$ "is replaced by" $\mathbf{f}(\mathbf{u})$.

If s is a scalar field on M, then its *pullback* by the manifold mapping ϕ is $s \circ \phi$. In order to think of this as an operation on s, one uses the notation

$$\phi^* s = s \circ \phi. \tag{6.134}$$

Suppose s is expressed in coordinates as $s = h(\mathbf{x})$. Then this is

$$(\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u}))^* h(\mathbf{x}) = h(\mathbf{x})(\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u})) = h(\mathbf{f}(\mathbf{u})). \tag{6.135}$$

Example: Consider the manifold mapping $(x, y) \leftarrow (u^2 + v, uv)$. The pullback of $s = x^2 y$ is $(u^2 + v)^2 uv = u^5 v + 2u^3 v^2 + uv^3$. ||

Remark: The manifold mapping $\phi = (\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u}))$ is the same as $\phi = \mathbf{x}^{-1} \circ \mathbf{f} \circ \mathbf{u}$. Why not write it this way? The reason is that practical computations seldom make explicit mention of the inverse of a coordinate mapping. On the other hand, the formula $h \circ \mathbf{x} \circ \phi = h \circ \mathbf{f} \circ \mathbf{u}$ makes no reference to an inverse. Furthermore, the idea expressed by this formula is often written in some form resembling $h(\mathbf{x})|_{\mathbf{x}=\mathbf{f}(\mathbf{u})} = h(\mathbf{f}(\mathbf{u}))$ or perhaps $h(\mathbf{x})(\mathbf{x} = \mathbf{f}(\mathbf{u})) = h(\mathbf{f}(\mathbf{u}))$. The problem with this is that the $=$ sign in $\mathbf{x} = \mathbf{f}(\mathbf{u})$ is not really an equal sign, but rather an instruction to replace $\mathbf{x}$ by $\mathbf{f}(\mathbf{u})$. For this the notation $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u})$ seems natural. We thus arrive at $h(\mathbf{x})(\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u})) = h(\mathbf{f}(\mathbf{u}))$. ||

There are various interpretations of a manifold mapping. If $k < n$, then the manifold mapping ϕ may be thought of as a *parametrized k-surface* in M. When $k = 1$ it is a *parametrized curve*. When $k = n - 1$ it is a *parametrized hypersurface*.

If $k > n$ then the manifold mapping may be thought of as a *implicit* $(k - n)$ *surface* in N. When $n = 1$ this is an *implicit hypersurface*.

When $k = n$ there are several possibilities. One is that N and M are different spaces. Then calculations are done the same way as for a parametrized surface.

When $k = n$ and $N = M$ there are two cases. In the case of a *passive tranformation* there are the same number of x_i coordinates as u_j coordinates, and the mapping $\mathbf{x} = \mathbf{f}(\mathbf{u})$ is just giving a different description of the same points. The transformation ϕ given by $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u})$ is the identity. All that is happening is that $\mathbf{x}$ coordinates are being expressed in terms of $\mathbf{u}$ coordinates. In this case it is appropriate to write $\mathbf{x} = \mathbf{f}(\mathbf{u})$ as an actual equality of functions on M.

The other case when $M = N$ is that of an *active transformation*. The transformation ϕ given by $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{x})$ makes sense. In that case the transformation can be iterated. This is equivalent to iterating the function $\mathbf{f}$, since $\mathbf{x}(\mathbf{x} \leftarrow \mathbf{f}(\mathbf{x}))^n = \mathbf{f}^n(\mathbf{x})$. The mapping $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{x})$ is not the same as the function $\mathbf{f}$. In fact, a common notation for the function $\mathbf{f}$ is $\mathbf{x} \mapsto \mathbf{f}(\mathbf{x})$ with the arrow going the other direction.

Remark: A manifold mapping ϕ defined by

$$\begin{aligned} x &\leftarrow u^2 + v \\ y &\leftarrow uv \\ z &\leftarrow u^2 v. \end{aligned} \tag{6.136}$$

has a clear meaning. The coordinates u, v are defined on the domain manifold patch. The coordinates x, y, z are defined on the target manifold patch. The left arrow is replacement. The mapping is a function ϕ that sends points in the domain to points in the target. It is another name for the same thing: $\phi = ((x, y, z) \leftarrow (u^2 + v, uv, u))$.

The same information is conveyed by the assertion

$$\begin{aligned} x \circ \phi &= u^2 + v \\ y \circ \phi &= uv \\ z \circ \phi &= u^2 v. \end{aligned} \tag{6.137}$$

Each side of each equation is a function from the domain to the real numbers. Shorter version: $(x, y, z) \circ \phi = (u^2 + v, uv, u)$. ||

Remark: Here are some perhaps dangerous variant notations. It is common to define the same object by

$$\begin{aligned} x &= u^2 + v \\ y &= uv \\ z &= u^2 v. \end{aligned} \tag{6.138}$$

In mathematics it is customary to think of the = sign as denoting equality. With this interpretation the equations are conditions on pairs of points taken from the domain and target. The pairs that satisfy these conditions are the graph of the mapping ϕ. (If the underlying manifold map is not of particular concern, then the equations could also be interpreted as defining a particular function from $\mathbf{R}^2$ to $\mathbf{R}^3$.)

One could also write

$$\begin{aligned} u^2 + v &= x \\ uv &= y \\ u^2 v &= z. \end{aligned} \tag{6.139}$$

Since equality = is symmetric this should be the same thing. The reason it looks odd is due to an ambiguity in mathematical notation where = often slips into denoting replacement. ||

Replacement is a natural operation; composition of replacements is straightforward. This book will recognize this and attempt to use the explicit replacement notation as much as possible. The equality notation may occasionally creep in.

Remark: The situation is different when the map ϕ is from a manifold patch to itself. The Hénon map is described in coordinates x, y by

$$\begin{aligned} x &\leftarrow 1 - ax^2 + y \\ y &\leftarrow bx. \end{aligned} \tag{6.140}$$

Even more succinctly: $\phi = ((x, y) \leftarrow (1 - ax^2 + y, bx))$.

Alternatively

$$\begin{aligned} x \circ \phi &= 1 - ax^2 + y \\ y \circ \phi &= bx. \end{aligned} \tag{6.141}$$

In this situation people may replace the above system by something like

$$\begin{aligned} x' &= 1 - ax^2 + y \\ y' &= bx. \end{aligned} \tag{6.142}$$

This makes sense as a way of defining the graph of ϕ. For a situation like this where the map has an inverse, the pair x', y' is another coordinate system. The

map ϕ is defined by giving two coordinate systems on the same manifold patch. ||

Remark: This book has attempted to maintain a distinction between scalar fields x, y, u, v, z, w (meaningful quantities in the application) and numerical functions f, g, h. This is partially abandoned by writers who introduce expressions such as

$$y = y(u, v) \tag{6.143}$$

to indicate that the quantity y is being regarded as a function of the u and v coordinate quantities. One way to make sense of this is as a declaration that in the future an expression such as $y(z + w, w^2)$ will be regarded as meaning the replacement $y((u, v) \leftarrow (z+w, w^2))$. The abbreviated notation $y(z+w, w^2)$ depends heavily on context.

The apparent motivation for this notation is a desire to express everything in terms of scalar fields. It avoids explicit mention of the numerical function f with

$$y = f(u, v) \tag{6.144}$$

and of other expressions involving it such as $f(z + w, w^2)$. However this can already be done with the replacement notation.

Notational confusion between scalar fields and numerical functions is an unnecessary sacrifice of clarity. ||

6.5.2 Manifold mappings and syntactic replacement

The pullback of $h(\mathbf{x})$ under $\mathbf{x} \leftarrow \mathbf{g}(\mathbf{u})$ is $h(\mathbf{g}(\mathbf{u}))$. This is expressed symbolically as

$$(\mathbf{x} \leftarrow \mathbf{g}(\mathbf{u})^* h(\mathbf{x}) = h(\mathbf{x})(\mathbf{x} \leftarrow \mathbf{g}(\mathbf{u})) = h(\mathbf{g}(\mathbf{u})). \tag{6.145}$$

The effect of a manifold mapping on a scalar field is to perform a *replacement* of a variable by an expression. This is also called the *substitution* of the expression for the variable. The $\mathbf{x} \leftarrow \mathbf{g}(\mathbf{u})$ may be thought as a mathematical function from one manifold patch to another. In computations it may also be regarded as a syntactic (symbolic) operation.

The replacement operation has other names. In mathematics it often occurs as a change of variables. In computer science there is a related notion called *assignment*. An assignment $x \leftarrow g(t)$ makes a change in the machine state. It takes the number stored under the label t, computes $g(t)$, and then stores this result under the label x.

Similar distinctions are made in various areas of algebra and elementary calculus. There are two common situations:

Numerical function A numerical function is a function that sends numbers to numbers. Examples are the sine function sin and the square root function $\sqrt{\ }$. The sine function may be written $u \mapsto \sin(u)$ or $w \mapsto \sin(w)$. The function that squares, adds one, and then takes the square root may be written $y \mapsto \sqrt{y^2 + 1}$ or $x \mapsto \sqrt{x^2 + 1}$. In such expressions the variables

are only place markers. In logic such a variable is called a *bound variable*.

The composition of numerical functions is again a numerical function. For instance, $(y \mapsto f(y)) \circ (u \mapsto g(u))$ is the same as $t \mapsto f(g(t)$, which itself is just the composition $f \circ g$.

One could imagine going entirely without variables. The notation $\sqrt{(\ )^2 + 1}$ makes sense, or one could use the convention of the computer language Mathematica and write $\sqrt{\#^2 + 1}$ for the same function. (Mathematic allows #1 and #2 and so forth for functions depending on several inputs. This may be less readable than notations involving variables.)

Replacement A replacement sends expressions to expressions. Expressions are built out of variables. In logic this corresponds to the notion of *free variable*. For instance, $\sin(z)$ and $\sin(t)$ are different expressions, because they have different free variables z and t. The most important operation is replacement of a free variable by an expression. An example is $u \leftarrow \sin(t)$. This means to substitute $\sin(t)$ for u. As an example, $u^2(u \leftarrow \sin(t)) = \sin^2(t)$. The general pattern is that $h(x)(x \leftarrow g(t)) = h(g(t))$.

Replacements may also be composed. The result is another replacement. Thus $(x \leftarrow g(t))(t \leftarrow f(u)) = (x \leftarrow f(g(u)))$.

Remark: There seems to be no standard notation in mathematics for the notion of replacement. The most common practice is to use an = sign. For instance, one might write

$$\frac{dg(f(t))}{dt} = \frac{dg(x)}{dx}|_{x=f(t)} \frac{df(t)}{dt}. \tag{6.146}$$

There are two = signs in the displayed equation; the first is actual equality (a symmetric relation) and the second is not. (It would look odd to write $f(t) = x$.)

There is a good case for reserving = for equality and using a symbol like $\leftarrow$ for replacement. Then the above equation could be written

$$\frac{dg(f(t))}{dt} = \frac{dg(x)}{dx}(x \leftarrow f(t)) \frac{df(t)}{dt}. \tag{6.147}$$

This is better both logically and typographically. ||

Remark: Computer languages like C use = for assignment and == for equality. An expression like `x = f(t)` means take the value of `t` from the machine, compute `f(t)`, and change the machine state so that `x` has that value. Similarly, `x = x + 1` means take the value of `x` from the machine, add 1, then change the machine state so that `x` has this new value. Example: If the computer language admits exact calculation, the commands `t = pi/4` followed by `u = sin(t)` followed by `u ∧ 2` gives an output `1/2`. After that `u ∧ 2 == 1/2` should give the output `True`.

The computer language R is more friendly to mathematics; it uses $< -$ for assignment. In R the same example would take the form `t` $< -$ `pi/4` followed by `u` $< -$ `sin(t)` followed by `u ∧ 2`. ||

Remark: There is another way to think of the similarities and differences between $x \mapsto f(x)$ and $x \leftarrow f(x)$. The variables x is a coordinate on a manifold patch M. The meaning of $x \mapsto f(x)$ is $f(x) \circ x^{-1}$. The meaning of $x \leftarrow f(x)$ is $x^{-1} \circ f(x)$. The composition is in the opposite order.

The first is independent of the choice of coordinate system, since $(x \mapsto f(x)) = f \circ x \circ x^{-1} = f$. The second is a manifold mapping that is also independent of the coordinate system. However in a new coordinate system it takes the form $(x \leftarrow f(x)) = (u \leftarrow g(u))$ with a different g. (If $u = h \circ x$, then $g = h \circ f \circ h^{-1}$.) ||

6.5.3 Pullback of a differential 1-form

The pullback $\phi^* s$ of a scalar field s has been defined as $\phi^* s = s \circ \phi$. Define the pullback of an exact 1-form du by

$$\phi^* \, du = d\phi^* u. \tag{6.148}$$

The pullback of a product $s \, du$ is the product of the pullbacks. Also, the pullback of a sum of such products is the sum of the pullbacks of the individual products. This leads to the following formula for the *pullback* of a differential 1-form.

Proposition 6.23 (Coordinate expression of pullback of a 1-form) *Say that the 1-form ω is given in coordinates by*

$$\omega = \sum_{j=1}^{n} p_j \, dx_j. \tag{6.149}$$

The pullback by the manifold map $\phi : N \to M$ is

$$\phi^* \omega = \sum_{j=1}^{k} \left(\sum_{i=1}^{n} (p_i \circ \phi) \frac{\partial (x_i \circ \phi)}{\partial u_j} \right) du_j. \tag{6.150}$$

Say that $p_i = h_i(\mathbf{x})$ and that ϕ is given in coordinates by $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u})$. Then in coordinates the pullback is

$$\phi^* \omega = \sum_{j=1}^{k} \left(\sum_{i=1}^{n} h_i(\mathbf{f}(\mathbf{u})) f'_{i,j}(\mathbf{u}) \right) du_j. \tag{6.151}$$

The $h_i(\mathbf{f}(\mathbf{u}))$ are the components of a row vector, and $f'_{i,j}(\mathbf{u})$ is a matrix. The row vector acts on the left.

Example: Here is a simple illustration in dimension one. Consider the differential form $du^2 = 2u \, du$ and the manifold mapping $u \leftarrow \sin(t)$. Then $(u \leftarrow \sin(t))^* 2u \, du = 2 \sin(t) d \sin(t) = 2 \sin(t) \cos(t) \, dt$. This is consistent with $(u \leftarrow \sin(t))^* du^2 = d \sin^2(t) = 2 \sin(t) \cos(t) \, dt$. ||

Example: Here is an example in dimension 3. The manifold mapping is given by $(x, y, z) \leftarrow (u^2 + v, uv, u^2 v)$. The differential 1-form is given by $\omega = xy^2 \, dx + yz^2 \, dy + xyz \, dz$. The pullback is $(u^2+v)(uv)^2 \, d(u^2+v) + (uv)(u^2 v)^2 \, d(uv) + (u^2 + v)(uv)(u^2 v) \, d(u^2 v)$. This can be further simplified by $d(u^2 + v) = 2u \, du + dv$ and $d(uv) = du \, v + v \, du$ and $d(u^2 v) = 2uv \, du + u^2 \, dv$. ||

6.6 Pullback and pushforward

6.6.1 Vector fields and differential forms along a mapping

The idea of vector field along a mapping $\phi : N \to M$ is that it describes change at $\phi(\nu)$ in M as a function of ν in N.

Suppose that ϕ is a manifold mapping (parametrized surface) from N to M. A *vector field along the mapping* ϕ assigns to each point ν in N a vector $Y_{[\nu]}$ tangent to M at the point $\phi(\nu)$.

It is also possible to think of Y as a differential operator that assigns to each scalar field s on M a scalar field $Y\,s$ on N. The value of $Y\,s$ on ν in N is the value of $Y_{[\nu]}$ on s at the point $\phi(\nu)$.

Proposition 6.24 (Vector field along a mapping) *Consider the manifold mapping $\phi = (\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u}))$. A vector field along this mapping is of the form*

$$Y = \sum_j a_j(\mathbf{u}) \frac{\partial}{\partial x_j}{}_{[\phi]}. \tag{6.152}$$

Here the partial derivative acts on scalar field $h(\mathbf{x})$ on M according to

$$\frac{\partial}{\partial x_j}{}_{[\phi]} h(\mathbf{x}) = h'_{,j}(\mathbf{f}(\mathbf{u})). \tag{6.153}$$

The result is a scalar field on N.

Example: Consider a parametrized curve ϕ with values in an affine space A. The tangent vector to the curve is a vector field along the curve. The acceleration vector is another vector field along the curve. ||

The next definition is a rather boring repetition of the same idea, but for forms instead of vector fields. A *differential 1-form along a mapping* is defined as follows. A differential form η along ϕ sends each point ν in N to a linear function $\eta_{[\nu]}$ on tangent vectors to M at the point $\phi(\nu)$. The action of a differential form η along ϕ on a vector field Y along ϕ at a given point is to take ν in N and map it to the value of $\eta_{[\nu]}$ on $Y_{[\nu]}$ at $\phi(\nu)$. In other words,

$$\langle \eta \mid Y \rangle(\nu) = \langle \eta_{[\nu]} \mid Y_{[\nu]} \rangle \tag{6.154}$$

Thus $\langle \eta \mid Y \rangle$ is a scalar field on N.

Proposition 6.25 (Differential 1-form along a mapping) *Consider the manifold mapping $\phi = (\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u}))$. A differential form along ϕ is of the form*

$$\omega = \sum_j q_j(\mathbf{u})\, dx_{j\,[\phi]}. \tag{6.155}$$

Here

$$\left\langle dx_{j\,[\phi]} \mid \frac{\partial}{\partial x_i}{}_{[\phi]} \right\rangle = \delta_{ij}. \tag{6.156}$$

The pairing

$$\langle \eta, Y \rangle = \sum_{j=1}^{n} q_j(\mathbf{u}) a_j(\mathbf{u}) \tag{6.157}$$

is a scalar field on N.

Example: Consider a parametrized 2-dimensional surface ϕ in a three dimensional space. Suppose that the parameters are u_1 and u_2. Define tangent vectors

$$X_\alpha = \sum_{i=1}^{3} \frac{\partial x_i \circ \phi}{\partial u_\alpha} \frac{\partial}{\partial x_i}{}_{[\phi]}. \tag{6.158}$$

Each of these is a vector field along ϕ. Let ν_j be $(-1)^{j-1}$ times the determinant of the 2 by 2 matrix obtained by deleting the jth row from the 3 by 2 matrix with entries $\partial(x_i \circ \phi)/\partial u_\alpha$. Let η be defined by

$$\eta = \sum_{j=1}^{3} \nu_j \, dx_{j\,[\phi]}. \tag{6.159}$$

Then η is a 1-form along ϕ that is zero on each tangent vector. ||

6.6.2 Restriction along a mapping

The following discussion is dull; it consists of some necessary definitions having to do with restrictions of vector fields and differential forms. However it is necessary to have this on record.

At the outset it may be helpful to say that for a scalar field pullback and restriction are the same thing. These two operations make sense for differential forms, but in this case they are not the same. For vector field the two main operations are pushforward and restriction. Again these are not the same.

Say that X is a vector field on M and $\phi : N \to M$ is a manifold map. The *restricted vector field* is a differential operator that sends each scalar field on M to a scalar field on N. It is defined by

$$X_{[\phi]}\, s = (X\, s) \circ \phi. \tag{6.160}$$

The resulting $X_{[\phi]}$ is a vector field along the mapping ϕ. This may also be considered as a mapping that sends each ν in N to the corresponding vector $X_{[\phi(\nu)]}$ at $\phi(\nu)$. (Some authors call this the pullback of the vector field.)

Proposition 6.26 (Coordinate expression of a restricted vector field) *Consider the manifold map $\phi = (\mathbf{x} \leftarrow \mathbf{f}(\mathbf{t}))$. Suppose the vector field is*

$$X = \sum_j r_j(\mathbf{x}) \frac{\partial}{\partial x_j}. \tag{6.161}$$

Then

$$X_{[\phi]} = \sum_j r_j(\mathbf{f}(\mathbf{t})) \, \frac{\partial}{\partial x_j}{}_{[\phi]}. \tag{6.162}$$

Here

$$\frac{\partial}{\partial x_j}{}_{[\phi]} \, h(\mathbf{x}) = \frac{\partial h(\mathbf{x})}{\partial x_j}(\mathbf{x} \leftarrow \mathbf{f}(\mathbf{t})) = h'_{,j}(\mathbf{f}(t)). \tag{6.163}$$

A vector field Y along ϕ need not be a restriction from a vector field on M. The general form of Y is

$$Y = \sum_j s_j(t) \, \frac{\partial}{\partial x_j}{}_{[\phi]}. \tag{6.164}$$

Suppose, for instance, that ϕ is a parametrized curve whose image crosses itself, as in the number 8. Then it is possible that two different parameter values lead to the same crossing point, and these can give rise to two different vectors at this point. This cannot happen with a restriction.

Say that ω is a differential 1-form on M and $\phi : N \to M$ is a manifold map. The *restricted differential form* $\omega_{[\phi]}$ sends each point ν of N to the linear function $\omega_{[\phi(\nu)]}$. Then $\langle \omega_{[\phi]} \mid Y \rangle$ is a scalar field on N with

$$\langle \omega_{[\phi]} \mid Y \rangle(\nu) = \langle \omega_{[\phi(\nu)]} \mid Y_{[\nu]} \rangle. \tag{6.165}$$

The resulting restricted $\omega_{[\phi]}$ is a differential field along the mapping ϕ. The term restricted differential form distinguishes this from the more important concept of pullback of a differential form.

Proposition 6.27 *If $\phi : N \to M$ is a manifold map, and X is a vector field on M, and ω is a 1-form on M, then*

$$\langle \omega_{[\phi]} \mid X_{[\phi]} \rangle = \langle \omega \mid X \rangle \circ \phi. \tag{6.166}$$

Proposition 6.28 (Coordinate expression of a restricted differential form) *Consider the manifold map $\phi = (\mathbf{x} \leftarrow \mathbf{f}(\mathbf{t}))$. Suppose the differential form is*

$$\omega = \sum_j r_j(\mathbf{x}) \, dx_j \tag{6.167}$$

Then

$$\omega_{[\phi]} = \sum_j r_j(\mathbf{f}(\mathbf{t})) \, dx_{j\,[\phi]}, \tag{6.168}$$

where $\left\langle dx_{j\,[\phi]} \mid \frac{\partial}{\partial x_j}{}_{[\phi]} \right\rangle = \delta_{ij}$.

Suppose ϕ sends N to M. There are several kinds of differential forms.

- A differential form ω on M. It is dual to vector fields on M. For each such vector field X the value $\langle \omega \mid X \rangle$ is a scalar field on M.

- A differential form η along ϕ. It is dual to vector fields along ϕ. For each such vector field Y along ϕ the value $\langle \eta \mid Y \rangle$ is a scalar field on N. If ω is a differential form on M, then its restriction $\omega_{[\phi]}$ is a differential form along ϕ.
- A differential form α on N. It is dual to vector fields on N. For each such vector field Z the value $\langle \alpha \mid Z \rangle$ is a scalar field on N. More importantly, if ω is a differential form on M, then its pullback $\phi^*\omega$ is a differential form on N.

6.6.3 Pushforward of a vector field

The discussion in the last two sections leads to a genuinely interesting and important concept, that of the pushforward of a vector field. While the pullback of a differential 1-form by a mapping is another differential 1-form, the pushforward of a vector field by a mapping is not a vector field in the ordinary sense. Instead, it is a vector field along the mapping.

If $\phi : N \to M$, and Z is a vector field on N, then $\phi_* Z$ is a vector field along the mapping ϕ. This *pushforward* is a differential operator that sends each scalar field on M to a scalar field on N. It is defined by

$$(\phi_* Z)\, s = Z\,(s \circ \phi). \tag{6.169}$$

Remark: The convention is that superscripts denote pullback and subscripts denote pushforward. Here are some common abbreviations. Say that there is a manifold mapping from N to M. If s is a scalar field on M, then its pullback to N is s^*. If ω is a 1-form on M, the pullback $\phi^*\omega$ to N is ω^*. In particular, $ds^* = (ds)^*$. If Z is a vector field on N, its pushforward $\phi_* Z$ to a vector field along the mapping is Z_*. ||

Proposition 6.29 (Coordinate representation of pushforward of a vector field) *Say that ϕ is the mapping $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u})$. Let*

$$Z = \sum_{\alpha=1}^{k} a_\alpha \frac{\partial}{\partial u_\alpha} \tag{6.170}$$

be a vector field on N. Then $\phi_* Z$ is given by

$$\phi_* Z = \sum_{i=1}^{n} \left(\sum_{\alpha=1}^{k} \frac{\partial x_i \circ \phi}{\partial u_\alpha} a_\alpha \right) \frac{\partial}{\partial x_i}{}_{[\phi]}. \tag{6.171}$$

More explicitly this is

$$\phi_* Z\, h(\mathbf{x}) = \sum_{i=1}^{n} \left(\sum_{\alpha=1}^{k} f'_{i,\alpha}(\mathbf{u}) a_\alpha \right) h'_{,i}(\mathbf{f}(\mathbf{u})). \tag{6.172}$$

These formulas are valid for arbitrary coordinate systems.

Notice that the $f'_{i,\alpha}(\mathbf{u})$ is a matrix acting on the column vector with components a_α.

With this notation it is possible to define vectors

$$X_\alpha = \phi_* \frac{\partial}{\partial u_\alpha} = \sum_{i=1}^{n} \frac{\partial (x_i \circ \phi)}{\partial u_\alpha} \frac{\partial}{\partial x_i}{}^{[\phi]}. \tag{6.173}$$

These are vectors tangent to the surface.

Example: Consider the manifold mapping ϕ given by $(x, y, z) \leftarrow (u^2 + v, uv, u^2 v)$. The pushforwards of the coordinate vector fields are given by

$$\phi_* \frac{\partial}{\partial u} = 2u \frac{\partial}{\partial x}{}^{[\phi]} + v \frac{\partial}{\partial y}{}^{[\phi]} + 2uv \frac{\partial}{\partial z}{}^{[\phi]} \tag{6.174}$$

and

$$\phi_* \frac{\partial}{\partial v} = \frac{\partial}{\partial x}{}^{[\phi]} + u \frac{\partial}{\partial y}{}^{[\phi]} + u^2 \frac{\partial}{\partial z}{}^{[\phi]}. \tag{6.175}$$

The vectors on the right hand side are restricted to the surface. If for example they are used to differentiate a scalar field like $x^3 z$, a typical result is

$$\frac{\partial}{\partial x}{}^{[\phi]} x^3 z = 3(u^2 + v)^2 u^2 v. \tag{6.176}$$

In practice they are just used as convenient basis vectors at point on the surface. ||

Suppose ϕ sends N to M. There are several kinds of vector fields.

- A vector field X on M. For each such vector field X and scalar field s on M the resulting $X\, s$ is another scalar field on M.
- A vector field Y along ϕ. For each such vector field Y along ϕ and scalar field s on M the value $Y\, s$ is a scalar field on N. If X is a vector field on M, then its restriction $X_{[\phi]}$ is a vector field along ϕ, given by $X_{[\phi]}\, s = (X\, s) \circ \phi$. More importantly, if Z is a vector field on N, then its pushforward $\phi_* Z$ is a vector field along ϕ, given by $(\phi_* Z)\, s = Z\, (s \circ \phi)$.
- A vector field Z on N. For each such vector field Z and scalar field r on N the resulting $Z\, r$ is another scalar field on N.

6.6.4 Pushforward by an invertible mapping

There is another kind of pushforward that is much nicer, since it sends a vector field to another vector field. However it only works for an invertible mapping $\phi : N \to M$. If Z is a vector field on N, then $\phi_* Z$ is a vector field on M defined by

$$\phi_* Z\, s = (Z\, s \circ \phi) \circ \phi^{-1}. \tag{6.177}$$

If $\phi = (\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u}))$ and $Z = \sum_j g_j(\mathbf{u}) \frac{\partial}{\partial u_j}$, then

$$\phi_* Z = \sum_k \left(\sum_j f'_{k,j}(\mathbf{f}^{-1}(\mathbf{x})) g_j(\mathbf{f}^{-1}(\mathbf{x})) \right) \frac{\partial}{\partial x_k}. \tag{6.178}$$

6.6.5 Duality of pullback and pushforward

Theorem 6.30 (Duality along a manifold mapping) *Let $\phi : N \to M$ be a manifold mapping. Let ω be a differential 1-form on M, and let Z be a vector field on N. Then*

$$\langle \phi^* \omega \mid Z \rangle = \langle \omega_{[\phi]} \mid \phi_* Z \rangle. \tag{6.179}$$

Proof: In the special case $\omega = ds$ this works out to be the identity $Z\,(s \circ \phi) = (\phi_* Z)\,s$, which is the definition of pushforward. The result then extends to every multiple $r\,ds$. Finally, it works for linear combinations of such forms. □

Proposition 6.31 (Coordinate representation of duality) *Suppose that $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u})$ is a manifold mapping. Let*

$$\omega = \sum_{i=1}^{n} h_i(\mathbf{x})\, dx_i. \tag{6.180}$$

and

$$Z = \sum_{\alpha=1}^{k} w_\alpha(\mathbf{u}) \frac{\partial}{\partial u_\alpha}. \tag{6.181}$$

Then

$$\phi^* \omega = \sum_{\beta=1}^{k} \sum_{i=1}^{n} h_i(\mathbf{f}(u)) f'_{i,\beta}(\mathbf{u})\, du_\beta. \tag{6.182}$$

Furthermore,

$$\omega_{[\phi]} = \sum_{i=1}^{n} h_i(\mathbf{f}(\mathbf{u}))\, dx_{i\,[\phi]} \tag{6.183}$$

and

$$\phi_* Z = \sum_{j=1}^{n} \sum_{\alpha=1}^{k} f'_{j,\alpha}(\mathbf{u}) w_\alpha(\mathbf{u}) \frac{\partial}{\partial x_i}_{[\phi]}. \tag{6.184}$$

So the duality says that

$$\sum_{\beta=1}^{k} \sum_{i=1}^{n} h_i(\mathbf{f}(\mathbf{u})) f'_{i,\beta}(\mathbf{u}) w_\beta(\mathbf{u}) = \sum_{j=1}^{n} h_j(\mathbf{f}(\mathbf{u})) \sum_{\alpha=1}^{k} f'_{j,\alpha}(\mathbf{u}) w_\alpha(\mathbf{u}). \tag{6.185}$$

In these equations the $h_i(\mathbf{x})$ form a row vector, the $f'_{j,\alpha}(\mathbf{u})$ is the matrix of a linear transformation, and the $w_\alpha(\mathbf{u})$ form a column vector.

Corollary 6.32 (Duality for an exact form) *Let $\phi : N \to M$ be a manifold mapping. Let s be a scalar field on M, and let Z be a vector field on N.*

$$\langle d(s \circ \phi) \mid Z \rangle = \langle ds_{[\phi]} \mid \phi_* Z \rangle. \tag{6.186}$$

There is an important distinction in the interpretation. On the left hand $d(s \circ \phi)$ is a form on the parameter space N and is paired with Z on the same parameter space. On the right hand side $ds_{[\phi]}$ has been restricted to the surface, but still acts in M, while $\phi_* Z$ has been pushed forward to also act in M. In other words, the quantity on the left is internal to the k-dimensional parameter space, while the quantity on the right depends on how the surface sits in the n-dimensional target space.

This identity can also be written in terms of differential operators in the form

$$Z\,(s \circ \phi) = (\phi_* Z)\,s. \tag{6.187}$$

This is just the definition of pushforward.

Proposition 6.33 (Coordinate representation of exact form duality) *Suppose that ϕ given by $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u})$ is a manifold mapping. Let $s = g(\mathbf{x})$ be a scalar field. Then the pullback*

$$d(s \circ \phi) = dg(\mathbf{f}(\mathbf{u})) = \sum_{\beta=1}^{k} (g \circ \mathbf{f})'_{,\beta}(\mathbf{u})\, du_\beta. \tag{6.188}$$

Furthermore, the restriction

$$ds_{[\phi]} = dg(\mathbf{x})_{[\phi]} = \sum_{i=1}^{n} g'_{,i}(\mathbf{f}(\mathbf{u}))\, dx_{i\,[\phi]}. \tag{6.189}$$

Also, the pushforward of $\partial/\partial u_\alpha$ is

$$\phi_* \frac{\partial}{\partial u_\alpha} = \sum_{j=1}^{n} f'_{j,\alpha}(\mathbf{u}) \frac{\partial}{\partial x_j}{}_{[\phi]}. \tag{6.190}$$

So the identity says that

$$(g \circ \mathbf{f})'_{,\alpha}(\mathbf{u}) = \sum_{j=1}^{n} g'_{,j}(\mathbf{f}(\mathbf{u})) f'_{j,\alpha}(\mathbf{u}). \tag{6.191}$$

Remark: Suppose X is a vector field on M, and $\phi : P \to M$ is a manifold mapping from the time parameter space P to M. Then ϕ is a solution of the differential equation corresponding to X when

$$\phi_* \frac{d}{dt} = X_{[\phi]}. \tag{6.192}$$

This says that the pushforward defining the tangent vectors to the solution curve coincides with the restriction of the vector field to the curve. A particle is moving along a trajectory; its velocity each time is given by the vector field at the corresponding point on the trajectory.

The deeper meaning of such an identity may be seen by seeing how a scalar field changes with time along the trajectory. We can think of the scalar field s as something like temperature. Then the identity says that

$$\langle d(s \circ \phi) \mid \frac{d}{dt} \rangle = \langle ds_{[\phi]} \mid X_{[\phi]} \rangle \tag{6.193}$$

On the left hand the computation is internal to the particle; the expression merely tells how the temperature changes with time. On the right hand side the computation takes place in the space M, the change in s is determined by the velocity vector of the particle in this bigger space. Since $\langle ds_{[\phi]} \mid X_{[\phi]} \rangle = X_{[\phi]}\, s = (X\, s) \circ \phi$, this is equivalent to the usual form of the differential equation

$$\frac{d}{dt}(s \circ \phi) = (X\, s) \circ \phi. \tag{6.194}$$

Either expression gives a description of how the temperature at the particle location changes with time. The left side is internal; the right side is external. ||

Remark: If $\phi : N \to M$ is a manifold map, then there is a derivative ϕ' that takes vector fields on N to vector fields along ϕ. If $\phi = (\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u}))$, then

$$\phi' = \sum_{j=1}^{n} \sum_{\alpha=1}^{k} f'_{j,\alpha}(\mathbf{u}) \frac{\partial}{\partial x_j}{}_{[\phi]}\, du_\alpha. \tag{6.195}$$

This is a somewhat complicated object. Most of the previous concepts (push-forward of a vector field, pullback of a form) may be defined in terms of this derivative. For instance, $\phi_* Y = \langle \phi' \mid Y \rangle$ and $\phi^* \omega = \langle \omega_{[\phi]} \mid \phi' \rangle$. ||

6.7 Regular surfaces

6.7.1 Regular surfaces and implicit definition

In the following it will be useful to distinguish various types of surface (or curve) objects. In both cases these are thought of as belonging to an n-dimensional space, but there is often a lower dimensions

- A *parametrized surface* is a mapping from a k-dimensioal parameter space to an n-dimensional space, where typically $k < n$. It is typically visualized as the image of the mapping in the n-dimensional space.

- A *labeled surface family* is a mapping from an n-dimensional space to a $(n-k)$-dimensional label space, where typically $k < n$. An *implicit surface* is the inverse image of a point in the label space. It is visualized as a subset of the n-dimensional space. A labeled surface family is usually visualized in terms of typical implicit surfaces belonging to the family.

- A *surface* (perhaps given implicitly or explicitly) is regarded as a subset of the n-dimensional space, without regard to a particular parametrization. Similarly, a *surface family* consists of a family of surfaces without any particular labeling.

Example: These concepts are most easily illustrated by an example where the surfaces are curves. Consider the family of circles centered at a fixed point in the plane. Let x, y be Cartesian coordinates that vanish at this point. The circles form a labeled surface family given by $x^2 + y^2 = a^2$, where a^2 labels the curves in the family. Another labeling could be $\sqrt{x^2 + y^2} = a$. A particular surface is given implicitly by specifying a value of a. Each circle of radius $a > 0$ may be given parametrically by $x \leftarrow a\cos(t), y \leftarrow a\sin(t)$. ||

Let M be an n-dimensional manifold patch. Let S be a subset of M. Then S is a *regular surface* if for each point in S there is an open set $U \subseteq M$ including the point with the following property: There are coordinates $u_1, \ldots, u_n$ for U such that $N = S \cap U$ is the subset of U where $u_{k+1} = 0, \ldots, u_n = 0$. (Another name for a regular surface is *regular submanifold*.)

When N is one-dimensional a regular surface is a regular curve. When N is $(n-1)$-dimensional a regular surface is often called a regular hypersurface.

In many cases the natural coordinates for M may be some other system $x_1, \ldots, x_n$. On the other hand, N is a k-dimensional manifold patch with coordinates $u_1, \ldots, u_k$. The *injection* of N into M is the manifold mapping ϕ that sends ν in N to the same point $\phi(\nu)$ in M. Since the coordinates $x_1, \ldots, x_n$ for U are not related in a simple way to the $u_1, \ldots, u_k$ for N, this injection $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u})$ may be given by a rather complicated function $\mathbf{f}$.

Suppose that $\mathbf{w}$ is a smooth function from the n dimensional manifold patch M to $\mathbf{R}^m$, where $m \leq n$. Then $d\mathbf{w}$ is an m-tuple of differential froms on M. A point in M is a *regular point* if these differential forms are linearly independent at the point. Otherwise it is a *critical point*. A point in $\mathbf{R}^m$ is a *regular value* if every point in its inverse image is a regular point. It is a *critical value* if some point in its inverse image is a critical point.

Let $w_{k+1}, \ldots, w_n$ be scalar fields on M. An *implicit surface* S is a surface in M that is the solution of the equations $w_{k+1} = c_{k+1}, \ldots, w_n = c_n$ for some given constants. An implicit surface S is a *regular implicit surface* if at each point of S the differential forms $dw_{k+1}, \ldots, dw_n$ are linearly independent. In other words, every point of S is a regular point, or, equivalently, the point in $\mathbf{R}^{n-k}$ defining the surface is a regular value.

Theorem 6.34 *Suppose that S is a regular implicit surface. Then S is a regular surface.*

Proof: Write $w_j = g_j(\mathbf{x})$ in the coordinate system $\mathbf{x}$. The condition defining the surface may be written $\mathbf{g}(\mathbf{x}) = \mathbf{c}$. The condition on this function is that at each point $\mathbf{g}'(\mathbf{x})$ is a $n-k$ by n matrix with $n-k$ linearly independent rows. In other words, it has row rank $n-k$. It follows that it has column rank $n-k$. Suppose that the $\mathbf{x}$ coordinates are permuted so that the last $n-k$ columns are linearly independent.

Define $h_i(\mathbf{x}) = x_i$ for $i = 1, \ldots, k$, and $h_i(\mathbf{x}) = g_i(\mathbf{x})$ for $i = k+1, \ldots, n$. Then $\mathbf{h}(\mathbf{x})$ has derivative $\mathbf{h}'(\mathbf{x})$ that is an invertible n by n matrix. By the inverse function theorem, near every point it has an inverse function. Thus near every point $\mathbf{u} = \mathbf{h}(\mathbf{x})$ is a coordinate system for M. Furthermore, near that point S is specified by specifying that the last $n - k$ coordinates are zero. □

6.7.2 Regular parametrized surfaces

Suppose P and M are manifold patches of dimensions k and n. Suppose that $\phi : P \mapsto M$ is a manifold mapping. In the case when $k \leq n$ it is sometimes natural to think of ϕ as a surface in M. The idea is that P is a parameter space, and the image $\phi(P)$ is a geometric object that is a subset of M.

When P is one-dimensional a parametrized surface is a parametrized curve. When P is $(n-1)$-dimensional the parametrized surface is often called a parametrized hypersurface. In the following it will be understood that such special cases are included.

For a general parametrized surface this image can be quite complicated. Here are some possibile restrictions that go toward making the image have better behavior.

Parameterized surface A general parametrized surface $\phi : P \to M$ may have a degenerate image.

Immersed parametrized surface The parametrized surface $\phi : P \to M$ is an *immersion* if at every point ν of P the map that sends tangent vectors to P at ν to corresponding pushforward tangent vectors to M at $\phi(\nu)$ is one-to-one.

One-to-one immersed parametrized surface The surface is not only an immersion, but the manifold map ϕ is one-to-one. (This is an instance of the general concept of *immersed submanifold*.)

Regular parametrized surface The surface is a one-to-one immersed parametrized surface, and in addition the map $\phi : P \to \phi(P)$ is a homeomorphism. This imposes the restriction that the inverse map $\phi^{-1} : \phi(P) \subseteq M \to P$ is continuous. Points in the image that are close in M must have parameters that are also close. (This is an instance of the notion of *embedded submanifold*.)

The continuity requirement for a regular parametrized surface says that for every open subset $W \subseteq P$ there is an open subset $V \subseteq M$ such that $\phi(W) = \phi(P) \cap V$. Alternatively, for every point ν in the parameter space P and every open subset $W \subseteq P$ with ν in W, there exists an open subset $V \subseteq M$ with $\phi(\nu)$ in V and with $\phi(W) \subseteq \phi(P) \cap V$.

How can this fail? It is sufficient that there exists a point ν in P and an open subset $W \subseteq P$ j with ν in P with the following property. For every open subset $V \subset N$ with $\phi(\nu)$ in V there exists a point ν' not in W with $\phi(\nu')$ in V. In other words, there are points near $\phi(\nu)$ that do not come from points near ν.

Spivak [45] gives an curve that provides an example. The parameter space P is an open interval, and it maps into a image figure that looks like the number 6. A point ν in the parameter space maps to the point $\phi(\nu)$ where the image almost touches itself. Consider a small parameter interval W around the point ν. It maps to part of the left hand side of the 6. But there are points ν' near the end of the parameter interval that are not in W but map arbitrarily close to $\phi(\nu)$.

Theorem 6.35 *Let $\phi : P \to M$ be a regular parametrized surface with image $N = \phi(P)$. Then N is a regular surface.*

Proof: Consider a point on the surface. Suppose the parametric representation near this point is $x_i = f_i(u_1, \ldots, u_k)$. Without loss of generality suppose that the partial derivatives with respect to the variables $x_1, \ldots, x_k$ form a k by k invertible matrix. Define a new function $x_i = g_i(u_1, \ldots, u_n)$ as follows. Take $x_i = g_i(\mathbf{u}) = f_i(u_1, \ldots, u_k)$ for $i = 1, \ldots, k$, and take $x_i = g_i(\mathbf{u}) = f_i(u_1, \ldots, u_k) + u_i$ for $i = k+1, \ldots, u_n$. Then the derivative of $\mathbf{g}(\mathbf{u})$ is an invertible n by n matrix. So we may locally express $\mathbf{u} = \mathbf{g}^{-1}(\mathbf{x})$ by the inverse function theorem. We would like to show that near the given point on the surface it is obtained by setting $u_{k+1} = 0, \ldots, u_n = 0$. Clearly when this is satisfied we have $x_i = f_i(u_1, \ldots, u_k)$, and so the corresponding point is on the surface. On the other hand, we have the self-avoiding condition. Consider a parameter region W around the point so small that it is in the region where the inverse function theorem applies. Then there is an open subset V in N near the point such that that every $\mathbf{x}$ in V that is also in $N \cap U$ is of the form $\mathbf{x} = \mathbf{f}(u_1, \ldots, u_k)$ for $u_1, \ldots, u_k$ in W. In other words, $\mathbf{x} = \mathbf{g}(u_1, \ldots, u_k, 0, \ldots, 0)$. Since $\mathbf{g}$ is one-to-one, this means that nearby points $\mathbf{x}$ on the surface $N \cap U$ have u_i coordinates satisfying $u_{k+1} = 0, \ldots, u_n = 0$. $\square$

In the other direction, a regular surface is locally a regular parametrized surface. That is, if S is a regular surface in M, then for each point in S there is an open set $U \subseteq M$ such that $N = S \cup U$ is a manifold patch and the injection $\phi : N = S \cap U \to M$ is a regular parametrized surface.

Example: Here is an example of a one-to-one immersed parametrized surface for which every point in the image has close points in the image with very different parameter values.

Fix $0 < a < b$. Consider a plane with parameters u, v and a map ϕ into three-dimensional space with coordinates x, y, z defined by

$$\begin{aligned} x &\leftarrow (b + a\cos(u))\cos(v) \\ y &\leftarrow (b + a\cos(u))\sin(v) \\ z &\leftarrow a\sin(u). \end{aligned} \tag{6.196}$$

This maps the plane onto a *torus*. Notice that the mapping is periodic with period 2π both in u and in v.

Now fix real real numbers r_1 and r_2 and consider the map

$$u \leftarrow r_1 t$$
$$v \leftarrow r_2 t. \tag{6.197}$$

This is a curve whose image is a line in the plane.

The composition is a curve in the torus. Suppose that either r_1 is a rational multiple of r_2 or r_2 is a rational multiple of r_1. Then $m_1 r_1 = m_2 r_2$ for some integers m_1, m_2, not both zero. Let $t = 2\pi m_1 m_2$. Then the corresponding $u = 2\pi m_1$ and $v = 2\pi m_1$ are both multiples of 2π. This shows that the curve in the torus is periodic.

Suppose on the other hand that neither r_1 nor r_1 is a rational multiple of the other. Then the curve is one-to-one. It winds around in the torus, forming a dense subset of the torus. In this case two points on the curve that are very close can correspond to very different parameter values. ||

6.7.3 The tangent space of a regular surface

Let M be a manifold patch of dimension n with some coordinate system $\mathbf{x} = (x_1, \ldots, x_n)$. Consider a regular parametrized surface S of dimension k with some coordinate system $\mathbf{u} = (u_1, \ldots, u_k)$. The injection of S into M may be written $\phi = (\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u}))$.

Remark: Here are some notations. Suppose that ϕ is the injection of S into M. If z is a scalar field on M the restriction $z \circ \phi$ is the scalar field z_S on S. Similarly, if ω is a 1-form on M, the restriction $\omega_{[\phi]}$ along the injection is written $\omega_{[S]}$. (Note that the pullback $dz_S = dz^* = (dz)^*$ is not the same as the restriction $(dz)_{[S]}$.) If X is a vector field on M, its restriction $X_{[\phi]}$ along the injection is written $X_{[S]}$. ||

The surface S has tangent vectors in the larger space. These are spanned by

$$X_\alpha = \phi_* \frac{\partial}{\partial u_\alpha} = \sum_{i=1}^{n} \frac{\partial x_S}{\partial u_\alpha} \frac{\partial}{\partial x_i}_{[S]}. \tag{6.198}$$

This may also be written

$$X_\alpha = \sum_{i=1}^{n} f'_{i,\alpha}(\mathbf{u}) \frac{\partial}{\partial x_i}_{[S]}. \tag{6.199}$$

Each X_α is a *vector field along the parametrized surface S.*

Theorem 6.36 *Consider a regular parametrized surface S and the injection ϕ of S into M. At each point ν in S the k linearly independent tangent vectors $\frac{\partial}{\partial u_\alpha}$ are mapped into k linearly independent vectors $X_\alpha = \phi_* \frac{\partial}{\partial u_\alpha}$ in the n dimensional space of vectors tangent to M at ν.*

The span of the vectors X_α is called the *tangent space of the surface* at the point. It is a k-dimensional subspace of the n-dimensional space of tangent vectors at the point

Theorem 6.37 *Consider a regular implicit surface S that is also a regular parametrized surface. Suppose the regular implicit surface is defined by $w_j = c_j$ for $j = k+1, \ldots, n$. Then the $n-k$ differential forms dw_j each satisfy*

$$\langle dw_{j\,[S]} \mid X_\alpha \rangle = 0. \tag{6.200}$$

Furthermore, at each point they form a basis for the $n-k$ dimensional space of differential forms that vanish on the k dimensional space of tangent vectors at the point.

This theorem says that very small changes X_α tangent to the surface do not make a significant change in the constraints defining the surface.

Proof: Let ϕ be the injection of S into M. For each j the composition $w_j \circ \phi = 0$. Hence $d(w_j \circ \phi) = 0$. By the duality identity

$$\langle d(w_j \circ \phi) \mid \frac{\partial}{\partial u_\alpha} \rangle = \langle dw_{j\,[\phi]} \mid X_\alpha \rangle = 0. \tag{6.201}$$

Since there are k independent tangent vectors and $n-k$ independent differential forms, the differential forms $dw_{j\,[\phi]}$, $j = k+1, \ldots, n$ form a basis for the space of differential forms that vanish on the tangent vectors. □

Here is the same story more concrete language. Write the constraint functions $w_j = g_j(\mathbf{x})$ in coordinates. For the parametrized surface S the constraining equation is

$$g_j(\mathbf{f}(\mathbf{u})) = c_j \tag{6.202}$$

for $j = k+1, \ldots, n$. Differentiation with respect to $\mathbf{u}_\alpha$ gives

$$\sum_{i=1}^{n} g'_{j,i}(\mathbf{f}(\mathbf{u})) f'_{i,\alpha}(\mathbf{u}) = 0. \tag{6.203}$$

Since there are k independent tangent vectors and $n-k$ independent differential forms, the differential forms $dg_j(\mathbf{x})_{[S]}$, $j = k+1, \ldots, n$ form a basis for the space of differential forms restricted to points of the surface that vanish on the tangent vectors.

Example: Consider the unit sphere S given by $x^2 + y^2 + z^2 = 1$. One possible parameterization that covers most (but not all) of the sphere is $(x, y, z) \leftarrow (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$. Here $0 < \theta < \pi$ and $0 < \phi < 2\pi$. This may be though of as giving the injection from this part of the sphere S to Euclidean space. The basis tangent vectors corresponding to the colatitude θ and the longitude ϕ are

$$C = \cos(\theta)\cos(\phi)\frac{\partial}{\partial x}_{[S]} + \cos(\theta)\sin(\phi)\frac{\partial}{\partial y}_{[S]} - \sin(\theta)\frac{\partial}{\partial z}_{[S]}. \tag{6.204}$$

and

$$L = -\sin(\theta)\sin(\phi)\frac{\partial}{\partial x}_{[S]} + +\sin(\theta)\cos(\phi)\frac{\partial}{\partial y}_{[S]}. \tag{6.205}$$

The differential of the constraining equation is $2x\,dx + 2y\,dy + 2z\,dz$. Its restriction to the sphere is

$$\eta = 2\sin(\theta)\cos(\phi)\,dx_{[S]} + 2\sin(\theta)\sin(\phi)\,dy_{[S]} + 2\cos(\theta)\,dz_{[S]} \tag{6.206}$$

It is easy to see that $\langle \eta \mid C \rangle = 0$ and $\langle \eta \mid L \rangle = 0$. ||

6.8 Lagrange multipliers*

6.8.1 Lagrange multipliers: first differential*

The topic in this section is *constrained optimization*. The problem is to maximize or minimize a function restricted to a surface. This *constraint surface* is given implicitly.

Theorem 6.38 (Lagrange multiplier theorem) *Let M be an n-dimensional manifold patch. Let w_j be scalar fields on M. Consider a regular k-surface S given implicitly by $w_j = c_j$, $j = k+1, \ldots, n$. The differentials $dw_{j\,[\nu]}$ are linearly independent at each point ν of S. Suppose that s is a scalar field whose restriction to the surface has a local minimum or a local maximum at a certain point ν. Then there are unique coefficients $\lambda_1, \ldots, \lambda_p$ such that*

$$ds_{[\nu]} = \sum_{j=k+1}^{n} \lambda_j\, dw_{j\,[\nu]}. \tag{6.207}$$

at that point.

Proof: Write ϕ for the injection of S into M. The hypothesis is that $s \circ \phi$ has a local minimum or a local maximum at ν. By the first derivative test this implies that $d(s \circ \phi)$ is zero at ν. But

$$\left\langle d(s \circ \phi), \frac{\partial}{\partial u_\alpha} \right\rangle = \langle ds_{[\phi]} \mid X_\alpha \rangle. \tag{6.208}$$

Evaluating at ν shows that $ds_{[\nu]}$ is zero on tangent vectors to the surface at ν. Therefore it must be a linear combination of the $dw_{j\,[\nu]}$. □

The coefficients λ_j are called *Lagrange multipliers*. This result is intuitive. It says that if s defined on the bigger space has a local maximum when restricted to the constraint surface S, then the only way it can be made larger is by moving off S. This can only be done by relaxing the constraint that the surface is defined by constants c_j. The Lagrange multiplier λ_j itself is the partial derivative of the critical value with respect to a change in the parameter c_j.

The number of equations is correct. There are n variables and $n-k$ Lagrange multipliers. To determine these there are n minimization conditions with $n-k$ constraint equations.

Example: Say that we want to maximize or minimize $u = x + y + 2z$ subject to $v = x^2 + y^2 + z^2 = 1$. The manifold in this case is the unit sphere. The Lagrange multiplier condition says that

$$du = dx + dy + 2\,dz = \lambda dv = \lambda(2x\,dx + 2y\,dy + 2z\,dz). \tag{6.209}$$

Thus $1 = 2\lambda x$, $1 = 2\lambda y$, and $2 = 2\lambda z$. Insert these in the constraint equation $x^2 + y^2 + z^2 = 1$. This gives $(1/4) + (1/4) + 1 = \lambda^2$, or $\lambda = \pm\sqrt{3/2}$. So the solution is $(x, y, z) \leftarrow (\pm\sqrt{2/3}/2, \pm\sqrt{2/3}/2, \pm\sqrt{2/3})$. ||

Example: Say that we want to maximize or minimize $u = x - 4y + 3z + z^2$ subject to $v = x - y = 0$ and $w = y - z = 0$. The manifold in this case is just a line through the origin. The Lagrange multiplier condition says that

$$dx - 4\,dy + (3 - 2z)\,dz = \lambda(dx - dy) + \mu(dy - dz). \tag{6.210}$$

Thus $1 = \lambda$, $-4 = -\lambda + \mu$, and $(3 - 2z) = -\mu$. When we solve we get $\mu = -3$ and so $z = 0$. The solution may be expressed as $(x, y, z) \leftarrow (0, 0, 0)$.

We could also solve this example without Lagrange multipliers. Since the manifold is $x = y = z$, the function to be maximized or minimized is $u = z^2$, and this has its minimum at the point where $z = 0$. The utility of the Lagrange multiplier technique in more complicated problems is that it is not necessary to do such a preliminary elimination before solving the problem. ||

Example: Here is a simple example to emphasize the point that the Lagrange multiplier technique is coordinate independent. Say that one wants to maximize or minimize x subject to $x^2 + y^2 + z^2 = 1$. The Lagrange multiplier method says to write $dx = \lambda(2x\,dx + 2y\,dy + 2z\,dz)$. This says that $2\lambda x = 1$ and $y = z = 0$. From the constraint equation $x = \pm 1$. The solution is the point $(x, y, z) \leftarrow (1, 0, 0)$ or the point $(x, y, z) \leftarrow (-1, 0, 0)$.

In spherical polar coordinates this would be the problem of maximizing $r\sin(\theta)\cos(\phi)$ subject to $r^2 = 1$. The desired solution is away from the coordinate singularities at $\theta = 0$ and $\theta = \pi$. The equations are $\sin(\theta)\cos(\phi)\,dr + r\cos(\theta)cos(\phi)\,d\theta - r\sin(\theta)\sin(\phi)\,d\phi = \lambda 2r\,dr$. Solutions include $\theta = \pi/2$ and $\phi = 0, \pi$. The solution is the point $(r, \theta, \phi) \leftarrow (1, \pi/2, 0)$ or the point $(r, \theta, \phi) \leftarrow (1, \pi/2, \pi)$. These are the same points as before. ||

Remark: Here is a general strategy for finding points where maximum and minimum values are assumed. Every continuous real function on a non-empty compact set has has a maximum and a minimum. There is a common situation when the compact set has a boundary, and this boundary is an implicitly defined regular surface that itself has no boundary. For example, the set could be a closed ball, and then the boundary is a sphere. The search for the points where the maximum and minimum is assumed can begin by looking for critical points in the interior of the set. Other candidates are critical points of the function restricted to the boundary; these may be found by the method of Lagrange multipliers. If there are only finitely many critical points of either sort, then the maximum and minimum may be found by inspecting the values at these points.

This method may require modification for functions that are defined on non-compact sets. For example, if s is a continuous function, and there is a real

number c such that the set where $s \leq c$ is compact, and if there is some point where $s < c$, then it makes sense to look for a minimum point in the interior of the set where $s \leq c$. In this situation it is quite possible that there is no maximum. ||

6.8.2 Lagrange multipliers: second differential*

There are Lagrange multiplier techniques corresponding to the second derivative test, but they are more awkward. This section gives a sketch of a simple version that works in many cases.

The setting for the Lagrange multiplier theorem is an implicitly defined surface S defined by $w_1 = c_1, \ldots, w_{n-k} = c_{n-k}$, where the differentials dw_r are linearly independent at each point. The theorem says that if s restricted to S has a critical point, then at that point there are numbers λ_r such that $ds = \sum_r \lambda_r \, dw_r$ at the point. In other words, to first order the only possible change in s is by moving off the surface. The problem in devising a second differential test is that the second differential of s is only defined at points where ds is zero. This problem is solved by introducing another scalar field L such that $L = s$ on the surface S and dL is zero at the relevant point. This L will have a well-defined second differential d^2L. (This is not the second differential in the sense of exterior algebra; it is a symmetric quantity involving a Hessian matrix.)

Theorem 6.39 *Let s be a scalar field on M. Consider an implicitly defined surface S defined by $w_1 = c_1, \ldots, w_{n-k} = c_{n-k}$, where the differentials dw_r are linearly independent at each point. Consider the scalar field*

$$L = s - \sum_r \lambda_r (w_r - c_r). \tag{6.211}$$

This has the same restriction to S as the original scalar field s. Its differential

$$dL = d(s - \sum_r \lambda_r w_r) = ds - \sum_r \lambda_r dw_r \tag{6.212}$$

is zero at the critical point. Its second differential

$$d^2L = d^2(s - \sum_r \lambda_r w_r) \tag{6.213}$$

is thus defined at this point. A vector X at this point is tangent to S if each

$$\langle dw_r \mid X \rangle = 0. \tag{6.214}$$

Suppose that on each non-zero tangent vector X the second differential d^2L at the critical point is strictly positive (strictly negative). Then s restricted to S has a local minimum (local maximum).

Example: Here is a simple problem to illustrate the fact that the technique works with arbitrary coordinates. Take $a > 0$. Say that we want to maximize x with the constraint that $x^2 + y^2 = a^2$. The Lagrange condition is that $dL = dx - \lambda(2x\,dx + 2y\,dy) = 0$. The only solution is $y = 0$. Consider the solution where x has the value a. Then $\lambda = \pm 1/(2a)$. The second differential is $d^2L = -2\lambda((dx)^2 + (dy)^2) = -(1/a)(dx)^2 + (dy)^2)$. This is strictly negative definite, so its restriction to vectors tangent to the circle is also strictly negative definite. So this is the maximum.

It is instructive to do this example in polar coordinates. The problem is to maximize $r\cos(\theta)$ with the constraint $r^2 = a^2$. The Lagrange condition is that $dL = \cos(\theta)\,dr - r\sin(\theta)\,d\theta - \lambda 2r\,dr = 0$. It follows that θ is 0 or π. In the first case the second differential is $d^2L = -2\sin(\theta)\,dr\,d\theta - r\cos(\theta)\,(d\theta)^2 - \lambda 2\,(dr)^2)$. At the critical point this is $dL = -a\,(d\theta)^2 - (1/a)\,(dr)^2 = -(1/a)(a^2\,(d\theta)^2 + (dr)^2)$. This is precisely the same second differential.

The comparison of these two calculations shows that while the individual terms in the second differential may be quite different, the final sum is the same in both coordinates systems. ||

Example: Here is a simple example to illustrate the strictly negative definiteness condition. The problem is to maximize y subject to $y - x + x^2 = 0$. The Lagrange condition is that $dL = dy - \lambda(dy - dx - 2x\,dx) = 0$. The solution is $\lambda = 1$ and $x = 1/2$. The second differential is $d^2L = -2(dx)^2$. This is negative definiite but not strictly negative definite. However the differential of the constraint equation at the critical point is dy. The non-zero vectors in the null space of dy are of the form $t\partial/\partial x$ with $t \neq 0$ The value of the second differential on such vectors is $-2t^2 < -0$. So on the null space this is strictly negative definite, and so this is passes the test for a local maximum. ||

Example: This repeats an earlier example. Say that we want to maximize or minimize $u = x - 4y + 3z + z^2$ subject to $v = x - y = 0$ and $w = y - z = 0$. The manifold in this case is just a line through the origin. The differential $dL = u - \lambda_1 dv - \lambda_2 dw$ becomes $dL = dx - 4dy + 3dz + 2z\,dz - \lambda_1(dx - dy) - \lambda_2(dy - dz)$. The multipliers are $\lambda_1 = 1$ and $\lambda_2 = -3$. The solution is $(x, y, z) \leftarrow (0, 0, 0)$. The second differential is $d^2L = 2\,(dz)^2$. This is positive definite but not strictly positive definite. However the constraints give $dx - dy = 0$ and $dy - dz = 0$. If X satisfies these constraints, then it is of the form $t(\partial/\partial x + \partial/\partial y + \partial/\partial z)$. In other words, it points along the line. In particular, it has a non-zero z component. The value of the second differential $d^2L = 2\,(dz)^2$ on such a vector is $2t^2 > 0$. This indicates a local minimum. ||

Example: Consider again the problem of finding the open box with maximum volume. This is maximizing $V = uvw$ with $A = uv + 2uw + 2vw$ held constant. The equation for the critical point is obtained by setting $dL = dV - \lambda\,dA$ equal to zero. This is

$$dL = vw\,du + uw\,dv + uv\,dw - \lambda((v + 2w)\,du + (u + 2w)\,dv + (2u + 2v)\,dw) = 0. \tag{6.215}$$

The solution is $2w = u = v = 4\lambda$.

The second differential at this point works out to be

$$d^2L = 2\lambda\, du\, dv + 4\lambda\, du\, dw + 4\lambda dw\, dv. \tag{6.216}$$

Apply this to a vector $X = a\partial/\partial u + b\partial/\partial v + c\partial/\partial w$. This gives $4\lambda\,(ab + 2ac + 2bc)$ which is not definite. This is seen by taking $c = 0$; the remaining ab can take either sign.

The situation is different when the vectors are restricted. At the critical point $dA = 8\lambda(du + dv + 2\,dw)$. The condition $\langle dA \mid X\rangle = 0$ restricts the tangent vector X to satisfy $a+b+2c = 0$. The restricted form is then $-4\lambda(a^2+b^2+ab)$. Since $a^2+b^2+ab = \frac{1}{2}a^2 + \frac{1}{2}b^2 + \frac{1}{2}(a+b)^2$ the restricted form is strictly negative definite. This shows that the volume is a local maximum. ||

6.9 Vector fields and flows

6.9.1 Flow of a vector field

Every vector field defines a manifold mapping called a *flow*.

Theorem 6.40 (Existence and uniqueness of flow) *Suppose M is a manifold patch, and X is a vector field on M. For every point in M there exists an open neighborhood U and a number a with $0 < a \leq +\infty$ with the following properties. There is a unique manifold mapping Φ that sends a pair (ν, t) with ν in U and $-a < t < a$ to a point $\Phi_t(\nu)$ in M.*

- *For every point ν in U the mapping $t \mapsto \Phi_t(\nu)$ is the solution curve for the vector field X starting at ν when t is zero.*
- *For each t with $-a < t < a$ the manifold mapping $\Phi_t : U \to M$ that sends ν in U to $\Phi_t(\nu)$ in M is a diffeomorphism of U onto an open subset of M.*

This theorem is proved in texts on differential equations. There is an excellent treatment in [1]. It is clear that Φ_0 is the identity map. Since X does not depend on time, it is also true that $\Phi_{t+t'} = \Phi_t \circ \Phi_{t'}$. One immediate but important consequence of this is that $\Phi_t \circ \Phi_{t'} = \Phi_{t'} \circ \Phi_t$.

Proposition 6.41 (Coordinate representation of flow) *Say that the vector field is*

$$X = \sum_{i=1}^{n} g_i(\mathbf{x}) \frac{\partial}{\partial x_i}. \tag{6.217}$$

Write the corresponding differential equation as

$$\frac{d\hat{\mathbf{x}}}{dt} = \mathbf{g}(\hat{\mathbf{x}}) \tag{6.218}$$

and specify that $\hat{\mathbf{x}}$ *must be* $\mathbf{x}$ *when* t *is zero. The solution of this equation* $\hat{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ *satisfies the differential equation*

$$\frac{d\mathbf{f}(\mathbf{x}, t)}{dt} = \mathbf{g}(\mathbf{f}(\mathbf{x}, t)) \tag{6.219}$$

with the initial condition $\mathbf{f}(\mathbf{x}, 0) = \mathbf{x}$. *The flow* Φ_t *is given by* $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{x}, t)$, *so*

$$\mathbf{x} \circ \Phi_t = \mathbf{f}(\mathbf{x}, t). \tag{6.220}$$

The notation $\hat{\mathbf{x}}$ is intended to suggest that the ordinary differential equation is supposed to be solved in terms of a function not only of time t but also of the initial condition.. The actual solution is $\mathbf{x} \circ \Phi_t = \mathbf{f}(\mathbf{x}, t)$, which is indeed a function of a point in $U \subseteq M$ as well as of time t.

Theorem 6.42 (Straightening out theorem) . *If*

$$X = \sum_{i=1}^{n} g_i(\mathbf{x}) \frac{\partial}{\partial x_i} \neq 0 \tag{6.221}$$

is a vector field that is non-zero near some point, then near that point there is another coordinate system $u_1, \ldots, u_n$ *in which it has the form*

$$X = \frac{\partial}{\partial u_j}. \tag{6.222}$$

Proof: Since the vector field is not zero at the point, at least one component must be non-zero. Suppose for simplicity of notation that this is the nth component. Suppose that at that point $x_n = c$.

Let $(x_1, \ldots, x_n, t) \mapsto \mathbf{f}(x_1, \ldots, x_n)$ be the function that gives the flow of X in the original coordinates. Consider the function that send $(u_1, \ldots u_n)$ to $(x_1, \ldots, x_n)$ given by

$$x_i = f_i(u_1, \ldots, u_{n-1}, c, u_n) \tag{6.223}$$

In particular, for $i < n$

$$x_i = f_i(u_1, \ldots, u_{n-1}, c, 0) = u_i \tag{6.224}$$

and

$$x_n = f_i(u_1, \ldots, u_{n-1}, c, 0) = c. \tag{6.225}$$

To compute $\partial x_i / \partial u_j$ for $j < n$ at the point, one has to hold u_n constant, that is, $u_n = 0$. This shows that

$$\frac{\partial x_i}{\partial u_j} = \delta_{ij} \tag{6.226}$$

at the point for $j < n$.

On the other hand,

$$\frac{\partial x_i}{\partial u_n} = g_i(\mathbf{x}) \tag{6.227}$$

at the point, and $g_n(\mathbf{x}) \neq 0$. This shows that at this point the determinant of $\partial x_i/\partial u_j = g_n(\mathbf{x}) \neq 0$.

By the inverse function theorem there is a smooth solution for $\mathbf{u}$ in terms of $\mathbf{x}$. This shows that these are equivalent coordinate systems near the point.

By the chain rule and the differential equation

$$\frac{\partial}{\partial u_n} = \sum_j \frac{\partial x_j}{\partial u_n} \frac{\partial}{\partial x_j} = \sum_j g_j(\mathbf{x}) \frac{\partial}{\partial x_j}. \tag{6.228}$$

Notice that the role of the new variables is that $u_1, \ldots, u_{n-1}$ are the initial conditions, and u_n is time. □

Example: Consider the vector field

$$-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \tag{6.229}$$

away from the origin. The corresponding system is

$$\frac{dx}{dt} = -y \tag{6.230}$$

$$\frac{dy}{dt} = x. \tag{6.231}$$

The coordinates in which the straightening out takes place are polar coordinates r, θ. Thus if we write $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we have

$$-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \frac{\partial}{\partial \theta}, \tag{6.232}$$

where the partial derivative with respect to θ is taken with r held fixed. ||

Example: Consider the Euler vector field

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = r\frac{\partial}{\partial r}, \tag{6.233}$$

where the partial derivative with respect to r is taken with fixed θ. We need to stay away from the zero at the origin. If we let $t = \ln(r)$, then this is

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = r\frac{\partial}{\partial r} = \frac{\partial}{\partial t}, \tag{6.234}$$

where the t derivative is taken with θ fixed. ||

Example: Consider the one-dimensional vector field $X = x^2\frac{\partial}{\partial x}$. The flow is $x \leftarrow f(x,t) = x/(1-xt)$. This example shows that the solution may exist only for small value of t. Is it true that this formula gives $f(x, t+t') = f(f(x,t),t')$? It should be true, but does it work out? ||

Example: Consider the two-dimensional vector field $X = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$. The flow is $(x,y) \leftarrow \mathbf{f}(x,y,t) = (x(1-xt)^{-1}, y(1-xt)^{-1})$. ||

6.9.2 Time evolution of a scalar field*

A vector field

$$X = \sum_j g_j(x) \frac{\partial}{\partial x_j} \tag{6.235}$$

defines a flow

$$\Phi_t = (\mathbf{x} \leftarrow \mathbf{f}(\mathbf{x}, t)). \tag{6.236}$$

with $\mathbf{f}(\mathbf{x}, 0) = \mathbf{x}$. For every scalar field $u = h(\mathbf{x})$ this satisfies an ordinary differential equation

$$\frac{\partial u \circ \Phi_t}{\partial t} = (X\, u) \circ \Phi_t. \tag{6.237}$$

This may also be written

$$\frac{\partial h(\mathbf{f}(\mathbf{x}, t))}{\partial t} = \sum_j g_j(\mathbf{f}(\mathbf{x}, t)) h'_{,j}(\mathbf{f}(\mathbf{x}), t)). \tag{6.238}$$

For instance, if $h(\mathbf{x}) = x_k$ is a coordinate function, then this says that

$$\frac{\partial f_k(\mathbf{x}, t)}{\partial t} = g_k(\mathbf{f}(\mathbf{x}, t)). \tag{6.239}$$

While these last two equations have been written using partial derivative notation, they do not involve derivatives with respect to the x_k.

The property

$$\Phi_{s+t} = \Phi_s \circ \Phi_t. \tag{6.240}$$

says that addition of the time displacements is mapped to composition of the manifold mappings. (In the language of algebra, this is a group homomorphism.) With the coordinate language, this property has the perhaps less intuitive form

$$\mathbf{f}(\mathbf{x}, s + t) = \mathbf{f}(\mathbf{f}(\mathbf{x}, t), s). \tag{6.241}$$

Theorem 6.43 *Let X be a vector field on on M with flow Φ. For sufficiently small t each $\Phi_t : U \to M$ maps points in an open subset $U \subseteq M$ to M. Define a scalar field depending on t by*

$$u_t = u \circ \Phi_t. \tag{6.242}$$

This satisfies the linear partial differential equation

$$\frac{\partial u_t}{\partial t} = X\, u_t \tag{6.243}$$

with initial condition is $u_0 = u$.

Proof: Use

$$\frac{\partial u_t}{\partial t} = \frac{\partial u_{t+t'}}{\partial t'}(t' \leftarrow 0). \tag{6.244}$$

Then fix t and calculate

$$\frac{\partial u \circ \Phi_{t+t'}}{\partial t'} = \frac{\partial u \circ \Phi_t \circ \Phi_{t'}}{\partial t'} = (X\,(u \circ \Phi_t)) \circ \Phi_{t'}. \tag{6.245}$$

Setting $t' \leftarrow 0$ gives the result. □
Remark: Another notation for the partial differential equation is

$$\frac{\partial u_t}{\partial t} = X \lrcorner\, du_t. \tag{6.246}$$

This gives an explicit reminder that on the right hand side u_t is differentiated with respect to the space variables. ||

It is important to distinguish the ordinary differential equation $d(s \circ \Phi_t)/dt = (X\,s) \circ \Phi_t$, where X acts on a fixed scalar field s, from the partial differential equation $\partial u_t/\partial t = X\,u_t$, where X acts on $u_t = u \circ \Phi_t$ depending on time.

Theorem 6.44 *Every solution u_t of this partial differential equation with $u_0 = u$ is of the form $u_t = u \circ \Phi_t$.*

Proof: Suppose that u_t satisfies the linear partial differential equation with the initial condition.. The idea is to find a quantity that does not change in time. Use the product rule and the ordinary differential equation to compute

$$\frac{d}{dt'}(u_{t'} \circ \Phi_{-t'}) = \frac{\partial u_{t'}}{\partial t'} \circ \Phi_{-t'} - (X\,u_{t'}) \circ \Phi_{-t'} = 0. \tag{6.247}$$

Thus $u_{t'} \circ \Phi_{-t'}$ is constant as a function of time t'. In particular $u_t \circ \Phi_{-t} = u$, which implies $u_t = u \circ \Phi_t$. □
Example: Take the case of a constant vector field $X = c\partial/\partial x$.The ordinary differential equation is $d\hat{x}/dt = c$ with solution $\hat{x} = x + tc$. So the flow is $x \leftarrow x + ct$. The solution of the equation $\partial h(x,t)/\partial t = c\partial h(x,t)/\partial x$ is $h(x,0)(x \leftarrow x + ct) = h(x + ct, 0)$. The function moves in the opposite direction from the direction given by c. ||

Proposition 6.45 (Scalar field equation in coordinates) *Suppose*

$$X = \sum_j g_j(\mathbf{x}) \frac{\partial}{\partial x_j}. \tag{6.248}$$

Write $\hat{\mathbf{x}}$ for $\mathbf{x} \circ \Phi_t = \mathbf{f}(\mathbf{x}, t)$. The corresponding differential equation is

$$\frac{d\hat{\mathbf{x}}}{dt} = \mathbf{g}(\hat{\mathbf{x}}) \tag{6.249}$$

with initial condition that $\hat{\mathbf{x}}$ is $\mathbf{x}$ when t is zero. The solution $\hat{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ of this equation gives the corresponding flow

$$\Phi_t = \mathbf{x} \leftarrow \mathbf{f}(\mathbf{x}, t). \tag{6.250}$$

If $u = h(\mathbf{x})$ is the initial condition, then the equation

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = \sum_j g_j(\mathbf{x}) \frac{\partial h(\mathbf{x},t)}{\partial x_j} \tag{6.251}$$

is solved by

$$u_t = h(\mathbf{x},t) = h(\mathbf{f}(\mathbf{x},t)). \tag{6.252}$$

The proof that the solution $h(\mathbf{x},t)$ has the required form reduces to showing that $h(\mathbf{f}(\mathbf{x},t'),t')$ is independent of t'. In this context the parametrized curve given by $\mathbf{f}(\mathbf{x},t')$ is called a *characteristic curve*. This technique is called the method of characteristics.

Example: Suppose that

$$X = x^2 \frac{\partial}{\partial x}. \tag{6.253}$$

The differential equation is

$$\frac{d\hat{x}}{dt} = \hat{x}^2 \tag{6.254}$$

with initial condition $\hat{x} = x$ when $t = 0$. The solution is

$$\hat{x} = f(x,t) = \frac{1}{\frac{1}{x} - t}. \tag{6.255}$$

So the flow is

$$x \leftarrow \frac{1}{\frac{1}{x} - t}. \tag{6.256}$$

The solution with function $u = h(x)$ as initial condition is

$$u_t = s\left(x \leftarrow \frac{1}{\frac{1}{x} - t}\right) = h\left(\frac{1}{\frac{1}{x} - t}\right) = h\left(\frac{x}{x - xt}\right). \tag{6.257}$$

||

Example: Consider the two-dimensional vector field $X = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$. The solution is is $u_t = h(x(1-xt)^{-1}, y(1-xt)^{-1})$. ||

The equation $du_t/dt = Xu_t$ is called a *backward equation*. It defines a function $u_t = u \circ \Phi_t$ that expresses a future reward at time t given by u in terms the present point. Thus $u_t = \Phi_t^* u$ is the pullback of u from the future to the present.

Often the situation is different; a scalar field r gives the present situation, and the goal is to to find the corresponding future r_t after time t has elapsed. This is governed by the *forward equation* $dr_t/dt = -Xr_t$ with solution $r_t = r \circ \Phi_{-t}$. Thus $r_t = \Phi_{-t}^* r$ is the pullback from the present to the future. Often such a forward equation is called a *transport equation*, since describes a quantity transported forward in time by the vector field X.

Example: Consider the equation $\partial r_t/\partial t + a\partial r_t/\partial y = 0$, regarded as a forward equation. The solution is $r_t = k(y,t) = k(y - at, 0)$. The solution moves in direction given by a. ||

Example: Consider the equation $\partial r_t/\partial t + y\partial r_t/\partial y = 0$. To find the solution, solve $d\hat{y}/dt = \hat{y}$ with initial condition y. The solution is $\hat{y} = e^t y$. So the flow is given by $f(y,t) = e^t y$. If $r_t = h(y,t)$, then the solution is $h(y,t) = h(e^{-t}y, 0)$. This solution moves outward from the origin. ||

One way to see the relation between these two situations is to consider $w \circ \Phi_{s-t}$. As a function of s it satisfies a backward equation, while as a function of t it satisfies a forward equation. The distinction between forward and backward equation may seem fussy, but the next subsection will describe a situation where the distinction makes more sense.

6.9.3 Time-dependent vector fields*

A tme-dependent vector field X_t is a vector field that depends on time. In coordinates it has the form

$$X_t = \sum_j g_j(\mathbf{x}, t) \frac{\partial}{\partial x_j}. \tag{6.258}$$

This defines manifold mappings Φ^X_{t,t_0} that takes points at time t_0 and moves them along according to the time-dependent vector field to get points at time t.

The ordinary differential equations are

$$\begin{aligned} \frac{\partial}{\partial t}(u \circ \Phi_{t,s}) &= (X_t\, u) \circ \Phi_{t,s}, \\ \frac{\partial}{\partial s}(u \circ \Phi_{t,s}) &= -(X_s\, u) \circ \Phi_{t,s}. \end{aligned} \tag{6.259}$$

In coodinates the scalar u may be written $u = h(\mathbf{x})$, and the flow may be written

$$\Phi_{t,s} = (\mathbf{x} \leftarrow \mathbf{f}(\mathbf{x}, t, s)). \tag{6.260}$$

Then the ordinary differential equations say that

$$\begin{aligned} \frac{\partial}{\partial t} h(\mathbf{f}(\mathbf{x}, t, s)) &= \sum_j g_j(\mathbf{f}(\mathbf{x}, t, s)) h'_{,j}(\mathbf{f}(\mathbf{x}, t, s) \\ \frac{\partial}{\partial s} h(\mathbf{f}(\mathbf{x}, t, s)) &= -\sum_j g_j(\mathbf{f}(\mathbf{x}, t, s)) h'_{,j}(\mathbf{f}(\mathbf{x}, t, s). \end{aligned} \tag{6.261}$$

A mapping $\Phi_{t,s}$ that maps time s data to time t data may be called a *process*. The inverse process is given by

$$\Phi^{-1}_{t,s} = \Phi_{s,t}. \tag{6.262}$$

A process must satisfy

$$\Phi_{t_2,t_0} = \Phi_{t_2,t_1} \circ \Phi_{t_1,t_0}. \tag{6.263}$$

(If $\mathbf{R}^2$ is regarded as a groupoid, with the product of (t_2, t_1) with (t_1, t_0) equal to (t_2, t_0), then this is a groupoid homomorphism.)

A flow is a special case of a process. If Φ_t is a flow arising from a constant vector field, then $\Phi_{t,s} = \Phi_{t-s}$ is a process.

Proposition 6.46 *Every solution of the forward equation*

$$\frac{\partial}{\partial t} u_t + X_t \, u_t = 0. \tag{6.264}$$

must satisfy

$$u_t = u_s \circ \Phi_{s,t}. \tag{6.265}$$

Proof: The chain rule and the ordinary differential equation give

$$\frac{d}{ds'}(u_{s'} \circ \Phi_{s',t}) = \left(\frac{\partial}{\partial s'} u_{s'} \right) \circ \Phi_{s',t} + (X_{s'} \, u_{s'}) \circ \Phi_{s',t}. \tag{6.266}$$

The partial differential equation says that this is zero. It follows that $u_{s'} \circ \Phi_{s',t}$ is constant in s'. In particular its value at t is the same as its value at s. □

Proposition 6.47 *The function*

$$u_t = u_s \circ \Phi_{s,t}. \tag{6.267}$$

satisfies the forward equation

$$\frac{\partial}{\partial t} u_t + X_t \, u_t = 0. \tag{6.268}$$

Proof: The proof uses

$$\frac{\partial}{\partial t} u_t = \frac{\partial}{\partial t'} u_{t'}(t' \leftarrow t). \tag{6.269}$$

Then

$$\frac{\partial}{\partial t'} u_{t'} = \frac{\partial}{\partial t'}(u_s \circ \Phi_{s,t'}) = \frac{\partial}{\partial t'}(u_s \circ \Phi_{s,t} \circ \Phi_{t,t'}). \tag{6.270}$$

From the ordinary differential equation this is

$$-(X_{t'} \, (u_s \circ \Phi_{s,t})) \circ \Phi_{t,t'} = -(X_{t'} \, u_t) \circ \Phi_{t,t'}. \tag{6.271}$$

The replacement $t \leftarrow t'$ gives the result. □

This proof gives the practical way of solving the forward equation. Say that $X = \sum_j g_j(\mathbf{y}, t) \partial / \partial y_j$. The flow starting with $\mathbf{y}$ at t is obtained by integrating into the past. Use a different variable $\hat{\mathbf{y}}$ to denote $\mathbf{y} \circ \Phi_{s,t}$. So the differential equation is

$$\frac{d\hat{\mathbf{y}}}{ds} = \mathbf{g}(\hat{\mathbf{y}}, s) \tag{6.272}$$

with the condition that $\hat{\mathbf{y}}$ is $\mathbf{y}$ when t is s. The solution to this problem is

$$\hat{\mathbf{y}} = \mathbf{f}(\mathbf{y}, s, t). \tag{6.273}$$

So the flow is $\mathbf{y} \leftarrow \mathbf{f}(\mathbf{y}, s, t)$. The key fact in solving for $h(\mathbf{y}, t)$ is that $h(\mathbf{f}(\mathbf{y}, s', t), s')$ is constant in s'. This is the method of characteristics.

Example: Consider the forward equation

$$\frac{\partial u}{\partial t} + (y+t)\frac{\partial u}{\partial y} = 0 \tag{6.274}$$

with $u_t = h(y,t)$. The problem is to express $h(y,t)$ in terms of $h(y,s)$. The ordinary differential equation is

$$\frac{d\hat{y}}{ds} = x + s \tag{6.275}$$

with $\hat{y} = y$ when $s = t$. This has solution

$$\hat{y} = f(y,s,t) = e^{s-t}(y+t+1) - s - 1. \tag{6.276}$$

So

$$h(y,t) = h(y,s)(y \leftarrow e^{s-t}(y+t+1)-s-1) = h(e^{s-t}(y+t+1)-s-1, s). \tag{6.277}$$

||

The results for the backward equation are almost a direct repetition of the story for the forward equation. The small distinctions can be confusing, so it is useful to write again the details.

Proposition 6.48 *Every solution of the backward equation*

$$\frac{\partial}{\partial s}u_s = X_s\, u_s \tag{6.278}$$

must satisfy

$$u_s = u_t \circ \Phi_{s,t}. \tag{6.279}$$

Proof: The chain rule and the ordinary differential equation give

$$\frac{d}{dt'}(u_{t'} \circ \Phi_{s,t'}) = \left(\frac{\partial}{\partial t'}u_{t'}\right) \circ \Phi_{s,t'} - (X_{t'}\, u_{t'}) \circ \Phi_{s,t'}. \tag{6.280}$$

The partial differential equation says that this is zero. It follows that $u_{t'} \circ \Phi^X_{s,t'}$ is constant in t'. In particular its value at s is the same as its value at t. □

If we write $u_s(\mathbf{x}) = h(\mathbf{x}, s)$, then the above proof says that $h(\mathbf{f}(\mathbf{x}, s, t'), t')$ is constant in t'. This is again the method of characteristics.

Proposition 6.49 *The function*

$$u_s = u_t \circ \Phi_{s,t}. \tag{6.281}$$

satisfies the backward equation

$$\frac{\partial}{\partial s}u_t = X_s\, u_s. \tag{6.282}$$

Proof: The proof uses the strange looking identity $\Phi_{s',t} = \Phi_{s,t} \circ \Phi_{s',s}$. This identity is discussed in the remark below. Start with

$$\frac{\partial}{\partial s} u_s = \frac{\partial}{\partial s'} u_{s'}(s' \leftarrow s). \tag{6.283}$$

Then

$$\frac{\partial}{\partial s'} u_{s'} = \frac{\partial}{\partial s'} (u_t \circ \Phi_{s',t} = \frac{\partial}{\partial s'} \left(u_t \circ \Phi_{s,t} \circ \Phi_{s',s}\right). \tag{6.284}$$

From the ordinary differential equation this is

$$\left(X_{s'} \left(u_t \circ \Phi_{s,t}\right)\right) \circ \Phi_{s',s} = \left(X_{s'} \, u_s\right) \circ \Phi_{s',s}. \tag{6.285}$$

The replacement $s' \leftarrow s$ gives the result. $\square$

Remark: The identity $\Phi^X_{s',t} = \Phi^X_{s,t} \circ \Phi^X_{s',s}$.in the proof is equivalent to the identity $\Phi^X_{t,s'} = \Phi^X_{s,s'} \circ \Phi^X_{t,s}$ for the inverse maps. This in turn is equivalent to the natural identity $\Phi^{-X}_{s',t} = \Phi^{-X}_{s',s} \circ \Phi^{-X}_{s,t}$ for the time-reversed vector field. ||

Example: Consider the backward equation

$$\frac{\partial u}{\partial s} = -\frac{s}{x} \frac{\partial u}{\partial s} \tag{6.286}$$

with $u_s = h(x, s)$. The problem is to express $h(x, s)$ in terms of $h(x, t)$. The ordinary differential equation is

$$\frac{d\hat{x}}{dt} = -\frac{t}{\hat{x}} \tag{6.287}$$

with $\hat{x} = x$ when $t = s$. If we suppose $\hat{x} > 0$ this has solution

$$\hat{x} = f(x, s, t) = \sqrt{t^2 + x^2 + s^2}. \tag{6.288}$$

So

$$h(x, t) = h(x, s)(x \leftarrow \sqrt{t^2 + x^2 + s^2}) = h(\sqrt{t^2 + x^2 + s^2}, s). \tag{6.289}$$

||

Remark: Solutions of backward and forward equations can easily be recovered from each other. Consider the time dependent vector field Y_t that defines the flow $\Phi^Y_{t.s}$ for the forward equation and the time-dependent vector field X_t that defines the flow $\Phi^X_{s.t}$ for the backward equation. Then $X_t = -Y_t$. The solution of the forward equation is $w_t = w_s \circ \Phi^Y_{s,t}$, and the solution of the backward equation is $u_s = u_t \circ \Phi^X_{s,t}$. Then $u_s = u_t \circ \Phi^{-X}_{t,s} = u_t \circ \Phi^Y_{t,s}$, which solves a forward equation. ||

6.9.4 Vector fields versus differential forms

Here is a brief summary of the contrast between vector fields and differential 1-forms.

- A vector field X measures the rate of change of each scalar field. The rate of change of scalar field s along X is the scalar field

$$X\, s = \langle ds \mid X \rangle = X \lrcorner\, ds. \tag{6.290}$$

- A differential 1-form ω has a rate of change along every vector field. But when the form is not exact this need not be represented as the rate of change of a particular scalar field. The rate of change given by ω along X is the scalar field

$$\langle \omega \mid X \rangle = X \lrcorner\, \omega \tag{6.291}$$

The behavior of these quantities X and ω are quite different.

- At a point where the vector field X is zero, there are eigenvalues (and a Jordan form) given by its linearization at the point. These eigenvalues are independent of the coordinate system in which the vector field is expressed. Real eigenvalues represent rates of attraction and repulsion. Complex eigenvalues occur in conjugate pairs; the real part represent a rate of attraction or repulsion, the imaginary part represent a rate of spiral motion.

- A differential form ω may be exact, or it may not be exact. This is independent of coordinate system. There is no such distinction for vector fields, at least not one that is independent of coordinate system or some other special structure.

- At a point where the exact differential form ds is zero, there are numbers (each equal either to $+1$, -1, or 0) that characterize a quadratic form at that point. These numbers are independent of the coordinate system. They may be used to characterize local minima (only $+1$ values occuring), local maxima (only -1 values occuring), or saddle points (some $+1$ and some -1 values occuring).

Mapping properties also contrast.

- Under a mapping mapping a differential form is pulled back by replacement. The result is another differential form.

- Under a manifold mapping a vector field is pushed forward to a vector field along the mapping. This is not a vector field of the usual kind, since it only produces vectors that are tangent to the image of the mapping.

The relation to differential equations is also different.

- A vector field is equivalent to an autonomous system of first order ordinary differential equations. The solutions are given by the flow of the vector field.

- An exact 1-form ds defines a scalar field s (up to an additive constant of integration). This scalar field defines a family of surfaces of dimension $n-1$.

- For an exact 1-form ds the integral along an oriented curve is the value of s at the final point minus the value of s at the initial point. For a general differential form ω the value of the integral depends on the oriented curve.

In general there is no natural correspondence between vector fields and differential 1-forms. The relation between them is duality: a vector field and a differential 1-form together determine a scalar field.

In Euclidean space there is a natural correspondence between vector fields and 1-forms. But the formulas expressing this correspondence are simple only in Cartesian coordinates.

In orthogonal coordinate systems on Euclidean space the correspondence can be made simpler by replacing coordinate basis vectors by normalized basis vectors and by replacing coordinate basis forms by normalized basis forms. However this introduces new complications. For example, the normalized basis forms need not be exact.

Remark: In two dimensions there is a close relation between the flow of a vector field and integrating a differential 1-form. A vector field X defines a flow. Suppose the vector field is never zero. Then there is a differential 1-form α that is never zero such that $\langle \alpha \mid X \rangle = 0$. Such a form is not unique: If r is a non-zero scalar field, then $r\alpha$ also satisfies this condition. It is always possible to choose the integrating factor r so that $r\alpha = dv$ is exact. In particular, $\alpha = 0$ on a curve if and only if $dv = 0$ on the curve. The parametrized curves that make up the flow of X have images that are the curves where v has constant value. ||

Problems

Differential 1-forms

1. (a) Is $(x^2 + y^2)\,dx + 2xy\,dy$ an exact differential form? If so, write it as the differential of a scalar.

   (b) Convert the form to polar coordinates and do the problem in this coordinate system.

2. Is $(1 + e^x)\,dy + e^x(y - x)\,dx$ an exact differential? If so, write it as the differential of a scalar.

3. Is $e^y\,dx + x(e^y + 1)\,dy$ an exact differential? If so, write it as the differential of a scalar.

4. Consider

$$\alpha = -y\,dx + x\,dy. \tag{6.292}$$

(a) Is this locally exact? Globally exact? Find the line integral around an oriented circle of radius a centered at the origin. How does it depend on a.

(b) Sketch closely spaced flux lines (contour lines) that illustrate how the line integral over an arc of the circle counts the number of lines passing through the arc. Where do these lines start?

5. Consider
$$\beta = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy. \tag{6.293}$$
(a) Is this locally exact? Globally exact? Find the line integral around an oriented circle of radius $a > 0$ centered at the origin. How does it depend on a.

(b) Sketch closely spaced flux lines (contour lines) that illustrate how the line integral over an arc of the circle counts the number of lines passing through the arc. Where do these lines start?

6. Solve the ordinary differential equation $(xy^2 + y)\,dx - x\,dy = 0$ by finding an integrating factor that depends only on y.

7. Consider the oriented curve C given by $x = \sin(y)$ as y runs from 0 to $\pi/2$. Evaluate
$$\int_C e^y\,dx + x(e^y + 1)\,dy. \tag{6.294}$$

8. Consider the oriented curve given parametrically by $(x \leftarrow \sin(t), y \leftarrow \cos^3(t))$ as t runs from 0 to $\pi/2$. Evaluate
$$\int_C (1 + e^x)\,dy + e^x(y - x)\,dx. \tag{6.295}$$

Constraint surfaces and Lagrange multipliers

1. (a) Use Lagrange multipliers to find the critical points of xy restricted to the surface given implicitly by $x^2 + y^2 = 2$. Where are the minimum and maximum values assumed?

   (b) Do the same problem in polar coordinates.

2. Use Lagrange multipliers to find the critical points of xyz with the constraints $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 - z^2 = 0$. Where are the minimum and maximum values assumed?

3. Let $0 < a < b$. Consider the scalar field $w = \left(\sqrt{x^2+y^2} - b\right)^2 + z^2$.

   Consider also the parametrized surface ϕ given by
$$\begin{aligned} x &\leftarrow (b + a\cos(u))\cos(v) \\ y &\leftarrow (b + a\cos(u))\sin(v) \\ z &\leftarrow a\sin(u). \end{aligned} \tag{6.296}$$

(a) Show that ϕ parameterizes the implicit surface $w = a^2$. (in other words, show that $w \circ \phi = a^2$.) This surface is a *torus*.

(b) Calculate the pushforwards via ϕ of $\partial/\partial u$ and $\partial/\partial v$. These are tangent vectors to the surface.

(c) Show that these tangent vectors are linearly independent.

(d) Calculate the restriction of dw to the surface.

(e) Show by explicit computation that the restriction of dw is zero on the tangent vectors.

4. Consider the surface in the previous problem. Find the critical points of x restricted to $w = a^2$. Give the corresponding (x, y, z) values for each critical point. Classify them as maxima, minima, and saddles. (Hint: There is no need for Lagrange multipliers when a parameterization is known.)

Chapter 7

Multilinear Differential Forms

7.1 Exterior algebra pictures*

7.1.1 Exterior algebra*

This section is for readers who like pictures to go with formulas. The idea is to start with a real vector space V of dimension n. It has a *dual space* V^* that is also a vector space of dimension n. This consists of all linear transformations from V to the real numbers. An object in the dual space is thus called a *linear form*.

An example to keep in mind is when M is an n-dimensional manifold patch, μ is a point in M, and V is the space of tangent vectors at μ. Then V^* is the space of differential 1-forms at the same point.

From these vector spaces it is possible to construct vector spaces $\Lambda^k(V)$ and $\Lambda^k(V^*)$ for $k = 0, 1, \ldots, n$. The vector space $\Lambda^k(V)$ is called the space of k-vectors, and it has dimension $\binom{n}{k}$. The vector space $\Lambda^k(V^*)$ is called the space of k-forms, and it also has dimension $\binom{n}{k}$. When the number k is not specified an object in one of these spaces may be called a *multivector* or a *multiform*. In practice, a multiform is often simply called a *form*.

For $k = 0$ the spaces $\Lambda^0(V)$ and $\Lambda^0(V^*)$ are the same space, namely the 1-dimensional space of real numbers. For $k = 1$ the spaces $\Lambda^1(V) = V$ and $\Lambda^1(V^*) = V^*$. In general, the space $\Lambda^k(V^*)$ is the dual space of $\Lambda^k(V)$. There are also algebraic operations on these spaces, of which the most important is the *exterior product*. There is also a useful *interior product*.

The purpose of this section is to develop an intuition for multivectors and multiforms and for the exterior and interior product. For this purpose it is convenient to specialize to dimensions $n = 2$ and $n = 3$. For this case there are helpful geometric pictures. The ones given here are inspired by those in the book by Weinreich [51].

The central objects in the rest of this book are the k-forms, and these are the objects that are the most difficult to visualize. Roughly speaking, the result is the following. Suppose that V is an n-dimensional vector space.

- A 1 form in $\Lambda^1(V^*) = V^*$ is visualized as a closely spaced family of parallel $(n-1)$-dimensional hyperplanes in V.
- A $(n-1)$ form in $\Lambda^{n-1}(V^*)$ is visualized as a closely spaced family of parallel lines in V.
- A n-form in $\Lambda^n(V^*)$ is visualized as a closely spaced cloud of points in V.

This is not the complete story (in particular, orientation needs to be take into account). The next two sections give this picture in more detail for the cases $n = 2$ and $n = 3$. Then it is not only possible to draw the objects, but also to illustrate the way they are combined by operations such as the exterior product.
Remark: Some of the features of two or three dimensions are special, and this simplifies the description. In particular, all the multivectors and multiforms are *decomposable*, meaning that they are exterior products of 1-vectors or of 1-forms. ||

Remark: It is common to regard k-vectors as attached to the origin of the vector space V. In the following the convention is that they may be translated to other positions. This makes the geometric constructions somewhat more convenient and natural. ||

7.1.2 Exterior algebra in 2 dimensions*

The two-dimensional case is simple, so this is a good starting point. Here is the geometric representation of multivectors in two dimensions.

1. A 1-vector X is represented by an oriented line segment (an arrow).

2. A 2-vector A is represented by an oriented parallelogram.

Two 1-vectors are regarded as being the same if the segments are parallel and have the same length and orientation. There are two orientations, corresponding to the two possible directions for the arrow. In the case of 2-vectors two parallelograms are regarded to be the same if they have the same area and the same orientation. There are two orientations, which may be given by specifying an orientation on each edge, in such a way that they cancel at the corners. (The head of the arrow is canceled by the tail of the following arrow.) Figure 7.1 shows a typical 1-vector and a typical 2-vector.

