"Every activity worth doing has a learning curve ."

Seth Gordin

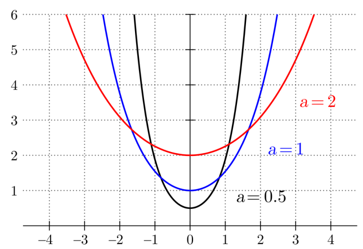
1

Curves

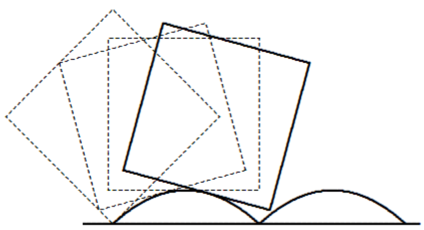
Natural Curves

Spiderweb segments dangle in the shape of catenary curves, exemplifying aspects of the general theory of curves presented in this handout. In physics and geometry, a catenary is the curve that an idealized hanging chain or cable assumes under its own weight when supported only at its ends.

The equation of a catenary has the form: $y = a\cosh\left(\frac{x}{a}\right) = \frac{a(e^{\frac{x}{a}} + e^{-\frac{x}{a}})}{2}$ All catenary curves are similar to each other. Changing the parameter a is equivalent to a uniform scaling of the curve.



The problem of the square wheel: what should be the shape of the road in order for a square wheel to roll smoothly?



Answer: A square wheel rolls smoothly on inverted catenaries.

Some Theoretical Background:

- in this handout we will focus more on space curves

Famous 3D Curves

• the parametric equations of a 3D curve are:

$$c:\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \quad , \qquad t \in I \subset \mathbb{R},$$

where $x(t), y(t), z(t)$ are functions of t.

• sometimes the parametric equations are given in the form:

$$c: \quad \boldsymbol{\bar{r}(t)} = x(t) \cdot \bar{i} + y(t) \cdot \bar{j} + z(t) \cdot \bar{k}$$

where $\bar{r}(t)$ is the position vector of an ordinary point $M(t)$ of the curve.

• we call a curve $c : I \to \mathbb{R}^3$ regular if $\boldsymbol{r'(t)} \neq 0, \quad \forall t \in I$.

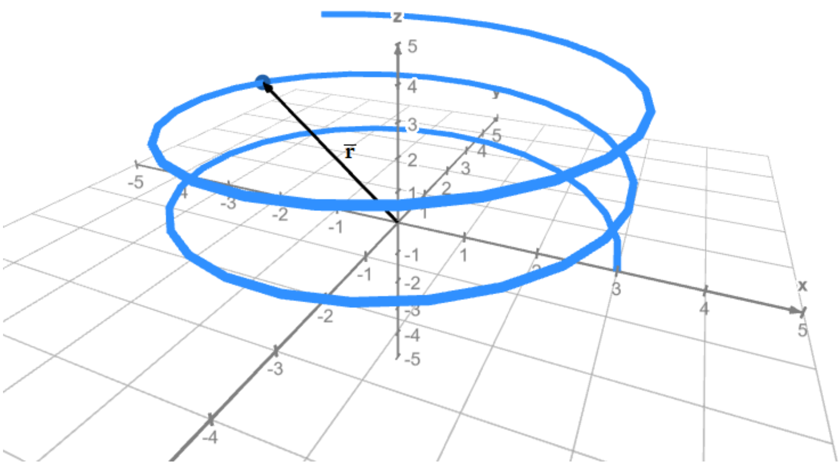
• a $3D$ curve can also be given as an intersection of surfaces (the implicit equations of a curve):

$$c: \begin{cases} F(x,y,z) = 0 \\ G(x,y,z) = 0 \end{cases}$$

where $F(x,y,z) = 0$ and $G(x,y,z) = 0$ are the implicit equations of the surfaces.

