Title: On a Bottcher functional equation and its application for evaluation of a well-known improper integral

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ON A BÖTTCHER FUNCTIONAL EQUATION AND ITS APPLICATION FOR EVALUATION OF A WELL-KNOWN IMPROPER INTEGRAL

§ 1. We consider Böttcher's functional equation in the form

$$\varphi(ax) = \varphi(x)^p, \tag{1}$$

where $\varphi(x)$ is an unknown function, $\varphi: R \to R$, whilst a and p are real positive numbers different from one. Write $r = log_a p$. Equation (1) will have the form

$$\varphi(ax) = \varphi(x)^{a^r}. \tag{2}$$

Every solution of equation (1) must fulfil the condition $\varphi(o) = \varphi(o)^p$ (this follows immediately from (1) on setting $x = 0$). Hence we have that $\varphi(o) = 0$ or $\varphi(o) = 1$ or, for some rational numbers p, $\varphi(o) = -1$ (in the latter case, if the function $\varphi(x)$ satisfies equation (1), then the function $-\varphi(x)$ also satisfies (1) — therefore we may leaveout the case $\varphi(o) = -1$).

Remark 1. Considering equation (1) we may suppose that $a < 1$ or $a > 1$ (according to the need).

In fact, if $a < 1$, then we may write equation (2) in the form $\varphi(x)^{1/a^r} = \varphi\left(\frac{1}{a}x\right)$, thus we receive this equation in the form (2) with the coefficient $\frac{1}{a} > 1$. If $a > 1$, we proceed similarly.

Remark 2. If a function $\varphi(x)$ satisfies equation (1) and p is irrational, or rational in the form $\frac{m}{l}$, where m and l are relatively prime integers (i.e. $(m, l) = 1$) and 1 is even, then $\varphi(x) \geqslant 0$, since otherwise equation (1) would be meaningless. Furthermore, if m is even, then from equation (1) we see that similarly $\varphi(x) \geqslant 0$.

Further suppose that $r > 0$.

LEMMA 1. *If a function $\varphi(x)$ continuous at zero satisfies equation* (1) *and fulfils the condition $\varphi(0) = 0$, then $\varphi(x) \equiv 0$.*

Proof. We may suppose that $a > 1$. We write equation (2) in the form

$$\varphi(x) = \varphi\left(\frac{1}{a} x\right) \varphi\left(\frac{1}{a} x\right)^{a^r - 1}. \tag{3}$$

The function $\varphi(x)$ is continuous at zero, so there exists a neighbourhood U of zero such that $|\varphi(x)| \leqslant 1$ for every $x \in U$. Since $a^r - 1 > 0$, we have $|\varphi(x)|^{a^r - 1} < 1$ for every $x \in U$. Consequently, it follows from equality (3) that $|\varphi(x)| \leqslant \left|\varphi\left(\frac{1}{a} x\right)\right|$ for every $x \in U$. Thus by induction we have

$|\varphi(x)| \leq \left|\varphi\left(\frac{1}{a^n} x\right)\right|$ for $x \in U$ and $n = 1, 2, \ldots$ It follows from the continuity of the function $\varphi(x)$ at zero that $\lim\limits_{n\to\infty} \varphi(a^{-n}x) = \varphi(0) = 0$, so $\varphi(x) = 0$ for $x \in U$, and hence it follows that $\varphi(x) \equiv 0$ in the interval $(-\infty, \infty)$.

LEMMA 2. *If the function $\varphi(x)$ is continuous at zero and satisfies equation (1), and $\varphi(x_o) = 0$ for a certain x_o, then $\varphi(x) \equiv 0$.*

Proof. Now we assume that $a < 1$. It follows by induction from equation (2) that

$$\varphi(a^n x) = \varphi(x)^{a^{nr}}. \tag{4}$$

If $\varphi(x_o) = 0$, then $\varphi(a^n x_o) = 0$. From the continuity of the function $\varphi(x)$ at zero we have $\varphi(0) = 0$. Hence and from Lemma 1 we get $\varphi(x) \equiv 0$.

LEMMA 3. *If a function $\varphi(x)$ continuous at zero satisfies eqution (1) and fulfils the condition $\varphi(0) = 1$, then $\varphi(x) > 0$.*

Proof. Suppose that $a < 1$ and that $p = \frac{m}{l}$, where m and l are odd integers such that $(m, l) = 1$. It follows from equation (1) by induction that $\varphi(a^n x) = \varphi(x)^{p^n}$. Hence we have that if there existed an x_1 such that $\varphi(x_1) \leqslant 0$, then $\varphi(a^n x_1) \leqslant 0$ for all non-negatives integers n. By the continuity of the function $\varphi(x)$ at zero we have $\varphi(0) = \lim\limits_{n\to\infty} \varphi(a^n x_1) \leqslant 0$, which is a contradiction with the condition $\varphi(0) = 1$. If p is of the previous form, then by Remark 2 and Lemma 2 we get that $\varphi(x) > 0$.