The exterior product of two 1-vectors is a 2-vector defined as follows. If X and Y are 1-vectors, then $X \wedge Y$ is the parallelogram determined by X and Y. The orientation is determined by the order in which X and Y appear in the product. Thus $X \wedge Y = -Y \wedge X$, where the minus sign indicates opposite orientation. See Figure 7.2 for the picture.

The space of 1-vectors is closed under addition. The same is true for the space of 2-vectors. The addition operation in these two cases is illustrated in Figure 7.3. For the 2-vector case the shapes of the parallelograms may have to be adjusted so that they have a common side. This can be accomplished by writing $A = X \wedge Z$ and $B = Y \wedge Z$, so that $A + B = (X + Y) \wedge Z$. It is possible for the parallelograms to have opposite orientations; in that case they can partially cancel.

Here is the geometric representation of multiforms in two dimensions. Choose a small number $\delta > 0$.

1. A 1-form α is a linear form defined on 1-vectors. It is pictured by closely spaced parallel lines. Each line is parallel to the vectors on which α vanishes. Also, each line has the same transverse orientation. The value of the 1-form on a 1-vector that goes from one line to the next according to the orientation is δ.

2. A 2-form ω is a linear form defined on 2-vectors. It is pictured by a cloud of points closely arranged in the plane in a regular pattern of small oriented parallelograms. This pattern may be rearranged provided that the parallelograms keep the same area and the same orientation. The 2-form

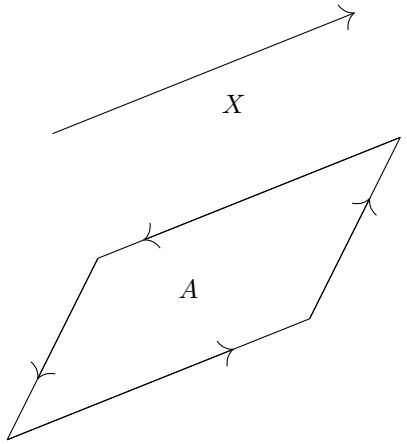

Figure 7.1: 1-vector X, 2-vector A

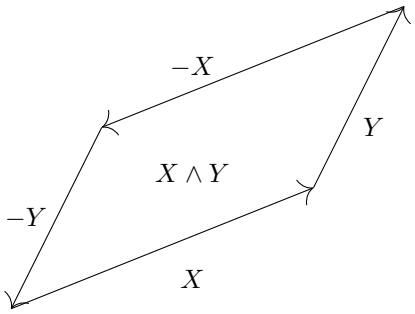

Figure 7.2: Exterior product of 1-vectors X, Y giving 2-vector $X \wedge Y$

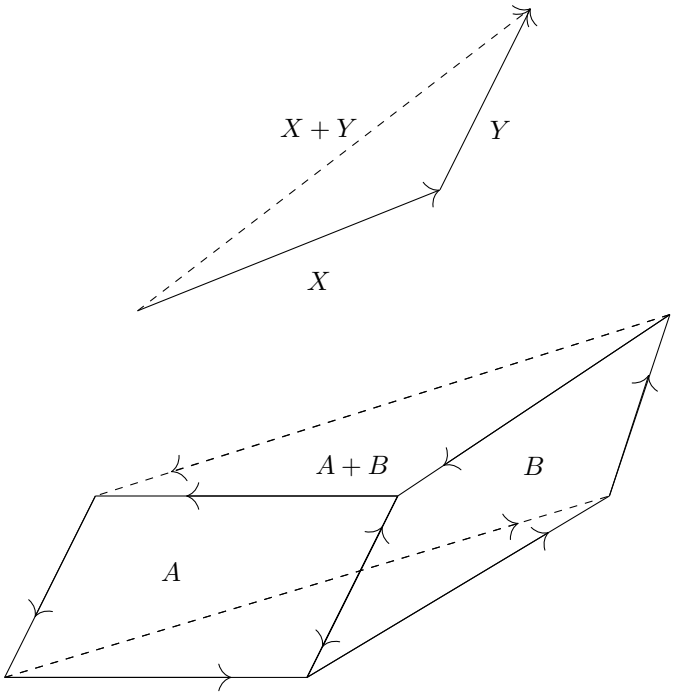

Figure 7.3: 1-vector $X + Y$, 2-vector $A + B$ (2d)

assigns to each small 2-vector parallelogram with the same orientation the value δ^2.

Figure 7.4 illustrates these objects.

The space of 1-forms is dual to the space of 1-vectors. Similarly, the space of 2-forms is dual to the space of 2-vectors; the multiform together with the multivector gives a numerical value. This pairing is an instance of an interior product.

1. Each value $\langle \alpha \mid X \rangle$ is a number. It is obtained by taking the number of lines traversed by X and multiplying by $\pm\delta$, where the sign depends on the relative orientations.

2. Each $\langle \omega \mid A \rangle$ is a number. This value is the number of 2-form parallelograms that belongs to the 2-vector parallelogram, times $\pm\delta^2$, with sign depending on the relative orientations.

These pairings are illustrated in Figure 7.5.

The exterior product of two 1-forms α and β is a 2-form $\alpha \wedge \beta$ defined as follows. Consider 1-forms α and β. Let X and Y be vectors such that $\langle \alpha \mid Y \rangle = 0$ and $\langle \beta \mid X \rangle = 0$. Then

$$\langle \alpha \wedge \beta \mid X \wedge Y \rangle = \langle \alpha \mid X \rangle \langle \beta \mid Y \rangle. \tag{7.1}$$

The picture is that the two families of lines giving α and β have a cloud of intersection points that gives $\alpha \wedge \beta$. Switching the order of the vectors changes the orientation: $\alpha \wedge \beta = -\beta \wedge \alpha$. The picture of the exterior product appears in Figure 7.6.

Addition of 1-forms is pictured in Figure 7.7. The forms are given by lines $\alpha = a$ and $\beta = b$ for equally spaced values of a and b. In a typical situation these generate a pattern of parallelograms. The sum is pictured by lines that run between diagonals of these parallelograms. One can pass from one sum line to the next either by incrementing a by a fixed small amount and leaving b alone, or by incrementing b by a fixed small amount and leaving b alone. Thus the increment in $\alpha + \beta$ is the fixed small amount.

For $k = 2$ the sum of 2-forms ω and σ, each defined by a family of points, is $\omega + \sigma$ given by a third family of points, with density given by the sum or difference of the densities, taking orientation into account.

Another useful definition is the interior product of a vector X with a 2-form ω. The formula for the resulting 1-form $X \lrcorner \omega$ is given by the formula

$$\langle X \lrcorner \omega \mid Y \rangle = \langle \omega \mid X \wedge Y \rangle. \tag{7.2}$$

The value of the interior product on X is $\langle X \lrcorner \omega \mid X \rangle = 0$. This means that the contour lines of $X \lrcorner \omega$ are parallel to X. The spacing of these lines is determined by the formula. Figure 7.8 gives an illuminating picture. In this picture the 2-form is pictured by points in a pattern where one direction is aligned with the vector. The resulting 1-form is indicated by the parallel lines.

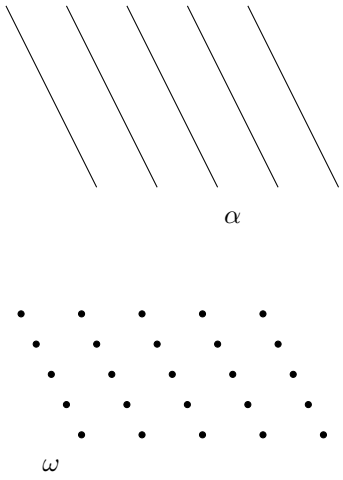

Figure 7.4: 1-form α, 2-form ω (2d)

It may help to think of the 2-form ω as two-dimensional "dirt." The dirt is swept up in the direction of the arrow X into one-dimensional dirt whose distribution is described by the 1-form $X \lrcorner \omega$.

7.1.3 Exterior algebra in 3 dimensions*

The pictures in two dimensions to a large part also govern the pictures in three dimensions. Here is the geometric representation of multivectors in three dimensions.

1. A 1-vector is represented by an oriented line segment (an arrow).
2. A 2-vector is represented by an oriented parallelogram.
3. A 3-vector is represented by an oriented parallelepiped.

Two 1-vectors are regarded as being the same if the arrows are parallel and have the same length. In the case of 2-vectors two parallelograms are regarded to be the same if they are parallel and have the same area and the same orientation. (An orientation of a parallelogram may be given by orienting each edge in such a way that there is cancellation at each corner.) For 3-vectors the rule is that two parallelepipeds are the same if they have the same volume and the same orientation. (A parallelepiped also has one of two possible orientations. It may be specified by giving an orientation of each face, in such a way that the orientations cancel on each edge.)

The most important operation is the exterior product.

1. The exterior product of two 1-vectors X and Y is a 2-vector $X \wedge Y$ pictured as an oriented parallelogram.

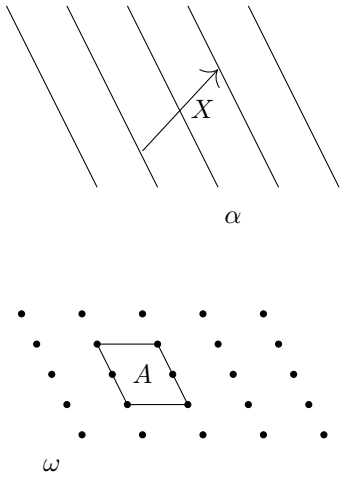

Figure 7.5: Pairings $\langle \alpha \mid X \rangle$, $\langle \omega \mid A \rangle$ (2d)

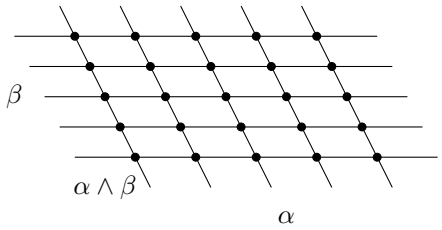

Figure 7.6: Exterior product of 1-forms α, β giving 2-form $\alpha \wedge \beta$ (2d)

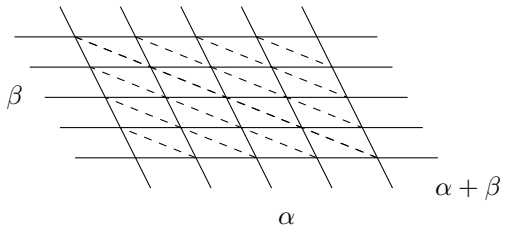

Figure 7.7: 1-form sum $\alpha + \beta$ (2d)

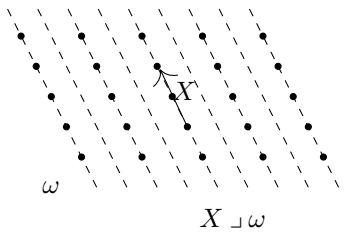

Figure 7.8: Interior product $X \lrcorner \omega$ (2d)

2. The exterior product of a 1-vector X with a 2-vector E is a 3-vector $X \wedge E$ pictured as an oriented parallelepiped.

We have $X \wedge Y = -Y \wedge X$, where the minus sign indicates opposite orientation. However $X \wedge E = E \wedge X$.

The space of k-vectors is closed under addition. For the case $k = 2$ A determines a plane and B determines a plane. These planes intersect in a line. Let Z be a vector in the direction of this line, and write $A = X \wedge Z$ and $B = Y \wedge Z$. Then $A + B = (X + Y) \wedge Z$. Thus the addition of 2-vectors reduces to the addition of 1-vectors.

Here is the geometric representation of multiforms in three dimensions. Choose a small number $\delta > 0$.

1. A 1-form α is a linear form defined on 1-vectors It is pictured by by a family of closely spaced parallel planes. These planes are parallel to every vector on which α vanishes. Each plane has the same transverse orientation.

2. A 2-form ω is a linear form defined on 2-vectors. It is pictured by a family of closely spaced parallel lines, arranged so that their cross-sections form a regular pattern of oriented parallelograms. These lines are parallel to every 2-vector on which ω vanishes.

3. A 3-form γ is a linear form defined on 3-vectors. It is pictured by a cloud of points closely arranged in space in a regular pattern of oriented parallelepipeds.

Again there is a pairing of k-forms with k-vectors.

1. Each value $\langle \alpha \mid X \rangle$ is a number. It is obtained by taking the number of planes traversed by X and multiplying by $\pm\delta$, where the sign depends on the relative orientations.

2. Each $\langle \omega \mid A \rangle$ is a number. The value depends on the intersections of the lines with the 2-vector parallelogram. It is the number of 2-form parallelograms between the intersection points that belong to the 2-vector parallelogram A, times $\pm\delta^2$, with sign depending on the relative orientations.

3. Each $\langle \gamma \mid E \rangle$ is a number. This value is the number of 3-form parallelepipeds that belongs to the 3-vector parallelepiped E, times $\pm\delta^3$, with sign depending on the relative orientations.

The new feature is the pairing of 2-form with 2-vector. This is illustrated in Figure 7.9. In this picture the only lines in the family corresponding to the 2-form ω are those relevant to pairing with the given 2-vector A.

The exterior product follows a familiar pattern.

1. The exterior product of two 1-forms α and β, each given by a family of planes (not parallel), is a 2-form $\alpha \wedge \beta$ defined as the family of intersection lines, with appropriate orientations.

2. The exterior product of a 1-form α, given by a family of planes, with a 2-form ω, given by a family of lines, is a 3-form $\alpha \wedge \omega$ given by the family of intersection points, with appropriate orientations.

For 1-forms the order matters: $\alpha \wedge \beta = -\beta \wedge \alpha$. For a 1-form and a 2-form the relation is $\alpha \wedge \omega = \omega \wedge \alpha$.

The addition of k-forms is pictured as follows.

1. The 1-forms α and β are given by two families of parallel planes. Consider an arbitrary plane that is perpendicular to each of these families. In this arbitrary plane α and β appear as two families of parallel lines. Then may be added as in the 2-dimensional case to get a third family of parallel lines in the arbitrary plane. This in turn defines a family of planes in space, representing the sum $\alpha + \beta$.

2. The 2-forms ω and σ are given by two families of parallel lines. There is a 1-form λ that is zero on both families. This 1-form defines a special family of parallel planes, and the lines can be rearranged to lie in these planes. This amounts to writing $\omega = \alpha \wedge \lambda$ and $\sigma = \beta \wedge \lambda$. The forms α and β each define a family of planes that intersect the special planes in these lines. Their sum $\alpha + \beta$ defines a third family of planes that intersect the special planes in lines. These lines give the sum $\omega + \sigma = (\alpha + \beta) \wedge \lambda$.

3. The sum of 3-forms defined by families of points is another family of points with density given by the sum or difference of the densities, depending on relative orientations.

The interior product of a vector X with a k-form is given as follows.

1. For a 1-form α the interior product is the number $X \lrcorner \alpha = \langle \alpha \mid X \rangle$.

2. For a 2-form ω the interior product $X \lrcorner \omega$ is defined as a 1-form whose planes are parallel to the planes defined by X and by the lines determining ω. The spacing of the planes is determined by $\langle X \lrcorner \omega \mid Y \rangle = \langle \omega \mid X \wedge Y \rangle$.

3. For a 3-form γ the interior product $X \lrcorner \gamma$ is defined as a 2-form whose lines are parallel to X. The spacing of the lines is determined by the formula $\langle X \lrcorner \gamma \mid Y \wedge Z \rangle = \langle \gamma \mid X \wedge Y \wedge Z \rangle$.

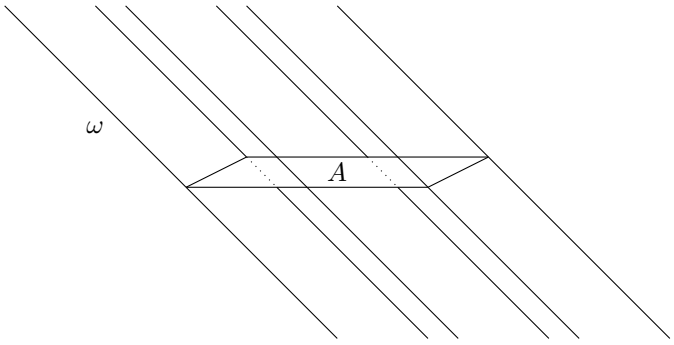

Figure 7.9: Pairing $\langle \omega \mid A \rangle$ of 2-form with 2-vector (3d)

Remark: It is possible to define k-forms as anti-symmetric multilinear functions on the vector space V. This bypasses the need to speak of k-vectors. For $k = 2$ or 3 the formulas are the left hand side in

$$\begin{aligned} \langle \omega \mid X, Y \rangle &= \langle \omega \mid X \wedge Y \rangle, \\ \langle \gamma \mid X, Y, Z \rangle &= \langle \omega \mid X \wedge Y \wedge Z \rangle. \end{aligned} \tag{7.3}$$

This will be the practice in most of what follows. ||

7.2 Differential k-forms

7.2.1 Algebra of differential k-forms

The subject of this chapter is differential k-forms and their integrals over k-dimensional surfaces. This leads to the ultimate generalization of the fundamental theorem of calculus: Stokes's theorem. This remarkable result makes no use of Euclidean geometry, in fact, it takes the same form in an arbitrary coordinate system.

The algebra and differential calculus of differential k-forms may be unfamiliar, but fortunately it is an easy subject. The main properties stated in the following sections may be proved by checking that the definitions take the same form after a change of coordinate system. These proofs are often dull; many will be omitted. The book of Rudin [42] takes such a computational approach, but he does not stress the invariance under coordinate changes, which is the most wonderful aspect of the subject. There are alternate proofs that are more interesting and conceptual, but they are also more abstract. More advanced books [4, 33] give an idea of such an approach.

Example: We begin with a quick summary of the algebraic aspect of differential forms. Consider the case of a three-dimensional space with arbitrary coordinates u, v, w. The 0-forms are the scalars. The 1-forms are $p\,du + q\,dv + r\,dw$. These are integrated over oriented curves. The 2-forms are $a\,dvdw + b\,dw\,du +$

$c\,du\,dv$. These are integrated over oriented surfaces. The 3-forms are $s\,du\,dv\,dw$. These are good for integrals over oriented 3-dimensional regions. However for the moment we are only concerned with the differential forms, not with their integrals.

These forms have an algebra. The fundamental law is the anticommutative law for 1-forms. Thus for instance $dw\,du = -\,du\,dw$. Since 1-forms anticommute with themselves, we have $du\,du = 0$, and so on.

The algebra here is called the exterior product. In theoretical discussions it is denoted by a wedge symbol, so that we would write $dv \wedge dw$ instead of the shorter form $dv\,dw$. Sometimes it is a good idea to use such a notation, since it reminds us that there is a rather special kind of algebra, different from ordinary multiplication. Practical computations tend to leave it out. ||

A theoretical approach to the definitions relates differential forms to vector fields. A *differential k-form* ω on a coordinate patch is a quantity that depends on k vector fields. We write it as $\langle \omega \mid X_1, \ldots, X_k \rangle$. One way to get such a k-form is to multiply together k 1-forms and then anti-symmetrize. The multiplication operation that accomplishes this is often written $\wedge$ and is called the *exterior product*. In the simplest case of a differential 2-form $\omega = \alpha \wedge \beta$ this is given by the determinant

$$\langle \alpha \wedge \beta \mid X, Y \rangle = \det \begin{bmatrix} \langle \alpha \mid X \rangle & \langle \beta \mid X \rangle \\ \langle \alpha \mid Y \rangle & \langle \beta \mid Y \rangle \end{bmatrix} \tag{7.4}$$

When we write this out we get

$$\langle \alpha \wedge \beta \mid X, Y \rangle = \langle \alpha \mid X \rangle \langle \beta \mid Y \rangle - \langle \alpha \mid Y \rangle \langle \beta \mid X \rangle. \tag{7.5}$$

This can be thought of as a kind of signed area attached to the vectors X, Y. This product anti-commutes: For 1-forms α, β we always have

$$\alpha \wedge \beta = -\beta \wedge \alpha. \tag{7.6}$$

In particular

$$\alpha \wedge \alpha = 0. \tag{7.7}$$

The general formula for a product of k 1-forms is also given by a determinant

$$\langle \alpha_1 \wedge \cdots \wedge \alpha_k \mid X_1, \ldots, X_k \rangle = \det \left(\langle \alpha_i \mid X_j \rangle \right). \tag{7.8}$$

This can be thought of as a kind of signed volume attached to the vectors $X_1, \ldots, X_k$.

The general definition of a k-form ω is that it associates to each $X_1, \ldots, X_k$ a number $\langle \omega \mid X_1, \ldots, X_k \rangle$. This expression is suppose to be multi-linear, that is, it is linear in each X_i with the other X_j for $j \neq i$ held fixed. It is also supposed to be alternating, in that interchanging two vectors gives a minus sign.

There is also a general definition of the exterior product of forms. If θ is a p-form and λ is a q-form, then $\theta \wedge \lambda$ is a $p + q = k$ form given by

$$\langle \theta \wedge \lambda \mid X_1, \ldots, X_k \rangle = \sum_{\sigma} \operatorname{sign}(\sigma) \langle \theta \mid X_{\sigma(1)}, \ldots, X_{\sigma(p)} \rangle \langle \lambda \mid X_{\sigma(p+1)}, \ldots, X_{\sigma(k)} \rangle, \tag{7.9}$$

where the sum is over all permutations such that $\sigma(1), \dots, \sigma(p)$ are in increasing order and $\sigma(p+1), \dots, \sigma(k)$ are in increasing order. See [4].

There is a more compact and memorable way of writing this result. Let $U = \{1, \dots, p+q\}$. If $S \subseteq U$, write X_S for the list of X_j with j in S, taken in the original order. Then

$$\langle \theta \wedge \lambda \mid X_U \rangle = \sum_{|S|=p} \text{sign}(S) \langle \theta \mid X_S \rangle \langle \lambda \mid X_{U \setminus S} \rangle. \tag{7.10}$$

The coefficient $\text{sign}(S)$ depends on the number of interchanges needed to bring X_U to the order $X_S, X_{U \setminus S}$; there is a factor of -1 for each interchange.

Example: Consider the product of a 2-form θ with a 1-form β. Then from X, Y, Z one can select the 2-element subset X, Y or X, Z or Y, Z. This gives

$$\langle \theta \wedge \beta \mid X, Y, Z \rangle = \langle \theta \mid X, Y \rangle \langle \beta, Z \rangle - \langle \theta \mid X, Z \rangle \langle \beta, Y \rangle + \langle \theta \mid Y, Z \rangle \langle \beta, X \rangle. \tag{7.11}$$

The minus sign in the second term comes from the fact that the permutation that sends XYZ to XZY interchanges Z with Y. The plus sign in the final term comes from the permutation that sends XYZ to YZX, which can be obtained by taking XYZ to YXZ and then to YZX. ||

The multiplicative properties are summarized as follows.

Associative law

$$(\omega \wedge \sigma) \wedge \tau = \omega \wedge (\sigma \wedge \tau). \tag{7.12}$$

Distributive law

$$\omega \wedge (\sigma + \tau) = \omega \wedge \sigma + \omega \wedge \tau. \tag{7.13}$$

Commutative (anticommutative) law For k-form ω and ℓ-form σ

$$\omega \wedge \sigma = (-1)^{k\ell} \sigma \wedge \omega. \tag{7.14}$$

Even degree forms commute with everything. On the other hand, odd degree forms anticommute with each other. In particular, if ω has odd degree, then $\omega \wedge \omega = 0$.

Here is a summary in the language of modules. Given a n-dimensional manifold patch M, the module $\mathcal{E}$ of all vector fields on M has a dual module $\mathcal{E}^*$ consisting of all 1-forms on M. A natural notation for the module of all k-forms on M is $\Lambda^k \mathcal{E}^*$. The rank of $\Lambda^k \mathcal{E}^*$ is the binomial coefficient $\binom{n}{k}$. Special cases are $\Lambda^0 \mathcal{E}^* = \mathcal{R}$, the ring of scalar fields on M, and $\Lambda^1 \mathcal{E}^* = \mathcal{E}^*$. This goes up to the top rank $\Lambda^n \mathcal{E}^*$.

There is an even larger module $\Lambda \mathcal{E}^*$ consisting of all sums of k-forms for varying k. This has rank 2^n. It has the structure of an algebra, where the product is $\wedge$. In the next section we shall see that there is an operation $d : \Lambda \mathcal{E}^* \to \Lambda \mathcal{E}^*$ that differentiates a form and then manipulates it algebraically in such a way that the result is again a form.

7.2.2 The exterior derivative on differential k-forms

Example: Say that we have a three-dimensional space with coordinates x, y, z. If we have a scalar like $x^2 z$, then we know how to take its differential. In this case we get

$$d(x^2 z) = 2xz\,dx + x^2\,dz. \tag{7.15}$$

||
Example: Say that we have a 1-form like $x^2\,dz$ with the scalar on the left. Then the differential is

$$d(x^2\,dz) = d(x^2)\,dz = 2x\,dx\,dz. \tag{7.16}$$

However if we wrote the scalar on the right, this would be

$$d(dz\,x^2) = -dz d(x^2) = -dz\,2x\,dx = -2x\,dz\,dx = 2x\,dx\,dz \tag{7.17}$$

which is the same thing. The rule is that each time the d is moved past another d there is a new minus sign. ||
Example: Say that we have a differential form like $x^2 z\,dy\,dz$. Compute the differential by putting the scalar on the left. The derivative is obtained by taking the differential of the scalar part. Thus

$$d(x^2 z\,dy\,dz) = d(x^2 z)\,dy\,dz. \tag{7.18}$$

When we compute this, we get

$$d(x^2 z\,dy\,dz) = (2xz\,dx + x^2\,dz)\,dy\,dz = 2xz\,dx\,dy\,dz + x^2\,dz\,dy\,dz. \tag{7.19}$$

But $dz\,dy\,dz = -dy\;dz\,dz = 0$ since $dz\,dz = 0$. So the final result is

$$d(x^2 z\,dy\,dz) = 2xz\,dx\,dy\,dz. \tag{7.20}$$

||

Now to a more theoretical treatment. We already know that the differential of a scalar is

$$du = \frac{\partial u}{\partial x_1}\,dx_1 + \cdots + \frac{\partial u}{\partial x_n}\,dx_n. \tag{7.21}$$

Every k form may be written as a sum of forms of the form

$$\omega = u\,dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \tag{7.22}$$

We can define the *exterior derivative* or *differential* by

$$d\omega = du \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \tag{7.23}$$

It is important that the du goes on the left.

Here are the main properties of the exterior derivative.

Additivity

$$d(\omega + \sigma) = d\omega + d\sigma. \tag{7.24}$$

Product property If ω is a k-form and σ is an ℓ-form, then

$$d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^k \omega \wedge d\sigma. \tag{7.25}$$

Differential of a differential

$$dd\omega = 0. \tag{7.26}$$

If we think of d as a degree one quantity, then the sign in the product property makes sense. Also, in this context the differential of a differential property also makes sense.

Theorem 7.1 *There is a unique operation d that takes k-forms to $(k+1)$-forms, agrees with the d operation for every scalar field (0-form), and satisfies the above properties.*

One proof of the theorem uses the coordinate form for $d\omega$ given below. There is another proof that uses sophisticated properties of vector fields. See [34] or [4] for further discussion.

A k-form σ is called *closed* if $d\sigma = 0$. A k-form σ is called *exact* in an open set if $\sigma = d\alpha$ for some $(k-1)$-form α in the open set. It follows from the differential of a differential property that every exact form is closed. (A 0-form is closed if it is constant on each connected component; it is exact if it is constant.)

It is useful to look at these quantities in low dimensions. For instance, in three dimensions one might have a differential 2-form such as

$$\sigma = a\, dy \wedge dz + b\, dz \wedge dx + c\, dx \wedge dy. \tag{7.27}$$

Here x, y, z are arbitrary coordinates, and a, b, c are smooth functions of x, y, z. Similarly, in three dimensions a typical 3-form might have the form

$$\tau = s\, dx \wedge dy \wedge dz. \tag{7.28}$$

Notice that these forms are created as linear combinations of exterior products of 1-forms.

Since these expressions are so common, it is customary in many contexts to omit the explicit symbol for the exterior product. Thus the forms might be written

$$\sigma = a\, dy\, dz + b\, dz\, dx + c\, dx\, dy \tag{7.29}$$

and

$$\tau = s\, dx\, dy\, dz. \tag{7.30}$$

The exterior derivative of an r-form α is an $(r+1)$-form $d\alpha$. It is defined by taking the differentials of the coefficients of the r-form. For instance, for the 1-form

$$\alpha = p\, dx + q\, dy + r\, dz \tag{7.31}$$

the differential is

$$d\alpha = dp\,dx + dq\,dy + dr\,dz. \tag{7.32}$$

This can be simplified as follows. First, note that

$$dp = \frac{\partial p}{\partial x}\,dx + \frac{\partial p}{\partial y}\,dy + \frac{\partial p}{\partial z}\,dz. \tag{7.33}$$

Therefore

$$dp\,dx = \frac{\partial p}{\partial y}\,dy\,dx + \frac{\partial p}{\partial z}\,dz\,dx = -\frac{\partial p}{\partial y}\,dx\,dy + \frac{\partial p}{\partial z}\,dz\,dx. \tag{7.34}$$

Therefore, the final answer is

$$d\alpha = \left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}\right) dy\,dz + \left(\frac{\partial p}{\partial z} - \frac{\partial r}{\partial x}\right) dz\,dx + \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx\,dy. \tag{7.35}$$

Similarly, suppose that we have a 2-form

$$\sigma = a\,dy\,dz + b\,dz\,dx + c\,dx\,dy. \tag{7.36}$$

Then

$$d\sigma = da\,dy\,dz + db\,dz\,dx + dc\,dx\,dy = \frac{\partial a}{\partial x}\,dx\,dy\,dz + \frac{\partial b}{\partial y}\,dy\,dz\,dx + \frac{\partial c}{\partial z}\,dz\,dx\,dy. \tag{7.37}$$

This simplifies to

$$d\sigma = d(a\,dy\,dz + b\,dz\,dx + c\,dx\,dy) = \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}\right) dx\,dy\,dz. \tag{7.38}$$

Theorem 7.2 (Coordinate form for exterior derivative) *The exterior derivative of the k-form*

$$\omega = \sum_{i_1<i_2<\cdots<i_k} p_{i_1 i_2 \cdots i_k}\,dx_{i_1}\,dx_{i_2}\cdots dx_{i_k} \tag{7.39}$$

is the $(k+1)$-form

$$d\omega = \sum_{i_1<i_2<\cdots<i_k} \sum_j \frac{\partial p_{i_1 i_2 \cdots i_k}}{\partial x_j}\,dx_j\,dx_{i_1}\,dx_{i_2}\cdots dx_{i_k}. \tag{7.40}$$

The inner sum is over all $j \neq i_1, \ldots, i_k$.

7.2.3 Pullback of a differential k-form

Now suppose that N is a manifold patch of dimension k and M is a manifold patch of dimension n. Suppose $\phi : N \to M$ is a smooth function. We shall call such a ϕ a *manifold mapping*. Sometimes we may just say *mapping*.

Here is a framework relating the notion of manifold mapping to other concepts:

- $\mathbf{u} : N \to \mathbf{R}^k$ is a coordinate system on N.
- $\mathbf{x} : M \to \mathbf{R}^n$ is a coordinate system on M,
- $\mathbf{f}$ is a function from an open set in $\mathbf{R}^k$ to $\mathbf{R}^n$.
- $\mathbf{f}(\mathbf{u}) : N \to \mathbf{R}^n$ is a function from N to $\mathbf{R}^n$.
- $\phi : N \to M$ is the manifold mapping equal to $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u})$.

The notation $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u})$ is a way of defining the manifold mapping ϕ in terms of coordinate systems. Thus ϕ takes a point in N, reads the numbers $\mathbf{u}$, computes the numbers $\mathbf{f}(\mathbf{u})$, and then finds the point in M where $\mathbf{x}$ has this value.

Let s be a scalar field on M. Define the *pullback* $\phi^* s = s \circ \phi$ as a scalar field on N. Thus if $s = h(\mathbf{x})$, then the pullback is

$$(\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u}))^* h(\mathbf{x}) = h(\mathbf{x})(\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u})) = h(\mathbf{f}(\mathbf{u})). \tag{7.41}$$

Similarly, define the pullback of an exact 1-form ds by $\phi^* \, ds = d\phi^* s$. Thus

$$(\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u}))^* dh(\mathbf{x}) = dh(\mathbf{f}(\mathbf{u})) = \sum_{j=1}^{n} \sum_{i=1}^{m} h'_{,i}(\mathbf{f}(\mathbf{u})) f'_{i,j}(\mathbf{u}) \, du_j. \tag{7.42}$$

In particular,

$$(\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u}))^* dx_i = d(f_i(\mathbf{u})) = \sum_{j=1}^{n} f'_{i,j}(\mathbf{u}) \, du_j. \tag{7.43}$$

Every k form may be written as a sum of forms of the form

$$\omega = h(\mathbf{x}) \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \tag{7.44}$$

We can define the *pullback* by

$$(\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u}))^* \omega = h(\mathbf{f}(\mathbf{u})) \wedge df_{i_1}(\mathbf{u}) \wedge \cdots \wedge df_{i_k}(\mathbf{u}). \tag{7.45}$$

This can then be written in terms of products of the du_j. If the products are arranged in order, then the resulting coefficient is a determinant.

Example: Consider the mapping $\phi = ((z, w) \leftarrow (x^2 y, xy^3))$. Then

$$\phi^*(dz \, dw) = (2xy \, dx + x^2 \, dy)(y^3 \, dx + 3xy^2 \, dy) = 5x^2 y^3 \, dx \, dy. \tag{7.46}$$

The $5x^2y^3$ factor is the determinant of the derivative of the transformation that relates z, w to x, y. We have already encountered similar formulas when we change coordinates. In this special case we think of the mapping ϕ as an equality, and we write the result as an equality. However in general we want to keep the possibility that z, w describe one set of objects and x, y describe another set of objects. In this case we must write ϕ explicitly to describe how one goes from the objects described by x, y to the objects described by z, w. ||

The general properties of the pullback are extraordinarily nice.

Additivity

$$\phi^*(\omega + \sigma) = \phi^*\omega + \phi^*\sigma. \tag{7.47}$$

Product property

$$\phi^*(\omega \wedge \sigma) = \phi^*\omega \wedge \phi^*\sigma. \tag{7.48}$$

Derivative

$$\phi^*(d\omega) = d(\phi^*\omega). \tag{7.49}$$

Theorem 7.3 (Pullback theorem) *Given the manifold mapping $\phi : N \to M$, there is a unique pullback operation that sends a differential forms ω on M to differential forms $\phi^*\omega$ on N, agrees with the usual pullback of scalar fields, and satisfies these general properties.*

A proof of this theorem may be found in [34] or [4]. Another nice property of pullback is its behavior under composition of mappings:

$$(\phi \circ \chi)^*\omega = \chi^*\phi^*\omega. \tag{7.50}$$

Theorem 7.4 (Coordinate expression of pullback of a m-form) *Say that the m-form ω is given in coordinates by*

$$\omega = \sum_I p_I \, d\mathbf{x}_I. \tag{7.51}$$

Here I ranges over increasing sequences of m indices taken from the n indices. The pullback by the manifold map $\phi : N \to M$ is

$$\phi^*\omega = \sum_J \left(\sum_I (p_I \circ \phi) \det \frac{\partial (x_I \circ \phi)}{\partial u_J} \right) d\mathbf{u}_J. \tag{7.52}$$

Here J ranges over increasing sequences of m indices taken from the k indices. Say that $p_I = h_I(\mathbf{x})$ and that ϕ is given in coordinates by $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u})$. Then in coordinates the pullback is

$$\phi^*\omega = \sum_J \left(\sum_I h_I(\mathbf{f}(\mathbf{u})) \det(\mathbf{f}'_{I,J}(\mathbf{u})) \right) d\mathbf{u}_J. \tag{7.53}$$

There are several interpretations of pullback. The most typical one is where the coordinates x_i are describing one situation, and the coordinates u_j are describing some other situation. For instance, the u_j could be parameters for a surface in the space described by the x_i. There is no reason to require that N and M have the same dimension. The transformation $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u})$ takes points of N to points of a different space M. In this case it is best to write the pullback explicitly.

There is one situation that can be particularly confusing. This is when N is a subset of M, and ϕ sends each point in N into the same point, but now

regarded as a point in M. In this situation it is possible to write $\mathbf{x}_N = \mathbf{f}(\mathbf{u})$. This means that the restriction of $\mathbf{x}$ to N is $\mathbf{f}(\mathbf{u})$.

Consider the case $M = N$. The pullback has two special interpretations. In the case of a *passive tranformation* the transformation ϕ given by $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u})$ is the identity. All that is happening is that $\mathbf{x}$ coordinates are being expressed in terms of $\mathbf{u}$ coordinates. In this case it is appropriate to write $\mathbf{x} = \mathbf{f}(\mathbf{u})$ as an actual equality of functions on M. One writes everything as an equality, for instance,

$$dx_i = \sum_j \frac{\partial x_i}{\partial u_j}\, du_j. \tag{7.54}$$

In the case of an *active transformation* with $M = N$, the transformation ϕ given by $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{x})$ makes sense and need not be the identity transformation. One can write $\mathbf{x}' = \mathbf{x} \circ \phi = \mathbf{f}(\mathbf{x})$. Then

$$dx'_i = \sum_j \frac{\partial x'_i}{\partial x_j}\, dx_j. \tag{7.55}$$

This represents the change after one iteration in response to a change in the initial condition.

Remark: There is a relation between pullback of a differential k-form and the pushforward of vector fields. Say that ω is a k-form on M, $Z_1, \ldots, Z_k$ are vector fields on N, and $\phi : N \to M$ is a manifold mapping. Then the pullback $\phi^*\omega$ of ω to N is

$$\langle \phi^*\omega \mid Z_1, \ldots, Z_k \rangle = \langle \omega_{[\phi]} \mid \phi_* Z_1, \ldots, \phi_* Z_k \rangle. \tag{7.56}$$

The right hand side is the pairing between the restricted form along ϕ and the pushforward vector fields along ϕ. ||

7.2.4 The Poincaré lemma*

A k-form ω is *exact* if $\omega = d\alpha$ for some $k-1$ form α. If this is to happen, then ω must satisfy $d\omega = 0$. The integral α is not uniquely defined. In fact, if $\beta = d\gamma$ is an exact $k-2$ form, then also $\omega = d(\alpha + \beta)$. In some contexts the $(k-1)$-form α is called a *potential* for $\omega = d\alpha$. It may give a simpler description of α.

The most elementary example of this is when ω is a 1-form. If ω is exact with $\omega = ds$, then also $\omega = d(s + c)$, where c is a constant. (A 0-form is considered to be exact if it is constant.) In some applications another terminology is used in which a 1-form ω with $d\omega = 0$ is said to be *irrotational*. A function s with $ds = \alpha$ is then called a *scalar potential*.

When a 2-form ω is exact, then $\omega = d\alpha$ with α a 1-form. Such an α is called a *1-form potential* for ω. It is defined up to addition of an exact 1-form $\beta = ds$. So the ambiguity in the 1-form potential is captured in the choice of the scalar field s. Sometimes the replacement of α by $\alpha + ds$ is called a *gauge transformation*.

(In two dimensions the 1-form potential always has an integrating factor. In three dimensions it need not have an integrating factor, but it may be possible to

make a gauge transformation to replace it by one which does have an integrating factor.)

Example: The 2-form $\omega = z^2\,dx\,dz + (1 - e^x)\,dx\,dy$ satisfies $\omega = d(xz^2\,dz - (1 - e^x)y\,dx)$. It also satisfies $\omega = d(xz^2\,dz + x\,dy + e^x y\,dx)$. So there are two integrals. This explained because they differ by $x\,dy + y\,dx = d(xy)$. ||

Example: For a 3-form an integral (if it exists) is defined only up to an exact 2-form. Take $\omega = 2xz\,dx\,dy\,dz$. This is $\omega = d(x^2 z\,dy\,dz)$. Also $\omega = d(xz^2\,dx\,dy)$. The difference $xz^2\,dx\,dy - x^2 z\,dy\,dz = xz^2\,dx\,dy + x^2 z\,dz\,dy = (1/2)d(x^2 z^2\,dy)$. ||

In the following it will be convenient to have a notion of a set in which nothing interesting can happen. A subset U of $\mathbf{R}^n$ will here be called a *nice region* if it is diffeomorphic to an open ball. This is not standard terminology, but it will be convenient here.

Proposition 7.5 *The following are nice regions, that is, diffeomorphic to the open ball B centered at zero with radius one.*

1. *The space $\mathbf{R}^n$.*
2. *The space $\mathbf{R}^n_+$ consisting of all points in $\mathbf{R}^n$ with strictly positive coordinates.*
3. *The open cube C_n with all coordinates between 0 and 1.*
4. *The interior of the simplex Δ_n consisting of all points $\mathbf{x}$ in C_n with $\sum_{i=1}^n x_i < 1$.*
5. *The open subset of $\mathbf{R}^2$ excluding the points with first component negative. [This subset is not convex.]*

Proof:

1. There is a map from $\mathbf{x}$ in B to $\mathbf{y}$ in $\mathbf{R}^n$ given by $\mathbf{y} = \mathbf{x}/\sqrt{1 - |\mathbf{x}|^2}$. The inverse map is given by $\mathbf{x} = \mathbf{y}/\sqrt{1 + |\mathbf{y}|^2}$.
2. There is a map from $\mathbf{R}^n$ to $\mathbf{R}^n_+$ given by $z_i = e^{y_i}$. The inverse map is $y_i = \log(z_i)$.
3. There is a map from C_n to $\mathbf{R}^n_+$ given by $z_i = u_i/\sqrt{1 - u_i^2}$. The inverse map is $u_i = z_i/\sqrt{1 + z_i^2}$.
4. There is a map from Δ_n to $\mathbf{R}^n_+$ given by $\mathbf{z} = \mathbf{x}/(1 - \sum_i x_i)$. The inverse map is $\mathbf{x} = \mathbf{z}/(1 + \sum_i z_i)$.
5. There is a map from $\mathbf{R}^2_+$ to the open right half plane obtained by doubling the angle and then rotating. This may be followed by a map from the open right half plane to the open subset with excluded half-line; again one doubles the angle.

□

An n-dimensional *local manifold patch* is a manifold patch with a coordinate system $\mathbf{x} : M \to U$, where $U \subseteq \mathbf{R}^n$ is a nice region. In the case $n = 0$ a local manifold patch is a single point. The terminology used here is not standard, but the idea is that a local manifold patch has no interesting global features. In fact, for each n it is an essentially unique object.

Example: An example of a 2-dimensional manifold patch with global features is one modeled on a plane with a point removed. Another example is one modeled on a plane with two points removed. These last two examples are not only not diffeomorphic to the plane, but they are also not diffeomorphic to each other. In fact, they are very different as global objects. ||

Example: If ω is defined on a manifold patch, that is, if $\omega = d\alpha$ in the region, then ω is closed: $d\omega = 0$. The converse is not true in general. Here is a two dimensional example. Let

$$\omega = \frac{1}{x^2 + y^2}(x\,dy - y\,dx). \tag{7.57}$$

in the plane with the origin removed. Then ω is closed, but not exact. If we remove a line running from the origin to infinity, then the resulting region is a local manifold patch. In this smaller region ω is exact, in fact, $\omega = d\theta$, where $x = r\cos(\theta)$ and $y = r\sin(\theta)$. ||

What is true is that if ω is closed, then ω is locally exact. In fact, it is exact on every local manifold patch. This will be proved in the following famous *Poincaré lemma*. The treatment here is adopted from Flanders [14]; a perhaps more concrete treatment along the same line appears in Spivak [45]. The book Rudin [42] gives an inductive proof.

Theorem 7.6 (Poincaré lemma) *Consider a local manifold patch of dimension n. Let $1 \leq k \leq n$ and suppose that the form ω is a closed k-form, that is, $d\omega = 0$. Then ω is exact on this local manifold patch, that is, there is a form α defined on it with $\omega = d\alpha$.*

Proof: We may as well assume that the local manifold patch is an open ball B centered at the origin with Cartesian coordinates $\mathbf{x}$. The idea is to introduce time dependence. Let C be a space with coordinates $(t, \mathbf{x})$, where $0 \leq t \leq 1$ and the $\mathbf{x}$ are coordinates on B. A differential k-form σ on C involves differentials dt and some of the dx_j. It may be written $\sigma = \sigma_0 + \sigma_1$, where σ_0 is the static part, each term of which involves the product of k differentials dx_j but with no factors of dt, and σ_1 is the remaining dynamic part, with a factor of dt preceding a product of $k - 1$ differentials dx_j. Define

$$K(\sigma) = \int_0^1 \sigma_1. \tag{7.58}$$

Then $K(\sigma)$ is a differential $(k-1)$-form on B obtained by integrating out the time dependence. It involves only products of k differentials dx_i.

This integral operation K has two important properties. The first is

$$dK(\sigma_1) = -K(d\sigma_1). \tag{7.59}$$

To see this, write $\sigma_1 = dt\,\beta_0$. The calculation $d\sigma_1 = -dt\,d\beta_0$ introduces a crucial minus sign. So the equation to be proved is $dK(dt\,\beta_0) = K(dt\,d\beta_0)$. The terms in $d\beta_0$ that involve dt do not contribute to $dt\,d\beta_0$. So this follows from differentiating under the integral sign.

The second important property is that

$$K(d\sigma_0) = \sigma_0(t \leftarrow 1) - \sigma_0(t \leftarrow 0). \tag{7.60}$$

The terms in $d\sigma_0$ that do not involve dt contribute nothing to $K(d\bar{\omega}_0)$. The terms that do involve dt involve the partial derivative of the form with respect to t. So this follows from the fundamental theorem of calculus.

Define the shrinking mapping $\phi : C \to B$ to be $tx \leftarrow (t, x)$. As t varies from 1 to 0 this shrinks the ball to a point. If ω is a k-form on B, then we may obtain a k-form $\bar{\omega} = \phi^*\omega$ on C by substituting tx_i for x_i everywhere. In particular expressions dx_i become $d(tx_i) = dt\,x_i + t\,dx_i$. Write $\bar{\omega} = \bar{\omega}_0 + \bar{\omega}_1$. Thus if $\omega = f(\mathbf{x})\,d\mathbf{x}_I$ is a monomial, then this is

$$\bar{\omega} = \bar{\omega}_0 + \bar{\omega}_1 = f(t\mathbf{x})t^k\,d\mathbf{x}_I + \sum_j (-1)^{j-1} f(t\mathbf{x}) dt\,x_j\,t^{k-1}\,d\mathbf{x}_{I\setminus\{j|}. \tag{7.61}$$

The next step is the identity

$$dK(\bar{\omega}) + K(d\bar{\omega}) = \omega. \tag{7.62}$$

The argument for this startes with $K(\bar{\omega}_0) = 0$. This gives

$$dK(\bar{\omega}) + K(d\bar{\omega}) = dK(\bar{\omega}_1) + K(d\bar{\omega}_1) + K(d\bar{\omega}_0).. \tag{7.63}$$

By previous results this is

$$K(d\bar{\omega}_0) = \bar{\omega}_0(t \leftarrow 1) - \bar{\omega}_0(t \leftarrow 0) = \omega. \tag{7.64}$$

To complete the proof, suppose that ω is closed, that is, $d\omega = 0$. Then $d\bar{\omega} = d\phi^*\omega = \phi^* d\omega = 0$. The identity gives

$$dK(\bar{\omega}) = \omega. \tag{7.65}$$

Thus $\omega = d\alpha$ must be exact, with $\alpha = K(\bar{\omega})$. $\square$

Remark: The algorithm is simple, provided that one can do the integrals. Start with a closed differential form ω defined in terms of $x_1, \ldots, x_n$. Replace x_i by tx_i everywhere, including in differentials. Collect all terms that begin with dt. Put the dt in front. Integrate from 0 to 1 (with respect to t, keeping everything else fixed). The result is a form α with $d\alpha = \omega$. ||

Example: Consider the closed 1-form $\omega = x\,dy + y\,dx$. Then $\bar{\omega} = t^2 x\,dy + t^2 y\,dx + 2xyt\,dt$. The integral of $2xyt\,dt$ is $\alpha = xy$. ||

Example: Consider the closed 2-form $\omega = dx\,dy$. Then $\bar{\omega} = t^2 dx\,dy + tx\,dt\,dy - ty\,dt\,dx$. The integral of $t\,dt\,(x\,dy - y\,dx)$ is $\alpha = (1/2)(x\,dy - y\,dx)$. ||

7.2.5 Integrating factor conditions

A non-zero 1-form ω is exact when $\omega = ds$. In that case there are surfaces defined implicitly by $s = C$ for various constants C. The pullback of ω to such a surface is zero.

There is a somewhat more general situation. Consider a non-zero 1-form. If r is a non-zero scalar field with $r\omega = ds$, then r is called an *integrating factor*. Again the pullback of ω to a surface $s = C$ is zero.

We say that the form ω satisfies the *integrating factor condition* if $\omega \wedge d\omega = 0$. This is automatic in dimension two. In higher dimensions it is not at all typical.

Proposition 7.7 *Suppose that ω is a non-zero 1-form with an integrating factor. Then ω must satisfy the integrating factor condition.*

Proof: If ω has an integrating factor, then there are scalar fields u and s with $\omega = u\,ds$. Then

$$d\omega = du \wedge ds = \frac{1}{u}\, du \wedge \omega. \tag{7.66}$$

Since $\omega \wedge \omega = 0$, the result follows. □

The integrating factor condition is always satisfied for $n = 2$. However, for $n = 3$ there are examples like $x\,dy + dz$ that do not satisfy this condition.

There is a special case of the *Frobenius theorem* that implies that $\omega \neq 0$ and $\omega \wedge d\omega = 0$ implies that in some coordinte system $\alpha = dx$. A more general *Pfaff theorem* gives even stronger results, the first of which is that $\omega \neq 0$ and $\omega \wedge d\omega \wedge d\omega = 0$ implies that in some coordinate system $\omega = dx - y\,dz$. See [5] for a discussion.

Example: It is relatively rare in three dimensions that the subspaces where a 1-form is zero define a family of surfaces. Here is an example. Let $\omega = 3yz\,dx + 2xz\,dy + xy\,dz$. Then $\omega \wedge d\omega = 0$, so it is plausible that ω has an integrating factor. In fact, $x^2y\omega$ is exact. The curves on which it vanishes are of the form $x^3y^2z = C$. ||

Example: It may seem puzzling tht $x\,dy + dz$ does not have an integrating factor, and there is no associated family of surfaces. However it is a fact that $\omega \wedge d\omega$ is not zero. In such a situation there is a plane at each point on which the form is zero. In this particular case the plane is spanned by $\partial/\partial x$ and $\partial/\partial y - x\partial/\partial z$. If there were a surface, then these planes would fit nicely together in layers. However in such a case the planes are shuffled together in a way that does not allow them to be tangent to a surface. The book [3, Fig. 5.1] illustrates the geometry of this form. ||

7.3 Vector fields and differential k-forms

7.3.1 The interior product

There is an operation called *interior product* (or *contraction*). In the case of interest to us, it is a way of defining the product of a vector field X with a

k-form ω to get a $(k-1)$-form $X \lrcorner \omega$. (The exterior product is also called the wedge product; the interior product is sometimes called the hook product.) The definition of interior product that is used in practice is very simple and may be illustrated by an example. Consider a basis vector field $\frac{\partial}{\partial u_j}$ and a differential form $du_i\, du_j\, du_k$. The rule is to move the du_j to the left and then remove it. So the result in this case is

$$\frac{\partial}{\partial u_j} \lrcorner\, du_i\, du_j\, du_k = -\frac{\partial}{\partial u_j} \lrcorner\, du_j\, du_i\, du_k = -du_i\, du_k. \tag{7.67}$$

There is a general theoretical definition that is also illuminating. For $k \geq 1$ the interior product of the vector field X with the k-form σ is defined by

$$\langle X \lrcorner \sigma \mid Y_1, \ldots, Y_{k-1} \rangle = \langle \sigma \mid X, Y_1, \ldots, Y_{k-1} \rangle. \tag{7.68}$$

For a 1-form α the interior product $X \lrcorner \alpha$ is the scalar field $\langle \alpha \mid X \rangle$. For a scalar field s we take $X \lrcorner s = 0$.

One interesting property of the interior product is that if α is an r-form and β is an s-form, then

$$X \lrcorner (\alpha \wedge \beta) = (X \lrcorner \alpha) \wedge \beta + (-1)^r \alpha \wedge (X \lrcorner \beta). \tag{7.69}$$

This follows the pattern of a derivation.

Apply this when $r = 1$ and $s = n$. Since β is an n-form, it follows that $\alpha \wedge \beta = 0$. Hence in this special case

$$\langle \alpha \mid X \rangle \beta = \alpha \wedge (X \lrcorner \beta). \tag{7.70}$$

Another application is with two 1-forms β and γ. In this case it gives

$$X \lrcorner (\beta \wedge \gamma) = \langle \beta \mid X \rangle \gamma - \langle \gamma \mid X \rangle \beta. \tag{7.71}$$

So the interior product of a vector with $\beta \wedge \gamma$ is a linear combination of β and γ.

Later we shall see the connection with classical vector algebra in three dimensions. The exterior product $\beta \wedge \gamma$ is an analog of the cross product, while $\alpha \wedge \beta \wedge \gamma$ is an analog of the triple scalar product. The combination $-X \lrcorner (\beta \wedge \gamma)$ will turn out to be an analog of the triple vector product.

For a vector field X and 1-forms $\alpha_1, \ldots, \alpha_k$

$$X \lrcorner (\alpha_1 \wedge \cdots \wedge \alpha_k) = \sum_{i=1}^{k} (-1)^{i-1} \langle \alpha \mid X \rangle \alpha_1 \wedge \cdots \wedge \alpha_{i-1} \wedge \alpha_{i+1} \wedge \cdots \wedge \alpha_k. \tag{7.72}$$

In a coordinate representation this implies the following identity for the interior product of a vector field with a k-form:

$$\left(\sum_{j=1}^{k} a_j \frac{\partial}{\partial u_j} \right) \lrcorner\, du_1 \cdots du_k = \sum_{i=1}^{k} a_i (-1)^{i-1}\, du_1 \cdots\, du_{i-1}\, du_{i+1} \cdots\, du_k. \tag{7.73}$$

Define the *characteristic subspace* of a non-zero k form ω to be the set of vector fields X for which $X \lrcorner \omega = 0$. (The characteristic subspace may also be called the *nullspace* of the form.) Thus ω vanishes on every k-tuple $X, Y_1, \ldots, Y_{k-1}$ to which X belongs.

Proposition 7.8 *Every non-zero k-form with $k = 1$ or $k = n - 1$ has characteristic subspace of dimension $n - k$ at every point. That is, a 1-form has characteristics subspace of dimension $n-1$, while a $(n-1)$-form has characteristic subspace of dimension 1.*

Proof: The harder case is that of an $(n-1)$-form ω. Consider a non-zero n-form σ. This volume form allows a correspondence that sends $X \to X \lrcorner \sigma$. This correspondence is invertible, so it is possible to write $\omega = X \lrcorner \sigma$ with $X \neq 0$. This X is in the characteristic subspace of ω. If X' is also in the characteristic subspace of ω, then $\langle X' \lrcorner \omega \mid Y_1, \ldots, Y_{n-2} \rangle = \langle \sigma \mid X, X', Y_1, \ldots, Y_{n-2} \rangle = 0$. However if X, X' were linearly independent, it would be possible to make this non-zero. So X' must be a multiple of X. $\square$

Remark: There are differential k-forms that have characteristic subspace of dimension strictly less than $n - k$. The simplest example is a 2-form in 4 dimensions: $dx\,dy + dz\,dw$. For more information, including the relation to indecomposable forms, see the books by Crampin and Pirani [10] and by Şuhubi [48]. ||

Proposition 7.9 *Suppose that ω is a non-zero 1-form with an integrating factor. If X and Y are in the characteristic space of ω, then the Lie product $[X, Y]$ must also be in the characteristic subspace.*

Proof: Let $\omega = r\,ds$. Since $r\,ds$ and ds have the same characteristic subspace, it is sufficient to prove this for ds. Suppose $\langle ds \mid X \rangle = 0$ and $\langle ds \mid Y \rangle = 0$. Then $Xs = 0$ and $Ys = 0$. it follows immediately that $[X, Y]s = 0$. Hence $\langle ds \mid [X, Y] \rangle = 0$. $\square$

Example: Consider again the example $\omega = 3yz\,dx + 2xz\,dy + xy\,dz$. For this example x^2y is an integrating factor. The special feature is seen by examining the characteristic subspaces. These are spanned by $2x\partial/\partial x - 3y\partial/\partial y$ and by $x\partial/\partial x - 3z\partial/\partial z$. The Lie product of these vector fields is zero and hence is in the characteristic subspace. ||

Example: As we have seen, the form $x\,dy + dz$ does not have an integrating factor. The characteristic subspaces are spanned by $\partial/\partial x$ and $\partial/\partial y - x\partial/\partial z$. The Lie product of these vector fields is $-\partial/\partial z$, which is not in the characteristic subspace. Moving in the direction of $\partial/\partial x$ amounts to a flow $x \leftarrow x + t$. This changes the other basis vector to $\partial/\partial y - (x + t)\partial/\partial z$. This is equal to $\partial/\partial y - x\partial/\partial z - t\partial/\partial z$. The flow moves it out of the characteristic subspace. ||

7.3.2 Flux curves for a $(n-1)$-form

Theorem 7.10 *Suppose that*

$$\omega = \sum_{j=1}^{n} (-1)^{j-1} p_j \, du_1 \cdots du_{j-1} \, du_{j+1} \cdots du_n \tag{7.74}$$

is a non-zero $n-1$ form. For a vector field in the characteristic space one can take

$$X = \sum_{k=1}^{n} p_k \frac{\partial}{\partial u_k}. \tag{7.75}$$

This vector field is not unique; an arbitrary non-zero scalar field multiple would also work.

Proof: Let $X = \sum_k a_k \frac{\partial}{\partial u_k}$. The equation $X \lrcorner \omega = 0$ turns out to be equivalent to $a_k p_j = a_j p_k$ for all k, j. This can be seen by writing

$$\omega = \sum_{j} (-1)^{j-1} p_j \, du_1 \cdots \widehat{du_j} \cdots du_n, \tag{7.76}$$

where the hat represents a missing factor. Then

$$X \lrcorner \omega = \sum_k \sum_{k<j} a_k p_j \, du_1 \cdots \widehat{du_k} \cdots \widehat{du_j} \cdots du_n - \sum_k \sum_{k>j} a_k p_j \, du_1 \cdots \widehat{du_j} \cdots \widehat{du_k} \cdots du_n. \tag{7.77}$$

This is the same as

$$X \lrcorner \omega = \sum_k \sum_j 1_{k<j} (a_k p_j - a_j p_k) \, du_1 \cdots \widehat{du_k} \cdots \widehat{du_j} \cdots du_n. \tag{7.78}$$

Hence $a_k p_j - a_j p_k = 0$ for $k < j$. If $a_k p_j = a_j p_k$ for all $k < j$, then the same equation results for all $k \neq j$, and hence for all k, j.

There is some r with $p_r \neq 0$, so this implies in particular that $a_k = (a_r/p_r) p_k$ for all k. Thus each vector in the characteristic subspace is a scalar multiple of the vector given in the statement of the theorem.

There is an easier way to see that vectors X of this form belong to the characteristic subspace. Let $\sigma = du_1 \cdots du_n$. Then $\omega = X \lrcorner \sigma$, so $X \lrcorner \omega = X \lrcorner (X \lrcorner \sigma) = 0$, since a repeated interior product always gives zero. $\square$

The picture of interest is thus not the actual vector field, but the corresponding solution curves. They obey the system of ordinary differential equations

$$\frac{du_1}{p_1} = \frac{du_2}{p_2} = \cdots = \frac{du_n}{p_n}. \tag{7.79}$$

This is a system of $n-1$ ordinary differential equations. (For instance, one can consider $u_2, \ldots, u_n$ as a function of u_1.) The solutions are flux curves. The density of end points is expressed by

$$d\omega = \left(\sum_{j=1}^{n} \frac{\partial p_j}{\partial x_j} \right) du_1 \cdots du_n. \tag{7.80}$$

Example: Here is the two-dimensional case. Consider $\omega = p\,dv - q\,du$. Then we can take $X = p\frac{\partial}{\partial u} + q\frac{\partial}{\partial v}$. The equation to solve is

$$\frac{du}{p} = \frac{dv}{q}. \tag{7.81}$$

This is the usual ordinary differential equation associated with the form. ||
Example: Consider three dimensions, so $\omega = p\,dv\,dw + q\,dw\,du + r\,du\,dv$. Then $X = p\frac{\partial}{\partial u} + q\frac{\partial}{\partial v} + r\frac{\partial}{\partial w}$. The equations for this case are

$$\frac{du}{p} = \frac{dv}{q} = \frac{dw}{r}. \tag{7.82}$$

There are only two independent equations. This is still some work, since these are two coupled nonlinear ordinary differential equations. ||

There is a variation on the notion of differential form called a twisted differential form. The main idea is that while the flux curves or point clouds for differential forms have external (transverse) orientation, the flux curves or point clouds for twisted differential forms have internal orientation. The correspondence between vector fields and $(n-1)$-forms given by a non-zero n-form has a corresponding version in which there a correspondence between vector fields and twisted $(n-1)$-forms given by a non-zero twisted n-form. This will be discussed in detail in a later chapter.

The terminology for a pictorial representation of a (perhaps twisted) $(n-1)$-form varies. In this work we use the term *flux curve*, since these are in fact curves. In physics it is common to say instead *flux line* or *line of flux*. The terms *field line* and *line of force* are also used. The line of force concept was introduced by Faraday in connection with electric and magnetic fields. An electric field line starts at positive electric charges and ends at negative electric charges, since at the charges the electric flux twisted 2-form has a differential that is a non-zero twisted 3-form. A magnetic flux is given by an exact 2-form, so a magnetic field line is always a closed curve.

In many circumstances there is a special (twisted) n-form σ of special importance, perhaps a form determining volume. Then the mapping that sends vector fields X to (twisted) $(n-1)$-forms $\omega = X \lrcorner \sigma$ translates between a vector field picture and a (twisted) $(n-1)$-form picture. The vector field picture is in terms of flow curves, for which there is a notion of tangent vector representing velocity. The $(n-1)$-form picture is in terms of flux curves, which have no notion of velocity, but for which the spacing is important. Given the density of points determined by σ, the spacing between flux curves near a point becomes smaller as the tangent vector gets larger. This is illustrated in Figure 7.10. The flux curves also give a pictorial description of $d\omega$, expressed by the density of end points.

It follows that in the case of electric and magnetic fields (in dimension $n = 3$) there is a choice of representation. The vector field representation is standard in most textbooks. Faraday's original interpretation was closer to the 2-form picture. This picture has the advantage that 2-forms are the objects that are

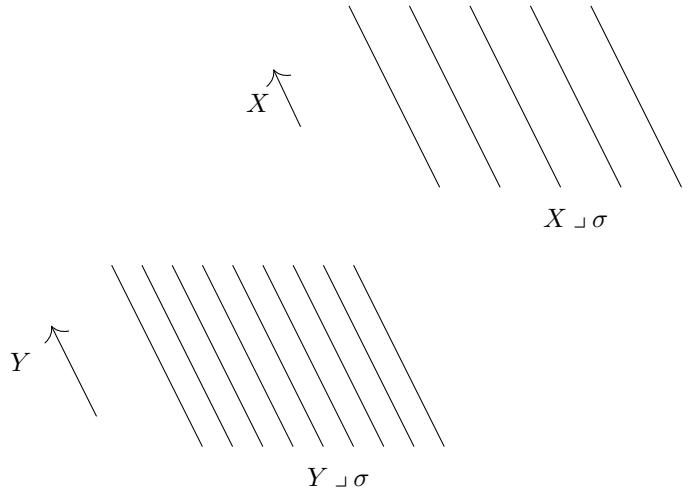

Figure 7.10: Interior product $X \lrcorner \sigma$ (2d)

integrated over surfaces in the integral representation of electromagnetism. In many ways this is the most intuitive description of the subject. There is a detailed description of these aspects of electromagnetism in the later chapter dealing with volume.

The following result is one of the most important structural properties of differential forms, and it is fundamental to the problem of visualizing differential forms. While it is an easy computation, many references on differential forms make no mention of it.

Theorem 7.11 *Let ω be a non-zero $(n-1)$-form. Then locally there are coordinates $x_1, \ldots, x_n$ and a scalar field $s \neq 0$ for which*

$$\omega = s\, dx_2 \cdots dx_n. \tag{7.83}$$

Proof: Let $\omega = \sum_j p_j (-1)^{j-1}\, du_1 \cdots du_{j-1}\, du_{j+1} \cdots du_n$ be a non-zero $(n-1)$-form, expressed in some arbitrary coordinates $u_1, \ldots, u_n$. Let $\sigma = du_1 \cdots du_n$ be the n-form that has coefficient 1 in these coordinates. Then there is a vector field X with $X \lrcorner \sigma = \omega$. It has the explicit form $X = \sum_j p_j \frac{\partial}{\partial u_j}$, expressed with the same coordinates.

Since X is also non-zero, by the straightening out theorem for vector fields there are coordinates $x_1, \ldots, x_n$ with $X = \frac{\partial}{\partial x_1}$. In these new coordinates $\sigma = s\, dx_1 \cdots dx_n$ for some non-zero scalar field s. It follows that in these new coordinates

$$\omega = X \lrcorner \sigma = s\, dx_2 \cdots dx_n. \tag{7.84}$$

The new coordinate system is the one that works to straighten out the vector field. There is more: the scalar factor is

$$s = \det\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right). \tag{7.85}$$

This is the determinant of the change of variable matrix. □

This result completes the picture of an non-zero $(n-1)$-form. The differential equations for the flux curves are $p_k\,dx_j = p_j\,dx_k$. Since $p_1 = s \neq 0$ and all other $p_k = 0$, this gives $dx_k = 0$ for all $k \neq 1$ in these coordinates.

The differential in these special coordinates is

$$d\omega = \frac{\partial s}{\partial x_1}\,dx_1 \cdots dx_n. \tag{7.86}$$

This is non-zero only when there is a non-trivial contribution from the scalar factor. In this case, when $d\omega \neq 0$, the mapping from $x_1, x_2, \ldots x_n$ to $s, x_2, \ldots, x_n$ is invertible, and so the latter is also a coordinate system.

Remark: If ω is an $(n-1)$ form with $d\omega \neq 0$, then it uniquely defines a vector field Y with $Y \lrcorner d\omega = \omega$. This may not always be the natural choice of vector field associated to ω. Often there is an n-form σ that is natural in the problem at hand, and then one can define a vector field X by $X \lrcorner \sigma = \omega$. Fortunately, this latter version can work even when ω is exact and $d\omega = 0$. ||

7.4 Surfaces and Forms

7.4.1 Orientation

The context is an n-dimensional connected manifold patch M. One way to define an *orientation* of such a manifold is with a basis $(X_1, \ldots, X_n)$ of vector fields. This basis determines an orientation. There are two orientations.

A more convenient way of giving an orientation is to give a never-zero n-form σ. This defines the same orientation as $(X_1, \ldots, X_n)$ if $\langle \sigma \mid X_1, \ldots, X_n \rangle > 0$. It defines the opposite orientation as $(X_1, \ldots, X_n)$ if $\langle \sigma \mid X_1, \ldots, X_n \rangle < 0$. Any two never-zero n forms that are positive multiples of each other define the same orientation. The orientation defined by σ may be denoted $[\sigma]$.

Here are some examples.

- For $n = 0$ the space M consists of a single point. The two orientions are called $+$ and $-$.
- For $n = 1$ the space M consists of an interval. The two orientations correspond to the two directions. If t is a coordinate on M, then the two orientations are defined by dt and $-dt$.
- For $n = 2$ the orientations are commonly called clockwise and counter-clockwise. However these are not absolute notions, since if one looks at a clock from the back its hands move in a counter-clockwise direction. If u, v is a coordinate system, then $du\,dv$ and $dv\,du$ define opposite orientations.
- For $n = 3$ the two orientations may be called right-handed and left-handed. Again these are not absolute notions. If v, w, y is a coordinate system, then one orientation is defined by $dv\,dw\,dy = dy\,dv\,dw = dw\,dy\,dw$ and the other orientation is defined by $dw\,dv\,dy = dv\,dy\,dw = dy\,dw\,dv$.

For $n \geq 1$ the orientations are anonymous; there are two of them, but neither one has a preference over the other.

There are also notions of orientation for surfaces. Suppose that there is a k-dimensional connected regular surface S in the n-dimensional space M. Then S may have its own orientation, quite independent of that of M. This may be given by a basis $(Y_1, \ldots, Y_k)$ of tangent vector fields to S. Alternatively, it may be given by a k-form β that never vanishes on such a basis. This kind of orientation is sometimes called an *internal orientation*, to contrast it with the concept of external orientation (or transverse orientation) explained below.

Before considering this, we need to explore what it means for a p-form η to be zero on a space of vectors. The form η may be said to *vanish* on the subspace if for $X_1, \ldots, X_p$ each in this space the value $\langle \eta \mid X_1, \ldots, X_p \rangle = 0$.

There is a stronger notion of how a p-form η can be zero on a space of vectors. The form η may be said to be *transverse* to the subspace if for every X_1 in the space and for every $X_2, \ldots, X_n$ the value $\langle \eta \mid X_1, X_2, \ldots, X_p \rangle = 0$. This is equivalent to saying that for every X_1 in the space the $(p-1)$-form $X_1 \lrcorner \eta$ is the zero form. (This is also equivalent to saying that the space is contained in the characteristic subspace of η.)

Proposition 7.12 *If the form η is transverse to a subspace, then it vanishes on the subspace.*

The terminology used to state the above proposition is not standard, but it seems convenient. One important special case is when $p = 1$. Then $X \lrcorner \eta = \langle \eta \mid X \rangle$. So in this case η is transverse to the space precisely when η vanishes on the space. For 1-forms the two notions are the same.

The following proposition shows that the X with $X \lrcorner \eta = 0$ do not matter for computing η.

Proposition 7.13 *Suppose that the vectors $X_i \lrcorner \eta = 0$ for $i = 1, \ldots, p$. Then for every $Z_1, \ldots, Z_p$ we have*

$$\langle \eta \mid Z_1 + X_1, Z_2 + X_2, \ldots, Z_p + X_p \rangle = \langle \eta \mid Z_1, \ldots, Z_p \rangle. \tag{7.87}$$

Example: Consider four dimensional space with coordinates x, y, u, v. The vectors $\partial/\partial x$ and $\partial/\partial y$ span just two of the directions. An example of a 2-form that vanishes on these vectors is $dx\,dv$. In fact $\langle dx\,dv \mid \partial/\partial x, \partial/\partial y \rangle = 0$. On the other hand, $dx\,dv$ is not transverse to this subspace. This is because $\partial/\partial x \lrcorner dx\,dv = dv$ which is not zero. This last observation may also be expressed by noting that $\langle\, dx\,dv \mid \partial/\partial x, \partial/\partial v \rangle = 1$.

An example of a 2-form that is transverse to these vectors in the stronger sense is $du\,dv$. Clearly both $\partial/\partial x \lrcorner du\,dv = 0$ and $\partial/\partial y \lrcorner du\,dv = 0$ ||

A *transverse orientation* or *external orientation* of a connected regular surface $S \subseteq M$ may be given by vectors $Z_1, \ldots, Z_{n-k}$ along the injection of S into M such that $Z_1, \ldots, Z_{n-k}, Y_1, \ldots, Y_k$ is a basis of vector fields along the injection of S into M and $Y_1, \ldots, Y_k$ is a basis of vector fields tangent to S. Alternatively, it may be given by a transverse $(n-k)$-form η along the injection

of S into M, that is, one for which that $Y_j \lrcorner \eta$ for each Y_j tangent to S. There are two such orientations.

These definitions may also be formulated in terms of coordinates. Say that the surface S is defined by a coordinate system $x_1, \ldots, x_n$ by setting $x_{k+1}, \ldots, x_n$ to have constant values. Then an internal orientation for S can be given by a k-form that is a non-zero multiple of $dx_1 \cdots dx_k$ plus other terms with factors dx_j with $j > k$.These extra terms give zero when the form is restricted to S. An external (transverse) orientation for S can be given by a never-zero multiple of $dx_{k+1} \cdots dx_n$.

Example: Here are some examples.

- The coordinates are x, y and the curve is the circle $x^2 + y^2 = c^2$. An internal orientation of the circle is determined by $r^2 d\theta = x\,dy - y\,dx$, while an external orientation is determined by $dr^2 = x\,dx + y\,dy$.

- The coordinates are x, y, z and the surface is the sphere $x^2 + y^2 + z^2 = c^2$. An internal orientation is given by $x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$. An external orientation is given by $x\,dx + y\,dy + z\,dz$.

- The coordinates are x, y, z and the curve is the circle $x^2 + y^2 = c^2, z = 0$. An internal orientation is $x\,dy - y\,dx$, while an external orientation is $dz(x\,dx + y\,dy)$.

- The coordinates are x, y, z and the surface is the cylinder $x^2 + y^2 = c^2$. An internal orientation is $(x\,dy - y\,dx)dz$ and an external orientation is $x\,dy + y\,dx$.

||

There is an important relation between internal orientations and external (transverse) orientations of a surface and orientations of the space. The simplest formulation is in terms of the interior product of forms.

Proposition 7.14 *Suppose the connected regular surface S has internal orientation given by β and external orientation given by η. The requirement that η be an external orientation inplies that η is a form transverse to S, that is, for each Y tangent to S we have $Y \lrcorner \eta = 0$. Then $\sigma = \eta \wedge \beta$ determines an orientation for M. Furthermore, if $Z_1, \ldots, Z_{n-k}, Y_1, \ldots, Y_k$ is a basis of vector fields along the injection of S into M, and $Y_1, \ldots, Y_k$ is a basis of the space of vector fields tangent to S, then*

$$\langle \sigma \mid Z_1, \ldots, Z_{n-k}, Y_1, \ldots, Y_k \rangle = \langle \eta \mid Z_1, \ldots, Z_{n-k} \rangle \langle \beta \mid Y_1, \ldots, Y_k \rangle. \qquad (7.88)$$

The above proposition presents three forms η, β, σ describing three kinds of orientations of the surface. They are not independent; any two of them determines the third.

7.4.2 Orientation of a boundary

A *surface with boundary* is typically defined by a combination of equalities and inequalities. Here is a basic construction that produces such surfaces. Consider a k-dimensional manifold patch P. (For instance this could be part of a k-dimensional regular surface in some larger n-dimensional manifold patch M.) Consider a scalar field v defined on P. Suppose there is a a constant c and a subset $K \subseteq P$ consisting of the points where $v \leq c$. Suppose also that dv is never zero on the set ∂K where $v = c$. Then the boundary ∂K is a regular surface.

Example: A classic example in three dimensions is a three dimensional ball B defined by $x^2 + y^2 + z^2 \leq a^2$. Its boundary is the two-dimensional sphere $\partial B = S$ where $x^2 + y^2 + z^2 = a^2$. ||

Example: Take $0 < a < b$. Consider the three-dimensional solid torus S defined by $\left(\sqrt{x^2 + y^2} - b\right)^2 + z^2 \leq a^2$. Its boundary is the two-dimensional torus $\partial S = T$ given by $\left(\sqrt{x^2 + y^2} - b\right)^2 + z^2 = a^2$ ||

Example: Take the region Q where $0 \leq x \leq a, 0 \leq y \leq a, 0 \leq z \leq a$. This is a solid cube that does not fit in with the construction above. The problem is that it has corners, so the boundary is not a regular surface. The boundary ∂Q is the unions of the six two-dimensional square surfaces that satisfy the six equations $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$, $z = a$. The interiors of these squares give 6 regular surfaces. These are not the entire boundary, but they are most of it. ||

If the surface K is oriented, then its boundary ∂K has a *boundary orientation*. Consider a k-surface K with $(k-1)$-boundary ∂K. Suppose that near some part of the boundary there are independent vector fields $X_1, \ldots, X_k$ tangent to K such that $X_2, \ldots, X_k$ are tangent to the boundary and X_1 is an *outward transverse vector* that points from inside K to outside K. If $X_1, \ldots, X_k$ defines the orientation of K, the $X_2, \ldots, X_k$ defines the orientation of ∂K.

This definition may also be given via differential forms. Let ϵ be a never-zero 1-form such that $X \lrcorner \epsilon = \langle \epsilon \mid X \rangle = 0$ for every X tangent to the boundary ∂K, and such that ϵ is increasing leaving the boundary. Such an ϵ may be called an *outward transverse form*. For instance, if K is defined by $v \leq c$, then $\epsilon = dv$ is suitable.

Proposition 7.15 *Say that K has an orientation given by a never-zero k-form α. Let ϵ be an outward transverse 1-form. Let β be a $(k-1)$-form such that $\epsilon \wedge \beta = \alpha$. Then β orients ∂K. Furthermore, suppose that $X_1, \ldots, X_k$ are such that $X_2, \ldots, X_k$ are tangent to the boundary. Then*

$$\langle \alpha \mid X_1, \ldots, X_k \rangle = \langle \epsilon \mid X_1 \rangle \langle \beta \mid X_2, \ldots, X_k \rangle. \tag{7.89}$$

- For orienting a boundary surface the rule is to delete the outward direction in the first place.

Example: The three dimensional ball B defined by $x^2 + y^2 + z^2 \leq a^2$ has orientation given by $dx\,dy\,dz$. Vectors tangent to the boundary are annihilated

by $x\,dx + y\,dy + z\,dz$. This implies that the boundary is oriented by $x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$. This is because $(x\,dx+y\,dy+z\,dz)(x\,dy\,dz+y\,dz\,dx+z\,dx\,dy) = (x^2+y^2+z^2)\,dx\,dy\,dz$ defines the same orientation as $dx\,dy\,dz$. ||

Example: Take $0 < a < b$. Consider the three-dimensional solid torus S defined by $\left(\sqrt{x^2+y^2} - b\right)^2 + z^2 \leq a^2$ oriented by $\alpha = dx\,dy\,dz$. Its boundary is the two-dimensional torus $\partial S = T$ given by $\left(\sqrt{x^2+y^2} - b\right)^2 + z^2 = a^2$. In cylindrical coordinates $x = \rho\cos(\phi), y = \rho\sin(\phi), z = z$ the boundary is given by $(\rho - b)^2 + z^2 = a^2$. Take $\epsilon = (\rho - b)\,d\rho + z\,dz$. This annihilates the boundary. The solid orientation is given by $dx\,dy\,dz = \rho\,d\rho\,d\phi\,dz$. If $\beta = (\rho - b)\,d\phi\,dz$, then $\epsilon \wedge \beta = (\rho - b)^2\,d\rho\,d\phi\,dz$, so β orients the boundary. The variable ϕ is the angle the long way around the torus. Define a new variable θ on the boundary by $\rho - b = a\sin(\theta), z = a\cos(\theta)$. The variable θ is the angle the short way around the torus. Then $\beta = a^2\sin^2(\theta)\,d\theta\,d\phi$. So a nice way to describe the orientation of the boundary is by the form $d\theta\,d\phi$. ||

Example: Take the cube Q where $0 \leq x \leq a, 0 \leq y \leq a, 0 \leq z \leq a$. Suppose its orientation is given by $dx\,dy\,dz$. The forms ϵ on the six boundary squares are are $-dx, dx, -dy, dy, -dz, dz$. So the orientations of these squares are given by $dz\,dy, dy\,dz, dx\,dz, dz\,dx, dy\,dx, dx\,dy$. ||

There is also a notion of transverse boundary orientation. Consider a k-dimensional connected manifold patch P that is part of a k-dimensional connected regular surface in some larger n-dimensional connected manifold patch M. Suppose that there is a compact subset $K \subseteq P$ with regular boundary $\partial K \subseteq P$. A typical such subset may be defined by a scalar field v defined on P. The subset $K \subseteq P$ consisting of the points where for some constant c the scalar field v satisfies $v \leq c$. The subset ∂K is where $v = c$. If dv is never zero on the set ∂K, then ∂K is a regular surface.

Suppose that $Y_1, \ldots, Y_{n-k}$ give a transverse orientation of P and hence of K. Consider independent $X_1, \ldots, X_k$ tangent to P such that X_1 points outward across ∂K, while $X_2, \ldots, X_n$ are tangent to ∂K. Then the *transverse boundary orientation* of ∂K is given by $Y_1, \ldots, Y_k, X_1$.

The 1-form ϵ is a *outward transverse form* if ϵ vanishes on ∂K and increases across ∂K. In the above construction one could take $\epsilon = dv$.

Proposition 7.16 *Suppose a transverse orientation of K is given by a $(n-k)$-form τ such that $X \lrcorner \tau = 0$ for every X tangent to P. Let ϵ be an outward transverse form for ∂K such that $\tau \wedge \epsilon \neq 0$. Then the transverse orientation η of ∂K is given by the $(n-k+1)$-form $\eta = \tau \wedge \epsilon$. Furthermore, for the vectors $Y_1, \ldots, Y_{n-k}, X_1, \ldots, X_k$ described above*

$$\langle \eta \mid Y_1, \ldots, Y_{n-k}, X_1 \rangle = \langle \tau \mid Y_1, \ldots, Y_{n-k} \rangle \langle \epsilon \mid X_1 \rangle. \tag{7.90}$$

Proof: Suppose that the tangent vectors to P are $X_1, \ldots, X_k$ and that $X_2, \ldots, X_k$ are tangent to ∂K. Consider vectors $Y_1, \ldots, Y_{n-k}, X_1, \ldots, X_k$, where the Y_j are vectors that are not tangent to P. We need to calculate the result if $\tau \wedge \epsilon$ is applied to $n - k + 1$ of these vectors. This is a sum where τ is applied

to $n-k$ of the vectors and ϵ is applied to one of the other vectors. If an X_j appears with τ the result is zero, and if an X_j for $j > 1$ appears with ϵ the result is also zero. In fact, the only contribution to the sum is from $Y_1, \ldots, Y_{n-k}, X_1$. In particular, there are no contributions from any term involving a contribution taken from $X_2, \ldots, X_k$. $\square$

- For transverse orienting a boundary surface the rule is to insert the outward direction in the last place.

Example: The coordinates are x, y, z and the surface is the disk $x^2 + y^2 \leq c^2, z = 0$. A transverse orientation is given by dz. The form $\epsilon = x\,dx + y\,dy$ vanishes on the circle. So the corresponding transverse orientation of the circle is given by $dz(x\,dx + y\,dy)$. ||

The following result shows the consistency of the above definitions.

Proposition 7.17 *Consider a k-surface in n dimensions. Let α be a k-form giving a surface orientation, and let τ be an $(n-k)$-form giving a transverse surface orientation, so that $\tau \wedge \alpha = \sigma$ gives an orientation of the whole space. Let α' be the boundary orientation and τ' be the transverse boundary orientation, so that for some 1-form ϵ*

$$\begin{aligned} \alpha &= \epsilon \wedge \alpha' \\ \tau' &= \tau \wedge \epsilon. \end{aligned} \tag{7.91}$$

Then

$$\tau' \wedge \alpha' = \tau \wedge \alpha = \sigma. \tag{7.92}$$

7.4.3 Simple forms

Suppose that the n-dimensional manifold patch is M. A k-form ω will be called a *simple form* if there is a oordinate system $x_1, \ldots, x_n$ such that

$$\omega = s\,dx_1 \cdots dx_k \tag{7.93}$$

for some scalar field s that is never zero.

Consider a simple form ω. Then one can define a family of surfaces N for which $x_1, \ldots, x_k$ have constant values. Then ω restricted to each N is zero. Each surface N has a transverse orientation given by ω.

Proposition 7.18 *Consider a simple form ω. Then the family of surfaces N on each of which ω is zero is independent of the coordinate system.*

Proof: Suppose that $y_1, \ldots, y_n$ is another coordinate system such that

$$\omega = s\,dx_1 \cdots dx_k = r\,dy_1 \cdots dy_k. \tag{7.94}$$

Then

$$dy_i = \sum_{j=1}^{n} \frac{\partial y_i}{\partial x_j} dx_j. \tag{7.95}$$

Thus for $1 \leq i \leq k$

$$0 = dy_i \, dx_1 \cdots dx_k = \sum_{j=k+1}^{n} \frac{\partial y_i}{\partial x_j} dx_j \, dx_1 \cdots dx_k. \tag{7.96}$$

It follows that each coeffiencent $\partial y_i / \partial x_j = 0$ for $1 \leq i \leq k$ and $k+1 \leq j \leq n$. It follows that for $1 \leq i \leq k$ the $y_i = f_i(x_1, \ldots, x_k)$ depends only on $x_1, \ldots, x_k$.

It may be shown in the same way that for $1 \leq i \leq k$ the $x_i = g_i(y_1, \ldots, y_k)$ depends only on $y_1, \ldots, y_k$. So the surfaces N may be defined either by fixing $x_1, \ldots, x_k$ or by fixing $y_1, \ldots, y_k$. □

For a simple form ω there is a corresponding family of surfaces, and they have an orientation given by ω itself. If ω is a simple form, then the exact form $d\omega$ is also a simple form, and it has a corresponding family of closed boundary surfaces with transverse orientation given by $d\omega$.

A region in space is often described by its indicator function, equal to 1 on the region and 0 on its complement. Sometimes it is useful to think of a scalar field s as indicating a region in some approximate sense, so that large values of $|s|$ correspond to being approximately inside the region, and small values correspond to being approximately outside the region. With this interpretation the form

$$\epsilon = \frac{ds}{s} \tag{7.97}$$

is indicating the inward direction near the approximate boundary of the region. This may be called the *inward direction form*.

Proposition 7.19 *Suppose ω is a simple form. Then*

$$d\omega = \epsilon \wedge \omega, \tag{7.98}$$

where ϵ is a form in the inward direction.

- For transverse orienting the exterior derivative of a simple form the rule is to insert the inward direction form in the first place.

Example: Figure 7.11 shows the form $\omega = \exp(-x^2) \, dy$ in two dimensions. It is illustrated by a family of curves with constant values of y, with transverse orientation given by dy. The exterior derivative $d\omega = -2x \exp(-x^2) \, dx \, dy$. It is indicated by a family of points, mainly concentrated near the lines $x = \pm 1$. For the points near $x = -1$ the transverse orientation is counterclockwise, given by $dx \, dy$, while for the points near $x = 1$ the transverse orientation is clockwise, given by $-dx \, dy$.

The form $\epsilon = -2x \, dx$ is in the inward direction, and indeed $d\omega = \epsilon \wedge \omega$. ||

Remark: The orientation of the curve or surface associated with a simple differential form is a transverse orientation. We shall see in a later chapter that there is a concept of twisted differential form. The orientation of a curve or surface associated with a twisted simple differential form is an internal orientation. A simple differential k-form ω may be reoriented by a $(n-k)$-form η to produce a twisted simple differential k-form. In that context there will be another rule:

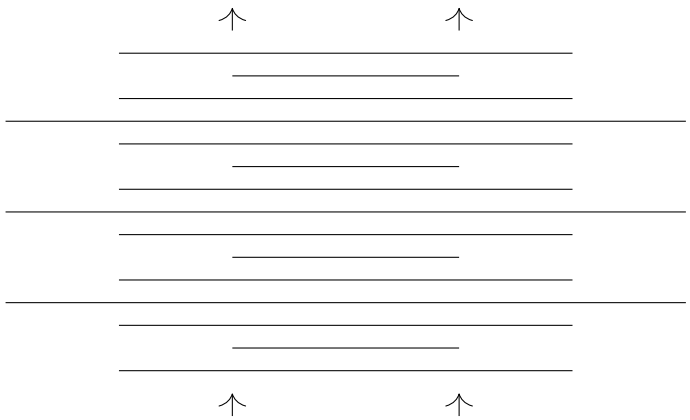

Figure 7.11: $\omega = \exp(-x^2)\,dy$; $d\omega = -2x\,dx \wedge \omega$

- The reorientation of the exterior derivative of a twisted simple form is obtained from the original reorientation by deleting an inward direction form in the last place.

These ideas will complete the story. ||

7.4.4 Pictures of simple forms

A simple k-form ω is pictured as a family of $(n-k)$-surfaces with k-dimensional transverse (external) orientation. These may have boundaries that are $(n-k-1)$ surfaces. In fact, the differential of a simple form is

$$d(s\,dx_1 \cdots dx_k) = \sum_{j=k+1}^{n} \frac{\partial s}{\partial x_j} dx_j\,dx_1 \cdots dx_k. \tag{7.99}$$

Each term in this sum has a structure very much like that of a simple form. Thus one can think of the boundary as consisting of a family of $(n-k-1)$-dimensional boundary surfaces.

Here are some special cases. First there are the positive results.

- A never-zero n-form is of the form $\omega = s\,dx_1 \cdots dx_n$ and so is simple. A never-zero n-form ω is pictured as closely spaced points each with an n-dimensional transverse (external) orientation.
- For never-zero $(n-1)$ forms the situation is very special, since every differential $(n-1)$-form is simple, of the form $\omega = s\,dx_1 \cdots dx_{n-1}$ for some suitable coordinate system. A never-zero differential $(n-1)$-form ω is pictured as closely spaced curves (possibly with end points) each with a $(n-1)$-dimensional external (transverse) orientation given by the form itself. These are called *circulatory flux curves*. The end points corresponding to $d\omega$ have n-dimensional orientations.

- If a 1-form ω has an integrating factor, then it is of the form $\omega = w\,dz$ and so is simple. Such a form may be pictured by the family of $(n-1)$-surfaces for which z is constant. These surfaces have one-dimensional transverse orientation. This applies to the important special case when $\omega = dz$ is exact; in this case the surfaces have no boundaries.

- It is always possible to picture a differential 1-form via a grid plot which shows the linear action of the form on typical tangent spaces. On each tangent space the form defines a family of $(n-1)$-spaces consisting of tangent vectors on which the form has constant values.

- The case of a never-zero 0-form (scalar field) s is very special. If $s > 0$ it seems natural to make pictures using the n-dimensional regions $s \geq c$ where the scalar field exceeds certain values. The orientation of these regions is transverse (external) and zero dimensional. For $s > 0$ this would be $+$. The differential ds would then be pictured by the $(n-1)$-surfaces for which $s = c$. The orientation would be in the uphill direction.

Then there are less attractive results.

- If $n \geq 4$, then a k-form form may not be decomposable. The simplest example is when the coordinate system is x, y, z, w and the form is $dx\,dy + dz\,dw$. Such a form is not simple.

- For $n \geq 3$ a 1-form ω typically does not have an integrating factor and is not simple.

The transverse orientations for circulatory flux curves associated with an $(n-1)$-form are shown in Figure 7.12 for the case of dimension $n = 2$ and in Figure 7.13 for the case of dimension $= 3$.

Example: Here are examples of differential forms with hints about the corresponding pictures.

- Consider the 1-form $y\,dx$ in two dimensions. This is represented by vertical contour lines that terminate at points in the plane. The density of these lines is greater as one gets farther from the x axis. The increase is to the right above the x axis, and it is to the left below the y axis. The differential of $y\,dx$ is $dy\,dx = -dx\,dy$. This 2-form represents the cloud of terminating points, which in this case has a uniform density. The usual convention is that the positive orientation $dx\,dy$ is counterclockwise. So the orientations of these source points $dy\,dx$ are clockwise. This is consistent with the direction of increase along the contours lines.

- If instead we consider the 1-form $y\,dx$ in three dimensions, then the representation is by contour half-planes with constant values of x. The boundary curves of the half-planes are lines in the z direction.

- Consider 3 dimensions. Say the 2-form is $dx\,dy$. Every multiple of the vector $\frac{\partial}{\partial z}$ has interior product zero with this form. So the flux curves are vertical lines.

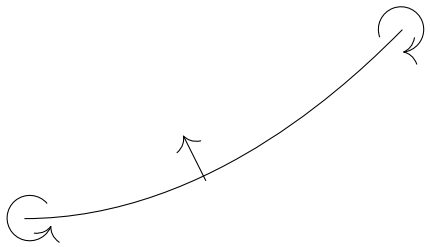

Figure 7.12: 1-form in two dimensions: typical circulatory flux curve, endpoint orientations shown

- On the other hand, say that the form is $z\,dx\,dy$. Then the vertical flux lines terminate in a uniformly distributed cloud of points.

- Consider the differential form $\omega = y\,dx - dy$ in the region where $y > 0$. This is not exact, but we can write $-e^{-x}\omega = d(ye^{-x})$. So we can represent ω by fragments of curves $s = ye^{-x} = C$. The contour curves of s are naturally spaced by taking small but equal increments of C. We have $d\omega = dy\,dx$, the end points of these curves are uniformly distributed and have clockwise sense. The density of the curves representing ω is related to the density of the contour curves of the exact form by the factor $-e^x$. The characteristic subspace is given by an arbitrary non-zero multiple of the vector $\partial/\partial x + y\partial/\partial y$. The transverse orientation is given by a vector that is not in this subspace and is pointing in the direction of increase. For example, $\partial/\partial x$ would do. Also $-\partial/\partial y$ works.

- Consider again $\omega = 3yz\,dx + 2xz\,dy + xy\,dz$. Then $x^2y\omega = d(x^3y^2z)$ is exact. If the contour surfaces of $x^3y^2z = C$ are equally spaced according to the values of C, then the contour surfaces of ω are spaced at a density obtained by multiplying these by $1/(x^2y)$. Because of this spacing, they terminate in contour curves. These curves are the flux curves of the two form $d\omega$.

- The characteristic subspaces of $x\,dy + dz$ are not associated with a family of surfaces. Nevertheless, at a particular point $(x, y, z) \leftarrow (c_1, c_2, c_3)$ one can think of planes parallel to and very near to the characteristic surface that have spacing like that of $c_1y + z = C$, as C is incremented by δ.

||

Figure 7.13: 2-form in three dimensions: typical circulatory flux curve

7.5 Integration of differential forms over parametrized surfaces

7.5.1 Integration of top-dimension differential forms

For a connected manifold patch P of dimension k there are two orientations. These may be determined by a coordinate system, by a basis of k vector fields, by a basis of k differential 1-forms, or by a single k-form.

Suppose P is a connected manifold patch. Let $\mathbf{u}$ and $\mathbf{y}$ be two coordinate systems for N. Then $\mathbf{u} = \mathbf{f}(\mathbf{y})$ for some smooth invertible $\mathbf{f}$. Write $d\mathbf{u}$ and $d\mathbf{y}$ for the k-forms obtained by multiplying the differentials of the coordinate functions. These k-forms are related by $d\mathbf{u} = \det(\mathbf{f}'(\mathbf{y}))\, d\mathbf{y}$. Since a non-zero k-form determies an orientation, the coordinate systems have the same orientation if $\det(\mathbf{f}'(\mathbf{y})) > 0$ at each point of P. They have opposite orientations if $\det(\mathbf{f}'(\mathbf{y})) < 0$ at each point.

Suppose that P is a k-dimensional connected manifold patch with a given orientation $\mathcal{O}$. Consider a differential k-form ω on P. This is a top-dimensional form, since the degree of ω is equal to the dimension P. Suppose that ω has compact support. Choose a coordinate system $u_1, \ldots, u_k$ and write

$$\omega = f(\mathbf{u})\, d\mathbf{u} = f(u_1, \ldots, u_k)\, du_1 \wedge \cdots \wedge du_k. \tag{7.100}$$

The definition is

$$\int_P \omega = \text{sign}(\mathcal{O}, [d\mathbf{u}])\ I(f). \tag{7.101}$$

where $\mathcal{O}$ is the given orientation of P, and $I(f)$ is the usual (unoriented) Riemann integral. The $\pm$ sign in the definition is the relative orientation.

Example: Suppose $k = 2$ and $dy\, du$ determines the orientation $\mathcal{O} = [dy\, du]$. The integral

$$\int_{0 \le y,u \le 1} u^2 y^3\, du\, dy = -\frac{1}{12}. \tag{7.102}$$

This is because the relative orientation of $dy\, du$ and $du\, dy$ is -1. ||

Proposition 7.20 *The definition of integral of a top dimensional differential form over an oriented manifold patch is independent of the coordinate system.*

Proof: Consider another coordinate system $y_1, \ldots, y_k$. Then

$$\omega = f(\mathbf{u})\, du_1 \wedge \cdots \wedge du_k = f(\mathbf{g}(\mathbf{y}))\det \mathbf{g}'(\mathbf{y})\, dy_1 \wedge \cdots \wedge dy_k. \tag{7.103}$$

The definition in the other coordinate system gives

$$\int_P \omega = \text{sign}(d\mathbf{y}, \mathcal{O}) I((f \circ \mathbf{g}) \det \mathbf{g}') = \text{sign}([d\mathbf{y}], \mathcal{O}) \,\text{sign}(\mathbf{g}'(\mathbf{y}))\, I((f \circ \mathbf{g})|\det \mathbf{g}'|). \tag{7.104}$$

But $\text{sign}([d\mathbf{y}], \mathcal{O})\ \text{sign}(\mathbf{g}'(\mathbf{y})) = \text{sign}([d\mathbf{u}], \mathcal{O})$ and

$$I(f) = I((f \circ \mathbf{g})|\det \mathbf{g}'|). \tag{7.105}$$

So this is the same as in the original coordinate system. □

This is a classic example of a passive transformation. The same differential form ω is integrated over the same oriented manifold patch. But the numerical expression of the integral is different in the two coordinate systems.