1) the hellix has the parametric equations:

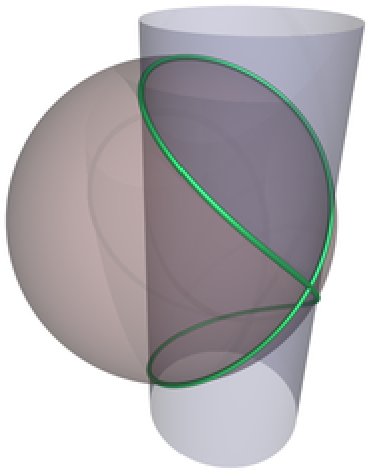
$$c:\begin{cases} x = a\cos t \\ y = a\sin t \\ z = bt \end{cases} \quad , \qquad a, b \ \text{ constants}$$

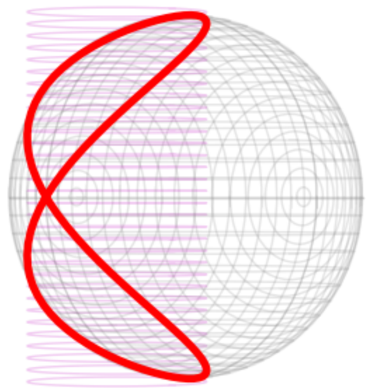


• It has a constant curvature and a constant torsion

• the helix obtained for $a = 3$ and $b = 1$ is drawn in the above figure.

2) Viviani's curve can be imagined as the intersection between a cylinder and a sphere. It looks like an eight symbol on a sphere:





If one considers the cylinder centered at $(a, 0, 0)$ of radius a:

$$(x-a)^2 + y^2 = a^2, \qquad (\ F(x,y,z) := (x-a)^2 + y^2 - a^2 \ )$$

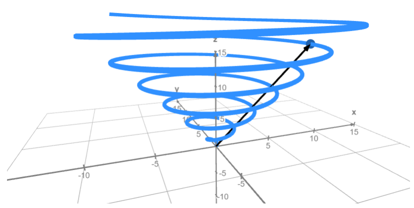
and the sphere:

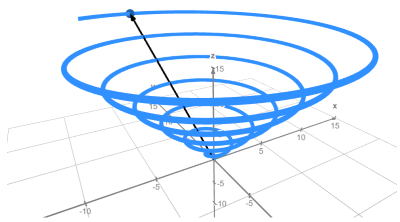
$$x^2 + y^2 + z^2 = 4a^2, \qquad (\ G(x,y,z) := x^2 + y^2 + z^2 - 4a^2 \ )$$

with center $(0, 0, 0)$ and radius $2a$, then their intersection will be Viviani's curve of parametric equations:

$$c : \begin{cases} x = a(1 + \cos t) \\ y = a \sin t \\ z = 2a \sin\left(\frac{t}{2}\right) \end{cases} \quad , \qquad a \text{ constant}$$

3) the conical helix is a three dimensional spiral:

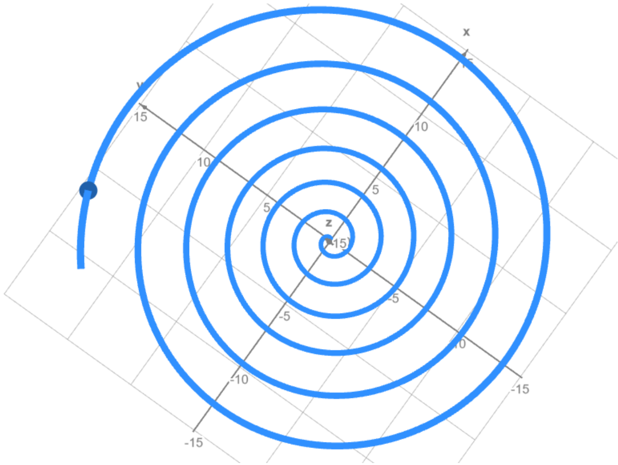




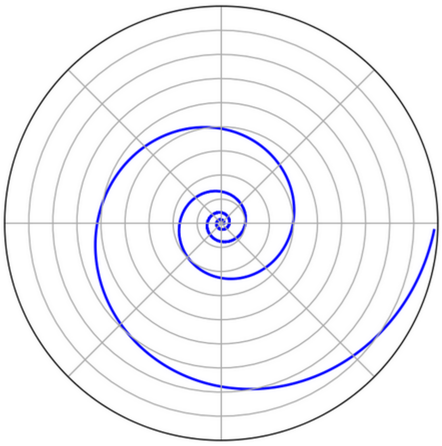
Possible parametric equations are:

$$c : \begin{cases} x = t\cos(at) \\ y = t\sin(at) \\ z = bt \end{cases} \quad , \qquad a, b, \text{ constants}$$

A view from above is given by:



In practice the most interesting 3D spirals are those called logarithmic spirals, which have a 2D projection of the following form:



More about spirals you can find here.

