1 Schur Decomposition

If matrix n by n matrix with complex entries, then

$$A = QUQ^{-1}$$

where Q is unitary matrix and U is upper triangle matrix

$$\textbf{If } \mathbf{A^*A = AA^*}$$

$$\textbf{Then } \mathbf{A = P^*\Lambda P}$$

$$\text{Proof}$$

$$\mathbf{A} \text{ is normal matrix}$$

$$\Rightarrow \mathbf{A = P^*\Delta P} \tag{1}$$

$$\Delta \text{ is upper triangle matrix } \textbf{(Schur decomposition)}$$

$$\mathbf{P} \text{ is unitary matrix}$$

$$\text{First we need to show if } \mathbf{A} \text{ is } \textbf{normal} \text{ then } \Delta \text{ is } \textbf{normal} \tag{2}$$

$$\mathbf{A = P^*\Delta P}$$

$$\Rightarrow \mathbf{A^* = (P^*\Delta P)^*}$$

$$\Rightarrow \mathbf{A^* = P^*(P^*\Delta)^*}$$

$$\Rightarrow \mathbf{A^* = P^*(\Delta^* P)} \tag{3}$$

$$\text{from (2) and (3)}$$

$$\Rightarrow \mathbf{AA^* = (P^*\Delta P)(P^*\Delta^* P)}$$

$$\Rightarrow \mathbf{AA^* = P^*\Delta(PP^*)\Delta^* P}$$

$$\mathbf{P} \text{ is unitary matrix}$$

$$\Rightarrow \mathbf{P^* = P^{-1}}$$

$$\Rightarrow \mathbf{AA^* = P^*\Delta\Delta^* P} \tag{4}$$

$$\Rightarrow \mathbf{A^*A = P^*\Delta^* PP^*\Delta P}$$

$$\Rightarrow \mathbf{A^*A = P^*\Delta^*\Delta P} \tag{5}$$

$$\textbf{From (4) and (5)}$$

$$\Rightarrow \mathbf{AA^* - A^*A = P^*\Delta\Delta^* P - P^*\Delta^*\Delta P}$$

$$\Rightarrow \mathbf{AA^* - AA^* = P^*(\Delta\Delta^* - \Delta^*\Delta)P = 0}$$

$$\Rightarrow \Delta\Delta^* - \Delta^*\Delta = \mathbf{0}$$

$$\Rightarrow \Delta\Delta^* = \Delta^*\Delta \tag{6}$$

$$\Rightarrow \Delta \textbf{ is normal matrix}$$

$$\textbf{From (6)}$$

$$\Rightarrow \vec{e_i}^*\Delta\Delta^*\vec{e_i} = \vec{e_i}^*\Delta^*\Delta\vec{e_i}$$

$$\Rightarrow \langle \Delta^*\vec{e_i}, \Delta^*\vec{e_i}\rangle = \langle \Delta\vec{e_i}, \Delta\vec{e_i}\rangle$$

$$\Rightarrow \|\Delta^*\vec{e_i}\|^2 = \|\Delta\vec{e_i}\|^2$$

$$\Rightarrow \|\Delta^*\vec{e_i}\| = \|\Delta\vec{e_i}\|$$

$$\Rightarrow \textbf{The length of ith column and ith row in } \Delta \textbf{ are same} \tag{7}$$

$$\Delta \textbf{ is upper triangle matrix}$$

$$\textbf{Let i to be the first row with nonzero off-diagonal element}$$

$$A_{n,n} = \begin{pmatrix} a_{1,1} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & a_{2,2} & \cdots & 0 & \cdots & 0 \\ 0 & 0 & a_{i,i} & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ . & . & . & \vdots & \vdots & a_{n,n} \end{pmatrix}$$

$$\textbf{If } \Delta \textbf{ is not diagonal matrix, then the ith column is } |a_{i,i}| \neq |a_{i,i}| + |*|$$

$$\textbf{This contracts our previous (7), therefore } \Delta \textbf{ must be diagonal matrix}$$

$$\textbf{Therefore } \mathbf{AA^* = A^*A \Rightarrow A = P^*\Lambda P} \quad \square$$