Let $[s]$ denotes the entire part of the real number s. By K^s we denote the function $\varphi(x)$ defined in the interval $[0, \infty)$ and fulfilling the condition:

$$\varphi(x) = a_o + a_1 x + \ldots + a_{[s]} x^{[s]} + x^s g(x)$$

where $g(x)$ is continuous at zero. (If the number s is the integer then the function $\varphi(x)$ may be considered in the interval $(-\infty, \infty)$). We may assume that if $[s] = s$, then $g(0) = 0$.

THEOREM 1. *The functions $\varphi(x) = \exp cx^r$ are the unique solutions of equation (1) in the class K^r fulfilling the condition $\varphi(0) = 1$.*

Proof. Let a function $\varphi \in K^r$ satisfy equation (2) and fulfil the condition $\varphi(0) = 1$. First we show that the function $\varphi(x)$ has the form $\varphi(x) = 1 + x^r\bar{g}(x)$, where the function $\bar{g}(x)$ is continuous at zero. It follows from the condition $\varphi(0) = 1$ that $a_0 = 1$. Inserting the function $\varphi(x) = 1 + a_1 x + \ldots + a_{[r]} x^{[r]} + x^r g(x)$ into equation (1) we get the following equation

(5) $$1 + a_1(ax) + \ldots + a_{[r]}(ax)^{[r]} + a^r x^r g(ax) = \\ = (1 + a_1 x + \ldots + a_{[r]} x^{[r]} + x^r g(x))^p.$$

We have the well-known formula

(6 $$(1+y)^p = 1 + \sum_{k=1}^{\infty} \binom{p}{k} y^k$$

for $y \in (-1, 1)$. Let us put in (6) $y = a_1 x + \ldots + a_{[r]} x^{[r]} + x^r g(x)$. Hence, for x belonging to a certain neighbourhood of zero, we obtain the relation

(7) $$1 + a_1 ax + a_2(ax)^2 + \ldots + a_{[r]}(ax)^{[r]} + (ax)^r g(ax) = \\ = 1 + pa_1 x + pa_2 x^2 + \ldots + pa_{[r]} x^{[r]} + px^r g(x) + \\ + \binom{p}{2}(a_1 x + a_2 x^2 + \ldots + a_{[r]} x^{[r]} + x^r g(x))^2 + \\ \cdots\cdots\cdots\cdots\cdots \\ + \binom{p}{[r]}(a_1 x + a_2 x^2 + \ldots + x^r g(x))^{[r]} + o(x^{[r]})$$

where $\lim\limits_{x \to 0} \frac{o(x)}{x} = 0$.

Comparing the coefficients of x we have $a^r a_1 = a_1 a$ (note that $p = a^r$), whence it follows that $a_1 = 0$ (if $1 < r$), because $a \neq 1$ and $a > 0$. Next we put $a_1 = 0$ in equality (7), and comparing the coefficients of x^2 we get $a^r a_2 = a_2 a^2$, which similarly implies that $a_2 = 0$ (if $2 < r$). Proceeding in the same manner with the next coefficients, we get $a_i a^i = a_i a^r$ for all integers $i \leqslant [r]$. Hence we have $a_i = 0$ for $i < [r]$, and $a_{[r]} = 0$ whenever $r \neq [r]$. If, however, $r = [r]$, then $\varphi(x) = 1 + a_r x^r + x^r g(x)$, thus $\varphi(x) = 1 + x^r(a_r + g(x))$. Thus we have $\varphi(x) = 1 + x^r \bar{g}(x)$, where $\bar{g}(x) = a_r + g(x)$ is continuous at zero.

It is known that $\log(x+1) = x + o(x)$. It follows from Lemma 3 that $\varphi(x) > 0$. Put $\psi(x) = \log \varphi(x)$. Equation (2) may by written in the form

(8) $$\psi(ax) = a^r \psi(x).$$

Moreover, we have $\psi(x) = \log(1 + x^r g(x)) = x^r g(x) + o(x^r g(x))$. (In the sequel we write $g(x)$ instead of $\bar{g}(x)$). We have $\lim\limits_{x \to 0} \frac{o(x^r g(x))}{x^r} = 0$. Hence we may write $\psi(x) = x^r g(x) + o(x^r)$. Therefore we have $\lim\limits_{x \to 0} \frac{\psi(x)}{x^r} = g(0) \overset{\mathrm{df}}{=} c$. Consequently the function $\Phi(x) = \frac{\psi(x)}{x^r}$ for $x > 0$ and $\Phi(0) = c$ is continuous at zero. From equation (8) we obtain

$$\Phi(ax) = \Phi(x). \tag{9}$$

The unique solutions of this equation which are continuous at zero are constant functions (cf. [4] theorem 2.2 and comp. the remark at end of § 4 Chapter II, p. 53 in [4]). If $\Phi(0) = c$, then $\Phi(x) \equiv c$, and thus we get $\varphi(x) = exp\ cx^r$.