- possible parametric equations of a 3D logarithmic spiral are:

$$c: \begin{cases} x = a e^{bt} \cos t \\ y = a e^{bt} \sin t \\ y = ct \end{cases} \quad , \qquad a, b, c \text{ constants}$$

You can generate more 3D curves using this link

↳ after you choose the parametrization click on **Redraw Display**

↳ press and hold the left-click of the mouse to move the graph and get **different perspectives of the generated curve**

Curvature and Torsion

- the curvature κ is the amount by which a curve deviates from being straight

  ↳ the curvature of a line is 0 and the curvature of a circle of radius r is constant in every point: $\kappa = \frac{1}{r}$

• for a 2D curve $c: \quad \bar{r}(t) = x(t) \cdot \bar{i} + y(t) \cdot \bar{j}$ the curvature at an arbitrary point $M(t_0)$ is defined as:

$$\kappa(t_0) = \frac{|x'(t_0)y''(t_0) - y'(t_0)x''(t_0)|}{\left[(x'(t_0))^2 + (y'(t_0))^2\right]^{\frac{3}{2}}}$$

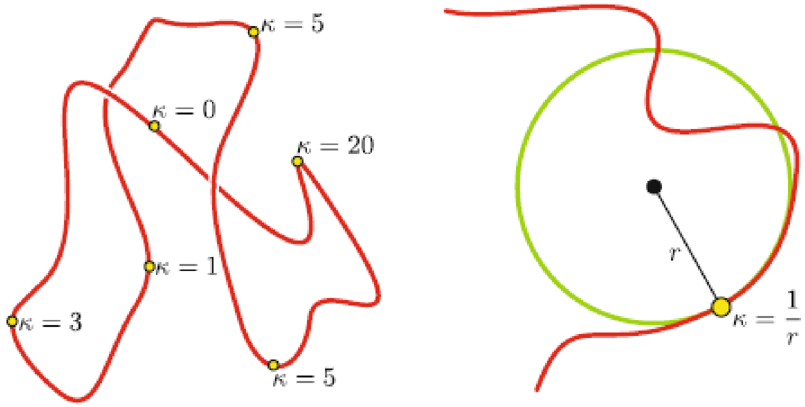


FIGURE 1 The curvature, κ, should measure how sharply the trace bends, compared to circles in $\mathbb{R}^2$

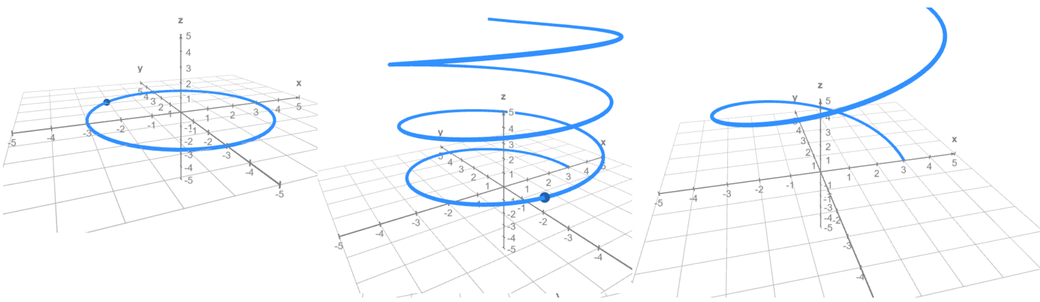
• at an arbitrary point $M(t_0)$ of a regular 3D curve c the curvature is defined as:

$$\kappa(t_0) = \frac{\|\bar{r}'(t_0) \times \bar{r}''(t_0)\|}{\|\bar{r}'(t_0)\|^3}$$

• the torsion τ of a curve is the amount by which a curve deviates from being a plane curve.

• at an arbitrary point $M(t_0)$ is defined as:

$$\tau(t_0) = \frac{\left|\left(\bar{r}'(t_0), \bar{r}''(t_0), \bar{r}'''(t_0)\right)\right|}{\|\bar{r}'(t_0) \times \bar{r}''(t_0)\|^2}$$

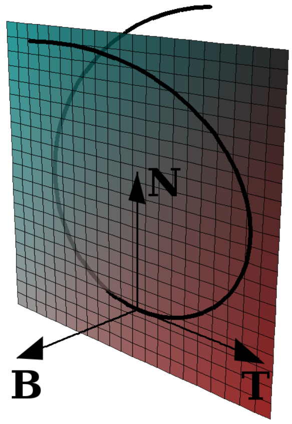


a) zero torsion b) medium torsion c) high torsion

The Frenet-Serret Frame

From now on we'll consider only curves $c : I \to \mathbb{R}^3$ that are C^2-differentiable curves for which $\bar{\mathbf{r}}'(\mathbf{t}) \times \bar{\mathbf{r}}''(\mathbf{t}) \neq \mathbf{0}$, for all $t \in I$. Consider also an ordinary point $M(t_0)$ on the curve c.