7.5.2 Fubini's theorem for differential forms*

This section is a rapid glance at issues related to iterated integrals. All the differential forms are top degree on the appropriate space. First comes a sketch of Fubini's theorem for product forms. Then there is a very brief introduction to a more general notion of fiber integration.

Suppose that $M = M_1 \times M_2$ is a product space. Then there are projections $\pi_1 : M \to M_1$ and $\pi_2 : M \to M_2$. Suppose M_1 has coordinates system $\mathbf{x} = x_1, \ldots, x_k$ and M_2 has coordinate system $\mathbf{y} = y_1, \ldots, y_\ell$. Then M has a coordinate system $\mathbf{x}'\,\mathbf{y}' = x'_1, \ldots, x'_k, y'_1, \ldots y'_\ell$, where $x'_i = x_i \circ \pi_1$ and $y'_j = y \circ \pi_2$. So the product of manifold patches is a manifold patch. Furthermore, if each of M_1 and M_2 has an orientation, then these determine an orientation of M (for instance by combining coordinate systems).

Suppose that α is a k-form on M_1 and β is a ℓ-form on M_2. Let α' be the pullback of α via π_1 and β' be the pullback of β via π_2. Then α' and β' are both forms on M, and the product $\alpha' \wedge \beta'$ is a $k + \ell$ form on M.

If the forms α and β have compact support, then they may be integrated over M_1 and M_2, respectively. Furthermore $\alpha' \wedge \beta'$ may be integrated over M. Let s be a scalar field on M. Fubini's theorem then says that

$$\int_M s\, \alpha' \times \beta' = \int_{M_2} \left[\int_{M_1} s \circ \sigma_2\, \alpha\right] \beta = \int_{M_1} \left[\int_{M_2} s \circ \sigma_1\, \beta\right] \alpha. \tag{7.106}$$

Here σ_2 for each point μ_2 in M_2 is the function on M_1 that sends μ_1 to the point in M whose projections are μ_1 and μ_2. In other words, σ_2 fixes the second projection and then maps M_1 to M. Similarly, σ_1 fixes the first projection and then maps M_2 to M.

The same formula may be expressed in coordinates. Suppose $\alpha = k_1(\mathbf{x})\, d\mathbf{x}$ and $\beta = k_2(\mathbf{y})\, d\mathbf{y}$. Then $\alpha' \times \beta' = k_1(\mathbf{x}')k_2(\mathbf{y}')\, d\mathbf{x}'\ d\mathbf{y}'$. The result is

$$\int_M h(\mathbf{x}', \mathbf{y}')k_1(\mathbf{x}')k_2(\mathbf{y}')\, d\mathbf{x}'\ d\mathbf{y}' = \int_{M_2} \left[\int_{M_1} h(\mathbf{x}, \mathbf{y})k_1(\mathbf{x})\, d\mathbf{x}\right] k_2(\mathbf{y})\, d\mathbf{y}\, d\mathbf{x}, \tag{7.107}$$

and also

$$\int_M h(\mathbf{x}', \mathbf{y}') k_1(\mathbf{x}') k_2(\mathbf{y}') \, d\mathbf{x}' \, d\mathbf{y}' = \int_{M_1} \left[\int_{M_2} h(\mathbf{x}, \mathbf{y}) k_2(\mathbf{y}) \, d\mathbf{y} \right] k_1(\mathbf{x}) \, d\mathbf{x}. \tag{7.108}$$

Fubini's theorem starts with a product space, which is a special situation. There is another result on iterated integrals that is more general and geometrical. This is *fiber integration*. The setting is a manifold patch M and a manifold mapping $\phi : M \to N$. For each point in N the fiber over the point is the surface consisting of all points in M that map to the given point. This family of surfaces in M as a function on N may be denoted $\phi^{-1}(\cdot)$. The idea is to integrate over N an inner integral that is taken over $\phi^{-1}(\cdot)$.

Let M have dimension n and N have dimension $n - k$. Under appropriate hypotheses each fiber should be a surface of dimension k. The formula starts with a n-form ω on M and an $(n-k)$-form λ on N. Since $\phi : M \to N$, the form λ has a pullback $\phi^* \lambda$ to a $(n-k)$-form on M. The next step is to attempt to find a k-form γ on M such that $\omega = \phi^* \lambda \wedge \gamma$. The formula should then say that

$$\int_M \omega = \int_M \phi^* \lambda \wedge \gamma = \int_N \left[\int_{\phi^{-1}(\cdot)} \gamma \right] \lambda. \tag{7.109}$$

Each inner integral is defined by pulling back the k-form γ on the n dimensional space M to the appropriate k-dimensional fiber.

In this discussion the spaces M and N are assumed to be oriented. The orientation of the space N may be given by a $(n-k)$-form τ. Then $\phi^* \tau$ is an $(n-k)$-form on M with the property that the interior product of every tangent vector to a fiber with this form is zero. In other words, the form $\phi^* \tau$ gives a transverse orientation of each fiber. However an orientation of M together with a transverse (external) orientation of a fiber determines an (internal) orientation of the fiber. So the inner integrals over the fibers are also over an oriented surfaces.

The difficult part of this procedure is the factorization $\omega = \phi^* \lambda \wedge \gamma$. This is best understood by writing everything in coordinates. Let $\omega = h(\mathbf{x}) \, d\mathbf{x}$. Also take $\lambda = k(\mathbf{z}) \, d\mathbf{z}$. Let the map ϕ be given by $\mathbf{z} \leftarrow G(\mathbf{x})$. The idea is to make a change of coordinates from the original n-dimensional $\mathbf{x}$ coordinates to a new system $\mathbf{u}, \mathbf{y}$, where $\mathbf{y} = G(\mathbf{x})$. Then we have

$$\mathbf{x} = F(\mathbf{y}, \mathbf{u}) \tag{7.110}$$

as the change of coordinates in one direction, and

$$\begin{aligned} \mathbf{y} &= G(\mathbf{x}) \\ \mathbf{u} &= K(\mathbf{x}) \end{aligned} \tag{7.111}$$

for some function K in the other direction. In these coordinates the n-form $\omega = h(F(\mathbf{y}, \mathbf{u})) \det F'(\mathbf{y}, \mathbf{u}) \, d\mathbf{y} \, d\mathbf{u}$ and the mapping $\phi : M \to N$ is $\mathbf{z} \leftarrow \mathbf{y}$. Also the $(n-k)$-form $\phi^* \lambda = k(\mathbf{y}) \, d\mathbf{y}$.

If this change of coordinates can be carried out, then we can identify $\gamma = (1/k(\mathbf{y}))h(F(\mathbf{u},\mathbf{y}))\det F'(\mathbf{u},\mathbf{y})\,d\mathbf{u}$. The formula becomes

$$\int_M h(\mathbf{x})\,d\mathbf{x} = \int_N \left[\int_{\mathbf{y}=\mathbf{z}} (1/k(\mathbf{y}))h(F(\mathbf{u},\mathbf{y}))\det F'(\mathbf{u},\mathbf{y})\,d\mathbf{u}\right] k(\mathbf{z})\,dz. \quad (7.112)$$

The $\mathbf{u}$ restricts to a coordinate system on each fiber $\mathbf{y} = \mathbf{z}$. So the inner integral is over each fiber, and the outer integral is over the target space.

7.5.3 Integration over compact-parametrized surfaces

Let M be an n-dimensional manifold patch. Let P be a k-dimensional connected manifold patch. A *parametrized surface* is a smooth manifold map $\chi : P \to M$. Such a map is sometimes called a singular parametrized surface, since the image may be degenerate or cross itself. There is a special case when P is a regular surface in M and χ is the injection of P into M. In this case one might think of χ as being the surface itself.

If ω is a differential k-form on M, then under the map χ the form ω pulls back to a k-form $\chi^*\omega$ defined on P. In coordinates, $\mathbf{u}$ could be a coordinate system on P, while $\mathbf{x}$ is a coordinate system on M. The form ω may be expressed in terms of the coordinates $\mathbf{x}$. The map χ may be given by a function $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u})$. Then $\chi^*\omega$ is expressed in terms of $\mathbf{u}$.

Let M be an n-dimensional manifold patch. Let P be an k-dimensional connected manifold patch. Let $K \subseteq P$ be a Jordan measurable compact subset. A *compact-parametrized surface* χ in M is defined to be a smooth function

$$\chi : K \to M. \quad (7.113)$$

In this context to say that χ is smooth is to say that it is a restriction of a manifold mapping from P to M.

Suppose M is an n-dimensional manifold patch. Suppose $K \subseteq P$ and $\chi : K \to M$ is a compact-parametrized k-surface in M. The mapping χ may be thought of as oriented by a choice of orientation $\mathcal{O}$ of P. Suppose ω is a k-form on M. The integral of ω over χ is defined by

$$\int_\chi \alpha = \int_K \chi^*\alpha. \quad (7.114)$$

The integral on the right is the integral of a top-dimensional form.

Proposition 7.21 *The integral of a k-form*

$$\alpha = \sum_I g_I(\mathbf{x})\,d\mathbf{x}_I \quad (7.115)$$

with respect to the compact-parametrized surface

$$\chi = (\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u})) \quad (7.116)$$

with orientation $[d\mathbf{u}]$ *is*

$$\int_\chi \alpha = \int_K \sum_I g_I(\mathbf{f}(\mathbf{u})) \det \mathbf{f}_I'(\mathbf{u})\, d\mathbf{u}. \tag{7.117}$$

Example: Consider the parametrized curve ψ given by $(x, y) \leftarrow (t^2, t^3)$, defined for $1 \le t \le 2$. The orientation $[dt]$ is in the direction of increasing t. We want to integrate the differential form $\omega = y\,dx + 2xy\,dy$ along this oriented curve. If we pull it back by ψ we get $2t^4\,dt + 6t^7\,dt = (2t^4 + 6t^7)\,dt = d(\frac{2}{5}t^5 + \frac{3}{4}t^8)$. Thus the integral has the value $(\frac{2}{5}2^5 + \frac{3}{4}2^8) - (\frac{2}{5} + \frac{3}{4})$. ||

Example: Consider the quarter cylinder $x^2 + y^2 = 25$ with $0 \le x, 0 \le y, 0 \le z \le 3$. Give it the orientation determined by $dy\,dz = dz\,dx$ on the cylinder. Say that one wants the integral of

$$\alpha = z\,dy\,dz + 2x\,dz\,dx - 4x^2 z\,dx\,dy \tag{7.118}$$

on this surface. One way to calculate this is to parameterize the surface by the corresponding rectangle in the x, z plane, with $0 \le x \le 5, 0 \le z \le 3$. The calculation amounts to pulling back by $y \leftarrow \sqrt{25 - x^2}$. The pullback is the form

$$\alpha^* = \left(\frac{zx}{\sqrt{25 - x^2}} + 2x \right) dz\,dx. \tag{7.119}$$

This can be integrated over the rectangle to give $195/2$. ||

Example: Let S be the unit sphere in Euclidean space. Say that we want to integrate $x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$ over the sphere. We can parameterize by $(x, y, z) \leftarrow \sin(u)\cos(v), \sin(u)\,\sin(v), \cos(u)$. The parameter region P is the rectangle where u goes from 0 to π and v goes from 0 to 2π. The orientation is determined by the form $du\,dv$.

$$\int_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy = \int_P \sin(u)\,du\,dv = 4\pi. \tag{7.120}$$

Later we shall see how to do this integral via Gauss's theorem. ||

An oriented compact-parametrized surface consists of a mapping $\chi : P \to M$ together with an orientation $\mathcal{O}$ of P. Sometimes this pair is denoted by χ; in that case $-\chi$ represents the oriented compact-parametrized surface with the opposite orientation. This gives

$$\int_{-\chi} \omega = -\int_\chi \omega. \tag{7.121}$$

Theorem 7.22 (Change of variables) *Let χ be an oriented compact-parametrized k-surface mapping to M of dimension n. Let ϕ be a manifold mapping of M to another N of dimension p. Let ω be a k-form defined on N. Then*

$$\int_{\phi\circ\chi} \omega = \int_\chi \phi^*\omega. \tag{7.122}$$

Proof: The proof is easy.

$$\int_{\phi\circ\chi} \omega = \int_K (\phi\chi)^*\omega = \int_K \chi^*\phi^*\omega = \int_\chi \phi^*\omega. \tag{7.123}$$

□

Proposition 7.23 (Change of variables in coordinates) *Let χ be a compact-parametrized k-surface mapping to M of dimension n. Let ϕ be a manifold mapping of an M to another N of dimension p. Suppose that ϕ is given by $\mathbf{y} \leftarrow \mathbf{g}(\mathbf{x})$. Let ω be a k-form*

$$\omega = \sum_I h_I(\mathbf{y})\, d\mathbf{y}_I \tag{7.124}$$

defined on N. Then

$$\int_{\phi\circ\chi} \sum_I h_I(\mathbf{y})\, d\mathbf{y}_I = \int_\chi \sum_J \sum_I h_I(\mathbf{g}(\mathbf{x})) \det \mathbf{g}'_{I,J}(\mathbf{x})\, d\mathbf{x}_J. \tag{7.125}$$

What happens if $k = n = p$? Then the form is $\omega = h(\mathbf{y})d\mathbf{y}$. So the theorem says

$$\int_{\phi\circ\chi} f(\mathbf{y})\, d\mathbf{y} = \int_\chi f(\mathbf{g}(\mathbf{x})) \det \mathbf{g}'(\mathbf{x})\, d\mathbf{x}. \tag{7.126}$$

This looks like the classic change of variables theorem, but without the absolute value sign. The reason the absolute value is not needed is that the integrals are defined with respect to parameterizations χ and $\phi\chi$, and these are oriented integrals.

Remark: There is a seeming paradox at the heart of the discussion of oriented integrals. Everyone agrees that in an oriented integral one should use the convention that $dx\, dy = -dy\, dx$. However, represent the integral as a Riemann sum

$$\int f(x,y)\, dx\, dy \approx \sum f(x,y)\, \Delta x\, \Delta y. \tag{7.127}$$

Then Δx and Δy are numbers. Should they not commute?

The resolution of this paradox is that Δx and Δy are indeed numbers, but they are obtained by a prescription that is two-dimensional and for which the order is important. Think of an oriented rectangle with oriented boundary. On the horizontal sides Δy is zero, and on the vertical sides Δx is zero. Now take two successive sides along the boundary and evaluate Δx along the first side and Δy along its successor. There are four possible answers, two of which are zero. Take either of the non-zero answers. This gives Δx and Δy that can be used in the Riemann sum. Notice that this prescription does not uniquely define the numbers Δx and Δy, but it does define their product. If instead the same prescription is applied with $\Delta y\, \Delta x$, then the sign is changed. So the answer to the paradox is that these are indeed numbers, but their definition depends on where they come from. ||

7.6 Stokes' theorem

7.6.1 Chains of parametrized surfaces

Recall that a compact-parametrized surface is defined by $K \subseteq P$ and $\chi : K \to M$. Such a surface may be regarded as oriented by specifying an orientation of P. It is often helpful to be able to combine oriented compact-parametrized surfaces. A *chain of oriented compact-parametrized surfaces* is a finite linear combination of oriented compact parametrized k-surfaces with integer coefficients. Thus, for instance, $2\chi_1 - \chi_2$ consists of two copies of the chain χ_1 and one copy of the chain oppositely oriented from χ_2. The integral of ω over $2\chi_1 - \chi_2$ is defined to be twice the integral of ω over χ_1 minus the integral of ω over χ_2.

Two chains of oriented compact-parametrized k-surfaces are said to be equivalent if for every k-form the integrals are the same. Equivalent chains may be regarded as equal for the purposes of integration.

Example: Let χ_1 be $(x, y) \leftarrow (\cos(t), \sin(t))$ for $0 \leq t \leq 2\pi$. Let χ_2 be $(x, y) \leftarrow (\cos(2t), \sin(2t)0$ for $0 \leq t \leq 2\pi$. Then $\chi_2 = 2\chi_1$. ||

Example: Let χ_1 be $x \leftarrow t(1-t)$ for $0 \leq t \leq \frac{1}{2}$. Let χ_2 be $x \leftarrow t(1-t)$ for $\frac{1}{2} \leq t \leq 1$. Let χ be $x \leftarrow t(1-t)$ from $0 \leq t \leq 1$. These curves all have the same image, the interval $0 \leq x \leq \frac{1}{4}$. As oriented curves they are different. In fact, $\chi_1 + \chi_2 = \chi = 0$. Here one can think of 0 as a curve on which every 1-form has integral zero. Such a curve is given by $x \leftarrow 0$. ||

Example: A change of parameters that preserves orientation will also give equivalent surfaces. Let χ_1 be $x \leftarrow t(1-t)$ for $0 \leq t \leq \frac{1}{2}$ with orientation $[dt]$. Let ϕ_1 be $x \leftarrow u$ for $0 \leq u \leq \frac{1}{4}$ with orientation $[du]$. Consider a differential 1-form $f(x)\,dx$. In the first case it pulls back to $f(t(1-t))(1-2t)\,dt$. In the second case it pulls back to $f(u)\,du$. The two integrals have the same value $F(\frac{1}{4}) - F(0)$, where $F' = f$. This establishes that $\chi_1 = \phi_1$. ||

For Stokes' theorem we use a more restricted notion of compact-parametrized k-surface. Consider a k-dimensional connected manifold patch P with a coordinate system $u_1, \ldots, u_k$. Choose a compact subset Q realized as the points whose coordinates satisfy $a_j \leq u_j \leq b_j$ for $j = 1, \ldots, k$. Here $a_j < b_j$ for each j. (When $k = 0$ we can take Q to be a single point.) We call Q a k-dimensional *cell*. A *cell-parametrized surface* χ is a smooth mapping $\chi : Q \to N$. An *oriented cell-parametrized surface* is a pair $\chi : Q \to N$ together with an orientation $\mathcal{O}$ of P.

For each $i = 1, \ldots, k$, define the $(k-1)$-dimensional parameter cell ∂Q_i^+ to be the points whose coordinates satisfy $u_i = b_i$ and $a_j \leq u_j \leq b_j$ for $j \neq i$. Similarly, define the $k-1$ dimensional parameter cell ∂Q_i^- to be the points whose coordinates satisfy $u_i = a_i$ and $a_j \leq u_j \leq b_j$ for $j \neq i$.

For simplicity assume that the coordinate system $u_1, \ldots, u_k$ has been chosen compatible with the orientation. Consider Q oriented by $u_1, \ldots, u_k$. The boundaries ∂Q_i^+ and ∂Q_i^- have orientations defined by $u_1, \ldots, u_{i-1}, u_{i+1}, \cdots, u_k$. However these are not the correct orientations for them as boundaries. To get the correct orientations, multiply the orientation for ∂Q_i^+ by $(-1)^{i-1}$ and the orientation for ∂Q_j^- by $-(-1)^{i-1}$.

A *chain of oriented cell-parametrized surfaces* is a finite integer combination of oriented cell-parametrized surfaces. The boundary may be written as a chain

$$\partial Q = \sum_{i=1}^{k} (-1)^{i-1} (\partial Q_i^+ - \partial Q_i^-) \tag{7.128}$$

This is a $(k-1)$-dimensional oriented cell-parametrized chain with $2k$ summands. Each cell provides its own parametrization.

Given an oriented cell-parametrized k-surface $\chi : Q \to N$, the *boundary chain* is the cell surface chain

$$\partial \chi = \sum_{i=1}^{k} (-1)^{i-1} (\chi_i^+ - \chi_i^-), \tag{7.129}$$

Here $\chi_i^\pm$ is the restriction of χ to $\partial Q_i^\pm$.

For a cell-parametrized k-surface $\chi : Q \to M$ given by $\mathbf{x} \leftarrow \mathbf{g}(\mathbf{u})$ the faces in the i direction, $i = 1, \ldots, k$, are given as the pullbacks

$$\begin{aligned} \chi_i^+ &= (\mathbf{x} \leftarrow \mathbf{g}(u_1, \ldots, u_{i-1}, b_i, u_{i+1}, \ldots, u_k)) \\ \chi_i^- &= (\mathbf{x} \leftarrow \mathbf{g}(u_1, \ldots, u_{i-1}, a_i, u_{i+1}, \ldots, u_k)). \end{aligned} \tag{7.130}$$

If $\chi = \sum_j m_j \chi_j$ is a k-dimensional oriented cell-surface chain (an integer combination of cell-parametrized k-surfaces), then its boundary $\partial \chi = \sum_j m_j \partial \chi_j$ is a $(k-1)$ cell-surface chain (the corresponding integer combination of the boundaries of the surfaces).

Example: The circle. The circle may be obtained by mapping the interval $0 \leq t \leq 2\pi$ by $x \leftarrow \cos(t), y \leftarrow \sin(t)$. There are two boundary points, corresponding to $t = 0$ and $t = 2\pi$. These have opposite orientations, so they add to zero. The circle has no boundary. ||

Example: The annulus. Many surfaces may be obtained by mapping cells. A good first example is an annulus, the region between two concentric circles. Here polar coordinates on the parameter space are useful. The parameter region is $0 < a \leq r \leq b, 0 \leq \theta \leq 2\pi$. The mapping χ is $(x, y) \leftarrow (r\cos(\theta), r\sin(\theta))$. The boundary chain is four curves, corresponding to $\theta = 0, r = b, \theta = 2\pi, r = a$. The corresponding orientations are given by $dr, d\theta, -dr, -d\theta$. When these orientations are taken into account the first and third of these sum to zero. So the boundary in fact consists of the two circles, with opposite orientations. ||

Example: The disk. This is slightly more degenerate than the previous example. The parameter region is $0 \leq r \leq b, 0 \leq \theta \leq 2\pi$. The mapping χ is $(x, y) \leftarrow (r\cos(\theta), r\sin(\theta))$. The boundary is four curves, corresponding to $\theta = 0, r = b, \theta = 2\pi, r = 0$. When orientation is taken into account the first and third of these sum to zero. The last curve is also zero, so the boundary is just a circle. ||

Example: The triangle. Some treatments like to build regions from triangles. However a rectangular region may be mapped to a triangular region. Take

$0 \leq u \leq 1$ and $-1 \leq v \leq -1$. The mapping is $x \leftarrow u, y \leftarrow uv$. The segment where $u = 0$ is mapped to a single point. ||

Example: The region under a curve. More complicated regions are possible. Here is the easiest example. Consider a function $f(x) > 0$. Take $a \leq u \leq b$ and $0 \leq v \leq 1$. Take $x \leftarrow u$ and $y \leftarrow vf(u)$. This maps to the region where $a \leq x \leq b$ and $0 \leq y \leq f(x)$. ||

Example: Star-shaped region. This is a more subtle example, made possible by the use of polar coordinates in the target space. Take $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$. Again $f(x) > 0$. The mapping is $r \leftarrow uf(v)$ with $\theta \leftarrow v$. The boundary has four curves, corresponding to $v = 0, u = 1, v = 2\pi, u = 0$. The first and third have opposite orientations and sum to zero, while the last is already zero. So the remaining boundary is where $r = f(\theta)$. ||

Example: Sphere. The obvious device is to use spherical polar coordinates on the parameter space. Thus $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. The mapping is $x \leftarrow \sin(\theta)\cos(\phi)$, $y \leftarrow \sin(\theta)\sin(\phi)$, $x \leftarrow \cos(\theta)$. The four boundaries are where $\phi = 0, \theta = \pi, \phi = 2\pi, \theta = 2\pi$. The first and third of these correspond to curves that run from the north pole to the south pole at the same longitude, but in opposite directions. These sum to zero. The second and third are curves that map to the south pole and north pole. These are each zero. The sphere has no boundary. ||

7.6.2 Stokes' theorem for chains of parametrized surfaces

Theorem 7.24 (Stokes' theorem: parametrized surfaces) *Let M be an n-dimensional manifold patch. Let $\chi = \sum_j m_j \chi_j$ be a chain of k-dimensional oriented cell-parametrized surfaces. Let $\partial\chi = \sum_j m_j \partial\chi_j$ be the corresponding boundary chain of $(k-1)$ dimensional oriented cell-parametrized surfaces. Let ω be a $(k-1)$-form on M with k-form exterior derivative $d\omega$. Then*

$$\int_\chi d\omega = \int_{\partial\chi} \omega. \tag{7.131}$$

Proof: It is sufficient to prove this when χ is an oriented cell-parametrized surface. Thus $\chi : Q \to M$ and $\mathcal{O}$ orients Q. The proof has two steps. The first step is to prove the theorem in the special case when the integral of $d\omega$ is over the k-dimensional parameter cell Q. In this case the $(k-1)$-form ω that occurs on the right hand side is pulled back to the boundary chain ∂Q and then integrated.

The general $k-1$ form on a k-dimensional space is

$$\omega = \sum_{j=1}^{k} \omega_j = \sum_{j=1}^{k} f_j(u_1, \ldots, u_k)\, du_1 \wedge \ldots \wedge du_{j-1} \wedge du_{j+1} \wedge \ldots \wedge du_k. \tag{7.132}$$

In that case

$$d\omega = \sum_{j=1}^{k} d\omega_j = \sum_{j=1}^{k} (-1)^{j-1} \frac{\partial f_j(u_1, \ldots, u_k)}{\partial u_j}\, du_1 \wedge \ldots \wedge du_k. \tag{7.133}$$

Hence

$$\int_Q d\omega = \sum_{j=1}^{k} (-1)^{j-1} \int_Q \frac{\partial f_j(u_1, \ldots, u_k)}{\partial u_j} \, du_1 \ldots \, du_k. \tag{7.134}$$

Consider the jth term in this sum. Use Fubini's theorem to write the integral on the right hand side as an iterated integral, with the du_j integral as the inside integral. The fundamental theorem of calculus gives

$$\int_{Q_j} [f_j(u_1, \ldots, b_j, \ldots, u_k) - f_j(u_1, \ldots, a_j, \ldots, u_k)] \, du_1 \ldots \, du_{j-1} \, du_{j+1} \ldots \, du_k. \tag{7.135}$$

Here Q_j is the relevant $(k-1)$-cell where $a_i \leq u_i \leq b_i$ for all $i \neq j$. In other words,

$$\int_Q d\omega = \sum_{j=1}^{k} (-1)^{j-1} [\int_{Q_j^+} \omega_j - \int_{Q_j^-} \omega_j]. \tag{7.136}$$

In the j term we may replace ω_j by ω. (All the other terms in ω are zero when pulled back to $Q_j^{\pm}$.) So this is

$$\int_Q d\omega = \sum_{j=1}^{k} (-1)^{j-1} [\int_{Q_j^+} \omega - \int_{Q_j^-} \omega] = \int_{\partial Q} \omega, \tag{7.137}$$

where ∂Q is the boundary chain with the boundary orientation. This completes the proof for the cell.

It remains to give the proof for a map $\chi : Q \to M$ of the cell to a target space. This second step uses the fact that the integral of a differential form may be expressed by pulling back to a parameter cell. It also depends on the result that the pullback of a differential is the differential of the pullback, that is, $\chi^*(d\omega) = d\chi^*\omega$. This gives

$$\int_\chi d\omega = \int_Q \chi^*(d\omega) = \int_Q d\chi^*\omega = \int_{\partial Q} \chi^*\omega = \int_{\partial \chi} \omega. \tag{7.138}$$

This completes the proof. □

The properly formulated result is rather simple; it follows from the case of the cell and from the remarkable transformation properties of differential forms.

7.6.3 Stokes' theorem for regular surfaces

There is another more sophisticated version of Stokes' theorem. It does not apply to singular surfaces. It does work for regular surfaces while making no explicit mention of parametrization.

Theorem 7.25 (Stokes' theorem: regular surfaces) *Let M be an n-dimensional manifold patch. Let S be an oriented compact regular k-surface in M with boundary regular surface ∂S. The boundary has the orientation coming from S. Let*

ω be a $(k-1)$-form on M with k-form exterior derivative $d\omega$. Then

$$\int_S d\omega = \int_{\partial S} \omega. \tag{7.139}$$

This theorem requires an understanding of the concept of regular surface with boundary. The required definitions and proof may be found in standard references [4, 33, 45]. Here is an example of a typical compact regular surface with boundary.

Example: Suppose that $x_1, \ldots, x_n$ are coordinates, and $u = f(x_1, \ldots, x_k)$ is a function that goes to $+\infty$ as $x_1, \ldots, x_k$ go to infinity. The surface S is defined as the k-surface where $u \leq c$ and $x_{k+1} = c_k, \ldots, x_n = c_n$. This is closed and bounded, hence compact. The boundary $(k-1)$-surface ∂S is where $u = c$ and $x_{k+1} = c_k, \ldots, x_n = c_n$. It is assumed that $du \neq 0$ on ∂S, so ∂S is a regular surface. ||

One problem with the standard proof is that it uses a concept (partition of unity) that is not very helpful in suggesting how to compute the integrals. The practical solution to this problem is to break the surfaces into parts that can be parametrized. There are several issues.

- There may be many ways of writing an regular surface as a chain of parametrized surfaces.
- The orientations must be compatible, but otherwise the results should not depend on the parametrizations.
- Where the parametrized surfaces overlap, the integrals must occur with opposite orientation and cancel.

There are few authors with the courage to attempt this connection. The lecture notes by Conrad [9, How to compute integrals] consider the questions seriously.

The general formulation of Stokes' theorem is the result of a long evolution. See [23] for some of the history. It says that the cumulative effect of circulation in the surface is captured by circulation in the boundary. A later chapter willl present a variant of Stokes' theorem for twisted forms and surfaces with transverse orientation. This version says that the cumulative effect of sources within the surface is captured by flux through the boundary. This version often occurs in applications.

Example: Take the 1-form $\alpha = x\,dy - y\,dx = r^2\,d\theta$ in two dimensions. This form is pictured by a family of half-lines lines with constant θ. Since $d\alpha = 2\,dx\,dy = 2r\,dr\,d\theta$ the end points have constant density. Each half-line has an external orientation in the direction of increasing θ. Consider $x_1 < 0 < x_2$ and $y_2 < 0 < y_2$. The region S is defined by $x_1 \leq x \leq x_2$ and $y_1 < y < y_2$. The orientation of S is defined by $dx\,dy$. One thinks of α as indicating some sort of subjective change in vertical distance. The integral along ∂S is $x_2(y_2 - y_1) - y_2(x_1 - x_2) + x_1(y_1 - y_2) + y_1(x_2 - x_1)$. The change is uphill along the entire closed curve. Since $d\alpha = 2\,dx\,dy$, the theorem says that the total uphill climb must be equal to $2(x_2 - x_1)(y_2 - y_2)$. This is indeed the case.

In essence this is the famous picture of the Penrose stairs. The most famous illustration of these stairs is the Ascending and Descending lithograph print by Escher. ||

Example: It is important that the boundary ∂S is the entire boundary of the region S where the form is defined. The classical example is the closed 1-form $\alpha = (x\,dy - y\,dx)/(x^2 + y^2) = d\theta$ defined in the plane with the origin removed. This 1-form is described by radial (constant angle) half-lines from the origin. Say that $S = A$ is the annulus between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ with $0 < a < b$. The orientation is $dx\,dy$. Then ∂S consists of the two circles, with opposite orientations. Each line that enters the annulus on one circle leaves it on the other circle. So there is complete cancellation, and the integral of α over ∂S is zero. Since $d\alpha = 0$, this is consistent with the theorem.

It is also important that α be smooth in S. With the same form α one could attempt to apply the theorem with $S = D$, the disk bounded by the circle $x^2 + y^2 = b^2$. The theorem does not apply because α is singular at the origin.

One could modify this attempt by taking $S = D^\bullet$, the disk with the origin removed. Again the theorem does not apply. Even though $d\alpha = 0$ in S, the region S cannot be constructed as the image of finitely many compact cells. In fact, the integral of α over ∂S is 2π. ||

7.6.4 The Gauss theorem for $(n-1)$-forms

The Gauss theorem is a special case of Stokes' theorem, but it is worth examining in detail.

Theorem 7.26 (Gauss) *Suppose that M is n-dimensional, and W is a n-dimensional oriented region in M with $(n-1)$-dimensional boundary regular surface ∂W. Then ∂W has the corresponding boundary orientation. Say that in some coordinate system*

$$\omega = \sum_j (-1)^{j-1} p_j\, dx_1 \cdots dx_{j-1}\, dx_{j+1} \cdots dx_n. \tag{7.140}$$

With this choice of signs

$$d\omega = \left(\sum_j \frac{\partial p_j}{\partial x_j} \right) dx_1 \cdots dx_n. \tag{7.141}$$

Then the Gauss theorem says that

$$\int_W d\omega = \int_{\partial W} \omega. \tag{7.142}$$

Parameterize ∂W as a regular parametrized surface by a mapping $\chi : P \to M$ given as $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u})$. Here P has the orientation $[du_1 \cdots du_{n-1}]$. Define *penetration coefficients* ν_j by

$$\nu_j\, du_1 \cdots du_{n-1} = (-1)^{j-1} \chi^*(dx_1 \cdots dx_{j-1}\, dx_{j+1} \cdots dx_n). \tag{7.143}$$

Theorem 7.27 (Gauss with parametrized boundary 1) *Suppose that the orientation of W is given by $dx_1 \cdots dx_n$. Let β be an $(n-1)$-form that gives the orientation of the boundary ∂W. Let $\chi : P \to \partial W$ parametrize the boundary, where the orientation of P is given by $du_1 \cdots du_{n-1}$. Finally suppose that this orientation is the same as the orientation on P given by $\chi^* \beta$. Then*

$$\int_W \left(\sum_j \frac{\partial p_j}{\partial x_j} \right) dx_1 \cdots dx_n = \int_P \sum_j (p_j \circ \chi) \nu_j \, du_1 \cdots du_{n-1}. \tag{7.144}$$

Define the *penetration form* along χ to be by

$$\nu = \sum_j \nu_j dx_j. \tag{7.145}$$

Here it is understood that the dx_j are restricted differentials; they act on tangent vectors to M, but only at points in the range of χ.

Proposition 7.28 *The penetration form ν is zero on each tangent vector to ∂W. Therefore it defines a transverse orientation of $\partial \Omega$.*

Proof: The form $\nu = \sum_j \nu_j \, dx_{j\,[\phi]}$ is non-zero and is zero on every tangent vector to ∂W. In coordinates this says that for each α we have

$$\sum_j \nu_j \frac{\partial (x_j \circ \chi)}{\partial u_\alpha} = 0. \tag{7.146}$$

The result may be seen by considering determinants. There is a a n by $n-1$ matrix with entries $\partial (x_j \circ \chi)/\partial u_\beta$. Select a particular value of α, and consider the n by n matrix obtained by putting the column $\partial (x_j \circ \chi)/\partial u_\alpha$ as the first column. This matrix has determinant zero, since it has a repeated column. The cofactor of the j entry in the first column is ν_j. So the result follows from the cofactor expansion of the determinant. □

Theorem 7.29 (Gauss with parametrized boundary 2) *Suppose that the orientation of W is given by $dx_1 \cdots dx_n$. Let $\chi : P \to \partial W$ parametrize the boundary, where the orientation of P is given by $du_1 \cdots du_{n-1}$. Finally suppose that the penetration form ν is an outward transverse form.*

$$\int_W \left(\sum_j \frac{\partial p_j}{\partial x_j} \right) dx_1 \cdots dx_n = \int_P \sum_j (p_j \circ \chi) \nu_j \, du_1 \cdots du_{n-1}. \tag{7.147}$$

Proof: The Gauss theorem involves the boundary orientation. Choose $\sigma = dx_1 \cdots dx_n$ to give the orientation $[\sigma]$ of W. Then there is an $(n-1)$-form β that gives the boundary orientation $[\beta]$ on ∂W. Let ϵ be a never-zero 1-form along ∂W that increases in the outward direction and is zero every tangent

vector to ∂W. This is an outward transverse orientation form. The form giving the orientation β of ∂W is defined by

$$\epsilon \wedge \beta = \sigma. \tag{7.148}$$

Example: If W is given implicitly by $s \leq c$, then ∂W is given by $s = c$, and we can take $\epsilon = ds$. ||

Lemma 7.30 *Suppose the outward transverse orientation form of ∂W is*

$$\epsilon = \sum_j c_j \, dx_j. \tag{7.149}$$

Then the form given by

$$\beta = \sum_j (-1)^{j-1} c_j dx_1 \cdots dx_{j-1} \, dx_{j+1} \cdots dx_n. \tag{7.150}$$

defines the boundary orientation for ∂W.

Proof: The form $\epsilon \wedge \beta = (\sum_j c_j)^2 dx_1 \cdots dx_n$ defines the same orientation as $\sigma = dx_1 \ldots dx_n$. □

Lemma 7.31 *The penetration form ν is zero on each tangent vector to ∂W. This is also true for the outer transverse orientation form ϵ restricted to be a form $\epsilon_{[\chi]}$ along χ. It follows that the penetration form $\nu = \lambda \epsilon_{[\chi]}$, where λ is a non-zero scalar field. In components*

$$\sum_j \nu_j \, dx_j = \lambda \sum_j (c_j \circ \chi) \, dx_j, \tag{7.151}$$

where it is understand that the dx_j are restricted differentials.

Proof: The form $\epsilon_{[\chi]}$ is non-zero and is zero on every tangent vector to ∂W. In coordinates this says that

$$\sum_j (c_j \circ \chi) \frac{\partial x_j \circ \chi}{\partial u_\alpha} = 0. \tag{7.152}$$

Since both ν and ϵ annihilate the tangent space, it follows that $\nu = \lambda \epsilon_{[\chi]}$ for some never-zero scalar function λ on P. □

Lemma 7.32 *The $(n-1)$-form*

$$\chi^* \beta = \sum_j (c_j \circ \chi) \nu_j \, du_1 \cdots du_{n-1} = \lambda \sum_j (c_j \circ \phi)^2 \, du_1 \cdots du_{n-1} \tag{7.153}$$

defines the same orientation on P as $\lambda \, du_1 \cdots du_{n-1}$. Therefore the boundary orientation of ∂W given by β is equal to $\pm$ the orientation given by $\chi : P \to \partial W$ (with orientation given by $du_1 \cdots du_{n-1}$ on P), depending on the sign of λ.

The last two lemmas above lead to the result. The condition that the penetration form is in the outward direction is $\lambda > 0$. The condition that $\chi^* \beta$ gives the same orientation as $du_1 \cdots du_{n-1}$ is also $\lambda > 0$. So these two conditions are equivalent. This shows that the two statements of the Gauss theorem are making the same assertion. □

Example: Let $a > 0$. Suppose that W is the ball where $x^2 + y^2 + z^2 \leq a^2$, and ∂W is the sphere where $x^2 + y^2 + z^2 = a^2$. Suppose the orientation of W is given by $\sigma = dx\, dy\, dz$. An outward orientation form is given by $\epsilon = x\, dx + y\, dy + z\, dz$. Then the orientation of ∂W may be given by $\beta = x\, dy\, dz + y\, dz\, dx + z\, dx\, dy$.

The orientation of ∂W may also be specified by tangent vectors, but this is more awkward. Tangent vectors to the sphere are be given by the restrictions of $X_1 = xz\partial/\partial x + yz\partial/\partial - (x^2 + y^2)\partial/\partial z$ and $X_2 = -y\partial/\partial x + x\partial/\partial y$. (These span the space except at the poles.) The form ϵ vanishes on both these vectors. A computation shows that $\langle \beta \mid X_1, X_2 \rangle = (x^2 + y^2)(x^2 + y^2 + z^2) > 0$. So the orientation is also determined by the ordered pair (X_1, X_2).

Take the parametrization to be $(x, y, z) \leftarrow (a \sin(u) \cos(v), a \sin(u) \sin(v), a \cos u)$. The pullbacks of $dx\, dy$, $dy\, dz$, and $dz\, dx$ are $a^2 \sin^2(u) \cos(v)\, du\, dv$), $a^2 \sin^2(u) \sin(v)\, du\, dv$, $a^2 \sin(u) \cos(u)\, du\, dv$. So the penetration form is $\nu = a^2 \sin^2(u) \cos(v)\, dx + a^2 \sin^2(u) \sin(v)\, dy + a^2 \sin(u) \cos(u)\, dz$.

Another basis of tangent vectors to the sphere is given by by pushing forward $\partial/\partial u$ and $\partial/\partial v$. The resulting tangent vectors are $Y - 1 = a \cos(u) \cos(v) \partial u/\partial x + a \cos(u) \sin(v) \partial/\partial v - a \sin(u) \partial/\partial z$ and $Y_2 = -a \sin(u) \sin(v) \partial/\partial x + a \sin(u) \cos(v) \partial/\partial y$. It is easy to check that the penetration form vanishes on these vectors.

The penetration form may be written as $\nu = a \sin(u)(a \sin(u) \cos(v)\, dx + a \sin(u) \sin(v)\, dy + a2 \cos(u)\, dz)$. The form $a \sin(u) \cos(v)\, dx + a \sin(u) \sin(v)\, dy + a \cos(u)\, dz$ is the restriction of ϵ to the sphere. The multiplicative factor is $a \sin(u) > 0$. So the parameterization gives the correct orientation.

The form $\omega = x^3\, dy\, dz + y^3\, dz\, dx + z^3\, dz\, dx$ gives a simple example of Gauss's theorem at work. The differential $d\omega = 3(x^2 + y^2 + z^2)\, dx\, dy\, dz = 3r^2\, r^2 \sin(\theta)\, dr\, d\theta\, d\phi$. The angular integrals give 4π. The radial integral gives 3 times $a^5/5$. So the answer is $12\pi a^5/5$. ||

7.6.5 Classical versions of Stokes' theorem

In the following less formal terminology may be used. A top dimensional surface with boundary may be called a *region*. A lower dimensional surface with boundary is a *surface*. A one-dimensional surface with boundary is a *curve*. For the purposes of integration of differential forms these must all be oriented.

Here are some classical forms of Stokes' theorem. The integral of a 1-form along an oriented curve is frequently called a *line integral*. In general such an integral must be computed by pulling back to a parameter interval. However for an exact form there is a shortcut. The *fundamental theorem of calculus* is the case relating scalars to 1-forms. It says that for every scalar field s and every parametrized curve C we have

$$\int_C ds = \Delta s. \tag{7.154}$$

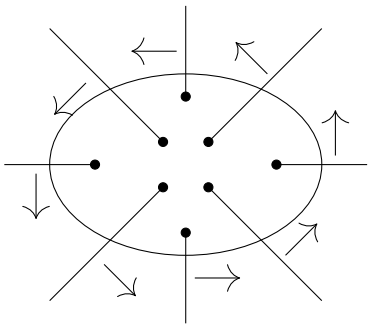

Figure 7.14: Green's theorem (2d): Curves are 1-forms α with arrows indicating transverse orientation; points are 2-forms $d\alpha$ (with 2d orientations, not shown).

Here C is an oriented path from one point to another point, and Δs is the value of s at the final point minus the value of s at the initial point. Notice that the result does not depend on the choice of path. This is because ds is an exact form.

Example: Consider the form $y^2\,dx + 2xy\,dy$. Since it is exact, we have

$$\int_C y^2\,dx + 2xy\,dy = \int_C d(xy^2) = \Delta(xy^2) \tag{7.155}$$

independent of the path. ||

The most common version of Stokes' theorem relates 1-forms to 2-forms. The 2-dimensional version of the theorem is *Green's theorem* . This is the result for a 1-form $\alpha = p\,dx + q\,dy$. It says that

$$\int_R \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)\,dx\,dy = \int_{\partial R} p\,dx + q\,dy. \tag{7.156}$$

Here R is an oriented region in two dimensional space, and ∂R is the curve that is its oriented boundary.

Figure 7.14 illustrates the mechanism of Green's theorem. The 2-form points (with 2d orientations) in the oriented region generate curves (with 1d transverse orientations) that pass through the oriented oundary. The integral of the 1-form α around the boundary counts these curves, which is the same thing as counting the points represented by the 2-form $d\alpha$. Of course all of this must take the signs (arising from relative orientations) into account.

Example: A classical application of Green's theorem is the computation of area via

$$\int_R dx\,dy = \frac{1}{2}\int_{\partial R} x\,dy - y\,dx. \tag{7.157}$$

In polar coordinates this takes the form

$$\int_R r\,dr\,d\theta = \frac{1}{2}\int_{\partial R} r^2\,d\theta. \tag{7.158}$$

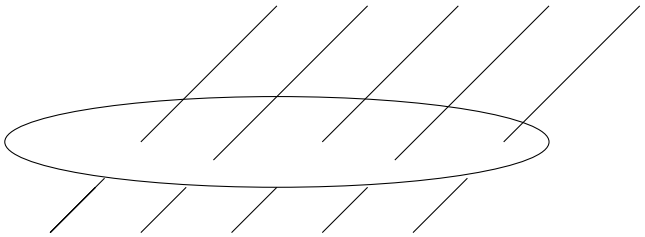

Figure 7.15: Stokes' theorem (3d): $d\alpha$ is 2-form shown as curves (2d transverse orientations not shown) penetrating surface with boundary ∂S.

A typical parameter region R for the integral on the left hand side is the set of r where $r \leq f(\theta)$. The boundary is where $r = f(\theta)$. This method may be implemented by a gadget called a *planimeter*. ||

The integral of a 2-form over a surface is called a *surface integral*. The *classical Stokes's theorem* applies to a 1-form $\alpha = p\,dx + q\,dy + r\,dz$. It says that for an oriented two dimensional surface S in a three dimensional space with oriented boundary curve ∂S we have

$$\int_S \left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}\right) dy\,dz + \left(\frac{\partial p}{\partial z} - \frac{\partial r}{\partial x}\right) dz\,dx + \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx\,dy = \int_{\partial S} (p\,dx + q\,dy + r\,dz). \tag{7.159}$$

Figure 7.15 illustrates Stokes' theorem in three dimensions. It may be difficult to picture the 1-form α, but $d\alpha$ is represented by circulatory flux curves that pass through a surface with boundary ∂S.

Figure 7.16 and Figure 7.17 represent two oriented surfaces S_1 and S_2 with the same oriented boundary ∂S. Restricting these forms α and $d\alpha$ to the surface makes the picture looks like that of Green's theorem. The restricted 1-forms are represented by curves in the surface, and the restricted 2-forms are shown as the intersection points of the circulatory flux curves with the surfaces. So the result appears as a consequence of Green's theorem. In brief, it says that the flux of $d\alpha$ through the surfaces is equal to the circulation of α around the boundary. The three dimensional character comes in from the fact that the circulatory flux curves (without end points) match up the intersection points in S_1 with the corresponding intersection points in S_2.

This result for 2-forms has an obvious analog in n dimensions. This case of Stokes' theorem has important consequences for line integrals of closed forms.

Theorem 7.33 (Integral over a boundary) *Let ω be a closed differential 1-form. Let R be a surface on which ω is smooth, and let ∂R be the curve that is its oriented boundary. Then*

$$\int_{\partial R} \omega = 0. \tag{7.160}$$

Theorem 7.34 (Integrals over a boundary) *Let ω be a closed differential 1-form. Let R be a surface on which ω is smooth, and let ∂R be the curve that*

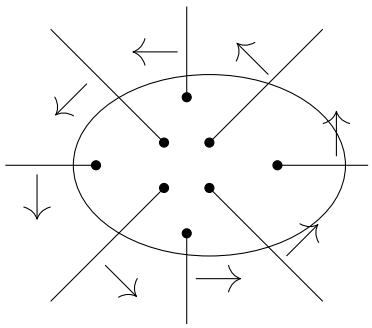

Figure 7.16: Stokes' theorem (3d) for $S_1, \partial S_1 = \partial S$: Curves are 1-forms α on S_1; points are 2-forms $d\alpha$ on S_1.

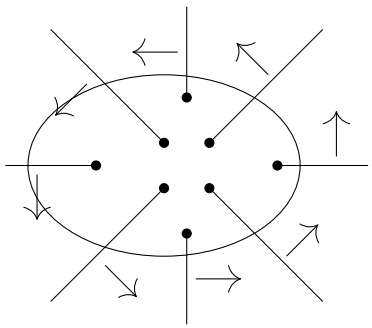

Figure 7.17: Stokes' theorem (3d) for $S_2, \partial S_2 = \partial S$: Curves are 1-forms α on S_2; points are 2-forms $d\alpha$ on S_2.

is its oriented boundary. Suppose that C_1 and C_2 are oriented curves such that $C_1 - C_2 = \partial R$. Then

$$\int_{C_1} \omega = \int_{C_2} \omega. \tag{7.161}$$

Theorem 7.35 (Exactness of closed forms in special regions) *Suppose ω is a closed differential 1-form that is smooth in an open set U. Suppose U has the property that whenever C is a closed curve in U, then $C = \partial R$ for some region in U. Then ω is exact in U.*

Proof: There is no loss of generality in assuming that U is connected. Choose an initial point in U. For each final point in U there is a path C from the initial point to the final point. If C_1 and C_2 are two such paths, then by assumption $C_1 - C_2$ is a boundary, so the integral over C_1 is equal to the integral over C_2. Let s be the integral ω over any C from the initial point to a final point, with s considered as a function of the final point.

Choose a fixed point in U and a coordinate system $\mathbf{x}$ so that this point is the origin. It is enough to show that $ds = \omega$ in some ball about the origin. Write ω as

$$\omega = \sum_i h_i(\mathbf{x})\, dx_i. \tag{7.162}$$

Fix j. Consider the integral along a line from the origin to to the point with coordinates $(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n)$. This has some value c_j that does not depend on x_j. This is followed by an integral from this point to $\mathbf{x}$. Therefore

$$s = s_0 + c_j + \sum_i \int_0^1 h_j(x_1, \ldots, x_{j-1}, t x_j, x_{j+1}, \ldots, x_n) x_j \, dt. \tag{7.163}$$

It follows that

$$\frac{\partial s}{\partial x_j} = \int_0^1 \frac{\partial}{\partial x_j} \left(h_j(x_1, \ldots, x_{j-1}, t x_j, x_{j+1}, \ldots, x_n) x_j \right) \, dt. \tag{7.164}$$

However by the product rule and chain rule

$$\begin{aligned} &\frac{\partial}{\partial x_j} \left(h_j(x_1, \ldots, x_{j-1}, t x_j, x_{j+1}, \ldots, x_n) x_j \right) \\ &= h'_{j,j}(x_1, \ldots, x_{j-1}, t x_j, x_{j+1}, \ldots, x_n) x_j t + h_j(x_1, \ldots, x_{j-1}, t x_j, x_{j+1}, \ldots, x_n) \\ &= \frac{d}{dt} \left(h_j(x_1, \ldots, x_{j-1}, t x_j, x_{j+1}, \ldots, x_n) t \right). \end{aligned} \tag{7.165}$$

So by the fundamental theorem of calculus

$$\frac{\partial s}{\partial x_j} = h_j(\mathbf{x}). \tag{7.166}$$

This shows that $ds = \omega$. $\square$

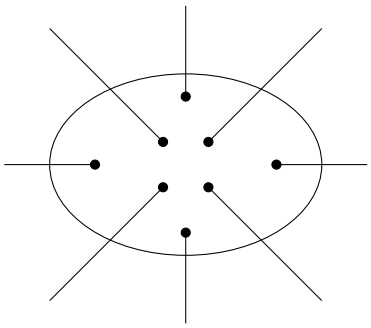

Figure 7.18: Gauss's theorem (3d): Curves are 2-forms ω (2d transverse orientations not shown); points are 3-forms $d\omega$ (3d orientions not shown).

In two dimensions the last three theorems are a consequence of Green's theorem. Even this case is significant. Much of what is interesting in complex variables depends on the fact that

$$\alpha = \frac{x\,dy - y\,dx}{x^2 + y^2} \tag{7.167}$$

is a form (defined in the plane with one point removed) that is closed but not exact. If one considers the plane with an entire half-line from the origin removed, then this form is exact in that smaller region, in fact, $\alpha = d\phi$, where ϕ is a suitable angle. But the interest is in what happens with curves that go entirely around the origin. Since such a curve is not a boundary, it is not surprising that the result can be a non-zero multiple of 2π.

Gauss's theorem is the case relating $(n-1)$-forms to n-forms. The classical case is when $n = 3$, so that it relates 2-forms to 3-forms. The 2-form is of the form $\omega = a\,dy\,dz + b\,dz\,dx + c\,dx\,dy$. Let W be an oriented three dimensional region, and let ∂W be the oriented surface that forms its boundary. Then the three dimensional version of Gauss's theorem states that

$$\int_W \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}\right) dx\,dy\,dz = \int_{\partial W} a\,dy\,dz + b\,dz\,dx + c\,dx\,dy. \tag{7.168}$$

Figure 7.18 shows Gauss's theorem for an oriented three-dimensional region R with two-dimensional oriented boundary surface ∂R. In this case the 2-form ω is represented by circulatory flux curves that leave the region, and the 3-form $d\omega$ is represented by the source points. The circulatory flux curves carry 2-d transverse orientations, while the source points carry 3-d orienations. The flux through the surface is represented by the (signed) count of the curves crossing the boundary, and this the total source is represented by the (signed) count of the points within. One has to use imagination to think of this as a three-dimensional picture, but the mechanism should be clear.

Example: Let S be the unit sphere in Euclidean space. Say that we want to integrate $\omega = x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$ over the sphere. We can apply Gauss's theorem. Since $d\omega = 3\,dx\,dy\,dz$ the result is 3 times the volume of the unit ball, that is,

$$\int_S \omega = \int_B 3\,dx\,dy\,dz = 3\frac{4}{3}\pi = 4\pi. = 4\pi. \tag{7.169}$$

The factor of 3 in the divergence is just what is needed to get the result that was obtained earlier via a parameterization. ||

A final word concerns spelling. For names that end with an s sound the usual possessive ending 's can seem cumbersome; some writers prerfer the apostrophe by iteself. In any case, the theorems most typically appear as Green's theorem, Stokes' theorem, and Gauss's theorem.

7.6.6 Picturing Stokes' theorem

Stokes' theorem states that for a k-form ω and an oriented $(k+1)$-surface W

$$\int_W d\omega = \int_{\partial W} \omega. \tag{7.170}$$

According to the theorem, these integrals depend on ω only through $d\omega$; the form ω may be replace $\omega + \sigma$ with $d\sigma = 0$, and both sides remain the same. Similarly, they depend only on the oriented boundary ∂W; how it is filled in by the oriented surface W does not matter.

- A 0-form (scalar field) $\omega = s$ has a differential ds that is an exact 1-form. This 1-form is represented by closely spaced $(n-1)$-dimensional surfaces. In this case W is an oriented curve. Stokes's theorem (the fundamental theorem of calculus) says that the sum of the contributions obtained from going from one surface to the next is equal to the difference in values of s at the end points.

- A 1-form ω in 2 dimensions is represented by contour curves that may have end points. The curves have 1-dimensional transverse orientation that indicates an up direction. The end points have 2-dimensional orientation. The integral of ω counts how many contour curves are crossed upward and downward. Since these contour curves have the points as sources, Green's theorm can be stated as saying that

  $$\text{total source from } d\omega \text{ in region } = \text{ circulation of } \omega \text{ around boundary.} \tag{7.171}$$

- A $(n-2)$-form ω has a differential $d\omega$ that is an exact $(n-1)$-form. (The case of greatest interest is $n = 3$.) Thus $d\omega$ is pictured as circulatory flux curves that do not terminate. Each such curve may be pictured as a tiny tube with an orientation around the tube. The integral of $d\omega$ over the oriented surface W is the flux through the surface. It depends on

the relative orientation of the tube and the surface. Each tube can be thought of as carrying one unit of circulation. This is equal to the integral of ω over the closed boundary ∂W. This quantity is sometimes called the circulation of ω. This terminology is particularly appropriate for $n = 3$. In this case the 1-form α pulls back to a 1-form on the surface W. The intersections of the tubes with the surface are sources for curves in the surface representing α. In summary, Stoke's theorem says

$$\text{flux of } d\omega \text{ through surface } = \text{ circulation of } \omega \text{ around boundary.} \tag{7.172}$$

- A $(n-1)$-form ω is represented by circulatory flux curves with an $(n-1)$-dimensional transverse orientation. These can be thought of as very narrow tubes carrying the orientation. The differential $d\omega$ is an n-form represented by the cloud of end points of these curves, with appropriate n-dimensional orientations. Stokes'theorem (Gauss's theorem) says that the contribution of the source points inside the region is equal to the contribution of the flux density curves crossing the oriented boundary surface. In summary:

$$\text{total source from } d\omega \text{ in region } = \text{ flux of } \omega \text{ through boundary.} \tag{7.173}$$

This story applies also when $n = 2$, so this gives an alternate intuition for Green's theorem.

Problems

Integration of differential k-forms

1. The differential 2-form $\omega = (2xy - x^2)\,dx\,dy$ is of the form $\omega = d\alpha$, where α is a 1-form. Find such an α. Hint: This is too easy; there are many solutions.

2. The differential 3-form $\sigma = (yz + x^2z^2 + 3xy^2z)\,dx\,dy\,dz$ is of the form $\sigma = d\omega$, where ω is a 2-form. Find such an ω. Hint: Many solutions.

3. Let $\sigma = xy^2z\,dy\,dz - y^3z\,dz\,dx + (x^2y + y^2z^2)\,dx\,dy$. Show that this 2-form σ satisfies $d\sigma = 0$.

4. The previous problem gives hope that $\sigma = d\alpha$ for some 1-form α. Find such an α. Hint: This may require some experimentation. Try α of the form $\alpha = p\,dx + q\,dy$, where p, q are functions of x, y, z. With luck, this may work. Remember that when integrating with respect to z the constant of integration is allowed to depend on x, y.

Integrating factors

1. Say that the differential 1-form $\alpha = p\,dx + q\,dy + r\,dz$ has an integrating factor $\mu \neq 0$ such that $\mu\alpha = ds$. Prove that $\alpha \wedge d\alpha = 0$. Also, express this condition as a condition on p, q, r and their partial derivatives.

2. Show that $\alpha = dz - y\,dx - dy$ has no integrating factor.

3. Show that the differential 1-form $\alpha = yz\,dx + xz\,dy + dz$ passes the test for an integrating factor.

4. In the previous problem it might be difficult to guess the integrating factor. Show that $\mu = e^{xy}$ is an integrating factor, and find s with $\mu\alpha = ds$.

Optimization without Lagrange multipliers

Suppose that $w_1 = c_1, \ldots, w_k = c_k$ give the implicit equations of a regular surface. To look for the critical points of s restricted to this surface, one ordinarily looks for solutions of $ds = \sum_{j=1}^{k} \lambda_j\,dw_k$. Exterior algebra gives an alternate method. We should have $dw_1 \wedge \cdots \wedge dw_k \neq 0$. In fact, at most points we should also have $ds \wedge dw_1 \wedge \cdots \wedge dw_k \neq 0$. However at a critical point

$$ds \wedge dw_1 \wedge \cdots \wedge dw_k = 0. \tag{7.174}$$

This gives equations to solve for the critical point that make no reference to the Lagrange multipliers λ_j.

1. (a) Use exterior algebra to find the critical points of $s = xy$ restricted to the surface given implicitly by $w = x^2 + y^2 = 2$.

   (b) Do the same problem in polar coordinates.

Stokes' theorem

1. Let C be the curve $x^2 + y^2 = 1$ in the first quadrant from $(1, 0)$ to $(0, 1)$. Evaluate
$$\int_C xy\,dx + (x^2 + y^2)\,dy. \tag{7.175}$$

2. Let C be a curve from $(2, 0)$ to $(0, 3)$. Evaluate
$$\int_C 2xy\,dx + (x^2 + y^2)\,dy. \tag{7.176}$$

3. Consider the problem of integrating the differential form
$$\alpha = -\frac{y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy \tag{7.177}$$
from $(1, 0)$ to $(-1, 0)$ along some curve avoiding the origin. There is an infinite set of possible answers, depending on the curve. Describe all such answers.

4. Let R be the region $x^2 + y^2 \leq 1$ with $x \geq 0$ and $y \geq 0$. Let ∂R be its boundary (oriented counterclockwise). Evaluate directly

$$\int_{\partial R} xy\,dx + (x^2 + y^2)\,dy. \tag{7.178}$$

5. This continues the previous problem. Verify Green's theorem in this special case, by explicitly calculating the appropriate integral over the region R.

Stokes' theorem (continued)

1. Let

$$\alpha = -y\,dx + x\,dy + xy\,dz. \tag{7.179}$$

Fix $a > 0$. Consider the surface S that is the hemisphere $x^2 + y^2 + z^2 = a^2$ with $z \geq 0$. Integrate α over the boundary ∂S of this surface (a counterclockwise circle in the x, y plane).

2. This continues the previous problem. Verify Stokes's theorem in this special case, by explicitly calculating the appropriate integral over the surface S.

3. Let $\sigma = xy^2 z\,dy\,dz - y^3 z\,dz\,dx + (x^2 y + y^2 z^2)\,dx\,dy$. Integrate σ over the sphere $x^2 + y^2 + z^2 = a^2$. Hint: This should be effortless.

4. There are two basic ways to use Stokes' theorem. The first is to try to integrate α over some oriented S. Check if $d\alpha = 0$. If this is so, then one has hope that $\alpha = d\omega$, with possibly many choices of ω. Then

$$\int_S \alpha = \int_S d\omega = \int_{\partial S} \omega. \tag{7.180}$$

Take $\alpha = (x+z^2)\,dy\,dz - (1+z)\,dx\,dy$. Integrate over S given by $z = x^2 + y^2$ with $z \leq 1$. Take counterclockwise orientation when viewed from above (from positive z). Use Stoke's theorem. Hint: Take $\omega = y\,dx - xz\,dy + yz^2\,dz$.

5. The other basic way to use Stokes' theorem is to integrate ω over a closed curve C. Thus $C = \partial S$, with possibly many choices of surface S. Then

$$\int_C \omega = \int_{\partial S} \omega = \int_S d\omega. \tag{7.181}$$

Integrate over C given by $4x^2 + 9y^2 = 36, z = 2$ with counterclockwise orientation viewed from above (from positive z). The form is

$$\omega = (\cos^6(x)\cos(z) + xy)\,dx + (\sin^8(y)e^z + x^3)\,dy + \sin(x^2 + y^2)e^{y^2+z^2}\,dz. \tag{7.182}$$

The angle form

In a vector space the *radial vector*

$$E = \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} \tag{7.183}$$

is rotationally symmetric. In a Euclidean vector space the volume form

$$\text{vol} = dx_1 \cdots dx_n \tag{7.184}$$

is also rotationally symmetric. It follows that the $n = 1$ form

$$\omega = E \lrcorner \text{vol} = \sum_{j=1}^{n} (-1)^{j-1} x_j dx_1 \cdots dx_{j-1}\, dx_{j+1} \cdots dx_n. \tag{7.185}$$

is also rotationally symmetric. If $r^2 = x_1^2 + \cdots + x_n^2$, then the $n-1$ form

$$\sigma = \frac{1}{r^n}\omega \tag{7.186}$$

is rotationally symmetric and invariant under change of scale. This is the *angle form*.

1. Find $d\omega$ away from the origin. This is the source of the radial flux.
2. Find the integral of ω over the sphere centered at the origin with radius $a > 0$.
3. Find the integral of σ over the sphere centered at the origin with radius $a > 0$.
4. Show that $dr^2 \wedge \omega$ is a multiple of the volume form (by a function of r). And so is $dr \wedge \omega$.
5. Show that the volume in polar coordinates is $\text{vol} = r^{n-1}\, dr \wedge \sigma$.
6. Find $d\sigma$ away from the origin.

7. Consider the sphere centered at the origin with radius $a > 0$. Fix k and consider the coordinates $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots x_n$ on a hemisphere. Thus x_k is a function of these coordinates. Prove that the pullback ω^* is

$$\omega^* = (-1)^{k-1} \frac{a^2}{x_k} dx_1 \cdots dx_{k-1}\, dx_{k+1} \cdots dx_n. \tag{7.187}$$

Hint: Write $\omega*$ as a sum over terms with $j \neq k$ along with the k term considered separately. In the terms with $j \neq k$ use the restriction to the sphere to express dx_k in terms of the other dx_i.

8. Show that on the sphere of radius a we have

$$\frac{x_k^2}{a^2}\omega^* = (-1)^{k-1} x_k dx_1 \cdots dx_{k-1}\, dx_{k+1} \cdots dx_n. \tag{7.188}$$

This explains the meaning of the individual terms in the original expression for ω. In fact,

$$\omega^* = \sum_k \frac{x_k^2}{a^2}\omega^* \tag{7.189}$$

is the decomposition along the coordinate directions.

9. Show that when $n = 3$ and the angle form is expressed in spherical polar coordinates, then the decomposition is

$$\sigma = \sin(\theta)\, d\theta\, d\phi = \sin^2(\theta)\cos^2(\phi)\sin(\theta)\, d\theta\, d\phi + \sin^2(\theta)\sin^2(\phi)\sin(\theta)\, d\theta\, d\phi + \cos^2(\theta)\sin(\theta)\, d\theta\, d\phi. \tag{7.190}$$

Chapter 8

Volume Form

8.1 Vector fields and volume

8.1.1 Volume form versus volume element

The setting for the following discussion is an n-dimensional connected manifold patch M. A *volume form* is a n-form on M that never vanishes. Each volume form σ defines an orientation $[\sigma]$ of M.

A *twisted differential k-form* is determined by a pair $(\alpha, \mathcal{O})$, where α is a differential k-form and $\mathcal{O}$ is an orientation of M. The pair $(-\alpha, -\mathcal{O})$ determines the same twisted differential k-form:

$$(\alpha : \mathcal{O}) = (-\alpha : -\mathcal{O}). \tag{8.1}$$

If $[\sigma] = \mathcal{O}$, we can denote the twisted differential form by

$$(\alpha : \mathcal{O}) = (\alpha : [\sigma]). \tag{8.2}$$

It might even be reasonable to write $(\alpha : \sigma)$. These notations are a slight variation of the notations used in [6].

Example: In the two-dimensional case, consider a coordinate system z, w. Then $dz\,dw$ is a 2-form. It determines a orientation $[dz\,dw]$. An example of a twisted 1-form is $((1 - wz)\,dw : [dz\,dw])$. This is the same twisted 1-form as $((wz - 1)\,dw : [dw\,dz])$. ||

Here are three basic constructions for operating on twisted differential forms:

- $\alpha \wedge (\omega : \mathcal{O}) = (\alpha \wedge \omega : \mathcal{O})$,
- $X \lrcorner (\omega : \mathcal{O}) = (X \lrcorner \omega : \mathcal{O})$,
- $d(\omega : \mathcal{O}) = (d\omega : \mathcal{O})$.

If $(\alpha : \mathcal{O}) = (-\alpha : -\mathcal{O})$ is a never-zero twisted differential k-form, then there may be an $(n - k)$-form η so that $[\eta \wedge \alpha] = [\sigma] = \mathcal{O}$. The form η is called an *reorienting form* for the twisted differential form. In that case it is convenient to make the definition

$$\alpha|_\eta = (\alpha : [\eta \wedge \alpha]). \tag{8.3}$$

This notation is not standard, but it is often quite useful for giving an intuition for twisted forms. It will indicate that the twisted form has lost the usual transverse orientation of the form α and instead has taken on an orientation determined by η.

Proposition 8.1 *A twisted differential k-form $\alpha|_\eta$ defined by reorienting a differential k-form α via an $(n - k)$-form η satisfies*

$$\alpha|_\eta = (-\alpha)|_\eta = -(\alpha|_{-\eta}). \tag{8.4}$$

In particular, it does not depend on the sign of α.

The key to computations involving the exterior differential is the following theorem.

Theorem 8.2 *Let $\omega|_\eta$ be a twisted differential form with orienting form η. An orienting form τ for the differential $d(\omega|_\eta) = (d\omega)|_\tau$ must satisfy (up to a strictly positive multiple)*

$$\tau \wedge d\omega = \eta \wedge \omega. \tag{8.5}$$

We shall see that there may be a wide choice of solutions τ.

Example: Consider the twisted 1-form $((1 - wz)\, dw : [dz\, dw])$ of the previous example in a region where it is never zero. It may be denoted

$$\omega|_\eta = ((1 - wz)\, dw)|_\eta \tag{8.6}$$

where $\eta = (1 - wz)\, dz$.

The differential of this twisted 1-form is the twisted 2-form

$$d(\omega|_\eta) = (d\omega)|_\tau = (-w\, dz\, dw)|_\tau, \tag{8.7}$$

where $\tau = -w$. It is positive (or negative) in the regions where $w < 0$ (or $w > 0$).

A sketch in the z, w plane is helpful. First consider the region where $zw < 1$. The flux lines have w constant and are oriented in the direction of increasing z. When $w > 0$ they terminate at negative points, while when $w < 0$ they begin at positive points.

There are also the two regions where $zw > 1$. The flux lines have w constant and are oriented in the direction of decreasing z. When $w > 0$ they terminate at negative points, while when $w < 0$ they begin at positive points. ||

Consider a volume form σ. There are two twisted differential n-forms associated with σ. These are

$$\begin{aligned} \sigma|_+ &= (\sigma : [\sigma]), \\ \sigma|_- &= (\sigma : [-\sigma]). \end{aligned} \tag{8.8}$$

Such twisted n-forms are each pictured as a cloud of signed points. In the first case the signs are $+$, while in the second case the signs are $-$. A twisted form $\sigma|_+$ is called a *volume element* or *volume density*.

Another case that is almost as important is that of a twisted $(n-1)$-form

$$\omega|_\eta = (\omega : [\eta \wedge \omega]). \tag{8.9}$$

Here η is an reorienting 1-form. The twisted $(n-1)$ form is pictured as a family of curves (possibly with end points). These curves have (internal) orientations which are determined by restricting η to each curve. Such a curve is called a *transport flux curve*. The idea is that while for an ordinary $(n-1)$-form the orientations are transverse to the curve, for the twisted $(n-1)$-form the orientations are along the curve. Thus the twisted form represents transport along the direction in the curves given by the reorientations.

Example: Suppose that in 2 dimensions there is a 1-form $\alpha = p\, dy - q\, dx$. There is also a twisted 1-form $(\alpha : [dx\, dy])$. For this twisted 2-form there is an

orienting 1-form $\eta = p\,dx + q\,dy$. (This is only one possible choice among many.) The reason this works is that

$$\eta \wedge \alpha = (p\,dx + q\,dy) \wedge (p\,dy - q\,dx) = (p^2 + q^2)\,dx\,dy. \tag{8.10}$$

Since $p^2 + q^2 > 0$ (except at points where the form is zero), the right hand defines the same orientation as $dx\,dy$. So

$$(p\,dy - q\,dx : [dx\,dy]) = (p\,dy - q\,dx)|_{(p\,dx+q\,dy)} \tag{8.11}$$

The second expression is geometrically more meaningful, since the orientation of the curves associated to this twisted form are not given by the 2-form, but instead by the 1-form $p\,dx + q\,dy$. ||

Example: Suppose that in 3 dimensions there is a 2-form $\omega = p\,dy\,dz + q\,dz\,dx + r\,dx\,dy$. There is also a twisted 2-form $(\omega : [dx\,dy\,dz])$. For this twisted 2-form there is an orienting 1-form $\eta = p\,dx + q\,dy + r\,dz$. The calculation gives

$$\eta \wedge \omega = (p\,dx + q\,dy + r\,dz) \wedge (p\,dy\,dz + q\,dz\,dx + r\,dx\,dy) = (p^2 + q^2 + r^2)\,dx\,dy\,dz. \tag{8.12}$$

The right hand defines the same orientation as $dx\,dy\,dz$. So

$$(p\,dy\,dz + q\,dz\,dx + r\,dx\,dy : [dx\,dy\,dz]) = (p\,dy\,dz + q\,dz\,dx + r\,dx\,dy)|_{(p\,dx+q\,dy+r\,dz)} \tag{8.13}$$

||

Figure 8.1 shows a typical oriented curve associated with a 1-form in two dimensions and with a twisted 1-form in two dimensions. For the 1-form the circulatory flux curve has a transverse orientation to one side, while with the twisted 1-form the transport flux curve has an orientation in one direction along the curve. In this picture the orientations at the end points are shown. For the 1-form the (transverse) orientations at the end points are 2-dimensional. For the twisted 1-form the orientations at the end points ares 0-dimensional, that is, just a $\pm$ sign.

Figure 8.2 shows a typical oriented curve associated with a 2-form in three dimensions and with a twisted 2-form in three dimensions. For the 2-form the circulatory flux curve has a transverse orientation around the curve, while for the twisted 2-form the transport flux curve has orientation along the curve.

Example: Say that $X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$ is a vector field. If $s\,dx\,dy\,dz$ is a 3-form, then the interior product

$$X \lrcorner\, s\,dx\,dy\,dz = s(a\,dy\,dz + b\,dz\,dx + c\,dx\,dy) \tag{8.14}$$

is a 2-form. Consider the twisted 3-form $(s\,dx\,dy\,dz : [dx\,dy\,dz])$. The interior product of X with this form is

$$X \lrcorner (s\,dx\,dy\,dz : [dx\,dy\,dz]) = s((a\,dy\,dz + b\,dz\,dx + c\,dx\,dy) : [dx\,dy\,dz]). \tag{8.15}$$

This can also be written in with reorienting forms as

$$X \lrcorner\, s(dx\,dy\,dx)|_+ = s(a\,dy\,dz + b\,dz\,dx + c\,dx\,dy)|_{a\,dx+b\,dy+c\,dz}. \tag{8.16}$$

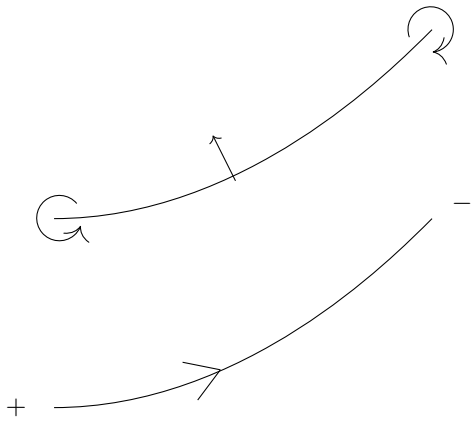

Figure 8.1: typical flux curve in 2d: 1-form; twisted 1-form

||

Example: Figure 8.3 shows the twisted form $(\omega : \mathcal{O}) = (\exp(-x^2)\,dy : dx\,dy) = \exp(-x^2)\,dy|_{dx}$ in two dimensions. It is illustrated by a family of curves with constant values of y, with orientation given by dx. The exterior derivative $d(\omega : \mathcal{O}) = (-2x\exp(-x^2)\,dx\,dy : dx\,dy) = -2x\exp(-x^2)(dx\,dy)|_+$. It is indicated by a family of points, mainly concentrated near the lines $x = \pm 1$. The reorienting 0-form may be taken to be $-x$. For the points near $x = -1$ the orientation is $+$, while for the points near $x = 1$ the orientation is $-$. ||

Once it is understood that the the object in a particular calculation is a twisted form, it may be convenient to do all the calculations with ordinary forms. The following *reference orientation convention* may be used. Choose a particular orientation $[\sigma]$ and declare it to be the reference orientation. In the subsequent discussion a twisted form $(\alpha, [\sigma])$ will be denoted by α. It is to be understood without further discussion that if the reference orientation had been chosen to be $[-\sigma]$, then the twisted form would have been denoted by $-\alpha$.

Example: Say that $X = p\frac{\partial}{\partial x} + q\frac{\partial}{\partial y} + r\frac{\partial}{\partial z}$ is a vector field. Consider the volume element $(dx\,dy\,dz : [dx\,dy\,dz])$. Choose the reference orientation to be $[dx\,dy\,dz]$. Then the volume element is represented by the volume form $dx\,dy\,dz$. The interior product of X with the volume element is represented by

$$X \lrcorner\, dx\,dy\,dz = p\,dy\,dz + q\,dz\,dx + r\,dx\,dy. \tag{8.17}$$

It is understood that the right hand side is a 2-form that is actually being used to represent a twisted 2-form. So it is the same object as the one in the earlier example. ||

This reference orientation convention can be useful, if one picks the reference form and uses it consistently. It helps to provide commentary about which forms are ordinary forms and which forms are twisted forms.

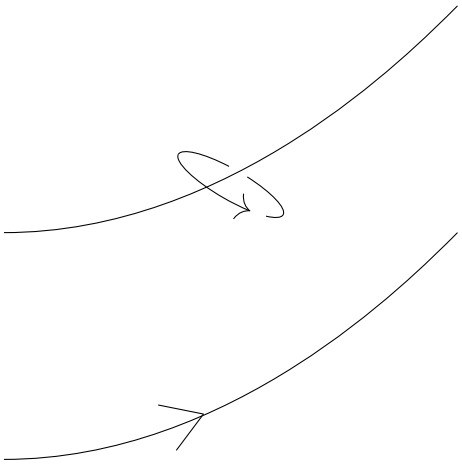

Figure 8.2: typical flux curve in 3d: 2-form; twisted 2-form

Example: Point source. In this example and the following examples take the reference orientation given by $dx\,dy = r\,dr\,d\theta$. Consider the twisted form represented by

$$\omega = \frac{-y\,dx + x\,dy}{x^2 + y^2} = d\theta. \tag{8.18}$$

The contour curves are lines from the origin with orientation away from the origin (with reorienting form dr.) The form is exact in any region for which the angle can be consistently defined. The integral over every closed curve that does not surround the origin is zero. The origin acts as a (positive) point source that makes this form not exact in a global sense. An integral over a closed curve that surrounds the origin can have a non-zero value. The twisted form is locally exact but not globally exact. ||

Example: Extended source everywhere. Consider the twisted form represented by

$$-y\,dx + x\,dy = r^2 d\theta. \tag{8.19}$$

The contour curves are lines whose end point are a scattering of points throughout the plane. These points have positive orientation, and the lines are oriented away from the origin. This form is very far from being exact. One can think of the entire plane as an extended source of non-exactness. ||

Example: Extended source in a disk. Consider the twisted form given by

$$\alpha = \min(r^2, 1)\,d\theta. \tag{8.20}$$

The contour curves are lines whose end points are a scattering of positively oriented points throughout the disk of radius 1 about the origin. This is also an example of an extended source of non-exactness, but the source is confined to this disk. Outside this disk the form is locally exact. ||

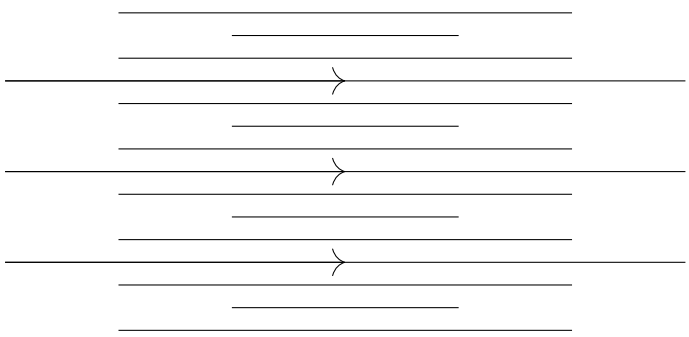

Figure 8.3: $(\omega : \mathcal{O}) = \exp(-x^2)\, dy|_{dx}$; $d(\omega : \mathcal{O}) = -2x \exp(-x^2)(dx\, dy)|_+$

Example: Opposite point sources. Let $a > 0$. Consider the twisted form

$$\omega_- = \frac{-y\, dx + (x+a)\, dy}{(x+a)^2 + y^2} - \frac{-y\, dx + (x-a)\, dy}{(x-a)^2 + y^2}. \tag{8.21}$$

This is locally exact except at two points. The contour curves are complicated. There are two point sources, one at $-a$ and one at a, with positive and negative orientations. There are curves going from the positively oriented source on the left to the negatively oriented source on the right. On the line where $x = 0$, the form ω_- restricts to $2a\, dy/(a^2 + y^2)$, so there it is reoriented by dx. ||

Example: Equal point sources. Let $a > 0$. Consider the twisted form

$$\omega_+ = \frac{-y\, dx + (x+a)\, dy}{(x+a)^2 + y^2} + \frac{-y\, dx + (x-a)\, dy}{(x-a)^2 + y^2}. \tag{8.22}$$

This is locally exact except at two points. There are two point sources, one at $-a$ and one at a, with the same positive orientation. Each curve from either source remains in its own half plane. On the line where $x = 0$, the form ω_+ restricts to $-2y\, dx/(a^2 + y^2)$. Thus this line is one of the curves; it separates the two regions, and it is reoriented by dy. At great distances this form looks like $2\, d\theta$, reoriented by dr. ||

There is an article by Gratus [18] that presents the basics of manifolds and orientations and differential forms almost entirely through pictures. In particular it contrasts differential forms and twisted differential forms. This is a wonderful and much needed approach; it would make an excellent subject for an entire book.

8.1.2 Integration of volume densities

Consider a connected manifold patch P of dimension k. Let $\mathbf{u} = (u_1, \ldots, u_k)$ be a coordinate system for P. A twisted k-form defined on P is of the form

$$f(\mathbf{u})(du_1 \wedge \cdots \wedge du_k)_+ = f(\mathbf{u})(du_1 \wedge \cdots \wedge du_k : [du_1 \wedge \cdots \wedge du_k]) \tag{8.23}$$

The integral over a manifold patch is defined to be

$$\int_P f(\mathbf{u})(du_1 \wedge \cdots \wedge du_k)|_+ = I(f). \tag{8.24}$$

Proposition 8.3 *The integral of a volume element over a manifold patch is independent of the coordinate system.*

Proof: Suppose that $\mathbf{y}$ is another coordinate system. Then $\mathbf{u} = \mathbf{g}(\mathbf{y})$. The volume element has the expression

$$f(\mathbf{u})(du_1 \cdots du_k : [du_1 \ldots du_k]) = f(\mathbf{g}(\mathbf{y})) \det \mathbf{g}'(\mathbf{y})(dy_1 \cdots dy_k : [du_1 \cdots du_k]). \tag{8.25}$$

This gives in turn

$$f(\mathbf{u})(du_1 \cdots du_k : [du_1 \ldots du_k]) = f(\mathbf{g}(\mathbf{y}))|\det \mathbf{g}'(\mathbf{y})|(dy_1 \cdots dy_k : [dy_1 \cdots dy_k]). \tag{8.26}$$

Integrating each side according to the definition gives $I(f) = I((f \circ \mathbf{g})|\det \mathbf{g}'|)$. This is a correct identity, so this proves the consistency of the definitions. □

Example: The integral over the cell Q where $0 \le u \le 1$ and $0 \le y \le 1$ is

$$\int_Q u^2 y^3 (du\, dy)_+ = \int_Q u^2 y^3 (dy\, du)_+ = \frac{1}{12}. \tag{8.27}$$

||

Remark: This result suggests the notation

$$f(u_1, \ldots, u_k)(du_1 \wedge \cdots \wedge du_k)|_+ = f(u_1, \ldots, u_k)|du_1 \wedge \cdots \wedge du_k| \tag{8.28}$$

with absolute value on the right hand side. This gives the correct hint about how to change to a new coordinate system. Thus

$$f(\mathbf{u})|du_1 \wedge \cdots \wedge du_k| = f(\mathbf{g}(\mathbf{y}))|\det \mathbf{g}'(\mathbf{y})|\,|dy_1 \wedge \cdots \wedge dy_k|. \tag{8.29}$$

||

Remark: The two theories of integration are oriented (for top-dimensional differential forms) and unoriented (for top-dimensional twisted differential forms). These are related by

$$\int_{\mathcal{O}} \sigma = \int_+ (\sigma : \mathcal{O}). \tag{8.30}$$

This notation exhibits explicitly the orientation of the region of integration on the left hand side. On the right hand the transversal orientation of the region of integration is written explicitly as $+$. Say that $k(\mathbf{u})$ does not change sign. In coordinates this identity may be written

$$\int_{[k(\mathbf{u})\, du_1 \cdots du_k]} f(\mathbf{u})\, du_1 \cdots du_k = \int_+ (f(\mathbf{u})\, du_1 \cdots du_k : [k(\mathbf{u})\, du_1 \cdots du_k]). \tag{8.31}$$

The right hand may also be expressed as

$$\int_+ (f(\mathbf{u})\, du_1 \cdots du_k : [k(\mathbf{u})\, du_1 \cdots du_k]) = \operatorname{sign}(k(\mathbf{u})) \int_+ f(\mathbf{u})(du_1 \cdots du_k)|_+. \tag{8.32}$$

From this it is evident that the transfer from one theory to the other is almost trivial. Conceptually they are quite different. There is no notion of positivity or negativity for differential forms. In the unoriented theory the integrand may be thought of as possibly positive or negative, according to the sign of $f(\mathbf{u})$, without any relation to an orientation. In the following it will appear that the two theories go almost in parallel. Exception: There is an extra difficulty in the twisted theory when considering pullbacks. ||

8.2 The Gauss theorem for twisted $(n-1)$-forms

The twisted version of the Gauss theorem is almost the same as the original Gauss theorem, but the intuitive interpretation is very different.

Theorem 8.4 (twisted Gauss) *Suppose that M is n-dimensional, and W is an n-dimensional region in M with the $+$ transverse orientation. The $(n-1)$-dimensional boundary regular surface ∂W then has transverse orientation given by an outward transverse orientation form $+\epsilon = \epsilon$. Consider a coordinate system $x_1, \ldots, x_n$ and let $\sigma = dx_1 \cdots dx_n$. Let*

$$(\omega : [\sigma]) = \left(\sum_j (-1)^{j-1} p_j dx_1 \cdots dx_{j-1}\, dx_{j+1} \cdots dx_n) \right) \Bigg|_{\sum_j p_j\, dx_j} . \tag{8.33}$$

Then

$$d(\omega : [\sigma]) = \left(\sum_j \frac{\partial p_j}{\partial x_j} \right) (dx_1 \cdots dx_n)|_+. \tag{8.34}$$

The Gauss theorem says that

$$\int_W d(\omega : [\sigma]) = \int_{\partial W} (\omega : [\sigma]). \tag{8.35}$$

The outward transverse orienting form ϵ is fixed. The n-forms $\pm\sigma$ on M determine $(n-1)$-forms $\pm\beta$ by $\epsilon \wedge \beta = \sigma$. Parameterize ∂W by a regular parametric surface mapping $\chi : P \to M$ given as $\mathbf{x} \leftarrow \mathbf{f}(\mathbf{u})$. The integral over the boundary is defined by

$$\int_{\partial W} (\omega : [\sigma]) = \int_P (\chi^* \omega : [\chi^* \beta]). \tag{8.36}$$

Define *penetration coefficients* ν_j by

$$\nu_j\, du_1 \cdots du_{n-1} = \chi^* (-1)^{j-1} dx_1 \cdots dx_{j-1}\, dx_{j+1} \cdots dx_n. \tag{8.37}$$

Define the *penetration form* along χ to be by

$$\nu = \sum_j \nu_j dx_j \tag{8.38}$$

Here it is understood that the dx_j are restricted differentials; they act on tangent vectors to M, but only at points in the range of χ. It was shown in the section on the Gauss theorem that the penetration form $\nu = \lambda\epsilon_{[\chi]}$, where λ is a non-zero scalar field. It was also shown that the $(n-1)$-form $\chi^*\beta$ defines the same orientation on P as $\lambda\, du_1 \cdots du_{n-1}$.

Theorem 8.5 *Suppose that the penetration form ν given by the parameterization is a positive multiple of the outward transverse orientation form ϵ of ∂W. Then the twisted Gauss theorem says that*

$$\int_W \left(\sum_j \frac{\partial p_j}{\partial x_j} \right) (dx_1 \cdots dx_n)|_+ = \int_P \sum_j (p_j \circ \chi)\nu_j (du_1 \cdots du_{n-1})|_+. \tag{8.39}$$

The twisted Gauss theorem works on with an arbitrary coordinate system $x_1, \ldots, x_n$ and requires no geometrical structure. At first it looks very close to the Gauss theorem. There are two crucial differences:

- It makes sense to say that a twisted-form like $(d\omega : [\sigma])$ is positive or negative. Similarly, it makes sense to say that the corresponding flux through the boundary is outward or inward.
- The pictures are drastically different. The sources (or sinks) in W have $\pm$ signs. The flux curves starting (or ending) at these sources (or sinks) are transport flux curves, with an (internal) orientation. The interpretation is that something is flowing along these curves in a particular direction.

This contrasts with the situation for differential forms, where there is no definition of positive or negative (except for scalar fields), and where the flux curves are curves of circulation, with transverse orientation indicating the sense of circulation.

Figure 8.4 gives pictures of the theorems in dimension 2. In this case the Gauss theorem is usually called Green's theorem. The upper picture shows the usual Green's theorem for a differential 1-form in the plane. The sources in the interior are 2-dimensionally oriented, and the circulatory flux curves that come from these sources have a transverse orientation. The integral around the boundary measures how many curves are crossed, keeping track of whether the crossings are up or down. The lower picture shows the Green's theorem for a twisted differential 1-form in the plane. The sources in the interior are are 0-dimensionally oriented (that is, signed), and the transport flux curves that come from these sources have an internal orientation. The integral on the boundary measures how many curves cross the boundary, keeping track of whether they are from inside to outside or the other way.

The special case when the $(n-1)$ form comes from a vector field is illuminating and useful.

Corollary 8.6 (twisted Gauss for vector field) *Suppose that M is n-dimensional, and W is an n-dimensional region in M with the + transverse orientation. The $(n-1)$-dimensional boundary regular surface ∂W then has transverse orientation given by an outward transverse orientation. Let X be a vector field and let $(\gamma : \mathcal{O})$ be a twisted n-form. Then the Gauss theorem may be stated in the form*

$$\int_W d(X \lrcorner (\gamma : \mathcal{O})) = \int_{\partial W} X \lrcorner (\gamma : \mathcal{O}). \tag{8.40}$$

The intuition is that $(\gamma : \mathcal{O})$ is a twisted n=form representing mass (or possibly charge). The vector field X has components that represent velocity. Then $X \lrcorner (\gamma : \mathcal{O}) = (X \lrcorner \gamma : \mathcal{O})$ represents transport flux. The twisted n-form integrand $d(X \lrcorner (\gamma : \mathcal{O})) = (d(X \lrcorner \gamma) : \mathcal{O})$ is the rate of production of of mass (or charge). The net production within W is equal to the next flux through the boundary ∂W.

The vector field X is represented geometrically by parametrized curves that represent the flow. The twisted $(n-1)$-form $(X \lrcorner \gamma : \mathcal{O})$ is represented by oriented curves that originated or terminate at points where its differential is non-zero. The orientations of the curves representing X and $(X \lrcorner \gamma : \mathcal{O})$ are the same or opposite, depending on the sign of $(\gamma : \mathcal{O})$.

With the reference form convention the twisted Gauss theorem for vector fields may be written

$$\int_W d(X \lrcorner \gamma) = \int_{\partial W} X \lrcorner \gamma. \tag{8.41}$$

The n-form $d(X \lrcorner \gamma)$ is called the *Lie derivative* of the n-form γ along the vector field X. This is the natural formula for describing how a vector field pushes around a n-form.

8.2.1 Volume

Up to this point the emphasis has been on quantities that are defined on an arbitrary manifold patch and are independent of the coordinate system. These include vector fields and differential k-forms, and also twisted vector fields and twisted differential k-forms.

For some purposes this is not enough structure. This chapter presents a setting where the structure is a given volume element. This will be called $\mathrm{vol}|_+$. In coordinates it takes the form

$$\mathrm{vol}|_+ = (w\, du_1 \wedge \cdots \wedge du_n)|_+. \tag{8.42}$$

where the weight $w > 0$ is strictly positive at each point.

A volume element may be determined by specifying a particular volume form

$$\mathrm{vol} = \pm w\, du_1 \wedge \cdots \wedge du_n, \tag{8.43}$$

Either sign will do, since the result is the same:

$$\mathrm{vol}|_+ = (\mathrm{vol} : [\mathrm{vol}]) = (w\, du_1 \wedge \cdots \wedge du_n : du_1 \wedge \cdots \wedge du_n). \tag{8.44}$$

Example: Consider the case of three dimensional Euclidean space. A volume form is

$$\text{vol} = dx\,dy\,dz = r^2\sin(\theta)\,dr\,d\theta\,d\phi. \tag{8.45}$$

The convention here for spherical polar coordinates is that θ is the co-latitude measured from the north pole, while ϕ is the longitude. This could also be written

$$\text{vol} = -\,dz\,dy\,dx = -r^2\sin(\theta)\,d\phi\,d\theta\,dr. \tag{8.46}$$

For this volume form the sign can change when the differentials are taken in a different order. The corresponding volume element is

$$\text{vol}|_+ = (dx\,dy\,dz)|_+ = r^2\sin(\theta)(dr\,d\theta\,d\phi)|_+. \tag{8.47}$$

This does not depend on the order of the differentials, and the weight factor is positive. The weight factor for Cartesian coordinates is 1, while the weight factor for spherical polar coordinates is $r^2\sin(\theta)$. ||

Example: In two dimensions it is perhaps more natural to call this area. So in Cartesian coordinates area $= dx\,dy$, while in polar coordinates area $= r\,dr\,d\theta$. There is a corresponding volume element area$|_+$. ||

The volume element produces a correspondence between scalar fields and twisted n-forms that sends s to $s\,\text{vol}|_+$. This operation and its inverse are usually denoted by $*$. Thus

$$\begin{aligned} *s &= s\,\text{vol}|_+ \\ *(s\,\text{vol}|_+) &= s. \end{aligned} \tag{8.48}$$

The volume form also produces a correspondence between vector fields and twisted $(n-1)$-forms. A convenient notation for this correspondence is c, for *contraction*. This is given by

$$\mathsf{c}X = X \lrcorner \text{vol}|_+ = X \lrcorner (\text{vol} : \text{vol}) = (X \lrcorner \text{vol} : \text{vol}) \tag{8.49}$$

The form $\mathsf{c}X$ is the *transport flux*. It is a twisted $(n-1)$-form used to describe the extent to which the vector field X is penetrating a given $(n-1)$-dimensional surface. The inverse operation c^{-1} sends a twisted $(n-1)$-form to the corresponding vector field. The pairing of vector fields and transport flux $(n-1)$-forms is the *vector field-transport flux correspondence*.

In an arbitrary coordinate system with

$$X = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} \tag{8.50}$$

and

$$\text{vol} = w\,dx_1 \cdots dx_n \tag{8.51}$$

the formula for $X \lrcorner \text{vol}$ is

$$X \lrcorner \text{vol} = \left(\sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}\right) \lrcorner\ \text{vol} = \sum_{i=1}^{n} (-1)^{i-1} a_i w\,dx_1 \cdots\, dx_{i-1}\,dx_{i+1} \cdots\, dx_n. \tag{8.52}$$

Then

$$\mathsf{c}X = X \lrcorner \operatorname{vol}|_+ = (X \lrcorner \operatorname{vol})|_{\mathsf{g}X}, \tag{8.53}$$

where $\mathsf{g}X$ is a differential 1-form that satisfies $\langle \mathsf{g}X \mid X\rangle > 0$. In that case

$$\mathsf{g}X \wedge (X \lrcorner \operatorname{vol}) = \langle \mathsf{g}X \mid X\rangle \operatorname{vol} \tag{8.54}$$

defines the same orientation as vol. One way to construct such a form is to choose strictly positive definite matrices g_{ij} at each point and take $\mathsf{g}X = \sum_i \sum_j g_{ij} a_i \, dx_j$. The identity matrix is will work for this purpose, but later on we shall see that other choices are natural.

It is not difficult to compute transport flux. Consider two dimensions with arbitrary coordinate system x, y. Aa vector field is of the form

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}. \tag{8.55}$$

The area form is

$$\text{area} = w \, dx \, dy. \tag{8.56}$$

The corresponding flux is

$$\mathsf{c}X = X \lrcorner \text{area}|_+ = w(a \, dy - b \, dx)|_{a \, dx + b \, dy}. \tag{8.57}$$

In three dimensions a vector field is of the form

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}. \tag{8.58}$$

The volume form is

$$\text{vol} = w \, dx \, dy \, dz. \tag{8.59}$$

The corresponding flux is

$$\mathsf{c}X = X \lrcorner \operatorname{vol}|_+ = w(a \, dy \, dz + b \, dz \, dx + c \, dx \, dy)|_{a \, dx + b \, dy + c \, dz} \tag{8.60}$$

There is a geometrical picture of the vector field-transport flux correspondence. A vector field is considered as a field of arrows, attached to each point. A transport flux is pictured as *transport flux curves*, which may have signed end points. When the arrows are long, the corresponding transport flux curves are close together. These curves are along the integral curves of the vector field. The cloud of signed end points is dense in regions where the divergence of the vector field is far from zero.

Example: Consider two dimensions with area form area $= dx \, dy$. The vector field $X = x\partial/\partial x$ corresponds to the twisted 1-form that is $(X \lrcorner \text{area} : [\text{area}]) = (x \, dy : [dx \, dy])$. This form is oriented by $x \, dx$, since $x \, dx \, x \, dy = x^2 \, dx \, dy$ defines the same orientation as $dx \, dy$.

For the vector field the flow is $(x, y) \leftarrow (e^t x, y)$. The integral curves are along lines of constant y value. The arrows of the vector field get longer as x moves away from where x is zero.

