Bilinear Forms over a field F

Let V be a vector space. A *bilinear form* on V is a set map $B : V \times V \longrightarrow F$ which is linear in each slot. This means that for $\lambda \in F$ and $x, x', y, y' \in F$ we have

$$\begin{aligned}\lambda B(x,y) &= B(\lambda x, y) = B(x, \lambda y),\\ B(x+x', y) &= B(x,y) + B(x', y)\\ B(x, y+y') &= B(x,y) + B(x,y').\end{aligned}$$

We call B *symmetric* if $B(x,y) = B(y,x)$ and *alternating* if $B(x,y) = -B(y,x)$.

The most famous symmetric bilinear form is the dot product $B(x,y) = x \cdot y$ on F^n. Another symmetric bilinear form is the "Lorentz metric" on $\mathbb{R}^4$: $B(x,y) = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$. The most famous alternating bilinear form is the cross product $B(x,y) = x \times y = x_1y_2 - x_2y_1$ on F^2. Every $n \times n$ matrix $A = (a_{ij})$ gives rise to a bilinear form on the vector space F^n of column vectors by the formula $B(x,y) = x^tAy = x \cdot Ay$.

A variation of this definition is often used for vector spaces over $\mathbb{C}$. Let λ^* denote the complex conjugate of $\lambda \in \mathbb{C}$. We call a set map $B : V \times V \longrightarrow F$ *sesquilinear* if it is linear in the second slot and anti-linear in the first; that is we replace the scalar condition for a bilinear form by the condition

$$\lambda B(x,y) = B(\lambda^* x, y) = B(x, \lambda y)$$

We call B *hermitian* if $B(x,y) = B(y,x)^*$. The most famous hermitian form on $\mathbb{C}^n$ is given by $x^* \cdot y = \sum x_i^* y_i$.

Any square matrix A gives rise to a sesquilinear form: $B(x,y) = x^* \cdot Ay$. If A is a *hermitian* matrix (a matrix with $A^t = A^*$) then this is a hermitian form. Our discussion will concentrate on bilinear forms, because the sesquilinar/hermitian cases are all proven the same way (with conjugation thrown in where needed).

Proposition. *If* $\dim(V) = n$*, there is a 1-1 correspondence between bilinear forms and* $n \times n$ *matrices. The symmetric and alternating bilinear forms correspond to symmetric and alternating matrices.*

There is also a 1-1 correspondence between sesquilinear forms and $n \times n$ *matrices, in which the hermitian forms correspond to hermitian matrices.*

To make this correspondence, choose a basis $e_1 \ldots, e_n$ for V. The matrix $A = (a_{ij})$ associated to a bilinear form B has $a_{ij} = B(e_i, e_j)$. The formula $B(x,y) = x^tAy$ follows from bilinearity of B.

Change of basis. A change of basis for V is carried out by an invertible matrix P. Writing $x = Px_0, y = Py_0$ we see that $x^tAy = x_0^t(P^tAP)y_0$. Thus the change of basis replaces the matrix A by the matrix P^tAP.

Warning: the use of A to describe a linear transformation and a bilinear form result in two distinct equivalence relations on matrices: A is *similar* to $P^{-1}AP$ as a linear transformation, and is *equivalent* to P^tAP as a bilinear form (or to $P^{*t}AP$ as a sesquilinear form.)

We call a form B *non-degenerate* if its corresponding matrix A has a nonzero determinant. Note that $\det(A)$ is only well-defined up to a square, since $\det(P^tAP) = \det(A)\det(P)^2$; $\det(A)$ is called the *discriminant* of B. If B is nondegenerate, the discriminant is well-defined in F^*/F^{*2}.