Tangent bundle, vector bundles and vector fields

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1 The Tangent bundle and vector bundle

The aim of this section is to introduce the tangent bundle TX for a differential manifold X. Intuitively this is the object we get by gluing at each point $p \in X$ the corresponding tangent space T_pX. The differentiable structure on X induces a differentiable structure on TX making it into a differentiable manifold of dimension $2\dim(X)$. The tangent bundle TX is the most important example of what is called a **vector bundle** over X(see the definition below).

1.1. Review of the tangent space T_pX:

Let X be a smooth differential manifold of dimension m and let $p \in X$. The tangent space T_pX is a collection of tangent vectors v_p to X at the point p. A tangent vector v_p is a map $v_p : C^\infty(X) \to \mathbf{R}$ such that (i) $v_p(af + bg) = av_p(f) + bv_p(g)$, (ii) $v_p(fg) = f(p)v_p(g) + g(p)v_p(f)$. Let (U, ϕ) a local coordinate for X at p, let $\mathbf{R}^m = \mathbf{R}^m_{u^1,\dots,u^m}$ and write $\phi = (x^1, \dots, x^m)$. Then we have special tangent vectors $\{\frac{\partial}{\partial x^k} |_p, 1 \le k \le m\}$ (called the **partial derivatives**)

$$\frac{\partial}{\partial x^k} |_p: C^\infty(X) \to \mathbf{R}$$

defined by

$$\frac{\partial}{\partial x^k} |_p (f) = \frac{\partial (f \circ \phi^{-1})}{\partial u^k} |_{\phi(p} .$$

Then $\{\frac{\partial}{\partial x^k} |_p, 1 \le k \le m\}$ forms a basis for T_pX, moreover(from the proof), for every $v_p \in T_pX$, we can write

$$v_p = \sum_{k=1}^m v_p(x^k) \frac{\partial}{\partial x^k} |_p .$$

1.2. Construction of the tangent bundle TX:

Let X be a smooth differential manifold of dimension m. Let

$$TX = \cup_{p\in X} T_pX = \{(p, v) \mid p \in X, v \in T_pX\}.$$

Let $\pi : TX \to X$ be the natural projection map with $\pi : (p, v) \mapsto p$. For a given point $p \in X$ the fiber $\pi^{-1}(\{p\})$ of π is the m-dimensional tangent space T_pX at p. The triple (TX, X, π) is called the **tangent bundle** of X. We can put a differentiable structure on TX making it into a differentiable manifold of dimension $2\dim(X)$ as follows:

Let X be a differential manifold with maximal atlas $\mathcal{A}$. Let $x : U \to \mathbf{R}^m$ in $\mathcal{A}$ be a chart for X and define

$$\tilde{\psi}_U : \quad \pi^{-1}(U) \to \mathbf{R}^m \times \mathbf{R}^m$$

by

$$\tilde{\psi}_U : \quad (p, \ \sum_{k=1}^{m} a_k \frac{\partial}{\partial x^k} \mid_p) \mapsto (x(p), (a_1, \ldots, a_m)).$$

Then it is easy to check that $\tilde{\psi}_U$ is one-to one and $\tilde{\psi}_U(\pi^{-1}(U))$ is an open set in $\mathbf{R}^m \times \mathbf{R}^m$. We now check that overlap(transition) maps are smooth maps. In fact, Let (U, x) and (V, y) be two charts in $\mathcal{A}$ such that $p \in U \cap V$. Then the overlap(transition) map

$$\tilde{\psi}_V \circ (\tilde{\psi}_U)^{-1} : \quad \tilde{\psi}_U(\pi^{-1}(U \cap V)) \to \mathbf{R}^m \times \mathbf{R}^m$$

is given by

$$(a, b) \mapsto (y \circ x^{-1}(a), \quad \sum_{k=1}^{m} \frac{\partial y_1}{\partial x^k} \mid_{x^{-1}(a)} b_k, \ldots, \sum_{k=1}^{m} \frac{\partial y_m}{\partial x^k} \mid_{x^{-1}(a)} b_k).$$

Since X is a smooth manifold, $y \circ x^{-1}$ is smooth, hence $\tilde{\psi}_V \circ (\tilde{\psi}_U)^{-1}$ is also smooth. Let

$$\mathcal{A}^* = \{(\pi^{-1}(U), \tilde{\psi}_U) \mid (U, x) \in \mathcal{A}\},$$

then $\mathcal{A}^*$ is a C^∞ atlas. So TX is an $2m$ smooth manifold. It is trivial that the projection map $\pi : TX \to X$ is also smooth.