For the twisted differential form the differential is $(dx\,dy : [dx\,dy])$ which is oriented by $+$. There is a constant density of end points, all with positive transverse orientation. The transport flux lines start at these end points and move with constant y away from where x is zero. The orientation is given by $x\,dx$, so it is in the direction of increasing y where $x > 0$ and is in the direction of decreasing y when $x < 0$. The density of the transport flux lines increases as x moves away from zero. ||

Consider a volume n-form vol. If X is a vector field, its *divergence* is a scalar field $\operatorname{div} X$ defined by

$$\operatorname{div} X \operatorname{vol} = d(X \lrcorner \operatorname{vol}). \tag{8.61}$$

In the definition of divergence the result is the same if vol is replaced by $-$vol. It follows that the divergence depends only on the volume element twisted n-form $\operatorname{vol}|_+$. So the definition could also be written in terms of the transport flux $X \lrcorner \operatorname{vol}|_+$ by

$$\operatorname{div} X \operatorname{vol}_+ = d(X \lrcorner \operatorname{vol}|_+) \tag{8.62}$$

An elegant way of summarizing this definition is

$$\operatorname{div} X = *d\mathsf{c}X. \tag{8.63}$$

Theorem 8.7 (Divergence in arbitrary coordinates) *Suppose that the $w > 0$ is a scalar field, and the volume element is $(w\,dx_1 \cdots dx_n)|_+$. Consider a vector field*

$$X = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}. \tag{8.64}$$

Then the divergence of this vector field is the scalar field

$$\operatorname{div}\, X = \sum_{j=1}^{n} \frac{1}{w} \frac{\partial w a_j}{\partial x_j}. \tag{8.65}$$

The reference orientation convention is very convenient in the context of a volume element. If the volume element is $\operatorname{vol}|_+$, then there are two choices of volume form that may be used to define the positive volume element. The two choices differ only by sign. The *reference volume form convention* is to choose a particular volume form vol and use the corresponding orientation [vol]. Thus a twisted differential form is

$$(\alpha : [\operatorname{vol}]) = (-\alpha : [-\operatorname{vol}]) \tag{8.66}$$

The convention is to say at the outset of a particular discussion that the reference volume form is vol. When a twisted form like the one above is encountered it is specified by the form α.

As an example, consider the definition of divergence. Say that some choice of volume form vol is used to define the volume element. It is declared as the reference form. Then the twisted $(n-1)$-form defined by X may be written $X \lrcorner \operatorname{vol}$. We than have

$$\operatorname{div}(X) \operatorname{vol} = d(X \lrcorner \operatorname{vol}) \tag{8.67}$$

Both sides of this equation are twisted n-forms. Everything looks simpler. But one needs to remember that the twisted n-form vol on the left hand side has reorienting form given by the zero-dimensional $+$. Similarly, the twisted $(n-1)$-form on the right has a 1-dimensional reorienting form η such that $\langle \eta \mid X \rangle > 0$.
Example: Calculations involving the divergence work in arbitrary coordinate systems. Consider parabolic coordinates given by $x = (1/2)(u^2 - v^2)$ and $y = uv$. Then $dx = u\,du - v\,dv$ and $dy = v\,du + u\,dv$. The area form is then

$$\text{area} = dx\,dy = (u\,du - v\,dv)(v\,du + u\,dv) = (u^2 + v^2)\,du\,dv. \tag{8.68}$$

The divergence is

$$\operatorname{div}\left(a\frac{\partial}{\partial u} + b\frac{\partial}{\partial b}\right) = \frac{1}{u^2+v^2}\left(\frac{\partial (u^2+v^2)a}{\partial u} + \frac{\partial (u^2+v^2)b}{\partial v}\right). \tag{8.69}$$

||
Example: Here is an example from mechanics, in the special case of two dimensional *phase space*. There is a coordinate system q, p on this space such that the volume form $dq\,dp$ plays a special role. One can think of q as position and p and momentum. Consider a scalar field H, which is thought of as energy. Its differential

$$dH = \frac{\partial H}{\partial p}\,dp + \frac{\partial H}{\partial q}\,dq \tag{8.70}$$

is a 1-form. The volume form has an orientation, so this 1-form corresponds to a vector field X with $X \lrcorner\, dq\,dp = dH$. This vector field is

$$X = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}. \tag{8.71}$$

Then

$$\langle dH \mid X \rangle = \langle X \lrcorner\, dq\,dp \mid X \rangle = \langle\, dq\,dp \mid X, X \rangle = 0. \tag{8.72}$$

So H does not change along the flow of the vector field. Furthermore,

$$\operatorname{div} X\, dq\,dp = d(X \lrcorner\, dq\,dp) = d(dH) = 0. \tag{8.73}$$

The divergence is zero; the flow in phase space preserves the volume form. ||

8.2.2 The divergence theorem

Consider an n-dimensional region W with transverse orientation $+$. Then its boundary ∂W has transverse orientation given by $\epsilon = dr$, where r increases from the interior of W to the exterior of W.

The general divergence theorem then takes the following form.

Theorem 8.8 (Divergence theorem for vector fields) *Consider an n-dimensional region W with transverse orientation $+$. Suppose there is a given volume element $\text{vol}|_+$. For every vector field X we have $X \lrcorner \text{vol}|_+ = (X \lrcorner \text{vol})|_{\mathrm{g}X}$, where $\mathrm{g}X$ is a reorienting 1-form such that $\langle \mathrm{g}X \mid X \rangle > 0$. Then*

$$\int_W \operatorname{div}\, X\, \text{vol}|_+ = \int_{\partial W} (X \lrcorner \text{vol})|_{\mathrm{g}X} \tag{8.74}$$

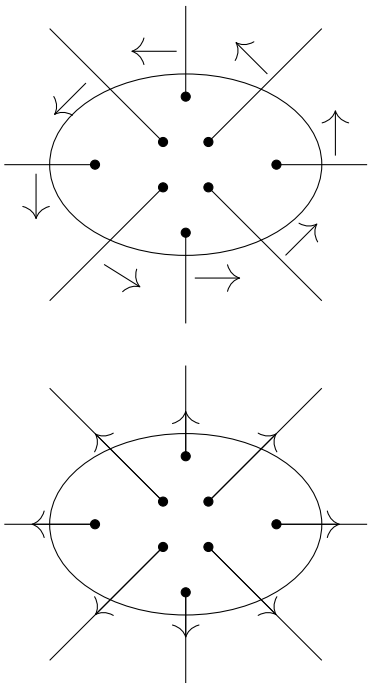

Figure 8.4: Green's theorem (2d): Form with transverse orientation; twisted form with internal orientation

The right hand integrand in the divergence theorem is called the *tranport flux* of the vector field through the surface. The left hand suggests that the source of the flux is the divergence of the vector field in the interior. In short,

$$\text{total source from } \operatorname{div} X \text{ in region } = \text{ flux of vectors } X \text{ through boundary.} \tag{8.75}$$

The theorem may also be stated with the reference volume form convention.

Theorem 8.9 (Divergence theorem for vector fields) *Consider an n-dimensional region W with transverse orientation $+$. Suppose there is a given volume element* $\mathrm{vol}|_+$. *Make the convention that the n-form* vol *is the reference volume form. Then for every vector field X.*

$$\int_W \operatorname{div} X \,\mathrm{vol} = \int_{\partial W} X \lrcorner \,\mathrm{vol}. \tag{8.76}$$

There is an important corollary when X is replaced by sX. Since $sX \lrcorner \mathrm{vol} = X \lrcorner (s\mathrm{vol})$, the result is:

Corollary 8.10 (Divergence theorem with vector field and scalar field) *Consider an n-dimensional region W with transverse orientation $+$. Suppose there is a given volume element* $\mathrm{vol}|_-+$. *Make the convention that the n-form* vol *is the reference volume form. Then for every vector field X.*

$$\int_W \operatorname{div}(sX)\,\mathrm{vol} = \int_{\partial W} X \lrcorner (s\,\mathrm{vol}). \tag{8.77}$$

The idea of this corollary is that s represents mass density, while X is a vector field that represents velocity. The quantity $\operatorname{div}(sX)$ describes a density of mass production. We shall see below that $\operatorname{div}(sX) = X \lrcorner ds + \operatorname{div}(X)s$. This useful formula separates the mass production into two parts.

In the following discussion the reference volume form is $\mathrm{vol} = w\,dx_1 \cdots dx_n$. Suppose that

$$X = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}. \tag{8.78}$$

To compute the integral over the boundary we have to pull back $X \lrcorner \mathrm{vol}$ to the parameter space P. Say that $u_1, \ldots, u_{n-1}$ are the parameters. Define ν_j on the parameter space by

$$\phi^*((-1)^{j-1} dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_n) = \nu_j \, du_1 \cdots du_{n-1}. \tag{8.79}$$

Then it pulls back to

$$\phi^*(X \lrcorner \mathrm{vol}) = \sum_{j=1}^{n} (a_j \circ \phi)\, \nu_j (w \circ \phi)\, du_1 \cdots du_{n-1}. \tag{8.80}$$

The explicit form of the theorem is then

$$\int_W \sum_{j=1}^{n} \frac{1}{w} \frac{\partial w a_j}{\partial x_j}\, w\, dx_1 \cdots dx_n = \int_P \sum_{j=1}^{n} (a_j \circ \phi)\, \nu_j\, (w \circ \phi)\, du_1 \cdots du_{n-1}. \tag{8.81}$$

There is no need to normalize the components ν_j, and there is also no need to compute the surface area of ∂W.

The most familiar versions of the divergence theorem are in two and three dimensions. In two dimensions we can make the reference form convention that area $= w\,dx\,dy$. The divergence theorem says that

$$\int_R \frac{1}{w}\left(\frac{\partial wa}{\partial x} + \frac{\partial wb}{\partial y}\right) \text{area} = \int_{\partial R} w\,(a\,dy - b\,dx). \tag{8.82}$$

The coefficients in the vector field are expressed with respect to a coordinate basis. We shall see in the next part of this book that this is not the only possible choice.

A marvelous application of the divergence theorem in two dimensions is the formula

$$\int_R dx\,dy = \frac{1}{2}\int_{\partial R} x\,dy - y\,dx. \tag{8.83}$$

This says that one can determine the area by walking around the boundary. It is perhaps less mysterious when one realizes that $x\,dy - y\,dx = r^2\,d\phi$.

In three dimensions the reference form convention can be vol $= w\,dx\,dy\,dz$. The divergence theorem says that

$$\int_W \frac{1}{w}\left(\frac{\partial wa}{\partial x} + \frac{\partial wb}{\partial y} + \frac{\partial wc}{\partial z}\right) \text{vol} = \int_{\partial W} w\,(a\,dy\,dz + b\,dz\,dx + c\,dx\,dy). \tag{8.84}$$

Here the volume form is $w\,dx\,dy\,dz$. Again the coefficients a, b, c of the vector field are expressed in terms of the coordinate basis vectors $\partial/\partial x, \partial/\partial y, \partial/\partial z$. This is not the the only possible kind of basis for a vector field, so in some treatments the formulas will appear differently. They will be ultimately equivalent in terms of their geometrical meaning.

The divergence theorem says that the integral of the divergence of a vector field X over W with respect to the volume element is the integral of the transport flux of the vector field across the bounding surface ∂W. It may help to think of the example where X is the velocity of a fluid, and vol represents mass of the fluid. The velocity X has dimension T^{-1}, while its coefficients have dimension LT^{-1}. The vol has dimension M, with coefficient having dimension ML^{-n}. Both sides of the divergence theorem have dimension MT^{-1}, mass per time. The rate of mass production in the interior is equal to the rate of transport through the boundary.

A particularly famous application of the divergence theorem in physics is when the vector field represents the electric flux density,, and the divergence represents the density of charge. The amount of charge in the region determines the electric flux through the boundary.

An important identity for a differential $(n-1)$-form ω and a scalar field s is

$$d(s\omega) = ds \wedge \omega + s\,d\omega. \tag{8.85}$$

This gives an *integration by parts* formula

$$\int_W ds \wedge \omega + \int_W s\,d\omega = \int_{\partial W} s\omega. \tag{8.86}$$

Here W has an orientation and ∂W is its oriented boundary.

Now take vol as the reference volume form. Apply the integration by parts identity to $\omega = X \lrcorner \text{vol}$ and use $ds \wedge (X \lrcorner \text{vol}) = \langle ds \mid X \rangle \text{vol}$. This gives the divergence identity

$$\text{div}(s\,X) = \langle ds \mid X \rangle + s\ \text{div}\ X. \tag{8.87}$$

From this we get another important integration by parts identity. This is important enough to state as a theorem:

Theorem 8.11 *Consider a volume element determined by a reference volume form* vol. *Then for every scalar field s and vector field X we have*

$$\int_W \langle ds \mid X \rangle \,\text{vol} + \int_W s\ \text{div}\ X\,\text{vol} = \int_{\partial W} s\,X \lrcorner \text{vol}. \tag{8.88}$$

Here W has positive transverse orientation, and ∂W has outward transverse orientation.

Remark: In the discussion of the divergence theorem here the hypothesis has always been that the vector field X is smooth enough, for instance C^1. Then $\text{div}(X)$ is continuous, and there is no problem defining the integrals.

In dimension one the divergence theorem is just a restatement of the fundamental theorem of calculus. However the fundamental theorem of calculus holds true in considerably more general circumstances. Suppose that f is a (Lebesgue) integrable function, and F is the indefinite integal of f, that is,

$$F(x) = \int_a^x f(u)\,du, \tag{8.89}$$

Then the fundameltal theorem of calculus holds in the form

$$\int_a^b f(u)\,du = F(b) - F(a). \tag{8.90}$$

In this case F is continuous (in fact, absolutely continuous), but there may be points at which it does not have a derivative. (The simplest example is when f is a step function.)

This raises the question of more general formulations of the divergence theorem in higher dimensions. This is a rather technical matter. The interested reader will find a survey of results in [32]. Another version with some interesting history is in [8]. ||

8.2.3 Vector field and twisted $(n-1)$-form pictures

In what follows it is assumed that there is a given volume element. The volume form vol is the reference form. Once the volume element is specified, the vector field and the twisted $(n-1)$-form pictures are somehow equivalent. The question is how best to picture these objects. A common situation is that in at least part

of the region the vector field X has divergence zero; equivalently, the twisted $(n-1)$-form $X \lrcorner \omega$ has exterior derivative zero.

In many ways the transport flux curve picture associated with the twisted $(n-1)$-form ω is more transparent. The reason is that one can see whether the form is exact: there are no end points. Furthermore, an integral over an $(n-1)$ surface is explained by the (signed) number of end points enclosed by the surface, that is, by a source twisted n-form $d\omega$. The relevant theorem is the Gauss theorem for twisted forms:

$$\int_W \omega = \int_{\partial W} d\omega. \tag{8.91}$$

Looking at the arrow picture of the vector field, there is no evident clue about whether the divergence is zero in some region. Furthermore, the way that a source produces a field is more obscure. On the other hand, if one thinks of the vector field as force or as velocity, the length of the vectors gives information that is more direct than looking at the density of the flux. The relevant theorem is the divergence theorem

$$\int_{\partial W} X \lrcorner \operatorname{vol} = \int_W \operatorname{div}\, X \operatorname{vol}. \tag{8.92}$$

The nice feature is that div X is a scalar field. But the device of comparing everything to the volume makes it harder to arrive at a simple picture.

Figures 8.5 and 8.6 compare the two pictures in an example in three dimensions. The pictures show two dimensional cross-sections of a rotationally symmetric field corresponding to a uniform source in the unit ball.

Figure 8.5 shows the twisted $(n-1)$-form

$$\omega = \frac{x\,dy\,dz + y\,dz\,dx + z\,dx\,dy}{r^3} = \sin(\theta)\,d\theta\,d\phi \tag{8.93}$$

for $r \geq 1$, that is, outside the source. It is clear from the second expression that $d\omega = 0$. For $r \leq 1$ the form is $r^3\omega$. The source inside the ball satisfies $d(r^3\omega) = 3r^2\,\omega = 3\,dx\,dy\,dz$. The transport flux line picture clearly shows that the source is localized inside the ball and is uniform.

Figure 8.6 gives the equivalent description by a vector field. This is

$$X = \frac{1}{r^3}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}. \tag{8.94}$$

for $r \geq 1$, that is, outside the source. This is the famous $1/r^2$ law. It is clear from the second expression that $\operatorname{div} X = 0$. For $r \leq 1$ the vector field is $r^3 X$. The source inside the ball satisfies $\operatorname{div}(r^3 X) = 3$. The vector field picture is not nearly as transparent, but the calculation does show that it is an equivalent description.

Summary: The relation between these pictures is $X \lrcorner \operatorname{vol} = \omega$. The vector field X describes how fast something will move. The 2-form ω describes penetration of a surface. The correspondence between source and field is clearer in the form description.

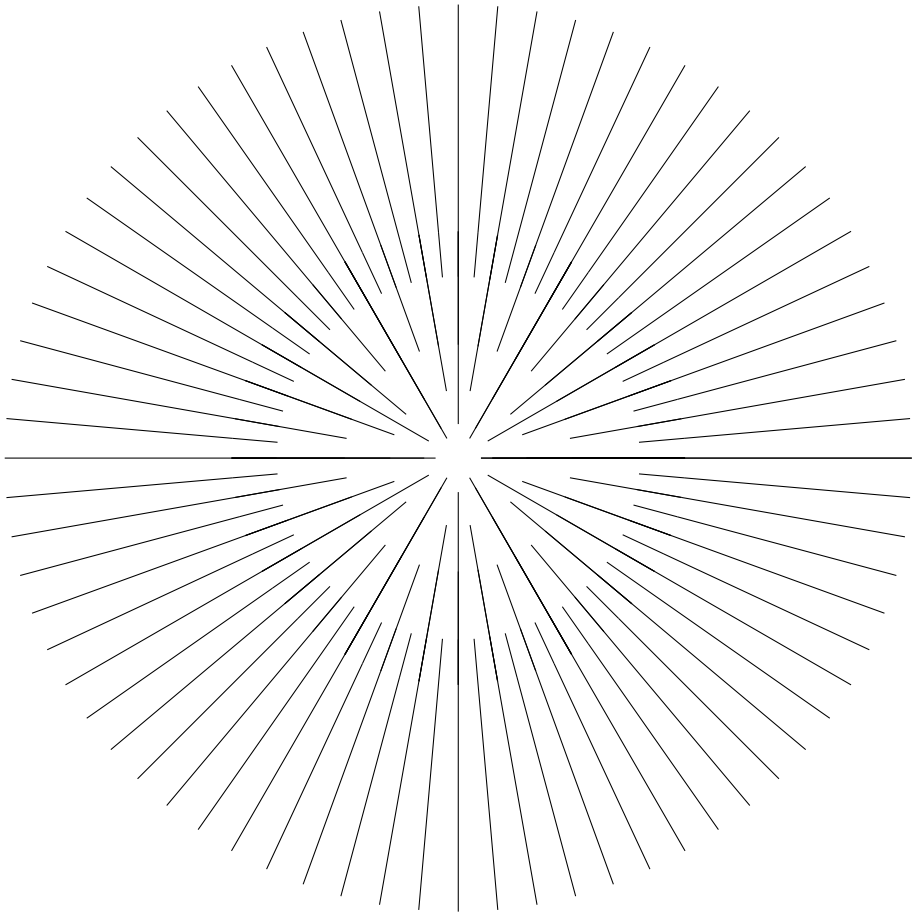

Figure 8.5: The 2-form $\omega = X \lrcorner \text{vol}$ with extended source (3 dimensional)

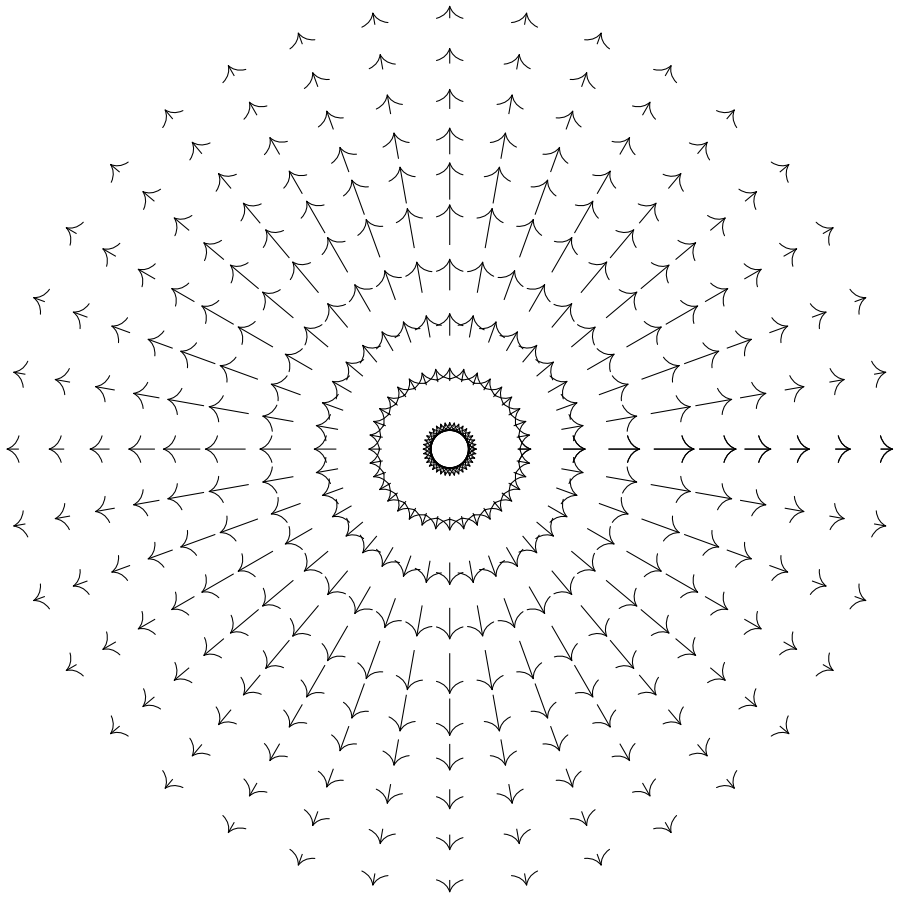

Figure 8.6: The vector field X with extended source (3 dimensional)

8.3 Conservation laws*

8.3.1 Transport of a volume element by a vector field*

A vector field X describes a time evolution given by a flow Φ_t. This satisfies an ordinary differential equation: For each scalar field u (independent of time)

$$\frac{d}{dt}(u \circ \Phi_t) = (X \lrcorner du) \circ \Phi_t. \tag{8.95}$$

If

$$X = \sum_j a_j \frac{\partial}{\partial y_j} \tag{8.96}$$

is the vector field in coordinates, then this says in particular that

$$\frac{d}{dt}(y_j \circ \Phi_t) = a_j \circ \Phi_t. \tag{8.97}$$

If we define $u_t = \Phi_t^* u = u \circ \Phi_t$, then it was shown in an earlier chapter that this time-dependent scalar field satisfies a backward partial differential equation

$$\frac{\partial}{\partial t} u_t = X \lrcorner du_t. \tag{8.98}$$

Similarly, if $w_t = \Phi_{-t}^* w = w \circ \Phi_{-t}$, then this satisfies a forward partial differential equation

$$\frac{\partial}{\partial t} w_t + X \lrcorner dw_t = 0. \tag{8.99}$$

This section describes corresponding results for the time evolution of a volume element. The time evolution is defined by the vector field X. The convention is that a reference orientation is chosen. A volume element is thus described by an n-form γ. The following result identifies the relevant form of the differential operator for this case.

Theorem 8.12 (Lie derivative) *Suppose that X is a vector field with flow Φ_t. Consider an n-form γ. Then*

$$\frac{\partial \Phi_t^* \gamma}{\partial t}(t \leftarrow 0) = d(X \lrcorner \gamma). \tag{8.100}$$

This is the natural way that a vector field measures the change of a n-form. It takes the same form in every coordinate system. It is called the *Lie derivative* of the n-form γ with respect to the vector field X.

Proof: A vector field X acts on a n-form γ by the interior product $X \lrcorner \gamma$, which is an $(n-1)$-form. This in turn defines an n-form $d(X \lrcorner \gamma)$. In an arbitrary coordinate system with X as given above and

$$\gamma = r\, dy_1 \cdots dy_n, \tag{8.101}$$

then

$$d(X \lrcorner \gamma) = \sum_j \frac{\partial}{\partial y_j}(r a_j)\, dy_1 \cdots dy_n. \tag{8.102}$$

The evolution acts according to

$$\Phi_t^* \gamma = (r \circ \Phi_t)\, d(y_1 \circ \Phi_t) \cdots d(y_n \circ \Phi_t). \tag{8.103}$$

Differentiate with respect to t and use the product rule. Then set t to zero. The time derivative applied to $y_j \circ \Phi_t$ is $a_j \circ \Phi_t$. The time derivative applied to $d(y_j \circ \Phi_t)$ is $\sum_k (\partial(a_j \circ \Phi_t)/\partial y_k)\, dy_k$. The result of the calculation is

$$\frac{\partial}{\partial t}\Phi_t^* \gamma(t \leftarrow 0) = (X \lrcorner dr)\, dy_1 \cdots dy_n + \sum_j r(\partial a_j/\partial y_j)\, dy_1 \cdots dy_n. \tag{8.104}$$

This is equivalent to the statement of the theorem. □

Sometimes it is helpful to write out such a proof in a language that makes the functional dependence more explicit. The reader may want to write $a_j = g_j(\mathbf{y})$ and $r = h(\mathbf{y})$ and $y_j \circ \Phi_t = f_j(\mathbf{y}, t)$ and do the calculations in this version.

Theorem 8.13 (Time evolution of an n-form) *Consider n-form γ and vector field X generating flow Φ_t. Let $\gamma_t = \Phi_{-t}^* \gamma$. This satisfies the forward partial differential equation*

$$\frac{\partial \gamma_t}{\partial t} + d(X \lrcorner \gamma_t) = 0. \tag{8.105}$$

Proof: Compute

$$\frac{\partial \Phi_{-t}^* \gamma}{\partial t} = \frac{\partial \Phi_{-t-t'}^* \gamma}{\partial t'}(t' \leftarrow 0). \tag{8.106}$$

Now use $\Phi_{-t-t'} = \Phi_{-t}\Phi_{-t'}$. From the formula for the Lie derivative this is

$$\frac{\partial \Phi_{-t'}^* \Phi_{-t}^* \gamma}{\partial t'}(t' \leftarrow 0) = d(X \lrcorner d\Phi_{-t}^* \gamma). \tag{8.107}$$

□

In the most common interpretation the n-form γ represents mass at time zero, while $\Phi_{-t}^* \gamma$ is the mass at time t in the future. This is a forward equation. It implies the *conservation law*

$$\frac{d}{dt}\int_W \gamma_t + \int_{\partial W} (X \lrcorner \gamma_t) = 0. \tag{8.108}$$

This is valid if there is no production or destruction of mass. It says that the increase of mass in the region plus the outward transport flux through its boundary must sum to zero.

In a typical application γ_t is a (twisted) n-form for each t, representing how mass in kilograms is distributed at any time. It is often written as a scalar

density, measured in kilograms per cubic meter, times a n-form measured in cubic meters.

The vector field X represents velocity. Since the components of X are measured in meters per second, and the basis vectors are in measured in inverse meters, the units of X itself are inverse seconds. The mass transport flux $X \lrcorner \gamma_t$ has the units of kilograms per second. It is oftten written in terms of basis forms that have the dimensions of square meters, so its coefficients have units kilograms per second per square meter.

Theorem 8.14 (Lie derivative and divergence) *Suppose there is a given volume form* vol *and corresponding notion of divergence. Write an arbitrary n-form as $\gamma = u\,\mathrm{vol}$, where u is scalar. Then the Lie derivative of γ is*

$$d(X \lrcorner (u\mathrm{vol})) = \mathrm{div}(uX)\mathrm{vol} = (X \lrcorner du + \mathrm{div}(X)u)\mathrm{vol}. \tag{8.109}$$

The operation on n-forms is reduced to operations that produce a scalar times the volume form.

Proof: For the first equality, write $X \lrcorner (u\mathrm{vol}) = uX \lrcorner \mathrm{vol}$ and take the exterior derivative.

For the second equality, start with $X \lrcorner (u\,\mathrm{vol}) = u(X \lrcorner \mathrm{vol})$. Then

$$d(u(X \lrcorner \mathrm{vol})) = du \wedge (X \lrcorner \mathrm{vol}) + u\, d(X \lrcorner \mathrm{vol}). \tag{8.110}$$

But $0 = X \lrcorner (du \wedge \mathrm{vol}) = (X \lrcorner du)\mathrm{vol} - du \wedge (X \lrcorner \mathrm{vol})$. So this is

$$d(u(X \lrcorner \mathrm{vol})) = (X \lrcorner du)\mathrm{vol} + u\ \mathrm{div}(X)\mathrm{vol}. \tag{8.111}$$

□

Theorem 8.15 (Explicit solution for the time evolution) *Suppose that $\gamma = u\mathrm{vol}$. Then the solution $\gamma_t = \Phi_{-t}^*\gamma$ of the forward equation is*

$$\gamma_t = \Phi_{-t}^*(u\,\mathrm{vol}) = (u \circ \Phi_{-t}) \exp\left(-\int_0^t (\mathrm{div}(X) \circ \Phi_{-s})\, ds\right) \mathrm{vol}. \tag{8.112}$$

Proof: Write $\gamma_t = u_t\mathrm{vol}$. Then

$$\frac{\partial u_t}{\partial t} + \mathrm{div}(u_t X) = 0. \tag{8.113}$$

This says that

$$\frac{\partial u_t}{\partial t} + X \lrcorner du_t = -\,\mathrm{div}(X)u_t. \tag{8.114}$$

Fix t and consider $u_{t'} \circ \Phi_{t'-t}$ as a function of t'. By the chain rule

$$\frac{d}{dt'}(u_{t'} \circ \Phi_{t'-t}) = (\frac{\partial}{\partial t'} u_{t'}) \circ \Phi_{t'-t} + (X \lrcorner du_{t'}) \circ \Phi_{t'-t} = -(\mathrm{div}(X) \circ \Phi_{t'-t})(u_{t'} \circ \Phi_{t'-t}). \tag{8.115}$$

This equation is easy to solve. Take

$$\frac{d}{dt'} \log(u_{t'} \circ \Phi_{t'-t}) = -(\operatorname{div}(X) \circ \Phi_{t'-t}) \tag{8.116}$$

and integrate from 0 to t. This gives

$$u_t = (u \circ \Phi_{-t}) \exp\left(- \int_0^t (\operatorname{div}(X) \circ \Phi_{t'-t})\, dt' \right). \tag{8.117}$$

This is equivalent to the statement of the theorem. □

Remark: It may help to write the above proof in coordinates. A hint for this may be found in the proof of the theorem of the following subsection. ||

The above proof starts by writing the conservation equation in two ways. The first way is the the usual *divergence form* of a conservation law. In the second way the left hand side is a material derivative, that is, the change in density following a typical particle. This suggests the method of solving the differential equation by integrating the vector field to find particle trajectories. The change in density along a particle trajectory is driven by the right hand side, which is a measure of how fast particles are compressing together.

Remark: The equation for scalar field s associated with a flow involves $X\, s = \langle ds \mid X \rangle = X \lrcorner\, ds$. The n-form equation for γ involves $d(X \lrcorner\, \gamma)$. Is there a corresponding equation for a k-form ω? The answer is yes: the equation involves $X \lrcorner\, d\omega + d(X \lrcorner\, \omega)$. This expression is called the Lie derivative of ω along X. It is clear from this that the degree 0 and degree n cases are special, in that for the former $X \lrcorner\, \omega = 0$, while for the latter $d\omega = 0$. In these two extreme cases the Lie derivative has only one term. ||

Example: Consider the vector field

$$X = x\, \frac{\partial}{\partial x}. \tag{8.118}$$

The divergence of this vector field with respect to dx is the constant 1. The flow is $x \leftarrow xe^t$. The solution of the conservation law is

$$r(x,t)\, dx = (x \leftarrow xe^{-t})^*(r(x))\, dx) = r(xe^{-t})\, d(xe^{-t}) = r(xe^{-t})e^{-t}\, dx. \tag{8.119}$$

||

Example: Consider the vector field

$$X = x^2\, \frac{\partial}{\partial x} + xy\, \frac{\partial}{\partial y}. \tag{8.120}$$

The problem is to solve the conservation law with this X. The divergence of this with respect to $dx\, dy$ is $3x$. The flow is $(x,y) \leftarrow (x/(1-tx), y/(1-tx))$. So the solution is $r(x/(1+tx), y/(1+tx))\, d(x/(1+tx))\, d(y/(1+tx))$. The differential part is $d(x/(1+tx))d(y/(1+tx)) = 1/(1+tx)^3\, dx\, dy$. This factor is consistent with

$$\exp(-\int_0^t 3x/(1+sx)\, ds) = \exp(-3\log(1+tx)) = 1/(1+tx)^3. \tag{8.121}$$

||

8.3.2 Time-dependent vector fields*

A time-dependent vector field may be thought of as a velocity that is changing in time and space. It is of the form

$$X_t = \sum_j g_j(\mathbf{x}, t)\frac{\partial}{\partial x_j}. \tag{8.122}$$

It defines a flow $\Phi_{t',t}$ that takes a particle with a position at time t and moves it to its position at time t'. It is convenient to look at the effect of this flow as a function of t'. The ordinary differential equation it satisfies is

$$\frac{\partial(x_j \circ \Phi_{t,t'})}{\partial t'} = g_j(\mathbf{x} \circ \Phi_{t',t}, t'). \tag{8.123}$$

If the flow is written

$$\Phi_{t',t} = (\mathbf{x} \leftarrow \mathbf{f}(\mathbf{x}, t', t)) \tag{8.124}$$

then the ordinary differential equation says that

$$\frac{\partial}{\partial t'} f_j(\mathbf{x}, t', t) = g_j(\mathbf{f}(\mathbf{x}, t', t), t'). \tag{8.125}$$

with $f_j(\mathbf{x}, t, t) = \mathbf{x}$.

The conservation law for ρ_t has the form

$$\frac{\partial \rho_{t'}}{\partial t'} + \operatorname{div}(\rho_{t'} X_{t'}) = 0 \tag{8.126}$$

This may also be written

$$\frac{\partial \rho_{t'}}{\partial t'} + X_{t'} \lrcorner\, d\rho_{t'} = -\operatorname{div}(\ X_{t'})\rho_{t'}. \tag{8.127}$$

Theorem 8.16 (Solution of conservation law) *Suppose that the conservation law has a solution $\rho_t = r(\mathbf{x}, t)$ with initial condition at time t_0 equal to $\rho_{t_0} = r(\mathbf{x}, t_0)$. Suppose also that the flow of the vector field exists and satisfies the equations above. Then the solution of the conservation law has the explicit form*

$$\rho_t = (\rho_{t_0} \circ \Phi_{t_0,t}) \exp(-\int_{t_0}^{t} \operatorname{div}(X'_t) \circ \Phi_{t',t}\, dt'). \tag{8.128}$$

This is the solution for $\rho_t = r(\mathbf{x}, t)$ at any point given by $\mathbf{x}, t$. The particles at time t have been transported from where they were at time t_0. They may be more (or less) spread out because they are diverging (or converging), and this will decrease (or increase) the density.

Proof: The proof is by the method of characteristics. Solve the ordinary differential equation

$$\begin{aligned}\frac{d}{dt'}\rho_{t'} \circ \Phi_{t',t} &= \frac{\partial \rho_{t'}}{\partial t'} \circ \Phi_{t',t} + (X_{t'} \lrcorner\, d\rho_{t'}) \circ \Phi_{t',t} \\ &== -(\operatorname{div}(\ X_{t'})\rho_{t'}) \circ \Phi_{t',t}.\end{aligned} \tag{8.129}$$

The first equality uses the chain rule and the ordinary differential equation defined by the vector field. The second equality uses the conservation law. The ordinary differential equation is a linear equation whose solution is elementary.

To clarify the use of chain rule in the first step, write the equation as

$$\frac{d}{dt'} r(\mathbf{f}(\mathbf{x}, t', t), t') = r'_{,n+1}(\mathbf{f}(\mathbf{x}, t', t), t') + \sum_{j=1}^{n} r'_{,j}(\mathbf{f}(\mathbf{x}, t', t), t') \frac{\partial}{\partial t'} f_j(\mathbf{x}, t', t). \tag{8.130}$$

By the ordinary differential equation the partial derivative with respect to t' is $g_j(\mathbf{f}(\mathbf{x}, t', t), t')$. $\square$

Example: Consider the conservation law

$$\frac{\partial \rho_t}{\partial t} + \frac{\partial (x+t)\rho_t}{\partial x} = 0 \tag{8.131}$$

with $\rho_0 = r(x, 0)$ given at time zero. The ordinary differential equation is

$$\frac{dx}{dt'} = x + t' \tag{8.132}$$

with solution

$$f(x_0, t', t) = x_0 e^{t'} + (e^{t'} - t' - 1). \tag{8.133}$$

Here x_0 is determined by

$$x = f(x_0, t, t) = x_0 e^t + (e^t - t - 1). \tag{8.134}$$

As a consequence,

$$f(x, t, 0) = x_0 = (x + t)e^{-t} + e^{-t} - 1. \tag{8.135}$$

So the solution is

$$\rho_t = r(x, t) = r((x + t)e^{-t} + e^{-t} - 1), 0)e^{-t}. \tag{8.136}$$

||

Example: Consider an exploding star, at a scale at which the original star is a point. If the point is the origin, and the explosion was at time zero, then the particles found at $\mathbf{x}, t$ will have travelled a distance $\mathbf{x}$ in time t and will then have (constant) velocity $\mathbf{v} = \mathbf{x}/t$. If we look at particles at the same time but at greater or lesser distance, then they will not be the same particles, so they will have correspondingly greater or lesser velocities.

The explosion need not be symmetric; the profile at time $t > 0$ is $\rho_t = r(\mathbf{x}, t)$. The divergence of the vector field is div $\mathbf{v} = n/t$. The exponential factor is thus $(t_0/t)^n$. (For the exploding star example the dimension of space is $n = 3$, but it is just as easy to do the calculation for arbitrary dimension.) For fixed $\mathbf{x}$ and t the solution of the ordinary differential equation $d\mathbf{x}'/dt' = \mathbf{x}'/t'$ is $f(\mathbf{x}, t', t) = (\mathbf{x}/t)t'$. So if $\rho_{t_0} = r(\mathbf{x}, t_0)$, the solution to the partial differential equation is

$$\rho_t = r(\frac{\mathbf{x}}{t} t_0, t_0) \left(\frac{t_0}{t}\right)^n. \tag{8.137}$$

It consists of a profile that is the same for all particles with the same constant velocity $\mathbf{x}/t$, modified by a factor that says that they are flying apart. ||

8.4 Electric and magnetic fields*

8.4.1 Electric fields*

Electric and magnetic fields provide a wonderful illustration of differential form concepts. The subject may be formulated in terms of four integal theorems, two for electric fields and two for magnetic fields. Each integral theorem dictates the particular kind of differential form that is appropriate. The two manifestations of an electric field are as a 1-form and as a twisted 2-form. The corresponding objects for a magnetic field are a 2-form and a twisted 1-form.

The power of the differential forms concept is that it works in spaces without much structure, other than a notion of differentiability. Euclidean space is quite special in this regard, in that there is considerable structure, in particular notions such as length and orthogonality. For applications of forms to problems involving Euclidean space, it is desirable to use basic forms that are of unit length and are mutually orthogonal. The most obvious example is to take Cartesian coordinates, where the basic forms are dx, dy, and dz, taken in that order. Once we have such basic forms, we can convert from 1-forms to vector fields, just by taking the same components. We can also convert from 1-forms to twisted 2-forms, by replacing dx, dy and dz by $dy\,dz$, $dz\,dx$, and $dx\,dy$. This is called the *Hodge dual*. The Hodge dual may also be used to convert from twisted 1-forms to 2-forms.

This also works in other coordinate systems. The most important example is spherical polar coordinates r, θ, ϕ, where $x = r\cos(\phi)\sin(\theta)$, $y = r\sin(\phi)\sin(\theta)$, and $z = r\cos(\theta)$. Here $r^2 = x^2 + y^2 + z^2$. This is the distance from the origin, so surfaces of constant r are spheres. The quantities θ and ϕ are co-latitude and longitude. The basic forms are dr, $r\,d\theta$, and $r\sin(\theta)\,d\phi$.

Sometimes it is useful to use cylindrical coordinates ρ, ϕ, z. Here $x = \rho\cos(\phi)$, $y = \rho\sin(\phi)$, and $z = z$. This time $\rho^2 = x^2 + y^2$. This is the distance from the z axis, so surfaces of constant ρ are cylinders. The basic forms are $d\rho$, $\rho\,d\phi$, and dz.

There is also another kind of Hodge dual that takes 1 to the twisted 3-form $*1 = \mathrm{vol} = dx\,dy\,dz$. This volume element may be expressed in other coordinate systems; in particular $\mathrm{vol} = r^2\,dr\,d\theta\,d\phi$ and $\mathrm{vol} = \rho\,d\rho\,d\phi\,dz$.

If X is a vector field, then there are two ways of obtaining a twisted differential form. The obvious conversion in Cartesian coordinates is a 1-form $\mathsf{g}X$. There is also a twisted 2-form $\mathsf{c}X = *\mathsf{g}X$. The symbol c indicates that this is the same as contraction (interior product) with respect to the twisted 3-form vol, that is, $\mathsf{c}X = X \lrcorner \mathrm{vol}$. This operation may also be used to convert a twisted vector field to a 2-form.

We are familiar with some units in physics, such as the newton as the unit of force, or the joule (newton-meter) as the unit of energy. In the theory of electric and magnetic fields there are a number of less familiar units. The unit of electric charge is the coulomb. A volt is a joule per coulomb. The unit of electric charge is the ampere, or coulomb per second. Magnetic flux is measured in webers, or joules per ampere.

An *electric field intensity* is most often modeled as a vector field **E**. However in this section we will think of it as a 1-form E with $E = \mathsf{g}\mathbf{E}$. If we write $E = E_1\,dx + E_2\,dy + E_3\,dz$, then the form E is measured in volts. The coefficients E_1, E_2, E_3 are measured in volts per meter. In electrostatics E is an exact 1-form, so

$$E = -d\phi, \tag{8.138}$$

where ϕ is a 0-form called the *electric potential*. The electric potential is measured in volts. (Note: This electric potential ϕ is to be distinguished from the spherical polar coordinate in r, θ, ϕ.)

Example: The electric field of a point charge at the origin is (up to a constant multiple)

$$E = \frac{1}{4\pi r^2}\frac{x\,dx + y\,dy + z\,dz}{r}. \tag{8.139}$$

This can also be written in spherical polar coordinates as

$$E = \frac{1}{4\pi r^2}\,dr. \tag{8.140}$$

The electric potential is

$$\phi = \frac{1}{4\pi r}. \tag{8.141}$$

The surfaces of constant potential are spheres. In most physics texts the electric field **E** is represented by the gradient vector field, which gives vectors in the radial direction, orthogonal to these spheres. ||

There is another kind of electric field often considered in physics. It is called the *electric flux density*. Another term is *electric displacement*. It is ordinarily considered to be a vector field **D**, but here we take it to the the twisted 2-form D defined by $D = \mathsf{c}\mathbf{D}$. (One can think of D for displacement.) If $D = D_1\,dy\,dz + D_2\,dz\,dx + D_3\,dx\,dy$, then D is measured in coulombs. The coefficients D_1, D_2, D_3 are measured in coulombs per square meter. The relation between E and D is that there is a scalar ϵ so that D is ϵ times the Hodge dual of E.

Example: For the electric field of a point charge this is (up to a multiple)

$$D = \frac{1}{4\pi r^2}\frac{x\,dy\,dz + y\,dz\,dx + z\,dx\,dy}{r}. \tag{8.142}$$

This has a nice expression in spherical polar coordinates as

$$D = \frac{1}{4\pi r^2} r\,d\theta\; r\sin(\theta)\,d\phi = \frac{1}{4\pi}\sin(\theta)\,d\theta\,d\phi. \tag{8.143}$$

This is represented geometrically by flux lines coming radially from the origin. ||

The fundamental equations of electrostatics are

$$dE = 0 \tag{8.144}$$

and

$$dD = R. \tag{8.145}$$

Here $R = \rho\,\mathrm{vol}$ is charge, a twisted 3-form. The charge density ρ is allowed to assume values that are positive, negative, or zero. (Note: This charge density ρ is to be distinguished from the cylindrical coordinate in ρ, ϕ, z.)

These equation have integral forms. The first equation says that for every oriented surface S the integral around the oriented closed curve ∂S that is its boundary is zero:

$$\int_{\partial S} E = 0. \tag{8.146}$$

The static electric field has no circulation. The second says that for every region W the (transport) flux over the (transversally oriented) surface ∂W of D is equal to the total charge in the region:

$$\int_{\partial W} D = \int_W R. \tag{8.147}$$

The first of the equations is consistent with

$$\int_C E = -\langle \phi, \partial C \rangle, \tag{8.148}$$

where the right hand side is the difference of the electric potential ϕ at the end points of the oriented curve C.

Example: As an illustration of these ideas, here is the computation of the electric displacement field of a charge that is uniform on a ball of radius ϵ about the origin. For D we have the same formula as before for $r \geq \epsilon$. However for $r \leq \epsilon$ we have

$$D_{\text{in}} = \frac{r^3}{\epsilon^3} D. \tag{8.149}$$

One way to see that this works is to compute the exterior derivative. This is for $r \leq \epsilon$

$$d\, D_{\text{in}} = \frac{3r^2}{\epsilon^3}\, dr\, \frac{1}{4\pi} \sin(\theta)\, d\theta\, d\phi = \left(\frac{4\pi\epsilon^3}{3}\right)^{-1} r^2 \sin(\theta)\, dr\, d\theta\, d\phi = \left(\frac{4\pi\epsilon^3}{3}\right)^{-1} dx\, dy\, dz. \tag{8.150}$$

Indeed this a constant charge density within the ball with total charge equal to one. Thus the radial lines representing the electric displacement field begin inside this ball. If we restrict the form to the region outside the charged ball, then D is a closed twisted 2-form that is not exact.

The corresponding electric field for $r \leq \epsilon$ is

$$E_{\text{in}} = \frac{r^3}{\epsilon^3} E = \frac{1}{4\pi} \frac{r}{\epsilon^3}\, dr. \tag{8.151}$$

The potential is

$$\phi_{\text{in}} = \frac{1}{4\pi\epsilon^3} \frac{1}{2} (3\epsilon^2 - r^2). \tag{8.152}$$

The constant term makes the potential continuous at $r = \epsilon$. ||

The physical effect of an electric field is to exert a force on a charged particle. If the particle has charge q (measured in coulombs), the work form is $F = F_1\,dx + F_2\,dy + F_3\,dz$. This is measured in terms of the unit of energy, the joule. The coefficients F_1, F_2, F_3 are the forces, measured in newtons. The equation is

$$F = qE = -qd\phi. \tag{8.153}$$

The volt unit is equivalent to a joule per coulomb. The effect of the force is to accelerate the particle in the direction of the electric field.

8.4.2 Magnetic fields*

A *magnetic flux density* form is most often modeled as a twisted vector field $\mathbf{B}$, but in many ways it is more natural to model it as a differential 2-form defined by $B = \mathbf{cB}$. This is $B = B_1\,dy\,dz + B_2\,dz\,dx + B_3\,dx\,dy$. The quantity B is measured in webers, that is, in joules per ampere. The coefficients B_1, B_2, B_3 are measured in webers per square meter. This unit is also called a tesla.

The magnetic flux density B is an exact 2- form. It has a 1-form *magnetic potential* A with $dA = B$. Such a potential is of course not unique, since one may add to it a 1-form of the form ds and get the same magnetic flux density.
Example: Consider a wire bearing current along the z axis. The magnetic flux density form in this case is most easily expressed in cylindrical coordinates via

$$B = \frac{1}{2\pi}\,dz\,\frac{x\,dx + y\,dy}{x^2 + y^2} = \frac{1}{2\pi\rho}\,dz\,d\rho. \tag{8.154}$$

The curves of magnetic flux density circle around the wire; their density drops off with distance from the wire. The vector potential may be taken to be

$$A = -\frac{1}{2\pi}\log(\rho)\,dz. \tag{8.155}$$

||

There is another kind of magnetic field, sometimes called the *magnetic field intensity*. It is often written as a twisted vector field $\mathbf{H}$, but here is is considered as a twisted 1-form $H = \mathbf{gH}$. It is $H = H_1\,dx + H_2\,dy + H_3\,dz$. Here H is measured in amperes, and the coefficients H_1, H_2, H_3 are measured in amperes per meter. The relation between H and B is that there is a scalar μ such that B is μ times the Hodge dual of H.

For the magnetic field intensity of the wire we have

$$H = \frac{1}{2\pi}\frac{x\,dy - y\,dx}{x^2 + y^2} = \frac{1}{2\pi\rho}\,\rho\,d\phi = \frac{1}{2\pi}\,d\phi \tag{8.156}$$

in cylindrical coordinates. The surfaces of constant ϕ are planes with one side on the wire. Notice that this is not an exact 1-form, since the integral around a closed curve surrounding the wire is 1. In physics the magnetic field intensity is

often represented by a vector field orthogonal to the planes, again circling the wire.

The fundamental equations of magnetostatics are

$$dB = 0 \tag{8.157}$$

and

$$dH = J. \tag{8.158}$$

Here J is a twisted 2-form called the current form. Thus $J = J_1\, dy\, dz + J_2\, dz\, dx + J_3\, dx\, dy$. The form J is measured in amperes, while the coefficients J_1, J_2, J_3 are measured in amperes per square meter. An ampere is the same as a coulomb per second. The fact that J is exact represents current conservation: the lines representing the 2-form J have no end points. The B field is an exact 2-form, so it may be pictured by circulatory flux curves (with transverse orientations) that have no end points. The H field is a twisted 1-form that is locally exact in regions where the current J is zero. In such regions it may be pictured by surfaces (with internal orientations).

These equations have integral forms. The first says that for every oriented region W the (circulatory) flux over the oriented surface ∂W of B is zero.

$$\int_{\partial W} B = 0. \tag{8.159}$$

Magnetic flux density curves never end. The second equation says that for every transversally oriented surface S the integral of twisted 1-form H around the closed transversally oriented curve ∂S that is its boundary is the (transport) flux of current passing through the surface.

$$\int_{\partial S} H = \int_S J. \tag{8.160}$$

The first of these equations is compatible with

$$\int_W B = \int_{\partial W} A. \tag{8.161}$$

Example: Here is the computation of the magnetic intensity field of a current that is uniform on a cylinder of radius ϵ about the z axis. For H we have the same formula as before for $\rho \geq \epsilon$. However for $\rho \leq \epsilon$ we have

$$H_{\text{in}} = \frac{\rho^2}{\epsilon^2} H. \tag{8.162}$$

One way to see that this works is to compute the exterior derivative. This is for $r \leq \epsilon$

$$d\, H_{\text{in}} = \frac{2\rho}{\epsilon^2}\, d\rho\, \frac{1}{2\pi}\, d\phi = \left(\pi\epsilon^2\right)^{-1} \rho\, d\rho\, d\phi = \left(\pi\epsilon^2\right)^{-1} dx\, dy. \tag{8.163}$$

electric field intensity	$\mathbf{E}$	vector field
electric flux density	$\mathbf{D} = \epsilon\mathbf{E}$	vector field
magnetic flux density	$\mathbf{B}$	twisted vector field
magnetic field intensity	$\mathbf{H} = \frac{1}{\mu}\mathbf{B}$	twisted vector field
charge density	ρ	scalar field
current density	$\mathbf{J}$	vector field

Table 8.1: Electric and magnetic fields as vector fields and twisted vector fields

Indeed this is a constant current density within the cylinder with total current equal to one. Thus the planes representing the H field end in lines inside the cylinder. If we restrict H to the region outside the wire, then it is an example of a closed 1-form that is not exact.

The corresponding magnetic flux density for $\rho \leq \epsilon$ is

$$B_{\text{in}} = \frac{\rho^2}{\epsilon^2} B = \frac{1}{2\pi} \frac{\rho}{\epsilon^2} \, dz \, d\rho. \tag{8.164}$$

The vector potential outside the wire may be taken to be

$$A = -\frac{1}{2\pi} \log(\frac{\rho}{\epsilon}) \, dz. \tag{8.165}$$

The reason for writing it with the $\epsilon > 0$ is that it is convenient to have the magnetic potential be zero at the surface of the wire. The corresponding expression inside is then

$$A_{\text{in}} = -\frac{1}{2\pi} \frac{1}{2} \left(\frac{\rho^2}{\epsilon^2} - 1 \right) \, dz. \tag{8.166}$$

This also is zero at the surface of the wire. ||

The relation between magnetic field and force is more complicated. Say that the particle with charge q has velocity vector $\mathbf{v}$. Then

$$F = q\mathbf{v} \lrcorner B. \tag{8.167}$$

Here $\mathbf{v} = v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + v_3 \frac{\partial}{\partial z}$. Here $\mathbf{v}$ is a rate measured in inverse seconds. The coefficients v_1, v_2, v_3 are measured in meters per second. The work form F is measured in joules.

The interior product is explained elsewhere. Here it suffices to say that if X is a vector field and α and β are 1-forms, then $X \lrcorner \alpha \wedge \beta = \langle X, \alpha \rangle \beta - \langle X, \beta \rangle \alpha$. Thus, for instance, the interior product of $\partial/\partial x$ with $dx \, dy$ is dy, with $dz \, dx$ is $-dz$, and with $dy \, dz$ is zero. The same pattern is repeated if x, y, z is replaced by y, z, x or z, x, y.

As a consequence, the interior product of $\mathbf{v}_1 \partial/\partial x$ with $B_1 \, dx \, dy$ is $v_1 B_1 \, dy$. The force is in the plane of the magnetic flux density and is perpendicular to the velocity. This kind of force encourages the particle go around in circles in the plane of the magnetic flux density.

Most textbooks on electromagnetism treat the fields as vector fields or as twisted vector fields. Table 8.1 lists these conventional descriptions. The other

electric field intensity	E	$E = \mathsf{gE}$	1-form	$dE = 0$
electric flux density	$D = \epsilon * E$	$D = \mathsf{cD}$	twisted 2-form	$dD = R$
magnetic flux density	B	$B = \mathsf{cB}$	2-form	$dB = 0$
magnetic field intensity	$H = \frac{1}{\mu} * B$	$H = \mathsf{gH}$	twisted 1-form	$dH = J$
charge density	R	$R = *\rho$	twisted 3-form	
current density	J	$J = \mathsf{cJ}$	twisted 2-form	$0 = dJ$

Table 8.2: Electric and magnetic fields as forms and twisted forms

choice is to write the fields as differential forms or as twisted differential forms. This makes the integral forms of these equations immediate. Table 8.2 summarizes this way of realizing the fields.

It may seem confusing to have four kinds of fields. The forces are expressed by E and B. On the other hand it is their Hodge duals D and H that depend on the sources R and J. In relativity E and B are combined into one field, as are D and H. The equations in the last column of Table 8.2 are the equations of electrostatics, that is, for the situation when there is no dependence on time. Among other things, they say that charge R is a source of electric flux D, while current J is is a source of magnetic circulation of H.

Quadratic expressions in the fields are also important. Thus $\frac{1}{2}E \wedge D$ and $\frac{1}{2}H \wedge B$ are twisted 3-forms representing electric and magnetic energy. Also $S = E \wedge H$ is the *Poynting energy flux*, a twisted 2-form of great importance in electromagnetic theory. The equation $dS = -E \wedge J$ describes energy transfer in the static situation. Energy is transferred when electric field and current align. See for instance [50] for much more on these topics. This source also contains nice pictures.

8.4.3 Maxwell's equations*

The Maxwell equations with time dependence represent one of the greatest scientific achievements of all time. Here they are written with the sources on the right hand side:

$$\begin{aligned} dE + \frac{\partial B}{\partial t} &= 0 \\ dB &= 0 \\ dH - \frac{\partial D}{\partial t} &= J \\ dD &= R. \end{aligned} \tag{8.168}$$

The first two equations are for the differential forms E, B, and there are no sources. The final two equations are for the twisted differential forms H, D, and the sources J, R are also twisted differential forms. As before, $D = \epsilon * E$ and $B = \mu * H$. There is also the equation relating charge and current

$$dJ + \frac{\partial R}{\partial t} = 0. \tag{8.169}$$

The integral forms of the Maxwell equations are more intuitive.

$$\begin{aligned}
\int_{\partial S} E &= -\frac{d}{dt}\int_S B \\
\int_{\partial W} B &= 0 \\
\int_{\partial S} H &= \int_S J + \frac{d}{dt}\int_S D \\
\int_{\partial W} D &= \int_W R.
\end{aligned} \tag{8.170}$$

And also there is

$$\int_{\partial W} J = -\int_W R. \tag{8.171}$$

These integral forms each have a story.

- The first Maxwell equation says that the circulation of the electric field intensity E around the boundary of the surface S is the negative of the time derivative of the magnetic flux B in S. This is Faraday's law of induction. It says that a changing magnetic field produces an electric field.

- The second equation says that the flux curves of the magnetic flux B are closed curves, so that whatever enters W also leaves W.

- The third Maxwell equation says that the circulation of the magnetic field intensity H around the boundary of S is the sum of two terms. The first term involves the flux of the current J through S. This is the law of Ampère; it says that a current can produce a magnetic field. The second term is the time derivative of the electric flux through S. This "displacement current" term was found by Maxwell. A changing electric field produces a magnetic field.

- The final equation says that the flux curves of the electric flux D start and end where there is a non-zero charge R.

The marvelous thing about the time dependent Maxwell equations is that they predict that electric and magetic fields that are both changing can propagate in a region distant from any sources. The following theorem gives some insight about how this happens.

Theorem 8.17 (Wave equation for electric field) *Suppose $\epsilon > 0$ and $\mu > 0$ are constants. Then the electric field satisfies the 1-form equation*

$$\epsilon\mu\frac{\partial^2 E}{\partial t^2} - \nabla^2 E = -\frac{1}{\epsilon} d * R - \mu\frac{\partial}{\partial t} * J. \tag{8.172}$$

This theorem shows that the electric field satisfies the wave equation with a source that depends on the charge 3-form R and the current 2-form J. When J is varying very rapidly, then the current term gives rise to a wave that can be felt at very long distances. According to the Kirchoff formula below, the waves travel at speed c given by

$$c = \frac{1}{\sqrt{\epsilon\mu}}. \tag{8.173}$$

This remarkable formula shows that the speed of light may be computed from quantities known from electrostatics.

Lemma 8.18 *The Laplace operator ∇^2 for three-dimensional Euclidean space acting on k-forms is given by*

$$\nabla^2 = (-1)^k(*d*d - d*d*). \tag{8.174}$$

The lemma will be useful in the proof. It will emerge as a special case of formulas in the final chapter.

Proof: Compute

$$\begin{aligned}
dE &= -\frac{\partial}{\partial t}B \\
*dE &= -\mu\frac{\partial}{\partial t}H \\
d*dE &= -\mu\frac{\partial}{\partial t}dH = -\mu\frac{\partial^2}{\partial t^2}D - \mu\frac{\partial}{\partial t}J \\
*d*dE &= -\mu\epsilon\frac{\partial^2 E}{\partial t^2} - \mu\frac{\partial}{\partial t}*J \\
-\nabla^2 E + d*d*E &= -\mu\epsilon\frac{\partial^2 E}{\partial t^2}E - \mu\frac{\partial}{\partial t}*J \\
-\nabla^2 E + \frac{1}{\epsilon}d*dD &= -\mu\epsilon\frac{\partial^2 E}{\partial t^2} - \mu\frac{\partial}{\partial t}*J \\
-\nabla^2 E + \frac{1}{\epsilon}d*R &= -\mu\epsilon\frac{\partial^2 E}{\partial t^2} - \mu\frac{\partial}{\partial t}*J.
\end{aligned} \tag{8.175}$$

This can be rearranged to give the statement of the theorem. □

There is a similar result for the magnetic field. There is only one source term, and it does not involve a time derivative.

Theorem 8.19 (Wave equation for magnetic field) *Suppose $\epsilon > 0$ and $\mu > 0$ are constants. Then the magnetic field satisfies the 2-form equation*

$$\epsilon\mu\frac{\partial^2 B}{\partial t^2} - \nabla^2 B = \mu d*J. \tag{8.176}$$

Proof: There is a chain of equalities:

$$\begin{aligned}\frac{\partial^2 B}{\partial t^2} &= -d\frac{\partial E}{\partial t} \\ &= -\frac{1}{\epsilon} d * \frac{\partial D}{\partial t} \\ &= -\frac{1}{\epsilon} d * (dH - J) \\ &= -\frac{1}{\epsilon\mu} d * d * B + \frac{1}{\epsilon} d * J \\ &= \frac{1}{\epsilon\mu}\nabla^2 B + \frac{1}{\epsilon} d * J. \end{aligned} \tag{8.177}$$

This uses the lemma and $dB = 0$. $\square$

- Summary: A changing 2-form magnetic field B is a source of circulation of electric field E, while a changing twisted 2-form electric flux D produces a second source of circulation of magnetic field H. This reciprocal action is the explanation for electromagnetic waves (radio, light, and other parts of the spectrum). This is one of the greatest syntheses in the entire history of science.

8.4.4 The Kirchoff solution of the wave equation*

The wave equation with a source term is

$$\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = s(\mathbf{x}, t). \tag{8.178}$$

The following theorem may be found in texts on partial differential equations. See for instance [13].

Theorem 8.20 (Kirchoff formula) *A particular solution of the wave equation with source in three space dimensions is given by*

$$u = f(\mathbf{x}, t) = \int_{|\mathbf{x}'-\mathbf{x}|\leq ct} \frac{1}{4\pi|\mathbf{x}' - \mathbf{x}|} s\left(\mathbf{x}', t - \frac{|\mathbf{x}' - \mathbf{x}|}{c}\right) d^3\mathbf{x}'. \tag{8.179}$$

This particular solution is where the value of the solution of at $\mathbf{x}, t$ is an integral over the ball about $\mathbf{x}$ of radius ct. The integrand on the sphere at radius r is $1/(4\pi r)$ times the source evaluated at the previous time $t - r/c$. This is because it took time r/c for the influence to travel the distance r from the point on the sphere to the point $\mathbf{x}$.

The fact the decay is as slow as $1/(4\pi r)$ means that the influence of the source is felt over very long distance. This applies in particular to the electric field generated by a time dependent current.

In the limit that c goes to infinity (letting μ go to zero while keeping ϵ fixed), the solution becomes an integral over all space:

$$u = f(\mathbf{x}, t) = \int \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} s\,(\mathbf{x}', t)\, d^3\mathbf{x}'. \tag{8.180}$$

This at first seems to indicate that in the static electric field situation the source is again felt strongly over long distance. However in this case the source $s_i(\mathbf{x}, t) = -(1/\epsilon)\partial\rho(\mathbf{x}, t)/\partial x_i$ is exact, so integrating by parts gives

$$u_i = f_i(\mathbf{x}, t) = \int \frac{1}{4\pi\epsilon|\mathbf{x} - \mathbf{x}'|^2} \frac{x_i - x_i'}{|\mathbf{x} - \mathbf{x}'|} \rho\,(\mathbf{x}', t)\, d^3\mathbf{x}'. \tag{8.181}$$

In the static situation the influence decays like $1/(4\pi\epsilon r^2)$, so has has short range relative to the case of the Kirchoff formula.

8.5 Twisted differential forms*

8.5.1 Properties of twisted differential forms*

The remainder of this chapter is devoted to twisted differential forms. This is supplementary material that might seem to deserve less priority, but there is an important issue. The properties of differential forms make them very natural objects for calculus. The properties of twisted differential forms seem almost as nice. Maybe twisted differential forms are deserving of the same attention. Here the argument is that twisted differential forms are indeed useful and important, but they have a fundamental defect: less attractive properties under pullback by a manifold mapping. Making all this clear takes a considerable amount of discussion.

The context is an n-dimensional connected manifold patch M. A *twisted differential k-form* (or *k-pseudoform*) is an odd function from the 2-element set of orientations of M to the set of differential k-forms. Thus if it sends the orientation $\mathcal{O}$ to ω, then it sends the opposite orientation $-\mathcal{O}$ to $-\omega$. A twisted differential form will be denoted by

$$(\omega : \mathcal{O}) = (-\omega : -\mathcal{O}). \tag{8.182}$$

In practice it is convenient to use a never-zero n form σ to determine an orientation $[\sigma]$. This allows the twisted differential form to be denoted by

$$(\omega : [\sigma]) = (-\omega : [-\sigma]). \tag{8.183}$$

In this representation σ may be replaced by $s\sigma$, where s is positive and never zero.

Twisted differential forms are combined using simple and natural definitions. Among these are:

- $\alpha \wedge (\omega : \mathcal{O}) = (\alpha \wedge \omega : \mathcal{O})$

- $(\alpha : \mathcal{O}) \wedge \omega = (\alpha \wedge \omega : \mathcal{O})$
- $(\alpha : \mathcal{O}) \wedge (\omega : \pm\mathcal{O}) = \pm\alpha \wedge \omega$.
- $X \lrcorner (\omega : \mathcal{O}) = (X \lrcorner \omega : \mathcal{O})$
- $d(\omega : \mathcal{O}) = (d\omega : \mathcal{O})$.

There is a local representation for a non-zero twisted differential k-form in terms of an $(n-k)$-form η. This is called a *reorienting form*. The notation is

$$\omega|_\eta = (\omega : [\eta \wedge \omega]). \tag{8.184}$$

This notation makes sense in a region where ω is never zero. Notice that

$$(-\omega)|_\eta = \omega|_\eta, \tag{8.185}$$

while

$$(\omega)|_{-\eta} = -(\omega)|_\eta. \tag{8.186}$$

This is not a standard notation, but it should be.
Example: Consder a two-dimensional manifold patch with coordinates y, u. The twisted forms of degree 0, 1, and 2 are spanned by

$$\begin{aligned}(1 : [dy\,du]) &= 1|_{dydu}\\ (dy : [dy\,du]) &= -dy|_{du}\\ (du : [dy\,du]) &= du|_{dy}\\ (dy\,du : [dy\,du]) &= (dy\,du)|_+.\end{aligned} \tag{8.187}$$

||
Example: Consider a three-dimensional manifold patch with coordinate y, u, w. Then

$$(-dw)|_{dydu} = (dw)|_{dydu} \tag{8.188}$$

while

$$-(dw|_{dydu}) = dw|_{dudy} \tag{8.189}$$

||
Example: Consider a three-dimensional manifold patch with coordinate system x, y, z. (These need not be Cartesian coordinates.)

$$\begin{aligned}(dx : [dx\,dy\,dz]) &= dx|_{dy\,dz}\\ (dy : [dx\,dy\,dz]) &= dy|_{dz\,dx}\\ (dz : [dx\,dy\,dz]) &= dz|_{dx\,dy}.\end{aligned} \tag{8.190}$$

The intuition is that an expression like $dz|_{dx\,dy}$ should be pictured as planes of fixed z with orientations determined by $dx\,dy$. This contrasts with the form dz which is pictured as planes of fixed z, but with a direction across the planes

determined by increasing z. In general the notation $[dz, dx\,dy\,dz]$ is more convenient for computations, while the notation $dz|_{dx\,dy}$ gives a better picture of the particular form.

Similarly,

$$\begin{aligned}(dx\,dy : [dx\,dy\,dz]) &= (dx\,dy)|_{dz}\\ (dy\,dz : [dx\,dy\,dz]) &= (dy\,dz)|_{dx}\\ (dz\,dx : [dx\,dy\,dz]) &= (dz\,dx)|_{dy}.\end{aligned} \tag{8.191}$$

An expression like $(dx\,dy)|_{dz}$ should be pictured as lines of fixed x and y. These have orientations determined by dz. This contrasts with $dx\,dy$, which is also pictured as lines of fixed x and y, but with orientations transverse to the lines.

There are also extreme cases, such at the twisted 0-form

$$(1 : [dx\,dy\,dz]) = 1|_{dx\,dy\,dz}. \tag{8.192}$$

This has value ± 1 depending on whether the orientation agrees or disagrees with $[dx\,dy\,dz]$.

There is also

$$(dx\,dy\,dz : [dx\,dy\,dz]) = (dx\,dy\,dz)|_{+} \tag{8.193}$$

This may be pictured as points with + signs. By contrast, $dx\,dy\,dz$ is pictured as points with 3-dimensional orientations. ||.

There is a construction that works for never-zero n-forms. Say that σ is such a form. There are always two special twisted forms

$$\begin{aligned}\sigma|_{+} &= (\sigma : [\sigma]) = (-\sigma : [-\sigma])\\ \sigma|_{-} &= (\sigma : [-\sigma]) = (-\sigma : [\sigma]).\end{aligned} \tag{8.194}$$

Then $\sigma|_{+}$ may thought of as a cloud of points with + signs. Such a twisted n-form will be called a *volume element*. Similarly, $\sigma|_{-}$ is pictured as a cloud of points with $-$ signs.

8.5.2 Pullback of twisted forms*

This section deals with the main defect of twisted differential forms. Given a mapping and a differential form on the target space, there is always a pullback form on the domain space. For a twisted form extra conditions are required.

Let N be k-dimensional and M be n-dimensional with $k \leq n$. Suppose $\phi : N \to M$ is a manifold mapping. If $(\gamma : \mathcal{O})$ is a twisted p-form on M, then under certain circumstances there should be a pullback p-form $(\phi^*\gamma : \phi^*\mathcal{O})$ on N. The problem is to define the pullback $\phi^*\mathcal{O}$ of the n-dimensional orientation of M to a k-dimensional orientation of M. It will turn out that this is possible, provided that two conditions are satisfied:

- ϕ is an immersion;
- ϕ has a transverse (external) orientation.

These concepts are explained in the following.

The mapping $\phi : N \to M$ is an *immersion* if it pushes forward each basis $X_1, \ldots, X_k$ of tangent vectors to N to linearly independent vectors $\phi_* X_1, \ldots, \phi_* X_k$ along the mapping.

Suppose that $\phi : N \to M$ is an immersion. A *transverse orientation* is determined by a collection of vector fields $Z_1, \ldots, Z_{n-k}$ along the mapping such that $Z_1, \ldots, Z_{n-k}, \phi_* X_1, \ldots, \phi_* X_k$ are linearly independent.

This gives a first way to pull back the orientation of M to an (internal) orientation of N. Choose $X_1, \ldots, X_k$ so that $Z_1, \ldots, Z_{n-k}, \phi_* X_1, \ldots, \phi_* X_k$ defines an orientation that agrees with the orientation of M. Then this choice of $X_1, \ldots, X_k$ defines the pullback of the orientation to N.

The second way to proceed is via differential forms. A transverse (external) orientation of ϕ is defined by a differential $(n-k)$-form along the mapping η with the property that $\phi_* X_i \lrcorner \eta = 0$ for each X_i in the basis. (This is just the condition that η is transverse to the tangent space of the image.) It is also required that $\langle \eta \mid Z_1, \ldots, Z_{n-k} \rangle$ is non-zero. On the other hand, an (internal) orientation of N is given by a k-form on N.

Since ϕ is an immersion, there exists a k-form α on M that pulls back to an orientation on N. It can be found by taking $\langle \alpha \mid \phi_* X_1, \ldots, \phi_* X_k \rangle > 0$ and giving it arbitrary values on sequences that include some of the Z_j. Then the pullback is

$$\langle \phi^* \alpha \mid X_1, \ldots, X_k \rangle = \langle \alpha \mid \phi_* X_1, \ldots, \phi_* X_k \rangle. \tag{8.195}$$

Proposition 8.21 *Let $\phi : N \to M$ be the manifold mapping with transverse orientation given by η along ϕ as above. Let α along ϕ be a form that determines an orientation of N as above. Consider an n-form σ along ϕ given by*

$$\sigma = \eta \wedge \alpha \tag{8.196}$$

Then

$$\langle \sigma \mid Z_1, \ldots, Z_{n-k}, X_1, \ldots, X_k \rangle = \langle \eta \mid Z_1, \ldots, Z_{n-k} \rangle \langle \alpha \mid X_1, \ldots, X_k \rangle. \tag{8.197}$$

Proof: The formula for $\sigma = \eta \wedge \alpha$ applied to the vectors on the left is a sum for each subset that may be assigned to the $(n-k)$ slots for η. But if any of the vectors in the subset are X_j, then the contribution is zero. □

This gives another way of pulling back an orientation from M to N. Take a form defining the orientation of M and restrict it to a form σ along ϕ. Suppose that $\sigma = \eta \wedge \alpha$ as above. Then $\phi^* \alpha$ defines the corresponding orientation of N.

Proposition 8.22 *Let N be a k-dimensional manifold patch, and let M be an n-dimensional manifold patch. Assume that $k \leq n$. Suppose ϕ is a manifold mapping $\phi : N \to M$ that is an immersion. Suppose that ϕ has a given transverse orientation. Then this defines a correspondence between orientations of M and orientations of N.*

Theorem 8.23 *Let N be a k-dimensional manifold patch, and let M be an n-dimensional manifold patch. Take $k \leq n$. Suppose ϕ is a manifold mapping*

$\phi : N \to M$ that is an immersion. Suppose that ϕ has a given transverse orientation given by an $(n-k)$-form η. Then every twisted differential p-form $(\omega : [\sigma])$ on M has a well-defined pullback via ϕ (with this transverse orientation) to a twisted differential form on N. This pullback is the form $(\phi^*\omega : [\phi^*\alpha])$, where $\sigma = \eta \wedge \alpha$.

All these considerations apply when S is a surface in M and $\phi : S \to M$ is the injection. It is required that S be regular and have a transverse orientation.
Example: Let M be 3-dimensional and take the surface S to be the sphere $r^2 = x^2 + y^2 + z^2 = c^2$. An external orientation is given by the form $\eta = (1/2)dr^2 = r\,dr$. The two orientations of M are given by $\pm dx\,dy\,dz$, while the two orientations of S are given by $\pm\alpha = \pm\sin(\theta)\,d\theta\,d\phi$. Since $\sigma = \eta \wedge \alpha = r\sin(\theta)\,dr\,d\theta\,d\phi$ determines the same orientation as $dx\,dy\,dz = r^2\sin(\theta)\,dr\,d\theta\,d\phi$, the mapping takes $\pm[dx\,dy\,dz]$ to $\pm[\alpha]$. ||
Example: Here is a simple example that illustrates the case of an immersion. Here N is one-dimensional with coordinate t, while M is two-dimensional with coordinates x, y. The mapping ϕ is given by $x \leftarrow (t^2 - 1)$ and $y \leftarrow t(t^2 - 1)$. Thus for $t < -1$ the image has $x > 0, y < 0$, for $-1 < t < 0$ the image has $x < 0, y > 0$, for $0 < t < 1$ the image has $x < 0, y < 0$, while for $1 < t$ the image has $x > 0$, $y > 0$. The points where t has the values ± 1 both map to the point where $x = y = 0$. The image is not a regular surface. The point where t has value 1 maps to the point where $x = -1, y = 0$.

The pushforward of $\partial/\partial t$ is the vector field along the mapping given by $X = 2t\frac{\partial}{\partial x} + (3t^2 - 1)\frac{\partial}{\partial y}$. Since this never vanishes, the mapping is an immersion. The points where $t = \mp\frac{1}{\sqrt{3}}$ are interesting; these are points where X switches from having an upward component to a downward component and then switches back the other way. At these points $x = -\frac{2}{3}$ and $y = \pm\frac{1}{\sqrt{3}}\frac{2}{3}$.

The mapping can be given an external orientation, for example by the 1-form along the mapping expressed by $\eta = (3t^2 - 1)\,dx - 2t\,dy$. For each point in N this η is zero on X. This is true even at the points where $t = \pm 1$.

This external orientation can be used to transport an orienting form on M to an orienting form on N. Say that M has orientation given by $dx\,dy$. Then if we take $\alpha = 2t\,dx + (3t^2 - 1)\,dy$ we get $\eta \wedge \alpha = ((2t)^2 + (3t^2 - 1)^2)\,dx\,dy$, which defines the same orientation as $dx\,dy$. But α also defines an orientation of N by pullback. The pullback of α is $((2t)^2 + (3t^2 - 1)^2)\,dt$ which defines the same orientation on N as dt.

This is the mechanism that works to pull back twisted differential forms. Consider the simple example

$$dy|_{dx} = (dy : [dx\,dy]). \tag{8.198}$$

This pulls back to

$$((3t^2 - 1)\,dt : [dt]) = (3t^2 - 1)(dt)_+. \tag{8.199}$$

The whole process runs smoothly. ||

Example: Here is an even simpler example that illustrates the case where the immersion property fails. Here N is one-dimensional with coordinate t, while M is two-dimensional with coordinates x, y. The mapping ϕ is given by $x \leftarrow 0$ and $y \leftarrow t(t^2 - 1)$. Thus the y values move up to $\frac{1}{\sqrt{3}}\frac{2}{3}$ at $t = -\frac{1}{\sqrt{3}}$ and then down to $-\frac{1}{\sqrt{3}}\frac{2}{3}$ at $t = \frac{1}{\sqrt{3}}$ and then up again. All this is on the vertical line where $x = 0$.

The pushforward of $\partial/\partial t$ is the vector field along the mapping given by $X = (3t^2 - 1)\frac{\partial}{\partial y}$. At the points of N where $3t^2 = 1$ the mapping fails to be an immersion. The space N breaks up into three parts, each of which may be treated separately.

One strategy would be to have the transverse orientation given by $\eta = (3t^2 - 1)\,dx$. Say that M has orientation given by $dx\,dy$. Then if we take $\alpha = (3t^2-1)\,dy$ we get $\eta\wedge\alpha = (3t^2-1)^2\,dx\,dy$, which defines the same orientation as $dx\,dy$. But α also defines an orientation of N by pullback. The pullback of α is $(3t^2 - 1)^2\,dt$ which defines the same orientation as dt. So this choice works just like the previous example. The twisted form

$$dy|_d x = (dy : [dx\,dy]) \tag{8.200}$$

pulls back to

$$((3t^2 - 1)\,dt : [dt]) = (3t^2 - 1)\,(dt)_+. \tag{8.201}$$

The image curve is just the line where $x = 0$. It might seem natural to think of a transverse orientation of this curve, for instance one could take the transverse orientation to be dx. Then for $\sigma = dx\,dy$ and $\alpha = dy$, the corresponding orientation of N is $(3t^2 - 1)\,dt$ which for $3t^2 < 1$ is the same as the orientation $-dt$ of N. On this interval the twisted form

$$dy|_{dx} = (dy : [dx\,dy]). \tag{8.202}$$

pulls back to

$$((3t^2 - 1)\,dt : -[dt]) = (3t^2 - 1)\,(dt)_- = -(3t^2 - 1)\,(dt)_+. \tag{8.203}$$

The formulas are not the same on the three intervals. ||

Example: The last example has a variant that is relevant to the usual calculus in one dimension. Here N is one-dimensional with coordinate t, while M is one-dimensional with coordinates y. The mapping ϕ is given by $y \leftarrow t(t^2 - 1)$. The pushforward of $\partial/\partial t$ is the vector field along the mapping given by $X = (3t^2 - 1)\frac{\partial}{\partial y}$. At the points of N where $3t^2 = 1$ the mapping fails to be an immersion. The space N breaks up into three parts, each of which may be treated separately.

One strategy would be to have the transverse orientation given by the sign of the 0-form $\eta = (3t^2 - 1)$. Say that M has orientation given by dy. Then if we take $\alpha = (3t^2 - 1)\,dy$ the result is that $\eta \wedge \alpha$ defines the same orientation as $dx\,dy$. But α also defines an orientation of N by pullback. The pullback of α is

$(3t^2-1)^2\,dt$ which defines the same orientation as dt. So this choice works just like the previous example. The twisted form

$$dy|_+ = (dy : [dy]) \tag{8.204}$$

pulls back to

$$((3t^2-1)\,dt : [dt]) = (3t^2-1)\,(dt)_+. \tag{8.205}$$

Another strategy gives a different result. Some calculus approaches might suggest that one should always take the transverse orientation to be $+$. Then for $3t^2 < 1$ the orientation $dx\,dy$ of M is sent to the orientation $-dt$ of N. On this interval the twisted form

$$dy|_+ = (dy : [dy]) \tag{8.206}$$

pulls back to

$$((3t^2-1)\,dt : -[dt]) = (3t^2-1)\,(dt)_- = -(3t^2-1)\,(dt)_+. \tag{8.207}$$

This result is that the pullback of $(dy)|_+$ is $|3t^2-1|\,(dt)|_+$ on all three intervals. This formula has the absolute value of the derivative. With twisted forms in calculus the pullback operation can be awkward. ||

8.5.3 Integration of twisted forms*

Let $(\tau : [\sigma])$ be a twisted k-form on M. Here τ is a k-form, and $[\sigma]$ is the orientation of M defined by the n-form σ. Let $\chi : Q \to M$ map the k-dimensional parameter space Q to M. Suppose that χ is an immersion, and suppose that it has a transversal orientation given by an $(n-k)$-form γ along the mapping. The corresponding integral is defined by

$$\int_{\chi,[\gamma]} (\tau : [\sigma]) = \int_Q (\chi^*\tau : [\chi^*\alpha]) \tag{8.208}$$

Here $\gamma \wedge \alpha = \sigma$ are regarded as forms along the mapping χ. (More precisely, the σ in the last equation is the restriction of the σ defining the orientation.) According to the pullback construction the form $\chi^*\alpha$ defines an orientation of Q depending on α and hence on σ. So the right hand side is the integral of a twisted k-form over a k-dimensional surface, which is defined as an unoriented integral.

The definition above is the natural definition of the integral of a twisted form. It is a somewhat awkward, because of the necessity of introducing a transverse orientation of the immersion mapping.

It is also possible to express the integral of the twisted k-form in terms of the integral of a form over an oriented surface. With the same notation as above, the formula is

$$\int_{\chi,[\gamma]} (\tau : [\sigma]) = \int_{Q,[\chi^*\alpha]} \chi^*\tau = \int_{\chi,[\chi^*\alpha]} \tau. \tag{8.209}$$

This method is useful for someone who wants to reduce facts about twisted forms to facts about forms. However it is not particularly natural, and is still requires the same complicated apparatus as part of the definition.

These definitions apply to the special case of a regular surface, when χ is the injection of the surface into M. Even in that case it is necessary to have the transverse orientation of the surface. The integral of a form over a surface describes circulation within the surface. The integral of a twisted form describes flux through the surface. The distinction is worth maintaining.

8.5.4 Pictures of twisted forms*

Contrast the pictures of forms and of twisted forms.

- A simple k form is never zero and is

$$\omega = s\, dx_1 \cdots dx_k \tag{8.210}$$

  in some coordinate system. It is pictured in terms of the family of surfaces $x_1 = c_1, \ldots, x_k = c_k$ (possibly with boundaries). Each surface has transverse (external) orientation $[\omega]$.

- A simple twisted k form is of the form

$$\omega|_\eta = (\omega : [\eta \wedge \omega]) \tag{8.211}$$

  where ω is a simple k-form and η is a reorienting orienting $(n-k)$-form. It is pictured in terms of the family of surfaces $x_1 = c_1, \ldots, x_k = c_k$ (possibly with boundaries). Each surface has (internal) orientation $[\eta]$.

Example:

- A never-zero n-form is of the form $\omega = s\, dx_1 \cdots dx_n$ and so is simple. In this case each surface is a point, and its transverse orientation is an orientation of the entire space.

- A never-zero twisted n-form $\omega|_+$ (or $\omega|_-$) has orienting 0-form given by $+1$ or by -1. Each surface is a point with $+$ sign (or $-$ sign) internal orientation.

||
Example:

- A never-zero $(n-1)$ forms is locally of the form $\omega = s\, dx_1 \cdots dx_{n-1}$ for some suitable coordinate system and so is simple. It is pictured as closely spaced curves (possibly with end points). Each transverse orientation is $(n-1)$-dimensional. These are called *circulatory flux curves*. The end points have n-dimensional orientations.

- A twisted differential $(n-1)$-form $\omega|_\eta$ is pictured as closely spaced curves (possibly with end points) Each has 1-dimensional internal orientation given by the 1-form η. These are called *transport flux curves*. The end points have $\pm$ signs, determined so that the oriented curves go from $+$ to $-$.

||

Example: Consider the twisted 1-form $y\,dx$ with orienting form dy in two dimensions. This is represented by vertical contour lines that terminate at points in the plane. The density of these lines is greater as one gets farther from the x axis. The (internal) orientation of these lines is in the direction of increasing y. This twisted form may also be written $(y\,dx : [y\,dy\,dx])$. Its differential is $(dy\,dx : [y\,dy\,dx])$. This is the same as $dy\,dx$ oriented by the sign of y. This twisted 2-form represents the cloud of terminating points, which in this case has a uniform density. The points where $y > 0$ have (internal) orientation $+$ and are source points. The points where $y < 0$ have (internal) orientation $-$ and are sink points. ||

8.5.5 Boundaries and exterior derivatives*

This section discusses orientations of boundaries of surfaces. It also treats the geometric interpretation of the exterior derivative, which acts something like the operation of taking a boundary.

Consider an oriented k-surface S with oriented $(k-1)$-boundary ∂S. There are two ways to define the orientation of the boundary.

- Suppose the orientation of S is given by independent vector fields $X_1, \ldots, X_k$ that are tangent to S. Suppose that X_1 at the boundary is pointing outward and $X_2, \ldots, X_k$ are tangent to ∂S. Then $X_2, \ldots, X_n$ orients ∂S.

- Suppose the orientation of S is given by a k-form α restricted to S. Suppose ϵ is a 1-form field defined near the boundary that is zero on tangent vectors to ∂S and is strictly positive for a vector pointing outward. Let β be a $(k-1)$-form on ∂S such that

$$\epsilon \wedge \beta = \alpha \tag{8.212}$$

  on ∂S. Then the internal orientation of ∂S is the orientation defined by β.

- Summary: The boundary orientation of an oriented surface is obtained by deleting the outer direction in the first position.

The following proposition shows that these definitions are consistent.

Proposition 8.24 *With the above definitions*

$$\langle \alpha \mid X_1, \ldots, X_m \rangle = \langle \epsilon \mid X_1 \rangle \langle \beta \mid X_2, \ldots, X_n \rangle. \tag{8.213}$$

Proof: The general formula for $\alpha = \epsilon \wedge \beta$ is

$$\langle \alpha \mid X_1, \dots, X_n \rangle = \sum_j (-1)^{j-1} \langle \epsilon \mid X_j \rangle \langle \beta \mid X_1, \dots, \hat{X}_j, \dots, X_k \rangle. \tag{8.214}$$

Since $\langle \epsilon \mid X_j \rangle = 0$ for $j > 1$, only one term survives. $\square$

Consider a transversally oriented k-surface S with transversally oriented $(k-1)$-boundary ∂S. There are two ways to define the orientation of the boundary.

- Consider a basis $Z_1, \dots, Z_{n-k}, X_1, \dots, X_k$ of vector fields, where $X_1, \dots, X_n$ are tangent vectors to S and $X_2, \dots, X_n$ are tangent vectors to ∂X. Suppose that X_1 points outward from ∂S. If S has exterior orientation given by $Z_1, \dots, Z_{n-k}$, then the boundary ∂S has orientation given by $Z_1, \dots, Z_{n-k}, X_1$.

- Consider a k-surface S with $(k-1)$-boundary ∂S. Suppose that it has an external orientation given by a $(n-k)$-form γ whose interior product $X \lrcorner \gamma$ with vectors X tangent to S is zero. (That is, γ is transverse to the tangent spaces.) Suppose ϵ is a 1-form field defined near the boundary that is zero on tangent vectors to ∂S and is strictly positive for a vector pointing outward. Define the $(n-k+1)$-form

  $$\delta = \gamma \wedge \epsilon. \tag{8.215}$$

  This form δ vanishes on ∂S and gives the external orientation for the boundary.

- Summary: The transverse boundary orientation of a transversally oriented surface is obtained by inserting the outer direction in the final position.

The following proposition show the equivalence of these two definitions.

Proposition 8.25 *Suppose the above conditions are satisfied. The only $n-k+1$ element subset of the n-element set $Z_1, \dots, Z_{n-k}, X_1, \dots, X_k$ on which δ is non-zero is $Z_1, \dots, Z_{n-k}, X_1$, and*

$$\langle \delta \mid Z_1, \dots, Z_{n-k}, X_1 \rangle = \langle \gamma \mid Z_1, \dots, Z_{n-k} \rangle \langle \epsilon \mid X_1 \rangle. \tag{8.216}$$

Proof: The left hand side of the equation is the sum over a chosen element of the set $Z_1, \dots, Z_{n-k}, X_1, \dots, X_k$ of the product of ϵ on the chosen element and γ on some $(n-k)$ element subset of the remaining elements. The contributions when the chosen element is from $X_2, \dots, X_k$ is zero from the ϵ factor. On the other hand, if the chosen element is some Z_j, then at least one of the other $X_1, \dots, X_k$ appears in the γ factor, and this also gives zero. The only remaining possibility is that the chosen element is X_1. The only non-zero contribution in that case is when the γ factor has no other X_j. Thus it can only involve $Z_1, \dots, Z_{n-k}$. $\square$

Theorem 8.26 *Suppose a transverse (external) orientation of a surface followed by an (internal) orientation of the same surface determines an orientation of the entire space. Then the transverse (external) orientation of the boundary followed by the (internal) orientation of the boundary determine the same orientation of the whole space.*

Proof: This may be seen via vector fields. If $Z_1, \ldots, Z_{n-k}$ followed by $X_1, \ldots, X_k$ gives an orientation, then $Z_1, \ldots, Z_{n-k}, X_1$ followed by $X_2, \ldots, X_k$ gives the same orientation.

It may also be proved using differential forms. Adopt the notation used in the previous discussion. Thus γ determines the external orientation, while α determines the internal orientation, and $\gamma \wedge \alpha = \sigma$. Suppose that δ is the external orientation of the boundary, with if $\delta = \gamma \wedge \epsilon$. Suppose also that β is the internal orientation of the boundary, so $\alpha = \epsilon \wedge \beta$. Then also $\delta \wedge \beta = \gamma \wedge \epsilon \wedge \beta = \gamma \wedge \alpha = \sigma$. □

A differential form ω has an exterior derivative $d\omega$. For a simple form ω the exterior derivative has a picture in terms of simple forms that resemble boundaries with transverse orientations. Suppose there is a coordinate system $w_1, \ldots, w_{n-k}, y_1, \ldots, y_k$ such that the differential form

$$\omega = s\, dy_1 \cdots dy_k$$

for some never zero scalar-field s. Then ω may be thought of as an external (transverse) orientation for a family of $(n-k)$ surfaces. Let

$$\epsilon = \frac{1}{s} \sum_{j=1}^{n-k} \frac{\partial s}{\partial w_j} dw_j \tag{8.217}$$

be a 1-form in the inner direction.

Proposition 8.27 *A simple differential form may be regarded as defining a transverse orientation of its associated surfaces. If ω is a simple differential form, then its differential $d\omega$ is an exact differential form with*

$$d\omega = \epsilon \wedge \omega. \tag{8.218}$$

- Summary: The transverse boundary orientations associated with the exterior derivative of a simple differential form are obtained by inserting an inner direction form in the first position.

A twisted differential form also has an exterior derivative. For simple twisted forms there is a picture of the exterior derivative in terms of (internally) oriented boundaries. Let ω be as above, and let

$$\eta = s\, dw_1 \cdots dw_{n-k}. \tag{8.219}$$

Write $\sigma = s^2 dw_1 \cdots dw_{n-k} dy_1 \cdots dy_k$. Then

$$\eta \wedge \omega = \sigma. \tag{8.220}$$

The pair

$$(\omega : [\sigma]) = \sigma|_\eta \tag{8.221}$$

defines a twisted form. The internal orientation of the $(n-k)$ surfaces corresponding to this twisted form is given by the form η.

The exterior derivative of this twisted form is

$$(d\omega : [\sigma]) = d\omega|_{\kappa_i} \tag{8.222}$$

where

$$\kappa_i = (-1)^{n-k-i} \frac{\partial s}{\partial w_i}\, dw_1 \cdots dw_{i-1}\, dw_{i+1} \cdots dw_{n-k}. \tag{8.223}$$

Here i must be chosen so that $\partial s/\partial w_i \neq 0$. This works because

$$\kappa_i \wedge d\omega = \left(\frac{1}{s}\frac{\partial s}{\partial w_i}\right)^2 \sigma = p_i \sigma \tag{8.224}$$

where $p_i > 0$. With these notations, it is easy to establish the following proposition.

Proposition 8.28 *Suppose that ω is a simple differential k-form and that $\omega|_\eta$ is the twisted differential k-form obtained by reorienting with the $(n-k)$-form η. Let ϵ be the 1-form in the inner direction defined above. The $(n-k-1)$-form κ_i defined above satisfies*

$$\kappa_i \wedge \epsilon = p_i \eta, \tag{8.225}$$

where $p_i > 0$. The differential of $\omega|_\eta$ is the twisted differential $(k+1)$-form that is $d\omega$ reoriented by κ_i:

$$d(\omega|_\eta) = (d\omega)|_{\kappa_i}. \tag{8.226}$$

To go from the η that reorients ω to the κ_i that reorients $d\omega$, delete $\epsilon_i = \frac{1}{s}\frac{\partial s}{\partial w_i}\, dw_i$ in the final position. The ultimate consequence of these identities is

$$\kappa_i \wedge d\omega = \kappa_i \wedge \epsilon \wedge \omega = p_i \eta \wedge \omega = p_i \sigma. \tag{8.227}$$

- Summary: The boundary reorientations associated with the exterior derivative of a twisted simple differential form are obtained by deleting an inner direction form from the final position.

8.5.6 Stokes' theorem for twisted forms*

This section contrasts Stokes' theorem for forms and twisted forms, and it relates them to each other.

Consider a k-surface S with a 1-form ϵ that vanishes on its boundary ∂S and increases in the outward direction. Let α be a k-form that orients S, and β be a $(k-1)$-form that orients ∂S. The forms α and β are related by $\alpha = \epsilon \wedge \beta$ on the boundary. Stokes' theorem says that for every $(k-1)$-form ω

$$\int_{S,[\alpha]} d\omega = \int_{\partial S,[\beta]} \omega. \tag{8.228}$$

It says that circulation in the interior may be expressed in terms of circulatory flux within the boundary.

Consider a k-surface S with a 1-form ϵ that vanishes on its boundary ∂S and increases in the outward direction. Let γ be a $(n-k)$-form that transversally orients S, and δ be a $(k-1)$-form that transversally orients ∂S. The forms γ and δ are related by $\delta = \gamma \wedge \epsilon$ on the boundary.

Theorem 8.29 (Stokes' theorem: twisted forms) *Consider a compact regular k-surface S with boundary. Suppose the boundary ∂S is also a compact regular k-surface. The 1-form ϵ vanishes on ∂S and increases in the outward direction. Suppose that S has a given transverse orientation given by reorienting form η. Give ∂S the induced transverse orientation with reorienting form $\delta = \gamma \wedge \epsilon$. Then for every twisted $(k-1)$-form $(\omega : [\sigma])$*

$$\int_{S,[\eta]} d(\omega : d[\sigma]) = \int_{\partial S,[\delta]} (\omega : [\sigma]). \tag{8.229}$$

The theorem says that transport flux across the surface may be expressed in terms of an integral around the boundary.

One way to prove Stokes' theorem for twisted forms is to reduce it to the case of forms. If we take $\gamma \wedge \alpha = \sigma$, then the left hand sides are the same in the two theorems. But then it follows that $\delta \wedge \beta = \sigma$, so the two right hand sides are the same.

There is a similar story for singular surfaces. Let M be n-dimensional and Q be k-dimensional with $k \leq n$, and $\chi : Q \to M$. The first step is to characterize the integral of a twisted k-form $(\tau : [\sigma])$ on M over the mapping $\chi : Q \to M$. The mapping must be an immersion and have a given transverse orientation. Suppose that $X_1, \ldots, X_k$ are linearly independent tangent vectors to Q. The transverse orientation of the mapping is given by vector fields $Z_1, \ldots, Z_{n-k}$ along the mapping such that $Z_1, \ldots, Z_{n-k}, \phi_* X_1, \ldots, \phi_* X_k$ are independent. Alternatively, it can be given by a $(n-k)$-form γ along the mapping χ such that $\langle \gamma \mid Z_1, \ldots, Z_{n-k} \rangle > 0$ and γ is zero when any Z_j is replaced by $\phi_* X_i$.

One convenient characterization of the integral is via a reduction of the integral of the twisted form to the integral of a form. Let α be a k-form along the mapping such that $\gamma \wedge \alpha = \sigma$. The formula that gives the reduction is

$$\int_{\chi,[\gamma]} (\tau : [\sigma]) = \int_{\chi,[\chi^* \alpha]} \tau. \tag{8.230}$$

For Stokes' theorem it is necessary to deal with boundaries. The mapping $\chi : Q \to M$ has boundary mapping $\partial\chi : \partial Q \to M$. Near any boundary point of Q take the independent vector fields $X_1, \ldots, X_k$ on Q so that X_1 points outward. The transverse orientation of the mapping is given by vector fields $Z_1, \ldots, Z_{n-k}$ along the mapping such that $Z_1, \ldots, Z_{n-k}, \phi_* X_1, \ldots, \phi_* X_k$ are independent. The transverse orientation of the boundary mapping is given by $Z_1, \ldots Z_{n-k}, \phi_* X_1$.

