

# The First Fundamental Form

## Outline

### 1. The First Fundamental Form

Let  $S$  be a regular surface, and let  $\vec{X}: U \rightarrow S$  be a regular parameterization of an open subset of  $S$ . The **first fundamental form** of  $S$  with respect to  $\vec{X}$  is the matrix

$$g = (d\vec{X})^T(d\vec{X})$$

Since  $d\vec{X}$  depends on  $u$  and  $v$ , the matrix  $g$  also depends on  $u$  and  $v$ . The entries of  $g$  are as follows:

$$g = \begin{bmatrix} \vec{X}_u \cdot \vec{X}_u & \vec{X}_u \cdot \vec{X}_v \\ \vec{X}_v \cdot \vec{X}_u & \vec{X}_v \cdot \vec{X}_v \end{bmatrix}.$$

### 2. Dot Products and Lengths

We can use the first fundamental form to take the dot product of tangent vectors to the surface. If  $\vec{a}_1$  and  $\vec{a}_2$  are vectors at the same point in the  $uv$ -plane, and  $\vec{t}_1 = d\vec{X} \vec{a}_1$  and  $\vec{t}_2 = d\vec{X} \vec{a}_2$  are the corresponding tangent vectors to the surface, then

$$\vec{t}_1 \cdot \vec{t}_2 = (\vec{a}_1)^T g \vec{a}_2.$$

We can also use  $g$  to find lengths of tangent vectors to the surface. If  $\vec{a}$  is a vector based at a point in the  $uv$ -plane and  $\vec{t} = d\vec{X} \vec{a}$  is the corresponding tangent vector to the surface, then

$$\|\vec{t}\| = \sqrt{\vec{a}^T g \vec{a}}.$$

### 3. Lengths of Curves

If  $\vec{x}(t)$  (for  $a \leq t \leq b$ ) is a curve in the  $uv$ -plane, and  $\vec{y}(t) = \vec{X}(\vec{x}(t))$  is the corresponding curve on the surface, then the length of  $\vec{y}$  is

$$\int_a^b \sqrt{\vec{x}'(t)^T g(\vec{x}(t)) \vec{x}'(t)} dt.$$

Stated differently, the length of a curve on the surface is  $\int ds$ , where

$$ds = \sqrt{\begin{bmatrix} du & dv \end{bmatrix} g \begin{bmatrix} du \\ dv \end{bmatrix}}.$$

This quantity  $ds$  is known as the **length element** (or **line element**). It can also be written in terms of the entries of the matrix  $g$ :

$$\text{If } g = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \quad \text{then} \quad ds = \sqrt{a du^2 + 2b du dv + c dv^2}.$$