1.3. The tangent bundle, cotangent bundle and the definition of general vector bundle.

For each point $p \in X$ the fiber $\pi^{-1}(\{p\})$ is the tangent space T_pX of X at p hence an m-dimensional vector space. For a chart $x : U \to \mathbf{R}^m$ is $\mathcal{A}$, we define $\psi_U : \quad \pi^{-1}(U) \to U \times \mathbf{R}^m$ by

$$\psi_U : \quad (p, \ \sum_{k=1}^{m} a_k \frac{\partial}{\partial x^k} \ |_p) \mapsto (p, (a_1, \ldots, a_m)).$$

Obviously ψ_U is a diffeomorphism. Further more, it has the following important property: the restriction of ψ_U to the tangent space T_pX, i.e. $\psi_p = \psi_U|_{T_pX} : T_pX \to \{p\} \times \mathbf{R}^m$ is given by

$$\psi_p : \ \sum_{k=1}^{m} a_k \frac{\partial}{\partial x^k} \ |_p \mapsto (a_1, \ldots, a_m),$$

so it is **a vector space isomorphism**. The map $\psi_U : \pi^{-1}(U) \to U \times \mathbf{R}^m$ is called **a bundle chart**.

In summary: For a smooth manifold X, we get a triple (TX, X, π), which is called the tangent bundle of X, where π is a continuous surjective map(natural projection), TX is a smooth differential manifold of dimension $2\dim(X)$. Further, it satisfies the following property:

(i) For each $p \in X$, the fiber $\pi^{-1}(\{p\}) = T_p(X)$ is an m-dimensional vector space.

(ii) For each $p \in X$ there exists a **bundle chart** $(\pi^{-1}(U), \psi_U)$ (some book called it *trivialization*) such that $\psi_U : \pi^{-1}(U) \to U \times \mathbf{R}^m$ is a smooth diffeomorphism and for all $q \in U$, the map $\psi_q = \psi_U|_{T_q(X)} : T_q(X) \to \{q\} \times \mathbf{R}^m$ is a vector space isomorphism.

A smooth map $v : X \to TX$ is called a **smooth vector field** (or **smooth section**) if $\pi \circ v(p) = p$ for each $p \in X$.

Finally, motivated by the above construction, we introduce the following general definition:

Definition 1: *Let E and X be smooth manifolds and $\pi : E \to X$ be a smooth surjective map. The triple (E, X, π) is called* **a (smooth) vector bundle of rank k over** X *if*

(i) For each $p \in X$, the fiber $E_p = \pi^{-1}(\{p\})$ is a k-dimensional vector space.

(ii) For each $p \in X$ there exists a **bundle chart** $(\pi^{-1}(U), \psi_U)$ *(some book called it trivialization) such that $\psi_U : \pi^{-1}(U) \to U \times \mathbf{R}^k$ is a smooth diffeomorphism and for all $q \in U$, the map $\psi_q = \psi_U|_{E_q} : E_q(X) \to \{q\} \times \mathbf{R}^k$ is a vector space isomorphism.*

Vector bundles of rank 1 is also called the **line bundle**.

The vector bundle of rank r over X is said to be **trivial** if there exists a global bundle chart $\psi : E \to X \times \mathbf{R}^k$.