Another way of giving the transverse orientation is with differential forms along the mapping. Let γ be a differential $(n-k)$- form along the mapping such

that $\langle \gamma \mid Z_1, \ldots, Z_{n-k} \rangle > 0$ and γ is zero when any Z_j is replaced by $\phi_* X_i$. This defines the transverse orientation of χ. Let ϵ be a differential 1-form along the boundary mapping such that $\langle \epsilon \mid \phi_* X_1 \rangle > 0$ and ϵ is zero on all other basis vectors. Let $\delta = \gamma \wedge \epsilon$. Then $\langle \delta \mid Z_1, \ldots, Z_{n-k}, \phi_* X_1 \rangle = \langle \gamma \mid Z_1, \ldots, Z_{n-k} \rangle \langle \epsilon \mid \phi_* X_1 \rangle > 0$, so δ defines the transverse orientation of the boundary mapping $\partial\chi$.

This leads to the twisted form version of Stokes' theorem for singular surfaces; the last chapter in this story of twisted forms.

Theorem 8.30 *Suppose the $(n-k)$-form γ along χ transversally orients χ. This determines an $(n-k+1)$-form $\delta = \gamma \wedge \epsilon$ along $\partial\chi$ that transversally orients $\partial\chi$. Then for every twisted $(k-1)$-form $(\omega : [\sigma])$*

$$\int_{\chi,[\gamma]} d(\omega : [\sigma]) = \int_{\partial\chi,[\delta]} (\omega : [\sigma]). \tag{8.231}$$

Proof: Let α be a k-form along χ with $\gamma \wedge \alpha = \sigma$. Similarly, let β be a $(k-1)$ form along $\partial\chi$ such that $\delta \wedge \beta = \sigma$. Then $\chi^*\alpha$ orients Q, while $(\partial\chi)^*\beta$ orients ∂Q, and in fact is the boundary orientation. Stokes' theorem for forms then says that

$$\int_{\chi,[\phi^*\alpha]} d\omega = \int_{\partial\chi.[(\partial\chi)^*\beta]} \omega. \tag{8.232}$$

The reduction of twisted form integrals to form integrals shows that this equality coincides with the inequality stated in the theorem. □

8.5.7 Forms and twisted forms: summary*

This subsection reviews some of the basic facts about forms and twisted forms. These are illustrated by an elementary example where the geometry is that of a rectangular surface in three dimensional space.

The main idea is the following:

- A differential k-form α is an object that can be integrated over an internally oriented k-dimensional surface.
- A twisted differential k-form $\alpha|_\eta$ is a differential form that has been reoriented by a $(n-k)$-form η so that it can be integrated over an externally (transversely) oriented k-dimensional surface.

Here is the story in more detail:

- A never-zero simple k-form α in n dimensions has a picture in terms of $(n-k)$ surfaces. These surfaces have k dimensional external (transverse) orientations given by α itself.
- A k-form α is integrated over a k-dimensional internally oriented surface.
- A never-zero simple twisted k-form $\alpha|_\eta$ in n-dimensions also has a picture in terms of the same $(n-k)$-surfaces. However in this case the surfaces have internal $(n-k)$ dimensional orientations given by the reorienting $(n-k)$-form η. They have no external orientation, since $(-\alpha)|_\eta = (\alpha)|_\eta$.

- A twisted k-form $\alpha|_\eta$ is integrated over a k-dimensional externally (transversely) oriented surface.

Here are the fundamental operations on these objects that play a role in Stokes' theorem:

- A k-forms α has a differential $d\alpha$ that is a $(k+1)$-forms. The terms in $d\alpha$ are obtained by inserting an inner 1-form in the first position.
- An internally oriented surface S has a boundary ∂S that is internally oriented. The orientation of the boundary is obtained by deleting an outer direction from the first position.
- A twisted k-forms $\alpha|_\eta$ has a differential $d(\alpha|_\eta) = (d\alpha)|_\tau$, where $\tau \wedge d\alpha = \eta \wedge \tau$ (up to a strictly positive scalar multiple). Such a reorienting form τ is obtained by deleting an inner 1-form from the final position.
- An externally (transversely) oriented surface S has a boundary ∂S that is externally oriented. The orientation of the boundary is obtained by inserting an outer direction in the final position.

These prescriptions are best understood in the context of examples. What follows are the simplest non-trivial examples: Stoke's theorem for surfaces with boundary curves in three-dimensional space.

Example: Take $n = 3$ and $\alpha = s\,dy = h(x,y,z)\,dy$. Then α is pictured by planar surfaces with y constant and with external orientations given by $s\,dy$. Then $d\alpha = (1/s)\,ds\,\alpha$ is obtained by inserting an inner form in the first position. Explicitly,

$$d\alpha = \left(\frac{\partial s}{\partial x}\,dx + \frac{\partial s}{\partial z}\,dz\right)dy = \frac{1}{s}\left(\frac{\partial s}{\partial x}\,dx + \frac{\partial s}{\partial z}\,dz\right)\alpha. \tag{8.233}$$

This is pictured by boundary curves in the planes with y constant that satisfy $\frac{\partial s}{\partial x}\,dx + \frac{\partial s}{\partial z}\,dz = 0$.

Let S be a rectangle in a plane with x between a_1 and b_1 and y between a_2 and b_2 and with $z = c$ constant. The orientation is given by $dx\,dy$. Then $\partial S = C_1 + C_2 + C_3 + C_4$, where C_1 is oriented by dx, C_2 is oriented by dy, C_3 is oriented by $-dx$, and C_4 is oriented by $-dy$. (These orientations are obtained by deleting $-dy, dx, dy, -dx$ from the first position in $dx\,dy = -dy\,dx$.) See Figure 8.7.

Then Stokes' theorem says that

$$\int_{S,dx\,dy} \frac{\partial h(x,y,c)}{\partial x}\,dx\,dy = \int_{C_2,dy} h(b_1,y,c)\,dy + \int_{C_4,-dy} h(a_1,y,c)\,dy. \tag{8.234}$$

The mismatch of orientations in the last term gives a minus sign. This formula says that the oriented sources in the interior of S correspond to the circulation due to change along the boundary. ||

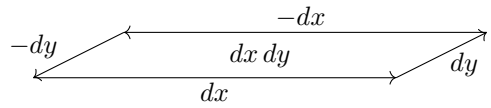

Figure 8.7: Internally oriented surface (3d)

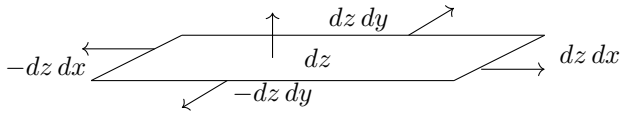

Figure 8.8: Externally oriented surface (3d)

Example: Take $n = 3$ and $\alpha = s\,dy = h(x, y, z)\,dy$, reoriented by $\eta = dz\,dx$. Then $\alpha|_\eta$ is pictured by planes with y constant and with internal orientation given by $dz\,dx$. The differential is

$$d(\alpha|_\eta) = (d\alpha)|_\tau = \left(\frac{\partial s}{\partial x}\,dx\,dy + \frac{\partial s}{\partial z}\,dz\,dy\right)|_\tau. \tag{8.235}$$

Here one can take $\tau = \frac{1}{s}\frac{\partial s}{\partial x}\,dz$ or $\tau = -\frac{1}{s}\frac{\partial s}{\partial z}\,dx$. Either of these satisfies $\tau \wedge d\alpha = \eta \wedge \alpha$ (up to a positive multiple), provided that the relevant partial derivative is not zero.

Either of these expressions for τ may be obtained by deletion of an inner 1-form of some sort from the last term of $\eta = dz\,dx = -dx\,dz$. For instance, if $\partial s/\partial x \neq 0$, then $dz\,dx$ gives the same orientation as $\frac{1}{s}\frac{\partial s}{\partial x}\,dz$ times $\frac{1}{s}\frac{\partial s}{\partial x}\,dx$. Deleting the final factor gives one expression for τ. The same type of computation with $-dx\,dz$ gives the other expression for τ.

Let S be a rectangle in a plane with x between a_1 and b_1 and y between a_2 and b_2 and with $z = c$ constant. The external orientation is given by dz. Then $\partial S = C_1 + C_2 + C_3 + C_4$, where C_1 is externally oriented by $dy\,dz$, C_2 is externally oriented by $dz\,dx$, C_3 is externally oriented by $dy\,dz$, and C_4 is externally oriented by $dx\,dz$. (These orientations are obtained by taking the product of dz with $-dy, dx, dy, -dx$ in the final position.) See Figure 8.8.

Then Stokes' theorem says that with $s = h(x, y, z)$

$$\int_{S,dz} \left(\frac{\partial h(x,y,z)}{\partial x}\,dx\,dy\right)\Big|_{\frac{1}{s}\frac{\partial s}{\partial x}\,dz} dx\,dy =$$

$$\int_{C_2,dz\,dx} (h(x, y, z)\,dy)|_{dz\,dx} + \int_{C_4,dx\,dz} (h(x, y, z)\,dy)|_{dz\,dx}. \tag{8.236}$$

The mismatch of orientations in the last term gives a minus sign. This formula says that the flux through the boundary corresponds to the contribution from the boundary with signs coming from matching orientations.

The formula above may be written even more explicitly by pulling back from from three dimensional space to the plane with its boundary curve. Then

$$\int_{S,+} \left(\frac{\partial h(x,y,c)}{\partial x} \, dx \, dy \right) \Big|_{+} = \int_{C_2,+} (h(b_1,y,c)\, dy)|_{+} - \int_{C_4,+} (h(a_1,y,c)\, dy)|_{+}. \tag{8.237}$$

The integrals are the usual (unoriented) integrals from calculus. ||

Remark: The twisted version of Stokes's theorem for a one-form in three dimensions can occur in practice. For example, if H is the twisted one-form representing a static magnetic field, then $dH = J$ is the twisted two-form representing current. This current has flux curves with internal orientation, since current flows in some definite direction. Furthermore, J must be integrated over an externally oriented surface, and it is this integral that gives the corresponding integral of H around the externally oriented boundary curve. ||

8.5.8 The fundamental theorem of calculus: two versions*

Even in elementary calculus we tend to make the distinction between two kinds of integrals. Let us look at the case of a 1-dimensional manifold patch M and a segment C in M between two points. Suppose that u is a coordinate system on M. Let $F(u)$ be a scalar field on M. Then $dF(u) = f(u)\, du$, where $f(u) = F'(u)$. This is a 1-form on M.

One version of the fundamental version of calculus requires that C be oriented. The theorem then states

$$\int_C f(u)\, du = F(u_{\text{final}}) - F(u_{\text{initial}}). \tag{8.238}$$

Here the initial and final values are at the points determined by the orientation of C. This is the version that works as nicely as possible for calculations in calculus. In this case, if v is another coordinate with $u = g(v)$, then

$$\int_C f(u)\, du = \int_C f(g(v))g'(v)\, dv. \tag{8.239}$$

The other version works when C is not oriented. In this case, the integrand is a twisted 1-form on M. In our notation $(du)|_+$ denotes the twisted form that is du reoriented to be positive. The theorem is

$$\int_C f(u)(du)|_{+} = F(u_{\max}) - F(u_{\min}). \tag{8.240}$$

As consequence, if $f(u) \geq 0$, then the integral is positive. This is the version that makes sense for calculations involving length, mass, probability, and so on. In this case, if v is another coordinate with $u = g(v)$, then

$$\int_C f(u)(du)|_{+} = \int_C f(g(v))|g'(v)|(dv)|_{+}. \tag{8.241}$$

The above observations are all that are needed for the mechanics of calculus. But it is illuminating to be more explicit about the twisted form version. The twisted form in question is the twisted scalar field $(F(u) : du)$ with differential

$$d(F(u) : du) = (d(F(u)) :, du) = (f(u)\, du : du) = f(u)(du : du) = f(u)(du)|_+. \tag{8.242}$$

One can also think of the twisted scalar field as an unsigned field $|F(u)|$ reoriented by $\mathrm{sign}(F(u))\, du$.

Similarly, the endpoints of C each have external orientation pointing outward. Let u_* be the value of u at one of these endpoints. The relative sign of the orientation is $\mathrm{sign}\, F(u_*)$ times the relative sign of du and the outward orientation. The contribution from an endpoint is thus $F(u_*)\,\mathrm{sign}(F(u_*) = F(u_*)$ times the relative sign of du and the outward orientation. But this relative sign is $+$ when $u_* = u_{\max}$ and $-$ when $u_* = u_{\min}$.

Problems

The divergence theorem

1. Let $r^2 = x^2 + y^2 + z^2$, and let

$$X = \frac{1}{r^3}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right). \tag{8.243}$$

   Let $\mathrm{vol} = dx\, dy\, dz$. Show that the *solid angle form*

$$\sigma = X \lrcorner \mathrm{vol} = \frac{1}{r^3}(x\, dy\, dz + y\, dz\, dx + z\, dx\, dy). \tag{8.244}$$

2. In the preceding problem, show directly that $d\sigma = 0$ away from $r = 0$.

3. Find σ in spherical polar coordinates. Hint: This can be done by blind computation, but there is a better way. Express $\mathbf{v}$ in spherical polar coordinates, using Euler's theorem

$$r\frac{\partial}{\partial r} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}. \tag{8.245}$$

   Then use $\mathrm{vol} = r^2 \sin(\theta)\, dr\, d\theta\, d\phi$ to calculate $\sigma = X \lrcorner \mathrm{vol}$.

4. In the preceding problem, show that $d\sigma = 0$ away from $r = 0$ by a spherical polar coordinate calculation.

5. Let S be the sphere of radius $a > 0$ centered at the origin. Calculate the integral of σ over S.

6. Let Q be the six-sided cube with side lengths $2L$ centered at the origin. Calculate the integral of σ over Q. Prove that your answer is correct. Hint: Given the result of the previous problem, this should be effortless.

Part III

Metric Geometry

Chapter 9

Prelude: Affine and Euclidean Space

9.1 Affine and Euclidean geometry*

9.1.1 Affine geometry*

Up to this point the emphasis has been on the geometry of featureless points, that is, the geometry of a manifold patch. This very general geometry is fundamental to mathematical modeling. It is remarkable that there is a concept of vector field that works in this setting.

An affine space is a much more specific setting. There is (up to isomorphism) only one such space A for each n. It is a space with points and straight lines and flat planes. To a reasonable approximation, we live in a three-dimensional affine space.

An n-dimensional *affine space* is a non-empty set A together with an n-dimensional vector space V that acts on A freely and transitively.

To say that V acts on A means that for each P in A and X in V there is a point $P + X$ in A. The requirements on this operation are:

- $P + 0 = P$,
- $P + (X + Y) = (P + X) + Y$.

For an affine space the vector space must act freely and transitively.

- To say that V acts freely on A means that $P+X = P+Y$ implies $X = Y$. In other words, for every pair P, Q there is at most one vector X with $P + X = Q$.
- To say that V acts transitively on A means that for every pair P, Q there is at least one X with $P + X = Q$.

A vector Y in V is called a *translation vector*. Sometimes a translation vector is called a *displacement vector*. The unique translation vector that takes P to Q is denoted $Q - P$.

Each translation vector Y determines a *translation* $\Phi^Y : A \to A$ that sends P to $P + Y$.

Each point P in A determines a mapping $F_P : V \to A$ that sends Y to $P + Y$. In this context each Y is called a *position vector* with origin P and is used to determine a corresponding point in A.

Remark: There are many related concepts of vector. An ordered pair P, Q of points in A is called a *bound vector* at P. A bound vector is usually pictured as an arrow from P to Q. Two bound vectors P, Q and R, S are said to be equivalent if $Q - P = S - R$. (An equivalent characterization in terms of affine combinations is $Q+R-P = S$. This says that S is the vertex of a parallelogram whose other vertices are P, Q, R.) The collection of all bound vectors equivalent to P, Q is called the *free vector* defined by P, Q. The free vector defined by P, Q is the graph of the translation mapping defined by $Q - P$. The space of free vectors is isomorphic to the space of translations, which in turn is isomorphic to the original space V of translation vectors. ||

There is also a useful notion of *velocity vector*. This is a vector X that when multiplied by a time t gives a displacement vector tX. The vector space of velocity vectors will be denoted $\dot{V}$. If X is in $\dot{V}$, then $\Phi_t^X : A \to A$ is the mapping that sends P to $P + tX$.

The fundamental operation on an affine space A is *affine combination*. For two points P, Q in A and for real numbers a, b with $a + b = 1$, there is a unique point in A called $aP + bQ$. This may be define either as

$$aP + bQ = P + b(Q - P) \tag{9.1}$$

or as

$$aP + bQ = Q + a(P - Q). \tag{9.2}$$

These are the same, since $P + b(Q - P) = P + (1 - a)(Q - P) = P + (Q - P) - a(Q - P) = Q + a(P - Q)$.

Similarly, for three points P, Q, R in A and for $a + b + c = 1$ there is a unique point $aP + bQ + cR$ in A. One possible definition is $aP + bQ + cR = P + b(Q - P) + c(R - P)$.

The most general structure of affine combination is the following. Let $P_1, \ldots, P_k$ be points in A. Let $c_1, \ldots, c_k$ be real numbers such that $c_1 + \cdots + c_k = 1$. Then the affine combination $c_1P_1 + \cdots + c_kP_k$ is a well-defined point in A. There is a beautiful axiomatic description of affine space based entirely on affine combinations of points [28, Appendix: Affine and Projective Spaces].

For two points P, Q the affine combinations are $aP + bQ$ with $a + b = 1$. If $P \neq Q$ these determine the line $\{aP + bQ \mid p + q = 1\}$ through P and Q. Similarly, for three points P, Q the affine combinations are $aP + bQ + cR$ with $a + b + c = 1$. If $P \neq Q$ and R is not on the line through P, Q, then these determine the plane $\{aP + bP + cQ \mid a + b + c = 1\}$ through P, Q, and R.

The space $\mathbf{R}^n$ is an affine space with the natural notion of affine combination. An affine space may have *affine coordinates* $\mathbf{x} : A \to \mathbf{R}^n$. This is a one-to-one transformation of A onto $\mathbf{R}^n$ that preserves affine combinations.

An affine coordinate system is not uniques. Two such systems are related in the following way. Say that $\mathbf{x} : A \to \mathbf{R}^n$ is an affine coordinate system. Let L be a constant invertible n by n real matrix, and let $\mathbf{b}$ be a constant in $\mathbf{R}^n$. Then $\mathbf{x}' = L\mathbf{x} + \mathbf{b} : A \to \mathbf{R}^n$ is another affine coordinate system. All affine coordinate systems are related in this way. In particular, $\mathbf{x} = L^{-1}\mathbf{x}' - L^{-1}\mathbf{b}$.

Consider affine coordinates $\mathbf{x}$ on A. A translation vector Y in V may be represented in terms of coordinates $\mathbf{c}$ on V by

$$Y = \sum_j c_j \frac{\partial}{\partial x_j}. \tag{9.3}$$

The corresponding translation is

$$\Phi^Y = (\mathbf{x} \leftarrow \mathbf{x} + \mathbf{c}). \tag{9.4}$$

In the same way a velocity vector X in $\dot{V}$ may be represented in terms of coordinates $\mathbf{a}$ on $\dot{V}$ by

$$X = \sum_j a_j \frac{\partial}{\partial x_j}. \tag{9.5}$$

The corresponding translation flow is

$$\Phi_t^X = (\mathbf{x} \leftarrow \mathbf{x} + t\mathbf{a}). \tag{9.6}$$

9.1.2 Euclidean geometry*

A *Euclidean space* is an affine space E such that the vector space V of translation vectors has an inner product. The distance between points P, Q in E is the length $|Q - P|$ of the vector $Q - P$ in V. To a reasonable approximation we live in a three-dimensional Euclidean space.

A velocity vector X for a Euclidean space also has a length. This length is called *speed*.

The space $\mathbf{R}^n$ may be regarded as a Euclidean space. A Euclidean space E has *Cartesian coordinates*. A Cartesian coordinate system is a function $\mathbf{x} : E \to \mathbf{R}^n$ that is one-to-one and preserves affine combinations and length.

Two Cartesian coordinate systems are related as as follows. Say that $\mathbf{u} : E \to \mathbf{R}^n$ is a Cartesian coordinate system. Let O be a constant orthogonal n by n real matrix, and let $\mathbf{b}$ be a constant in $\mathbf{R}^n$. Then $\mathbf{x} = O\mathbf{u} + \mathbf{b} : M \to \mathbf{R}^n$ is another Cartesian coordinate system, and all Cartesian coordinate systems are related in this way. In particular, $\mathbf{u} = O^{-1}\mathbf{x} - O^{-1}\mathbf{b}$.

For two points the coordinates are related by $\mathbf{x}_2 - \mathbf{x}_1 = O(\mathbf{u}_2 - \mathbf{u}_1)$. Since O is orthogonal, the distance given by the metric is $|\mathbf{x}_2 - \mathbf{x}_1| = |\mathbf{u}_2 - \mathbf{u}_1|$, the same in either coordinate system.

9.1.3 Affine space derivatives*

Consider an affine space A with associated vector space V of translation vectors. An *vector field on affine space* is a smooth function $Y : A \to V$. For each point P in A there is a corresponding translation vector Y_P. The vector field is pictured as giving an arrow at each point of A.

Sometimes the vector field is instead considered as a velocity field. This is a smooth function $X : A \to \dot{V}$. Each X_P is a velocity vector; it needs to be multiplied by a time value t to produce a translation vector. .

A scalar field is a smooth real function s defined on A. Given a velocity vector field X, the derivative of s along X is another scalar field $X \lrcorner \nabla s$ whose value at P is

$$(X \lrcorner \nabla s)_P = \lim_{t \to 0} \frac{s_{P+tX_P} - s_P}{t}. \tag{9.7}$$

This is a derivative of s at P in the direction X_P. Thus an affine vector field X gives a way of differentiating a scalar field s to get another scalar field. Sometimes it is convenient to think of X itself as the differential operator; then one can write $X \lrcorner \nabla s$ as $X\, s$.

Consider a vector field Y, a smooth function defined on A with values in V (translation vectors) or in $\dot{V}$ (velocity vectors). Given a velocity vector field X, there is another vector field $X \lrcorner \nabla Y$ whose value at P is

$$(X \lrcorner \nabla Y)_P = \lim_{t \to 0} \frac{Y_{P+tX_P} - Y_P}{t}. \tag{9.8}$$

This is a derivative of Y at P in the direction X_P. Thus an affine vector field X gives a way of differentiating an affine vector field Y to get another vector field $X \lrcorner \nabla Y$. Again this will sometimes be abbreviated as XY. In this notation X is regarded as a differential operator, while Y is simply a vector field on which this operator acts.

In the affine space setting it makes sense to talk of a *constant vector field*. If Y is a constant vector field, then $X \lrcorner \nabla Y = 0$.

These definition are special to the case of an affine space (or Euclidean space). The reason is that it relies on the notion of translation vector, which is not available in more general situations. Even though Y_{P+tX_P} is a translation vector associated with $P + tX_P$, it may be compared to the translation vector Y_P associated with P.

In the general setting of a manifold patch there is no obvious notion of differentiating vector fields. The problem is that vectors at two different points are not in the same vector space, so there is no way of forming their difference. We shall see that this difficulty can be overcome in the case when there is a metric tensor structure, but this is not an elementary matter.

Proposition 9.1 (Derivative of a scalar field on affine space) *Take* $\mathbf{x}$ *as affine coordintes. Let* $X = \sum_i g_i(\mathbf{x})\partial/\partial x_i$. *Suppose* $s = h(\mathbf{x})$. *Then the affine derivative in affine coordintes is*

$$X \lrcorner \nabla s = \sum_i g_i(\mathbf{x}) \frac{\partial h(\mathbf{x})}{\partial x_i}. \tag{9.9}$$

Proof: The definition of the derivative is

$$X \lrcorner \nabla s = \lim_{t \to 0} \sum_j \frac{h(\mathbf{x} + t\mathbf{g}(\mathbf{x})) - h(\mathbf{x})}{t} = \sum_i g_i(\mathbf{x}) h'_{,i}(\mathbf{x}). \tag{9.10}$$

This is equivalent to the statement of the theorem. □

Corollary 9.2 *The affine derivative*$X \lrcorner \nabla s$ *is the same as the usual derivative* $X \lrcorner ds = \langle ds \mid X \rangle$ *of a scalar* s *along a vector* X. *It is given by the above formula in an arbitrary coordinate system.*

Consider a vector field

$$Y = \sum_j h_j(\mathbf{x}) \frac{\partial}{\partial x_j} \tag{9.11}$$

expressed in affine coordinates. The coefficients $h_j(\mathbf{x})$ can be complicated functions. However each basis vector field $\frac{\partial}{\partial x_j}$ is a constant vector field.

Proposition 9.3 (Derivative of a vector field on affine space) *Take* $\mathbf{x}$ *as affine coordinates. Let* $X = \sum_i g_i(\mathbf{x})\partial/\partial x_i$ *and* $Y = \sum_j h_j(\mathbf{x})\partial/\partial x_j$*. Then the affine derivative in affine coordinates is*

$$X \lrcorner \nabla Y = \sum_i \sum_j g_i(\mathbf{x}) \frac{\partial h_j(\mathbf{x})}{\partial x_i} \frac{\partial}{\partial x_j}. \tag{9.12}$$

Proof: Since the basis vectors in affine coordinates are constant vector fields, the derivative is

$$X \lrcorner \nabla Y = \lim_{t \to 0} \sum_j \frac{h_j(\mathbf{x} + t\mathbf{g}(\mathbf{x})) - h_j(\mathbf{x})}{t} \frac{\partial}{\partial x_j} = \sum_j \sum_i g_i(\mathbf{x}) h'_{j,i}(\mathbf{x}) \frac{\partial}{\partial x_j}. \tag{9.13}$$

□

The operation $X \lrcorner \nabla Y$ is defined for arbitrary vector fields X and Y. This proposition gives a simple description of this in affine coordinates. The following discussion describes the considerably more complicated formulas that work for arbitrary coordinates.

9.1.4 The covariant derivative*

Using affine coordinates on an affine space makes calculations simple. However it is often useful to use other coordinate systems. In that case, the coordinate basis vectors vary as one moves from point to point in the affine space. One way to do the calculations in this case is to do the calculations in affine coordinates and then make the change of variable to the new coordinates. The convention in the following is that the partial derivatives on the left represent affine covariant derivatives, while the partial derivatives on the right merely represent basis vectors.

Example: Polar coordinates give a nice example. In this case the affine coordinates are Cartesian coordinates x, y. We have

$$\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) = -\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right). \tag{9.14}$$

This says that

$$\frac{\partial}{\partial \theta}\,\frac{\partial}{\partial \theta} = -r\frac{\partial}{\partial r} \tag{9.15}$$

Similarly

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right). \tag{9.16}$$

This says that

$$\frac{\partial}{\partial r}\,\frac{\partial}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial \theta} \tag{9.17}$$

These are intuitive if one thinks of the polar coordinate basis vectors as changing direction as they move from point to point.

This kind of calculation gives the general result for polar coordinates

$$\begin{aligned}\frac{\partial}{\partial r}\frac{\partial}{\partial r} &= 0\\ \frac{\partial}{\partial r}\frac{\partial}{\partial \theta} &= \frac{1}{r}\frac{\partial}{\partial \theta}\\ \frac{\partial}{\partial \theta}\frac{\partial}{\partial r} &= \frac{1}{r}\frac{\partial}{\partial \theta}\\ \frac{\partial}{\partial \theta}\frac{\partial}{\partial \theta} &= -r\frac{\partial}{\partial r}.\end{aligned} \tag{9.18}$$

These are the correct formulas for differentiating coordinate basis vectors. There is a symmetry property, reflected in this case by $(\partial/\partial r)(\partial/\partial\theta) = (\partial/\partial\theta)(\partial/\partial r)$.

In many circumstances it is more natural to use normalized basis vectors. For polar coordinates these are $\partial/\partial r$ and $(1/r)\partial/\partial\theta$. A simple calculation gives the result

$$\begin{aligned}\frac{\partial}{\partial r}\frac{\partial}{\partial r} &= 0\\ \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial \theta} &= 0\\ \frac{\partial}{\partial \theta}\frac{\partial}{\partial r} &= \frac{1}{r}\frac{\partial}{\partial \theta}\\ \frac{\partial}{\partial \theta}\frac{1}{r}\frac{\partial}{\partial \theta} &= -\frac{\partial}{\partial r}.\end{aligned} \tag{9.19}$$

These equation are geometrically natural and intuitive. Notice, however, that the symmetry property for coordinate bases is no longer evident. ||

There is a general theory of calculation involving non-constant basis vectors associated with an affine space. Sometimes this is called the *covariant derivative*. It uses rather complicated formulas for the coefficients called *Christoffel symbols*. For the polar coordinate example these appear as the coefficients $-r$ and $1/r$ in the expression in terms of coordinate basis vectors. In general, the Chrisfoffel symbols are the coordinate experssions of the covariant derivative.

Theorem 9.4 *Consider an affine space with affine coordinates* $\mathbf{x}$*. Let* $\mathbf{u}$ *be an arbitrary coordinate system for the affine space. Then*

$$\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j} = \sum_k \Gamma^k_{ij}\frac{\partial}{\partial u_k}, \tag{9.20}$$

where the Christoffel symbol Γ^k_{ij} *is given by*

$$\Gamma^k_{ij} = \sum_{\ell=1}^{n} \frac{\partial^2 x_\ell}{\partial u_i \partial u_j}\frac{\partial u_k}{\partial x_\ell}. \tag{9.21}$$

Proof: The left hand side is

$$\left(\sum_p \frac{\partial x_p}{\partial u_i}\frac{\partial}{\partial x_p}\right)\left(\sum_\ell \frac{\partial x_\ell}{\partial u_j}\frac{\partial}{\partial x_\ell}\right) = \sum_\ell\sum_p \frac{\partial^2 x_\ell}{\partial x_p \partial u_j}\frac{\partial x_p}{\partial u_i}\frac{\partial}{\partial x_\ell}. \tag{9.22}$$

This is

$$\sum_\ell \frac{\partial^2 x_\ell}{\partial u_i \partial u_j} \frac{\partial}{\partial x_\ell} = \sum_k \sum_\ell \frac{\partial^2 x_\ell}{\partial u_i \partial u_j} \frac{\partial u_k}{\partial x_\ell} \frac{\partial}{\partial u_k}. \tag{9.23}$$

This translates to the result in the theorem. □

This result has an immediate corollary, obtained by switching the roles of j and k in the second term. This is the way the formulas appear in most computations.

Corollary 9.5 *Suppose that*

$$Y = \sum_j b_j \frac{\partial}{\partial u_j} \tag{9.24}$$

is a vector field. Then

$$\frac{\partial}{\partial u_i} Y = \sum_j \left(\frac{\partial}{\partial u_i} b_j + \sum_k \Gamma^j_{ik} b_k \right) \frac{\partial}{\partial u_j}. \tag{9.25}$$

In the same way it is possible to apply a general vector field X acting as differentiating the vector field Y. The vector field XY is thought of as the result of differentiating the vector field Y in the direction of X. All the differentiating is done by X; the partial derivatives in Y are merely acting as basis vectors. This is not the same as applying Y to a scalar field and then applying X to the result. This would certainly define a transformation XY, but XY defined this way is not a vector field.

Remark: The covariant derivative works for affine space, but it extends to a much more general setting, a space equipped with a metric tensor. The metric tensor concept will be explored in later chapters. In this case the Christoffel symbols need not be given by the formula in the theorem of this subsection. Instead they are derived from the metric tensor by a moderately complicated formula. ||

9.1.5 The covariant differential*

The *covariant differential*

$$\nabla = \sum_i du_i \frac{\partial}{\partial u_i} \tag{9.26}$$

is an important mathematical object. In the expression the du_i on the left are basis forms, while the $\partial/\partial u_i$ on the right are covariant derivatives.

The covariant differential ∇ may be applied to a scalar field, in which case it is the same as the differential d. After that the most important operation with the covariant differential is to apply it to a vector field.

Theorem 9.6 (Covariant differential acting on vector field) *Consider a vector field*

$$Y = \sum_j b_j \frac{\partial}{\partial u_j}. \tag{9.27}$$

Its covariant differential of Y is a vector-valued form

$$\nabla Y = \sum_i du_i \frac{\partial}{\partial u_i} Y = \sum_i \sum_j du_i \left(\frac{\partial b_j}{\partial u_i} + \sum_k \Gamma^j_{ik} b_k \right) \frac{\partial}{\partial u_j}. \tag{9.28}$$

This is in an arbitrary coordinate system, but the Christoffel symbols may be quite complicated. For affine coordinates they are zero.

One can think of the coefficients of the covariant differential to be a matrix indexed by row index j and column index i. The natural thing to do with such a matrix is to multiply it by a column vector. Say

$$X = \sum_\ell a_\ell \frac{\partial}{\partial u_\ell}. \tag{9.29}$$

Then

$$X \lrcorner \nabla Y = \sum_i \sum_j a_i \left(\frac{\partial b_j}{\partial u_i} + \sum_k \Gamma^j_{ik} b_k \right) \frac{\partial}{\partial u_j}. \tag{9.30}$$

This is another vector field. It is describing how Y is changing along X. Other names for it are $\nabla_X Y$ amd the somewhat dangerous XY. In applied mathematics this has various names, such as *advective derivative*.

If s is a scalar field, then $\nabla s = ds$ is the usual differential. So ∇ has a dual role; it can act both on scalar and vector fields. It can be used to define yet another important quantity div $Y = \nabla \mid Y$. The divergence $\nabla \mid Y$ is the result of replacing $du_i(\partial/\partial u_j)$ by δ_{ij} in ∇Y. Only the diagonal elements with $i = j$ survive, so this is the trace of a matrix.

Theorem 9.7 *The divergence of a vector field is a scalar field given by*

$$\text{div } Y = \nabla \mid Y = \sum_j \left(\frac{\partial b_j}{\partial u_j} + \sum_k \Gamma^j_{jk} b_k \right). \tag{9.31}$$

Example: Here is the covariant differential in polar coordinates. In this case it is convenient to use normalized bases. Let

$$Y = b_1 \frac{\partial}{\partial r} + b_2 \frac{1}{r} \frac{\partial}{\partial \theta}. \tag{9.32}$$

Then

$$\nabla Y = \left(dr \frac{\partial}{\partial r} + d\theta \frac{\partial}{\partial \theta} \right) \left(b_1 \frac{\partial}{\partial r} + b_2 \frac{1}{r} \frac{\partial}{\partial \theta} \right). \tag{9.33}$$

This may be computed using covariant derivatives for polar coordinates. The result is that

$$\begin{aligned} \nabla Y = {} & dr \left(\frac{\partial b_1}{\partial r} \frac{\partial}{\partial r} + \frac{\partial b_2}{\partial r} \frac{1}{r} \frac{\partial}{\partial \theta} \right) \\ & + d\theta \left(\left(\frac{\partial b_1}{\partial \theta} - b_2 \right) \frac{\partial}{\partial r} + \left(\frac{\partial b_2}{\partial \theta} + b_1 \right) \frac{1}{r} \frac{\partial}{\partial \theta} \right). \end{aligned} \tag{9.34}$$

||

Example: Here is the divergence in polar coordinates. One way to get it is from the covariant differential. The only terms that contribute are the ones with $dr(\partial/\partial r)$ and $d\theta(\partial/\partial\theta)$. So

$$\nabla \mid Y = \frac{\partial b_1}{\partial r} + \frac{1}{r}\left(\frac{\partial b_2}{\partial \theta} + b_1\right) = \frac{1}{r}\frac{\partial (r b_1)}{\partial r} + \frac{1}{r}\frac{\partial b_2}{\partial \theta}. \tag{9.35}$$

An alternate method is to realize that the only terms that contribute are

$$\nabla \mid Y = \frac{\partial b_1}{\partial r} + \frac{1}{r}\frac{\partial b_2}{\partial \theta} + \Gamma^2_{21} b_1 \tag{9.36}$$

and use $\Gamma^2_{21} = 1/r$. ||

Remark: Many elementary treatments of vector calculus have an object called $\overrightarrow{\nabla}$ that plays an important role. It is usually described as a vector whose components are differential operators. This object is closely related to the covariant differential, which is a form whose components are differential operators. It will be discussed in a later chapter. To distinguish the two objects, we use ∇ for the covariant differential discussed here, and $\overrightarrow{\nabla} = \mathsf{g}^{-1}\nabla$ for the corresponding vector object. It will turn out that both these objects are defined in the more general situation when the covariant derivatives are determined by a metric tensor.

The usual name for an operator like ∇ or $\overrightarrow{\nabla}$ is *del*. The symbol ∇ is called *nabla*, a Greek word referring to a Phoenician harp. Since the mathematics of wave motion (including music) makes heavy use of ∇, it is fitting that the symbol has the shape of a harp. ||

9.1.6 The covariant differential and divergence*

The covariant differential ∇ applies to vector fields and differential forms defined on affine space. The covariant differential of a vector field Y is ∇Y. If Y has components b_j, then ∇Y is given by a matrix $\partial b_j/\partial u_i + \sum_k \Gamma^j_{ik} b_k$. The divergence of Y is defined as $\nabla \mid Y$, the trace of this matrix.

This section explores the relation between the definition of divergence using the covariant differential and the definition of divergence involving the volume form.

For every affine space there is a volume form given in affine coordinates $x_1, \ldots, x_n$ by $\mathrm{vol} = c\,dx_1 \cdots dx_n$. Here the reference orientation is $[dx_1 \cdots dx_n]$, and $c > 0$. The constant c is not determined by the affine structure. In the special situation of Euclidean space and Cartesian coordinates the natural choice is $c = 1$.

The following theorem shows how to take the covariant derivative of a differential form. For now it may be regarded as the definition of this covariant derivative. The proof is then a justification for this definition.

Theorem 9.8 (Covariant derivative of basis form) *The covariant derivative applied to a coordinate basis form du_j is given by*

$$\frac{\partial}{\partial u_i} du_m = -\sum_j \Gamma_{ij}^m \, du_j \tag{9.37}$$

Proof: Start with

$$\left\langle du_m \mid \frac{\partial}{\partial u_j} \right\rangle = \delta_{mj}. \tag{9.38}$$

Since this is constant, applying the covariant derivative $\partial/\partial u_i$ gives zero. The product rule for differentiation leads to

$$0 = \left\langle \frac{\partial}{\partial u_i} du_m \mid \frac{\partial}{\partial u_j} \right\rangle + \left\langle du_m \mid \sum_k \Gamma_{ij}^k \frac{\partial}{\partial u_j} \right\rangle. \tag{9.39}$$

It follows that

$$0 = \left\langle \frac{\partial}{\partial u_i} du_m \mid \frac{\partial}{\partial u_j} \right\rangle + \Gamma_{ij}^m. \tag{9.40}$$

□

Theorem 9.9 (Volume coefficient identity) *Consider a volume form* $\mathrm{vol} = w \, du_1 \cdots du_n$. *The identity*

$$\nabla \mathrm{vol} = 0. \tag{9.41}$$

is equivalent to

$$\frac{\partial w}{\partial u_i} = w \sum_k \Gamma_{ik}^k. \tag{9.42}$$

Proof: Compute

$$0 = \frac{\partial}{\partial u_i}(w \, du_1 \cdots du_n) = \frac{\partial w}{\partial u_i} \, du_1 \cdots du_n - w \sum_k \left(du_1 \cdots du_{k-1} \left(\sum_\ell \Gamma_{i\ell}^k \, dw_\ell \right) du_{k+1} \cdots du_n \right) \tag{9.43}$$

The only term that survives in the ℓ sum is $\ell = k$. The result is

$$0 = \frac{\partial w}{\partial u_i} \, du_1 \cdots du_n - w \sum_k \left(du_1 \cdots du_{k-1} \left(\Gamma_{ik}^k \, du_k \right) du_{k+1} \cdots du_n \right). \tag{9.44}$$

□

Theorem 9.10 (Covariant differential expression for divergence) *Consider* $\mathrm{vol} = w \, du_1 \cdots du_n$ *with* $\nabla \mathrm{vol} = 0$. *The covariant differential expression for the divergence of a vector field*

$$Y = \sum_j b_j \frac{\partial}{\partial u_j} \tag{9.45}$$

is

$$\nabla \mid Y = \sum_j \left(\frac{\partial b_j}{\partial u_j} + \frac{1}{w} \frac{\partial w}{\partial u_j} b_j \right) = \sum_j \frac{1}{w} \frac{\partial (w b_j)}{\partial u_j}. \tag{9.46}$$

Proof: This follows from the expression for $\nabla \mid Y$ in terms of Γ^k_{ik} coefficients and the volume coefficient identity. The right hand side is the usual coordinate expression for the divergence in terms of the volume form. $\square$

Remark: The formulas of this section are valid not only for affine space, but for any space equipped with a metric tensor. The metric tensor determines a volume form, and it will turn out that the volume coefficient identity is satisfied. In particular, the divergence defined by the volume form is the same as the divergence defined in terms of the covariant differential. These issues will be explored in detail in later chapters. ||

9.1.7 Fluid dynamics*

The Euler equations of fluid dynamics make sense in Euclidean space. If X is the fluid velocity, ρ is the density, and p is the pressure, then the law of conservation of mass is

$$\nabla \mid (\rho X) = 0. \tag{9.47}$$

The law of motion is then

$$\rho \left(\frac{\partial}{\partial t} X + X \lrcorner \nabla X \right) = - \operatorname{grad} p. \tag{9.48}$$

In Cartesian coordinates the gradient is

$$\operatorname{grad} p = \sum_j \frac{\partial p}{\partial x_i} \frac{\partial}{\partial x_i}. \tag{9.49}$$

In other coordinate systems it is more complicated. There should be another equation relating p and ρ. See [11] for a discussion.

The dimensional analysis of this is interesting. The ∇ operator is dimensionless. In three dimensions the dimension of ρ is ML^{-3}. The velocity X has dimension T^{-1}. The left hand side of the first equation has dimensions $ML^{-3}T^{-1}$; The dimension of ρX vol is MT^{-1}. The transport flux associated with conservation of mass is $\rho X \lrcorner$ vol which also has dimensions MT^{-1}. The law of conservation of mass says that for every boundary region the flux in is equal to the flux out. This is measured in mass per time.

The acceleration term $(\partial/\partial t)X + X \lrcorner \nabla X$ has dimensions T^{-2}. The left hand side thus has dimensions $ML^{-3}T^{-2}$. The pressure is force per area and has dimensions $ML^{-1}T^{-2}$. The gradient operator has dimensions L^{-2}, so the right hand side also has dimensions $ML^{-3}T^{-2}$.

The above analysis takes into account that the basis vectors have units. This analysis looks more familiar using Cartesian coordinates and taking components. The basis vectors have dimension L^{-1}, so the components get an extra dimension L. Suppose $X = \sum_i v_i \frac{\partial}{\partial x_i}$ in Cartesian coordinates. The equations become

$$\sum_i \frac{\partial}{\partial x_i} (\rho v_i) = 0 \tag{9.50}$$

and

$$\rho\left(\frac{\partial}{\partial t}v_j + \sum_i v_i \frac{\partial}{\partial x_i} v_j\right) = -\frac{\partial}{\partial x_j} p. \tag{9.51}$$

The v_j are velocity components and have dimension LT^{-1}. The dimensions for the components in the first equation are $ML^{-2}T^{-1}$, a momentum density. The dimensions for the components in the second equaton are $ML^{-2}T^{-2}$, a force density. This is simple enough, but the equations given before have the advantage that they are independent of the coordinate system.

Example: In non-Cartesian coordinate systems the formulas will be more complicated. Here is an example in plane polar coordinates. Consider a fluid with constant density ρ that is rotating with constant angular velocity ω. The velocity component with respect to the normalized basis vector will then be ωr. So

$$X = \omega r \frac{1}{r}\frac{\partial}{\partial \theta} = \omega \frac{\partial}{\partial \theta}. \tag{9.52}$$

Since ρ is constant and $\operatorname{div} X = 0$, the conservation law is satisfied. The Euler equation says that

$$\rho\, X \lrcorner \nabla X = -\rho\omega^2 r \frac{\partial}{\partial r} = -\frac{\partial p}{\partial r}\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial p}{\partial \theta}\frac{1}{r}\frac{\partial}{\partial \theta}. \tag{9.53}$$

This shows that $\partial p/\partial\theta = 0$, so p depends only on r. Then $\partial p/\partial r = \rho\omega^2 r$, so the final expression for the pressure is $p = p_0 + \frac{1}{2}\rho\omega^2 r^2$. The pressure has to increase away from the origin to hold the rotating fluid in place. ||

9.1.8 Acceleration*

In physics the most important quantities associated with a parameterized curve are *velocity* and *acceleration*. The velocity is always defined. The acceleration makes sense for motion in an affine space. When the coordinate system is an affine coordinate system, then the computation of the acceleration is easy. When it is a more general coordinate system, the formulas get complicated.

Suppose that ϕ is a time parameterized curve in an affine space. There are affine coordinates $\mathbf{x}$ and general coordinates $\mathbf{y}$. For the affine coordinates the velocity and acceleration are

$$\mathbf{v} = \sum_j \frac{d(x_j \circ \phi)}{dt}\frac{\partial}{\partial x_j}{}^{[\phi]} \tag{9.54}$$

and

$$\mathbf{a} = \sum_j \frac{d^2(x_j \circ \phi)}{dt^2}\frac{\partial}{\partial x_j}{}^{[\phi]}. \tag{9.55}$$

For the general coordinates the velocity is equally simple:

$$\mathbf{v} = \sum_j \frac{d(u_j \circ \phi)}{dt}\frac{\partial}{\partial u_j}{}^{[\phi]}. \tag{9.56}$$

The acceleration is another story.

Theorem 9.11 *Consider a time parameterized curve in affine space. The acceleration is*

$$\mathbf{a} = \sum_k \left(\frac{d^2(u_k \circ \phi)}{dt^2} + \sum_i \sum_j \frac{d(u_i \circ \phi)}{dt} \left(\Gamma_{ij}^k \circ \phi\right) \frac{d(u_j \circ \phi)}{dt} \right) \frac{\partial}{\partial u_k}{}^{[\phi]}. \quad (9.57)$$

Here the Γ_{ij}^ℓ are Christoffel symbols

$$\Gamma_{ij}^k = \sum_\ell \frac{\partial^2 x_\ell}{\partial u_i \partial u_j} \frac{\partial u_k}{\partial x_\ell}. \quad (9.58)$$

Proof: The acceleration is $\mathbf{a} = d\mathbf{v}/dt$. This is

$$\mathbf{a} = \frac{d}{dt} \sum_i \frac{d(u_i \circ \phi)}{dt} \frac{\partial}{\partial u_i}{}^{[\phi]} = \sum_i \sum_\ell \frac{d}{dt} \left(\frac{d(u_i \circ \phi)}{dt} \frac{\partial x_\ell}{\partial u_i} \circ \phi \right) \frac{\partial}{\partial x_\ell}{}^{[\phi]}. \quad (9.59)$$

Use the product rule. The first term is

$$\mathbf{a}_1 = \sum_i \sum_\ell \frac{d^2(u_i \circ \phi)}{dt^2} \frac{\partial x_\ell}{\partial u_i} \circ \phi \; \frac{\partial}{\partial x_\ell}{}^{[\phi]} = \sum_i \frac{d^2(u_i \circ \phi)}{dt^2} \frac{\partial}{\partial u_i}{}^{[\phi]}. \quad (9.60)$$

The second term is

$$\mathbf{a}_2 = \sum_i \sum_\ell \frac{d(u_i \circ \phi)}{dt} \frac{d}{dt} \left(\frac{\partial x_\ell}{\partial u_i} \circ \phi \right) \frac{\partial}{\partial x_\ell}{}^{[\phi]}. \quad (9.61)$$

This is

$$\mathbf{a}_2 = \sum_i \sum_\ell \sum_j \frac{d(u_i \circ \phi)}{dt} \frac{\partial^2 x_\ell}{\partial u_i \partial u_j} \circ \phi \; \frac{d(u_j \circ \phi)}{dt} \frac{\partial}{\partial x_\ell}{}^{[\phi]}. \quad (9.62)$$

However then

$$\mathbf{a}_2 = \sum_i \sum_\ell \sum_j \sum_k \frac{d(u_i \circ \phi)}{dt} \frac{\partial^2 x_\ell}{\partial u_i \partial u_j} \circ \phi \; \frac{d(u_j \circ \phi)}{dt} \frac{\partial u_k}{\partial x_\ell} \circ \phi \; \frac{\partial}{\partial u_k}{}^{[\phi]}. \quad (9.63)$$

Then $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$ works out to the formula in the statement of the theorem. □

Example: This example is about motion in the Euclidean plane. It is assumed that there has been a choice of origin and a choice of Cartesian coordinates referred to this origin. The polar coordinates for Euclidean space satisfy $x = r\cos(\theta)$ and $y = r\sin(\theta)$. These are not affine coordinates. The normalized basis vectors are $\partial/\partial r$ and $(1/r)\partial/\partial\theta$. These rotate in a particularly simple way:

$$\begin{aligned} \frac{\partial}{\partial\theta} \; \frac{\partial}{\partial r} &= \frac{1}{r} \frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial\theta} \; \frac{1}{r} \frac{\partial}{\partial\theta} &= -\frac{\partial}{\partial r}. \end{aligned} \quad (9.64)$$

Furthermore, they do not change as a function of r. This will be crucial below.

The position vector from the origin to a point in the plane is

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = r\frac{\partial}{\partial r}. \tag{9.65}$$

The motion may be described in terms of this position vector.

Consider a path ϕ. The Cartesian coordinates of this path as a function of time will be denoted $x^* = x \circ \phi$ and $y^* = y \circ \phi$. The corresponding normalized basis vectors along the path will be abbreviated $\left[\frac{\partial}{\partial x}\right] = \frac{\partial}{\partial x}[\phi]$ and $\left[\frac{\partial}{\partial y}\right] = \frac{\partial}{\partial y}[\phi]$.

Similarly, the polar coordinates along the path are $r^* = r \circ \phi$ and $\theta^* = \theta \circ \phi$. The corresponding normalized basis vectors along the path will be abbreviated $\left[\frac{\partial}{\partial r}\right] = \frac{\partial}{\partial r}[\phi]$ and $\left[\frac{1}{r}\frac{\partial}{\partial \theta}\right] = (\frac{1}{r}\frac{\partial}{\partial y})[\phi]$.

The position vector of the moving particle is

$$\mathbf{r} = x^* \left[\frac{\partial}{\partial x}\right] + y^* \left[\frac{\partial}{\partial y}\right] = r^* \left[\frac{\partial}{\partial r}\right]. \tag{9.66}$$

In Cartesian coordinates the velocity vector and acceleration vector are obtained by differentiating the coefficients with respect to t.

For polar coordinates the situation is more difficult, because the basis vectors depend on time. As one moves along the curve, the radial vector always points away from the origin, and so it keeps changing. This change must be taken into account.

To take the time derivatives we use the chain rule to write

$$\frac{d}{dt}\left[\frac{\partial}{\partial r}\right] = \frac{\partial}{\partial \theta}\left[\frac{\partial}{\partial r}\right]\frac{d\theta^*}{dt} + \frac{\partial}{\partial r}\left[\frac{\partial}{\partial r}\right]\frac{dr^*}{dt} = \frac{d\theta^*}{dt}\left[\frac{1}{r}\frac{\partial}{\partial \theta}\right]. \tag{9.67}$$

Also,

$$\frac{d}{dt}\left[\frac{1}{r}\frac{\partial}{\partial \theta}\right] = \frac{\partial}{\partial \theta}\left[\frac{1}{r}\frac{\partial}{\partial \theta}\right]\frac{d\theta^*}{dt} + \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial \theta}\right]\frac{dr^*}{dt} = -\frac{d\theta^*}{dt}\left[\frac{\partial}{\partial r}\right]. \tag{9.68}$$

The velocity in terms of normalized basis vectors is obtained by differentiating the position and using the product rule. It is

$$\mathbf{v} = \frac{dr^*}{dt}\left[\frac{\partial}{\partial r}\right] + r^*\frac{d\theta^*}{dt}\left[\frac{1}{r}\frac{\partial}{\partial \theta}\right]. \tag{9.69}$$

The angular momentum (divided by mass) is the (scalar) cross product

$$\mathbf{r} \times \mathbf{v} = r^{*2}\frac{d\theta^*}{dt}. \tag{9.70}$$

The acceleration is more complicated:

$$\mathbf{a} = \left(\frac{d^2r^*}{dt^2} - r^*\left(\frac{d\theta^*}{dt}\right)^2\right)\left[\frac{\partial}{\partial r}\right] + \left(r^*\frac{d^2\theta^*}{dt^2} + 2\frac{dr^*}{dt}\frac{d\theta^*}{dt}\right)\left[\frac{1}{r}\frac{\partial}{\partial \theta}\right]. \tag{9.71}$$

In physics the quadratic term in the radial acceleration is called the centripetal acceleration. Similarly, the quadratic term in the angular acceleration is called the Coriolis acceleration. This may also be written:

$$\mathbf{a} = \left(\frac{d^2 r^*}{dt^2} - r^* \left(\frac{d\theta^*}{dt} \right)^2 \right) \left[\frac{\partial}{\partial r} \right] + \frac{1}{r^*} \frac{d}{dt} \left(r^{*2} \frac{d\theta^*}{dt} \right) \left[\frac{1}{r} \frac{\partial}{\partial \theta} \right]. \tag{9.72}$$

This way of writing the equation displays the role of the angular momentum. ||

Example: Consider uniform circular motion given by ϕ equal to $(x, y) \leftarrow (a\cos(\omega t), a\sin(\omega t))$. This is the same ϕ as $(r, \theta) \leftarrow (a, \omega t)$. The motion is

$$\mathbf{r} = a\cos(\omega t) \frac{\partial}{\partial x}{}_{[\phi]} + a\sin(\omega t) \frac{\partial}{\partial y}{}_{[\phi]} = a \frac{\partial}{\partial r}{}_{[\phi]} \tag{9.73}$$

The velocity is

$$\mathbf{v} = -a\omega\sin(\omega t) \frac{\partial}{\partial x}{}_{[\phi]} + a\omega\omega\cos(\omega t) \frac{\partial}{\partial y}{}_{[\phi]} = -a\omega \frac{\partial}{\partial \theta}{}_{[\phi]}. \tag{9.74}$$

The acceleration is

$$\mathbf{a} = a\omega^2\cos(\omega t) \frac{\partial}{\partial x}{}_{[\phi]} - a\omega^2\sin(\omega t) \frac{\partial}{\partial y}{}_{[\phi]} = -a\omega^2 \frac{\partial}{\partial r}{}_{[\phi]}. \tag{9.75}$$

This is entirely centripetal acceleration. Even though the radius is not changing, the change in angle creates a radial acceleration. ||

Example: Instead consider the situation where $d\theta^*/dt = \omega$ is constant and $dr^*/dt = b$ is constant. Then the radial acceleration $-r^*\omega^2$ is still centripetal. However there is also an angular acceleration $2b\omega$ given by the Coriolis term. This says that outward radial motion results in faster angular motion. ||

Example: This is the solution of the Kepler problem for motion in a plane. Newton's equation is $m\mathbf{a} = \mathbf{F}$. The force is $\mathbf{F} = -k/r^2 \frac{\partial}{\partial r}$. Consider a solution ϕ. For brevity abbreviate $r \circ \phi$ by r^* and $\theta \circ \phi$ by θ^*. This θ^* is a coordinate on the parameter space; it is a replacement for the parameter t. The crucial fact in the following discussion is that if

$$\left(\frac{1}{r} \frac{\partial}{\partial \theta} \right)_{[\phi]} = -\sin(\theta^*) \frac{\partial}{\partial x}{}_{[\phi]} + \cos(\theta^*) \frac{\partial}{\partial y}{}_{[\phi]}, \tag{9.76}$$

then

$$\frac{d}{d\theta^*} \left(\frac{1}{r} \frac{\partial}{\partial \theta} \right)_{[\phi]} = -\cos(\theta^*) \frac{\partial}{\partial x}{}_{[\phi]} - \sin(\theta^*) \frac{\partial}{\partial y}{}_{[\phi]} = -\frac{\partial}{\partial r}{}_{[\phi]}. \tag{9.77}$$

As the unit tangent vector rotates, its change is given by the inward unit radial vector.

In this problem the angular acceleration is zero. This implies that a solution ϕ must satisfy

$$m(r^*)^2 \frac{d\theta^*}{dt} = L \tag{9.78}$$

for some constant L. The angular momentum is constant.

We have

$$\frac{dt}{d\theta^*} = \frac{m}{L}(r^*)^2. \tag{9.79}$$

Multiply the equation $d\mathbf{v}/dt = (1/m)\mathbf{F}$ by this quantity. This gives

$$\frac{d\mathbf{v}}{d\theta^*} = -\frac{k}{L}\frac{\partial}{\partial r}^{[\phi]} \tag{9.80}$$

This integrates to

$$\mathbf{v} = \frac{k}{L}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)_{[\phi]} + \mathbf{c}. \tag{9.81}$$

The first term is a vector of fixed length $k/(mL)$ in the angular direction. This term alone would give a parametric equation of a circle in velocity space centered at the origin. The second term is a constant of integration, so the actual motion in velocity space is a circle centered at fixed velocity $\mathbf{c}$. The velocity traces out a circle as a periodic function of parameter θ^* (and as a more complicated function of time t).

Remark: The fact that the velocity traces a circle in the plane is a miracle that deserves explanation. The hidden symmetry that makes this happen in described in [31]. The secret is to use stereographic projection to map the velocity plane to a sphere. The planar circles correspond to great circles on the sphere. ||

To proceed further, take the (2-dimensional scalar) cross product $m\mathbf{r} \times \mathbf{v}$. This is

$$L = \frac{km}{L}r^* + m\mathbf{r} \times \mathbf{c}. \tag{9.82}$$

The second term $m\mathbf{r} \times \mathbf{c} = mr^*|\mathbf{c}|\cos(\theta^* - \chi)$. The equation then has the form

$$L = \frac{km}{L}r^*(1 + e\cos(\theta^* - \chi)). \tag{9.83}$$

This shows that at every time the position is on a conic section. The simplest parametrization of this conic section is by θ^*. The solution ϕ is then

$$(r, \theta) \leftarrow (r^*, \theta^*) = \left(\frac{L^2}{km}\frac{1}{1 + e\cos(\theta^* - \chi)}, \theta^*\right). \tag{9.84}$$

There is a parametrization by t, but it is more complicated. ||

9.2 Vector calculus in three-dimensional Euclidean space

9.2.1 Twisted vector fields and forms in Euclidean space

Some vector concepts require going beyond affine space to Euclidean space; a good example is the gradient operator. This takes the differential ds, a 1-form,

and converts it to a vector field. In fact, there is a tradition that insists on working with vector fields, and this tradition has developed an apparatus of vector calculus that works in Euclidean space. In fact, a substantial part of it works only for three-dimensional Euclidean space.

This section explains this apparatus and gives a translation between the language of vector fields and the language of differential forms. The vector fields and the differential forms can be twisted, so this also needs to be taken into account.

Let M be a connected n-dimensional oriented manifold patch with two orientations, say $\mathcal{O}$ and $-\mathcal{O}$.

- A twisted vector field is an odd function from orientations to vector fields. Thus for some vector field X it sends $\mathcal{O}$ to X and $-\mathcal{O}$ to $-X$. The twisted vector field may be denoted $(X : \mathcal{O})$. Thus

$$(X : \mathcal{O}) = (-X : -\mathcal{O}). \tag{9.85}$$

- A twisted differential k-form is an odd function from orientations to differential k-forms. Thus for some differential k-form α it sends $\mathcal{O}$ to α and $-\mathcal{O}$ to $-\alpha$. The differential k-form may be denoted $(\alpha : \mathcal{O})$. Thus

$$(\alpha : \mathcal{O}) = (-\alpha : -Oc). \tag{9.86}$$

Remark: It is common to encounter terminology in which the terms for twisted vector and twisted form are *pseudovector* and *pseudoform*. The Greek prefix "pseudo" means "false", which seems overly harsh. The description "twisted" conveys the suggestion that it might have something do do with orientation. ||
Example: The classic example of a twisted vector field is given by the cross product in 3-dimensional Euclidean space. If X and Y are vector fields, then $X \times Y = (Z : \mathcal{O}) = (-Z : -\mathcal{O}))$, where the length Z at a point is the product of the length of X and Y at the same point, the direction of Z at a point is perpendicular to the direction of X and the direction of Y, and where (X, Y, Z) has orientation $\mathcal{O}$. If we use Cartesian coordinates x, y, z and apply this rule to the basis vectors $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ the result is

$$\frac{\partial}{\partial x} \times \frac{\partial}{\partial y} = \left(\frac{\partial}{\partial z} : [dx\, dy\, dz] \right) = \left(-\frac{\partial}{\partial z} : [dz\, dy\, dx] \right). \tag{9.87}$$

Equivalently,

$$\frac{\partial}{\partial y} \times \frac{\partial}{\partial x} = \left(-\frac{\partial}{\partial z} : [dx\, dy\, dz] \right) = \left(\frac{\partial}{\partial z} : [dz\, dy\, dx] \right). \tag{9.88}$$

||
Example: There is a construction for differential forms in 3-dimensional Euclidean space that sends 1-forms to twisted 2-forms and sends 2-forms to twisted

1-forms. This is called the Hodge dual. In three dimensions with Cartesian coordinates

$$\begin{aligned} *dx &= (dy\,dz : [dx\,dy\,dz]) = (dz\,dy : [dz\,dy\,dx]) \\ *dy &= (dz\,dx : [dx\,dy\,dz]) = (dx\,dz : [dz\,dy\,dx]) \\ *dz &= (dx\,dy : [dx\,dy\,dz]) = (dy\,dx : [dz\,dy\,dx]). \end{aligned} \tag{9.89}$$

Also

$$\begin{aligned} *(dx\,dy) &= (dz : [dx\,dy\,dz]) = (-dz : [dz\,dy\,dx]) \\ *(dy\,dz) &= (dx : [dx\,dy\,dz]) = (-dx : [dz\,dy\,dx]) \\ *(dz\,dx) &= (dy : [dx\,dy\,dz]) = (-dy : [dz\,dy\,dx]). \end{aligned} \tag{9.90}$$

There are similar formulas that send twisted 1-forms to 2-forms and twisted 2-forms to 1-forms. ||

In 3-dimensional Euclidean space there are forms like $dx\,dy\,dz = -dz\,dy\,dx$ that have something to do with computing volume. Such a form is called a *volume form*. The problem with a volume form is that it depends not only on the volume, but on the orientation defined by the coordinate system. However a *volume density* or *volume element* does not have this defect. This is the twisted 3-form

$$(dx\,dy\,dz, [dx\,dy\,dz]) = (dz\,dy\,dx : [dz\,dy\,dx]). \tag{9.91}$$

The volume form defines its own orientation, and this pair determines the volume density.

Example: Suppose $X = a\partial/\partial x + b\partial/\partial y + c\partial/\partial z$ is a vector field. Then

$$X \lrcorner\, (dx\,dy\,dz : [dx\,dy\,dz]) = (a\,dy\,dz + b\,dz\,dx + c\,dx\,dy : [dx\,dy\,dz]). \tag{9.92}$$

This is a twisted 2-form. ||

Pictures of twisted differential forms differ from the corresponding pictures for differential forms. A twisted n-form is pictured by a cloud of points; each point has a $\pm$ sign, with the signs varying continuously (changing only where the form is zero). A twisted $(n-1)$-form is pictured by *transport flux curves*. These are curves with an orientation (a direction along the curve). A transport flux curves may have end points; the direction along the curve goes from the $+$ end point to the $-$ end point.

There is a useful *reference orientation convention*. One chooses an orientation $\mathcal{O}$ and specifies at the outset of a disussion that this will be the reference orientation. In the following discussion the twisted form $(\alpha : \mathcal{O})$ is denoted α. It is important to keep in mind that α stands for a twisted form.

Example: Say that the reference orientation is $[dx\,dy\,dz]$. Then in three dimensions the Hodge dual interchanges forms and twisted forms with

$$\begin{aligned} *dx &= dy\,dz \\ *dy &= dz\,dx \\ *dz &= dx\,dy. \end{aligned} \tag{9.93}$$

In three dimensions the inverse of the $*$ operation is the $*$ operation. This simple statement summarizes everything in the previous example. ||

Example: Say that the reference orientation is $[dz\,dy\,dx]$. Then in three dimensions the Hodge dual interchanges forms and twisted forms with

$$\begin{aligned} *dx &= dz\,dy \\ *dy &= dx\,dz \\ *dz &= dy\,dx. \end{aligned} \tag{9.94}$$

This is entirely equivalent to the statement in the previous example. ||

Example: Suppose $X = a\partial/\partial x + b\partial/\partial y + c\partial/\partial z$ is a vector field. Take the reference orientation to be $[dx\,dy\,dz]$, so the volume density is represented by the volume form $dx\,dy\,dz$. Then

$$X \lrcorner\, dx\,dy\,dz = a\,dy\,dz + b\,dz\,dx + c\,dx\,dy. \tag{9.95}$$

This represents a twisted 2-form. ||

9.2.2 Differential forms as vector fields

It is generally wrong to think of differential 1-forms as being anything like vector fields. In many situations there is no relation between them; in other situations the relation is complicated. However in Euclidean space using Cartesian coordinates there is an easy translation.

Here is a quick summary. For Euclidean space there is a map g from vector fields to 1-forms. This sends every basis vector field $\partial/\partial x_i$ to the cooresponding basis form dx_i. For example, in three dimensions

$$\begin{aligned} \mathsf{g}\frac{\partial}{\partial x} &= dx \\ \mathsf{g}\frac{\partial}{\partial y} &= dy \\ \mathsf{g}\frac{\partial}{\partial z} &= dz \end{aligned} \tag{9.96}$$

The intuition is that at each point the vector (an arrow) is orthogonal to the hyperplanes where the form is constant.

There is also a map $*$ that interchanges forms and twisted forms. It sends k-forms to twisted $(n-k)$-forms. It also sends twisted k-forms to $(n-k)$-forms. This is the *Hodge dual* (or *Hodge star*). It is defined on basis k-forms as follows. Since twisted forms are part of the discussion, it is necessary to choose a reference orientation. In this case we use $[dx_1 \ldots dx_n]$. The formula is then

$$*dx_{i_1} \ldots dx_{i_k} = \frac{\partial}{\partial x_{i_k}} \lrcorner \cdots \frac{\partial}{\partial x_{i_1}} \lrcorner\, (dx_1 \ldots dx_n). \tag{9.97}$$

The intuition is that the k forms and $(n-k)$ forms are orthogonal. The intechange of forms and twisted form makes the result not depend on an arbitrary

choice of orientation. For instance, in three dimensions it should not depend on whether the coordinate system x, y, z is right-handed or left-handed.

Example: In three dimensions with reference orientation $[dx\,dy\,dz]$ we have these formula for going from forms to twisted forms

$$\begin{aligned} *dx &= \frac{\partial}{\partial x} \lrcorner\, (dx\,dy\,dz) = dy\,dz \\ *dy &= \frac{\partial}{\partial y} \lrcorner\, (dx\,dy\,dz) = dz\,dx \\ *dz &= \frac{\partial}{\partial z} \lrcorner\, (dx\,dy\,dz) = dx\,dy,. \end{aligned} \tag{9.98}$$

Also

$$\begin{aligned} *(dx\,dy) &= \frac{\partial}{\partial y} \lrcorner\, \frac{\partial}{\partial x} \lrcorner\, (dx\,dy\,dz) = dz \\ *(dy\,dz) &= \frac{\partial}{\partial z} \lrcorner\, \frac{\partial}{\partial y} \lrcorner\, (dx\,dy\,dz) = dx \\ *(dz\,dx) &= \frac{\partial}{\partial x} \lrcorner\, \frac{\partial}{\partial z} \lrcorner\, (dx\,dy\,dz) = dy. \end{aligned} \tag{9.99}$$

Finally

$$*1 = dx\,dy\,dz \tag{9.100}$$

and

$$*(dx\,dy\,dz) = \frac{\partial}{\partial z} \lrcorner\, \frac{\partial}{\partial y} \lrcorner\, \frac{\partial}{\partial x}(dx\,dy\,dz) = 1. \tag{9.101}$$

These formulas work when both sides of the equation are twisted forms or when both sides of the equations are forms. The only important thing is that the Hodge dual $*$ switches forms and twisted forms.

Warning: If, on the contrary, the reference orientation were taken to be $[dz\,dy\,dx]$, then all the expressions would need a minus sign on the right hand side to be correct. But the meanings of the equations would be the same. ||

Example: In two dimensions it is no longer possible to hide the signs. Take $[dx\,dy]$ to be the reference orientation. Then

$$\begin{aligned} *dx &= \frac{\partial}{\partial x} \lrcorner\, (dx\,dy) = dy \\ *dy &= \frac{\partial}{\partial y} \lrcorner\, (dx\,dy) = -dx. \end{aligned} \tag{9.102}$$

In two dimensions $*$ followed by $*$ has the effect of an overall minus sign. This gives the pattern in the general case of mappings between 1-forms and twisted $(n-1)$-forms. When n is odd the inverse map from $n-1$ forms to 1 forms is again $*$. When n is even this inverse map is $-*$. ||

There is a third map c from vector fields to twisted $(n-1)$ forms. This might be called *contraction*, since it is a special instance of a more general conception

of contraction from tensor algebra. (In fact, it is just contraction of a vector field with the volume density.) In Cartesian coordinates it is

$$\mathsf{c}(X) = X \lrcorner (dx_1 \cdots dx_n : [dx_1 \cdots dx_n]) = (X \lrcorner (dx_1 \cdots dx_n) : [dx_1 \cdots dx_n]). \tag{9.103}$$

With the reference orientation $[dx_1 \cdots dx_n]$ this could be written simply as

$$\mathsf{c}(X) = X \lrcorner (dx_1 \cdots dx_n). \tag{9.104}$$

On basis vectors this is

$$\mathsf{c}(\frac{\partial}{\partial x_j}) = (-1)^{j-1} dx_1 \cdots dx_{j-1}\, dx_{j+1} \cdots dx_n. \tag{9.105}$$

Proposition 9.12 *The maps from vectors fields to twisted $(k-1)$-forms are related by*

$$\mathsf{c}(X) = *\mathsf{g}(X). \tag{9.106}$$

The same formula works for mapping twisted vector fields to $(k-1)$-forms.

See Table 9.1 for the relation between these various operations.

Example: In three dimensions choose the reference orientation $[dx\, dy\, dz]$. Then

$$\begin{aligned} \mathsf{c}(\frac{\partial}{\partial x}) &= dy\, dz \\ \mathsf{c}(\frac{\partial}{\partial y}) &= dz\, dx \\ \mathsf{c}(\frac{\partial}{\partial z}) &= dx\, dy. \end{aligned} \tag{9.107}$$

||

Example: In two dimensions declare the reference orientation $[dx\, dy]$. Then

$$\begin{aligned} \mathsf{c}(\frac{\partial}{\partial x}) &= dy \\ \mathsf{c}(\frac{\partial}{\partial y}) &= -dx. \end{aligned} \tag{9.108}$$

This should be compared with

$$\begin{aligned} \mathsf{g}(\frac{\partial}{\partial x}) &= dx \\ \mathsf{g}(\frac{\partial}{\partial y}) &= dy. \end{aligned} \tag{9.109}$$

||

There are two very different ways of going from a vector field to a differential form. The first is to use g to go from a vector field to a 1-form. The second is to use c to go from a vector field to a twisted $(n-1)$-form. These are easy to to confuse in dimension 2.

$$\begin{array}{ccccc} & V & & & \\ \mathsf{g} & \downarrow & \searrow \mathsf{c} & & \\ & 1 & \longrightarrow & (n-1)^\sim & \\ & & * & & \end{array}$$

Table 9.1: From vectors to 1 forms to twisted $n-1$ forms

9.2.3 Vector calculus in Euclidean space

Consider the case of three dimensional Euclidean space with Cartesian coordinates. There is a standard vector calculus that describes operations of gradient (on scalar fields) and of curl and divergence (on vector fields). They are determined by

$$\begin{aligned} \text{grad } s &= \mathsf{g}^{-1}\, ds \\ \text{curl } X &= \mathsf{c}^{-1} d(\mathsf{g}\, X) \\ \text{div } X &= *d(\mathsf{c}\, X). \end{aligned} \tag{9.110}$$

The only one of these operations that switches vector fields with twisted vector fields is curl.

These give rise to exact differential forms via

$$\begin{aligned} \mathsf{g}\ \text{grad } s &= ds \\ \mathsf{c}\ \text{curl } X &= d(\mathsf{g}\, X) \\ *\,\text{div } X &= d(\mathsf{c}\, X). \end{aligned} \tag{9.111}$$

They often appear in this guise. If X is considered as a vector field (rather than a twisted vector field), then the third identity is an identity for twisted 3-forms. If X is a twisted vector field, then the third identity is an identity for 3-forms.

In calculus textbooks these differential forms might be written in another notation. First, define

$$\begin{aligned} d\vec{r} &= (dx, dy, dz) \\ d\vec{A} &= (dy\, dz, dz\, dx, dx\, dy) \\ dV &= dx\, dy\, dz. \end{aligned} \tag{9.112}$$

The second one of these could be thought of as the Hodge dual of the first one. The third one could be thought of as the Hodge dual of 1.

For a vector field $\vec{\mathbf{v}}$ with components (v_1, v_2, v_3) we have

$$\begin{aligned} \text{grad } s \cdot d\vec{r} &= ds \\ \text{curl } \vec{\mathbf{v}} \cdot d\vec{A} &= d(\vec{\mathbf{v}} \cdot d\vec{r}) \\ \text{div } \vec{\mathbf{v}}\, dV &= d(\vec{\mathbf{v}} \cdot d\vec{A}). \end{aligned} \tag{9.113}$$

The first two are identities between 1-forms and 2-forms; the third is an identity between twisted 3-forms.

$$\begin{array}{ccccccc} & \text{grad} & & \text{curl} & & \text{div} & \\ S & \longrightarrow & V & \longrightarrow & V^\sim & \longrightarrow & S^\sim \\ \downarrow & \mathsf{g} & \downarrow & \mathsf{c} & \downarrow & * & \uparrow \\ 0 & \longrightarrow & 1 & \longrightarrow & 2 & \longrightarrow & 3 \\ & d & & d & & d & \end{array}$$

$$\begin{aligned} \text{grad} &= \mathsf{g}^{-1}\, d \\ \text{curl} &= \mathsf{c}^{-1}\, d\, \mathsf{g} \\ \text{div} &= *\, d\, \mathsf{c}. \end{aligned} \tag{9.114}$$

Table 9.2: Exterior derivative vs vector operations: 3 dimensions

$$\begin{array}{ccccccc} & \text{grad} & & & & \text{div} & \\ S & \longrightarrow & V & \longrightarrow & V & \longrightarrow & S \\ \downarrow & \mathsf{g} & \downarrow & \mathsf{c} & \downarrow & * & \uparrow \\ 0 & \longrightarrow & 1 & \longrightarrow & 2^\sim & \longrightarrow & 3^\sim \\ & d & & * & & d & \end{array}$$

$$\nabla^2 = \text{grad div} = *\, d\, *\, d. \tag{9.115}$$

Table 9.3: The Laplace operator: 3 dimensions

There are, however, important distinctions. First, the differential form formulas take the same form in arbitrary coordinate systems. This is not true for the formulas for the gradient, curl, and divergence. The reason is that the usual definitions of gradient, curl, and divergence are as operations on vector fields, not on differential forms. This leads to a much more complicated theory, except for the very special case of Cartesian coordinates on Euclidean space. Later on we shall examine this issue in detail.

Second, the differential form formulas have natural formulations for manifolds of arbitrary dimension. While the gradient and divergence may also be formulated in arbitrary dimensions, the curl only works in three dimensions.

This does not mean that notions such as gradient of a scalar (a vector field) or divergence of a vector field (a scalar) are not useful and important. Indeed, in some situations they play an essential role.

The relation between differential form and vector field operations in dimension 3 is summarized in Table 9.2. The numbers are the degree of the forms. In this table $V^\sim$ stands for pseudovector (twisted vector field), and $S^\sim$ means pseudoscalar (twisted scalar field).

The vector operations are obtained by following the arrows. These formulas are true in arbitrary coordinate systems once the g and c and $*$ operations have been found for these coordinate systems. This will be the subject of later developments.

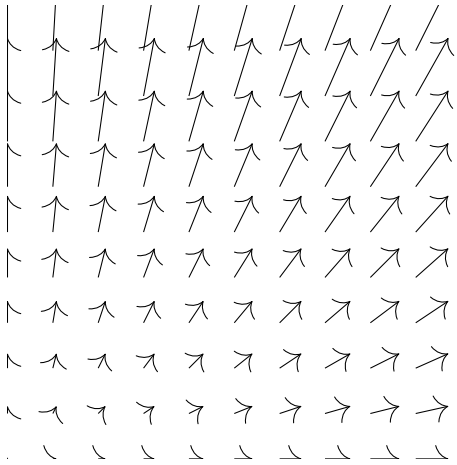

Figure 9.1: Vector field $X = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$

The most important combination of these vector operations is the *Laplace operator*

$$\nabla^2 = \text{div grad}\,. \tag{9.116}$$

The Laplace operator may be expressed in various ways, as shown in Table 9.3. In this table $2^\sim$ and $3^\sim$ indicate pseudoforms (twisted forms). The Laplace operator also may be written $*\,d\,\mathsf{c}\,\mathsf{g}^{-1}\,d$ which is the same as $*\,d\,*\,d$. This means that the action of the Laplace operator may be read either along the S V V S path at the top or along the S 0 1 $2^\sim$ $3^\sim$ S path that runs along the bottom.

Example: There are pictures associated with the operations. Figure 9.1 shows a vector field $X = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$ for $x > 0$ and $y > 0$. This particular X has curl equal to 0, so it is not a completely typical example. On the other hand, it has non-zero divergence. The corresponding orbits of X are shown in Figure 9.2.

The most common way of associating a differential form to X is to take $\mathsf{g}\,X = x\,dx + 2y\,dy$, for which the contour curves are orthogonal to the vectors. This form is illustrated in Figure 9.3. It is also special, since it is exact.

The other way is to take $\mathsf{c}\,X = (-2y\,dx + x\,dy : [dx\,dy])$, a twisted differential form. (The picture here is that the transport flux curves are oriented along the curve, in this case in the direction of increasing x along each curve. The end points represent positive sources.) This twisted form is pictured in Figure 9.4. It is not exact, in fact its exterior derivative is $(3\,dx\,dy : [dx\,dy])$. This should indicate the density of end points in the picture. The divergence of X is the Hodge dual of this twisted 2-form, namely the scalar $\text{div}(X) = 3$. ||

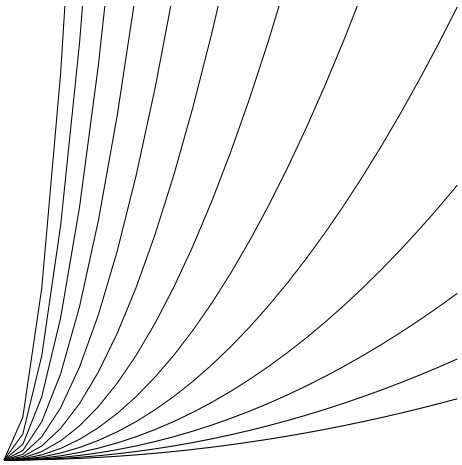

Figure 9.2: Orbits of $X = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$

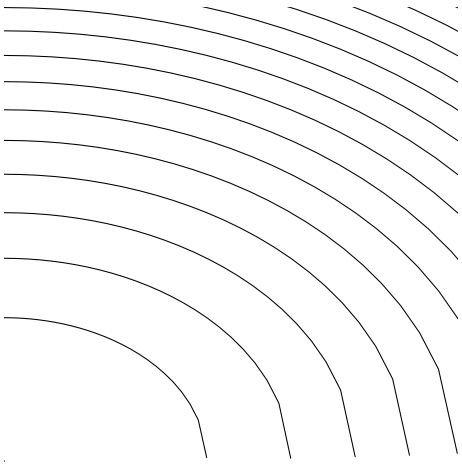

Figure 9.3: Differential form $\mathbf{g}X = \omega = x\,dx + 2y\,dy = d(\frac{1}{2}x^2 + y^2)$

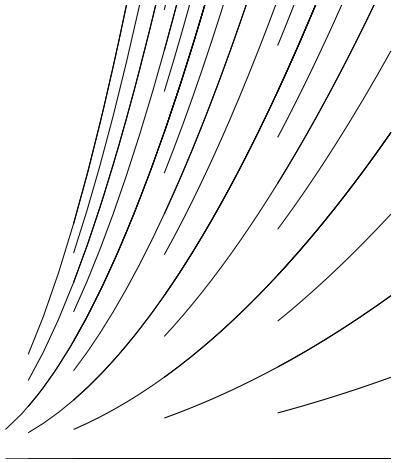

Figure 9.4: Twisted differential form $\mathsf{c}X = *\omega = (-2y\,dx + x\,dy : [dx\,dy])$

9.2.4 Vector versions of Stoke's theorem

Many texts present the classical three-dimensional versions of Stokes' theorem in vector form. This section states these results for the record. The first statement is in a form that will continue to make sense in arbitrary coordinate systems.

The fundamental theorem of calculus is

$$\int_C \mathsf{g}\,\operatorname{grad} s = \sum_{\partial C} s. \tag{9.117}$$

Stokes's theorem says that

$$\int_S \operatorname{curl} X \lrcorner \operatorname{vol} = \int_{\partial S} \mathsf{g}\, X. \tag{9.118}$$

The divergence theorem states

$$\int_R \operatorname{div} X \operatorname{vol} = \int_{\partial R} X \lrcorner \operatorname{vol}. \tag{9.119}$$

The next statement shows how these might appear in calculus texts.

The fundamental theorem of calculus is

$$\int_C \operatorname{grad} s \cdot d\vec{r} = \sum_{\partial C} s. \tag{9.120}$$

Stokes's theorem says that

$$\int_S \operatorname{curl} \mathbf{v} \cdot\ d\vec{A} = \int_{\partial S} \mathbf{v} \cdot d\vec{r}. \tag{9.121}$$

The divergence theorem states

$$\int_R \operatorname{div} \mathbf{v}\, dV = \int_{\partial R} \mathbf{v} \cdot d\vec{A}. \tag{9.122}$$

Chapter 10

Metric Tensor

10.1 The metric tensor

10.1.1 The metric

In geometry one measures distance in a straight line using the Euclidean distance. In coordinates this is computed by using the theorem of Pythagoras. However with some calculus one can also measure distance along a curve. It is even possible to do this in arbitrary coordinate systems. This motivates a very general notion of squared distance, given by the *metric tensor* g. This associates to every ordered pair of vector fields X, Y a scalar field $\langle X \mid \mathsf{g} \mid Y \rangle$. This is linear in each variable and is symmetric: $\langle X \mid \mathsf{g} \mid Y \rangle = \langle Y \mid \mathsf{g} \mid X \rangle$. Furthermore, it is positive: $\langle X \mid \mathsf{g} \mid X \rangle \geq 0$. It is also required to be strictly positive at every point where X is not the zero vector.

For a given coordinate system $x_1, \ldots, x_n$ it is customary to define

$$g_{ij} = \left\langle \frac{\partial}{\partial x_i} \mid \mathsf{g} \mid \frac{\partial}{\partial x_j} \right\rangle. \tag{10.1}$$

This notation does not indicate explicitly the dependence on the coordinate system, but it is very convenient. If $X = \sum_i a_j \frac{\partial}{\partial x_i}$ and $Y = \sum_j b_j \frac{\partial}{\partial x_j}$ are vector fields expressed in terms of coordinate bases, then

$$\langle X \mid \mathsf{g} \mid Y \rangle = \sum_i \sum_j g_{ij} a_i b_j. \tag{10.2}$$

The same equation can also be written

$$\langle X \mid \mathsf{g} \mid Y \rangle = \sum_i \sum_j g_{ij} \langle dx_i \mid X \rangle \, \langle dx_j \mid Y \rangle. \tag{10.3}$$

The short form for this is

$$\mathsf{g} = \sum_i \sum_j g_{ij} \, dx_i \, dx_j \tag{10.4}$$

Here the product $dx_i \, dx_j$ merely multiplies the values. It needs to be carefully distinguished from the antisymmetric exterior product $dx_i \wedge dx_j$.

The metric tensor is determined in a particular coordinate system by functions g_{ij} forming the matrix of coefficients. This matrix is required to be symmetric and positive definite. The distance along a regular parametrized curve C is then given by

$$\text{distance} = \int_C \sqrt{\sum_{i=1}^n \sum_{j=1}^n g_{ij} \, dx_i \, dx_j}. \tag{10.5}$$

This can be computed via a parametrization ϕ of C via

$$\text{distance} = \int_{t_0}^{t_1} \sqrt{\sum_{i=1}^n \sum_{j=1}^n (g_{ij} \circ \phi) \frac{d(x_i \circ \phi)}{dt} \frac{d(x_j \circ \phi)}{dt}} \, dt. \tag{10.6}$$

Because of the square root the integrals can be nasty. In this expression it is helpful to think of $\frac{dx_i}{dt}$ as the components of the *velocity*. The square root expression is then interpreted as the *speed* $\frac{ds}{dt}$. So the computation of *arc length* of the curve comes down to integrating ds.

If we have the metric tensor in one set of coordinates, then we have it in any other set of coordinates. Thus

$$\mathsf{g} = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{\alpha=1}^{n}\frac{\partial x_i}{\partial u_\alpha}\,du_\alpha\right) g_{ij} \left(\sum_{\beta=1}^{n}\frac{\partial x_j}{\partial u_\beta}\,du_\beta\right) = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial x_i}{\partial u_\alpha} g_{ij} \frac{\partial x_j}{\partial u_\beta}\right) du_\alpha\,du_\beta. \tag{10.7}$$

So the matrix in the new coordinates is

$$g_{\alpha\beta} = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial x_i}{\partial u_\alpha} g_{ij} \frac{\partial x_j}{\partial u_\beta}. \tag{10.8}$$

The case $g_{ij} = \delta_{ij}$ is the special case when the x_i are Cartesian coordinates. In this case

$$\mathsf{g} = \sum_{j=1}^{n} dx_i^2. \tag{10.9}$$

However even if we start with Cartesian coordinates, the coefficients of the metric tensor takes a more complicated form in other coordinate systems. If the coefficients come from Cartesian coordinates via a change of coordinates, then the metric is said to be a *flat metric*.

The notion of metric tensor goes at least part way to erasing the distinction between differential 1-forms and vector fields. It provides a correspondence, one that is sometimes awkward, but that always exists. Recall that a bilinear form may also be regarded as a linear transformation from a vector space to its dual space. In the present instance this is $\langle \mathsf{g}X \mid Y\rangle = \langle X \mid \mathsf{g} \mid Y\rangle$. This may also be expressed in coordinates. Let g_{ij} be the coefficients of the metric tensor for the coordinate system $u_1, \ldots, u_n$. If

$$X = \sum_{k=1}^{n} a_k \frac{\partial}{\partial u_k} \tag{10.10}$$

is a vector field, then the associated differential 1-form is

$$\mathsf{g}X = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} g_{ij}a_j\right) du_i. \tag{10.11}$$

One can also go the other direction. If

$$\omega = \sum_{k=1}^{n} p_k\,du_k \tag{10.12}$$

is a differential 1-form, then the associated vector field is

$$\mathsf{g}^{-1}\omega = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} p_i g^{ij} \right) \frac{\partial}{\partial u_j}. \tag{10.13}$$

Here we are using the perhaps unfamiliar notation that g^{ij} is the inverse matrix to g_{ij}. (This notation is standard in this context.)

Example: A familiar example of a flat metric is the metric in three dimensional Euclidean space. Then we have

$$\mathsf{g} = dx^2 + dy^2 + dz^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2(\theta)\, d\phi^2. \tag{10.14}$$

The latter is in spherical polar coordinates. The usual spherical polar coordinate normalized basis forms are dr, $r\, d\theta$, and $r \sin(\theta)\, d\phi$. The volume form is the product of normalized basis forms, so it is $dx \wedge dy \wedge dz = r^2 \sin(\theta)\, dr \wedge d\theta \wedge d\phi$. ||

Example: Perhaps the simplest example of a metric that is not flat is the induced metric on the two-dimensional sphere of radius a. Then

$$\mathsf{g} = a^2\, d\theta^2 + a^2 \sin^2(\theta)\, d\phi^2. \tag{10.15}$$

The normalized basis forms are $a\, d\theta$, and $a \sin(\theta)\, d\phi$. The area form is the product $a^2 \sin(\theta)\, d\theta \wedge d\phi$.

It is well known from attempts to map the surface of the earth that there is no coordinate system on the two-dimensional sphere that expresses the metric g in the form $\mathsf{g} = a^2(du^2 + dv^2)$. (On the other hand, one can express the area form as $a^2 du \wedge dv$ by taking $u = -\cos(\theta), v = \phi$.)

It is possible to try to simplify the representation of the metric in some way, taking account that it will never be flat. One method is to find a coordinate system u, v with normalized basis vectors $a \sin(\theta)\, du$ and $a \sin(\theta)\, dv$, so that the coefficients of du and dv are the same. One can accomplish this by taking $d\theta = \sin(\theta)\, du$ and $dv = d\phi$. These relations may be solved to obtain $\tanh(u) = -\cos(\theta)$ and $v = \phi$. Here θ ranges from 0 to π, while u ranges from $-\infty$ to $+\infty$. One consequence is that $\operatorname{sech}(u) = \sin(\theta)$. Thus the normalized forms are $a \operatorname{sech}(u)\, du$ and $a \operatorname{sech}(u)\, dv$. The metric is

$$\mathsf{g} = a^2 \operatorname{sech}^2(u)(du^2 + dv^2). \tag{10.16}$$

This is the Mercator representation of the metric on the sphere; it looks flat except for distortion near $u = \pm\infty$, that is near the poles $\theta = 0$ and $\theta = \pi$. The area form in these coordinates is $a^2 \operatorname{sech}^2(u)\, du \wedge dv$. ||

10.1.2 The volume element

Integration on a space with a metric tensor g is usually discussed in terms of a volume n-form

$$\text{vol} = \sqrt{g}\, dx_1 \wedge \cdots \wedge dx_n. \tag{10.17}$$

Here g denotes the determinant of the matrix g_{ij} associated with the coordinate system $x_1, \ldots, x_n$. Strictly speaking, this does not make sense without introducing further conventions. The problem is that it is meaningless to say that vol is positive or negative. However both vol and $-$vol determine the same volume element

$$\text{vol}|_+ = \sqrt{g}(dx_1 \wedge \cdots \wedge dx_n)|_+. \tag{10.18}$$

This object is a twisted n-form. An integral with respect to the volume element is defined in the standard way. Suppose that $s = f(u_1, \ldots, u_n)$ is a scalar field. Write $g = k(u_1, \ldots, u_n)$. Then

$$\int_M s\,\text{vol}|_+ = \int_M f(u_1, \ldots, u_n)\sqrt{k(u_1, \ldots, u_n)}(du_1 \wedge \cdots \wedge du_n)|_+ = I(f\sqrt{h}). \tag{10.19}$$

The fact that vol$|_+$ is a twisted n-form means that an orientation for M is not required. (Technically, M has a transverse orientation that is taken as $+$.)

The transformation property for the coefficient $\sqrt{g}$ may be seen as follows. Consider two coordinate systems related by $\mathbf{x} = \mathbf{h}(\mathbf{u})$. Suppose that G is the matrix of the metric in the $\mathbf{x}$ system. Then

$$\mathsf{g} = d\mathbf{x}^T G\, d\mathbf{x} = d\mathbf{u}^T(\mathbf{h}'(\mathbf{u})^T G \mathbf{h}'(\mathbf{u}))\, d\mathbf{u}. \tag{10.20}$$

So the matrix of the metric in the $\mathbf{u}$ system is $\mathbf{h}'(\mathbf{u})^T G \mathbf{h}'(\mathbf{u})$. Furthermore,

$$\det(\mathbf{h}'(\mathbf{u})^T G \mathbf{h}'(\mathbf{u})) = \det(\mathbf{h}'(\mathbf{u}))^2 \det G. \tag{10.21}$$

So

$$\text{vol}|_+ = \sqrt{\det G}(dx_1 \wedge \cdots dx_n)|_+ = \sqrt{\det G}\,|\det \mathbf{h}'(\mathbf{u})\,|(du_1 \wedge \cdots \wedge du_n)|_+. \tag{10.22}$$

Its coefficient transforms as the coefficient of a twisted n-form should.

The above formula for the volume element in terms of coordinates is simple, but it would be nice to have a deeper understanding of where it comes from. The remainder of this section is an attempt to gain this perspective.

Suppose that M is n-dimensional and σ is a volume form, that is, a n form on M that never vanishes. Then σ has an associated orientation $[\sigma]$. There is then a twisted n-form

$$\sigma|_+ = (\sigma : [\sigma]) \tag{10.23}$$

that is the associated volume element. Thus $\sigma|_+$ is σ oriented by itself. (It is also $-\sigma$ oriented by itself.)

Given a twisted n-form volume element $\sigma|_+$ and a list of vector fields $X_1, \ldots, X_n$ there is a corresponded twisted scalar field

$$\langle \sigma|_+ \mid X_1, \ldots, X_n\rangle = (\langle \sigma \mid X_1, \ldots, X_n\rangle : [\sigma]). \tag{10.24}$$

The sign of the scalar field $\langle \sigma \mid X_1, \ldots, X_n\rangle$ is positive or negative depending on whether the orientation of $X_1, \ldots, X_n$ agrees with that of σ or not. One way to describe a volume element is through the twisted scalar fields $\langle \sigma|_+ \mid X_1, \ldots, X_n\rangle$.

In general the exterior product of two twisted forms is a form. In particular, the product of two twisted scalar fields is a scalar field. This is the basis of the following result.

Theorem 10.1 *Suppose that M is n-dimensional and g is a metric tensor. Then there is an associated twisted n-form volume element $\mathrm{vol}|_+$ such that for every list of vector fields $X_1, \ldots, X_n$ the twisted scalar field $\langle \mathrm{vol}|_+ \mid X_1, \ldots, X_n \rangle$ has a square that is the scalar field*

$$(\langle \mathrm{vol}|_+ \mid X_1, \ldots, X_n \rangle)^2 = \langle \mathsf{g}X_1 \wedge \cdots \wedge \mathsf{g}X_n \mid X_1, \ldots, X_n \rangle. \tag{10.25}$$

Proof: The straightforward approach is to compute with coordinates $u_1, \ldots, u_n$. In the proof expressions like $du_1 \wedge \cdots \wedge du_n$ are abbreviated $du_1 \cdots du_n$. Let

$$X_i = \sum_j a_{ij} \frac{\partial}{\partial u_j}. \tag{10.26}$$

Since

$$\mathsf{g}\frac{\partial}{\partial u_j} = \sum_k g_{jk}\, du_k, \tag{10.27}$$

it follows that

$$\mathsf{g}X_i = \sum_k \sum_j a_{ij} g_{jk}\, du_k = \sum_k \bar{a}_{ik}\, du_k \tag{10.28}$$

where $\bar{a}_{ik} = \sum_j a_{ij} g_{jk}$. This last equation is a matrix product $\bar{A} = AG$.

The next task is to compute the exterior product. This is the n-form

$$\prod_{i=1}^n (\mathsf{g}X_i) = \prod_{i=1}^n \sum_k \bar{a}_{ik}\, du_k = \sum_{k_1,\ldots,k_n} \prod_{i=1}^n \bar{a}_{ik_i}\, du_{k_i}. \tag{10.29}$$

This in turn is

$$\prod_{i=1}^n (\mathsf{g}X_i) = \sum_{k_1,\ldots,k_n} \prod_{i=1}^n \bar{a}_{ik_i} \epsilon_{k_1,\ldots k_n}\, du_1 \ \cdots\ du_n. \tag{10.30}$$

In summary

$$\prod_{i=1}^n (\mathsf{g}X_i) = \det(\bar{A})\, du_1 \cdots du_n. = g \det(A)\, du_1 \ \cdots\ du_n. \tag{10.31}$$

Here g is the standard notation for $\det(G)$.