• the elements of the Frenet-Serret frame, or the TNB frame, are:



the unit tangent vector at M:

$$\bar{t} = \frac{\bar{r}'(t_0)}{\|\bar{r}'(t_0)\|}$$

the unit binormal vector at M:

$$\bar{b} = \frac{\bar{r}'(t_0) \times \bar{r}''(t_0)}{\|\bar{r}'(t_0) \times \bar{r}''(t_0)\|}$$

the principal normal vector at M:

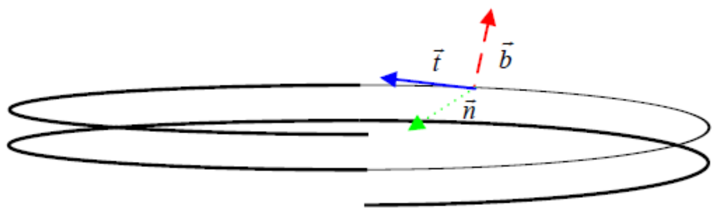
$$\bar{n} = \bar{b} \times \bar{t} = \frac{(\bar{r}'(t_0) \times \bar{r}''(t_0)) \times \bar{r}'(t_0)}{(\bar{r}'(t_0) \times \bar{r}''(t_0)) \times \bar{r}'(t_0)}$$

The axes:

• the tangent line to c at $M(t_0)$:
↳ has the direction given by $\bar{r}'(t_0)$

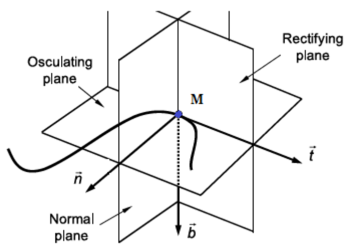
• the binormal line to c at $M(t_0)$:
↳ has the direction given by $\bar{r}'(t_0) \times \bar{r}''(t_0)$

• the principal normal to c at $M(t_0)$:
↳ direction given by $(\bar{r}'(t_0) \times \bar{r}''(t_0)) \times \bar{r}'(t_0)$



Visualization of the Frenet–Serret frame

The planes:



• the osculating plane at M is defined by M and the normal vector $\bar{b}(t_0)$

• the normal plane at M is defined by M and the normal vector $\bar{t}(t_0)$

• the rectifying plane at M is defined by M and the normal vector $\bar{n}(t_0)$

Solved Problems

Problem 1. *Consider the curve given by the parametric equations:*

$$c: \begin{cases} x = 2\cos t \\ y = 2\sin t \\ z = 3t \end{cases} \quad , \quad t \in [0, 2\pi]$$

i) Find the elements of the Frenet-Serret frame at $M(2,0,0)$.

ii) Find the length of the chord AB, where $A(0)$ and $B(\pi)$

Solution: The parameter corresponding to M will be $t_0 = 0$ since $2\cos 0 = 2$, $2\sin 0 = 0$ and $3 \cdot 0 = 0$. The parametric vectorial equations of c are:

$$\bar{r}(t) = 2\cos t\bar{i} + 2\sin t\bar{j} + 3t\bar{k}.$$

We get $\bar{r}'(0) = 2\bar{j} + 3\bar{k}$ and $\bar{r}''(0) = -2\bar{i}$. The unit tangent vector in M will be:

$$\bar{t}_M = \frac{\bar{r}'(0)}{\|\bar{r}'(0)\|} = \frac{2\bar{j} + 3\bar{k}}{\sqrt{0^2 + 2^2 + 3^2}} = \frac{1}{\sqrt{13}}(0,2,3)$$

For the unit binormal vector one needs:

$$\bar{r}'(0) \times \bar{r}''(0) \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 2 & 3 \\ -2 & 0 & 0 \end{vmatrix} = -6\bar{j} + 4\bar{k}$$