Definition 2: *Let (E, X, π) be a vector bundle over X. A smooth map $\sigma : X \to E$ is said to be a* **smooth section** *of the bundle (E, X, π) if $\pi \circ \sigma(p) = p$ for every $p \in X$. The set of all smooth sections is denoted by $\Gamma(X, E)$ or just $\Gamma(E)$.*

Definition 3: *Let (E, X, π) be a vector bundle of rank k over X. Let $\{U_\alpha\}$ be an open covering of X and let $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbf{R}^k$ be the trivialization. Then, on $U_\alpha \cap U_\beta \neq \emptyset$, the composition map*

$$\psi_\alpha \circ \psi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbf{R}^k \to (U_\alpha \cap U_\beta) \times \mathbf{R}^k$$

is of the form, for every $p \in U_\alpha \cap U_\beta$ and $b \in \mathbf{R}^k$,

$$\psi_\alpha \circ \psi_\beta^{-1}(p, b) = (p,\ g_{\alpha\beta}(p)(b))$$

for some smooth map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbf{R})$ where $GL(k, \mathbf{R})$ is the set of $k \times k$ non-singular matrices. The smooth $GL(k, \mathbf{R})$-valued maps $\{g_{\alpha\beta}\}$ are called the **transition functions** *for a vector bundle E.*

Examples

1. Let $E = X \times \mathbf{R}^k$. Then E is a vector bundle of rank k. In this case, the trivialization map is an identity map. This bundle is called the trivial bundle.

2. Let $E = TX = \cup_{p\in X} T_p(X)$. It is called the tangent bundle, denoted by TX. The rank of this bundle is m (the dimension of TX as a manifold is $2m$), where $\dim X = m$. Let (U, ϕ_U) be a chart of X with coordinate functions $x^1, \ldots, x^m$. Then it defines a trivialization $\psi_U : \quad \pi^{-1}(U) \to U \times \mathbf{R}^m$ by

$$\psi_U : \quad (p, \ \sum_{k=1}^m a_k \frac{\partial}{\partial x^k} \mid_p) \mapsto (p, (a_1, \ldots, a_m)).$$

We now calculate the transition functions. Let$(U, \phi_U), (V, \phi_V)$ two charts on X, with coordinate functions $x^1, \ldots, x^m$ and $y^1, \ldots, y^m$ respectively, where $U \cap V \neq \emptyset$. For every $b = (b_1, \ldots, b_m) \in \mathbf{R}^m$, and $p \in U \cap V$,

$$\psi_V^{-1}(p, b) = (p, \ \sum_{i=1}^m b_i \frac{\partial}{\partial y^i} \mid_p).$$

Since, on $U \cap V$,

$$\frac{\partial}{\partial y^i} \mid_p = \sum_{j=1}^m \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j} \mid_p,$$

we conclude that, on $U \cap V$,

$$\psi_V^{-1}(p, b) = (p, \ \sum_{i=1}^m b_i \frac{\partial}{\partial y^i} \mid_p) = (p, \sum_{j=1}^m \sum_{i=1}^m b_i \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j} \mid_p).$$

Hence,

$$\psi_U \circ \psi_V^{-1}(p, b) = (p, \sum_{i=1}^m b_i \frac{\partial x^1}{\partial y^i}, \ldots, \sum_{i=1}^m b_i \frac{\partial x^m}{\partial y^i}).$$

This means the transition map g_{UV} is, for every $p \in U \cap V$,

$$g_{UV}(p) = \left(\frac{\partial x^i}{\partial y^j} \right)_{1 \le i,j \le m} \mid_{\phi_V(p)}.$$

3. Besides the tangent bundle TX above, we also have the cotangent bundle T^*X as follows: Consider a smooth manifold X of dimension m. The dual space to the tangent space T_pX, $p \in X$, is called the **cotangent space** to X at p, denoted by T_p^*X. Suppose that $x : U \to \mathbf{R}^m$ be a local coordinates for X at p, then $\{\frac{\partial}{\partial x^k} \mid_p, 1 \leq k \leq m\}$ forms a basis for T_pX, i.e. for every $v_p \in T_pX$. The dual basis to $\{\frac{\partial}{\partial x^k} \mid_p, 1 \leq k \leq m\}$ is traditionally denoted by $\{dx^k \mid_p, 1 \leq k \leq m\}$. (Sometimes, I may drop the subscript p from the notation.) Thus an arbitrary element of T_p^*X is expressed as $\sum_{k=1}^{m} a_k dx^k \mid_p$ for some $a_k \in \mathbf{R}$. The disjoint union of all the cotangent space

$$T^*X = \cup_{p \in X} T_p^*X$$

is called the **cotangent bundle** of X. The cotangent bundle can be given a smooth structure making it into a manifold of dimension $2\dim(X)$ by an argument very similar to the one for the tangent bundle as above). It is easy to verify that the transition functions for T^*X is

$$g_{UV}(p) = \left(\frac{\partial y^i}{\partial x^j}\right)_{1 \leq i,j \leq m} \mid_{\phi_U(p)}.$$