The computation continues with

$$\langle du_1 \cdots du_n \mid X_1, \ldots X_n \rangle = \sum_{j_1,\ldots,j_n} \prod_{i=1}^n a_{ij_i} \epsilon_{j_1,\ldots,j_n} = \det A. \tag{10.32}$$

The final result is

$$\langle \mathsf{g}X_1 \wedge \cdots \wedge \mathsf{g}X_n \mid X_1, \ldots, X_n \rangle = g \det(A)^2. \tag{10.33}$$

This suggests that the twisted scalar field must be

$$\langle \mathrm{vol}|_+ \mid X_1 \ldots, X_n \rangle = (\sqrt{g}\det(A) : [du_1 \cdots du_n]). \tag{10.34}$$

Indeed, the sign of $\det(A)$ indicates the relative orientation of $X_1, \ldots, X_n$ with $\partial/\partial u_1, \ldots, \partial/\partial u_n$. $\square$

Corollary 10.2 *In coordinates the volume element is given by*

$$\mathrm{vol}|_+ = \sqrt{g}\,(du_1 \cdots du_n)|_+. \tag{10.35}$$

Proof: According to the proof of the theorem, $\langle \mathrm{vol}|_+ \mid X_1 \ldots, X_n \rangle$ is given by

$$(\sqrt{g}\det(A) : [du_1 \cdots du_n]) = (\sqrt{g}\langle du_1 \cdots du_n \mid X_1, \ldots X_n \rangle : [du_1 \cdots du_n]). \tag{10.36}$$

This leads to

$$\langle \mathrm{vol}|_+ \mid X_1 \ldots, X_n \rangle = \langle (\sqrt{g}\, du_1 \cdots du_n : [du_1 \cdots du_n]) \mid X_1, \ldots X_n \rangle. \tag{10.37}$$

The conclusion is that

$$\mathrm{vol}|_+ = (\sqrt{g}\, du_1 \cdots du_n : [du_1 \cdots du_n]). \tag{10.38}$$

$\square$

A byproduct of the discussion of volume element is a remarkable formula for the n-tuple twisted scalar product. This generalizes the triple twisted scalar product of elementary vector algebra.

Theorem 10.3 *Suppose that M is n-dimensional and g is a metric tensor. Then there is a twisted scalar field identity*

$$*(\mathsf{g}X_1 \wedge \cdots \wedge \mathsf{g}X_n) = \langle \mathrm{vol}|_+ \mid X_1, \ldots, X_n \rangle. \tag{10.39}$$

Lemma 10.4 *The twisted scalar field identity is equivalent to the n-form identity*

$$\mathsf{g}X_1 \wedge \cdots \wedge \mathsf{g}X_n = \langle \mathrm{vol}|_+ \mid X_1, \ldots, X_n \rangle\, \mathrm{vol}|_+. \tag{10.40}$$

Proof: By the lemma it is sufficient to show that

$$\langle \mathsf{g}X_1 \wedge \cdots \wedge \mathsf{g}X_n \mid Y_1, \ldots, Y_n \rangle = \langle \mathrm{vol}|_+ \mid X_1, \ldots, X_n \rangle\, \langle \mathrm{vol}|_+ \mid Y_1, \ldots, Y_n \rangle \tag{10.41}$$

Let A be the matrix that expresses the X_i in turns of cooordinate basis vectors, and let B be the matrix that expresses that Y_j in terms of coordinate basis vectors. Then the same kind of computation as in the previous theorem gives

$$\langle \mathsf{g}X_1 \wedge \cdots \wedge \mathsf{g}X_n \mid Y_1, \ldots, Y_n \rangle = g\det(A)\det(B) = \sqrt{g}\det(A)\sqrt{g}\det(B). \tag{10.42}$$

This translates to the desired result. $\square$

10.1.3 Pullback of a metric tensor

There is a very important construction that produces new metrics. Suppose that the n-dimensional space has coordinates $x_1, \ldots, x_n$, and there is a k-dimensional immersed surface S with coordinate $u_1, \ldots, u_k$. Start with the metric

$$\mathsf{g} = \sum_{i=1}^{n} \sum_{j=1}^{n} dx_i \, g_{ij} \, dx_j. \tag{10.43}$$

Let $\phi : S \to M$ be an immersion. The pullback by ϕ of the metric to S is

$$\mathsf{g}^* = \phi^* \mathsf{g} = \sum_{i=1}^{n} \sum_{j=1}^{n} d(x_i \circ \phi) \, (g_{ij} \circ \phi) \, d(x_j \circ \phi). \tag{10.44}$$

The condition that the mapping is an immersion ensures that the pullback quadratic form continues to be non-degenerate.

Inserting

$$d(x_i \circ \phi) = \sum_{\alpha=1}^{k} \frac{\partial (x_i \circ \phi)}{\partial u_\alpha} \, du_\alpha \tag{10.45}$$

gives the following result.

Proposition 10.5 (Pullback metric in coordinates) *Let ϕ be an immersion from S to M. The pullback $\mathsf{g}^* = \phi^* \mathsf{g}$ of*

$$\mathsf{g} = \sum_{i=1}^{n} \sum_{j=1}^{n} dx_i \, g_{ij} \, dx_j \tag{10.46}$$

by ϕ is

$$\mathsf{g}^* = \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} du_\alpha \, g^*_{\alpha\beta} \, du_\beta, \tag{10.47}$$

where

$$g^*_{\alpha\beta} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial (x_i \circ \phi)}{\partial u_\alpha} (g_{ij} \circ \phi) \frac{\partial (x_j \circ \phi)}{\partial u_\beta}. \tag{10.48}$$

A simple example is the pullback of the Euclidean metric given above to the sphere $x^2 + y^2 + z^2 = a^2$. The metric pulls back to

$$\mathsf{g}^* = a^2 \, d\theta^2 + a^2 \sin^2(\theta) \, d\phi^2. \tag{10.49}$$

This is not a flat metric. Even if one only considers a small open subset of the sphere, it is still impossible to find coordinates u, v such that $\mathsf{g}^* = du^2 + dv^2$.

Remark: The words *tensor* and *tensor field* can refer to a number of kinds of objects. Strictly speaking, a tensor is defined at a particular point, and a tensor field is a function that assigns to each point a tensor at that point. More precisely, a tensor of type (p, q) at a point is a real multi-linear function whose

inputs consist of q vectors in the tangent space at the point and p vectors in the dual space to the tangent space at the point (covectors). When $p = 0$ and all the inputs are vectors it is called a *covariant tensor*. When $q = 0$ and all the inputs are covectors it is called a *contravariant tensor*. When both kinds of vectors are allowed as inputs, it is called a *mixed tensor*. A tensor field assigns (in a smooth way) to each point in a manifold patch a corresponding tensor at that point. People often use the word tensor to mean tensor field.

The most basic tensor fields are scalar fields of type (0,0), vector fields of type (1,0), and differential 1-forms of type (0,1) There are also more complicated tensor fields. A differential k-form assigns to each point a real multi-linear function on k-tuples of tangent vectors at the point, so it is of type $(0, k)$. A metric tensor field assigns to each pair an inner product on tangent vectors at the point, so it is of type $(0, 2)$. For the more complicated tensors one can also impose symmetry conditions. Thus one distinguishes between the *antisymmetric tensor* case (differential k-forms) and the *symmetric tensor* case (the metric tensor). The metric tensor is required not only to be symmetric, but also strictly positive definite. The inverse of the metric tensor is a tensor of type $(2, 0)$; it is also symmetric and strictly positive definite.

The main example in these lectures of a mixed tensor is a (1,1) tensor, that is, a linear transformation. An example is the linear transformation associated with a vector field at a zero. This should be distinguished from the quadratic form associated with a scalar field at a critical point, which is a symmetric covariant tensor of type (0,2).

The study of tensors at a point is called *tensor algebra*, while the study of tensor fields is *tensor calculus*. The metric tensor field provides a particularly rich ground to explore. A choice of metric tensor field is the beginning of a subject called *Riemannian geometry*. The metric tensor field and related objects are fundamental to Einstein's general relativity. ||

10.1.4 Gaussian curvature*

A major topic in differential geometry is *curvature*. It is a complicated subject, even in the two dimensional case. In this case the curvature defined by the metric tensor is measured by a scalar field called the *Gaussian curvature*. When the Gaussian curvature is non-zero, then the metric tensor cannot be flat.

Example: It is easy to generate examples. One way is to consider a two-dimensional surface in three-dimensional space with the induced metric. For instance, the sphere with metric $\mathsf{g} = d\theta^2 + \sin^2\theta\, d\phi^2$ is not flat. (For a quick proof, see [41].) Another example where the metric is not flat is the upper half plane $y > 0$ with the Poincaré metric $\mathsf{g} = (dx^2 + dy^2)/y^2$. ||

These two examples are among the simplest, since they each have a constant Gaussian curvature. To see this, we need a formula for Gaussian curvature. There is one situation where the formula is not so complicated, namely, when the metric has the form $\mathsf{g} = h_1^2\, du_1^2 + h_2^2\, du^2$. In this situation u_1, u_2 form

orthogonal coordinates. The forms $h_1\,du_1$ and $h_2\,du_2$ are the normalized basis forms. The area form is $h_1 h_2\,du_1 \wedge du_2$.

Begin with the change in the unit basis forms given by

$$\begin{aligned} d(h_1\,du_1) &= -\frac{1}{h_2}\frac{\partial h_1}{\partial u_2} du_1 \wedge h_2\,du_2 \\ d(h_2\,du_2) &= -\frac{1}{h_1}\frac{\partial h_2}{\partial u_1}\,du_2 \wedge h_1\,du_1. \end{aligned} \tag{10.50}$$

Use these coefficients to define the 1-form

$$\omega = \frac{1}{h_2}\frac{\partial h_1}{\partial u_2}\,du_1 - \frac{1}{h_1}\frac{\partial h_2}{\partial u_1}\,du_2. \tag{10.51}$$

This form represent the rate of change of the unit basis forms as one moves along some curve. It is created in a clever way so that

$$\begin{aligned} d(h_1\,du_1) &= -\omega \wedge h_2\,du_2 \\ d(h_2\,du_2) &= \omega \wedge h_1\,du_1. \end{aligned} \tag{10.52}$$

The Gaussian curvature K is defined by

$$d\omega = K\,h_1\,du_1 \wedge h_2\,du_2. \tag{10.53}$$

This is a special case of the *Brioschi formula* for Gaussian curvature in two dimensions. Since

$$\int_R K\text{area} = \int_{\partial R} \omega, \tag{10.54}$$

the Gaussian curvature in a region R measures the total amount of turning around the closed curve ∂R that forms its boundary. Formulas of this type are treated in [47].

Example: For the sphere example $\mathsf{g} = d\theta^2 + \sin^2(\theta)\,d\phi^2$. Thus $h_1 = 1$ and $h_2 = \sin(\theta)$. Only the second term in ω contributes. The form is $\omega = -\cos(\theta)\,d\phi$. Thus $d\omega = \sin(\theta)\,d\theta \wedge d\phi$, so the Gaussian curvature is the constant 1. On curves of constant longitude (which are great circles) there is no turning. On curves of constant latitude the turning is larger near the north and south poles.

For the Poincaré upper half plane $\mathsf{g} = (dx^2 + dy^2)/y^2$, so $h_1 = h_2 = 1/y$. Only the first term in ω contributes. The 1-form $\omega = -1/y\,dx$. Then $d\omega = 1/y^2\,dy \wedge dx = -1/y^2\,dx \wedge dy$. So the Gaussian curvature is the constant -1. ||

Example: If one chooses a metric tensor with no special structure, then it is quite likely that the curvature is non-zero. It is amusing to look at the parabolic coordinate example $\mathsf{g} = (u^2 + v^2)(du^2 + dv^2)$. We know this is flat, since it is just $\mathsf{g} = dx^2 + dy^2$ in disguise. However it is not immediately obvious from the formula involving u, v that the Gaussian curvature is zero. ||

Example: As another example, take the helicoid surface given parametrically by $(x, y, z) \leftarrow (u\cos v, u\sin v, v)$. The metric on this surface is the pullback of $dx^2 + dy^2 + dz^2$ which is $d(u\cos v)^2 + d(u\sin v)^2 + dv^2$. This works out to be $du^2 + (u^2+1)\,dv^2$, so $h_1 = 1$ and $h_2 = (u^2+1)^{\frac{1}{2}}$. It follows that

$\omega = -u(u^2+1)^{-\frac{1}{2}}\,dv$. The turning goes along with vertical motion. Then $d\omega = -(u^2+1)^{-\frac{3}{2}}\,du \wedge dv = -1/(u^2+1)^2(u^2+1)^{\frac{1}{2}}\,du \wedge dv$. So the curvature $K = -1/(u^2+1)^2$. It is greatest near the z axis. ||

10.2 Vector calculus

10.2.1 Vector algebra conventions

At this point there will be a change in notation. In mathematics it is common to denote a vector field by letters X, Y, Z and so on. In applied mathematics it is more common to use letters such $\mathbf{u}, \mathbf{v}, \mathbf{w}$ to denote a vector field. From now on the applied mathematics convention will be in frequent use. In particular, the action of a vector field $\mathbf{v}$ differentiating a scalar field s will be written in a notation like $\mathbf{v} \lrcorner\, ds$ that makes the differentiation explicit.

If α is a 1-form and $\mathbf{v}$ is a vector field, then we write

$$\alpha\mathbf{v} = \langle \alpha \mid \mathbf{v} \rangle \tag{10.55}$$

and

$$\mathbf{v}\alpha = \mathbf{v} \lrcorner\, \alpha, \tag{10.56}$$

so in particular $\alpha\mathbf{v} = \mathbf{v}\alpha$. Along with this, we may indicate the action of a vector field $\mathbf{v}$ on a scalar field s as

$$\mathbf{v}\; ds = \mathbf{v} \lrcorner\, ds. \tag{10.57}$$

Suppose that g is a metric tensor. Then it is consistent with these notations to write the inner product of two vector fields as

$$\mathbf{v} \cdot \mathbf{w} = \mathsf{g}\mathbf{v}\; \mathbf{w} = \mathbf{v}\; \mathsf{g}\mathbf{w}. \tag{10.58}$$

Here are the explicit formulas for vector algebra in Euclidean space. The notation vol refers to an n-form that is supposed by some convention to express a corresponding twisted n-form. The notation is that $*$ switches between scalar s and n-form $s\,\mathrm{vol}$. Also $\mathsf{c}\mathbf{v} = \mathbf{v} \lrcorner \mathrm{vol}$. A scalar or vector whose definition depends on the choice of volume form is called a twisted scalar (pseudoscalar) or twisted vector (pseudovector).

There are two expressions for the scalar product (inner product, dot product). They are

$$\mathbf{u} \cdot \mathbf{v} = \mathsf{g}\mathbf{u}\; \mathbf{v} = \mathbf{u} \lrcorner\, \mathsf{g}\mathbf{v} \tag{10.59}$$

$$\mathbf{u} \cdot \mathbf{v} = *(\mathsf{g}\mathbf{u} \wedge \mathsf{c}\mathbf{v}). \tag{10.60}$$

The first is the usual interior product expression for the scalar product. The second exterior product expression comes from looking at how the vector $\mathbf{u}$ stands out from the plane normal to $\mathbf{v}$.

Proposition 10.6 *The interior product and the exterior product expressions for the scalar product are equal.*

Proof: Since $\mathbf{gv} \wedge \text{vol} = 0$, the product rule (with appropriate sign) gives

$$0 = \mathbf{v} \lrcorner (\mathbf{gu} \wedge \text{vol}) = (\mathbf{v} \lrcorner \mathbf{gu})\text{vol} - \mathbf{gu} \wedge (\mathbf{u} \lrcorner \text{vol}). \tag{10.61}$$

This implies the that the two expressions give the same result. □

In three dimensions there are also two expressions for the vector product (cross product). They are

$$\mathbf{u} \times \mathbf{v} = \mathsf{c}^{-1}(\mathbf{gu} \wedge \mathbf{gv}) \tag{10.62}$$

$$\mathbf{u} \times \mathbf{v} = -\mathsf{g}^{-1}(\mathbf{u} \lrcorner \mathsf{c}\mathbf{v}). \tag{10.63}$$

The first is the exterior product representation; the second is the interior product representation.

Remark: The map c sends untwisted or twisted vectors to twisted or untwisted 2-forms. As a consequence the vector product interchanges vectors and twisted vectors (pseudovectors) in a predictable way. ||

Proposition 10.7 *The exterior product and interior product expressions for the vector product in three dimensions are equal.*

Proof: This result is more subtle. One path to a proof is to show that with either definition the scalar expressions $\mathbf{w} \cdot (\mathbf{u} \cdot \mathbf{v})$ are the same, for arbitrary $\mathbf{w}$.

With the exterior product definition the computation begins with the exterior product representation of the scalar product. This leads to $*(\mathbf{gw} \wedge \mathbf{gu} \wedge \mathbf{gv})$.

With the interior product definition the interior product representation of the scalar product gives the appropriate formula. This is $-\mathbf{w} \lrcorner (\mathbf{u} \lrcorner (\mathbf{v} \lrcorner \text{vol}))$. The formula works out to be $-\langle \mathbf{u} \lrcorner (\mathbf{v} \lrcorner \text{vol}) \mid \mathbf{w} \rangle = -\langle \mathbf{v} \lrcorner \text{vol} \mid \mathbf{u}, \mathbf{w} \rangle = -\langle \text{vol} \mid \mathbf{v}, \mathbf{u}, \mathbf{w} \rangle$. This final result may be written as $\langle \text{vol} \mid \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle$.

The result follows from the triple twisted scalar identity $*(\mathbf{gu} \wedge \mathbf{gv} \wedge \mathbf{gw}) = \langle \text{vol} \mid \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle$. The n-dimensional version of this identity was proved in a previous section. □

These expressions help explain some of the standard identities of vector algebra. For instance, we have just seen that in three dimensions the scalar triple product is

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = *(\mathbf{gu} \wedge \mathbf{gv} \wedge \mathbf{gw}) = \text{vol}(\mathbf{u}, \mathbf{v}, \mathbf{w}).. \tag{10.64}$$

This relates it to the volume of a parallelepiped. The vector triple product is

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = -\mathsf{g}^{-1}\left(\mathbf{u} \lrcorner (\mathbf{gv} \wedge \mathbf{gw})\right) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}. \tag{10.65}$$

This reflects the product rule property of the interior product.

There is also the famous identity

$$(\mathbf{u} \times \mathbf{z}) \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v})(\mathbf{z} \cdot \mathbf{w}) - (\mathbf{u} \cdot \mathbf{w})(\mathbf{z} \cdot \mathbf{v}). \tag{10.66}$$

This determinant is related to the square of the area of a parallelogram.

There are corresponding operations in two dimensions. One useful remark is that the natural notion of cross product in two dimensions is

$$\mathbf{u} \times \mathbf{v} = *(\mathbf{g}\mathbf{u} \wedge \mathbf{g}\mathbf{v}) = \langle \mathbf{c}\mathbf{u} \mid \mathbf{v} \rangle. \tag{10.67}$$

This two-dimensional cross product of two vectors is a twisted scalar (pseudoscalar).

10.2.2 Gradient, divergence, and Laplacian

In the presence of a metric tensor there are three important operators: the gradient grad, the divergence div, and the Laplacian

$$\nabla^2 = \operatorname{div} \operatorname{grad}. \tag{10.68}$$

Here is a relevant quotation from Edward Nelson [34].

> The Laplace operator in its various manifestations is the most beautiful and central object in all of mathematics. Probability theory, mathematical physics, Fourier analysis, partial differential equations, the theory of Lie groups, and differential geometry all revolve around this sun, and its light even penetrates such obscure regions as number theory and algebraic geometry.

Vector calculus makes sense for an arbitrary metric tensor field g. The volume form vol, the operation c (interior product with the volume form), and the Hodge dual $*$ are all defined in terms of the metric. The operator $*$ maps 1-forms to $(n-1)$-forms. The three operators are related by $\mathsf{c} = *\mathsf{g}$. There is also an operator $*$ that maps 0 forms to n forms by multiplying by vol. Its inverse going from n-forms to 0-forms is also denoted $*$.

The *gradient* of a scalar is the vector field

$$\operatorname{grad} s = \mathsf{g}^{-1}\, ds. \tag{10.69}$$

In coordinates the gradient has the form

$$\operatorname{grad} s = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} g^{jk} \frac{\partial s}{\partial u_k} \right) \frac{\partial}{\partial u_j}. \tag{10.70}$$

The other basic operation is the divergence treated in the previous chapter. The divergence of a vector field $\mathbf{v}$ is defined by

$$\operatorname{div} \mathbf{v} \operatorname{vol} = d(\mathbf{v} \lrcorner \operatorname{vol}). \tag{10.71}$$

Alternatively,

$$\operatorname{div} \mathbf{v} = *d\mathsf{c}\mathbf{v}. \tag{10.72}$$

Another notation is in common use. It is imagined that there is a vector of differential operators $\overrightarrow{\nabla}$ that combines with scalar fields and vector fields. In this notation

$$\begin{aligned} \text{grad } s &= \overrightarrow{\nabla} s \\ \text{div } \mathbf{v} &= \overrightarrow{\nabla} \cdot \mathbf{v}. \end{aligned} \tag{10.73}$$

Remark: For the moment $\overrightarrow{\nabla}$ is a convenient abbreviation. It will turn out that $\overrightarrow{\nabla} = \mathsf{g}^{-1}\nabla$, where ∇ is the covariant differential. ||

Example: In physics the force is the negative of the gradient of the potential energy, that is, $\mathbf{F}(\mathbf{x}) = -\overrightarrow{\nabla} V(\mathbf{x}) = -\mathsf{g}^{-1} dV(\mathbf{x})$. The law of motion is the vector equation

$$m\frac{d^2\mathbf{x}}{dt^2} + \kappa\frac{d\mathbf{x}}{dt} = -\overrightarrow{\nabla} V(\mathbf{x}). \tag{10.74}$$

Here $m > 0$ is mass, $\kappa > 0$ is the drag coefficient, and $V(\mathbf{x})$ is the potential energy function.

When $\kappa = 0$ this implies conservation of energy:

$$\frac{d}{dt}\left(\frac{1}{2}m\frac{d\mathbf{x}}{dt}\mathsf{g}\frac{d\mathbf{x}}{dt} + V(\mathbf{x})\right) = \left(m\frac{d^2\mathbf{x}}{dt^2} + \overrightarrow{\nabla} V(\mathbf{x})\right)\mathsf{g}\frac{d\mathbf{x}}{dt} = 0. \tag{10.75}$$

When $m = 0$ this is gradient descent, and the potential energy decreases:

$$\kappa\frac{d}{dt}V(\mathbf{x}) = -\overrightarrow{\nabla} V(\mathbf{x})\ \mathsf{g}\ \overrightarrow{\nabla} V(\mathbf{x}) \le 0. \tag{10.76}$$

||

Theorem 10.8 (Adjointness of gradient and divergence) *If s and $\mathbf{v}$ are a scalar field and a vector field defined in the bounded region Ω, then*

$$\int_\Omega s\ \text{div}\,\mathbf{v}\,\text{vol} + \int_\Omega \text{grad}\, s\,\mathsf{g}\,\mathbf{v}\,\text{vol} = \int_{\partial\Omega} s\,\mathbf{v}\,\lrcorner\,\text{vol}. \tag{10.77}$$

Equivalently,

$$\int_\Omega s\,\overrightarrow{\nabla}\cdot\mathbf{v}\,\text{vol} + \int_\Omega \overrightarrow{\nabla} s\,\mathsf{g}\,\mathbf{v}\,\text{vol} = \int_{\partial\Omega} s\,\mathbf{v}\,\lrcorner\,\text{vol}. \tag{10.78}$$

Proof: This is just integration by parts. By the product rule we have

$$d(s\,\mathbf{v}\,\lrcorner\,\text{vol}) = ds\wedge\mathbf{v}\,\lrcorner\,\text{vol} + s\,d(\mathbf{v}\,\lrcorner\,\text{vol}). \tag{10.79}$$

However $ds\wedge\mathbf{v}\,\lrcorner\,\text{vol} = \langle ds \mid \mathbf{v}\rangle\text{vol} = \text{grad}\ s\,\mathsf{g}\,\mathbf{v}\,\text{vol}$. So this is

$$d(s\,\mathbf{v}\,\lrcorner\,\text{vol}) = \text{grad}\ s\,\mathsf{g}\,\mathbf{v}\,\text{vol} + s\ \text{div}\ \mathbf{v}\,\text{vol}. \tag{10.80}$$

Integrate and use Stokes' theorem. □

The form $\mathbf{v}\,\lrcorner\,\text{vol}$ pulls back to zero on $\partial\Omega$ when $\langle\text{vol} \mid \mathbf{v}, X_1, X_2\rangle = 0$ for all vectors X_1, X_2 tangent to $\partial\Omega$. This says that $\mathbf{v}$ is itself tangent to $\partial\Omega$.

The identity is often used in cases when s or $\mathbf{v}$ satisfies a *boundary condition* that makes the boundary contribution zero. The most basic such conditions are:

$$\begin{array}{ccccccc} & \text{grad} & & & & \text{div} & \\ S & \longrightarrow & V & \longrightarrow & V & \longrightarrow & S \\ \downarrow & \mathsf{g} & \downarrow & \mathsf{c} & \downarrow & * & \uparrow \\ 0 & \longrightarrow & 1 & \longrightarrow & (n-1)^{\sim} & \longrightarrow & n^{\sim} \\ & d & & * & & d & \end{array}$$

$$\nabla^2 = \text{div grad} = *\, d\, \mathsf{c}\mathsf{g}^{-1}\, d = *\, d\, *\, d. \tag{10.81}$$

Table 10.1: The Laplace operator: n dimensions

1. s is zero on $\partial\Omega$.

2. $\mathbf{v}$ is tangent to $\partial\Omega$.

The *Laplacian* of s is the divergence of the gradient. This will be denoted

$$\nabla^2 s = \text{div grad } s. \tag{10.82}$$

The Hodge dual operation $*$ interchanges 0-forms and n-forms. It also interchanges 1-forms and $(n-1)$ form. From $\mathsf{c} = *\mathsf{g}$ we get

$$\nabla^2 s = *d\mathsf{c}\,\mathsf{g}^{-1} ds = *\, d\, *\, d\, s. \tag{10.83}$$

See Table 10.1 for a summary of these relations. In the table the symbols S, V stand for scalar field and vector field. The $\sim$ indicates for pseudo-form (or twisted form). Recall that a scalar field in S is the same as a 0-form. The div grad expression is obtained by following the top path S, V, S. The $*d\mathsf{c}\mathsf{g}^{-1}d$ version is obtained by following the winding path $0, 1, V, (n-1)^{\sim}, n^{\sim}, S$. The $*d*d$ formula uses the lower path $0, 1, (n-1)^{\sim}, n^{\sim}, S$.

Remark: The following chapter introduces another diffferential operator δ, called the *codifferential*. With the (perhaps unusual) sign convention in that chapter, the codifferential is $\delta = *d*$ and sends 1-forms to 0-forms. Thus the Laplace operator on 0-forms has yet another expresion $\nabla^2 = \delta d$. ||

In coordinates the Laplacian $\nabla^2 = \text{div grad}$ is

$$\nabla^2\, s = \frac{1}{\sqrt{g}} \sum_{i=1}^{n} \frac{\partial}{\partial u_i} \left(\sqrt{g} \sum_{k=1}^{n} g^{ik} \frac{\partial s}{\partial u_k} \right). \tag{10.84}$$

Example: Consider parabolic coordinates given by $x = (1/2)(u^2 - v^2)$, $y = uv$. The metric tensor is

$$\mathsf{g} = dx^2 + dy^2 = (u\, du - v\, dv)^2 + (v\, du + u\, dv)^2 = (u^2 + v^2)(du^2 + dv^2). \tag{10.85}$$

The gradient is

$$\text{grad } s = \frac{1}{u^2 + v^2} \left(\frac{\partial s}{\partial u} \frac{\partial}{\partial u} + \frac{\partial s}{\partial v} \frac{\partial}{\partial v} \right). \tag{10.86}$$

The Laplacian is

$$\nabla^2 s = \frac{1}{u^2+v^2}\left(\frac{\partial^2 s}{\partial u^2} + \frac{\partial^2 s}{\partial v^2}\right). \tag{10.87}$$

||

Example: Suppose that the density function ρ gives rise to a diffusive flux

$$\mathbf{J} = -\frac{1}{2}\sigma^2 \overrightarrow{\nabla}\rho, \tag{10.88}$$

where σ^2 is the diffusion constant. The continuity equation is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \tag{10.89}$$

These combine to give the *diffusion equation*

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\sigma^2 \nabla^2 \rho. \tag{10.90}$$

The solution of this equation for $t > 0$ is $\rho = f(\mathbf{x}, t)$, where

$$f(\mathbf{x}, t) = \int \delta_{\sigma^2 t}(\mathbf{x} - \mathbf{y}) f(\mathbf{y}, 0)\, d\mathbf{y}. \tag{10.91}$$

Here $\delta_\epsilon(\mathbf{x})$ is the Gaussian approximate delta function

$$\delta_\epsilon(\mathbf{x}) = \frac{1}{(2\pi\epsilon)^{\frac{n}{2}}} e^{-\frac{\mathbf{x}^2}{2\epsilon}}. \tag{10.92}$$

This says that the density spreads out at an average rate that is roughly $\sigma\sqrt{t}$.
||

Theorem 10.9 (Green's first identity) *If s and u are scalars defined in the bounded region Ω, then*

$$\int_\Omega s\, \nabla^2 u \,\mathrm{vol} + \int_\Omega \overrightarrow{\nabla} s \,\mathsf{g}\, \overrightarrow{\nabla} u \,\mathrm{vol} = \int_{\partial\Omega} s\, \overrightarrow{\nabla} u \lrcorner \,\mathrm{vol}. \tag{10.93}$$

Proof: This comes from the adjointness identity with $\mathbf{v} = \overrightarrow{\nabla} u$. □

The form $\overrightarrow{\nabla} u \lrcorner \,\mathrm{vol}$ pulls back to zero on $\partial\Omega$ when $\overrightarrow{\nabla} u$ is itself tangent to $\partial\Omega$. This is in turn equivalent to saying that $\overrightarrow{\nabla} u$ is orthogonal to N for N orthogonal to the surface. Finally, this says that $\langle du \mid N\rangle = 0$ for vectors N normal (orthogonal) to the surface. .

The identity is often used in cases when s satisfies a *boundary condition* that makes the boundary contribtution zero. The most basic such conditions are:

1. s is zero on $\partial\Omega$.
2. The normal derivative $\langle du \mid N\rangle$ is zero on $\partial\Omega$.

If either of these boundary conditions is satisfied, then

$$-\int_\Omega s\,\nabla^2 u\,\mathrm{vol} = \int_\Omega \overrightarrow{\nabla} s\,\mathsf{g}\,\overrightarrow{\nabla} u\,\mathrm{vol}. \tag{10.94}$$

In particular,

$$-\int_\Omega u\,\nabla^2 u\,\mathrm{vol} = \int_\Omega \overrightarrow{\nabla} u\,\mathsf{g}\,\overrightarrow{\nabla} u\,\mathrm{vol} \geq 0. \tag{10.95}$$

This suggests that in some sense $-\nabla^2$ is a positive operator. One possible interpretation is that this integral is the energy of the scalar field u.

Theorem 10.10 (Green's second identity) *If s and u are scalars defined in the bounded region Ω, then*

$$\int_\Omega s\,\nabla^2 u\,\mathrm{vol} = \int_\Omega \nabla^2 s\,u\,\mathrm{vol} + \int_{\partial\Omega} \left(s\,\overrightarrow{\nabla} u - u\overrightarrow{\nabla} s\right) \lrcorner\,\mathrm{vol}. \tag{10.96}$$

When the right hand side vanishes the Laplacian is adjoint to itself. This is the starting point for serious mathematics with the Laplace operator.

The remaining objects are in three dimensions. The *cross product* of two vectors $\mathbf{v}$ and $\mathbf{w}$ is defined as

$$(\mathbf{v} \times \mathbf{w}) = \mathsf{c}^{-1}(\mathbf{g}\mathbf{v} \wedge \mathbf{g}\mathbf{w}). \tag{10.97}$$

In other words, it is the unique vector $\mathbf{v} \times \mathbf{w}$ such that $(\mathbf{v} \times \mathbf{w}) \lrcorner \mathrm{vol} = \mathbf{g}\mathbf{v} \wedge \mathbf{g}\mathbf{w}$. It is the operation on vectors that corresponds to the exterior product on forms. The *curl* of a vector field $\mathbf{v}$ is defined by

$$\mathrm{curl}\ \mathbf{v} = \mathsf{c}^{-1}\,d(\mathbf{g}\mathbf{v}). \tag{10.98}$$

In the notation with the vector of differential operators it is also possible to write

$$\mathrm{curl}\,\mathbf{v} = \overrightarrow{\nabla} \times \mathbf{v}. \tag{10.99}$$

It is easy to see that curl grad $s = 0$ and that div curl $\mathbf{v} = 0$. This because the definitions are made so that these are instances of $d\,ds = 0$ and $d\,d\omega = 0$.

10.2.3 Vector algebra and calculus summary

The relation between differential form and vector field operations in dimension 3 is summarized in Table 10.2. The numbers are the degree of the forms. The S, V, $V^\sim$, $S^\sim$ stand for scalar, vector, pseudo-vector, pseudo-scalar. The pseudo quantities are the ones that depend on the orientation of space. These can be reversed; what is important is that the curl operation depends on orientation.

One can make a similar description for differential forms and vectors in dimension 2. This is given in Table 10.3. The situation is actually more complicated, since there are two choices of the vector operations. In the first one the gradient is a vector field in two dimensions, and the curl is then a pseudo-scalar.

$$\begin{array}{ccccccc} & & d & & d & & d & \\ 0 & \longrightarrow & 1 & \longrightarrow & 2 & \longrightarrow & 3 \\ \\ & & \text{grad} & & \text{curl} & & \text{div} & \\ S & \longrightarrow & V & \longrightarrow & V^{\sim} & \longrightarrow & S^{\sim} \end{array}$$

Table 10.2: Exterior derivative vs vector operations: 3 dimensions

$$\begin{array}{ccccccc} & & d & & d & & \\ 0 & \longrightarrow & 1 & \longrightarrow & 2 & & \\ \\ & & \text{grad} & & \text{curl} & & \\ S & \longrightarrow & V & \longrightarrow & S^{\sim} & & \\ \\ & & \text{curl}' & & \text{div} & & \\ S & \longrightarrow & V^{\sim} & \longrightarrow & S^{\sim} & & \end{array}$$

Table 10.3: Exterior derivative vs vector operations: 2 dimensions

In the second one the operation called curl' acts on a scalar as if it were the third component of a vector and produces a pseudo-vector.

The classical vector field calculus definitions may be summarized as follows. The expressions are valid with arbitrary metric tensor and in an arbitrary coordinate system. The subscript on curl indicates the dimension. (Thus curl_3 is the usual curl, while curl_2 is the curl with scalar values.) The operation curl' works in arbitrary dimension; it sends $(n-2)$-forms to vector fields. (In two dimensions curl' is the curl with scalar inputs.)

$$\begin{aligned} \text{grad} &= \mathsf{g}^{-1} d \\ \text{curl}_3 &= \mathsf{c}^{-1} d\, \mathsf{g} \\ \text{curl}_2 &= *d\, \mathsf{g} \\ \text{curl}' &= \mathsf{c}^{-1} d \\ \text{div} &= *d\, \mathsf{c}. \end{aligned} \tag{10.100}$$

The curl expressions all map scalar or vector fields into pseudoscalar or pseudovector fields. In order to use these operations in the context of Stokes' theorem, it is necessary to formulate them as statements about the exterior derivative d. In that form, these identities become

$$\begin{aligned} \mathsf{g}\ \text{grad}\ s &= d\, s \\ \mathsf{c}\ \text{curl}_3\ \mathbf{v} &= d\, \mathsf{g}\, \mathbf{v} \\ *\ \text{curl}_2\ \mathbf{v} &= d\, \mathsf{g}\, \mathbf{v} \\ \mathsf{c}\ \text{curl}'\ \omega &= d\, \omega \\ *\ \text{div}\ \mathbf{v} &= d\, \mathsf{c}\, \mathbf{v}. \end{aligned} \tag{10.101}$$

The most common vector field calculus identities follow immediately. In three dimensions they are

$$\begin{aligned} \operatorname{curl}_3 \operatorname{grad} &= 0 \\ \operatorname{div} \operatorname{curl}_3 &= 0 \\ \Delta_3 &= \operatorname{div} \operatorname{grad}. \end{aligned} \tag{10.102}$$

Here Δ_3 is the Laplacian acting on scalars in 3 dimensions.

In two dimensions they are

$$\begin{aligned} \operatorname{curl}_2 \operatorname{grad} &= 0 \\ \operatorname{div} \operatorname{curl}' &= 0 \\ \Delta_2 = \operatorname{div} \operatorname{grad} &= -\operatorname{curl}_2 \operatorname{curl}'. \end{aligned} \tag{10.103}$$

The Laplacian Δ_2 in 2 dimensions may be expressed in two quite different ways. This follows from the fact that in two dimensions $\mathsf{c}\mathsf{g}^{-1} = -\mathsf{g}\mathsf{c}^{-1}$.

There are also identities for the derivative of a product. Here are the basic ones for dimension 3. They are all valid with arbitrary metric and arbitrary coordinate system.

$$\begin{aligned} \operatorname{grad}(rs) &= (\operatorname{grad} r)s + r(\operatorname{grad} s) \\ \operatorname{div}(s\mathbf{v}) &= \operatorname{grad} s \cdot \mathbf{v} + s \operatorname{div} \mathbf{v} \\ \operatorname{curl}(s\mathbf{v}) &= \operatorname{grad} s \times \mathbf{v} + s \operatorname{curl} \mathbf{v} \\ \operatorname{div}(\mathbf{u} \times \mathbf{v}) &= \operatorname{curl} \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \operatorname{curl} \mathbf{v}. \end{aligned} \tag{10.104}$$

The most interesting of these identities is the last divergence identity one, since it has a minus sign. It is based on the differential of a product of two 1-forms. The minus sign is the usual one in a derivation involving a form of odd degree. (There are other identities for the derivative of a product; some of these may be valid only in Euclidean space.)

There are also Laplace operators acting on vector fields. In three and two dimensions these are

$$\begin{aligned} \overrightarrow{\Delta}_3 &= \operatorname{grad} \operatorname{div} - \operatorname{curl}_3 \operatorname{curl}_3 \\ \overrightarrow{\Delta}_2 &= \operatorname{grad} \operatorname{div} - \operatorname{curl}' \operatorname{curl}_2 . \end{aligned} \tag{10.105}$$

The motivation for these rather strange expressions will appear in the following chapter.

Proposition 10.11 (Vector versions of Stokes' theorem in 3 dimensions) *The vector field versions of the fundamental theorem of calculus, of Stokes' theorem, and of the divergence theorem are*

$$\int_C \mathsf{g} \operatorname{grad} s = s_1 - s_0, \tag{10.106}$$

and

$$\int_S \text{curl } \mathbf{v} \lrcorner \text{ vol} = \int_{\partial S} \mathbf{gv}, \tag{10.107}$$

and

$$\int_R \text{div } \mathbf{v} \, \text{vol} = \int_{\partial R} \mathbf{v} \lrcorner \text{vol}. \tag{10.108}$$

These are Stokes' theorem applied to s, $\mathbf{gv}$, *and* $\mathsf{cv} = \mathbf{v} \lrcorner \text{vol}$.

Another identity is

$$\int_R (\text{curl } \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \text{curl } \mathbf{v}) \text{vol} = \int_{\partial R} (\mathbf{u} \times \mathbf{v}) \lrcorner \text{vol}. \tag{10.109}$$

The vector field $\mathbf{u} \times \mathbf{v}$ defines an energy flux density $(\mathbf{u} \times \mathbf{v}) \lrcorner \text{vol}$. (An example is the Poynting vector field in electromagnetic theory.) This 2-form has an alternative representation $(\mathbf{u} \times \mathbf{v}) \lrcorner \text{vol} = \mathbf{gu} \wedge \mathbf{gv}$.

Proposition 10.12 (Vector versions of Stokes' theorem in 2 dimensions) *The vector field version of the fundamental theorem of calculus in 2 dimensions is either*

$$\int_C \mathsf{g} \text{ grad } s = s_1 - s_0 \tag{10.110}$$

or

$$\int_C \text{curl}' \, s \lrcorner \text{ area} = s_1 - s_0. \tag{10.111}$$

The vector field version of Green's theorem is either the Stokes version

$$\int_R \text{curl } \mathbf{v} \, \text{area} = \int_{\partial R} \mathbf{gv} \tag{10.112}$$

or the divergence version

$$\int_R \text{div } \mathbf{v} \, \text{area} = \int_{\partial R} \mathbf{v} \lrcorner \text{area}. \tag{10.113}$$

These are Stokes' theorem applied to s, $\mathbf{gv}$, *and* $\mathsf{cv} = \mathbf{v} \lrcorner \text{area}$.

Figure 10.1 illustrates the Stokes version in 2 dimensions, while Figure 10.2 illustrates the divergence version in 2 dimensions. Figure 10.3 is an attempt to show Stokes' theorem in 3 dimensions.

The classic vector calculus is in three dimensions; for the reader's convenience Table 10.4 summarizes the basic facts. The somewhat more complicated two-dimensional version is given in Table 10.5.

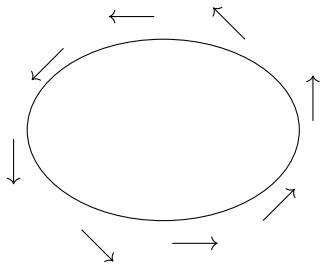

Figure 10.1: Stoke's theorem (2d): Arrows indicate vector field $\mathbf{v}$; curl($\mathbf{v}$) is twisted scalar field in interior.

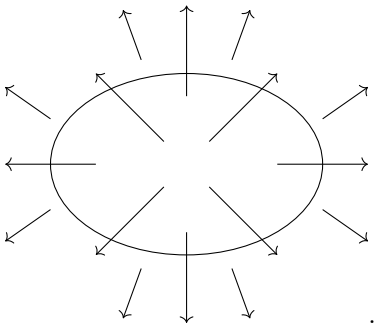

Figure 10.2: Divergence theorem (2d): Arrows indicate vector field $\mathbf{v}$; div($\mathbf{v}$) is scalar field in interior.

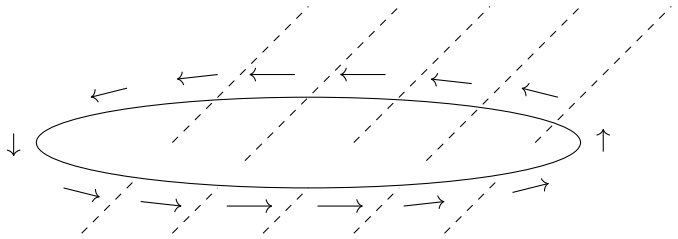

Figure 10.3: Stokes's theorem (3d): Arrows indicate vector field $\mathbf{v}$; broken curves (transverse orientations not shown) represent twisted vector field curl($\mathbf{v}$) .

10.2.4 A vector of differential operations*

Many treatments of vector analysis introduce a vector field whose values are certain differential operators. What is this object? This section provides an answer.

It is convenient to start with a different object that is a 1-form whose values are differential operators. This is the *covariant differential*

$$\nabla = \sum_i dx_i \frac{\partial}{\partial x_i}. \tag{10.114}$$

This should be independent of the coordinate system. On scalars this is familiar: $\nabla s = ds$. The new feature is that this is also supposed to act on vector fields and differential forms. For that the partial derivative on the right must be interpreted as a covariant derivative.

The covariant derivative applied to a coordinate basis vector field is

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x_k}. \tag{10.115}$$

The first partial derivative on the left hand side is a covariant derivative; the other partial derivatives denote basis vectors. The coefficients Γ^k_{ij} are given in terms of the metric by a complicated formula, to be considered later. For the case of Cartesian coordinates on Euclidean space they are identically zero. The coefficients Γ^k_{ij} are Christoffel symbols. They satisfy the symmetry identity $\Gamma^k_{ij} = \Gamma^k_{ji}$.

If we apply this to a vector field

$$Y = \sum_j b_j \frac{\partial}{\partial x_j} \tag{10.116}$$

and use the product rule, we get

$$\frac{\partial}{\partial x_i} Y = \sum_j \frac{\partial b_j}{\partial x_i}\frac{\partial}{\partial x_j} + \sum_j \sum_k \Gamma^k_{ij} b_j \frac{\partial}{\partial x_k}. \tag{10.117}$$

It is a trivial but confusing observation that this may also be written

$$\frac{\partial}{\partial x_i} Y = \sum_j \left(\frac{\partial b_j}{\partial x_i} + \sum_k \Gamma^j_{ik} b_k \right) \frac{\partial}{\partial x_j}. \tag{10.118}$$

The covariant derivative may also be applied to a differential k-form. It was shown in a previous chapter that it differentiates the coordinate basis forms according to the rule

$$\frac{\partial}{\partial x_i}\, dx_j = -\sum_k \Gamma^j_{ik}\, dx_k. \tag{10.119}$$

$$
\begin{aligned}
*s &= s\,\mathrm{vol}\ ;\ *(s\,\mathrm{vol}) = s \\
\mathsf{c}\mathbf{v} &= \mathbf{v} \lrcorner \mathrm{vol} \\
\mathbf{u}\cdot\mathbf{v} &= \mathbf{u} \lrcorner \mathsf{g}\mathbf{v} = *(\mathsf{g}\mathbf{u} \wedge \mathsf{c}\mathbf{v}) \\
\mathbf{u}\times\mathbf{v} &= \mathsf{c}^{-1}(\mathsf{g}\mathbf{u} \wedge \mathsf{g}\mathbf{v}) = -\mathsf{g}^{-1}(\mathbf{u} \lrcorner \mathsf{c}\mathbf{v}) \\
\mathrm{grad} &= \mathsf{g}^{-1} d \\
\mathrm{curl} &= \mathsf{c}^{-1} d \mathsf{g} \\
\mathrm{div} &= *d\mathsf{c} \\
\Delta &= \mathrm{div}\,\mathrm{grad}
\end{aligned}
$$

Table 10.4: Summary of 3-dimensional vector calculus

$$
\begin{aligned}
*s &= s\,\mathrm{area}\ ;\ *(s\,\mathrm{area}) = s \\
\mathsf{c}\mathbf{v} &= \mathbf{v} \lrcorner \mathrm{area} \\
\mathbf{u}\cdot\mathbf{v} &= \mathbf{u} \lrcorner \mathsf{g}\mathbf{v} = *(\mathsf{g}\mathbf{u} \wedge \mathsf{c}\mathbf{v}) \\
\mathbf{u}\times\mathbf{v} &= *(\mathsf{g}\mathbf{u} \wedge \mathsf{g}\mathbf{v}) = -\mathbf{u} \lrcorner \mathsf{c}\mathbf{v} \\
\mathrm{grad} &= \mathsf{g}^{-1} d \\
\mathrm{curl} &= *d\mathsf{g} \\
\mathrm{curl}' &= \mathsf{c}^{-1} d \\
\mathrm{div} &= *d\mathsf{c} \\
\Delta &= \mathrm{div}\,\mathrm{grad} = -\,\mathrm{curl}\,\mathrm{curl}'
\end{aligned}
$$

Table 10.5: Summary of 2-dimensional vector calculus

The derivative acts on more general forms using the product rule. For instance, on a 1-form it is

$$\frac{\partial}{\partial x_i}(\sum_j p_j\,dx_j) = \sum_j \frac{\partial p_j}{dx_i}\,dx_j - \sum_j \sum_k \Gamma^j_{ik} p_j\,dx_k. \tag{10.120}$$

In treatments of tensor analysis this is usually written in terms of the coefficients of the dx_j, so that the same result is expressed by changing the summation indices, leading to

$$\frac{\partial}{\partial x_i}(\sum_j p_j\,dx_j) = \sum_j \left(\frac{\partial p_j}{dx_i} - \sum_k \Gamma^k_{ij} p_k\right)\,dx_j. \tag{10.121}$$

While the expressions for the Christoffel symbols Γ^k_{ij} in terms of the metric tensor g_{mn} are complicated, there is one special relation where something simple results. This involves the volume form $\text{vol} = \sqrt{g}dx_1 \cdots dx_n$. It was shown in an earlier chapter that the identity $\nabla \text{vol} = 0$ is expressed in coordinates by

$$\frac{\partial \sqrt{g}}{\partial x_j} - \sqrt{g}\sum_k \Gamma^k_{kj} = 0. \tag{10.122}$$

This *volume coefficient identity* is important in understanding the formula for the divergence.

Theorem 10.13 *The relation of the covariant differential ∇ to the operations of vector calculus is given by*

$$\begin{aligned} \text{grad } s &= \mathsf{g}^{-1}\nabla s \\ \text{curl } \mathbf{v} &= \mathsf{c}^{-1}(\nabla \wedge \mathsf{g}\mathbf{v}) \\ \text{div } \mathbf{v} &= \nabla \mid \mathbf{v} \\ \nabla^2 s &= \nabla \mid \mathsf{g}^{-1}\nabla s. \end{aligned} \tag{10.123}$$

Proof: The result for the gradient follows from the fact that $\nabla s = ds$.

The result for the curl follows from the fact that $\nabla \wedge \omega = d\omega$. This is because

$$\nabla \wedge \omega = \left(\sum_i dx_i \frac{\partial}{\partial x_i}\right) \wedge (\sum_j p_j\,dx_j) = \sum_i \sum_j \left(\frac{\partial p_j}{\partial x_i} - \sum_k \Gamma^k_{ij} p_k\right)\,dx_i \wedge dx_j. \tag{10.124}$$

The terms with the Christoffel symbols give zero due to the symmetry property of the Christoffel symbols.

The notation $\nabla \mid \mathbf{v}$ denotes the natural way of combining the form components and vector components to get a scalar. This quantity does depend on the Christoffel symbols: First compute

$$\nabla \mathbf{v} = \sum_i \sum_j dx_i \left(\frac{\partial b_j}{\partial x_i} + \sum_k \Gamma^j_{ik} b_k\right) \frac{\partial}{\partial x_j} \tag{10.125}$$

Then

$$\nabla \mid \mathbf{v} = \sum_j \left(\frac{\partial b_j}{\partial x_j} + \sum_k \Gamma^j_{jk} b_k \right). \tag{10.126}$$

interchanging the j and k indices in the second term gives a more convenient expression

$$\nabla \mid \mathbf{v} = \sum_j \left(\frac{\partial b_j}{\partial x_j} + \sum_k \Gamma^k_{kj} b_j \right). \tag{10.127}$$

From the volume form coefficient identity this is

$$\text{div } \mathbf{v} = \nabla \mid \mathbf{v} = \sum_j \left(\frac{\partial b_j}{\partial x_j} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x_j} b_j \right) = \sum_j \frac{1}{\sqrt{g}} \frac{\partial (\sqrt{g} b_j)}{\partial x_j}. \tag{10.128}$$

□

The more standard (and awkward) practice is to introduce a vector whose values are differential operators. This is

$$\overrightarrow{\nabla} = \mathsf{g}^{-1} \nabla = \sum_k \sum_i g^{ki} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i}. \tag{10.129}$$

In this formula the partial derivative on the left is a coordinate basis vector, and the partial derivative on the right is a covariant derivative. The g^{ki} form the matrix of the inverse g^{-1}. The relation to the covariant derivative is

$$\mathbf{u} \lrcorner \nabla \mathbf{v} = \mathbf{u} \mathsf{g} \overrightarrow{\nabla} \mathbf{v} = \mathbf{u} \cdot \overrightarrow{\nabla} \mathbf{v}. \tag{10.130}$$

Theorem 10.14 *The relation of $\overrightarrow{\nabla}$ to the operations of vector calculus is given by*

$$\begin{aligned} \text{grad } s &= \overrightarrow{\nabla} s \\ \text{curl } \mathbf{v} &= \overrightarrow{\nabla} \times \mathbf{v} \\ \text{div } \mathbf{v} &= \overrightarrow{\nabla} \cdot \mathbf{v} \\ \nabla^2 s &= \overrightarrow{\nabla} \cdot \overrightarrow{\nabla} s. \end{aligned} \tag{10.131}$$

Proof: For curl the result is suggested by the identity $\mathbf{u} \times \mathbf{v} = \mathsf{c}^{-1}(\mathsf{g}\mathbf{u} \wedge \mathsf{g}\mathbf{v})$. Using $\overrightarrow{\nabla}$ in place of $\mathbf{u}$ gives

$$\overrightarrow{\nabla} \times \mathbf{v} = \mathsf{c}^{-1}(\mathsf{g} \overrightarrow{\nabla} \wedge \mathsf{g}\mathbf{v}) = \mathsf{c}^{-1}(\nabla \wedge \mathsf{g}\mathbf{v}). \tag{10.132}$$

But this is $\mathsf{c}^{-1} d(\mathsf{g}\mathbf{v})$ which is the definition of curl $\mathbf{v}$.

The expression for the divergence may be interpreted as $\overrightarrow{\nabla} \cdot \mathbf{v} = \mathsf{g} \overrightarrow{\nabla} | \mathbf{v} = \nabla | \mathbf{v}$. Another way to think about the divergence as $\overrightarrow{\nabla} \cdot \mathbf{v}$ is via the identity $\mathbf{u} \cdot \mathbf{v} = *(\mathsf{g}\mathbf{u} \wedge \mathsf{c}\mathbf{v})$. Using $\overrightarrow{\nabla}$ in place of $\mathbf{u}$ gives $\overrightarrow{\nabla} \cdot \mathbf{v} = *(\nabla \wedge \mathsf{c}\mathbf{v}) = *d(\mathsf{c}\mathbf{v})$. □

While ∇ is simpler and more elegant, $\overrightarrow{\nabla}$ is closer to what is found in vector calculus texts. The Laplace operator ∇^2 is the central object in the theory. It may be either in terms of ∇ or in terms of $\overrightarrow{\nabla}$. Thus

$$\nabla^2 = \nabla \mid \mathsf{g}^{-1} \nabla = \mathsf{g} \overrightarrow{\nabla} \mid \overrightarrow{\nabla}. \tag{10.133}$$

10.2.5 Orthogonal coordinates and normalized bases

In many applications it is possible to use orthogonal coordinates to simplify the calculations. This section presents some of the common formulas for this case. They are all consequences of the general theory.

It may be possible to choose coordinates so that the Riemannian metric is diagonal with respect to this coordinate system. Such a coordinate system is called a system of *orthogonal coordinates*. In this case it has the form

$$\mathsf{g} = h_1^2\, du_1^2 + h_2^2\, du_2^2 + \cdots + h_n^2\, du_n^2. \tag{10.134}$$

Here each coefficient h_i is a function of the coordinates $u_1, \ldots, u_n$.

Consider a manifold with a given Riemannian metric. (For instance, it could be a k dimensional surface in some Euclidean space of larger dimension n.) If the manifold has dimension at most three, then near every point there is always a new coordinate system that is a system of orthogonal coordinates. In the case of three dimensions this is not a particularly obvious fact, but it is a consequence of the Cartan-Kähler theorem. There is a treatment in the book of Bryant and coauthors [5].

With orthogonal coordinates it is tempting to make the basis vectors have length one. Instead of using the usual coordinate basis vectors $\frac{\partial}{\partial u_i}$ one uses the normalized basis vectors $\frac{1}{h_i}\frac{\partial}{\partial u_i}$. Similarly, instead of using the usual coordinate differential forms du_i one uses the normalized differentials $h_i\, du_i$. Then

$$\mathsf{g}\left(a_1 \frac{1}{h_1}\frac{\partial}{\partial u_1} + \cdots a_n \frac{1}{h_n}\frac{\partial}{\partial u_n}\right) = a_1 h_1\, du_1 + \cdots a_n h_n\, du_n \tag{10.135}$$

With normalized basis vectors the coefficients of vector fields and the corresponding differential forms are the same. This makes it very easy to confuse vector fields with differential forms.

In orthogonal coordinates the volume is given in terms of normalized differentials by

$$\mathrm{vol} = h_1\, du_1 \wedge \cdots \wedge h_n\, du_n. \tag{10.136}$$

A simple example of orthogonal coordinates is that of *polar coordinates* r, θ in the plane. These are related to Cartesian coordinates x, y by

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned} \tag{10.137}$$

The Riemannian metric is expressed as

$$\mathsf{g} = dr^2 + r^2\, d\theta^2. \tag{10.138}$$

The normalized basis vectors are $\frac{\partial}{\partial r}$ and $\frac{1}{r}\frac{\partial}{\partial \theta}$. The normalized basis forms are dr and $r\, d\theta$. The area form is $r\, dr \wedge d\theta$. Warning: Even though coordinate forms like $d\theta$ are closed forms, a normalized form like $r\, d\theta$ need not be a closed form. In fact, in this particular case $d(r\, d\theta) = dr \wedge d\theta \neq 0$.

Another example of orthogonal coordinates is that of *spherical polar coordinates* r, θ, ϕ. These are related to Cartesian coordinates x, y, z by

$$\begin{aligned} x &= r\sin(\theta)\cos(\phi) \\ y &= r\sin(\theta)\sin(\phi) \\ z &= r\cos(\theta) \end{aligned} \tag{10.139}$$

The Riemannian metric is expressed as

$$\mathsf{g} = dr^2 + r^2\, d\theta^2 + r^2 \sin^2(\theta)\, d\phi^2. \tag{10.140}$$

The normalized basis vectors are $\frac{\partial}{\partial r}$ and $\frac{1}{r}\frac{\partial}{\partial \theta}$ and $\frac{1}{r\sin(\theta)}\frac{\partial}{\partial \phi}$. The normalized basis forms are dr and $r\, d\theta$ and $r\sin(\theta)\, d\phi$. The volume form is $r^2 \sin(\theta)\, dr \wedge d\theta \wedge d\phi$.

If s is a scalar field, then its gradient is

$$\operatorname{grad} s = \mathsf{g}^{-1}\nabla s = \overrightarrow{\nabla} s = \sum_{i=1}^{n} \frac{1}{h_i}\frac{\partial s}{\partial u_i}\frac{1}{h_i}\frac{\partial}{\partial u_i}. \tag{10.141}$$

If $\mathbf{u}$ is a vector field, then its divergence is a scalar field. Say that $\mathbf{u}$ has an expression in terms of normalized basis vectors of the form

$$\mathbf{u} = \sum_{i=1}^{n} a_i \frac{1}{h_i}\frac{\partial}{\partial u_i}. \tag{10.142}$$

Then

$$\operatorname{div} \mathbf{u} = \nabla \mid \mathbf{u} = \nabla \cdot \mathbf{u} = \sum_{i=1}^{n} \frac{1}{h_1 \cdots h_n}\frac{\partial}{\partial u_i}\left(\frac{h_1 \cdots h_n}{h_i} a_i\right). \tag{10.143}$$

In coordinates the Laplacian has the form

$$\nabla^2 s = \frac{1}{h_1 \cdots h_n}\sum_{i=1}^{n} \frac{\partial}{\partial u_i}\left(\frac{h_1 \cdots h_n}{h_i^2}\frac{\partial s}{\partial u_i}\right) \tag{10.144}$$

For example, in three dimensions with Cartesian coordinates it is

$$\nabla^2 s = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2}. \tag{10.145}$$

In spherical polar coordinates it is usually written

$$\nabla^2 s = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial s}{\partial r} + \frac{1}{r^2}\left[\frac{1}{\sin(\theta)}\frac{\partial}{\partial \theta}\sin(\theta)\frac{\partial s}{\partial \theta} + \frac{1}{\sin^2(\theta)}\frac{\partial^2 s}{\partial \phi^2}\right]. \tag{10.146}$$

10.2.6 The covariant derivative in orthogonal coordinates*

The covariant derivative is the natural notion of directional derivative of a vector field in the direction of another vector field. It is encountered in applied mathematics under various names: *advective derivative*, *material derivative*, *convective derivative*, and so on. The common practice is to use orthogonal coordinates. So this formula is a practical tool.

In orthogonal coordinates there are natural unit basis vectors. Perhaps the simplest example is polar coordinates, where $\mathsf{g} = dr^2 + (r\,d\theta)^2$. The unit basis vectors are $\partial/\partial r$ and $(1/r)(\partial/\partial\theta)$.

The formula for covariant derivatives in orthogonal coordinates is

$$\frac{\partial}{\partial u_i}\left(\frac{1}{h_j}\frac{\partial}{\partial u_j}\right) = \frac{1}{h_j}\frac{\partial h_i}{\partial u_j}\left(\frac{1}{h_i}\frac{\partial}{\partial u_i}\right) \quad i \neq j. \tag{10.147}$$

This says that for $i \neq j$ the covariant derivative of the j unit vector in the i direction is in the (orthogonal) i direction. A picture of this formula is given by the polar coordinate relation $(\partial/\partial\theta)(\partial/\partial r) = +(1/r)(\partial/\partial\theta)$. Changing the angle of the radial vector produces a first-order change in the angular direction.

This must be supplemented by

$$\frac{\partial}{\partial u_j}\left(\frac{1}{h_j}\frac{\partial}{\partial u_j}\right) = -\sum_{k\neq j}\frac{1}{h_k}\frac{\partial h_j}{\partial u_k}\left(\frac{1}{h_k}\frac{\partial}{\partial u_k}\right). \tag{10.148}$$

This says tht the covariant derivative of the j unit vector in the j direction is orthogonal to the j direction. The polar coordinate picture in this case is $(\partial/\partial\theta)(1/r)(\partial/\partial\theta) = -(\partial/\partial r)$. Changing the angle of the angular vector gives a first-order change in the radial direction, in the inward direction.

These results may be combined in a single formula:

$$\frac{\partial}{\partial u_i}\left(\frac{1}{h_j}\frac{\partial}{\partial u_j}\right) = \frac{1}{h_j}\frac{\partial h_i}{\partial u_j}\left(\frac{1}{h_i}\frac{\partial}{\partial u_i}\right) - \delta_{ij}\sum_k \frac{1}{h_k}\frac{\partial h_j}{\partial u_k}\left(\frac{1}{h_k}\frac{\partial}{\partial u_k}\right). \tag{10.149}$$

This formula may be derived as a special case of general formulas in tensor calculus that express the covariant derivative in terms of the metric tensor. The formula is somewhat simpler than in the general case because of the special choice of orthogonal coordinates. See, for instance, [40]. The following theorem states this formula in a way that exhibits the symmetry hidden in this complexity. A proof of this result is given in the following chapter.

Theorem 10.15 (Coordinate covariant derivative in orthogonal coordinates) *Suppose coordinates $u_1, \ldots, u_n$ can be chosen so that the metric tensor has the diagonal form $\sum_j h_j^2 du_j^2$. Then the covariant derivatives of the unit basis vectors are given by*

$$\frac{\partial}{\partial u_i}\left(\frac{1}{h_j}\frac{\partial}{\partial u_j}\right) = \sum_k C_{ij}^k \left(\frac{1}{h_k}\frac{\partial}{\partial u_k}\right). \tag{10.150}$$

Here

$$C_{ij}^{k} = \frac{1}{h_j}\frac{\partial h_k}{\partial u_j}\delta_{ik} - \delta_{ij}\frac{1}{h_k}\frac{\partial h_j}{\partial u_k}. \tag{10.151}$$

Furthermore the coefficients satisfy an anti-symmetry property:

$$C_{ij}^{k} = -C_{ik}^{j}. \tag{10.152}$$

Let

$$\mathbf{v} = \sum_j b_j \frac{1}{h_j}\frac{\partial}{\partial u_j}. \tag{10.153}$$

Recall that

$$\nabla = \sum_i du_i \frac{\partial}{\partial u_i} \tag{10.154}$$

in an arbitrary coordinate system. Using the formula and making a simple change of summation variable leads to a formula for the covariant differential:

Corollary 10.16 (Covariant differential in orthogonal coordinates)

$$\nabla\mathbf{v} = \sum_i \sum_j \left(du_i \frac{\partial b_j}{\partial u_i} + \left(du_j \frac{1}{h_i}\frac{\partial h_j}{\partial u_i} - du_i \frac{1}{h_j}\frac{\partial h_i}{\partial u_j} \right) b_i \right) \frac{1}{h_j}\frac{\partial}{\partial u_j} \tag{10.155}$$

For the next result, let

$$\mathbf{u} = \sum_i a_i \frac{1}{h_i}\frac{\partial}{\partial u_i} \tag{10.156}$$

Corollary 10.17 (Covariant derivative in orthogonal coordinates)

$$\mathbf{u} \lrcorner \nabla\mathbf{v} = \sum_i \sum_j \left(\frac{a_i}{h_i}\frac{\partial b_j}{\partial u_i} + \frac{b_i}{h_i h_j}\left(a_j \frac{\partial h_j}{\partial u_i} - a_i \frac{\partial h_i}{\partial u_j} \right) \right) \frac{1}{h_j}\frac{\partial}{\partial u_j}. \tag{10.157}$$

Some authors might find it more conventional to write the covariant derivative as $\mathbf{u} \cdot \vec{\nabla}\mathbf{v}$, where in orthogonal coordinates

$$\vec{\nabla} = \sum_i \frac{1}{h_i}\frac{\partial}{\partial u_i}\frac{1}{h_i}\frac{\partial}{\partial u_i}. \tag{10.158}$$

The derivative on the right is the covariant derivative acting on scalar fields, vector fields, and so on. Since the inner product of vectors involves the factors h_i^2, this gives the same covariant derivative as before.

Remark: With orthogonal coordinates the covariant derivatives of the unit basis forms involve the same coefficients. The formula is

$$\frac{\partial}{\partial u_i}\left(h_j\, du_j\right) = -\sum_k C_{ik}^{j}\left(h_k\, du_k\right) = \sum_k C_{ij}^{k}\left(h_k\, du_k\right). \tag{10.159}$$

||

10.2.7 Spherical harmonics*

The Laplace operator in three dimensions often occurs in a situation where there is rotational symmetry. This suggests using spherical polar coordinates. On the other hand, the angular part of the Laplacian in spherical polar coordinates looks complicated. Nevertheless, spherical harmonics give a way of analyzing this situation.

Write the Laplacian as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2}\left(E(E+1) + L\right). \tag{10.160}$$

Here

$$E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} = r\frac{\partial}{\partial r} \tag{10.161}$$

is the Euler operator. Also

$$L = \frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta} + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2} \tag{10.162}$$

is the Laplace operator on the sphere.

The space of solid spherical harmonics of degree ℓ is the space of polynomials s in x, y, z characterized by the two equations

$$\nabla^2 s = 0 \tag{10.163}$$

and

$$Es = \ell s. \tag{10.164}$$

Both these equations are rotation invariant. In other words, a rotated solid spherical harmonic of degree ℓ is another solid spherical harmonic of degree ℓ.

A solid spherical harmonic s of degree ℓ may be written in the form $s = r^\ell S$. The function S is called a surface spherical harmonic of degree ℓ. It is usually expressed in terms of angular variables θ, ϕ from spherical polar coordinates. That is, the corresponding surface spherical harmonics are obtained by replacing x by $\sin(\theta)\cos(\phi)$, y by $\sin(\theta)\sin(\phi)$, and z by $\cos(\theta)$. The use of these angular variables leads to rather complicated formulas.

Instead, observe that $Er^\ell = \ell r^\ell$. It follows from the Laplace and Euler equations that $0 = \ell(\ell+1)s + Ls$. Dividing by r^ℓ gives

$$LS = -\ell(\ell+1)S. \tag{10.165}$$

This allows an analysis of spherically symmetric problems in terms of surface spherical harmonics.

Often one only needs to analyze low degree spherical harmonics, for instance $\ell = 0, 1, 2$, and possibly 3. Here are some examples. A solid spherical harmonic of degree ℓ is a polynomial in variables x, y, z that is homogeneous of degree ℓ and satisfies the Laplace equation. Here they are for small values of ℓ.

0 a

1 $ax + by + cz$

2 $ax^2 + by^2 + cz^2 + 2dyz + 2ezx + 2fxy$ with $a + b + c = 0$

3 $ax^3 + bx^3 + cx^3 + 3dx^2y + 3ex^2z + 3fy^2x + 3gy^2z + 3hz^2x + 3iz^2y + 6jxyz$ with $a + f + h = 0, b + d + i = 0, c + e + g = 0$.

The dimensions for $\ell = 0, 1, 2, 3$ are 1, 3, 5, 7. The general result is that the space of solid spherical harmonics of degree ℓ has dimension $2\ell + 1$. For $\ell = 0$ we have the constant functions. For $\ell = 1$ we get the linear functions. For $\ell = 2$ we get quadratic functions given by a symmetric matrix whose trace is zero.

Surface spherical harmonics occur in applications ranging from vibrations of stars and planets to the geometry of atoms. Sometimes the $\ell = 0, 1, 2$ harmonics are called monopoles, dipoles, and quadrupoles. In atomic theory the $\ell = 0, 1, 2, 3$ harmonics are called s,p,d,f.

Remark: It is natural to ask what these low degree spherical harmonics look like. For $\ell = 1$ the only invariant is $a^2 + b^2 + c^2$. By choosing new coordinates one can regard a typical $\ell = 1$ spherical harmonic to be cz, so the surface spherical harmonic is $c\cos(\theta)$. Thus it has maximum and minimum values in two opposite directions on the sphere. This is the reason for the term dipole.

The situation is more complicated for $\ell = 2$. The solid spherical harmonic is given by a quadratic form in x, y, z. The coefficients of this form are displayed in a 3 by 3 symmetric matrix Q. The invariants of such a matrix under rotation are $t_1 = \mathrm{tr}(Q)$, $t_2 = \mathrm{tr}(Q^2)$, and $t_3 = \mathrm{tr}(Q^3)$. There is an identity $6\det(Q) = t_1^3 - 3t_2t_1 + 2t_3$. Since in the present case $t_1 = 0$, this gives $3\det(Q) = \mathrm{tr}(Q^3)$. So the invariants may be taken to be $\mathrm{tr}(Q^2)$ and $\det(Q)$. The first is a simple measure of the magnitude of the spherical harmonic, while the second one helps characterize its shape.

By choosing coordinates that diagonalize the matrix one can take the $\ell = 2$ spherical harmonic to be $ax^2 + bx^2 + cx^2$ with $a + b + c = 1$. The invariants are $a^2 + b^2 + c^3$ and abc. The shape invariant may be taken to be abc divided by $(a^2 + b^2 + c^2)^{\frac{3}{2}}$.

A typical example where the shape invariant is zero is $a(x^2 - y^2)$. This is $a\sin^2(\theta)\cos(2\phi)$. A polar plot of the $\cos(2\phi)$ factor gives a four-leaf clover shape with four maximum and minimum values. This justifies the term quadrupole.

A typical example where the shape invariant is as far as possible from zero is $ax^2 + ay^2 - 2az^2$. The determinant is $-2a^3$. The surface spherical harmonic is $-a(2\cos^2(\theta) - \sin^2(\theta)) = -a(3\cos^2(\theta) - 1)$. A polar plot of $3\cos^2(\theta) - 1 = \frac{1}{2} + \frac{3}{2}\cos(2\theta)$ has a shape with four maximum and minimum values, but with less symmetry. It is still a quadrupole.

It is natural to ask about invariants for spherical harmonics of higher order. There is a discussion in [27]. ||

Remark: In computations it is sometimes convenient to pick bases for these spaces. Here are some possible examples. We write $r^2 = x^2 + y^2 + z^2$. Then typical choices are

0 1

1 x, y, z

2 $3z^2 - r^2, xz, yz, x^2 - y^2, xy.$

3 $z(5z^2 - 3r^2), x(5z^2 - r^2), y(5z^2 - r^2), (x^2 - y^2)z, xyz, x^3 - 3xy^2, 3x^2y - y^3.$

The surface spherical harmonics are obtained by replacing x, y, z, r by $\sin(\theta)\cos(\phi)$, $\sin(\theta)\sin(\phi)$, $\cos(\theta)$, 1. ||

It is possible to develop the theory of spherical harmonics in dimension n. See the book by Stein and Weiss [46] for a thorough analysis.

10.2.8 Potential flow in two dimensions*

Let $\mathbf{v}$ be a vector field in the plane that is irrotational and incompressible, that is, curl $\mathbf{v} = 0$ and div $\mathbf{v} = 0$. Then it should be possible to write $\mathbf{v} = \text{grad}\,\phi$ and also $\mathbf{v} = \text{curl}'\,\psi$. The scalar field ϕ is called the *velocity potential*. The scalar field ψ is called the *stream function*. A curve where the stream function is constant is called a *streamline*. The streamlines are invariant under the flow, since

$$\langle d\psi \mid \mathbf{v}\rangle = \langle d\psi \mid \text{curl}'\,\psi\rangle = \langle \mathbf{v} \lrcorner \text{area} \mid \mathbf{v}\rangle = \langle \text{area} \mid \mathbf{v}, \mathbf{v}\rangle = 0. \tag{10.166}$$

It follows also that

$$\nabla^2\phi = \text{div grad}\,\phi = 0 \tag{10.167}$$

and

$$\nabla^2\psi = -\,\text{curl curl}'\,\psi = 0. \tag{10.168}$$

Now specialize to Cartesian coordinates x, y in the Euclidean plane. Write the components of the vector field as u, v. Then $\partial\phi/\partial x = u$ and $\partial\phi/\partial y = v$. On the other hand, $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. In particular, ϕ and ψ satisfy the Cauchy-Riemann equations $\partial\phi/\partial x = \partial\psi/\partial y$ and $\partial\phi/\partial y = -\partial\psi/\partial x$. It follows that

$$w = \phi + i\psi = f(z) = f(x + iy) \tag{10.169}$$

defines an analytic function. The problem of finding potential flows reduces to finding appropriate analytic functions.

Example: Flow inside a corner. The region is the first quadrant $x > 0, y > 0$. The function $w = z^2$ gives $\phi = x^2 - y^2$ and $\psi = 2xy$. The flow has components $2x, -2y$. The streamlines are $2xy = c$. A fluid particle in the first quadrant descends and is deflected to the right. ||

Example: A doublet flow. This example is rather singular. The region is the plane with the origin removed. The function $w = 1/z$ gives $\phi = x/(x^2 + y^2)$ and $\psi = y/(x^2 + y^2)$. The velocity has components $(y^2 - x^2)/(x^2 + y^2)^2$ and $-2xy/(x^2 + y^2)^2$. The streamlines are given by $x^2 + y^2 = 2ay$ for constant a. This may be written in more transparent form as $x^2 + (y - a)^2 = a^2$. So the flow is in circles that all pass through the origin. In the upper half plane the flow is clockwise; in the lower half plane the flow is counterclockwise. ||

Example: Flow around a half-line. The region is the complement of the half-line $x \geq 0$. The function $w = \sqrt{z}$ gives $x = \phi^2 - \psi^2$ and $y = 2\phi\psi$. A streamline is given by $\psi = c > 0$. Then $x = y^2/(4c^2) - 1/c^2 = 1/(4c^2)(y^2 - c^4)$. This is the equation of a parabola. When c is large this is a shallow parabola far away from the half-line. When c is small it hugs close to the half-line and is very steep. ||

10.3 Surface area*

10.3.1 Area of a surface*

Say that we are in n dimensions with metric

$$\mathsf{g} = \sum_{i=1}^{n} \sum_{i=1}^{n} g_{ij}\, dx_i\, dx_j. \tag{10.170}$$

The n-dimensional volume is given by integrating

$$\text{vol} = \sqrt{g}\, dx_1 \cdots dx_n, \tag{10.171}$$

where G is the matrix with entries g_{ij}, and $g = \det G$.

Consider a k-dimensional regular parametrized surface S with parameters $u_1, \ldots, u_k$. This parametrization is one-to-one, so that $u_1, \ldots, u_k$ may be thought of as coordinates on the surface. It seems reasonable to compute the k-dimensional *surface area* using the pull-back of g to the surface. This is

$$\mathsf{g}^* = \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} g^*_{\alpha,\beta}\, du_\alpha\, du_\beta = \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} (g_{ij} \circ \phi) \frac{\partial (x_i \circ \phi)}{\partial u_\alpha} \frac{\partial (x_j \circ \phi)}{\partial u_\beta} \right) du_\alpha\, du_\beta. \tag{10.172}$$

Let $B^i_\alpha = \partial(x_i \circ \phi)/\partial u_\alpha$. These are the components of tangent vectors. Then

$$g^*_{\alpha\beta} = B^T_\alpha G B_\beta. \tag{10.173}$$

The matrix $g^*_{\alpha\beta}$ is a Gram matrix. Let g^* be the corresponding determinant. The area is given by integrating

$$\text{area} = \sqrt{g^*}\, du_1 \cdots du_k. \tag{10.174}$$

In some sense this is the end of the story. One computes a Gram determinant, takes the square root, and integrates.

10.3.2 The surface element*

There is an interesting way of looking at the surface integral in the divergence theorem. Consider an $(n-1)$-dimensional regular surface and its injection ϕ into the n dimensional space M. Let $u_1, \ldots, u_{n-1}$ be coordinates on the surface, and let $x_1, \ldots, x_n$ be coordinates on M. Define

$$\nu_j\, du_1 \cdots du_{n-1} = \phi^*((-1)^{j-1} dx_1 \cdots dx_{j-1}\, dx_{j+1} \cdots dx_n). \tag{10.175}$$

Consider the matrix whose entries are $\partial(x_i \circ \phi)/\partial u_\alpha$, where i ranges from 1 to n excluding j, and α runs from 1 to $n-1$. Then ν_j is $(-1)^{j-1}$ times the determinant of this matrix. The claim is that for each β from 1 to $n-1$ there is an orthogonality relation

$$\sum_j \nu_j \frac{\partial(x_j \circ \phi)}{\partial u_\beta} = 0. \tag{10.176}$$

The proof of the relation is simple. Consider the n by n matrix whose first column is $\partial(x_j \circ \phi)/\partial u_\beta$ and whose other columns are those of the n by $n-1$ matrix $\partial(x_j \circ \phi)/\partial u_\alpha$, where $\alpha = 1, \ldots, n-1$. The determinant of this matrix is zero, since the first column is also one of the other columns. Expand the determinant by cofactors of the first column. This gives the relation.

Define the *penetration form*

$$\nu = \sum_{j=1}^{n} \nu_j \, dx_{j\,[\phi]}. \tag{10.177}$$

The relation may be restated as a proposition. For a regular surface the basis tangent vectors are

$$X_\beta = \sum_{i=1}^{n} \frac{\partial(x_i \circ \phi)}{\partial u_\beta} \frac{\partial}{\partial x_i}{}_{[\phi]}. \tag{10.178}$$

Proposition 10.18 *For an $(n-1)$-dimensional regular surface the penetration form ν vanishes on the $n-1$ tangent basis vectors X_β, that is,*

$$\langle \nu \mid X_\beta \rangle = 0. \tag{10.179}$$

If

$$\mathbf{v} = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} \tag{10.180}$$

is a vector field, then its flux is

$$\mathbf{v} \lrcorner \operatorname{vol} = \sum_{j=1}^{n} a_j (-1)^{j-1} sqrtg \, dx_1 \cdots dx_{j-1} \, dx_{j+1} \cdots dx_n. \tag{10.181}$$

This pulls back to

$$\phi^*(\mathbf{v} \lrcorner \operatorname{vol}) = \sum_{j=1}^{n} (a^j \circ \phi) \nu_j \sqrt{g \circ \phi} \, du_1 \cdots du_{n-1}. \tag{10.182}$$

This can also be written

$$\phi^*(\mathbf{v} \lrcorner \operatorname{vol}) = \langle \nu \mid \mathbf{v}_{[\phi]} \rangle \sqrt{g \circ \phi} \, du_1 \cdots du_{n-1}. \tag{10.183}$$

The form ν is giving a numerical indication of the extent to which the vector field is failing to be tangent, that is, of the extent to which it is penetrating the surface.

This suggests a definition of *surface element* in terms of the penetration form. This

$$\text{element} = \nu \sqrt{g \circ \phi}\, du_1 \cdots du_{n-1}. \tag{10.184}$$

Define

$$\mathbf{v}_{[\phi]} \lrcorner \,\text{element} = \langle \nu \mid \mathbf{v}_{[\phi]} \rangle \sqrt{g \circ \phi}\, du_1 \cdots du_{n-1}. \tag{10.185}$$

Then

$$\phi^*(\mathbf{v} \lrcorner \,\text{vol}) = \mathbf{v}_{[\phi]} \lrcorner \,\text{element} \tag{10.186}$$

This says that the integrand in the surface integral is obtained by restricting $\mathbf{v}$ to the surface and pairing it with the surface element.

This element associated to the surface has a peculiar structure; it is the exterior product of the differential 1-form ν along ϕ (acting on vectors tangent to M) with a differential $(n-1)$-form belonging to the surface (acting on tangent vectors to the surface). The reason that this is a sensible object is that ν vanishes on tangent vectors to the surface. This mixed form element plays a natural role in the divergence theorem. It also turns out to have an application in the context of surface area.

10.3.3 Area of a hypersurface*

The penetration form ν has length given by

$$|\nu|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \nu_i (g^{ij} \circ \phi) \nu_j. \tag{10.187}$$

Since this is a form and not a vector, its norm is computed via the inverse of the metric tensor.

The following relation is valid

$$g^* = |\nu|^2 \, (g \circ \phi). \tag{10.188}$$

The calculation may be found in [25, §7.6, (6.6)]. This gives the following theorem

Theorem 10.19 (Area element of a hypersurface) *Consider an n-dimensional manifold patch and a regular parametrized $(n-1)$-surface with parameters $u_1, \ldots, u_{n-1}$. Consider also the surface element*

$$\text{element} = \nu \sqrt{g \circ \phi}\, du_1 \cdots du_{n-1} \tag{10.189}$$

with

$$\nu = \sum_{i=1}^{n} \nu_i \, dx_{i\,[\phi]}. \tag{10.190}$$

This form measures the extent to which a vector is transverse to the surface. Then the area element is the length

$$\text{area} = |\nu| \sqrt{g \circ \phi}\, du_1 \cdots du_{n-1}. \tag{10.191}$$

The textbook case is that where $n = 3$ and $k = 2$, that is, a surface in 3-space. The most common coordinates are Cartesian coordinates x, y, z for which the metric is given by $dx^2 + dy^2 + dz^2$. However we might want some other set of coordinates, so temporarily we think of x, y, z as some choice of orthogonal coordinates with metric

$$\mathsf{g} = h_x^2\, dx^2 + h_y^2\, dy^2 + h_z^2\, dz^2. \tag{10.192}$$

With Cartesian coordinates we simply have $h_x = h_y = h_z = 1$.

Say that we have a surface parametrized by u, v. Then the metric on this surface is

$$\mathsf{g}^* = E\, du^2 + 2F\, du\, dv + G\, dv^2. \tag{10.193}$$

Here E, F, G are functions of u, v. They of course depend on the choice of coordinates. What is required is that $E > 0, G > 0$ and the determinant $EF - G^2 > 0$. Explicitly, the coefficients are

$$E = h_x^2 \left(\frac{\partial x}{\partial u}\right)^2 + h_y^2 \left(\frac{\partial y}{\partial u}\right)^2 + h_z^2 \left(\frac{\partial z}{\partial u}\right)^2, \tag{10.194}$$

and

$$F = h_x^2 \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + h_y^2 \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + h_z^2 \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}, \tag{10.195}$$

and

$$G = h_x^2 \left(\frac{\partial x}{\partial v}\right)^2 + h_y^2 \left(\frac{\partial y}{\partial v}\right)^2 + h_z^2 \left(\frac{\partial z}{\partial v}\right)^2. \tag{10.196}$$

The formula for the area of a surface is

$$A = \int_S \text{area} = \int_S \sqrt{g^*}\, du\, dv = \int_S \sqrt{EG - F^2}\, du\, dv. \tag{10.197}$$

Here $g^* = EG - F^2$ is the determinant of the metric tensor.

The alternative expression for surface area is sometimes convenient. Let $\nu_x = J^{yz}$, $\nu_y = J^{zx}$, and $\nu_z = J^{xy}$. An expression such as J^{yz} indicates the determinant of the matrix of partial derivatives of y, z with respect to u, v. Then the surface element is

$$\text{element} = (\nu_x\, dx + \nu_y\, dy + \nu_x\, dz)\, h_x h_y h_z\, du\, dv. \tag{10.198}$$

The area is given by

$$A = \int_S \text{area} = \int_S \sqrt{\left(\frac{\nu_x}{h_x}\right)^2 + \left(\frac{\nu_y}{h_y}\right)^2 + \left(\frac{\nu_z}{h_z}\right)^2}\, h_x h_y h_z\, du\, dv. \tag{10.199}$$

Problems

Stereographic projection

In this problem $S_n \subseteq E_{n+1}$ is the n-dimensional unit sphere in $(n+1)$-dimensional Euclidean space. The coordinates on E_{n+1} are taken to be $\mathbf{y}, y$. The sphere S_n is characterized by $\mathbf{y}^2 + y^2 = 1$. The tangent space to S_n at a point with coordinates $\mathbf{y}, y$ may be characterized by $\mathbf{y} \cdot d\mathbf{y} + y\, dy = 0$.

The stereographic projection of S_n onto E_n is defined everywhere except at the north pole of S_n. The mapping sends $\mathbf{y}, y$ to the point in E_n where the line from the north pole $(0, 1)$ in the direction $(\mathbf{y}, y) - (0, 1)$ intersections E_n. This says that the point in E_n has coordinates $\mathbf{x}$, where

$$(0, 1) + t(\mathbf{y}, y - 1) = (\mathbf{x}, 0). \tag{10.200}$$

Thus

$$\mathbf{x} = \frac{1}{1 - y}\mathbf{y}. \tag{10.201}$$

One way to think of this is that the $\mathbf{x}$ are new coordinates on S_n (with the north pole removed).

1. Show that $\mathbf{x}^2 = \frac{1+y}{1-y}$.

2. Show that $y = \frac{\mathbf{x}^2-1}{\mathbf{x}^2+1} = 1 - \frac{2}{\mathbf{x}^2+1}$. Show that $1 - y = \frac{2}{\mathbf{x}^2+1}$.

3. Find $\mathbf{y}$ in terms of $\mathbf{x}$.

4. Spherical subsets $C \subseteq S_n$ of dimension $n - 1$ symmetric about the origin ("great circles") map to corresponding subsets $C' \subseteq E_n \cup \{\infty\}$. Describe these subsets C'.

5. Find dy in terms of $\mathbf{x}$ and $d\mathbf{x}$.

6. Find $d\mathbf{y}$ in terms of $\mathbf{x}$ and $d\mathbf{x}$. Hint: Use the product rule for differentiation.

7. Show that $(d\mathbf{y})^2 + (dy)^2 = \left(\frac{2}{\mathbf{x}^2+1}\right)^2 (d\mathbf{x})^2$.

8. The preceding results show that the ratio of distances in the two coordinate systems is $2/(\mathbf{x}^2 + 1)$. Thus the metric tensor on the sphere is

$$\mathsf{g} = \frac{4}{(\mathbf{x}^2 + 1)^2}\,(d\mathbf{x})^2 = \frac{4}{(x_1^2 + \cdots + x_n^2 + 1)^2}(dx_1^2 + \cdots + dx_n^2). \tag{10.202}$$

   Fine the corresponding ratio of volumes. Find the volume form on the sphere in the $\mathbf{x}$ coordinates.

9. Conclusion: The volume of the sphere is the integral of a rational function over Euclidean space. Check that this gives the correct answer for the area of a sphere of dimension $n = 2$.

Remark: Stereographic projection gives an amazing insight into the Kepler problem [31]. In particular, 4-dimensional symmetry explains why the velocity vectors in planetary motion trace out circles. (Similar insights in the context of quantum theory clarify the structure of the periodic table.) ||

The Laplacian

1. Let $\mathbf{x}$ be Cartesian coordinates for n-dimensional Euclidean space. Let $r = |\mathbf{x}|$ be the distance from the origin. Recall that the volume of the ball of radius r is $v_n r^n$. Define $a_n = n v_n$.

   (a) Show that the gradient of the quadratic scalar field $\frac{1}{2}r^2$ is the *radial vector field*

   $$E = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} = r\frac{\partial}{\partial r}. \tag{10.203}$$

   Find the divergence of E.

   (b) Show that the associated *radial flux* $(n-1)$-form $\omega = E \rfloor \text{vol}$ is

   $$\omega = \sum_{i=1}^{n} (-1)^{i-1} x_i \, dx_1 \cdots dx_{i-1} \, dx_{i+1} \cdots dx_n \tag{10.204}$$

   Find $d\omega$. Show that the integral of ω over a sphere S_δ of radius δ centered at the origin is $a_n \delta^n$.

2. From now on $n > 2$.

   (a) Show that the negative of the gradient of the scalar field given by $1/(n-2)(1/r^{n-2})$ is the *Newton vector field*

   $$N = \frac{1}{r^n} E = \frac{1}{r^{n-1}} \frac{\partial}{\partial r}. \tag{10.205}$$

   Find the divergence of N away from the origin.

   (b) Show that the associated $(n-1)$-form $\sigma = N \rfloor \text{vol}$ satisfies

   $$\text{vol} = r^{n-1} \, dr \, \sigma. \tag{10.206}$$

   This $n-1$ form σ is the *solid angle form*. Show that the integral of σ over a sphere S_δ of radius δ centered at the origin is a_n.

   (c) Find $d\sigma$ away from the origin.

3. From now on $\epsilon > 0$. Define a scalar field potential function $\phi_\epsilon(\mathbf{x})$ by

   $$\phi_\epsilon(\mathbf{x}) = \frac{1}{a_n \epsilon^n} \frac{1}{2}(C - r^2) \tag{10.207}$$

   for $r \leq \epsilon$ and

   $$\phi_\epsilon(\mathbf{x}) = \frac{1}{a_n} \frac{1}{n-2} \frac{1}{r^{n-2}} \tag{10.208}$$

for $r > \epsilon$.

(a) Find the value of C that makes this potential function continuous.

(b) Let X_ϵ be the negative of the gradient of $\phi_\epsilon(\mathbf{x})$. Calculate this vector field inside and outside the ϵ-sphere. Prove that it is continuous.

(c) Find the divergence of X_ϵ inside and outside of the sphere.

(d) Find the integral of the divergence of X_ϵ over all of Euclidean space.

(e) Find the integral of the flux density $X_\epsilon \rfloor \text{vol}$ over the sphere S_δ as a function of $\delta > 0$.

4. Show that

$$-\nabla^2 \phi_\epsilon(\mathbf{x}) = \delta_\epsilon(\mathbf{x}), \tag{10.209}$$

where the right hand side is an approximate δ function.

5. Define the potential of a point source to be

$$\phi(\mathbf{x}) = \frac{1}{a_n} \frac{1}{n-2} \frac{1}{r^{n-2}}. \tag{10.210}$$

Let $f(\mathbf{y})$ be a function on $\mathbf{R}^n$ that is C^2 and has compact support. Define the Poisson potential of $f(\mathbf{y})$ to be

$$u(\mathbf{x}) = \int \phi(\mathbf{x} - \mathbf{y}) f(\mathbf{y})\, d\mathbf{y}. \tag{10.211}$$

The problem is to show that $-\nabla^2 u(\mathbf{x}) = f(\mathbf{x})$. This is Poisson's equation that shows how an extended source $f(\mathbf{y})$ creates a potential $u(\mathbf{x})$.

(a) Show that $u(\mathbf{x}) = \int \phi(\mathbf{y}) f(\mathbf{x} - \mathbf{y})\, d\mathbf{y}$.

(b) Show that $\nabla^2_{\mathbf{x}} u(\mathbf{x}) = \int \phi(\mathbf{y}) \nabla^2_{\mathbf{x}} f(\mathbf{x} - \mathbf{y})\, d\mathbf{y}$.

(c) Show that $\nabla^2_{\mathbf{x}} u(\mathbf{x}) = \int \phi(\mathbf{y}) \nabla^2_{\mathbf{y}} f(\mathbf{x} - \mathbf{y})\, d\mathbf{y}$. In other words, differentiating the potential is equivalent to differentiating the source.

(d) Take $\epsilon > 0$. Show that $\int \phi_\epsilon(\mathbf{y}) \nabla^2_{\mathbf{y}} f(\mathbf{x}-\mathbf{y})\, d\mathbf{y} = \int \nabla^2_{\mathbf{y}} \phi_\epsilon(\mathbf{y}) f(\mathbf{x}-\mathbf{y})\, d\mathbf{y}$. Where is the fact that $f(\mathbf{x} - \mathbf{y})$ has compact support used?

(e) Show that $-\int \phi(\mathbf{y}) \nabla^2_{\mathbf{y}} f(\mathbf{x} - \mathbf{y})\, d\mathbf{y} = f(\mathbf{x})$.

(f) Show that $-\nabla^2_{\mathbf{x}} u(\mathbf{x}) = f(\mathbf{x})$.

Remark: This Laplacian calculation is fundamental in various areas of physics. It could model a planet of radius ϵ producing a gravitational field and gravitational potential. Or it could model a uniform charge of radius ϵ producing an electric field and electric potential. ||

The Laplacian (continued)

These problems are another approach to the identity involving the Newtonian potential and the approximate delta function. The method is to get an approximation $\phi_\epsilon(\mathbf{x})$ whose Laplacian is an approximate delta function $\delta_\epsilon(\mathbf{x})$. To this end, define $r_\epsilon = \sqrt{r^2 + \epsilon^2}$ and let

$$\phi_\epsilon(\mathbf{x}) = \frac{1}{n\alpha(n)} \frac{1}{n-2} \frac{1}{r_\epsilon^{n-2}}. \tag{10.212}$$

1. Prove that $dr_\epsilon^2 = dr^2$ and hence $dr_\epsilon/dr = r/r_\epsilon$.

2. (a) Prove that
$$-\nabla\phi_\epsilon(\mathbf{x}) = \frac{1}{n\alpha(n)} \frac{1}{r_\epsilon^n} E. \tag{10.213}$$
(b) Let $\omega = E \lrcorner \text{vol}$. Prove that
$$-\nabla\phi_\epsilon(\mathbf{x}) \lrcorner \text{vol} = \frac{1}{n\alpha(n)} \frac{1}{r_\epsilon^n} \omega. \tag{10.214}$$

3. Show that
$$-\nabla^2\phi_\epsilon(\mathbf{x}) = \delta_\epsilon(\mathbf{x}), \tag{10.215}$$
where $\delta_\epsilon(\mathbf{x})$ is a constant times a power of ϵ times an inverse power of r_ϵ. Find the explicit form of $\delta_\epsilon(\mathbf{x})$.

4. Show that
$$\delta_\epsilon(\mathbf{x}) = \delta_1\left(\frac{\mathbf{x}}{\epsilon}\right) \frac{1}{\epsilon^n}. \tag{10.216}$$
What is the explicit form for the function $\delta_1(\mathbf{x})$?

5. To show that this is an approximate delta function, we need to show that $\int \delta_\epsilon(\mathbf{x}) \,\text{vol} = 1$. For each a compute $\int_{B_a} \delta_\epsilon(\mathbf{x}) \,\text{vol}$ as an explicit function of a and ϵ and n. Hint: Use the divergence theorem.

6. Show that for fixed $\epsilon > 0$ we have
$$\int_{B_a} \delta_\epsilon(\mathbf{x}) \,\text{vol} \to 1 \tag{10.217}$$
as $a \to \infty$.

A final Laplacian problem

1. Let $\mathtt{g} = h^2(dx^2 + dy^2)$ with $h > 0$ a scalar field. Find its Gaussian curvature. Show that the Gaussian curvature vanishes if and only if $-\log h$ satisfies the Laplace equation (with respect to the Euclidean metric). More generally, the Gaussian curvature has the same sign as the Laplacian of $-\log h$.

2. Take $h = 2/(x^2 + y^2 + 1)$. Find the Gaussian curvature.

Chapter 11

Supplement: Differential and Codifferential

11.1 Differential, codifferential, and Laplacian

11.1.1 Signs for the codifferential and the Laplacian

The following warning is essential:

- The sign convention for the codifferential δ used in this book is such that for a 1-form ω it is the divergence: $\delta\omega = \operatorname{div} \mathsf{g}^{-1}\omega$. This is the opposite sign from that used in most treatments of advanced differential geometry.
- The sign convention for the Laplacian Δ used in this book is such that $\Delta s = \delta ds = \operatorname{div} \operatorname{grad} s$ for scalar fields. This is the opposite sign from that used in most treatments of advanced differential geometry. However it is the sign used in most other areas of mathematics and its applications.

11.1.2 Calculus on manifolds

The natural general framework for calculus in a geometrical setting is a manifold with a metric tensor. The presence of a metric tensor means that there is a natural correspondence between vector fields and differential 1-forms. This allows calculus to be formulated in several ways.

Vector fields The basic operations of classical vector calculus are the gradient and the divergence. In three dimensions the curl is also an important operation.

Differential forms The basic operations on differential forms are the differential, which sends k-forms to $(k+1)$-forms, and the codifferential, which sends k-forms to $(k-1)$ forms. An advantage of this framework is that Stokes' theorem is most naturally expressed in terms of differential forms.

Tensor fields Tensor calculus combines vector fields and differential forms and other geometrical quantities in a unified theory. There is a very general notion of covariant derivative that allows differentiation of arbitrary tensor fields.

There is a special situation when the manifold is Euclidean space. In that case the metric tensor in Cartesian coordinates has an extremely simple form. For other coordinate systems it may have a more complicated form, but in principle this is only the result of a change of coordinates from Cartesian coordinates. Of course there may be good reasons for using other coordinate systems. A typical situation is when this exhibits a useful symmetry in the problem under consideration.

It is easy to produce manifolds that are essentially different from Euclidean space. One way to do this is to take the Euclidean metric and pull it back via an immersion. Such a metric may have non-zero curvature: the most familiar example is the pullback of the metric on three-dimensional space to a two-dimensional spherical surface. In such situations there is no way at all to get rid of the complexities of the metric structure.

Finally, the formulas for a general metric give a deeper reason for the way that certain calculations turn out. It is useful to know which calcululations work for a manifold without metric, for a manifold with a metric that may or may not involve curvature, or only for Euclidean space. It is remarkable how much can be done in the first two of these three situations.