Thus:

$$\bar{b}_M = \frac{-6\bar{j} + 4\bar{k}}{\| -6\bar{j} + 4\bar{k}\|} = \frac{1}{\sqrt{52}}(-6\bar{j} + 4\bar{k})$$

Finally the principal normal vector is:

$$\bar{n}_M = \bar{b}_M \times \bar{t}_M = -\bar{i}$$

In the sequel we will find the equation of the lines and planes belonging to the Frenet-Serret frame at M. The tangent line passes through $M(2,0,0)$ and has the direction given by $\bar{r}'(0) = 2\bar{j} + 3\bar{k} = (0,2,3)$:

$$\frac{x-2}{0} = \frac{y-0}{2} = \frac{z-0}{3}$$

The binormal line passes through M and has the direction given by the vector $\bar{r}'(0) \times \bar{r}''(0) = (0,-6,4)$:

$$\frac{x-2}{0} = \frac{y-0}{-6} = \frac{z-0}{4}$$

The principal normal passes through M and has the direction given by the vector $\left(\bar{r}'(0)\times\bar{r}''(0)\right)\times\bar{r}'(0) = -26\bar{i}$:

$$\frac{x-2}{1} = \frac{y-0}{0} = \frac{z-0}{0}$$

Fact: if a vector v gives a direction then $c\cdot v$ gives the same direction. ($c = \frac{1}{-26}$)

The osculating plane passes through $M(2,0,0)$ and its normal vector is $\bar{b}_M$ so its equation will be:

$$-6(y-0)+4(z-0)=0 \iff -3y+2z=0$$

The normal plane passes through $M(2,0,0)$ and its normal vector is $\bar{t}_M$ so its equation will be:

$$2(y-0)+3(z-0)=0 \iff 2y+3z=0$$

The rectifying plane passes through $M(2,0,0)$ and its normal vector is $\bar{n}_M$ so its equation will be:

$$1(x-2)+0(y-0)+0(z-0)=0 \iff x=2$$

ii) The length of the chord between two points $M_1(t_1)$ and $M_2(t_2)$ is given by the formula:

$$\ell_{M_1M_2} = \int_{t_1}^{t_2} \sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}dt$$

Hence:

$$\ell_{AB} = \int_{0}^{\pi} \sqrt{(-2\sin t)^2+(2\cos t)^2+(3)^2}dt = \int_0^{\pi}\sqrt{13}dt = \sqrt{13}\pi.$$

Problem 2. *Find the curvature and the torsion of the curve given by the parametric equations:*

$$c: \begin{cases} x = e^t \\ y = e^{-t} \\ z = t\sqrt{2} \end{cases}, \quad t\in\mathbb{R}$$

Solution: For an arbitrary point $M(t_0)\in c$ the formulae of the curvature and torsion are:

$$\kappa(t_0) = \frac{\|\bar{r}'(t_0)\times\bar{r}''(t_0)\|}{\|\bar{r}'(t_0)\|^3}, \quad \tau(t_0) = \frac{\left|\left(\bar{r}'(t_0),\bar{r}''(t_0),\bar{r}'''(t_0)\right)\right|}{\|\bar{r}'(t_0)\times\bar{r}''(t_0)\|^2}$$

First of all, the **position vector** of a point $M(t)$ is given by:

$$\boldsymbol{\bar{r}(t)} = e^t\bar{i}+e^{-t}\bar{j}+t\sqrt{2}\bar{k}$$

Simple computations lead to:

$$\bar{r}'(t) = e^t\bar{i} - e^{-t}\bar{j} + \sqrt{2}\bar{k}$$

and:

$$\bar{r}''(t) = e^t\bar{i} + e^{-t}\bar{j}, \qquad \bar{r}'''(t) = e^t\bar{i} - e^{-t}\bar{j}$$

Thus:

$$\bar{r}'(t) \times \bar{r}''(t) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ e^t & -e^{-t} & \sqrt{2} \\ e^t & -e^{-t} & 0 \end{vmatrix} = -e^{-t}\sqrt{2}\bar{i} + e^t\sqrt{2}\bar{j} + 2\bar{k}$$

The necessary norms are:

$$\|\bar{r}'(t)\| = \sqrt{e^{2t} + e^{-2t} + 2} = e^t + e^{-t}, \quad \|\bar{r}'(t) \times \bar{r}''(t)\| = \sqrt{2e^{-2t} + 2e^{2t} + 4} = \sqrt{2}(e^{-t} + e^t)$$