4. Tensor Bundles. Consider the $(r,s)-$type tensor space

$$T_s^r(p) = T_p(x) \otimes T_p(X) \otimes \otimes T_p^*(X) \otimes \cdots \otimes T_p(X)^*,$$

where the first products for V is taken r times, and the second products for V^* is taken s times. Let

$$T_s^r = \cup_{p \in X} T_s^r(p).$$

Then, similar above, we can show that T_s^r is a vector bundle, which is called an *(r,s)-type tensor bundle on X.*

5. Bundle Operations. Let (E, X, π) be a vector bundle, we can define, in an obvious way, the dual bundle (E^*, X, π), which is called the *dual* bundle. Similarly, let E, E' be two vector bundles, we can define $E \oplus E'$ and $E \otimes E'$ the *direct sum* and the *the tensor product* bundles.

2 Smooth Vector fields

Let X be a smooth manifold of dimension m. A **vector field** v on X is a section of the tangent bundle TX, ie $v : X \to TX$ such that $\pi \circ v(p) = p$ for every $p \in X$. In other words. A **vector field** on X is a map v which assigns to each point $p \in X$ a tangent vector

$$v(p) = v_p \in T_p(X).$$

Let $x : U \to \mathbf{R}^m$ be a local chart of X, and $p \in U$, then

$$v(p) = \sum_{i=1}^{m} v_p(x^i) \frac{\partial}{\partial x^i} \mid_p .$$

The real-valued functions $v^i : U \to \mathbf{R}$ defined by $v^i(p) = v_p(x^i), 1 \leq i \leq m$, are called the **components** of v related to the local chart $x : U \to \mathbf{R}^m$.

A **vector field** is said to be smooth if v is a smooth section tangent bundle TX, i.e. v is smooth as a map. It can be checked (we omit it here) that v is smooth if and only if its components are smooth for all charts in some atlas for X. Denote by $\Gamma(X)$ the set of all smooth vector fields on X. We have the following algebra structure on $\Gamma(X)$:

For $v, w \in \Gamma(X), a \in \mathbf{R}$ and $f \in C^\infty(X)$:

(i) $v + w \in \Gamma(X)$, i.e. $(v + w)(p) = v(p) + w(p)$;

(ii) $av \in \Gamma(X)$, i.e. $(av)(p) = av(p)$;

(iii) $fv \in \Gamma(X)$, i.e. $(fv)(p) = f(p)v(p)$.

There is **another way** of thinking about vector fields: Recall that, for every $p \in X$, the tangent vector v_p map $v_p : C^\infty(X) \to \mathbf{R}$ such that (i) $v_p(af + bg) = av_p(f) + bv_p(g)$, (ii) $v_p(fg) = f(p)v_p(g) + g(p)v_p(f)$. Since v assigns at each point $p \in X$ a tangent vector v_p, we can define, for each $f \in C^\infty(X)$, $v(f)$ as a function on X by $v(f)(p) = v_p(f)$. If the vector field is smooth, then $v(f)$ is also a smooth function on X for every $f \in C^\infty(X)$. Hence, we **can think of** a smooth vector field $v \in \Gamma(X)$ as **a map**

$$v : C^\infty(X) \to C^\infty(X)$$

by

$$f \mapsto v(f),$$

where $(vf)(p) = v_p(f)$.

The smooth vector field

$$v : C^\infty(X) \to C^\infty(X)$$

also satisfies the following properties: For every $f, g \in C^\infty(X)$,

(i) $v(af + bg) = av(f) + bv(g)$,

(ii) $v(fg) = fv(g) + gv(f)$.