Remark: There is another algebraic structure that may be combined with tensor calculus. This is *Clifford algebra*. There is also a related notion of *spinor*. One version of the Clifford algebra story is sometimes presented under the name *geometric algebra*. None of this material will be covered here. ||

The previous chapter of this book placed the main emphasis on vector fields. The purpose of this chapter is to present the calculus of differential forms involving differential and codifferential. This gives a unified picture of the basic calculus operations that works on manifolds of arbitrary dimension. It also includes a discussion of the covariant differential, which is a much more general and powerful operation.

The next logical step would be tensor calculus, but an account of that subject deserves a book on its own. After that might come differential geometry and perhaps global analysis. These are even further from the scope of the present account.

11.1.3 The Hodge dual

The form calculus gives a beautiful and general story that works in any number of dimensions. The differential d is a remarkably simple operations that sends k forms to $(k+1)$-forms and satisfies $d^2 = 0$. The new feature is that in the presence of a metric tensor there is a codifferential δ that sends k forms to $(k-1)$-forms and satisfies $\delta^2 = 0$.

What makes this work is the *Hodge dual* (or Hodge star) operator $*$ that sends k-forms to (twisted) $(n-k)$-forms. It is closely related to the inner product on k-forms.

Proposition 11.1 *There is a unique inner product for k-forms with the property that for 1-forms $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_k$ the value of the inner product of the exterior products is*

$$(\alpha_1 \wedge \cdots \wedge \alpha_k) \bullet (\beta_1 \wedge \cdots \wedge \beta_k) = \det(\alpha_i \mathsf{g}^{-1} \beta_j). \tag{11.1}$$

Proposition 11.2 *Choose a volume form* vol *compatible with the metric. (There are two of them.) Then there is a unique linear map $*$ from k-forms to $(n-k)$-forms such that for every pair of k-forms α and β*

$$\alpha \wedge *\beta = (\alpha \bullet \beta)\text{vol}. \tag{11.2}$$

As a consequence, the bilinear expression $\alpha \wedge *\beta$ satisfies the symmetry property

$$\alpha \wedge *\beta = \beta \wedge *\alpha. \tag{11.3}$$

Furthermore, the map $*$ preserves the inner product: $*\alpha \bullet *\beta = \alpha \bullet \beta$.

These are standard results that are proved in advanced treaatments such as [33]. They are also treated in later sections below.

Example: Take the volume form to be $\text{vol} = \sqrt{g} du_1 \cdots du_n$. The Hodge dual of du_k is

$$* du_k = \sum_j g^{jk} (-1)^{j-1} \sqrt{g}\, du_1 \cdots du_{j-1}\, du_{j+1} \cdots du_n. \tag{11.4}$$

This can be seen by computing $du_\ell \wedge * du_k = g^{\ell k} \,\text{vol} = (du_\ell \bullet du_k)\text{vol}$. ||

There is a simple expression for the Hodge dual of the Hodge dual; it is the identity, up to a $\pm$ sign. This sign is easy to understand, as shown in the following corollary.

Corollary 11.3 *If β is a k-form, then*

$$** \beta = (-1)^{k(n-k)} \beta. \tag{11.5}$$

In particular, the inverse operator is $*^{-1} = (-1)^{k(n-k)} *$. It follows that

$$\begin{aligned} n \text{ odd } &\Rightarrow ** \beta = \beta \\ n \text{ even } &\Rightarrow ** \beta = (-1)^k \beta. \end{aligned} \tag{11.6}$$

Proof: Since $*$ is preserves the inner product and the inner product is symmetric, we have

$$*\alpha \wedge ** \beta = \alpha \wedge * \beta = \beta \wedge * \alpha. \tag{11.7}$$

Since β has degree k and $*\alpha$ has degree $n-k$, this gives

$$*\alpha \wedge ** \beta = (-1)^{k(n-k)} * \alpha \wedge \beta. \tag{11.8}$$

Since α is arbitrary, this gives the result. $\square$

With orthogonal coordinates the Hodge dual $*$ is easy to compute. The result is that it sends normalized basis vectors into normalized basis vectors, up to sign:

$$*(h_{i_1}\, dx_{i_1} \wedge \cdots \wedge h_{i_k}\, dx_{i_k}) = \pm\, h_{j_1}\, dx_{j_1} \wedge \cdots \wedge h_{j_{n-k}}\, dx_{j_{n-k}}. \tag{11.9}$$

The indices in the result are complementary to the original indices. The sign can be determined for such a basis vector β by $\beta \wedge * \beta = \text{vol}$.

Example: When $n = 3$ using Cartesian coordinates we have $* 1 = dx\, dy\, dz$. Also $* dx = dy\, dz$ and $* dy = dx\, dx$ and $* dz = dx\, dy$. Furthermore, for $n = 3$ $*$ is its own inverse. ||

Example: When $n = 2$ using Cartesian coordinates we have $* 1 = dx\, dy$. Also $* dx = dy$ and $* dy = -dx$. * On 1-forms the inverse of $*$ is $-*$. ||

Example: When $n = 2$ using polar coordinates we have $* 1 = r\, dr\, d\theta$. Also $* dr = r\, d\theta$ and $* r\, d\theta = -dr$. * On 1-forms the inverse of $*$ is $-*$. ||

There are important relations between interior product, exterior product, and Hodge dual.

Proposition 11.4 *If α is a $(k-1)$-form, and β is a k-form, and $\mathbf{u}$ is a vector field, then*

$$(\mathbf{gu} \wedge \alpha) \bullet \beta = \alpha \bullet (\mathbf{u} \lrcorner \beta). \tag{11.10}$$

Proof: Write $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_{k-1}$ and $\beta = \beta_1 \wedge \cdots \wedge \beta_k$. The inner product on the left is the determinant of a k by k matrix with first row $\mathbf{u} \lrcorner \beta_j$ and following rows $\alpha_i \mathsf{g}^{-1} \beta_j$. The inner product on the right is the sum of $(-1)^{-j-1} \mathbf{u} \lrcorner \beta_j$ times the determinant of the $k-1$ by $k-1$ matrix $\alpha_i \mathsf{g}^{-1} \beta_\ell$ with the j column removed. These two matrices are related by the cofactor expansion. □

This also has a useful corollary relating the Hodge dual with the interior product.

Corollary 11.5 *Suppose that β is a k-form and $\mathbf{u}$ is a vector. Then*

$$\mathbf{gu} \wedge *\beta = (-1)^{k-1} * (\mathbf{u} \lrcorner \beta). \tag{11.11}$$

Proof: Using the relation between the inner product and the Hodge * operator, the result of the proposition translates to the following assertion. Let α be a $k-1$ form. Then

$$(\mathbf{gu} \wedge \alpha) \wedge *\beta = \alpha \wedge *(\mathbf{u} \lrcorner \beta). \tag{11.12}$$

This can be written

$$\alpha \wedge \mathbf{gu} \wedge *\beta = (-1)^{k-1} \alpha \wedge *(\mathbf{u} \lrcorner \beta). \tag{11.13}$$

Since α is arbitrary, the result follows. □

The formula in the corollary may be specialized by taking β to be a 2-form $\mathbf{cv}$ or by taking β to be a 1-form $\mathbf{gv}$. These lead to the formulas for dot product and cross product given before.

Another specialization is to take $\beta = \text{vol}$. This gives the following important result:

Proposition 11.6 *1-forms map to $(n-1)$-forms by $* = \mathsf{cg}^{-1}$. In the other direction, $(n-1)$-forms map to 1-forms by $* = (-1)^{n-1} \mathsf{gc}^{-1}$.*

11.1.4 The codifferential

The *codifferential* δ mapping k-forms to $(k-1)$-forms is defined by

$$\delta\beta = (-1)^{k-1} *^{-1} d * \beta. \tag{11.14}$$

This sends the k-form β to the $(n-k)$-form $*\beta$ to the $(n-k+1)$-form $d * \beta$ to the $(k-1)$-form $\delta\beta$.

The motivation for the definition of codifferential is that if α is a $(k-1)$-form, then

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + \alpha \wedge *\delta\beta. \tag{11.15}$$

Warning: This definition is the negative of the definition most commonly encountered in advanced differential geometry.

Theorem 11.7 *The codifferential satisfies* $\delta\delta = 0$.

This theorem is immediate from the definition using the Hodge dual, but it disguises that fact that the codifferential δ is a complicated operator that does not necessarily share other properties with the differential d.

Sometimes it is convenient to not to deal with the inverse of the Hodge dual, so the following result is convenient.

Proposition 11.8 *In n dimensions the codifferential on k-forms satisfies* $\delta = (-1)^{n(k-1)} * d *$. *As a consequence*

$$\begin{aligned} n \text{ odd } &\Rightarrow \delta = (-1)^{k-1} * d * \\ n \text{ even } &\Rightarrow \delta = * d * . \end{aligned} \tag{11.16}$$

The proof of this proposition follows from the computation of the sign. This is $(k-1) + (k-1)(n-k+1) = (k-1)(n-k+2) = (k-1)n - (k-1)(k-2)$. The term $(k-1)(k-2)$ is always even.

The codifferential resembles the differential in some respects. In particular

$$\delta\delta\beta = 0. \tag{11.17}$$

This property $\delta^2 = 0$ suggests that the codifferential shares properties with the differential. As it happens, the codifferential and differential are related in a very simple way in Cartesian coordinates. On the other hand, with other coordinate systems or with a metric that is not flat the codifferential is a complicated operation. The differential remains simple and straightforward in every situation.

Example: The most important case is the codifferential on 1-forms. This acts somewhat like a divergence. Consider a 1-form $\alpha = \sum_k a_k \, du_k$. The Hodge dual $* \, du_k$ was computed in a previous example. The codifferential is

$$\delta\alpha = *d* \, \alpha = \frac{1}{\sqrt{g}} \sum_j \frac{\partial}{\partial u_j} \left(\sqrt{g} \sum_k g^{jk} a_k \right). \tag{11.18}$$

||

Example: Sometimes it is convenient to write the differential or the codifferential in terms of normalized bases. Here is a simple example. In polar coordinates the codifferential of a 1-form written in terms of coordinate basis forms is

$$\delta(a \, dr + b \, d\theta) = \frac{1}{r} \frac{\partial (ra)}{\partial r} - \frac{1}{r^2} \frac{\partial b}{\partial \theta}. \tag{11.19}$$

This uses the fact that the components of the inverse of the metric are 1 and $1/r^2$. In terms of normalized basis forms it is

$$\delta(p \, dr + q \, r \, d\theta) = \frac{1}{r} \frac{\partial (rp)}{\partial r} - \frac{1}{r} \frac{\partial q}{\partial \theta}. \tag{11.20}$$

||

A motivation for the definition of codifferential is provided by the following proposition.

Proposition 11.9 (Skew-adjointness identity) *Suppose that α is a $(k-1)$-form and β is a k-form. Then the differential $d\alpha$ and codifferential $\delta\beta$ are related by*

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + \alpha \wedge *\delta\beta. \tag{11.21}$$

This is an identity for n-forms. In terms of the inner product on forms it may also be written

$$d(\alpha \wedge *\beta) = (d\alpha \bullet \beta)\mathrm{vol}|_+(\alpha \bullet \delta\beta)\mathrm{vol}|_+. \tag{11.22}$$

Remark: With the other sign convention for the codifferential this would be an adjointness identity with a minus sign on the right.

Theorem 11.10 (Skew-adjointness theorem for differential and codifferential) *Suppose that α is a $(k-1)$-form and β is a k form. Then*

$$\int_R (d\alpha \bullet \beta)\mathrm{vol} + \int_R (\alpha \bullet \delta\beta)\mathrm{vol} = \int_{\partial R} \alpha \wedge *\beta. \tag{11.23}$$

This theorem is often used when the integral over the surface $S = \partial R$ is zero. This is ensured by imposing a *boundary condition*. A form α satisfies the *zero boundary condition* if the pullback of α to S is zero. Sometimes it is useful to express this in other equivalent ways.

- For α a 0-form the zero condition is that $\alpha = 0$ on S.
- For α a 1-form the zero condition is that $\mathsf{g}^{-1}\alpha$ is normal to S.
- For α an $(n-1)$-form the zero condition is that $\mathsf{c}^{-1}\alpha$ is tangent to S (has zero component normal to S).
- For α an n-form the zero condition is always satisfied.

The $(n-1)$-form case may require comment. Wrote

$$\langle \alpha \mid X_1, \ldots, X_{n-1}) = \langle \mathrm{vol} \mid \mathsf{c}^{-1}\alpha, X_1, \ldots, X_{n-1}\rangle. \tag{11.24}$$

The left hand size is zero when the vectors on the right hand side are linearly dependent.

A form β is said to satisfy the *dual zero boundary condition* if the pullback of $*\beta$ to S is zero. In special cases this has various statements.

- For β a 0-form the dual zero condition is always satisfied.
- For β a 1-form the dual zero condition is that $\mathsf{g}^{-1}\beta$ is tangent to S (has zero component normal to S).
- For β an $(n-1)$-form the dual zero condition is that $\mathsf{c}^{-1}\beta$ is normal to S.
- For β an n-form the dual zero condition is that β is 0 on S.

These follow from $\mathsf{c} = *\mathsf{g}$.

If one thinks of 1-form β as a fluid flow with vector field $\mathsf{g}^{-1}\beta$, then the dual zero condition says that the flow at the boundary is parallel to the boundary. The boundary does not distort the flow. This contrasts with the situation for the zero boundary condition. For a 1-form α this implies that at the boundary the component of the flow parallel to the boundary is zero. The flow is very much influenced by the presence of the boundary.

Example: Here is an example in Euclidean space with Cartesian coordinates. Suppose the surface is $z = 0$ and the form is $\alpha = p\,dx + q\,dy + r\,dz$. The corresponding vector is $\mathsf{g}^{-1}\alpha = p\partial/\partial x + q\partial/\partial y + r\partial/\partial z$. The pullback of α to the surface is $p\,dx + q\,dy = 0$. So the zero boundary condition is $p = q = 0$, or $\mathsf{g}^{-1}\alpha = r\partial/\partial z$. The Hodge dual is $*\alpha = p\,dy\,dz + q\,dz\,dz + r\,dx\,dy$ The pullback to the surface is $r\,dx\,dy = 0$. So the dual zero boundary condition is $r = 0$, or $\mathsf{g}\alpha = a\partial/\partial x + b\partial/\partial y$. ||.

Here is a special case: the divergence theorem may be formulated in the language of the codifferential. It says that for an 1-form β and n-dimensional region R we have

$$\int_R \delta\beta \,\mathrm{vol} = \int_{\partial R} *\beta. \tag{11.25}$$

This can be seen directly: the left hand integrand is $\delta\beta\,\mathrm{vol} = *\delta\beta = d * \beta$.

Example: With orthogonal coordinates the calculations for d and for δ are straightforward. Here are examples in three dimensions. Suppose the basis forms are $h_u\,du$, $h_v\,dv$, and $h_w\,dw$. Consider 0, 1, 2, and 3 forms s, $\alpha = a\,h_u\,du + b\,h_v\,dv + c\,h_c\,dw$, $\omega = p\,h_v h_w dv\,dw + q\,h_w h_u\,dw\,du + r\,h_u h_v\,du\,dv$, and $s\,h_u h_v h_w\,du\,dv\,dw$. The differentials are given by

$$ds = \left[\frac{1}{h_u}\frac{\partial s}{\partial u}\right]\,h_u\,du + \left[\frac{1}{h_v}\frac{\partial s}{\partial v}\right]\,h_v\,dv + \left[\frac{1}{h_w}\frac{\partial s}{\partial w}\right]\,h_w\,dw, \tag{11.26}$$

$$\begin{aligned} d\alpha = &\left[\frac{1}{h_v h_w}\left(\frac{\partial h_w c}{\partial v} - \frac{\partial h_v b}{\partial w}\right)\right] h_v h_w\,dv\,dw \\ &+ \left[\frac{1}{h_w h_u}\left(\frac{\partial h_u a}{\partial w} - \frac{\partial h_w c}{\partial u}\right)\right] h_w h_u\,dw\,du \\ &+ \left[\frac{1}{h_u h_v}\left(\frac{\partial h_v b}{\partial u} - \frac{\partial h_u a}{\partial v}\right)\right] h_u h_v\,du\,dv, \end{aligned} \tag{11.27}$$

and

$$d\omega = \left[\frac{1}{h_u h_v h_w}\left(\frac{\partial}{\partial u}(h_v h_w p) + \frac{\partial}{\partial v}(h_w h_u q) + \frac{\partial}{\partial w}(h_u h_v r)\right)\right] h_u h_v h_w\,du\,dv\,dw. \tag{11.28}$$

That is, on 0,1, and 2 forms the coefficients resemble those for grad, curl, and div.

On the other hand, the calculations for δ are

$$\delta(s\,\mathrm{vol}) = \left[\frac{1}{h_u}\frac{\partial s}{\partial u}\right]\,h_v h_w\,dv\,dw + \left[\frac{1}{h_v}\frac{\partial s}{\partial v}\right]\,h_w h_u\,dw\,du + \left[\frac{1}{h_w}\frac{\partial s}{\partial w}\right]\,h_u h_v\,du\,dv, \tag{11.29}$$

$$-\delta\omega = \left[\frac{1}{h_v h_w}\left(\frac{\partial h_w r}{\partial v} - \frac{\partial h_v q}{\partial w}\right)\right] h_u\, du + \left[\frac{1}{h_w h_u}\left(\frac{\partial h_u p}{\partial w} - \frac{\partial h_w r}{\partial u}\right)\right] h_v\, dv + \\ + \left[\frac{1}{h_u h_v}\left(\frac{\partial h_v q}{\partial u} - \frac{\partial h_u p}{\partial v}\right)\right] h_w\, dw, \tag{11.30}$$

and

$$\delta\alpha = \left[\frac{1}{h_u h_v h_w}\left(\frac{\partial h_v h_w a}{\partial u} + \frac{\partial h_w h_u b}{\partial v} + \frac{\partial h_u h_w c}{\partial w}\right)\right]. \tag{11.31}$$

The coefficients for the codifferential δ have almost the same form as those for the differential d; on 3, 2, and 1 forms they look like those for grad, $-$ curl, and div. The normalized basis vectors, on the other hand, are related by duality. ||

Example: The same kind of calculation works for $n = 2$. On 0-forms and 1-forms the differential d gives coefficients that resemble grad and curl. On 2-forms and 1-forms the codifferential δ gives coefficients that resemble $-\operatorname{curl}'$ and div. ||

Here is a summary of the form calculus:

- There are differential forms of degree 0 through n.
- The differential d sends k-forms to $(k+1)$-forms in the usual way.
- The Hodge dual operator $*$ is an algebraic operator that depends on the metric tensor. It sends k-forms to $(n-k)$-forms, and on k-forms it satisfies $**\omega = (-1)^{k(n-k)}\omega$. It is simple to compute if the coordinate system is orthogonal.
- The codifferential δ sends k-forms to $(k-1)$-forms. It is defined by

$$\delta\,\omega = (-1)^{k-1} *^{-1} d * \omega. \tag{11.32}$$

- The operators δ and d may be combined. For example, on scalar fields (0-forms) $\delta\, d\, s$ is the Laplacian of s, another scalar field.

11.1.5 The Laplace de Rham operator

The *Laplace de Rham operator* Δ is defined by

$$\Delta = -(d-\delta)^2 = d\,\delta + \delta\, d \tag{11.33}$$

Here δ is the codifferential, which with the convention adopted here is the negative of the adjoint. The operator $d-\delta$ sends k forms to sums of $k+1$ and $k-1$ forms. The Laplace de Rham operator sends k-forms to k-forms.

Remark: The Laplace de Rham operator is sometimes defined in advanced differential geometry with the opposite sign. The convention used here is standard in other areas of mathematics. ||

Remark: If $Q = d - \delta$, then $Q^2 = -\Delta$. So Q is a kind of square root of $-\Delta$, but a very strange one, since it sends even degree forms to odd degree forms,

and it sends odd degree forms to even degree forms. This is an instance of a much grander scheme called *supersymmetry*. See [52]. ||

The Laplace de Rham operator has a simple expression in Cartesian coordinates. In other coordinates it can be quite complicated. When the metric is not flat, the expression involves an extra curvature term. See [34] for the calculation in this case. There is also a detailed discussion at the end of this book.

Remark: A major advanced topic in this area of analysis is the Helmholtz decomposition and its generalizations. Here is a brief sketch in the simplest case.

Say that ω is a k form. Then ω is said to have a *Helmholtz decomposition* if there is a $(k-1)$-form α and a $(k+1)$-form β and a k-form σ with $\Delta\sigma = 0$ such that

$$\omega = d\alpha + \delta\beta + \sigma. \tag{11.34}$$

Proposition 11.11 *Suppose that $\Delta\sigma = 0$ and γ is a solution of the Poisson equation*

$$\Delta\gamma = \omega - \sigma. \tag{11.35}$$

Then ω has a Helmholtz decomposition.

Proof: This is a consequence of an ability to solve the Poisson equation. In this case one can take $\alpha = \delta\gamma$ and $\beta = d\gamma$. Then $d\alpha + \delta\beta = (d\delta + \delta d)\gamma = \Delta\gamma = \omega - \sigma$. □ ||

Because of the fundamental role of Stokes' theorem, the theory here is presented entirely in the language of differential forms. Classical vector analysis uses vector fields. The metric g takes vector fields to differential 1-forms. This translates classical vector analysis as a fragment of the Hodge calculus.

Here are the correspondences in n dimensions. The scalar product on vectors is related to the scalar product on forms by $\mathsf{g}^{-1}\alpha \cdot \mathsf{g}^{-1}\beta = *(\alpha \wedge *\beta)$. For the calculus operations there are the following relations.

- grad $= \mathsf{g}^{-1}d$ on scalar fields.
- div $= \delta\mathsf{g}$ on vector fields
- $\nabla^2 = \delta d$ on scalar fields.

11.1.6 Comparison of vector calculus and form calculus

Specialize to dimension $n = 3$. The translation from vector calculus to exterior calculus is simple: scalar fields, vector fields, pseudovector fields, and pseudoscalar fields go to 0-forms, 1-forms, 2-forms, and 3-forms. A scalar field is the same thing as a 0-form. For vector fields, pseudovector fields, and pseudoscalar fields the translations are given by g, c, and $*$. The last two depend on on orientation.

Pseudovector fields and pseudoscalar fields might seem strange, since they change sign when switiching from right handed to left handed orientation. Forms are defined independent of orientation. (While the Hodge $*$ depends on orientation, neither the derivation d nor the codifferential δ depend on orientation.)

Proposition 11.12 *Consider dimension 3. The classical vector calculation operations are*

$$\begin{aligned} \text{grad} &= \mathsf{g}^{-1} d = \mathsf{c}^{-1} \delta * \\ \text{curl} &= \mathsf{c}^{-1} d \mathsf{g} = -\mathsf{g}^{-1} \delta \mathsf{c} \\ \text{div} &= * d \mathsf{c} = \delta \mathsf{g}. \end{aligned} \tag{11.36}$$

We have curl grad $= 0$ *and* div curl $= 0$.

Proposition 11.13 *Consider dimension 3. The differential on 0, 1, and 2 forms is given by*

$$\begin{aligned} d &= \mathsf{g}\,\text{grad} \\ d &= \mathsf{c}\,\text{curl}\,\mathsf{g}^{-1} \\ d &= *\,\text{div}\,\mathsf{c}^{-1} \end{aligned} \tag{11.37}$$

For $n = 3$ the Hodge $*$ is equal to $\mathsf{c}\mathsf{g}^{-1}$ on 1-forms and to $\mathsf{g}\mathsf{c}^{-1}$ on 2-forms.

Proposition 11.14 *Consider dimension 3. The codifferential δ on 3-forms, 2-forms, and 1-forms is given by*

$$\begin{aligned} \delta &= \mathsf{c}\ \text{grad}\ * \\ \delta &= -\mathsf{g}\ \text{curl}\ \mathsf{c}^{-1} \\ \delta &= \text{div}\ \mathsf{g}^{-1}. \end{aligned} \tag{11.38}$$

These equations are valid with an arbitrary metric tensor in an arbitrary coordinate system.

Proposition 11.15 *Consider dimension 3. The Laplace de Rham operator on 0-forms, 1-forms, 2-forms, and 3-forms is related to divergence, curl, and gradient by*

$$\begin{aligned} \delta d &= \text{div grad} \\ \delta d + d\delta &= \mathsf{g}(-\,\text{curl curl} + \text{grad div})\mathsf{g}^{-1} \\ \delta d + d\delta &= \mathsf{c}(\text{grad div} - \text{curl curl})\mathsf{c}^{-1} \\ d\delta &= *\,\text{div grad}\,*. \end{aligned} \tag{11.39}$$

These equations are valid with an arbitrary metric tensor in an arbitrary coordinate system.

Remark: The most common version of a Helmholtz decomposition is for vector fields $\mathbf{v}$ in three dimensions. It is assumed that they satisfy a decay condition at infinity, so that the only solution of $\Delta\sigma = 0$ that satisfies the decay condition is zero. The result then says that

$$\mathbf{v} = -\,\text{grad}\ \phi + \text{curl}\ \mathbf{u}. \tag{11.40}$$

$$\begin{array}{ccccccc}
 & \overset{d}{\longrightarrow} & & \overset{d}{\longrightarrow} & & \overset{d}{\longrightarrow} & \\
0 & & 1 & & 2 & & 3 \\
 & \underset{\delta}{\longleftarrow} & & \underset{\delta}{\longleftarrow} & & \underset{\delta}{\longleftarrow} & \\
 & \overset{\text{grad}}{\longrightarrow} & & \overset{\text{curl}}{\longrightarrow} & & \overset{\text{div}}{\longrightarrow} & \\
S & & V & & V^{\sim} & & S^{\sim} \\
 & \underset{\text{div}}{\longleftarrow} & & \underset{-\text{curl}}{\longleftarrow} & & \underset{\text{grad}}{\longleftarrow} &
\end{array}$$

Table 11.1: Derivative and coderivative vs vector operations: 3 dimensions

The relation is that $-\mathsf{g}\mathbf{v} = \omega = d\phi + \delta\beta = d\phi - \mathsf{g}\,\text{curl}\,c^{-1}\beta = d\phi - \mathsf{g}\,\text{curl}\,\mathbf{u}$. When a Helmholtz decomposition is valid it gives a useful description of the structure of the vector field. ||

The relation between differential form and vector field operations in dimension 3 is summarized in Table 11.1. The numbers are the degree of the forms. The S, V, $V^{\sim}$, $S^{s}im$ stand for scalar, vector, pseudo-vector, pseudo-scalar. (These can be reversed; the important thing is that curl depends on orientation.) The general Laplace operator $\Delta = d\delta + \delta d$ takes specific forms for the vector fields. The corresponding Laplace operators in the vector framework are $\Delta_3 = \text{div}\,\text{grad}$ on scalar fields and $\vec{\Delta}_3 = \text{grad}\,\text{div} - \text{curl}\,\text{curl}$ on vector fields.

Specialize to dimension $n = 2$. The translation from vector calculus to exterior calculus is simple: scalar fields, vector fields, pseudovector fields, and pseudoscalar fields go to 0-forms, 1-forms, 1-forms, and 2–forms. For vector fields, pseudovector fields, and pseudoscalar fields the translations are given by g, c, and $*$.

In two dimensions a 1-form may arise from a vector field or from a pseudovector field. This is related to the fact that in two dimensions there is a map $* = \mathsf{c}\mathsf{g}^{-1} = -\mathsf{g}\mathsf{c}^{-1}$ from 1-forms to 1-forms that does depend on orientation.

Proposition 11.16 *For $n = 2$ the vector calculus operators on vector fields (or scalar fields) are*

$$\begin{aligned}
\text{grad} &= \mathsf{g}^{-1}d = \mathsf{c}^{-1}\delta* \\
\text{curl} &= *\,d\,\mathsf{g} = -\delta\mathsf{c} \\
\text{curl}' &= \mathsf{c}^{-1}d = -\mathsf{g}^{-1}\delta* \\
\text{div} &= *\,d\,\mathsf{c} = \mathsf{g}\delta.
\end{aligned} \tag{11.41}$$

The operator curl *acts on vector fields and has scalar field values, while* curl$'$ *acts on scalar fields and has vector field values. We have* curl grad $= 0$ *and* div curl$' = 0$.

Proposition 11.17 *Consider dimension 2. The differential d on 0-forms and 1-forms is given by*

$$d = \mathsf{g}\,\mathrm{grad} = \mathsf{c}\,\mathrm{curl}'$$
$$d = *\,\mathrm{div}\,\mathsf{c}^{-1} = *\,\mathrm{curl}\,\mathsf{g}^{-1}. \tag{11.42}$$

For $n = 2$ the Hodge star operator has values on 1-forms given by $* = \mathsf{c}\mathsf{g}^{-1} = -\mathsf{g}\mathsf{c}^{-1}$. Its inverse is given on 1-forms by $-\mathsf{c}\mathsf{g}^{-1} = \mathsf{g}\mathsf{c}^{-1}$.

Proposition 11.18 *Consider dimension 2. The codifferential δ on 2-forms and 1-forms is given by*

$$\delta = \mathsf{c}\ \mathrm{grad}\ * = -\mathsf{g}\ \mathrm{curl}'\ *$$
$$\delta = \mathrm{div}\ \mathsf{g}^{-1} = -\,\mathrm{curl}\ \mathsf{c}^{-1}. \tag{11.43}$$
$$\tag{11.44}$$

These formulas are valid with arbitrary metric tensor.

Proposition 11.19 *Consider dimension 2. The Laplace de Rham operator on 0-forms, 1-forms, and 2-forms is*

$$\delta d = \mathrm{div}\ \mathrm{grad} = -\,\mathrm{curl}\ \mathrm{curl}'$$
$$\delta d + d\delta = \mathsf{g}(-\,\mathrm{curl}'\ \mathrm{curl} + \mathrm{grad}\ \mathrm{div})\mathsf{g}^{-1}$$
$$d\delta = *\ \mathrm{div}\ \mathrm{grad}\ * = -\,*\ \mathrm{curl}\ \mathrm{curl}'\ *. \tag{11.45}$$

These formulas are valid with arbitrary metric tensor.

In two dimensions there are two different factorizations of the Laplacian on scalar fields.

The relation between differential form and vector field operations in dimension 2 is summarized in Table 11.2. The numbers are the degree of the forms. The S, V, $V^{\sim}$, $S^{\sim}$ stand for scalar, vector, pseudo-vector, pseudo-scalar. (These can be reversed; the important thing is that curl operations reverse orientation.) The general Laplace operator $\Delta = d\delta + \delta d$ again takes specific forms for the vector fields. The corresponding Laplace operators are $\Delta_2 = \mathrm{div}\ \mathrm{grad} = -\,\mathrm{curl}\ \mathrm{curl}'$ on scalar fields and $\vec{\Delta}_2 = \mathrm{grad}\ \mathrm{div} - \mathrm{curl}'\ \mathrm{curl}$ on vector fields. These do not depend on orientation.

11.1.7 The Laplace-de Rham energy

Recall that if α and β are p-forms, then $\alpha \wedge *\beta$ is a twisted n-form. Furthermore, the definition of the Hodge dual $*$ may be taken so that

$$\alpha \wedge *\beta = (\alpha \bullet \beta)\mathrm{vol}|_+. \tag{11.46}$$

where $\mathrm{vol}|_+$ is the volume element. In particular, it makes sense to say that

$$\alpha \wedge *\alpha = (\alpha \bullet \alpha)\mathrm{vol}|_+ \geq 0. \tag{11.47}$$

This quantity $\alpha \wedge *\alpha$ is a positive twisted form that measures the size of α at each point. Its integral is a number that gives an overall picture of its size.

	$\overset{d}{\longrightarrow}$		$\overset{d}{\longrightarrow}$	
0		1		2
	$\overset{\longleftarrow}{\delta}$		$\overset{\longleftarrow}{\delta}$	
	grad $\longrightarrow$		curl $\longrightarrow$	
S		V		$S^\sim$
	$\overset{\longleftarrow}{\text{div}}$		$\overset{\longleftarrow}{-\,\text{curl}'}$	

Table 11.2: Derivative and coderivative vs vector operations: 2 dimensions

Proposition 11.20 *Let α be a p-form. There is an $(n-1)$-form $\alpha \wedge *d\alpha$ whose exterior differential is the sum of a positive n-form with another n-form involving the Laplace-de Rham Laplacian Δ acting on α:*

$$d(\alpha \wedge *d\alpha + \delta\alpha \wedge *\alpha) = d\alpha \wedge *d\alpha + \delta\alpha \wedge *\delta\alpha + \alpha \wedge *\Delta\alpha. \tag{11.48}$$

The proposition is proved by using identities like $d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + \alpha \wedge *\delta\beta$.

Consider a bounded region R with boundary ∂R and a p-form α. We consider two kinds of boundary conditions.

1. Zero boundary condition on α: The forms α and $\delta\alpha$ pull back to zero on ∂R.

2. Dual zero boundary condition on α: The forms $*\alpha$ and $\delta * \alpha$ (or $*d\alpha$) pull back to zero on ∂R.

These are the zero and dual zero boundary conditions discussed before, applied to α and to $d * \alpha$ in the first case and to α and $d\alpha$ in the second case. See [44] for more discussion of these boundary conditions. This includes a description of what they mean in the context of vector calculus in three dimensions. See also [7] for a detailed account of the geometry in three dimensions.

Corollary 11.21 *Let α be a p-form. Suppose there is a bounded region R such that α satisfies either of the two boundary conditions on ∂R. Then the integral of the n-form associated with α and the Laplace-de Rham Laplacian is negative. This fact is expressed quantitatively by*

$$-\int_R \alpha \wedge *\Delta\alpha = \int_R (d\alpha \wedge *d\alpha + \delta\alpha \wedge *\delta\alpha) \geq 0. \tag{11.49}$$

In the language of inner products this is

$$-\int_R (\alpha \bullet \Delta\alpha)\text{vol}|_+ = \int_R (d\alpha \bullet d\alpha + \delta\alpha \bullet \delta\alpha)\text{vol}|_+ \geq 0. \tag{11.50}$$

The importance of this corollary is the following. The integrand on the right hand side is an obviously positive quantity that may be interpreted as a kind of energy. The left hand side may then be interpreted as saying that the operator $-\Delta$ is positive in some sense.

This is a version of *Green's first identity*. . The most elementary case is when $\alpha = s$ is a 0-form. In that case the zero boundary condition says that s is zero on ∂R. (For a 0-form δs is automatically zero.) The dual zero condition says that $\operatorname{grad} s = \mathsf{g}^{-1} ds$ is tangent to ∂R. This in turn says that $\langle ds \mid N \rangle = 0$, where N is the normal vector to ∂R. (For a 0-form $*s$ is an n-form, which automatically pulls back to zero on ∂R.)

Example: Here is an example in Euclidean space with Cartesian coordinates. Take the surface as $z = 0$. Then $*ds$ pulls back to $\partial s/\partial z \, dx \, dy$. So the dual zero condititon is that $\partial s/\partial z$ is zero on the boundary. ||

Here are the boundary conditions in 3 dimensions for a 1-form α with a 2-dimensional surface ∂R.

1. Zero boundary condition: The forms α and $\delta\alpha$ pull back to zero on ∂R if and only if $\mathsf{g}^{-1}\alpha$ is orthogonal to ∂R and $\operatorname{div}(\mathsf{g}^{-1}\alpha) = 0$ on ∂R.

2. Dual zero boundary condition: The forms $*\alpha$ and $*d\alpha$ pull back to zero on ∂R if and only if $\mathsf{g}^{-1}\alpha$ is tangent to ∂R and $\operatorname{curl}(\mathsf{g}^{-1}\alpha) = \mathsf{c}^{-1} d\alpha$ is orthogonal to ∂R.

If α is a 1-form and $\mathsf{g}^{-1}\alpha$ is considered as a fluid velocity, then the dual zero boundary condition says that the fluid cannot cross the boundary, but it can move quite freely parallel to the boundary. The zero boundary condition is satisfied when the velocity is zero on the boundary, a much more severe restriction on the flow. In the terminology of [44] the dual zero boundary conditions are *absolute*, and the zero boundary conditions are *relative* (to the boundary). The book by Frankel [15] describes the topological implications of these two kinds of boundary conditions.

11.2 Exterior product and metric interior product

11.2.1 The exterior product

This section is a discussion of algebraic operations on the exterior algebra generated by a vector space with given inner product. The inner product on the vector space gives rise to a new operation on the exterior algebra that is dual to the exterior product: the metric interior product (also called metric contraction).

The starting point for this discussion is an n-dimensional vector space V with an inner product. The inner product of vectors $\mathbf{a}$ and $\mathbf{b}$ will be written $\mathbf{a} \cdot \mathbf{b}$. There are two possible interpretations to keep in mind.

- The elements of V are tangent vectors at a point. The inner product is given by a metric g that sends vectors to 1-forms. Then $\mathbf{a} \cdot \mathbf{b} = \langle \mathsf{g}\mathbf{a} \mid \mathbf{b} \rangle$.
- The elements of V are 1-forms at a point, that is, they are linear functions on the space of tangent vectors. The inner product is given by the inverse metric g^{-1} that sends 1-forms to vectors. Then $\mathbf{a} \cdot \mathbf{b} = \langle \mathbf{a} \mid \mathsf{g}^{-1}\mathbf{b} \rangle$.

First recall the basic facts about exterior algebra. Given an n-dimensional real vector space V, there is a 2^n dimensional real vector space $\Lambda(V)$ called the *exterior algebra* of V. It also has a product $\wedge$ called the *exterior product*. The vector space is made from a sequence of $n+1$ vector spaces $\Lambda^k(V)$ with $k = 0, 1, \ldots, n$. The first in the list is the space of scalars $\Lambda^0(V)$, and the next one is the space $\Lambda^1(V)$ which is V itself. The space $\Lambda^k(V)$ consists of linear combinations of products of the form $\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k$, where each $\mathbf{a}_j$ are in V. According to the interpretation of V as consisting of tangent vectors or as 1-forms, an element of $\Lambda^k(V)$ is either called a *k-vector* or a *k-form*. In the present discussion thet k-vector terminology is used. The elements of $\Lambda(V)$ are sums of elements of $\Lambda^k(V)$. Such an element may be called a *multivector*.

The rules for computation are as follows. Suppose that $\mathbf{A}, \mathbf{A}'$ are in $\Lambda(V)$ and that $\mathbf{B}, \mathbf{B}'$ are in $\Lambda(V)$ and that c is a scalar. Then

$$\begin{aligned} (\mathbf{A} + \mathbf{A}') \wedge \mathbf{B} &= \mathbf{A} \wedge \mathbf{B} + \mathbf{A}' \wedge \mathbf{B} \\ \mathbf{A} \wedge (\mathbf{B} + \mathbf{B}') &= \mathbf{A} \wedge \mathbf{B} + \mathbf{A} \wedge \mathbf{B}' \\ c(\mathbf{A} \wedge \mathbf{B}) &= c\mathbf{A} \wedge \mathbf{B} = \mathbf{A} \wedge c\mathbf{B}. \end{aligned} \tag{11.51}$$

Furthermore, if $\mathbf{a}$ and $\mathbf{b}$ are in V, then they anticommute:

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}. \tag{11.52}$$

It follows from the anticommutatation that if $\mathbf{a}$ is in V and $\mathbf{B}$ is in $\Lambda^q(V)$, then

$$\mathbf{a} \wedge \mathbf{B} = (-1)^q \mathbf{B} \wedge \mathbf{a}. \tag{11.53}$$

This can be extended to general $\mathbf{B}$ in $\Lambda(V)$. Define $\hat{\mathbf{B}}$ as the result multiplying the qth order component of $\mathbf{B}$ by $(-1)^q$. Then

$$\mathbf{a} \wedge \mathbf{B} = \hat{\mathbf{B}} \wedge \mathbf{a}. \tag{11.54}$$

The result can be extended in another way. Thus if $\mathbf{A}$ is in $\Lambda^p(V)$ and $\mathbf{B}$ is in $\Lambda^q(V)$, then

$$\mathbf{A} \wedge \mathbf{B} = (-1)^{pq} \mathbf{B} \wedge \mathbf{A}. \tag{11.55}$$

An element of $\Lambda^k(V)$ that can be written $\mathbf{A} = \mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k$ is called a *decomposable* k-vector. A general k-vector in $\Lambda^k(V)$ is a linear combination of decomposable k-vectors. The space $\Lambda^k(V)$ of k-vectors is a vector space of dimension $\binom{n}{k}$.

A 1-vector is just a vector. A decomposable 2-vector $\mathbf{a} \wedge \mathbf{b}$ may be thought of as determined by the oriented parallelogram given by $\mathbf{a}, \mathbf{b}$. However different

pairs of vectors may give rise to the same 2-vector. In fact, if $\mathbf{c} = a\mathbf{a} + b\mathbf{b}$ and $\mathbf{d} = c\mathbf{a} + d\mathbf{b}$, then

$$\mathbf{c} \wedge \mathbf{d} = (ad - bc)(\mathbf{a} \wedge \mathbf{b}), \tag{11.56}$$

and so they give the same 2-vector when $ad - bc = 1$. Similarly, a decomposable 3-vector $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ may be thought of as determined by the parallelepiped given by $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

A non-zero decomposable k-vector determines a vector subspace $W \subseteq V$ of dimension k. If $\mathbf{A}$ is in the exterior algebra, then it is a sum of decomposable multivectors. If $\mathbf{B}$ is a decomposable k-vector, then it is natural to write $\mathbf{A} \subseteq \mathbf{B}$ to mean that every term in $\mathbf{A}$ determines a subspace that is contained in the subspace generated by $\mathbf{B}$.

There is another important operation on $\Lambda(V)$. If $\mathbf{A} = \mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k$, then the corresponding reversed element is defined to be $\tilde{\mathbf{A}} = \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_1$. This operation extends to an arbitrary element of the exterior algebra.

The two operations introduced above are associated with sign changes:

- If $\mathbf{A}$ has degree k, then the signed $\hat{\mathbf{A}} = (-1)^k \mathbf{A}$.
- If $\mathbf{A}$ has degree k, then the reversed $\tilde{\mathbf{A}} = (-1)^{\binom{k}{2}} \mathbf{A}$.

There is another useful relation between such signs. Suppose that $\mathbf{A}$ has degree k and $\mathbf{B}$ has degree ℓ. Since

$$(\tilde{\mathbf{A}} \wedge \tilde{\mathbf{B}})^{\sim} = \mathbf{B} \wedge \mathbf{A}, \tag{11.57}$$

it follows that

$$(-1)^{\binom{k+\ell}{2}} (-1)^{\binom{k}{2}} (-1)^{\binom{\ell}{2}} = (-1)^{k\ell}. \tag{11.58}$$

This equation may also be derived from the combinatorial identity

$$\binom{k+\ell}{2} = \binom{k}{2} + \binom{\ell}{2} + k\ell. \tag{11.59}$$

11.2.2 The metric interior product

The next topic is about an operation that is dual to the exterior product. The exposition uses material taken from the article [12] of Dorst. (This article is about geometric algebra, but that aspect is not considered here.)

Now use the fact that the n-dimensional real vector space V has an inner product. Then the exterior algebra $\Lambda(V)$ also has an inner product. This inner product will be denoted $\mathbf{A} \bullet \mathbf{B}$ and is defined as follows. The various spaces $\Lambda^k(V)$ of fixed degree are orthogonal to each other. Within a space $\Lambda^k(V)$ the inner product is determined by requiring that

$$(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k) \bullet (\mathbf{b}_1 \wedge \cdots \wedge \mathbf{b}_k) = \det(\mathbf{a}_i \cdot \mathbf{b}_j). \tag{11.60}$$

This definition works for decomposable k-vectors, but then extends to general k-vectors. In particular

$$(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k) \bullet (\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k) = \det(\mathbf{a}_i \cdot \mathbf{a}_j), \tag{11.61}$$

which is the square of the k-dimensional area of the paralellepiped spanned by the vectors.

This leads to the the following definition. Given elements $\mathbf{A}$ and $\mathbf{B}$ of $\Lambda(V)$, there is a new element $\mathbf{A} \rfloor \mathbf{B}$ in $\Lambda(V)$. This is called the *metric interior product* or *metric contraction* or more specifically *metric left contraction*. It is characterized by the identity

$$(\mathbf{A} \rfloor \mathbf{B}) \bullet \mathbf{C} = \mathbf{B} \bullet (\tilde{\mathbf{A}} \wedge \mathbf{C}). \tag{11.62}$$

where $\tilde{\mathbf{A}}$ reverses the order of the products in $\mathbf{A}$. In other words, $\mathbf{A}\rfloor$ is adjoint to $\tilde{\mathbf{A}}\wedge$.

The reason that the metric interior product exists and is unique follows from a standard representation theorem. This uses the fact that for each choice of $\mathbf{A}$ and $\mathbf{B}$ the function $\mathbf{C} \mapsto \mathbf{B} \bullet (\tilde{\mathbf{A}} \wedge \mathbf{C})$ is linear. The theorem says that every such linear function is represented by a vector in the space, acting by means of the inner product. In this case the vector is $\mathbf{A} \rfloor \mathbf{B}$.

The following theorem [12] shows how to calculate the metric interior product. The notation $(\mathbf{A})_0$ denotes the scalar part of $\mathbf{A}$.

Theorem 11.22 *Suppose that the metric interior product is defined by the previous equation. Then*

1. $a \rfloor \mathbf{B} = a\mathbf{B}$ *for all scalars* a; $\mathbf{A} \rfloor b = b(\mathbf{A})_0$ *for all scalars* b.
2. $\mathbf{a} \rfloor \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ *for all vectors* $\mathbf{a}, \mathbf{b}$.
3. $\mathbf{a} \rfloor (\mathbf{B} \wedge \mathbf{C}) = (\mathbf{a} \rfloor \mathbf{B}) \wedge \mathbf{C} + \hat{\mathbf{B}} \wedge (\mathbf{a} \rfloor \mathbf{C})$ *for all vectors* $\mathbf{a}$.
4. $(\mathbf{A} \wedge \mathbf{B}) \rfloor \mathbf{C} = \mathbf{A} \rfloor (\mathbf{B} \rfloor \mathbf{C})$.

The first two properties in the theorem follow easily from the definition. The fourth property also follows from the general definition. We have $\mathbf{D} \bullet (\mathbf{A} \wedge \mathbf{B}) \rfloor \mathbf{C} = \tilde{\mathbf{B}} \wedge \tilde{\mathbf{A}} \wedge \mathbf{D} \bullet \mathbf{C} = \mathbf{D} \bullet \mathbf{A} \rfloor (\mathbf{B} \rfloor \mathbf{C})$.

The third property is a kind of derivation property. It can be understood as follows. Consider the map $\mathbf{B} \mapsto \mathbf{a} \wedge \mathbf{B}$ from the exterior algebral to itself. This has an adjoint map $\mathbf{C} \mapsto \mathbf{a} \rfloor \mathbf{C}$ that by definition satisfies

$$(\mathbf{a} \wedge \mathbf{B}) \bullet \mathbf{C} = \mathbf{B} \bullet (\mathbf{a} \rfloor \mathbf{C}). \tag{11.63}$$

In order to compute with this, take $\mathbf{B} = \mathbf{b}_1 \wedge \cdots \wedge \mathbf{b}_{k-1}$ and $\mathbf{C} = \mathbf{c}_1 \wedge \cdots \wedge \mathbf{c}_k$. Then

$$(\mathbf{a}\wedge\mathbf{B})\bullet\mathbf{C} = \mathbf{B}\bullet\left(\sum_{j=1}^{k}(-1)^{j-1}(\mathbf{a}\cdot\mathbf{c}_j)\mathbf{c}_1\wedge\cdots\wedge\mathbf{c}_{j-1}\wedge\mathbf{c}_{j+1}\wedge\cdot\wedge\mathbf{c}_k\right). \tag{11.64}$$

This follows by expanding the determinant by the top row. This adjoint map satisfies the condition for the metric interior product $\mathbf{a} \rfloor \mathbf{C}$. The conclusion is that

$$\mathbf{a} \rfloor \mathbf{C} = \sum_{j=1}^{k}(-1)^{j-1}(\mathbf{a} \rfloor \mathbf{c}_j)\mathbf{c}_1 \wedge \cdots \wedge \mathbf{c}_{j-1} \wedge \mathbf{c}_{j+1} \wedge \cdot \wedge \mathbf{c}_k. \tag{11.65}$$

In other words, the metric interior product is given by a product rule with alternating signs. In the special cases when $k = 0$ and $k = 1$ the result is that $\mathbf{a} \rfloor 1 = 0$ and that $\mathbf{a} \rfloor \mathbf{c} = \mathbf{a} \cdot \mathbf{c}$.

The metric interior product formula that is used most in practice is

$$\mathbf{a} \rfloor (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \rfloor \mathbf{b})\mathbf{c} - (\mathbf{a} \rfloor \mathbf{c})\mathbf{b}. \tag{11.66}$$

This version has a generalization to a more general product rule identity:

$$\mathbf{a} \rfloor (\mathbf{B} \wedge \mathbf{C}) = (\mathbf{a} \rfloor \mathbf{B}) \wedge \mathbf{C} + \hat{\mathbf{B}} \wedge (\mathbf{a} \rfloor \mathbf{C}). \tag{11.67}$$

Here $\hat{\mathbf{B}}$ is $\mathbf{B}$ with minus signs attached to terms of odd degree.

The following remarks give an intuition for the metric interior product.

- If $\mathbf{B}$ is decomposable, then $\mathbf{A} \rfloor \mathbf{B} \subseteq \mathbf{B}$.
- If $\mathbf{A}$ in $\Lambda^k(V)$ with $k \geq 1$ is decomposable, then $\mathbf{A} \rfloor \mathbf{B}$ is orthogonal to $\mathbf{A}$.

In this algebraic system, there are only a few basic operations: There are the sum $\mathbf{A} + \mathbf{B}$ and exterior product $\mathbf{A} \wedge \mathbf{B}$ of multivectors. There is also the interior product $\mathbf{A} \rfloor \mathbf{B}$ of multivectors. The inner product of multivectors is a special case, since $\mathbf{A} \bullet \mathbf{B} = (\tilde{\mathbf{A}} \rfloor \mathbf{B})_0$.

Here are simple consequences of the last remark. If the degree of $\mathbf{A}$ is strictly greater than the degree of $\mathbf{B}$, then $\mathbf{A} \rfloor \mathbf{B} = 0$. If the degree of $\mathbf{A}$ is equal to the degree of $\mathbf{B}$, then $\tilde{\mathbf{A}} \rfloor \mathbf{B} = \mathbf{A} \rfloor \tilde{\mathbf{B}} = \mathbf{A} \bullet \mathbf{B}$. In particular, if the degree of both factors is k, then $\mathbf{A} \rfloor \mathbf{B} = (-1)^{\binom{k}{2}} \mathbf{A} \bullet \mathbf{B}$.

The content of the next theorem is a useful *multple interior product identity.*

Theorem 11.23 *A multiple metric interior product acting on an vector exterior product gives*

$$\mathbf{B} \rfloor (\mathbf{a} \wedge \mathbf{C}) = \mathbf{a} \wedge (\hat{\mathbf{B}} \rfloor \mathbf{C}) - (\mathbf{a} \rfloor \hat{\mathbf{B}}) \rfloor \mathbf{C}. \tag{11.68}$$

.

In the theorem above the interior product involving $\mathbf{B}$ on the left may be thought of as analogous to taking a derivative of higher order. The special feature is that $\mathbf{a}$ on the right hand side is a linear term. This prevents the terms in $\mathbf{B}$ from acting on $\mathbf{a}$ two or more times. One way to prove this identity is to start with the general product rule identity and take the adjoint.

There are more general formulas for multiple interior products that go under the name of Wick's theorem in the physics literature. However they typically generate more than two terms.

One consequence of the multiple interior product identity is the following *augmented interior product identity.*

Theorem 11.24 *Suppose that* $\mathbf{C}$ *is a decomposable multivector. Then*

$$\mathbf{A} \subseteq \mathbf{C} \Longrightarrow \mathbf{A} \wedge (\mathbf{B} \rfloor \mathbf{C}) = (\mathbf{A} \rfloor \mathbf{B}) \rfloor \mathbf{C}. \tag{11.69}$$

Proof: It is sufficient to prove the theorem when $\mathbf{A}$ is a decomposable. If $\mathbf{a}$ is a factor of $\mathbf{A}$, then $\mathbf{a}$ is in the space spanned by $\mathbf{C}$, and so $\mathbf{a} \wedge \mathbf{C} = 0$.

The idea of the proof is to suppose that the result is true for $\mathbf{A}$ and show that it follows that it is true for $\mathbf{a} \wedge \mathbf{A}$. The computation starts by using the inductive assumption to show that

$$\mathbf{a} \wedge \mathbf{A} \wedge (\mathbf{B} \rfloor \mathbf{C}) = \mathbf{a} \wedge ((\mathbf{A} \rfloor \mathbf{B}) \rfloor \mathbf{C}) \tag{11.70}$$

By the multiple interior product identity and property 4 this is

$$(\mathbf{a} \rfloor (\mathbf{A} \rfloor \mathbf{B})) \rfloor \mathbf{C} = ((\mathbf{a} \wedge \mathbf{A}) \rfloor \mathbf{B}) \rfloor \mathbf{C}. \tag{11.71}$$

This completes the induction step. □

Remark: There is a quantum physics interpretation of these formulas. Consider an orthonormal basis $\mathbf{e}_j$ of V, with $j = 1, \ldots, n$. For every subset $S \subseteq \{1, \ldots, n\}$ there there is a vector $\mathbf{e}_S = \prod_{j \in S} \mathbf{e}_j$ in $\Lambda^k(V)$. (The product is the exterior product taken in the order of the indices.) These $\binom{n}{k}$ vectors form an orthonormal basis for $\Lambda^k(V)$. The $\mathbf{e}_S$ for S of arbitrary size form an orthonormal basis for $\Lambda(V)$. The interpretation is there are n locations in which particles may be located, and $\mathbf{e}_S$ represents a situation in which there are particles at locations in S and not in the other locations. These particles are called *Fermi particles*, because this kind of particle has the property that there cannot be two particles at the same location.

This discussion motivates a terminology in which $\Lambda^k(V)$ cis called the k-particle space. The operator that sends C to $\mathbf{a} \wedge C$ is then called a *creation operator*, since it sends $\Lambda^k(V)$ to $\Lambda^{k+1}(V)$. Similarly, the operator that sends C to $\mathbf{a} \rfloor C$ is called an *annihilation operator*, since it sends $\Lambda^k(V)$ to $\Lambda^{k-1}(V)$.

The product rule identity in the form

$$\mathbf{a} \rfloor (\mathbf{b} \wedge C) + \mathbf{b} \wedge (\mathbf{a} \rfloor C) = (\mathbf{a} \rfloor \mathbf{b})C \tag{11.72}$$

then says that creation followed by annihilation plus annihilation followed by creation is a scalar contraction. This is called the *anticommutation relation.* ||

11.2.3 Computations with exterior and interior products

There are two multiplication operations: exterior product $\wedge$ and metric interior product $\rfloor$. Here are simple examples of how they are used.

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be 1-vectors, and $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be 2-vectors. Then we have:

- $\mathbf{a} \rfloor \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ is a scalar.
- $\mathbf{A} \rfloor \mathbf{B} = -\mathbf{A} \bullet \mathbf{B}$ is a scalar.
- $\mathbf{a} \rfloor \mathbf{B}$ is a 1-vector.
- $\mathbf{a} \wedge \mathbf{b}$ is a 2-vector.
- $\mathbf{a} \wedge \mathbf{B}$ is a 3-vector.

The most interesting of these operations is $\mathbf{a} \rfloor \mathbf{B}$. Suppose that $\mathbf{B}$ is a decomposable 2-vector, so that it defines a plane. Then the operation takes the vector $\mathbf{a}$, projects it onto the plane of $\mathbf{B}$, and then outputs a vector in that plane that is orthogonal to this projection.

There are triple product identities:

- $(\mathbf{a} \wedge \mathbf{b}) \rfloor \mathbf{C} = \mathbf{a} \rfloor (\mathbf{b} \rfloor \mathbf{C})$. (triple scalar product)
- $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = \mathbf{b} \wedge (\mathbf{c} \wedge \mathbf{a}) = \mathbf{c} \wedge (\mathbf{a} \wedge \mathbf{b})$. (triple 3-vector product)
- $\mathbf{a} \rfloor (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \rfloor \mathbf{b})\mathbf{c} - (\mathbf{a} \rfloor \mathbf{c})\mathbf{b}$. (right triple vector product)
- $(\mathbf{c} \rfloor \mathbf{B}) \rfloor \mathbf{A} = (\mathbf{B} \rfloor \mathbf{A})\mathbf{c} - \mathbf{B} \rfloor (\mathbf{c} \wedge \mathbf{A})$. (left triple vector product)
- $\mathbf{a} \wedge (\mathbf{b} \rfloor \mathbf{C}) = (\mathbf{a} \rfloor \mathbf{b})\mathbf{C} - \mathbf{b} \rfloor (\mathbf{a} \wedge \mathbf{C})$. (triple 2-vector product)

The right triple vector product identity is just the derivation property of the interior product. The triple 2-vector product identity is easy to derive from this same derivation property. The left triple product requires the multiple interior product identity.

Similarly, there are quadruple product identities. Two of the most basic ones are:

- $(\mathbf{a} \wedge \mathbf{b}) \rfloor (\mathbf{c} \wedge \mathbf{d}) = (\mathbf{b} \rfloor \mathbf{c})(\mathbf{a} \rfloor \mathbf{d}) - (\mathbf{a} \rfloor \mathbf{b})(\mathbf{b} \rfloor \mathbf{d})$. (quadruple scalar product)
- $(\mathbf{a} \rfloor \mathbf{B}) \rfloor (\mathbf{c} \wedge \mathbf{d}) = ((\mathbf{a} \rfloor \mathbf{B}) \rfloor \mathbf{c})\mathbf{c} - ((\mathbf{a} \rfloor \mathbf{B}) \rfloor \mathbf{d})\mathbf{c}$. (quadruple vector product)

The first is the of the inner product of the two 2-vectors. The second is a linear combination of two vectors with coefficients that are scalar triple products.

Finally, there are formulas that apply to three arbitrary vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. This first one represents a vector as a linear combination of these three vectors. The second one represents as 2-vector as a linear combination of exterior products of these three vectors.

- $\mathbf{D} \rfloor (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = ((\mathbf{b} \wedge \mathbf{c}) \rfloor \mathbf{D})\mathbf{a} + ((\mathbf{c} \wedge \mathbf{a}) \rfloor \mathbf{D})\mathbf{b} + ((\mathbf{a} \wedge \mathbf{b}) \rfloor \mathbf{D})\mathbf{c}$.
- $\mathbf{d} \rfloor (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = (\mathbf{d} \rfloor \mathbf{a})(\mathbf{b} \wedge \mathbf{c}) + (\mathbf{d} \rfloor \mathbf{b})(\mathbf{a} \wedge \mathbf{c}) + (\mathbf{a} \rfloor \mathbf{c})(\mathbf{a} \wedge \mathbf{b})$.

These identities, along with all the previous identities of this section, work for a vector space of any finite dimension.

11.2.4 Infinitesimal rotations

The space $\Lambda^2(V)$ of 2-vectors has an interpretation as infinitesimal rotations. More concretely, if $\mathbf{A}$ is a 2-vector, then the metric interior product defines a linear transformation $\mathbf{u} \mapsto \mathbf{u} \rfloor \mathbf{A}$.

Theorem 11.25 *Let $\mathbf{A}$ be a 2-vector in $\Lambda^2(V)$. Then the linear transformation $\mathbf{u} \mapsto \mathbf{u} \rfloor \mathbf{A}$ is skew adjoint, that is, it satisfies the antisymmetry property*

$$\mathbf{u} \rfloor \mathbf{A} \cdot \mathbf{v} = -\mathbf{u} \cdot \mathbf{v} \rfloor \mathbf{A}. \tag{11.73}$$

If we think of $\mathbf{A}$ as a linear transformation acting on vectors, then the equation in the theorem may be written

$$\mathbf{A}\mathbf{u} \cdot \mathbf{v} = -\mathbf{u} \cdot \mathbf{A}\mathbf{v}. \tag{11.74}$$

If $\mathbf{A}$ is a linear transformation of V to itself, then the exponential $\exp(t\mathbf{A})$ is an invertible linear transformation of V to itself with a positive determinant. The exponential may be defined by a power series expansion. The fact that $\exp(t\mathbf{A})$ has positive determinant follows from the observation that it is the square of $\exp(\frac{1}{2}t\mathbf{A})$.

Furthermore, if $\mathbf{A}$ is skew-adjoint, then it is an orthogonal transformation, that is, it preserves the inner product. This is because

$$\frac{d}{dt}\exp(t\mathbf{A})\mathbf{u}\cdot\exp(t\mathbf{A})\mathbf{v} = \mathbf{A}\exp(t\mathbf{A})\mathbf{u}\cdot\exp(t\mathbf{A})\mathbf{v} + \exp(t\mathbf{A})\mathbf{u}\cdot\mathbf{A}\exp(t\mathbf{A})\mathbf{v} = 0. \tag{11.75}$$

It follows easily that $\exp(t\mathbf{A})\mathbf{u} \cdot \exp(t\mathbf{A})\mathbf{v}$ is constant.

A *rotation* is an orthogonal transformation with positive determinant. The fact that the $\exp(t\mathbf{A})$ are rotations together with

$$\frac{d}{dt}\exp(t\mathbf{A}) = \mathbf{A}\exp(t\mathbf{A}) \tag{11.76}$$

justify considering $\mathbf{A}$ as an *infinitesimal rotation*.

Example: Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be orthogonal unit vectors, and let $\mathbf{A} = \mathbf{e}_1 \wedge \mathbf{e}_2$. By the derivation property $\mathbf{A}\mathbf{u} = (\mathbf{u}\cdot\mathbf{e}_1)\mathbf{e}_2 - (\mathbf{u}\cdot\mathbf{e}_2)\mathbf{e}_1$. Let the orthogonal projection P on the plane be defined by $P\mathbf{u} = (\mathbf{u}\cdot\mathbf{e}_1)\mathbf{e}_1 + (\mathbf{u}\cdot\mathbf{e}_2)\mathbf{e}_2$. Let J satisfy $J\mathbf{e}_1 = \mathbf{e}_2$ and $J\mathbf{e}_2 = -\mathbf{e}_1$ and be zero on vectors orthogonal to the plane. Then $\mathbf{A} = JP$. Hence

$$\exp(\theta\mathbf{A}) = \exp(\theta JP) = \exp(\theta J)P = (\cos(\theta)I + \sin(\theta)J)P. \tag{11.77}$$

This represents the rotation explicitly as a rotation in the plane by angle θ. ||

The natural operation on skew-adjoint linear transformations is the *Lie product* $[\mathbf{A}, \mathbf{B}] = \mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A}$. This is also a skew-adjoint linear transformation. The corresponding 2-vector satisfies

$$\mathbf{u} \rfloor [\mathbf{A}, \mathbf{B}] = (\mathbf{u} \rfloor \mathbf{B}) \rfloor \mathbf{A} - (\mathbf{u} \rfloor \mathbf{A}) \rfloor \mathbf{B}. \tag{11.78}$$

This way of combining 2-vectors to get another 2-vector is not itself an interior product or contraction. However it is a closely related construction. Define the *partial contraction* of $\mathbf{b}_1 \wedge \mathbf{b}_2$ with $\mathbf{a}_1 \wedge \mathbf{a}_2$ to be

$$(\mathbf{b}_1\wedge\mathbf{b}_2) \mid (\mathbf{a}_1\wedge\mathbf{a}_2) = (\mathbf{b}_1\cdot\mathbf{a}_2)(\mathbf{b}_2\wedge\mathbf{a}_1) - (\mathbf{b}_1\cdot\mathbf{a}_1)(\mathbf{b}_2\wedge\mathbf{a}_2) + (\mathbf{b}_2\cdot\mathbf{a}_1)(\mathbf{b}_1\wedge\mathbf{a}_2) - (\mathbf{b}_2\cdot\mathbf{a}_2)(\mathbf{b}_1\wedge\mathbf{a}_1). \tag{11.79}$$

That is, the inner products in the partial contractions are computed with a sign change whenever one factor is moved past another. If $\mathbf{B} = \mathbf{b}_1 \wedge \mathbf{b}_2$ and $\mathbf{A} = \mathbf{a}_1 \wedge \mathbf{a}_2$, then the Lie product $[\mathbf{A}, \mathbf{B}]$ is given by the partial contraction

$$[A, B] = (\mathbf{b}_1 \wedge \mathbf{b}_2) \mid (\mathbf{a}_1 \wedge \mathbf{a}_2). \tag{11.80}$$

Example: Here is an example from mechanics. It is a standard situation, except that here it is formulated in a way that does not depend on the dimension of space.

Consider a rigid body that is moving with one point stationary. Let the objects in the body have position vectors $\mathbf{r}_i$ with respect to this point. The angular momentum of an object is $m_i(\mathbf{r}_i \wedge \mathbf{v}_i)$, where m_i is the mass and $\mathbf{v}_i$ is the velocity vector. The total *angular momentum* is then

$$\mathbf{L} = \sum_i m_i(\mathbf{r}_i \wedge \mathbf{v}_i). \tag{11.81}$$

This is a 2-vector.

The velocity is

$$\mathbf{v}_i = \mathbf{r}_i \rfloor \Omega. \tag{11.82}$$

Here Ω is a 2-vector representing the *angular velocity* of the rigid body. Insert this in the formula for angular momentum and use the triple 2-vector product identity. This gives

$$\mathbf{L} = \sum_i m_i(\mathbf{r}_i \wedge (\mathbf{r}_i \rfloor \Omega)) = \sum_i m_i((\mathbf{r}_i \cdot \mathbf{r}_i)\Omega - \mathbf{r}_i \rfloor (\mathbf{r}_i \wedge \Omega)). \tag{11.83}$$

This can be written

$$\mathbf{L} = \mathbf{I}\Omega \tag{11.84}$$

where $\mathbf{I}$ is a linear transformation from 2-vectors to 2-vectors called *inertia*.

The total *kinetic energy* is

$$T = \frac{1}{2}\sum_i m_i \mathbf{v}_i^2. \tag{11.85}$$

This can be written

$$T = \frac{1}{2}\sum_i m_i(\mathbf{r}_i \rfloor \Omega) \rfloor (\mathbf{r}_i \rfloor \Omega) = \frac{1}{2}\sum_i m_i((\mathbf{r}_i \rfloor \Omega) \wedge \mathbf{r}_i) \rfloor \Omega. \tag{11.86}$$

The last equality is the triple scalar product identity. This in turn is

$$T = -\frac{1}{2}\sum_i m_i(\mathbf{r}_i \wedge (\mathbf{r}_i \rfloor \Omega)) \rfloor \Omega = -\frac{1}{2}\mathbf{L} \rfloor \Omega = \frac{1}{2}\Omega \bullet \mathbf{L}. \tag{11.87}$$

Finally, this gives

$$T = \frac{1}{2}\Omega \bullet \mathbf{I}\Omega. \tag{11.88}$$

This shows how the intertia and the angular velocity determine the kinetic energy. ||

11.2.5 Metric interior product and Hodge dual

The Hodge dual is a map $* : \Lambda(V) \to \Lambda(V)$ that sends each $\Lambda^k(V)$ to $\Lambda^{n-k}(V)$. It is defined by

$$*\mathbf{B} = \tilde{\mathbf{B}} \rfloor \text{ vol}. \tag{11.89}$$

Since there are two candidates for vol (depending on the orientation of V), there are two candidates for $*$. Alternatively, one can think of $*$ as sending k-vectors to pseudo $(n-k)$-vectors, or the other way around.

Theorem 11.26 *The Hodge dual and the exterior and metric interior products are related by*

$$\mathbf{A} \wedge *\mathbf{B} = *(\mathbf{A} \rfloor \tilde{\mathbf{B}})^{\sim}. \tag{11.90}$$

Proof: The left hand side is

$$\mathbf{A} \wedge *\mathbf{B} = \mathbf{A} \wedge (\tilde{\mathbf{B}} \rfloor \text{ vol}) = (\mathbf{A} \rfloor \tilde{\mathbf{B}}) \rfloor \text{ vol} = *(\mathbf{A} \rfloor \tilde{\mathbf{B}})^{\sim}. \tag{11.91}$$

The middle equality is because vol is in $\Lambda^n(V)$. □

There are several useful special cases. One is when $\mathbf{A} = \mathbf{a}$ is a vector. If $\mathbf{B}$ is of degree k, then

$$\mathbf{a} \rfloor \mathbf{B} = (-1)^{k-1} *^{-1} \mathbf{a} \wedge *B. \tag{11.92}$$

Another special case is when $\mathbf{A}$ and $\mathbf{B}$ are both k-vectors. Then the metric interior product in the theorem is closely related to the inner product. This particular case is so important that is should be stated in its own right.

Theorem 11.27 *If* $\mathbf{A}$ *and* $\mathbf{B}$ *both have degree* k*, then*

$$\mathbf{A} \wedge *\mathbf{B} = (\mathbf{A} \bullet \mathbf{B})\text{vol}. \tag{11.93}$$

Theorem 11.28 *The Hodge dual is (up to sign) its own inverse. If* $\mathbf{B}$ *has degree* k*, then*

$$** \mathbf{B} = (-1)^{k(n-k)}\mathbf{B} \tag{11.94}$$

In particular the inverse $*^{-1}$ *is given by* $*^{-1}\mathbf{B} = (-1)^{k(n-k)} * \mathbf{B}$.

Proof: We have

$$**\mathbf{B} = (\tilde{\mathbf{B}} \rfloor \text{vol})^{\sim} \rfloor \text{vol} = (-1)^{k(n-k)}(\mathbf{B} \rfloor \tilde{\text{vol}}) \rfloor \text{vol} = (-1)^{k(n-k)}\mathbf{B} \wedge (\tilde{\text{vol}} \rfloor \text{vol}) = (-1)^{k(n-k)}\mathbf{B}. \tag{11.95}$$

The reason for the factor $(-1)^{k(n-k)}$ is the identity involving reversal of factors. □

Theorem 11.29 *The Hodge dual preserves the inner product:*

$$*\mathbf{A} \bullet *\mathbf{B} = \mathbf{A} \bullet \mathbf{B}. \tag{11.96}$$

Proof: Suppose $\mathbf{A}$ and $\mathbf{B}$ both have degree k. Then

$$(*\mathbf{A}\bullet *\mathbf{B})\text{vol} = *\mathbf{A}\wedge **\mathbf{B} = (-1)^{k(n-k)} *\mathbf{A}\wedge\mathbf{B} = \mathbf{B}\wedge *\mathbf{A} = (\mathbf{A}\bullet\mathbf{B})\text{vol}. \quad (11.97)$$

□

The Hodge dual gives a key to understanding three dimensional vector analysis as a special case of constructions with the interior and exterior product.

Theorem 11.30 *In three dimensions*

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\rfloor\mathbf{b} = *(\mathbf{a}\wedge *\mathbf{b}) \quad (11.98)$$

and

$$\mathbf{a}\times\mathbf{b} = *(\mathbf{a}\wedge\mathbf{b}) = -\mathbf{a}\rfloor *\mathbf{b}. \quad (11.99)$$

Proof: The result for the scalar product follows from the fact that in three dimension $*$ is its own inverse.

For the vector product it makes sense to define $\mathbf{a}\times\mathbf{b} = *(\mathbf{a}\wedge\mathbf{b})$. The other expression follows from

$$*(\mathbf{a}\wedge\mathbf{b}) = (\mathbf{b}\wedge\mathbf{a})\rfloor\,\text{vol} = -(\mathbf{a}\wedge\mathbf{b})\rfloor\,\text{vol} = -\mathbf{a}\rfloor(\mathbf{b}\rfloor\,\text{vol}) = \mathbf{a}\rfloor *\mathbf{b}. \quad (11.100)$$

□

11.2.6 Classical vector algebra

Classical vector algebra takes place when the dimension of the vector space is $n = 3$. This section studies this subject in its most naive version. In this version the only spaces are the space S of scalars and the space V of three-dimensional vectors. The vector space operations are the product of a scalar with a vector and the sum of two vectors. There is also a *scalar product* (dot product, inner product) of two vectors. Finally there is a *vector product* (cross product) of two vectors.

It is defined by

$$\mathbf{a}\times\mathbf{b} = *(\mathbf{a}\wedge\mathbf{b}) = -\mathbf{a}\rfloor *\mathbf{b}. \quad (11.101)$$

The two possible defining equations reflect the fact that the vector product has two (or more) possible interpretations.

The scalar product satisfies

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\cos(\theta). \quad (11.102)$$

Here $0 \le \theta \le \pi$ is the angle between the two vectors.

The vector product $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$ and to $\mathbf{b}$ and satisfies

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,\sin(\theta). \quad (11.103)$$

The length of the vector is the area of the parallelogram spanned by the two vectors. Its definition also depends on an orientation of the three dimensional vector space V. The triple $\mathbf{a}, \mathbf{b}, \mathbf{a}\times\mathbf{b}$ is required to have the given orientation.

The vector product is bilinear and distributes over vector addition. It is anticommutative:

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}. \tag{11.104}$$

it also satisfies various identities. One is the *triple scalar product* identity involving a quantity $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ that can be defined in multiple ways:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}). \tag{11.105}$$

This represents (up to sign) the volume of the parallelepiped spanned by the three vectors.

There is also the *triple vector product* identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}. \tag{11.106}$$

This identity shows that the vector product is not associative. However it satisfies the Jacobi identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0. \tag{11.107}$$

There are also quadruple product identities. The *quadruple scalar product* is

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d}). \tag{11.108}$$

This is a determinant that is proportional to the areas of two parallelograms and to the cosine of the angle between them. The *quadruple vector product* is

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a}, \mathbf{b}, \mathbf{d}]\mathbf{c} - [\mathbf{a}, \mathbf{b}, \mathbf{c}]\mathbf{d}. \tag{11.109}$$

Finally there are two resuults for representing an arbitrary vector. The first is

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}]\mathbf{d} = [\mathbf{b}, \mathbf{c}, \mathbf{d}]\mathbf{a} + [\mathbf{c}, \mathbf{a}, \mathbf{d}]\mathbf{b} + [\mathbf{a}, \mathbf{b}, \mathbf{d}]\mathbf{c}. \tag{11.110}$$

The second is

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}]\mathbf{d} = (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \times \mathbf{c}) + (\mathbf{b} \cdot \mathbf{d})(\mathbf{c} \times \mathbf{a}) + (\mathbf{c} \cdot \mathbf{d})(\mathbf{a} \times \mathbf{b}). \tag{11.111}$$

Vector algebra may also be expressed in terms of an oriented orthonormal basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$. The algebra of these quantities is given by

$\times$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	0	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$-\mathbf{k}$	0	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	0

It may seem troubling that the vector product depends on an arbitrary choice of orientation. One way to avoid the difficulty is to note that in many cases the final answer may involve an even number of vector products. In this case, the answer does not depend on the orientation. A striking example of this is the triple vector product identity. Another example is the quadruple scalar product identity.

On the other hand, the triple scalar product does depend on the orientation. However, this actually helps in interpreting the equations for representing an arbitrary vector. Divide the equations by the triple scalar product on the left; the result in each case is an equation that does not depend on the orientation.

11.2.7 Vectors and pseudovectors

Classical vector analysis in three dimensions seems more complicated when the distinction is made between vectors and pseudovectors. At the same time the geometrical significance of the operations becomes clearer, and they no longer seem quite so arbitrary.

The 3-dimensional vector space V has two orientations. A *pseudoscalar* is a function from orientations to scalars, with the property that changing the orientation multiplies the scalar by -1. A *pseudovector* is a function from orientations to vectors, with the property that changing the orientation multiplies the vector by -1. The relevant spaces of scalars and vectors and pseudovectors and poseudoscalars are denoted S, V, PV, PS. These have dimensions 1, 3, 3, 1. From this point of view there are four kinds of data. This is more complicated, but it also gives a more accurate picture of the geometrical significance of the various quantities.

The reason for these definitions is that the most natural definition of the cross product sends $V \times V \to PV$, $V \times PV \to V$, $PV \times V \to V$, and and $PV \times PV \to PV$. This is because the cross product depends on a choice of orientation. This is contrasted with the case of the scalar product, which sends $V \times V \to S$, $V \times PV$ to PS, $PV \times V \to PS$, and $PV \times PV \to S$.

This definition makes the geometrical nature of vector calculus more transparent. A vector in V may be thought of as an arrow. On the other hand, a (non-zero) pseudovector in PV is determined by a plane through the origin, a real number representing an area, and an orientation of the plane. These are two-dimensional data. It is comparatively artificial to view the pseudovector as a function from orientations of V to vectors. This can be done, of course, by taking the vector to be orthogonal to the plane, have length equal to the area, and to have direction given so that the orientation of the plane together with the vector is the orientation of V.

Various geometrical interpretations result. The most obvious one is the product $\mathbf{a} \times \mathbf{b}$ of two vectors in V, which is a pseudovector in PV. These two vectors $\mathbf{a}, \mathbf{b}$ determine a parallelogram. This in turn gives the plane determined by the parallelogram as well as the area of the parallelogram. Finally, the two vectors taken in order determine an orientation of the plane. The scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ has a similar interpretation as a pseudoscalar in PS. The three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ determine a parallelepiped. This has a volume, but there is also a sign depending on whether the orientation of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is the same as the orientation of V.

There is a very different interpretation in the case when $\mathbf{a}$ is a vector and $\mathbf{b}$ is a pseudovector. In this case $\mathbf{a} \times \mathbf{b}$ is a vector. It is obtained as follows. Think of $\mathbf{b}$ in terms of the plane to which it is orthogonal. Then $\mathbf{a}$ has an orthogonal projection on this plane. The vector $\mathbf{a} \times \mathbf{b}$ is another vector in this plane that is obtained by rotating this orthogonal projection by a quarter turn. In other words, it gives a vector in this plane orthogonal to $\mathbf{a}$. As an example, take the case when $\mathbf{b} = \mathbf{c} \times \mathbf{d}$, where $\mathbf{c}$ and $\mathbf{d}$ are vectors. Then $\mathbf{a} \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{d})\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\mathbf{d}$.

This algebraic system may be written in terms of basis elements $1, \mathbf{i}, \mathbf{j}, \mathbf{k}, \hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}, \hat{1}$. The algebra is as before, except that now there are identities like $\mathbf{i} \times \mathbf{j} = \hat{\mathbf{k}}$ and $\mathbf{i} \times \hat{\mathbf{j}} = \mathbf{k}$ and $\mathbf{j} \cdot \hat{\mathbf{j}} = \hat{1}$. In other words, they are the same identities, except that they distinguish vectors from pseudovectors. Note particularly the difference between $\mathbf{i} \cdot (\mathbf{j} \times \mathbf{k}) = \hat{1}$ and $\hat{\mathbf{i}} \cdot (\hat{\mathbf{j}} \times \hat{\mathbf{k}}) = 1$.

This algebraic system has a subsystem that consists of S and PV. The cross product sends $PV \times PV \to PV$, while the scalar product sends $PV \times PV \to S$. The basis vectors in this case are $1, \hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$.

This view of vector algebra is considerably richer than classical vector algebra. This is explained by the fact that the operations of scalar product and vector product may be reduced to operations involving the interior and exterior products. Say that $\mathbf{a}$ and $\mathbf{b}$ are vectors, and $\mathbf{A}$ and $\mathbf{B}$ are pseudovectors. The pseudovectors may be interpreted as 2-vectors. Similarly, pseudoscalars are 3-vectors. The the translations of the scalar product operations are

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \rfloor \mathbf{b}$.
- $\mathbf{a} \cdot \mathbf{B} = \mathbf{a} \wedge \mathbf{B}$.
- $\mathbf{A} \cdot \mathbf{B} = -\mathbf{A} \rfloor \mathbf{B}$.

For the vector product the correspondences are

- $\mathbf{a} \times \mathbf{b} = \mathbf{a} \wedge \mathbf{b}$.
- $\mathbf{a} \times \mathbf{B} = -\mathbf{a} \rfloor \mathbf{B}$.
- $\mathbf{A} \times \mathbf{B} = [\mathbf{A}, \mathbf{B}]$.

Operations with pseudoscalars also fit into the scheme. Let $\mathbf{E}$ be a pseudoscalar, interpreted as a 3-vector. Then

- $\mathbf{a}\, \mathbf{E} = \mathbf{a} \rfloor \mathbf{E}$.
- $\mathbf{A}\, \mathbf{E} = \mathbf{A} \rfloor \mathbf{E}$.

This equivalence exhibits the rather complicated variety of geometrical interpretations of the scalar product and the vector product in the three dimensional case. The alternate picture, in terms of exterior and metric interior products, has the advantage of giving a clear and uniform geometrical picture in any number of dimensions.

Example: This is a repetition of the rigid body example, but now in the special notation that is customary in three dimensions.

Consider a rigid body that is moving with one point stationary. Let the objects in the body have position vectors $\mathbf{r}_i$ with respect to this point. The angular momentum of an object is $m_i(\mathbf{r}_i \times \mathbf{v}_i)$, where m_i is the mass and $\mathbf{v}_i$ is the velocity vector. The total *angular momentum* is then

$$\mathbf{L} = \sum_i m_i(\mathbf{r}_i \times \mathbf{v}_i). \tag{11.112}$$

This is a pseudovector.

The velocity is

$$\mathbf{v}_i = \omega \times \mathbf{r}_i. \tag{11.113}$$

Here ω is a pseudovector representing the *angular velocity* of the rigid body. Insert this in the formula for angular momentum and use the triple (pseudo)vector product identity. This gives

$$\mathbf{L} = \sum_i m_i(\mathbf{r}_i \times (\omega \times \mathbf{r}_i)) = \sum_i m_i((\mathbf{r}_i \cdot \mathbf{r}_i)\omega - \mathbf{r}_i(\mathbf{r}_i \cdot \omega)). \tag{11.114}$$

This can be written

$$\mathbf{L} = \mathbf{I}\omega \tag{11.115}$$

where $\mathbf{I}$ is a linear transformation from pseudovectors to pseudovectors called *inertia*.

The total *kinetic energy* is

$$T = \frac{1}{2}\sum_i m_i \mathbf{v}_i^2. \tag{11.116}$$

This can be written

$$T = \frac{1}{2}\sum_i m_i(\omega \times \mathbf{r}_i) \cdot (\omega \times \mathbf{r}_i) = \frac{1}{2}\sum_i m_i(\omega \cdot (\mathbf{r}_i \times (\omega \times \mathbf{r}_i)). \tag{11.117}$$

The last equality is the triple scalar product identity. This in turn is

$$T = \frac{1}{2}\omega \cdot \mathbf{L}. \tag{11.118}$$

Finally, this gives

$$T = \frac{1}{2}\omega \cdot \mathbf{I}\omega. \tag{11.119}$$

This shows how the inertia and the angular velocity determine the kinetic energy.

Almost all the steps in the derivation are the same as in the version with exterior and interior product. The details are different. Consider, for example, the pseudovector term $\mathbf{r}_i(\mathbf{r}_i \cdot \omega)$, which is a vector $\mathbf{r}_i$ times a pseudoscalar $\mathbf{r}_i \cdot \omega$. In the other version the corresponding term is a 2-vector $\mathbf{r}_i \rfloor (\mathbf{r}_i \wedge \Omega)$, which is the interior product of a vector $\mathbf{r}_i$ with a 3-vector $\mathbf{r}_i \wedge \Omega$. The translation process requires care, but is not difficult. ||

11.3 The codifferential as a divergence

11.3.1 The covariant derivative

When there is a metric tensor, there is a corresponding notion of *covariant derivative*. (This is related to the general concept of *connection*.) The key is to

find a formula for differentiating a coordinate basis vector or a coordinate basis form. Further calculations then use the product rule for differentiation.

For a vector field the fundamental formula is

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}. \tag{11.120}$$

The coefficient is called a *Christoffel coefficient*. There are two requirements that determine this operation:

- The symmetry property $\Gamma_{ij} = \Gamma_{ji}$.
- The covariant derivative of the metric tensor is zero.

This will be explained in detail below.

The symmetry property may also be written

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}. \tag{11.121}$$

In such an expression the partial differential operator on the left in each term is a covariant derivative, when the one on the right in the same term merely represents a coordinate basis vector.

Remark: One could imagine a situation where the there are two covariant derivatives following each other, perhaps acting on a third coordinate basis vector. For instance, one could have

$$\begin{aligned}
\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_\ell} &= \sum_k \frac{\partial \Gamma_{j\ell}^k}{\partial x_i}\frac{\partial}{\partial x_k} + \sum_k \sum_m \Gamma_{j\ell}^k \Gamma_{ik}^m \frac{\partial}{\partial x_m} \\
\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_\ell} &= \sum_k \frac{\partial \Gamma_{i\ell}^k}{\partial x_j}\frac{\partial}{\partial x_k} + \sum_k \sum_m \Gamma_{i\ell}^k \Gamma_{jk}^m \frac{\partial}{\partial x_m}.
\end{aligned} \tag{11.122}$$

It appears that the order in which the covariant derivatives are taken may be significant. This is related to the presence of curvature. ||

For differential forms the fundamental formula is

$$\frac{\partial}{\partial x_i} dx_j = -\sum_k \Gamma_{ik}^j \, dx_k. \tag{11.123}$$

The reason for this relation is that it is consistent with the product rule:

$$0 = \frac{\partial}{\partial x_i}\left\langle dx_j \mid \frac{\partial}{\partial x_k} \right\rangle = -\Gamma_{ik}^j + \Gamma_{ik}^j. \tag{11.124}$$

The minus sign comes from the fact that the change in dx_j must compensate for the change in $\partial/\partial x_k$.

The relation between the vector field and form pictures may also be written

$$\left\langle \frac{\partial}{\partial x_i}\, dx_k \mid \frac{\partial}{\partial x_j} \right\rangle = -\left\langle dx_k \mid \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} \right\rangle. \tag{11.125}$$

The symmetry property for the action on differential forms is different from the symmetry property for the action on vector fields. The action on a fixed coordinate basis form dx_j satisfies

$$\frac{\partial}{\partial x_k} \lrcorner \frac{\partial}{\partial x_i} dx_j = \frac{\partial}{\partial x_i} \lrcorner \frac{\partial}{\partial x_k} dx_j. \tag{11.126}$$

Here the partial derivative on the left in each term is a basis vector, while the other partial derivative in the same term is the covariant derivative acting on dx_j.

The covariant derivative of a differential form is computed by using the ordinary product rule. For instance,

$$\frac{\partial}{\partial x_i} p\, dx_j \wedge dx_k = \frac{\partial p}{\partial x_i}\, dx_j \wedge dx_k - \sum_r \Gamma^j_{ir}\, dx_r \wedge dx_k - \sum_s \Gamma^k_{is}\, dx_j \wedge dx_s. \tag{11.127}$$

The covariant derivative of a differential form is another differential form.

Proposition 11.31 *The condition that the covariant derivative of* g *is zero is expressed in coordinates by*

$$\frac{\partial}{\partial x_k} g_{ij} - \sum_r g_{jr}\Gamma^r_{ki} - \sum_r g_{ir}\Gamma^r_{kj} = 0. \tag{11.128}$$

Proof: Compute

$$\frac{\partial}{\partial x_k} \sum_i \sum_j dx_i\, g_{ij}\, dx_j = 0. \tag{11.129}$$

This is

$$-\sum_i \sum_j \sum_r \Gamma^i_{kr}\, dx_r\, g_{ij}\, dx_j + \sum_i \sum_j dx_i\, \frac{\partial g_{ij}}{\partial x_k}\, dx_j - \sum_i \sum_j \sum_r \Gamma^j_{kr}\, dx_i\, g_{ij}\, dx_r = 0. \tag{11.130}$$

Change the summation indices to write this in the form

$$-\sum_i \sum_j \sum_r \Gamma^r_{ki}\, dx_i\, g_{rj}\, dx_j + \sum_i \sum_j dx_i\, \frac{\partial g_{ij}}{\partial x_k}\, dx_j - \sum_i \sum_j \sum_r \Gamma^r_{kj}\, dx_i\, g_{ir}\, dx_j = 0. \tag{11.131}$$

Thus

$$\frac{\partial}{\partial x_k} g_{ij} - \sum_r g_{jr}\Gamma^r_{ki} - \sum_r g_{ir}\Gamma^r_{kj} = 0. \tag{11.132}$$

□

Theorem 11.32 (Levi-Civita connection) *Given a metric* g*, there is a unique covariant derivative operation defined with Christoffel symbols satisfying the symmetry condition* $\Gamma^k_{ij} = \Gamma^k_{ji}$ *and such that*

$$\frac{\partial}{\partial x_k} \mathsf{g} = 0. \tag{11.133}$$

Proof: The proof of this fundamental theorem of Riemannian geometry is given in standard texts [25]. One way to approach it is to define

$$\Gamma_{ikj} = \sum_r g_{ir}\Gamma^r_{kj}. \tag{11.134}$$

This quantity is symmetric in the last two indices. Using these new quantities the equation may be written

$$\frac{\partial}{\partial x_k} g_{ij} = \Gamma_{jki} + \Gamma_{ikj}. \tag{11.135}$$

Given the metric coefficients $g_{ij} = g_{ji}$, this is a system of equations that can be solved by the following trick. Write the equation three times as

$$\begin{aligned} -\frac{\partial g_{ij}}{\partial x_k} &= -\Gamma_{ijk} - \Gamma_{jik} \\ \frac{\partial g_{jk}}{\partial x_i} &= \Gamma_{jki} + \Gamma_{kji} \\ \frac{\partial g_{ki}}{\partial x_j} &= \Gamma_{kij} + \Gamma_{ikj} \end{aligned} \tag{11.136}$$

Add the three equations to get

$$2\Gamma_{kij} = \frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ki}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}. \tag{11.137}$$

Then

$$2\Gamma^m_{ij} = \sum_k g^{mk}\left(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ki}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right). \tag{11.138}$$

□

The final expression at the end of the above proof shows how the Christoffal symbols depend on the metric tensor coeffficients. The formula involves the inverse metric coefficients applied to the derivatives of the metric coefficients. One interesting consequence is that if the metric is multiplied by a constant $c > 0$ the covariant derivatives remain the same.

Theorem 11.33 *Let* $\mathsf{g} = \sum_{ij} g_{ij}\, dx_i\, dx_j$ *be a metric tensor. Let* g *be the determinant of the matrix* g_{ij}*, and let* $\mathrm{vol} = \sqrt{g}\, dx_1 \cdots dx_n$ *be the volume element. Then*

$$\frac{\partial}{\partial x_i}\mathrm{vol} = 0. \tag{11.139}$$

Proof: The quantity of interest is

$$\frac{\partial}{\partial x_i}\mathrm{vol} = \frac{\partial\sqrt{g}}{\partial x_i}\, dx_1 \cdots dx_n - \sqrt{g}\sum_k \Gamma^k_{ik}\, dx_1 \cdots dx_n. \tag{11.140}$$

In order to show this is zero, use Jacobi's formula for the derivative of a determinant. Thus

$$\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x_i} = \frac{1}{2}\frac{1}{g}\frac{\partial g}{\partial x_i} = \frac{1}{2}\sum_{jk} g^{kj}\frac{\partial g_{jk}}{\partial x_i}. \tag{11.141}$$

However

$$\frac{\partial g_{jk}}{\partial x_i} = \Gamma_{jki} + \Gamma_{kij}. \tag{11.142}$$

Inserting this gives

$$\frac{1}{2}\sum_k\sum_j g^{kj}\Gamma_{jki} + \frac{1}{2}\sum_j\sum_k g^{kj}\Gamma_{kij} = \frac{1}{2}\sum_k \Gamma^k_{ki} + \frac{1}{2}\sum_j \Gamma^j_{ij} = \sum_k \Gamma^k_{ki}. \tag{11.143}$$

This completes the calculation. The terms in the quantity of interest cancel. □

Example: The case of two dimensional Euclidean space with polar coordinates gives a simple illustration. The covariant derivatives of the coordinate basis forms are

$$\begin{aligned}
\frac{\partial}{\partial r}\,dr &= 0\\
\frac{\partial}{\partial r}\,d\theta &= -\frac{1}{r}\,d\theta\\
\frac{\partial}{\partial \theta}\,dr &= r\,d\theta\\
\frac{\partial}{\partial \theta}\,d\theta &= -\frac{1}{r}\,dr.
\end{aligned} \tag{11.144}$$

The same coefficients occur in the second and fourth equations; this is the symmetry property for coordinate bases. These equations are consistent with

$$\begin{aligned}
\frac{\partial}{\partial r}(dr^2 + r^2\,d\theta^2) &= 2r\,d\theta^2 + r^2\frac{\partial}{\partial r}\,d\theta\,d\theta + r^2\,d\theta\frac{\partial}{\partial r}\,d\theta = 0.\\
\frac{\partial}{\partial \theta}(dr^2 + r^2\,d\theta^2) &= \frac{\partial}{\partial \theta}\,dr\,dr + dr\,\frac{\partial}{\partial \theta}\,dr + r^2\frac{\partial}{\partial \theta}\,d\theta\,d\theta + r^2\,d\theta\frac{\partial}{\partial \theta}\,d\theta = 0.
\end{aligned} \tag{11.145}$$

Thus the covariant derivatives of a 1-form are

$$\begin{aligned}
\frac{\partial}{\partial r}(p\,dr + q\,d\theta) &= \frac{\partial p}{\partial r}\,dr + \frac{\partial q}{\partial r}\,d\theta - q\frac{1}{r}\,d\theta\\
\frac{\partial}{\partial \theta}(p\,dr + q\,d\theta) &= \frac{\partial p}{\partial \theta}\,dr + \frac{\partial q}{\partial \theta}\,d\theta + pr\,d\theta - q\frac{1}{r}\,dr.
\end{aligned} \tag{11.146}$$

The covariant derivative is a complicated operation, because it keeps track of how the coordinate basis forms are changing from point to point. ||

Remark: In an earlier chapter there was a formula for the covariant derivative of a vector field given in orthogonal coordinates. The formulas of this section lead to a proof of this formula.

In orthogonal coordinates the metric tensor has coefficients $g_{ij} = h_i^2 \delta_{ij}$. In this special case

$$\Gamma_{ij}^k = \frac{1}{h_k^2} \left(h_j \frac{\partial h_j}{\partial u_i} \delta_{jk} + h_k \frac{\partial h_k}{\partial u_j} \delta_{ki} - h_j \frac{\partial h_j}{\partial u_k} \delta_{ij} \right). \tag{11.147}$$

The coefficients for differentiating the normalized basis vectors are determined by

$$\frac{\partial}{\partial u_i} \left(\frac{1}{h_j} \frac{\partial}{\partial u_j} \right) = \sum_k C_{ij}^k \left(\frac{1}{h_k} \frac{\partial}{\partial u_k} \right). \tag{11.148}$$

This leads to the formula

$$C_{ij}^k = -\frac{1}{h_j} \frac{\partial h_j}{\partial u_i} \delta_{jk} + \frac{h_k}{h_j} \Gamma_{ij}^k. \tag{11.149}$$

The first term in this expression cancels with the contribution of the first term in Γ_{ij}^k. What remains comes from the second and third terms in Γ_{ij}^k:

$$C_{ij}^k = \frac{1}{h_j} \frac{\partial h_k}{\partial u_j} \delta_{ki} - \frac{1}{h_k} \frac{\partial h_j}{\partial u_k} \delta_{ij}. \tag{11.150}$$

This is antisymmetric under interchange of j and k.

The coefficients for differentiating the normalized basis forms are:

$$\frac{\partial}{\partial u_i} \left(h_j \, du_j \right) = -\sum_k C_{ik}^j \left(h_k \, du_k \right) = \sum_k C_{ij}^k \left(h_k \, du_k \right). \tag{11.151}$$

||

Example: The result for polar coordinates with normalized bases is

$$\begin{aligned} \frac{\partial}{\partial r} \, dr &= 0 \\ \frac{\partial}{\partial r} (r \, d\theta) &= 0 \\ \frac{\partial}{\partial \theta} \, dr &= +r \, d\theta \\ \frac{\partial}{\partial \theta} (r \, d\theta) &= -\, dr. \end{aligned} \tag{11.152}$$

For normalized bases there is antisymmetry in the coefficients; in this case the coefficients are just ± 1. ||

11.3.2 Covariant derivative and codifferential

If α and β are covectors, then it was shown in a previous section that there is a fundamental anticommutation relation

$$\alpha \rfloor (\beta \wedge \gamma) + \beta \wedge (\alpha \rfloor \gamma) = (\alpha \cdot \beta) \gamma. \tag{11.153}$$

The interior product in this formula is the metric interior product. In particular, on the right side $\alpha \cdot \beta = \alpha \rfloor \beta$.

There is another important formula that has a similar structure. This is the formula that gives the Laplacian of a differential form:

$$\delta d\omega + d\delta\omega = \Delta\omega. \tag{11.154}$$

Here the differential d and codifferential δ act on differential forms.

The main purpose of this section is to show that these formulas are related. In fact, the differential d may be written in terms of the exterior product $\wedge$, and the codifferential δ may be written in terms of the metric interior product $\rfloor$. These formulas that exhibit this relationship involve the covariant derivative of differential forms.

Theorem 11.34 *The differential and codifferential are related to the exterior and metric interior products by*

$$\begin{aligned} d\omega &= \sum_i dx_i \wedge \frac{\partial}{\partial x_i}\omega \\ \delta\omega &= \sum_i dx_i \rfloor \frac{\partial}{\partial x_i}\omega. \end{aligned} \tag{11.155}$$

In these formulas the derivatives on the right are covariant derivatives.