Thus the curvature at $M(t_0)$ will be:

$$\kappa(t_0) = \frac{\|\bar{r}'(t_0) \times \bar{r}''(t_0)\|}{\|\bar{r}'(t_0)\|^3} = \frac{\sqrt{2}(e^{-t_0} + e^{t_0})}{(e^{t_0} + e^{-t_0})^3} = \sqrt{2}(e^{t_0} + e^{-t_0})^{-2}$$

In order to compute the torsion one needs the triple product:

$$\big(\bar{r}'(t), \bar{r}''(t), \bar{r}'''(t)\big) = \begin{vmatrix} e^t & -e^{-t} & \sqrt{2} \\ e^t & e^{-t} & 0 \\ e^t & -e^{-t} & 0 \end{vmatrix} = -2\sqrt{2}$$

and the torsion at $M(t_0)$ will be:

$$\tau(t_0) = \frac{\left|\big(\bar{r}'(t_0), \bar{r}''(t_0), \bar{r}'''(t_0)\big)\right|}{\|\bar{r}'(t_0) \times \bar{r}''(t_0)\|^2} = \frac{|-2\sqrt{2}|}{2(e^{-t_0} + e^{t_0})^2} = -\kappa(t_0)$$

Problem 3. *Let us consider the curve:*

$$c : \begin{cases} x = 3\sin^2 t \\ y = 3\sin(2t) \\ z = 3\cos^2 t \end{cases}, \quad t \in \mathbb{R}$$

Show that c is a plane curve.

Solution: The torsion τ measures the amount to which c deviates from being a plane curve. In order to be a plane curve one has to have zero torsion at every arbitrary point $M(t_0)$ of this curve. Having in mind the formula of $\tau(t_0)$ it is enough to prove:

$$\big(\bar{r}'(t_0), \bar{r}''(t_0), \bar{r}'''(t_0)\big) = 0, \quad \forall\ t_0 \in \mathbb{R}$$

Straightforward one gets:

$$\bar{r}(t_0) = 3\sin^2 t\bar{i} + 3\sin(2t)\bar{j} + 3\cos^2 t\bar{k}$$

$$\bar{r}'(t_0) = 3\sin(2t)\bar{i} + 6\cos(2t)\bar{j} - 3\sin(2t)\bar{k}$$

$$\bar{r}''(t_0) = 6\cos(2t)\bar{i} - 12\sin(2t)\bar{j} - 6\cos(2t)\bar{k}$$

$$\bar{r}'''(t_0) = -12\sin(2t)\bar{i} - 24\cos(2t)\bar{j} + 12\sin(2t)\bar{k}$$

The triple product will be:

$$\big(\bar{r}'(t_0), \bar{r}''(t_0), \bar{r}'''(t_0)\big) = \begin{vmatrix} 3\sin(2t_0) & 6\cos(2t_0) & -3\sin(2t_0) \\ 6\cos(2t_0) & -12\sin(2t_0) & -6\cos(2t_0) \\ -12\sin(2t_0) & -24\cos(2t_0) & 12\sin(2t_0) \end{vmatrix} = 0, \quad \forall t_0 \in \mathbb{R}$$

because the first and the last row are linearly dependent. Finally the curve c will be a plane curve.

Problem 4. *Find the points lying on the curve:*

$$c: \begin{cases} x = 2t - 1 \\ y = t^3 \\ z = 1 - t^2 \end{cases}, \quad t \in \mathbb{R}$$

where the osculating plane is perpendicular to the plane:

$$\alpha: \ 7x - 12y + 5z = 0.$$

Solution: Let us suppose that $M(t_0) \in c$ is a point with the above property. We will try to get some restrictions on t_0 (equations) in order to find all the possible values of t_0. Two planes are perpendicular iff their normal directions are perpendicular.