An **exterior differential p-form** is a section of $\bigwedge^p T^*(X)$. In a local chart $(U, (x^1, \ldots, x^m))$, an exterior differential p-form

$$\omega = \sum_{1 \leq j_1 < \cdots < j_p \leq m} a_{j_1 \cdots < j_p} dx^{j_1} \wedge \cdots \wedge dx^{j_p},$$

where $a_{j_1 \cdots < j_p}$ are real valued functions on U.

$\Lambda^p(X)$ will denote the vector space of exterior differential p-forms.

$\Gamma(X)$ will denote the vector space of (smooth) vector fields.

$T_r^s(X)$ denote the vector space of (r, s)-tensor fields.

3 Lie bracket of smooth vector fields

Let v, w be two smooth vector fields (thought as maps from $C^\infty(X) \to C^\infty(X)$). Define the **Lie bracket** of v and w, denoted by

$$[v, w]: \ C^\infty(X) \to C^\infty(X),$$

by

$$[v, w](f) = v(w(f)) - w(v(f)).$$

The Lie bracket has the following properties:

(i) ($\mathbf{R}$-linearity): $[au + bw, v] = a[u, v] + b[w, v]$;

(ii) (skew-symmetry): $[v, w] = -[w, v]$;

(iii) (Jacobi-identity) $[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$;

(iv) for every $f, g \in C^\infty(X)$, $[fv, w] = f[v, w] - (w(f))v$, $[v, gw] = g[v, w] + (v(g))w$;

(v) In local coordinates, if $v = \sum_{i=1}^m v^i \frac{\partial}{\partial x^i}$, and $w = \sum_{i=1}^m w^i \frac{\partial}{\partial x^i}$, then

$$[v, w] = \sum_{i,j=1}^m \left(v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}.$$

4 Exterior Differentials

Recall that an exterior differential p-form is a section of $\bigwedge^p T^*(X)$. In a local chart $(U, (x^1, \ldots, x^m))$, an exterior differential p-form

$$\omega = \sum_{1 \leq j_1 < \cdots < j_p \leq m} a_{j_1 \cdots < j_p} dx^{j_1} \wedge \cdots \wedge dx^{j_p},$$

where $a_{j_1 \cdots < j_p}$ are real valued functions on U.

Let

$$\Lambda(X) = \bigoplus_{p=0}^{m} \Lambda^p(X).$$

It is called the *algebra of exterior differential forms*. Note that it has the structure of an algebra with respect to wedge product "$\wedge$".

We now define the *exterior differential* operator $d : \Lambda^p(X) \to \Lambda^{p+1}(X)$ as follows, for every $\omega \in \Lambda^p(X)$, and for every $X_1, \ldots, X_{p+1} \in \Gamma(X)$,

$$\begin{aligned} d\omega(X_1, \ldots, X_{p+1}) &= \sum_{i=1}^{p+1} X_i(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}). \end{aligned}$$

We also have a local expression, a local chart $(U, (x^1, \ldots, x^m))$, i.e. let

$$\omega = \sum_{1 \le j_1 < \cdots < j_p \le m} a_{j_1 \cdots < j_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_p}.$$

Then

$$d\omega = \sum_{1 \le j_1 < \cdots < j_p \le m} da_{j_1 \cdots < j_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_p}.$$

For this expression we see that, for any $\eta \in \Lambda^p(X)$,

$$d(\eta \wedge \xi) = d\eta \wedge \xi + (-1)^p \eta \wedge d\xi.$$

According to the definition, for $f \in \Lambda^0(X)$, we have $d(f)(X) = X(f)$, and for $\omega \in \Lambda^1(X)$, we have

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

Now let $F : M \to N$ be a smooth map, then we have $d(F^*\omega) = F^*(d\omega)$ for any differential form ω on N.

Note that we can define the **integration** of a k-form with compact support on a k-dimensional (orientable) manifold, and have the following statement of Stokes' formula: *Let*

M be a smooth compact manifold of dimension n with boundary, and ω is a $(n-1)$-form on M. Then

$$\int_M d\omega = \int_{\partial M} \omega.$$