Warning: The sign convention for the codifferential δ used here is the negative of the sign convention most common in advanced differential geometry.

The covariant derivative is defined relative to a metric. However, the $d\omega$ in the theorem must be independent of the metric. This can be see directly by noting that an expression like

$$\sum_i dx_i \wedge \sum_r \Gamma^j_{ir}\, dx_r = \sum_i \sum_r \Gamma^j_{ir}\, dx_i \wedge dx_j = 0. \tag{11.156}$$

This is because the Christoffel symbol is symmetric and the exterior product is antisymmetric.

The codifferential $\delta\omega$ very much depends on the metric, and it is a much more complicated object. The exception is Cartesian coordinates on Euclidean space, for which the metric is constant, the Christoffel symbols are all zero, and the covariant derivative reduces to the usual partial derivative acting on the coefficients. The general proof that the expression for δ in the theorem coincides with the expression given by the Hodge dual will be given in a later section as Theorem 11.43.

Example: For the case of polar coordinates we have $dr \rfloor dr = 1$ and $d\theta \rfloor d\theta = \frac{1}{r^2}$, and the cross terms are zero. So the codifferential of a 1-form is given by

$$\delta(p\,dr + q\,d\theta) = dr \rfloor \frac{\partial}{\partial r}(p\,dr + q\,d\theta) + d\theta \rfloor \frac{\partial}{\partial \theta}(p\,dr + q\,d\theta). \tag{11.157}$$

This works out to be

$$\delta(p\,dr + q\,d\theta) = \frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial q}{\partial \theta} + p\frac{1}{r} = \frac{1}{r}\frac{\partial}{\partial r}(rp) + \frac{1}{r^2}\frac{\partial q}{\partial \theta}. \tag{11.158}$$

The only Christoffel symbol that survives is the one that gives the $\frac{p}{r}$ term. ||
Example: It is instructive to compute the codifferential of a 1-form in the general situation. If the form is $\omega = \sum_j p_j\,dx_j$, then the covariant derivative is

$$\frac{\partial}{\partial x_i}\omega = \sum_j \frac{\partial p_j}{\partial x_i}\,dx_j - \sum_j p_j \sum_k \Gamma^j_{ik}\,dx_k. \tag{11.159}$$

Thus

$$\delta\omega = \sum_i dx_i \rfloor \frac{\partial}{\partial x_i}\omega = \sum_i \sum_j \left(g^{ij}\frac{\partial p_j}{\partial x_i} - p_j \sum_k g^{ik}\Gamma^j_{ik} \right). \tag{11.160}$$

This can be simplified using identities from tensor calculus. The equation that says that the covariant derivative of the inverse metric tensor is zero takes the form

$$\frac{\partial g^{kj}}{\partial x_i} + \sum_m g^{km}\Gamma^j_{mi} + \sum_m g^{jm}\Gamma^k_{mi} = 0. \tag{11.161}$$

This specializes to

$$\sum_i \frac{\partial g^{ij}}{\partial x_i} + \sum_i \sum_m g^{im}\Gamma^j_{mi} + \sum_i \sum_m g^{jm}\Gamma^i_{mi} = 0. \tag{11.162}$$

The identity that says that the covariant derivative of the volume form is zero is

$$\frac{1}{\sqrt{g}}\frac{\partial \sqrt{g}}{\partial x_i} - \sum_m \Gamma^m_{mi} = 0. \tag{11.163}$$

Multiply this by g^{ij} and sum over i. Add the equations. The last terms cancel to give

$$\frac{1}{\sqrt{g}}\sum_i \frac{\partial}{\partial x_i}\left(\sqrt{g}g^{ij}\right) + \sum_i \sum_k g^{ik}\Gamma^j_{ki} = 0. \tag{11.164}$$

This identity may be thought of as saying that the inverse metric tensor has divergence zero. Inserting this gives

$$\delta\omega = \sum_i \sum_j \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_i}\left(\sqrt{g}g^{ij}p_j\right). \tag{11.165}$$

The Christoffel symbols are gone, or at least well hidden. ||

The rest of this section is devoted to the calculation of the codifferential of an arbitrary p-form. The final result is stated as a theorem. Its statement is not intuitive, but the next section will go at least partway to explaining where it comes from.

An arbitrary p-form may be written in the form as a sum of terms $s\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. The scalar coefficient s will depend vary from term to term. It is convenient for notation reasons to renumber the coordinates and examine a term $s\,dx_1 \wedge \cdots \wedge dx_p$. Sometimes this will be written $\omega = s\,d\mathbf{x}_I$, where I is the index set.

With these conventions it is not too difficult to compute the codifferential $\delta\omega$. it is the sum of three terms

$$\delta\omega = A + B + C. \tag{11.166}$$

The first term is

$$A = \sum_i dx_i \rfloor \frac{\partial s}{\partial x_i} d\mathbf{x}_I = \sum_i \sum_\ell (-1)^{\ell-1} \frac{\partial s}{\partial x_i} g^{i\ell} d\mathbf{x}_{I\setminus\ell}. \tag{11.167}$$

The second term and third term come from

$$B + C = -\sum_i \sum_\ell \sum_k (-1)^{\ell-1} s\Gamma^\ell_{ik} dx_i \rfloor (dx_k \wedge d\mathbf{x}_{I\setminus\ell}). \tag{11.168}$$

Using again the fact that the metric interior product is a derivation leads to

$$B = -\sum_i \sum_\ell \sum_k (-1)^{\ell-1} s\Gamma^\ell_{ik} g^{ik} d\mathbf{x}_{I\setminus\ell} = \sum_\ell \sum_j (-1)^{\ell-1} s \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g} g^{j\ell})}{\partial x_j}\, d\mathbf{x}_{I\setminus\ell}. \tag{11.169}$$

These can be put together to get

$$A + B = \sum_i \sum_\ell (-1)^{\ell-1} \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g} g^{i\ell} s)}{\partial x_i}\, d\mathbf{x}_{I\setminus\ell}. \tag{11.170}$$

The remaining term is

$$\begin{aligned} C &= \sum_i \sum_\ell \sum_k (-1)^{\ell-1} s\Gamma^\ell_{ik}\, dx_k \wedge (dx_i \rfloor d\mathbf{x}_{I\setminus\ell}) \\ &= \sum_i \sum_\ell \sum_k \sum_{m<\ell} (-1)^{\ell-1}(-1)^{m-1} s\Gamma^\ell_{ik} g^{im}\, dx_k \wedge d\mathbf{x}_{I\setminus\ell,m} \\ &+ \sum_i \sum_\ell \sum_k \sum_{m>\ell} (-1)^{\ell-1}(-1)^{m-2} s\Gamma^\ell_{ik} g^{im}\, dx_k \wedge d\mathbf{x}_{I\setminus\ell,m}. \end{aligned} \tag{11.171}$$

The two terms may be combined to give

$$C = \sum_k \sum_{m<\ell} \sum_i (-1)^{\ell-m} s(\Gamma^\ell_{ik} g^{im} - \Gamma^m_{ik} g^{i\ell})\, dx_k \wedge d\mathbf{x}_{I\setminus\ell,m} \tag{11.172}$$

This difference of Christoffel coefficients looks formidable. However there is a simplification. Fix k, ℓ, m and consider the sum over i.

Lemma 11.35 *If g_{rs} is the matrix of the metric tensor, and g^{ij} is the matrix of the inverse metric tensor, then for each i, l, m the coefficient*

$$C_k^{\ell m} = \sum_i \left(\Gamma_{ik}^{\ell} g^{im} - \Gamma_{ik}^{m} g^{i\ell}\right) = \sum_p \sum_q g_{kq} \left(g^{mp} \frac{\partial g^{\ell q}}{\partial x_p} - g^{\ell p} \frac{\partial g^{mq}}{\partial x_p}\right). \quad (11.173)$$

Proof: Start with the equation that says that the covariant derivative of the inverse metric tensor is zero:

$$0 = \frac{\partial g^{\ell q}}{\partial x_p} + \sum_r g^{rq} \Gamma_{rp}^{\ell} + \sum_r g^{r\ell} \Gamma_{rp}^{q}. \quad (11.174)$$

Multiply by $g_{kq} g^{mp}$ and sum over p and q. This gives

$$0 = \sum_p \sum_q g_{kq} g^{mp} \frac{\partial g^{\ell q}}{\partial x_p} + \sum_p g^{mp} \Gamma_{kp}^{\ell} + \sum_p \sum_q \sum_r g_{kq} g^{mp} g^{r\ell} \Gamma_{rp}^{q} \quad (11.175)$$

The index k is a parameter, but the interesting feature belongs to the other indices ℓ, m. The last term remains the same when ℓ, m are interchanged. So we can subtract this equation from the equation where ℓ, m are interchanged, and the result is the lemma. □

This proposition shows that the final C term has a direct expression involving the inverse metric tensor and its partial derivatives.

Theorem 11.36 *If $\omega = s\, dx_1 \wedge \cdots \wedge dx_p$, then*

$$\delta\omega = \sum_i \sum_\ell (-1)^{\ell-1} \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g} g^{i\ell} s)}{\partial x_i}\, d\mathbf{x}_{I\setminus\ell} + \sum_k \sum_{m<\ell} (-1)^{\ell-m} s C_k^{\ell m}\, dx_k \wedge d\mathbf{x}_{I\setminus\ell,m}, \quad (11.176)$$

with the coefficient $C_k^{\ell m}$ of the lemma.

This looks at first rather ugly, because of the final term. However the final terms are expressed in terms of the metric tensor coefficients and their derivatives. In the next section we shall see that there is a simple reason for this.

11.3.3 Divergence and codifferential

The expression for the codifferential of a k-form in terms of the covariant derivative is rather complicated. This section shows that the codifferential has a relatively simple formula. The idea is to first derive a formula for the divergence of a k-vector field.

A k-vector field is a sum

$$\mathbf{X} = \sum_{i_1,\ldots,i_k} s_{i_1,\ldots,i_k} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_k}}. \quad (11.177)$$

The coefficients $s_{i_1,\ldots,a_k}$ are often taken to be antisymmetric in the indices, but this is awkward in some computations. This antisymmetryproperty is not assumed in the following. Its divergence is defined to be

$$\mathbf{div}\,\mathbf{X} = \sum_i dx_i \lrcorner \frac{\partial}{\partial x_i}\mathbf{X}. \tag{11.178}$$

Here $\frac{\partial}{\partial x_i}$ is the covariant derivative. It acts via the rule

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} = \sum_k \Gamma^k_{ij}\frac{\partial}{\partial x_k} \tag{11.179}$$

In this equation $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial x_k}$ are coordinate basis vectors. Furthermore, the dx_i acts on basis vectors $\partial/\partial x_j$ by

$$dx_i \lrcorner \frac{\partial}{\partial x_j} = \delta_{ij}. \tag{11.180}$$

Example: Suppose

$$\mathbf{X} = s\frac{\partial}{\partial x_j}\wedge\frac{\partial}{\partial x_k}. \tag{11.181}$$

Then

$$\frac{\partial}{\partial x_i}\mathbf{X} = \frac{\partial s}{\partial x_i}\frac{\partial}{\partial x_j}\wedge\frac{\partial}{\partial x_k} + s\sum_p \Gamma^p_{ij}\frac{\partial}{\partial x_p}\wedge\frac{\partial}{\partial x_k} + s\sum_q \Gamma^q_{ik}\frac{\partial}{\partial x_j}\wedge\frac{\partial}{\partial x_q}. \tag{11.182}$$

There are three terms in this equation. So

$$\mathbf{div}\,\mathbf{X} = \sum_i dx_i \lrcorner \frac{\partial}{\partial x_i}\mathbf{X} \tag{11.183}$$

has six terms

$$\begin{aligned}\mathbf{div}\,\mathbf{X} = &\frac{\partial s}{\partial x_j}\frac{\partial}{\partial x_k} - \frac{\partial s}{\partial x_k}\frac{\partial}{\partial x_j}\\ &+ s\sum_p \Gamma^p_{pj}\frac{\partial}{\partial x_k} - s\sum_p \Gamma^p_{kj}\frac{\partial}{\partial x_p}\\ &+ s\sum_q \Gamma^q_{jk}\frac{\partial}{\partial x_q} - s\sum_q \Gamma^q_{qk}\frac{\partial}{\partial x_j}.\end{aligned} \tag{11.184}$$

Since the Christoffel symbols are symmetric in the lower indices, the fourth and fifth terms cancel. Also, the third and sixth terms may be simplified using identities like

$$\sum_p \Gamma^p_{pj} = \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x_j}. \tag{11.185}$$

So

$$\mathbf{div}\,\mathbf{X} = \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}s)}{\partial x_j}\frac{\partial}{\partial x_k} - \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}s)}{\partial x_k}\frac{\partial}{\partial x_j}. \tag{11.186}$$

Each term has a factor that resembles the divergence of a vector field. The signs alternate. ||

The following theorem asserts that the pattern in the example is typical. This beautiful formula shows that it is possible to completely hide the Christoffel symbols.

Theorem 11.37 *The divergence of a k-vector field $\mathbf{X}$ is a $(k-1)$-vector field $\mathbf{div}\,\mathbf{X}$. All that is needed for its definition is the volume $\sqrt{g}dx_1\wedge\cdots\wedge dx_n$; there is no other dependence on the metric. Suppose one term is*

$$\mathbf{X} = s\frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_k}}. \tag{11.187}$$

Then the corresponding divergence is

$$\mathbf{div}\,\mathbf{X} = \sum_{\ell=1}^{k}(-1)^{\ell-1}\frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}s)}{\partial x_{i_\ell}}\frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{\ell-1}}} \wedge \frac{\partial}{\partial x_{i_{\ell+1}}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_k}}. \tag{11.188}$$

Proof: An arbitrary p-vector field may be written as a sum of terms

$$\mathbf{X} = s\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_p} = s\frac{\partial}{\partial \mathbf{x}_I}. \tag{11.189}$$

For convenience the coordinate system has been chosen so that the variables on $\mathbf{X}$ are the first p variables.

With these conventions it is not too difficult to compute the divergence $\mathbf{div}\,\mathbf{X}$. It is the sum of three terms

$$\mathbf{div}\,\mathbf{X} = A + B + C. \tag{11.190}$$

The first term is

$$A = \sum_i dx_i \lrcorner \frac{\partial s}{\partial x_i}\frac{\partial}{\partial \mathbf{x}} = \sum_i (-1)^{i-1}\frac{\partial s}{\partial x_i}\frac{\partial}{\partial \mathbf{x}_{I\setminus i}}. \tag{11.191}$$

The second term and third term come from

$$B + C = \sum_i\sum_k\sum_\ell (-1)^{\ell-1}s\Gamma^k_{i\ell}dx_i \lrcorner (\frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial \mathbf{x}_{I\setminus\ell}}). \tag{11.192}$$

Using again the fact that the interior product is a derivation leads to

$$B = \sum_\ell\sum_k (-1)^{\ell-1}s\Gamma^k_{k\ell}\frac{\partial}{\partial \mathbf{x}_{I\setminus\ell}} = \sum_\ell s\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x_\ell}\frac{\partial}{\partial \mathbf{x}_{I\setminus\ell}}. \tag{11.193}$$

These can be put together to get

$$A + B = \sum_i (-1)^{i-1} \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g}s)}{\partial x_i} \frac{\partial}{\partial \mathbf{x}_{I\setminus i}}. \tag{11.194}$$

The remaining term is

$$\begin{aligned} C &= \sum_i \sum_\ell \sum_k (-1)^\ell s\Gamma^k_{i\ell} \frac{\partial}{\partial x_k} \wedge \left(dx_i \lrcorner \frac{\partial}{\partial \mathbf{x}_{I\setminus \ell}} \right) \\ &= \sum_{i<\ell} \sum_k (-1)^\ell (-1)^{i-1} s\Gamma^k_{i\ell} \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial \mathbf{x}_{I\setminus \ell, i}} \\ &+ \sum_{i>\ell} \sum_k (-1)^\ell (-1)^{i-2} s\Gamma^k_{i\ell} \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial \mathbf{x}_{I\setminus \ell, i}}. \end{aligned} \tag{11.195}$$

Since $\Gamma^k_{i\ell} = \Gamma^k_{\ell i}$, the two terms combine to give $C = 0$.

The conclusion is that if $\mathbf{X} = s\frac{\partial}{\partial \mathbf{x}}$, then

$$\mathbf{div}\, \mathbf{X} = \sum_i (-1)^{i-1} \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g}s)}{\partial x_i} \frac{\partial}{\partial \mathbf{x}_{I\setminus i}}. \tag{11.196}$$

□

It is possible to map k-vector fields to k-forms fields by using the metric g. This map will be denoted g^k and is defined by

$$\mathsf{g}^k(\omega_1 \wedge \cdots \wedge \omega_k) = (\mathsf{g}\omega_1 \wedge \cdots \wedge \mathsf{g}\omega_k). \tag{11.197}$$

The inverse map from k-forms to k-vector fields is correspondingly denoted g^{-k}. The ultimate expression for the codifferential on k-forms is given by

$$\delta\omega = \mathsf{g}^{k-1}\, \mathbf{div}\, \mathsf{g}^{-k}\omega. \tag{11.198}$$

This will be proved as Theorem 11.45 in a later section.

Example: Here is an example in three-dimensional space with orthogonal oordinates u, v, w. Take the differential 2-form

$$\omega = ph_v\, dv \wedge h_w dw. \tag{11.199}$$

The volume factor is $\sqrt{g} = h_u h_v h_w$. The 2-vector field $\mathsf{g}^{-2}\omega$ has divergence

$$\mathbf{div}\left(ph_v^{-1}h_w^{-1} \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial w} \right) = \frac{1}{h_u h_v h_w} \frac{\partial(h_u p)}{\partial v} \frac{\partial}{\partial w} - \frac{1}{h_u h_v h_w} \frac{\partial(h_u p)}{\partial w} \frac{\partial}{\partial v} \tag{11.200}$$

So

$$\delta\omega = \mathsf{g}^1\, \mathbf{div}\, \mathsf{g}^{-2}\omega = h_u^{-1}h_v^{-1} \frac{\partial(h_u p)}{\partial v} h_w\, dw - h_u^{-1}h_w^{-1} \frac{\partial(h_u p)}{\partial w} h_v\, dv. \tag{11.201}$$

This agrees with a previous computation. ||

The fundamental operations of this calculus are d and δ. While d is very simple, δ is relatively complicated, with the exception of the case of Cartesian coordinates in Euclidean space. However it is as simple as it can possibly be. The corresponding operator div has a coordinate expression that only involves the metric via the factor $\sqrt{g}$ that occurs in the volume form. And δ is obtained from div via a completely straightforward use of the metric.

11.4 The covariant differential

11.4.1 The covariant differential on differential forms

From now on, write $\mathcal{A}^p$ for the space of differential p-forms, $p = 0, 1, \ldots, n$. (The letter $\mathcal{A}$ stands for alternating.) The Hodge theory centers about the operators $d : \mathcal{A}^p \to \mathcal{A}^{p+1}$ and $\delta : \mathcal{A}^p \to \mathcal{A}^{p-1}$. It turns out that these are both derived from an operator ∇, the covariant differential, that is at the same time simple and powerful. The starting point for the present discussion will be ∇ acting on the space $\mathcal{A}^p$ of differential p-forms.

If σ is in $\mathcal{A}^1$ and ω is in $\mathcal{A}^p$, then there is a *tensor product* $\sigma \otimes \omega$ that is in a new space $\mathcal{A}^{1,p}$. An element of this space may be called a $(1, p)$-form. The tensor product symbol $\otimes$ indicates that this product is not required to be antiisymmetric. It is natural to define the exterior product of an element of $\mathcal{A}^{1,p}$ of this form with an element ρ in $\mathcal{A}^q$ on the right by $(\sigma \otimes \omega) \wedge \rho = \sigma \otimes (\omega \wedge \rho)$. Similarly, define the exterior product by λ in $\mathcal{A}^q$ on the left by $\lambda \wedge (\sigma \otimes \omega) = \sigma \otimes (\lambda \wedge \omega)$.

The *covariant differential* ∇ in the case at hand is a differential operator

$$\nabla : \mathcal{A}^p \to \mathcal{A}^{1,p}. \tag{11.202}$$

In coordinates it takes the form

$$\nabla = \sum_i dx_i \otimes \frac{\partial}{\partial x_i}, \tag{11.203}$$

where the derivative is the covariant derivative. It acts on coefficients as ordinary differentiation, but acts on differentials dx_j according to

$$\frac{\partial}{\partial x_i} dx_j = -\sum_k \Gamma^j_{ik}\, dx_k. \tag{11.204}$$

This describes how the coordinate differentials dx_j change. Each Γ^j_{ik} is a *Christoffel coefficient*. It depends on the coordinate system and on the coefficients of the inverse metric tensor, in a way described below. The minus sign is standard in this context. It is required that the coefficients be symmetric in the sense that $\Gamma^j_{ik} = \Gamma^j_{ki}$.

For a 1-form

$$\omega = \sum_j s_j\, dx_j \tag{11.205}$$

this leads to the expression

$$\nabla\omega = \sum_i \sum_j dx_i \otimes \left(\frac{\partial s_j}{\partial x_i} \, dx_j - s_j \sum_k \Gamma^j_{ik} \, dx_k \right). \tag{11.206}$$

This is independent of the coordinate system. Sometimes it is convenient to write this in another way by interchanging indices:

$$\nabla\omega = \sum_i \sum_j dx_i \otimes \left(\frac{\partial s_j}{\partial x_i} - \sum_k s_k \Gamma^k_{ij} \right) dx_j. \tag{11.207}$$

The Christoffel symbols are determined by the requirement that for 1-forms σ and λ we have

$$\frac{\partial}{\partial x_i} \left(\sigma \mathsf{g}^{-1} \lambda \right) = \left(\frac{\partial}{\partial x_i} \sigma \right) \mathsf{g}^{-1} \lambda + \sigma \mathsf{g}^{-1} \left(\frac{\partial}{\partial x_i} \lambda \right). \tag{11.208}$$

In other words, the inverse metric tensor is required to behave like a constant during this kind of differentiation. If we take $\sigma = dx_j$ and $\lambda = dx_k$, this says

$$\frac{\partial}{\partial x_i} dx_j \mathsf{g}^{-1} dx_k = -\sum_\ell \Gamma^j_{i\ell} dx_\ell \mathsf{g}^{-1} dx_k - dx_j \mathsf{g}^{-1} \sum_\ell \Gamma^k_{i\ell} dx_\ell. \tag{11.209}$$

More explicitly,

$$\frac{\partial}{\partial x_i} g^{jk} = -\sum_\ell \Gamma^j_{i\ell} g^{\ell k} - \sum_\ell \Gamma^k_{i\ell} g^{j\ell}. \tag{11.210}$$

This is a system of equations that must be satisfies for every i, j, k. This shows how the Christoffel symbols must be related to the inverse metric tensor. It is just an elaborate way of saying that covariant differentiation treats the inverse metric tensor as constant.

The covariant differential on p-forms satisfies the derivation property

$$\nabla(\alpha \wedge \beta) = \nabla\alpha \wedge \beta + \alpha \wedge \nabla\beta. \tag{11.211}$$

This uses the fact that $\mathcal{A}^q$ acts on $\mathcal{A}^{1,p}$ both on the right and on the left. The more explicit form

$$\nabla(\alpha \wedge \beta) = \sum_i dx_i \otimes \left(\frac{\partial}{\partial x_i} \alpha \wedge \beta + \alpha \wedge \frac{\partial}{\partial x_i} \beta \right) \tag{11.212}$$

makes this clear.

Later on it will be useful to look at the covariant differential ∇ acting on the space $\mathcal{A}^{1,p}$. The values are then in the space $\mathcal{A}^{2,p}$ of $(2,p)$-forms. Such forms are generated by tensor products $\sigma \otimes \lambda \otimes \omega$, where σ and λ are 1-forms, and ω is a p-form. Though this is not strictly necessary, it will be convenient to denote the covariant differential with this domain by a slightly different symbol ∇', so

$$\nabla' : \mathcal{A}^{1,p} \to \mathcal{A}^{2,p}. \tag{11.213}$$

This covariant differential also satisfies a derivation property.

11.4.2 The anticommutation relation

The covariant differential works in combination with the metric interior product. Here is a quick review of this topic. This will lead to an important anticommutation relation that will be inportant in the following.

Suppose

$$\sigma = \sum_i r_i \, dx_i \tag{11.214}$$

and

$$\lambda = \sum_j s_j \, dx_j \tag{11.215}$$

are 1-forms. The inner product on 1-forms is determined by

$$\sigma \mathsf{g}^{-1} \lambda = \sum_i \sum_j r_i g^{ij} s_j. \tag{11.216}$$

This *inverse metric tensor* takes the same form in all coordinate systems. (In tensor notation the coefficients of the inverse metric tensor are conventionally written g^{ij} with superscripts.) This may also be thought of as 1-form σ acting on 1-form λ. In this case it is called the metric interior product on 1-forms and is written

$$\sigma \rfloor \lambda = \sigma \mathsf{g}^{-1} \lambda. \tag{11.217}$$

It takes the same form in all coordinate systems.

Note: The usual interior product is an action of a vector field on a p-form. Here a similar (but not identical) notation is used for the metric interior product, an action of a 1-form on a p–form.

This *metric interior product* extends to an antiderivation on p-forms. If σ is a 1-form and ω is a p-form, then $\sigma \rfloor \omega$ is a $(p-1)$-form. If $\alpha \wedge \beta$ is a product of a p-form α with q-form β, then

$$\sigma \rfloor (\alpha \wedge \beta) = (\sigma \rfloor \alpha) \wedge \beta + (-1)^p \alpha \wedge (\sigma \rfloor \beta). \tag{11.218}$$

Again this is a product rule where the $\sigma\rfloor$ operation gets a minus sign each time it is moved past a dx_j.

Proposition 11.38 (Anticommutation relation) *For 1-forms σ and λ and p-form ω there is the* anticommutation relation

$$\sigma \rfloor (\lambda \wedge \omega) + \lambda \wedge (\sigma \rfloor \omega) = (\sigma \mathsf{g}^{-1} \lambda) \omega. \tag{11.219}$$

The anticommutation relation above results from the derivation property $\sigma \rfloor (\lambda \wedge \omega) = (\sigma \rfloor \lambda) \wedge \omega - \lambda \wedge (\sigma \rfloor \omega)$ and the fact that $\sigma \rfloor \lambda = \sigma \mathsf{g}^{-1} \lambda$.

The relation between the covariant derivative and the metric interior product also extends. The following result shows how this works.

Proposition 11.39 *If σ is a 1-form and ω is a p-form, then*

$$\frac{\partial}{\partial x_i} (\sigma \rfloor \omega) = \frac{\partial}{\partial x_i} \sigma \rfloor \omega + \sigma \rfloor \frac{\partial}{\partial x_i} \omega. \tag{11.220}$$

Proof: Write $\omega_{[p]} = \omega_1 \wedge \cdots \wedge \omega_p$ as a product of 1-forms. Calculate the metric interior product on the left hand side: $\sigma \rfloor \omega = \sum_\ell (-1)^{\ell-1} (\sigma \rfloor \omega_\ell) \omega_{[p]\setminus\ell}$. Then calculate the derivative using $\partial/\partial x_i (\sigma \rfloor \omega_\ell) = (\partial/\partial x_i \sigma) \rfloor \omega_\ell + \sigma \rfloor (\partial/\partial x_i \omega_\ell)$. The contribution of the first term is

$$\sum_\ell (-1)^{\ell-1} (\frac{\partial}{\partial x_i} \sigma \rfloor \omega_\ell) \omega_{[p]\setminus\ell} = \frac{\partial}{\partial x_i} \sigma \rfloor \omega. \tag{11.221}$$

This is the first term on the right hand side. The other terms give

$$\sum_\ell (-1)^{\ell-1} (\sigma \rfloor \frac{\partial}{\partial x_i} \omega_\ell) \omega_{[p]\setminus\ell} + \sum_\ell (-1)^{\ell-1} (\sigma \rfloor \omega_\ell) \frac{\partial}{\partial x_i} \omega_{[p]\setminus\ell}. \tag{11.222}$$

This simplifies to give the second term on the right hand side. □

11.4.3 Covariant differential, exterior differential, and codifferential

Define the *left multiplication* map $m : \mathcal{A}^{1,p} \to \mathcal{A}^{p+1}$ by

$$m(\sigma \otimes \omega) = \sigma \wedge \omega. \tag{11.223}$$

Similarly, define the *metric trace* map $t : \mathcal{A}^{1,p} \to \mathcal{A}^{p-1}$ by

$$t(\sigma \otimes \omega) = \sigma \rfloor \omega. \tag{11.224}$$

Note: The trace defined by the usual interior product pairs a vector field with a p-form to produce a $(p-1)$-form. The metric trace used here is defined by the metric interior product: it pairs a 1-form with a p-form to produce a $(p-1)$-form.

These definitions are key to understanding the main objects of the theory, the differential d and the codifferential δ. The exterior differential d is related to the covariant differential by

$$d = m\nabla : \mathcal{A}^p \to \mathcal{A}^{p+1} \tag{11.225}$$

Since it does not depend on the metric, the contribution of the Christoffel coefficients $\Gamma^j_{ik} = \Gamma^j_{ki}$ must be zero. This is true because the symmetry of these coefficients is compensated by the antisymmetry of the exterior product. The *codifferential* is defined by

$$\delta = t\nabla : \mathcal{A}^p \to \mathcal{A}^{p-1}. \tag{11.226}$$

This map is a metric object, quite complicated, as we shall see. The pair d and δ are the ingredients of Hodge theory.

Warning: This definition of the codifferential used here has the opposite sign from the definition that is most common in advanced differential geometry. In that context the operator $\delta : \mathcal{A}^1 \to \mathcal{A}^0$ acts like the negative of the divergence. The definition here makes it act like the divergence.

It is an important fact that $\delta\delta = 0 : \mathcal{A}^p \to \mathcal{A}^{p-2}$. See [34] for the original proof by Weitzenböch. See below for the proof via Hodge duality. The action of the codifferential δ on an exterior product does not have the simple properties of the action of the differential d on an exterior product. In the simplest case of a scalar s times a p-form ω the result is $\delta(s\,\omega) = ds \rfloor \omega + s\,\delta\omega$.

11.4.4 Two definitions of codifferential

Lemma 11.40 *If β is a k-form in $\mathcal{A}^k$, then $\nabla * \beta = *\nabla\beta$ as an identity in $\mathcal{A}^{1,k}$.*

In the above lemma the Hodge $*$ on the right hand side of this equation only acts on the k-form side of the tensor product. The proof goes back to the definition of the Hodge dual. We have

$$(\alpha \wedge *\beta) = (\alpha \bullet \beta)\text{vol}. \tag{11.227}$$

Differentiating the left hand side gives

$$\nabla(\alpha \wedge *\beta) = \nabla\alpha \wedge *\beta + \alpha \wedge \nabla * \beta = (\nabla\alpha \bullet \beta)\text{vol} + \alpha \wedge \nabla * \beta. \tag{11.228}$$

Also, because the covariant differential treats the metric tensor as a constant, and also treats the volume as a constant, differentiating the right hand side gives

$$\nabla(\alpha \bullet \beta)\text{vol} = (\nabla\alpha \bullet \beta)\text{vol} + (\alpha \bullet \nabla\beta)\text{vol} = (\nabla\alpha \bullet \beta)\text{vol} + \alpha \wedge *\nabla\beta. \tag{11.229}$$

The conclusion is that $\alpha \wedge \nabla * \beta = \alpha \wedge *\nabla\beta$.

Lemma 11.41 *If γ is in $\mathcal{A}^1$ and ω is in $\mathcal{A}^k$, then $*(\gamma \rfloor \omega) = (-1)^{k-1}\gamma \wedge *\omega$.*

For α in $\mathcal{A}^{k-1}$ we have

$$\alpha \wedge *(\gamma \rfloor \omega) = (\alpha \bullet (\gamma \rfloor \omega))\text{vol} = ((\gamma \wedge \alpha) \bullet \omega)\text{vol} = \gamma \wedge \alpha \wedge *\omega. \tag{11.230}$$

This is because left multiplication and interior product are adjoint. This is in turn

$$\gamma \wedge \alpha \wedge *\omega = (-1)^{k-1}\alpha \wedge \gamma \wedge *\omega. \tag{11.231}$$

Lemma 11.42 *For β in $\mathcal{A}^{1,k}$ we have $*t\beta = (-1)^{k-1}m * \beta$.*

This can be seen by taking $\beta = \gamma \otimes \omega$. Then

$$*t\beta = *(\gamma \rfloor \omega) = (-1)^{k-1}\gamma \wedge *\omega = (-1)^{k-1}m(\gamma \otimes *\omega) = (1)^{k-1}m * \beta. \tag{11.232}$$

Theorem 11.43 *Define*

$$\delta = t\nabla : \mathcal{A}^k \to \mathcal{A}^{k-1}. \tag{11.233}$$

Then

$$\delta = (-1)^{k-1} *^{-1} d * . \tag{11.234}$$

Proof: This is

$$*\delta = *t\nabla = (-1)^{k-1} m * \nabla = (-1)^{k-1} m \nabla * = (-1)^{k-1} d * . \tag{11.235}$$

□

This theorem gives an easy proof that $\delta\delta = 0$.

11.4.5 Codifferential and divergence

Define the interior product of a 1-form α with a p-vector $\mathbf{X} = X_1 \wedge \cdots \wedge X_p$ by

$$\alpha \lrcorner \mathbf{X} = \sum_\ell (-1)^\ell \langle \alpha \mid X_\ell \rangle X_1 \wedge \cdots \wedge \widehat{X_\ell} \wedge \cdots \wedge X_p. \tag{11.236}$$

This allows the definition of trace

$$\mathrm{tr}(\alpha \otimes \mathbf{X}) = \alpha \lrcorner \mathbf{X}. \tag{11.237}$$

Finally, the divergence of a p-vector field is defined by letting the covariant differential ∇ act on p-vector fields. Explicitly:

$$\mathbf{div}\mathbf{X} = \mathrm{tr}\, \nabla \mathbf{X}. \tag{11.238}$$

The result is a $(p-1)$-vector field.

Contrast this with the codifferential δ mapping p-forms to $(p-1)$-forms. It is defined by $\delta = t\nabla$, where ∇ is the covariant differential acting on p=forms, and t is the metric trace involving the metric interior product of a 1-form with a p-form. It will now appear that these two objects are closely related.

Let g^p be the map from p-vectors to p-forms defined by the metric tensor. It may also be thought of as a map from 1-form with p-vector tensor products to 1-form with p-form tensor products. Let g^{-p} be the inverse map.

Lemma 11.44 $t\mathsf{g}^p = \mathsf{g}^{p-1}\,\mathrm{tr}$ *acting on 1-form with p-vector tensor products.*

The reason the lemma works is the following. Say that $\mathbf{X} = X_1 \wedge \cdots \wedge X_p$ is a p-vector field. Then $\mathsf{g}^p\mathbf{X} = \mathsf{g}X_1 \wedge \cdots \wedge \mathsf{g}X_p$. So

$$t(\alpha \otimes \mathsf{g}^p\mathbf{X}) = \sum_\ell (-1)^{\ell-1} (\alpha \rfloor \mathsf{g}X_\ell) \mathsf{g}X_1 \wedge \cdots \widehat{\mathsf{g}X_\ell} \cdots \wedge \mathsf{g}X_p. \tag{11.239}$$

Furthermore, $\alpha \rfloor \mathsf{g}X_\ell = \langle \alpha \mid X_\ell \rangle$. Similarly,

$$\mathrm{tr}(\alpha \otimes \mathbf{X}) = \sum_\ell (-1)^{\ell-1} \langle \alpha \mid X_\ell \rangle X_1 \wedge \cdots \widehat{X_\ell} \cdots \wedge X_p. \tag{11.240}$$

Apply g^{p-1} to this. The result is the same.

Theorem 11.45

$$\delta = \mathsf{g}^{p-1}\mathbf{div}\,\mathsf{g}^{-p}. \tag{11.241}$$

To prove the theorem, start from the fact that ∇ treats the metric tensor as a constant. This implies that $\nabla = \mathsf{g}^p \nabla \mathsf{g}^{-p}$. Then

$$t\nabla = t\mathsf{g}^p \nabla \mathsf{g}^{-p} = \mathsf{g}^{p-1}\,\mathrm{tr}\,\nabla \mathsf{g}^{-p}. \tag{11.242}$$

This is equivalent to the statement of the theorem.

11.5 Two Laplace operators

11.5.1 The Laplace-de Rham and Laplace-Beltrami operators

The *Laplace-de Rham operator* (also called the Hodge Laplacian) is defined by

$$\Delta\omega = \delta d\omega + d\delta\omega. \tag{11.243}$$

This map $\Delta : \mathcal{A}^p \to \mathcal{A}^p$ is the central object in Hodge theory.

Warning: This definition of the Laplace-de Rham operator used here has the opposite sign from the definition that is most common in advanced differential geometry. However the sign convention used here is standard in most other parts of mathematics; it involves second partial derivatives with no extra minus sign.

The Laplace-de Rham operator can also be written

$$\Delta = t\nabla m\nabla + m\nabla t\nabla. \tag{11.244}$$

Since t is related to metric interior product and m to multiplication, this remarkable formula reflects the anticommutation relation in some indirect way.

Let $\mathcal{A}^{2,p}$ be the space generated by $\sigma \otimes \lambda \otimes \omega$, where σ and λ are 1-forms, and ω is a p-form. Then the covariant differential sends $\mathcal{A}^{1,p}$ to $\mathcal{A}^{2,p}$. Here this will be denoted $\nabla' : \mathcal{A}^{1,p} \to \mathcal{A}^{2,p}$. The prime is a reminder that this is the covariant differential on a somewhat different domain.

The explicit form for this covariant differential $\nabla' : \mathcal{A}^{1,p} \to \mathcal{A}^{2,p}$ is

$$\nabla'(\sigma \otimes \omega) = \sum_i dx_i \otimes \left(\frac{\partial}{\partial x_i}\sigma \otimes \omega + \sigma \otimes \frac{\partial}{\partial x_i}\omega \right). \tag{11.245}$$

The inverse metric tensor $\mathsf{g}^{-1} : \mathcal{A}^{2,p} \to \mathcal{A}^p$ acts by *metric contraction*. This is defined by

$$\mathsf{g}^{-1}(\alpha \otimes \beta \otimes \omega) = (\alpha \mathsf{g}^{-1} \beta)\omega. \tag{11.246}$$

The *contracted covariant differential* is defined by

$$\nabla^\bullet = \mathsf{g}^{-1}\nabla' : \mathcal{A}^{1,p} \to \mathcal{A}^p. \tag{11.247}$$

On $\mathcal{A}^{1,0}$ this is closely related to the codifferential δ. In fact, if $j : \mathcal{A}^1 \to \mathcal{A}^{1,0}$ is the natural (identity) map, then $\delta = \nabla^\bullet j : \mathcal{A}^1 \to \mathcal{A}^0$. In general it is a quite different object, even more complicated. Perhaps one can think of ∇ as a generalized differential and $\nabla^\bullet$ as a generalized codifferential. In that case $\nabla^\bullet\nabla$ might be considered a generalized Laplacian. This leads to the following definition.

The *Laplace-Beltrami operator* (also called the Bochner Laplacian) is defined by

$$\nabla^\bullet\nabla = \mathsf{g}^{-1}\nabla'\nabla : \mathcal{A}^p \to \mathcal{A}^p. \tag{11.248}$$

The remainder of this article is devoted to the relation between the Laplace-Beltrami operator and the Laplace-de Rham operator. For Euclidean space they are the same; they give two ways of writing the same object in arbitrary coordinate systems. For curved spaces they can be different.

11.5.2 The exterior covariant differential and curvature

Thus far we have algebraic operators multiplication $m : \mathcal{A}^{1,p} \to \mathcal{A}^{p+1}$ and inverse metric trace $t : \mathcal{A}^{1,p} \to \mathcal{A}^{p-1}$. There is also the inverse metric operator $\mathsf{g}^{-1} : \mathcal{A}^{2,p} \to \mathcal{A}^{p}$. There are also covariant differential operators $\nabla : \mathcal{A}^{p} \to \mathcal{A}^{1,p}$ and $\nabla' : \mathcal{A}^{1,p} \to \mathcal{A}^{2,p}$. These lead to the differential $d = m\nabla : \mathcal{A}^{p} \to \mathcal{A}^{p+1}$ and the codifferential $\delta = t\nabla : \mathcal{A}^{p} \to \mathcal{A}^{p-1}$. Finally, there is the contracted covariant differential $\nabla^\bullet = \mathsf{g}^{-1}\nabla' : \mathcal{A}^{1,p} \to \mathcal{A}^{p}$.

Define another multiplication operator by $m' : \mathcal{A}^{2,p} \to \mathcal{A}^{1,p+1}$ by $m'(\sigma \otimes \lambda \otimes \omega) = \lambda \otimes (\sigma \wedge \omega)$.

Define the *exterior covariant differential* by

$$D = m'\nabla' : \mathcal{A}^{1,p} \to \mathcal{A}^{1,p+1}. \tag{11.249}$$

This looks much like the definition of the differential d, but unlike the differential, the exterior covariant differential depends on the metric. Explicitly,

$$D(dx_j \otimes \omega) = -\sum_i \sum_k dx_k \otimes \Gamma^j_{ik}\, dx_i \wedge \omega + dx_j \otimes d\omega. \tag{11.250}$$

If Ω is in $\mathcal{A}^{1,p}$ and λ is in $\mathcal{A}^{q}$, then $\Omega \wedge \lambda$ is in $\mathcal{A}^{1,p+q}$. There is an antiderivation property:

$$D(\Omega \wedge \lambda) = D\Omega \wedge \lambda + (-1)^p \Omega \wedge d\lambda. \tag{11.251}$$

While $dd\lambda$ is always zero, this is not necessarily true for $DD\Omega$. However it follows from the antiderivation property that

$$DD(\Omega \wedge \lambda) = (DD\Omega) \wedge \lambda. \tag{11.252}$$

In particular, if s is a scalar in $\mathcal{A}^0$, $DD(s\Omega) = sDD\Omega$. The operator DD commutes with scalar multiplication. This suggests that it is purely algebraic, that is, does not involve differentiation.

Proposition 11.46 (Curvature formula)

$$DD = R : \mathcal{A}^{1,p} \to \mathcal{A}^{1,p+2} \tag{11.253}$$

where R is the curvature operator. This operator has the explicit form on $\Omega = \sum_j dx_j \otimes \omega_j$ given by

$$R\Omega = \sum_i \sum_j dx_i \otimes r^j_i \wedge \omega_j \tag{11.254}$$

with the curvature matrix 2-form

$$r^j_i = \sum_k \sum_\ell \left(\frac{\partial \Gamma^j_{ik}}{\partial x_\ell} - \sum_m \Gamma^m_{\ell i} \Gamma^j_{km} \right) dx_k \wedge dx_\ell. \tag{11.255}$$

$m : \mathcal{A}^{1,p} \to \mathcal{A}^{p+1}$, $m' : \mathcal{A}^{2,p} \to \mathcal{A}^{1,p+1}$	multiplication
$j : \mathcal{A}^p \to \mathcal{A}^{1,p-1}$, $j' : \mathcal{A}^{1,p} \to \mathcal{A}^{2,p-1}$	tensor representation
$t : \mathcal{A}^{1,p} \to \mathcal{A}^{p-1}$, $t' : \mathcal{A}^{2,p} \to \mathcal{A}^{1,p-1}$	metric trace
$\mathsf{g}^{-1} : \mathcal{A}^{2,p} \to \mathcal{A}^p$	metric contraction
$\nabla : \mathcal{A}^p \to \mathcal{A}^{1,p}$, $\nabla' : \mathcal{A}^{1,p} \to \mathcal{A}^{2,p}$	covariant differential
$d = m\nabla : \mathcal{A}^p \to \mathcal{A}^{p+1}$	exterior differential
$D = m'\nabla' : \mathcal{A}^{1,p} \to \mathcal{A}^{1,p+1}$	exterior covariant differential
$\nabla^\bullet = \mathsf{g}^{-1}\nabla' : \mathcal{A}^{1,p} \to \mathcal{A}^p$	contracted covariant differential
$\delta = t\nabla = \nabla^\bullet j = tDj : \mathcal{A}^p \to \mathcal{A}^{p-1}$	codifferential

Table 11.3: Hodge calculus: d, δ related to $\nabla, \nabla^\bullet$

11.5.3 Anticommutation relations and decompositions of operators

This section and the next lead to an amazing formula relating the two Laplacians. The proof depends on a series of identities for various algebraic and differential operators. These operators are listed in Table 11.3. Certain identities arising from anticommutation relations give decompositions of the operators ∇ and $\nabla^\bullet$. These decompositions imply important integration by parts formulas.

The *tensor representation* map $j : \mathcal{A}^p \to \mathcal{A}^{1,p-1}$ is straightforward. One $\mathcal{A}^1$ it is the identity map $j : \mathcal{A}^1 \to \mathcal{A}^{1,0}$. Then it is extended to be an antiderivation: for α in $\mathcal{A}^p$ and β in $\mathcal{A}^q$ we have

$$j(\alpha \wedge \beta) = j(\alpha) \wedge \beta + (-1)^p \alpha \wedge j(\beta). \tag{11.256}$$

For instance, if α, β, γ are each in $\mathcal{A}^1$, then

$$j(\alpha \wedge \beta \wedge \gamma) = \alpha \otimes (\beta \wedge \gamma) - \beta \otimes (\alpha \wedge \gamma) + \gamma \otimes (\alpha \wedge \beta). \tag{11.257}$$

Lemma 11.47 $tj = 0 : \mathcal{A}^p \to \mathcal{A}^{p-1}$.

Proof: Write $\omega_{[p]} = \omega_1 \wedge \cdots \wedge \omega_p$. Then

$$j\omega_{[p]} = \sum_\ell (-1)^{\ell-1} \omega_\ell \otimes \omega_{[p]\setminus\{\ell\}}. \tag{11.258}$$

Then

$$\begin{aligned} tj\omega_{[p]} = &\sum_{r<\ell} (-1)^{r-1}(-1)^{\ell-1}(\omega_\ell \rfloor \omega_r)\omega_{[p]\setminus\{r,\ell\}} \\ &+ \sum_{r>\ell} (-1)^{r-2}(-1)^{\ell-1}(\omega_\ell \rfloor \omega_r)\omega_{[p]\setminus\{r,\ell\}}. \end{aligned} \tag{11.259}$$

The two sums are identical except for sign. □

The remainder of this account is about the relation between the pair d and δ and the pair ∇ and $\nabla^\bullet$. other hand. Recall that $\delta = t\nabla$; we shall also see that $\delta = \nabla^\bullet j$.

Define $j' : \mathcal{A}^{1,p} \to \mathcal{A}^{2,p-1}$ by $j'(\lambda \otimes \omega) = \lambda \otimes j(\omega)$.

Lemma 11.48 $j'\nabla = \nabla' j : \mathcal{A}^p \to \mathcal{A}^{2,p-1}$.

Proof: A computation works. Write

$$j\omega = \sum_\ell (-1)^{\ell-1} \omega_\ell \otimes \omega_{[p]\setminus\{\ell\}}. \tag{11.260}$$

Then

$$\begin{aligned}\frac{\partial}{\partial x_j} j\omega = &\sum_\ell (-1)^{\ell-1} \frac{\partial}{\partial x_j}\omega_\ell \otimes \omega_{[p]\setminus\{\ell\}} \\ &+ \sum_{r<\ell} (-1)^{\ell-1}(-1)^{r-1} \omega_\ell \otimes \frac{\partial}{\partial x_j}\omega_r \wedge \omega_{[p]\setminus\{\ell,r\}} \\ &+ \sum_{r>\ell} (-1)^{\ell-1}(-1)^{r-2} \omega_\ell \otimes \frac{\partial}{\partial x_j}\omega_r \wedge \omega_{[p]\setminus\{\ell,r\}}.\end{aligned} \tag{11.261}$$

For the other side, start with

$$\frac{\partial}{\partial x_j}\omega = \sum_\ell (-1)^{\ell-1} \frac{\partial}{\partial x_j}\omega_\ell \wedge \omega_{[p]\setminus\{\ell\}}. \tag{11.262}$$

Then

$$\begin{aligned} j\frac{\partial}{\partial x_j}\omega = &\sum_\ell (-1)^{\ell-1} \frac{\partial}{\partial x_j}\omega_\ell \otimes \omega_{[p]\setminus\{\ell\}} \\ &+ \sum_{r<\ell} (-1)^{\ell-1}(-1)^{r} \omega_r \otimes \frac{\partial}{\partial x_j}\omega_\ell \wedge \omega_{[p]\setminus\{\ell,r\}} \\ &+ \sum_{r>\ell} (-1)^{\ell-1}(-1)^{r+1} \omega_r \otimes \frac{\partial}{\partial x_j}\omega_\ell \wedge \omega_{[p]\setminus\{\ell,r\}}.\end{aligned} \tag{11.263}$$

The first terms are the same; the second and third terms correspond to the third and second terms. □

Lemma 11.49 *There is an identity*

$$jm + m'j' = 1 : \mathcal{A}^{1,p} \to \mathcal{A}^{1,p}. \tag{11.264}$$

Proof: It is enough to check this on terms of the form $\lambda \otimes \omega_{[p]}$, that is, on decomposable elements. First

$$jm(\lambda \otimes \omega_{[p]}) = j(\lambda \wedge \omega_{[p]}) = \lambda \otimes \omega_{[p]} + \sum_\ell (-1)^\ell \omega_\ell \otimes \lambda \wedge \omega_{[p]\setminus\{\ell\}}. \tag{11.265}$$

Compare this with $m'j'(\lambda \otimes \omega_{[p]}) = m'(\lambda \otimes j\omega_{[p]})$ which is

$$m'(\lambda \otimes \sum_\ell (-1)^{\ell-1} \omega_\ell \otimes \omega_{[p]\setminus\{\ell\}}) = \sum_\ell (-1)^{\ell-1} \omega_\ell \otimes \lambda \wedge \omega_{[p]\setminus\{\ell\}}. \tag{11.266}$$

Adding these gives $\lambda \otimes \omega_{[p]}$. □

Theorem 11.50 *The covariant differential $\nabla : \mathcal{A}^p \to \mathcal{A}^{1,p}$ has the decomposition*

$$\nabla = jd + Dj. \tag{11.267}$$

The proof of the theorem follows from the two lemmas. In fact, $\nabla = jm\nabla + m'j'\nabla = jm\nabla + m'\nabla' j = jd + Dj$.

Corollary 11.51 *The codifferential is expressed in terms of the exterior covariant differential by*

$$\delta = tDj : \mathcal{A}^p \to \mathcal{A}^{p-1}. \tag{11.268}$$

Define $t' : \mathcal{A}^{2,p} \to \mathcal{A}^{1,p-1}$ by $t'(\sigma \otimes \lambda \otimes \omega) = \sigma \otimes t(\lambda \otimes \omega)$.

Lemma 11.52 $t'\nabla' = \nabla t : \mathcal{A}^{1,p} \to \mathcal{A}^{1,p-1}$.

Proof: This proof uses the proposition on the codifferential of an interior product, which is a reflection of the fact that the codifferential treats the metric as constant. Using the product rule for ∇ and the fact that t' is linear leads to

$$t'\nabla'\Omega = \nabla\lambda \rfloor \omega + \lambda \rfloor \nabla\omega. \tag{11.269}$$

But $t\Omega = \lambda \rfloor \omega$. The proposition then gives

$$\nabla t\Omega = \nabla\lambda \rfloor \omega + \lambda \rfloor \nabla\omega. \tag{11.270}$$

This is exactly the same. □

Lemma 11.53 (Anticommutation relation) *There is an identity*

$$tm' + mt' = \mathsf{g}^{-1} : \mathcal{A}^{2,p} \to \mathcal{A}^p. \tag{11.271}$$

Proof: Check this on tensors $\sigma \otimes \lambda \otimes \omega$. First

$$mt'(\sigma \otimes \lambda \otimes \omega) = m(\sigma \otimes t(\lambda \otimes \omega)) = m(\sigma \otimes \lambda \rfloor \omega) = \sigma \wedge (\lambda \rfloor \omega). \tag{11.272}$$

Also

$$tm'(\sigma \otimes \lambda \otimes \omega) = t(\lambda \otimes (\sigma \wedge \omega)) = \lambda \rfloor (\sigma \wedge \omega). \tag{11.273}$$

By the anticommutation relation the sum is

$$(\sigma \mathsf{g}^{-1}\lambda)\omega = \mathsf{g}^{-1}((\sigma \otimes \lambda) \otimes \omega). \tag{11.274}$$

□

Theorem 11.54 *The contracted covariant differential $\nabla^\bullet : \mathcal{A}^{1,p} \to \mathcal{A}^p$ has the decomposition*

$$\nabla^\bullet = tD + dt. \tag{11.275}$$

The proof of the theorem follows from the lemmas. In fact, $\mathsf{g}^{-1}\nabla' = tm'\nabla' + mt'\nabla' = tm'\nabla' + m\nabla t = tD + dt$.

Corollary 11.55 *The codifferential is expressed in terms of the contracted covariant derivative by*

$$\delta = \nabla^\bullet j : \mathcal{A}^p \to \mathcal{A}^{p-1}. \tag{11.276}$$

Corollary 11.56 *The contracted covariant differential* $\nabla^\bullet : \mathcal{A}^{1,n} \to \mathcal{A}^n$ *applied to a top-dimensional form* ω *in* $\mathcal{A}^{1,n}$ *is exact:*

$$\nabla^\bullet \omega = dt\omega. \tag{11.277}$$

This last corollary is important for understanding the operator $\nabla^\bullet$, as will appear in the following subsection.

11.5.4 The Laplace-Beltrami energy

We begin with an integration by parts formula. Recall that if Ω is in $\mathcal{A}^{1,p}$, then the Hodge dual $*\Omega$ is in $\mathcal{A}^{1,n-p}$ and is obtained by taking the Hodge dual of the p-form factor.

If Ω and Σ are both in $\mathcal{A}^{1,p}$, then there is a product $\Omega(\mathsf{g}^{-1}\wedge)\Sigma$ in $\mathcal{A}^p$ obtained by taking the g^{-1} inner product for the 1-form factors and the usual $\wedge$ exterior product for the p-form factors .

Lemma 11.57 *Take* α *in* $\mathcal{A}^p$ *and* Ω *in* $\mathcal{A}^{1,p}$*. Then*

$$dt(\alpha \wedge *\Omega) = \nabla\alpha(\mathsf{g}^{-1}\wedge) * \Omega + \alpha \wedge *\nabla^\bullet\Omega. \tag{11.278}$$

Proof: Here is a sketch of a proof. Start with

$$\nabla'(\alpha \wedge *\Omega) = \nabla\alpha \wedge *\Omega + \alpha \wedge \nabla' * \Omega. \tag{11.279}$$

Apply g^{-1}. On the left the result is $\nabla^\bullet(\alpha \wedge *\Omega)$. Since this is a $(1, n)$-form, the left hand side is $dt(\alpha \wedge *\Omega)$. The first term on the right works out correctly. The second term on the right also works out, by the following argument. Note that $\nabla' * = *\nabla'$, by the same reasoning as in Lemma 11:40. Then $\alpha \wedge \mathsf{g}^{-1}\nabla' * \Omega = \alpha \wedge \mathsf{g}^{-1} * \nabla'\Omega = \alpha \wedge *\mathsf{g}^{-1}\nabla'\Omega$. Since $\mathsf{g}^{-1}\nabla' = \nabla^\bullet$, this is also correct. □

Proposition 11.58 *Let* α *be in* $\mathcal{A}^p$*. There is an* $(n-1)$*-form whose exterior differential is the sum of a positive* n*-form with an* n*-form associated with* α *and the Laplace-Beltrami Laplacian* $\nabla^\bullet\nabla$*:*

$$dt(\alpha \wedge *\nabla\alpha) = \nabla\alpha(\mathsf{g}^{-1}\wedge) * \nabla\alpha + \alpha \wedge *\nabla^\bullet\nabla\alpha. \tag{11.280}$$

Corollary 11.59 *Let* α *be in* $\mathcal{A}^p$*, so* $\alpha \wedge *\nabla\alpha$ *is in* $\mathcal{A}^{1,n}$*. Suppose there is a bounded region* R *such that the pullback of the form* $t(\alpha \wedge *\nabla\alpha)$ *in* $\mathcal{A}^{n-1}$ *to* ∂R *is zero. Then the integral of the* n*-form associated with* α *and the Laplace-Beltrami Laplacian is negative. This fact is expressed quantitatively by*

$$-\int_R \alpha \wedge *\nabla^\bullet\nabla\alpha = \int_R \nabla\alpha(\mathsf{g}^{-1}\wedge) * \nabla\alpha \geq 0. \tag{11.281}$$

The most elementary case is when $\alpha = s$ is a 0-form. In that case $t(s * \nabla s) = st(*\nabla s)$. But $\nabla s = ds \otimes 1$, and so $*\nabla s = ds \otimes \text{vol}$. Finally, $t(*\nabla s) = ds \rfloor \text{vol} = *ds$. So in that case the boundary condition can be that s is zero on ∂R or that the pullback of $*ds$ is zero on ∂R.

Again the integrand in the statement of the corollary is a positive quantity that may be interpreted as a kind of energy. The left hand side expresses it in terms of the second-order Laplace-Beltrami operator.

11.5.5 The Weitzenböch formula

Theorem 11.60 (Weitzenböck identity) *The Laplace-Beltrami operator is the Laplace-de Rham operator plus a zero order term.*

$$\nabla^\bullet \nabla = \delta d + d\delta + W : \mathcal{A}^p \to \mathcal{A}^p. \tag{11.282}$$

with $W = tRj$.

In the statement of the theorem $j : \mathcal{A}^p \to \mathcal{A}^{1,p-1}$ is the tensor representation, $R = D^2 : \mathcal{A}^{1,p-1} \to \mathcal{A}^{1,p+1}$ is the curvature operator, and $t : \mathcal{A}^{1,p+1} \to \mathcal{A}^p$ is the metric trace. In flat space $R = 0$.

The proof of the theorem is to expand

$$(tD + dt)(jd + Dj) = tDjd + dtDj + dtjd + tDDj \tag{11.283}$$

and use $tj = 0$, $tDj = \delta$, and $tDDj = tRj$.

Remark: The theorem could be written in the form

$$\mathsf{g}^{-1}\nabla'\nabla = t\nabla m\nabla + m\nabla t\nabla + tm'\nabla' m'\nabla' j, \tag{11.284}$$

A subtle interplay between ∇ and the pair m, t from the anticommutation relation leads to the extra curvature term. ||

Remark: The paper by Rummler[43] has additional discussion of these topics. That account introduces spaces $\mathcal{A}^p$ of differential p-forms and $\vec{\mathcal{A}}^p$ of vector-valued differential p-forms. There is a generalized gradient $\vec{\nabla} = \mathsf{g}^{-1}\nabla$ that maps from $\mathcal{A}^p$ to $\vec{\mathcal{A}}^p$ and a generalized divergence $\nabla| = \text{tr}\,\nabla$ that maps from $\vec{\mathcal{A}}^p$ to $\mathcal{A}^p$. The map $j : \mathcal{A}^p \to \vec{\mathcal{A}}^{p-1}$ is defined using the inverse metric tensor, while the map $\text{tr} : \vec{\mathcal{A}}^p \to \mathcal{A}^{p-1}$ is independent of the metric. The reason for this approach is that for ω in $\mathcal{A}^p$ the tensor $\nabla\omega$ in $\mathcal{A}^{1,p}$ is only alternating in the last p arguments. Changing the first argument to a vector (as in Rummler's $\vec{\mathcal{A}}^p$) exhibits the special role of the first argument. The conventions in the paper for the codifferential δ and the Laplace-de Rham operator Δ are the same as in this book. The Rummler paper should be consulted for integral formulas and applications to geometry. ||

Example: The simplest non-trivial example of the Weitzenböch formula is for 1-forms on the 2-sphere. The 2-sphere a has coordinates θ and ϕ. The convention here is that θ is co-latitude (measured from the north pole) and ϕ is longitude. The circles of constant ϕ are great circles. The circles of constant θ get small near the north and south poles.

Covariant derivatives are recovered as follows. Start with $d\theta \rfloor d\theta = 1$. Differentiating with respect to θ gives $(\partial/\partial\theta)d\theta = C\,d\phi$. Differentiating with respect to ϕ gives $(\partial/\partial\phi)d\theta = A\,d\phi$. The coefficients C and A are to be determined. However symmetry gives $(\partial/\partial\phi) \lrcorner (\partial/\partial\theta)d\theta = (\partial/\partial\theta) \lrcorner (\partial/\partial\phi)d\theta$, which says that $C = 0$.

Next start with $d\phi \rfloor d\phi = 1/\sin^2(\theta)$. Differentiating with respect to ϕ gives $(\partial/\partial\phi)d\phi = Bd\theta$.

Then use use $d\theta \rfloor d\phi = 0$. Differentiating with respect to θ gives $d\theta \rfloor (\partial/\partial\theta)d\phi = 0$, so $(\partial/\partial\theta)d\phi = B'd\phi$. Symmetry gives $(\partial/\partial\phi) \lrcorner (\partial/\partial\theta)d\phi = (\partial/\partial\theta) \lrcorner (\partial/\partial\phi)d\phi$, which gives $B' = B$.

The covariant derivatives must then have the form

$$\begin{aligned}
\frac{\partial}{\partial\theta}d\theta &= 0 \\
\frac{\partial}{\partial\phi}d\theta &= A\,d\phi \\
\frac{\partial}{\partial\theta}d\phi &= B\,d\phi \\
\frac{\partial}{\partial\phi}d\phi &= B\,d\theta.
\end{aligned} \tag{11.285}$$

It is instructive to think about the meaning of these formula; they should be intuitive, except perhaps for the values of the coefficients A and B.

A little more work gives the coefficients. By differentiating $d\theta \rfloor d\phi = 0$ with respect to ϕ one gets $A\,d\phi \rfloor d\phi + B\,d\theta \rfloor d\theta = 0$. This implies that $\sin(\theta)^{-2}A + B = 0$. From differentiating $d\phi \rfloor d\phi = 1/\sin^2(\theta)$ with respect to θ one gets $(\partial/\partial\theta)\sin(\theta)^{-2} = 2\sin(\theta)^{-2}B$. This gives $B = -\cot(\theta)$. The conclusion is that

$$\begin{aligned}
\frac{\partial}{\partial\theta}d\theta &= 0 \\
\frac{\partial}{\partial\phi}d\theta &= \cos(\theta)\sin(\theta)\,d\phi \\
\frac{\partial}{\partial\theta}d\phi &= -\cot(\theta)\,d\phi \\
\frac{\partial}{\partial\phi}d\phi &= -\cot(\theta)\,d\theta.
\end{aligned} \tag{11.286}$$

Finally, the covariant differential is

$$\nabla = \left(d\theta \otimes \frac{\partial}{\partial\theta} + d\phi \otimes \frac{\partial}{\partial\phi}\right), \tag{11.287}$$

where the partial derivatives act both on coefficients and on basis differentials. The result for coordinate basis differentials is

$$\begin{aligned}
\nabla\, d\theta &= \cos(\theta)\sin(\theta)\,d\phi \otimes d\phi \\
\nabla\, d\phi &= -\cot(\theta)(d\theta \otimes d\phi + d\phi \otimes d\theta).
\end{aligned} \tag{11.288}$$

The coefficients are negatives of Christoffel coefficients. Notice the symmetry on the right hand side. The $\cot(\theta)$ shows something problematic near the poles. The issue is that a small distance along a fixed latitude circle can represent a big change in ϕ.

These formulas are easier to interpret in terms of the normalized basis differentials $d\theta$ and $\sin(\theta)\,d\phi$ that measure distance on the sphere. The result is then

$$\begin{aligned}
\frac{\partial}{\partial\theta}d\theta &= 0 \\
\frac{\partial}{\partial\phi}d\theta &= \cos(\theta)\,(\sin(\theta)\,d\phi) \\
\frac{\partial}{\partial\theta}(\sin(\theta)\,d\phi) &= 0 \\
\frac{\partial}{\partial\phi}(\sin(\theta)\,d\phi) &= -\cos(\theta)\,d\theta.
\end{aligned} \tag{11.289}$$

The derivatives of normalized bases are zero along the great circles of constant latitude. But along circles of constant longitude near the poles the bases get rotated. Notice the antisymmetry in the coefficients $\cos(\theta)$ and $-\cos(\theta)$.

The codifferential acting on normalized bases is

$$\begin{aligned}
\nabla d\theta &= \cos(\theta)\,d\phi\otimes\sin(\theta)\,d\phi \\
\nabla\sin(\theta)d\phi &= -\cos(\theta)\,d\phi\otimes d\theta.
\end{aligned} \tag{11.290}$$

It is possible to compute in coordinates bases or in normalized bases. The coordinate basis elements are exact. The normalized basis elements are easier to interpret geometrically. It is a matter of choice.

These formula allow the computation of covariant differentials and codifferentials. Write

$$\omega = p\,d\theta + q\,d\phi. \tag{11.291}$$

The covariant differential is

$$\nabla\omega = \frac{\partial p}{\partial\theta}d\theta\otimes d\theta + p\sin(\theta)\cos(\theta)\,d\phi\otimes d\phi + \frac{\partial q}{\partial\phi}d\phi\otimes d\phi - q\cot(\theta)(d\theta\otimes d\phi + d\phi\otimes d\theta). \tag{11.292}$$

The codifferntial is then

$$\delta\omega = \frac{\partial p}{\partial\theta} + p\sin(\theta)\cos(\theta)\sin(\theta)^{-2} + \frac{\partial q}{\partial\phi}\sin(\theta)^{-2}. \tag{11.293}$$

This is

$$\delta\omega = \frac{1}{\sin(\theta)}\frac{\partial\sin(\theta)p}{\partial\theta} + \frac{1}{\sin^2(\theta)}\frac{\partial q}{\partial\phi}. \tag{11.294}$$

The result of this calculation is consistent with the general formula for the codifferential of a 1-form. The Laplacian on scalars is thus

$$\Delta s = \delta ds = \frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial s}{\partial\theta}\right) + \frac{1}{\sin^2(\theta)}\frac{\partial^2 s}{\partial\phi^2}. \tag{11.295}$$

The codifferential on 2-forms is also easy, particularly if the 2-form is expressed in terms of the normalized basis form $\sigma = \sin(\theta)\, d\theta\, d\phi$. Let $\lambda = s\sigma$. The codifferential works out to be

$$\delta\lambda = \frac{\partial s}{\partial \theta}\sin(\theta)\, d\phi - \frac{1}{\sin(\theta)}\frac{\partial s}{\partial \phi}\, d\theta. \tag{11.296}$$

The Laplacian is then $\Delta\lambda = d\delta\lambda = (\Delta s)\sigma$. It has much the same form as the scalar Laplacian.

On 1-forms the Laplace-Beltrami operator is not be the same as the Laplace-de Rham operator. The difference is the Weitzenböch term $W = tRj : \mathcal{A}^1 \to \mathcal{A}^1$, where $R = DD : \mathcal{A}^{1,0} \to \mathcal{A}^{1,2}$ is the curvature. The result for the curvature is

$$\begin{aligned} R(d\theta \otimes 1) &= -\sin^2(\theta)\, d\phi \otimes (d\theta \wedge d\phi). \\ R(d\phi \otimes 1) &= d\theta \otimes (d\theta \wedge d\phi). \end{aligned} \tag{11.297}$$

This is easier to interpret with normalized bases. Since R commutes with scalar multiplication, this is

$$\begin{aligned} R(d\theta \otimes 1) &= -(\sin(\theta)\, d\phi) \otimes (d\theta \wedge \sin(\theta)\, d\phi) \\ R(\sin(\theta)\, d\phi \otimes 1) &= d\theta \otimes (d\theta \wedge \sin(\theta)\, d\phi). \end{aligned} \tag{11.298}$$

Thus R is a 2-form whose values are antisymmetric linear transformations. The coefficients determining the transformations are constants in the normalized basis.

From this it is easy to compute

$$\begin{aligned} tRj(d\theta) &= tR(d\theta \otimes 1) = -t\sin^2(\theta)\, d\phi \otimes (d\theta \wedge d\phi) = d\theta \\ tRj(d\phi) &= tR(d\phi \otimes 1) = t\, d\theta \otimes (d\theta \wedge d\phi) = d\phi. \end{aligned} \tag{11.299}$$

For the sphere the Weitzenböch term is the identity operator. ||

Problems

The codifferential in orthogonal coordinates

The context for these problems is a two-dimensional space with orthogonal coordinates u, v and with metric $\mathsf{g} = h_u^2\, du^2 + h_v^2\, dv^2$. Recall that $*$ maps 1 to $h_u h_v\, du\, dv$, $h_u\, du$ to $h_v\, dv$, $h_v\, dv$ to $-h_u\, du$, and $h_u h_v\, du\, dv$ to 1.

1. Find the codifferential δ on 1-forms $\alpha = p\, du + q\, dv$ using coordinate basis vectors for 1-forms.

2. Find the codifferential δ on 1-forms $\omega = p\, h_u\, du + q h_v\, dv$ using normalized basis vectors for 1-forms.

3. Find the codifferential δ on 2-forms $\sigma = s\, du\, dv$ using coordinate basis vectors for 2-forms and for 1-forms.

4. Find the codifferential δ on 2-forms $\tau = s\, h_u h_v\, du\, dv$ using coordinate basis vectors for 2-forms and for 1-forms.

Covariant derivatives in orthogonal coordinates

The context for these problems is a two-dimensional space with orthogonal coordinates u, v and with metric $\mathsf{g} = h_u^2\,du^2 + h_v^2\,dv^2$. The covariant derivatives of the normalized basis 1-forms are

$$\begin{aligned}
\frac{\partial}{\partial u}(h_u\,du) &= -\frac{1}{h_v}\frac{\partial h_u}{\partial v}\,h_v\,dv \\
\frac{\partial}{\partial u}(h_v\,dv) &= \frac{1}{h_v}\frac{\partial h_u}{\partial v}\,h_u\,du \\
\frac{\partial}{\partial v}(h_v\,dv) &= -\frac{1}{h_u}\frac{\partial h_v}{\partial u}\,h_u\,du \\
\frac{\partial}{\partial v}(h_u\,du) &= \frac{1}{h_u}\frac{\partial h_v}{\partial u}\,h_v\,dv.
\end{aligned} \tag{11.300}$$

These are perhaps easier to remember in the pattern

$$\frac{\partial}{\partial u}\begin{bmatrix} h_u\,du & h_v\,dv \end{bmatrix} = \frac{1}{h_v}\frac{\partial h_u}{\partial v}\begin{bmatrix} -h_v\,dv & h_u\,du \end{bmatrix} \tag{11.301}$$

and the similar pattern when u and v are reversed.

1. Show that with these definitions

$$\begin{aligned}
\frac{\partial}{\partial u}\mathsf{g} &= 0 \\
\frac{\partial}{\partial v}\mathsf{g} &= 0.
\end{aligned} \tag{11.302}$$

   Hint: Write $\mathsf{g} = (h_u\,du)(h_u\,du) + (h_v\,dv)(h_v\,dv)$ and use the sum and product rules for differentiating.

2. Use the above definitions to evaluate the covariant derivatives of the coordinate basis 1-forms. There will be four results, each with two terms on the right. (The coefficients are negatives of Christoffel symbols.)

3. Consider the 1-form $\omega = \frac{1}{h_v}\frac{\partial h_u}{\partial v}\,du - \frac{1}{h_u}\frac{\partial h_v}{\partial u}\,dv$. Its exterior derivative is the 2-form $d\omega = K\,h_u\,du\,h_v\,dv$. Find the explicit formula for the curvature K.

4. Show that there is a quantity R such that

$$\begin{aligned}
\left(\frac{\partial}{\partial u}\frac{\partial}{\partial v} - \frac{\partial}{\partial v}\frac{\partial}{\partial u}\right) h_u\,du &= R\,h_v\,dv \\
\left(\frac{\partial}{\partial u}\frac{\partial}{\partial v} - \frac{\partial}{\partial v}\frac{\partial}{\partial u}\right) h_v\,dv &= -R\,h_u\,du.
\end{aligned} \tag{11.303}$$

   Hint: Compute using normalized basis 1-forms.

5. Show that the quantity R is closely related to curvature.

6. Show that if $\omega = ph_u\,du + qh_v\,dv$ is a 1-form, then

$$\delta\omega = \left(du \rfloor \frac{\partial}{\partial u} + dv \rfloor \frac{\partial}{\partial v}\right)\omega \tag{11.304}$$

Hint: The metric interior product satisfies $du \rfloor du = h_u^{-2}$ and $dv \rfloor dv = h_v^{-2}$ with cross products giving zero.

7. Show that if $\tau = sh_u h_v\,du \wedge dv$ is a 2-form, then

$$\delta\tau = \left(du \rfloor \frac{\partial}{\partial u} + dv \rfloor \frac{\partial}{\partial v}\right)\tau \tag{11.305}$$

Hint: Recall that the metric interior product acting on an exterior product of forms is an antiderivation.

The torus

Let $0 < a < b$. Consider the parameterized curve ϕ given by

$$x \leftarrow (b + a\cos(u))\cos(v) \tag{11.306}$$
$$y \leftarrow (b + a\cos(u))\sin(v) \tag{11.307}$$
$$z \leftarrow a\sin(u). \tag{11.308}$$

Its image is a torus. The functions $\cos(u)$ and $\sin(u)$ describe location on the smaller circles, while the functions $\cos(v)$ and $\sin(v)$ describe locations on the larger circles. The differentials du and dv describe corresponding changes.

The metric on this torus is $\mathsf{g} = a^2\,du^2 + (b + a\cos(u))^2\,dv^2$. The unit basis forms are $h_u\,du = a\,du$ and $h_v\,dv = (b + a\cos(u))\,dv$.

1. Show that the 1-form $\omega_1 = h_u/h_v = a/(b + a\cos(u))\,du$ satisfies $d\omega_1 = 0$ and $\delta\omega_1 = 0$.

2. Show that the 1-form $\omega_2 = dv$ satisfies $d\omega_2 = 0$ and $\delta\omega_2 = 0$.

3. Show that these 1-forms both satisfy Laplace's equation.

The sphere

Let $0 < a$. Consider the parameterized curve ϕ given by

$$x \leftarrow a\sin(u)\cos(v) \tag{11.309}$$
$$y \leftarrow a\sin(u)\sin(v) \tag{11.310}$$
$$z \leftarrow a\cos(u). \tag{11.311}$$

Its image is a sphere. The functions $\cos(u)$ describes location on circles of constant longitude, while the functions $\sin(u)\cos(v)$ and $\sin(u)\sin(v)$ describe

locations on circles of constant latitude. Similarly, the differentials du and $\sin(u)\,dv$ describe corresponding angular changes.

The metric tensor on this torus is $\mathtt{g} = a^2\,du^2 + a^2\sin^2(u)\,dv^2$. The unit basis forms are $a\,du$ and $a\sin(u)\,dv$. These describe corresponding changes in distance.

The Laplace operator is $d\delta + \delta d$, where δ depends on the metric tensor.

1. (a) Find $d\cos(u)$.

   (b) Find the Laplacian of the 0-form $\cos(u)$, that is, find $\delta d\cos(u)$. Show that it is a multiple of $\cos(u)$.

   (c) Find the Laplacian of the 1-form $d\cos(u)$, and show that it is a multiple of $d\cos(u)$.

2. (a) Find $d(\sin(u)\cos(v))$.

   (b) Find the Laplacian of the 0-form $\sin(u)\cos(v)$, and show that it is a multiple of the same form.

   (c) Find the Laplacian of the 1-form $d(\sin(u)\cos(v))$, and show that it is a multiple of the same form.

3. (a) Find $d(\sin(u)\sin(v))$.

   (b) Find the Laplacian of the 0-form $\sin(u)\sin(v)$, and show that it is a multiple of the same form.

   (c) Find the Laplacian of the 1-form $d(\sin(u)\sin(v))$, and show that it is a multiple of the same form.

4. The Laplacian acting on 0-forms on the sphere has eigenvalues $-\frac{1}{a^2}\ell(\ell+1)$ for $\ell = 0, 1, 2, 3, \ldots$. The eigenfunctions are surface spherical harmonics. Which of these eigenvalues are also eigenvalues for the Laplacian acting on 1-forms on the sphere? Explain.

Chapter 12

Reference material

12.1 Mathematical Notation

Linear Algebra	
$\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$	n-component column vectors
ω, μ	m-component row vectors
A, B, C	m by n matrices
A^T	transpose
A^{-1}	inverse
$\operatorname{tr}(A)$	trace
$\det(A)$	determinant
$\lvert\mathbf{x}\rvert = \sqrt{\mathbf{x}^T\mathbf{x}}$	Euclidean norm
$\lVert A\rVert$	Lipschitz norm
$\lVert A\rVert_2 = \sqrt{\operatorname{tr}(A^T A)}$	Euclidean norm
Multivariable functions	
$\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$	functions from $E \subseteq \mathbf{R}^n$ to $\mathbf{R}^m$
$\mathbf{x} \mapsto \mathbf{f}(\mathbf{x})$	same as $\mathbf{f}$
$\mathbf{f}'$	derivative matrix function from open $E \subseteq \mathbf{R}^n$ to m by n matrices
$\mathbf{x} \mapsto \mathbf{f}'(\mathbf{x})$	same as $\mathbf{f}'$
$\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$	variables in $\mathbf{R}^n$
$\mathbf{y} = \mathbf{f}(\mathbf{x})$	$\mathbf{y}$ as a function $\mathbf{f}(\mathbf{x})$ of $\mathbf{x}$
$y_i = f_i(\mathbf{x})$	y_i as a function $f_i(\mathbf{x})$ of $\mathbf{x}$
$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \mathbf{f}'(\mathbf{x})$	derivative matrix (Jacobian matrix)
$\frac{\partial y_i}{\partial x_j} = \frac{\partial f_i(\mathbf{x})}{\partial x_j} = f'_{i,j}(\mathbf{x})$.	entry of derivative matrix
$d\mathbf{x} = dx_1 \cdots dx_n$	product of differentials
$\frac{d\mathbf{y}}{d\mathbf{x}} = \det \frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \det \mathbf{f}'(\mathbf{x})$	determinant of derivative matrix (Jacobian determinant)
$\mathbf{g} \circ \mathbf{f}$	composite function
$(\mathbf{g} \circ \mathbf{f})(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x}))$	composite function of $\mathbf{x}$
$(\mathbf{g} \circ \mathbf{f})' = (\mathbf{g} \circ \mathbf{f})\mathbf{f}'$	chain rule
$(\mathbf{g} \circ \mathbf{f})'(\mathbf{x}) = \mathbf{g}'(\mathbf{f}(\mathbf{x}))\mathbf{f}'(\mathbf{x})$	chain rule as a function of $\mathbf{x}$
$\mathbf{p} = \mathbf{g}(\mathbf{u}), \mathbf{u} = \mathbf{f}(\mathbf{x})$	composite function expressed with variables
$\frac{\partial \mathbf{p}}{\partial \mathbf{x}} = \frac{\partial \mathbf{p}}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$	chain rule expressed with variables
f, g, h	functions from $E \subseteq \mathbf{R}^n$ to $\mathbf{R}$
$f_{,i\,j}(\mathbf{x})$	entries of Hessian matrix of second derivatives
Integration	
I, $m(I)$	cell, volume of cell
$\mathcal{P}$	partition into cells
f_I	restriction
$L(f, \mathcal{P})$, $U(f, \mathcal{P})$	lower sum, upper sum
$L(f)$, $U(f)$, $I(f)$	lower integral, upper integral, integral
1_A	indicator function of subset A
$m(A) = I(1_A)$	content (volume) of A
$\operatorname{int}(A)$	interior of subset A

$\mathrm{bdy}(A)$	boundary of subset A
$\mathrm{osc}_A(f)$	oscillation on set A
$\mathrm{osc}_{\mathbf{x}}(f)$	oscillation at point $\mathbf{x}$
$\mathrm{Disc}(f)$	discontinuity set $\{\mathbf{x} \mid \mathrm{osc}_{\mathbf{x}}(f) > 0\}$
$\delta_\epsilon(\mathbf{x})$	family of approximate delta functions
$v_n = \frac{1}{n}\frac{2\pi^{n/2}}{\Gamma(n/2)}$	volume coefficient
$B_n(\mathbf{a}, r)$	open n ball of volume $v_n r^n$
$a_n = nv_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$	area coefficient
$S_{n-1}(\mathbf{a}, r)$	$n-1$ sphere of area $a_{n-1}r^{n-1}$

Vector Fields

$\mathbf{x}, \mathbf{y}, \mathbf{u}$	coordinate systems
$s = h(\mathbf{x})$	scalar field
$X = \sum_{j=1}^n a_j \frac{\partial}{\partial x_j}$	vector field

Manifold mappings

$\phi = (\mathbf{x} \leftarrow \mathbf{g}(\mathbf{u}))$	manifold mapping (parametrized surface)
$\phi^* s = x \circ \phi$	pullback of a scalar field
$\phi^* h(\mathbf{x}) = h(\mathbf{g}(\mathbf{u}))$	pullback of a scalar field
$X_{[\phi]}$	restriction of a vector field
$\phi_* Y$	pushforward of a vector field
$(\phi_* Y)s = Y(s \circ \phi)$	pushforward acting on scalar field
$\phi_* \frac{\partial}{\partial u_\alpha} = \sum_{i=1}^k g'_{i,\alpha}(\mathbf{u}) \frac{\partial}{\partial x_i}[\phi]$	pushforward of a basis vector field
$X^i_\alpha = \frac{\partial (x_i \circ \phi)}{\partial u_\alpha} = g'_{i,\alpha}(\mathbf{u})$	components of αth tangent vector
$X_\alpha = \sum_{i=1}^n \frac{\partial (x_i \circ \phi)}{\partial u_\alpha} \frac{\partial}{\partial x_i}[\phi]$	αth tangent vector

Differential Forms

$ds = \sum_{i=1}^n \frac{\partial s}{\partial x_i}\, dx_i$	differential of a scalar (an exact 1-form)
$\omega = \sum_{i=1}^n p_i\, dx_i$	differential 1-form
$\langle \omega \mid X \rangle = X \lrcorner\, \omega = \sum_{i=1}^n p_i a_i$	scalar field from form and vector field
θ	differential k-form
$\langle \theta \mid X_1, \ldots, X_k \rangle$	scalar field from form and vectors
$X \lrcorner\, \theta$	interior product (a $(k-1)$-form)
$\theta \wedge \beta$	exterior product of k-form with ℓ-form
$d\theta$	exterior derivative of θ (a $(k+1)$-form)
$X \lrcorner\, ds$	Lie derivative of a acalar field s
$d(X \lrcorner\, \sigma)$	Lie derivative of n-form σ
$\phi^*\, dx_i = \sum_{\alpha=1}^k g'_{i,\alpha}(\mathbf{u})\, du_\alpha$	pullback of a basis differential
$\phi^* \theta$	pullback of a differential k-form
$\omega_{[\phi]}$	restriction of a differential form
χ	chain
$\partial\chi$	boundary of chain
$\int_\chi \theta$	integral of form over chain
S	surface

∂S	boundary of surface
Volume	
vol	volume form
$\operatorname{div}(X)\text{vol} = d(X \lrcorner \text{vol})$	divergence
$\operatorname{div}(sX)\text{vol} = d(X \lrcorner s\text{vol})$	Lie derivative of s vol
$\mathcal{O}$	orientation
$[\sigma]$	orientation determined by σ
$(\alpha : \mathcal{O})$	twisted differential form
$\alpha\|_\eta = (\alpha : [\eta \wedge \alpha])$	twisted differential form (reoriented)
$\sigma\|_+ = (\sigma : [\sigma])$	top-dimensional twisted form
$\text{vol}\|_+$	volume element
$*1 = \text{vol}\|_+$	Hodge dual
$*\text{vol}\|_+ = 1$	Hodge dual
$\mathsf{c}X = X \lrcorner \text{vol}\|_+$	contraction with volume
$\operatorname{div}(X)\text{vol}\|_+ = d(X \lrcorner \text{vol}\|_+)$	divergence
$\operatorname{div}(X) = *d\mathsf{c}X$	divergence
The Metric Tensor	
$\mathsf{g} = \sum_{i=1}^n \sum_{j=1}^n g_{ij}\, dx_i\, dx_j$	metric tensor
g_{ij}	matrix entries of metric tensor (inner product on vectors)
G	matrix of metric tensor
g^{ij}	matrix entries of inverse matrix (inner product on forms)
G^{-1}	inverse of matrix tensor matrix
$\sqrt{g} = \sqrt{\det G}$	volume factor
$\text{vol} = \sqrt{g}\, dx_1 \cdots dx_n$	volume form
$\mathsf{g}^* = \sum_{\alpha=1}^k \sum_{\beta=1}^k g^*_{\alpha\beta}\, du_\alpha\, du_\beta$	metric tensor on surface
$g^*_{\alpha\beta} = X_\alpha^T G X_\beta$	matrix entries of surface metric tensor
X	matrix of tangent vector components
$G^* = X^T G X$	Gram matrix of surface metric tensor
$\sqrt{g^*} = \sqrt{\det G^*}$	area factor
$\text{area} = \sqrt{g^*}\, du_1 \cdots du_k$	surface area form
element	hypersurface element determined by ϕ and vol
$\phi^*(X \lrcorner \text{vol}) = X_{[\phi]} \lrcorner \text{element}$	flux through a hypersurface
$\text{area} = \lvert\text{element}\rvert$	hypersurface area form
Vector Analysis	
$X \cdot Y = \mathsf{g}X\, Y = X \lrcorner \mathsf{g}Y$	scalar product
$X \times Y = \mathsf{c}^{-1}(\mathsf{g}X \wedge \mathsf{g}Y)$	vector product (3d)
$\Gamma^k_{ij} = \Gamma^k_{ji}$	Christoffel coefficient
$\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x_k}$	covariant derivative (vector field)
$\frac{\partial}{\partial x_i}\, dx_j = -\sum_k \Gamma^j_{ik}\, dx_k$	covariant derivative (differential form)
$\nabla = \sum_i dx_i \frac{\partial}{\partial x_i}$	covariant differential
$\overrightarrow{\nabla} = \mathsf{g}^{-1}\nabla$	vector of covariant derivatives

$\text{grad } s = \mathsf{g}^{-1} ds$	gradient
$\text{grad } s = \overrightarrow{\nabla} s$	gradient
$\text{div } X = *d\mathsf{c}X$	divergence
$\text{div } X = \nabla \rfloor X$	divergence (from contracted covariant differential)
$\text{div } X = \overrightarrow{\nabla} \cdot X$	divergence
$\nabla^2 s = \text{div grad } s$	Laplacian
$\nabla^2 s = *d\mathsf{c}\mathsf{g}^{-1} ds = *d*ds$	Laplacian
$\text{curl } X = \mathsf{c}^{-1} d\mathsf{g}X$	curl (3d)
$\text{curl } X = \overrightarrow{\nabla} \times X$	curl (3d)
h_j	normalization factor for orthogonal coordinates
$C^k_{ij} = -C^j_{ik}$	coefficient for orthogonal coordinates
$\frac{\partial}{\partial x_i}\left(\frac{1}{h_j}\frac{\partial}{\partial x_j}\right) = \sum_k C^k_{ij}\left(\frac{1}{h_k}\frac{\partial}{\partial x_k}\right)$	covariant derivative (vector field)
$\frac{\partial}{\partial x_i}(h_j\, dx_j) = \sum_k C^k_{ij}(h_k\, dx_k)$	covariant derivative (differential form)

Hodge calculus

$\wedge$	exterior product
$\rfloor$	metric interior product
d	exterior differential
$*$	Hodge dual (Hodge star) of k-form
$*^{-1} = (-1)^{k(n-k)}*$	inverse of Hodge dual (Hodge star) of k-form
$\delta = (-1)^{k-1} *^{-1} d*$	codifferential (negative of adjoint)
$\Delta = (d+\delta)^2 = d\delta + \delta d$	Laplace-de Rham operator on p-forms
m	left multiplication with $\wedge$
j	left tensor representation with $\otimes$
t	metric trace with $\rfloor$ (with $tj = 0$)
D	exterior covariant differential on $(1,p)$-forms
$\nabla = jd + Dj$	covariant differential on p-forms
$\nabla^\bullet = tD + dt$	contracted covariant differential on $(1,p)$-forms
$\nabla^\bullet\nabla$	Laplace-Beltrami operator on p=forms
$d = m\nabla$	exterior differential expressed with ∇
$\delta = t\nabla = \nabla^\bullet j = tDj$	codifferential expressed with $\nabla, \nabla^\bullet, D$
$R = D^2$	curvature operator
$W = tRj$	Weitzenböch operator
$\nabla^\bullet\nabla = \Delta + W$	Weitzennböch theorem

12.2 Antiderivation formulas

- $X \lrcorner (X \lrcorner \alpha) = 0$.
- $X \lrcorner (\alpha \wedge \beta) = (X \lrcorner \alpha) \wedge \beta + (-1)^k \alpha \wedge (X \lrcorner \beta)$.
- $dd\alpha = 0$.
- $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$.

Recall that $\operatorname{div}(X) \operatorname{vol} = d(X \lrcorner \operatorname{vol})$. Also, for α an $(n-2)$-form, $\operatorname{curl}' \alpha$ is a vector field satisfying $\operatorname{curl}' \alpha \lrcorner \operatorname{vol} = d\alpha$.

- $\operatorname{div} \operatorname{curl}' \alpha = 0$.
- $\operatorname{div}(sX) = X \lrcorner ds + \operatorname{div}(X) s$.

12.3 Picturing differential forms

- A k-form β may determine an orientation $[\beta]$ of a k-surface.
- A ℓ-form ϵ is "transverse" to a k-surface if $X \lrcorner \epsilon = 0$ for each tangent vector X to the surface.
- A $(n-k)$-form ϵ (transverse to the surface) may determine a transverse orientation $[\epsilon]$ of a k-surface.
- A differential k-form α is integrated over an oriented k-surface.
- A twisted differential k-form $(\alpha : \mathcal{O})$ is integrated over a transversally oriented k-surface.
- A simple differential k-form α may be pictured by a family of $(n-k)$-surfaces (possibly with boundaries) transversally oriented by $[\alpha]$.
- A twisted simple differential k-form $(\alpha : \mathcal{O})$ may be pictured by the same family of $(n-k)$ surfaces (possibly with boundaries), oriented by $[\eta]$, where η is a $(n-k)$-form with $[\eta \wedge \alpha] = \mathcal{O}$.
- The twisted simple differential k-form may be denoted $(\alpha : \mathcal{O}) = \alpha|_\eta$, where η is the "reorienting" $(n-k)$-form.
- A differential $(n-1)$-form α is represented by "circulatory flux curves" (possibly with endpoints) with $(n-1)$-dimensional transverse orientations $[\alpha]$.
- A twisted differential $(n-1)$ form $\alpha|_\eta$ is represented by "transport flux curves" (possibly with endpoints) with 1-dimensional orientations $[\eta]$.
- A differential n-form σ is represented by a cloud of points with n-dimensional transverse orientations $[\sigma]$.
- A twisted differential n-form $\sigma|_s$ is represented by a cloud of points with 0-dimensional orientations $[s] = \pm 1$.
- A vector field X is pictured by oriented segments or oriented integral curves.
- A twisted vector field $(X : \mathcal{O}) = X|_\eta$ is pictured by segments or integral curves with transverse $(n-1)$ -orientation $[\eta]$. .

12.4 Dimensions and Units

This appendix describes the notion of dimension as applied to various objects that occur in calculus. The conventions used here may seem unusual, so it is work recording them explicitly.

Physical quantities often have *dimensions* such as length L, time T, and mass M. They are expressed in terms of corresponding units, perhaps meters, seconds, and kilograms. This account describes how dimensions apply in the Euclidean space context, with particular attention on the length dimension L. The length dimension comes from the metric g. That is, a quantity has dimension L^k if it scales by this factor when g is scaled by L^2.

A displacement vector by itself has nothing to do with the metric, so it has dimension 1. Similarly, a velocity vector has dimension T^{-1}, while an acceleration vector has dimension T^{-2}. On the other hand, the length of a displacement vector has dimension L, and velocity and accceleration vectors have length LT^{-1} and LT^{-2}.

Consider, for example, a displacement vector in Eucliden space. This may be written as

$$X = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}. \tag{12.1}$$

Each basis vector $\frac{\partial}{\partial x_j}$ is a vector of length one in the appropriate units, so it must have dimensions L^{-1}. The coefficients a_j then have dimension L. The length of X is $\sqrt{\sum_j a_j^2}$ and also has dimension L.

For a velocity vector

$$V = \sum_{j=1}^{n} v_j \frac{\partial}{\partial x_j}. \tag{12.2}$$

with dimension T^{-1} the coefficients v_j have dimension LT^{-1}. The length of V also has dimension LT^{-1}.

Similarly, a differential form

$$\omega = \sum_{k=1}^{n} p_k \, dx_k \tag{12.3}$$

may have various dimensions, such as mass or temperature. However its length dimension is 1. The unit basis vectors dx_k have dimension L, and so the coefficients p_k have dimensions of mass or temperature or whatever, multiplied by L^{-1}.

All these calculations depend on the fact that the metric tensor

$$\mathsf{g} = \sum_{i=1}^{n} dx_i^2 \tag{12.4}$$

has dimension L^2. The inverse g^{-1} of the metric tensor, which is used to measure the size of forms, then has dimension L^{-2}.

Similar considerations apply to multilinear forms. The volume form

$$\text{vol} = dx_1 \cdots dx_n \tag{12.5}$$

has dimension L^n. A form

$$\gamma = \rho\, dx_1 \cdots dx_n \tag{12.6}$$

might have dimension M. Then the density ρ has dimension ML^{-n}.

Various operations on vector fields and differential forms have dimensions. The exterior product $\wedge$ is dimensionless. The exterior derivative d is also dimensionless. (However the action of the exterior derivative on coefficients has dimension L^{-1}, since it differentiates them.) Similarly the interior product $\lrcorner$ of a vector field with a k-form is also dimensionless. Integration of forms is also dimensionless.

Example: Consider the transport flux $V \lrcorner \gamma$ and the source $d(V \lrcorner \gamma)$. These both have dimension MT^{-1}. (The dimensions of the coefficients are $ML^{-(n-1)}T^{-1}$ and $ML^{-n}T$.) The integral of the former over a boundary is the flux through the boundary, while the integral of the source over the region is the production within the region. These two quantities are equal, both with dimension MT^{-1}. ||

The map g has dimension L^2. The map c is the interior product with the volume, so it has dimensions L^n. It follows from the definitions that grad $= \mathsf{g}^{-1}d$ has dimensions L^{-2} and div $= *d\mathsf{c}^{-1}$ is dimensionless. (On the other hand, in their action on components they just differentiate, so the dimensions in this sense are L^{-1} and L^{-1}.) The Laplacian $\nabla^2 =$ div grad has dimension L^{-2}. In the case $n = 3$ the operator curl $= \mathsf{c}d\,\mathsf{g}$ has dimension L^{-1}.

The Hodge dual acting on forms has dimension L^{n-2k}. As a consequence, the codifferential has dimension L^{-2}. The Laplacian on forms then has dimension L^{-2}.

The g metric interior product of a k-vector acting on an ℓ-vector (with $k \leq \ell$) has dimension L^{2k}. The Hodge dual operation acting on k-vectors has dimension L^{2k-n}. The g^{-1} metric interior product of a k-form acting on an ℓ-form (with $k \leq \ell$) has dimension L^{-2k}. The Hodge dual operation acting on k-vectors has dimension L^{n-2k}.

In the classical vector algebra for $n = 3$ the scalar product has dimension L^2 and the cross product has dimension L. For example, suppose that $a\partial/\partial x$ and $b\partial/\partial y$ are dimensionless vectors. Then $a\partial/\partial x \times b\partial/\partial y$ is equal to $ab\partial/\partial z$. This resulting vector has dimension L. This is a clue that it should be promoted from a one-dimensional vector to a two-dimensional pseudovector. In fact, it is most natural to consider it as $ab\partial/\partial x \wedge \partial/\partial y$ which is dimensionless (with coefficient ab having dimension L^2).

A more detailed analysis of vector algebra for $n = 3$ considers products involving vectors with vectors, vectors with pseudovectors, and pseudovectors with pseudovectors. The scalar product in these three cases has dimensions L^2, 1, and L^4. The vector product in these three cases has dimensions 1, L^2, and L^2.

In Euclidean space calculations the usual normalized basis vectors have dimension L^{-1}. Other kinds of basis vectors can occur. An example is angular velocity $\omega \frac{\partial}{\partial \theta}$, where ω has dimension T^{-1} and $\frac{\partial}{\partial \theta}$ is dimensionless.

The subject of quantities with physical dimensions touches on various areas of science and engineering. What happens can be quite complicated, especially when the problems involve several variables. There is an book by Hart [19] that poses the problem of finding which operations of matrix algebra are allowed when the entries have dimensions. The astonishing answer is that almost none are allowed. For arbitrary n by n matrices most familiar algebraic operations (products, determinants, eigenvalues, and so on) are not defined. There are certain very special classes of dimensional structures for which these operations make sense, and only these special forms occur in science and engineering applications.

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