The normal direction to the osculating plane is:

$$\bar{v} = \bar{r}'(t_0) \times \bar{r}''(t_0) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 3t_0^2 & -2t_0 \\ 0 & 6t_0 & -2 \end{vmatrix} = 6t_0^2\bar{i} + 4\bar{j} + 12t_0\bar{k}$$

because $\bar{b}(t_0)$ provides this direction. The normal direction to the plane α is $\bar{n} = (7, -12, 5)$. Thus the necessary condition $\bar{v} \perp \bar{n}$ becomes $\langle v, \bar{n}\rangle = 0 \implies 42t_0^2 - 48 + 60t_0 = 0$ with the roots $t_1 = -2$ and $t_2 = \frac{4}{7}$. As a consequence we get two points $M_1(t_1) = M_1(-5, -8, -3)$ and $M_2(t_2) = M_2(\frac{1}{7}, \frac{64}{343}, \frac{33}{49})$

Problem 5. *Find the unit vectors corresponding to the Frenet-Serret frame, at $M(-2, 4, -12)$, of the curve given by:*

$$c: \begin{cases} x^2 - y^2 - z = 0 \\ x^2 - y = 0 \end{cases}$$

Solution: Since the curve is given as an intersection of two surfaces we would like to find the parametric equations of c. Let us denote $F(x,y,z) = x^2 - y^2 - z$ and $G(x,y,z) = x^2 - y$. We can apply the Implicit Function Theorem since:

$$\frac{D(F,G)}{D(y,z)}|_{(-2,4,-12)} = \begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix}|_{(-2,4,-12)} = \begin{vmatrix} -2y & -1 \\ -1 & 0 \end{vmatrix}|_{(-2,4,-12)} = 1 \neq 0$$

Thus in a neighborhood of $(-2,4,-12) \in \mathbb{R}^3$ there exist implicit functions $y(x)$ and $z(x)$ such that $y = y(x)$ and $z = z(x)$. In this neighborhood the parametric equations will be:

$$\bar{\mathbf{r}}(x) = x\bar{i} + y(x)\bar{j} + z(x)\bar{k}$$

Denoting $x = t$ we get the usual form:

$$\bar{\mathbf{r}}(t) = t\bar{i} + y(t)\bar{j} + z(t)\bar{k}$$

In this parametrization the point M corresponds to a parameter t_0. For the unit vectors, of the Frenet-Serret frame at M, we have to find the vectors $\bar{r}'(t_0)$ and $\bar{r}''(t_0)$. First of all let us observe the identities:

$$t^2 - y^2(t) - z(t) = 0, \quad t^2 - y(t) = 0$$

Further one can differentiate these relations in order to find $y'(t_0), y''(t_0), z'(t_0)$, $z''(t_0)$ as in [Problem 2, Chapter VII] of the textbook of C. Ariesanu, or one can speculate the particular form of these identities and find t_0. It is easy to see that $y(t) = t^2$ and $z(t) = t^2 - t^4$. Thus M corresponds to the parameter $t_0 = -2$. In the sequel, we proceed as in Problem 1 using:

$$\bar{r}'(-2) = \bar{i} - 4\bar{j} + 28\bar{k}$$

$$\bar{r}''(-2) = 2\bar{j} - 46\bar{k}$$

and so forth...

Proposed Problems

Problem 1. *Find the points lying on the curve:*

$$c: \begin{cases} x = \frac{2}{t} \\ y = \ln t \\ z = -t^2 \end{cases}$$

for which the tangent line is parallel to the plane $\alpha: \; x - y + 8z - 1 = 0$.

Problem 2. *Write the equations of the unit vectors, lines and planes of the Frenet-Serret frame corresponding to the curve:*

$$c: \quad \bar{r}(t) = 2t \cdot \bar{i} + t^2 \cdot \bar{j} + \ln t \cdot \bar{k}, \qquad t \in (0, \infty)$$

at $M(t_0 = 1)$.

Problem 3. *Find the equations of the binormal lines corresponding to:*

$$c: \quad x^2 - y^2 = z, \ 2x = 3y^2$$

at those points M where these lines are parallel to the yOz-plane.

Hint: Denote $y = t$ in order to obtain the parametric equations of c.

Problem 4. *Compute the curvature at an arbitrary point of the curve:*

$$c: \quad \begin{cases} x = a(t + \sin t) \\ y = a(1 - \cos t) \end{cases} \quad , \quad t \in [0, 2\pi]$$

Problem 5. *Find the elements of the Frenet-Serret frame at $M(1, -1, 2)$ for:*

$$c: \quad \begin{cases} z = x^2 + y^2 \\ x + y + z = 2 \end{cases}$$

Problem 6. *Prove that the curve:*

$$c: \quad \begin{cases} x = 3 + 2t + 4t^3 \\ y = 4 + 3t + 2t^3 \\ z = 2 + 4t + 3t^3 \end{cases}$$

is a plane curve and find the equation of this plane.