Remark 3. If the function $\varphi(x)$ satisfies equation (2) with an integer r, and $\varphi(x)$ has derivatives up to the order r at zero, and fulfils the condition $\varphi(0) = 1$ then $\varphi(x) = exp\ cx^r$.

In fact, it follows by Taylor's formula with Peano's remainder that the function $\varphi(x)$ which has dreivatives up to the order r at zero belongs to the class K^r.

COROLLARY 1. The function $\varphi(x) = exp\ cx$ is the unique function differentiable at zero, satisfying the equation

$$\varphi(2x) = \varphi(x)^2, \tag{10}$$

and fulfilling the conditions $\varphi(0) = 1$ and $\varphi'(0) = c$.

It is possible to obtain this Corollary in another way (cf. [6]). Under the assumption that $\varphi \in C^1$ equation (10) was considered in paper [3].

COROLLARY 2. The function $\varphi(x) = exp\ cx^2$ is the unique function defined in the interval $(-\infty, \infty)$, twice differentiable at zero, satisfying the equation

$$\varphi(\sqrt{2}\, x) = \varphi(x)^2 \tag{11}$$

and fulfilling the conditions $\varphi(0) = 1$ and $\varphi''(0) = 2c$.

This Corollary follows also immediately from a theorem of E. Vincze (cf. [5], and [4] theorem 12.11).

Now we prove the following

THEOREM 2. *If $0 < s < r$, then equation (2) has in the class K^s the solution φ depending on an arbitrary function. More precisely, the solution may be expressed by the formula*

$$\varphi(x) = exp\ \{x^r \gamma(\log_a x)\}, \tag{12}$$

where γ is an arbitrary function defined in $R = (-\infty, \infty)$, periodic with period 1 and bounded in R; or $\varphi(x) \equiv 0$.

Proof. Suppose that a function $\varphi(x)$ satisfies equation (2) and belongs to the class K^s. If $\varphi \in K^s$, then $\varphi(x)$ is continuous at zero. If $|\varphi(0)| \neq 1$, then $\varphi(0) = 0$ and it follows from Lemma 1 that we have $\varphi(x) \equiv 0$. We shall assume further that $\varphi(0) = 1$. Using the notation from the proof of Theorem 1 we have

$$\psi(x) = x^s g(x) + o(x^s)$$

and $\psi(x)$ satisfies equation (8). Let us write $h(x) = g(x) + \dfrac{o(x)}{x^s}$ for $x > 0$

and $h(0) = g(0)$. Then the function $\psi(x)$ has the form $\psi(x) = x^s h(x)$, where the function $h(x)$ is continuous at zero. Let $\Phi(x) = \frac{\Psi(x)}{x^r}$ for $x > 0$. The function $\Phi(x)$ satisfies equation (9) except at zero. Since $x > 0$, we may write $x = a^t$ and $\gamma(t) = \Phi(a^t)$. The function $\gamma(t)$ satisfies the equation $\gamma(t+1) = \gamma(t)$. From the relation $\Phi(x) = \frac{\Psi(x)}{x^r}$ we obtain $\psi(x) = x^r \gamma(\log_a x)$, therefore $\varphi(x) = exp\,\{x^r \gamma(\log_a x)\}$.

We shall prove the boundedness of the function $\gamma(x)$ in the interval $(-\infty, \infty)$. By the continuity of the function $\psi(x)$ at zero there exists an $\varepsilon > 0$ such that the function $\psi(x)$ i bounded in the interval $[0, \varepsilon]$. Hence for each $\eta > 0$, $\eta < \varepsilon$, there exits an M such that $|\Phi(x)| < M$ for $x \in [\eta, \varepsilon]$. We may assume that $a > 1$ and we put $\eta = \frac{\varepsilon}{a}$. If $t \in [\log_a \frac{\varepsilon}{a}, \log_a \varepsilon]$, then $|\gamma(t)| = |\Phi(a^t)| < M_\eta$, since $a^t \in \left[\frac{\varepsilon}{a}, \varepsilon\right]$. Hence, from the fact that the function $\gamma(t)$ is periodic with period 1 and from the equality $\log_a \varepsilon - \log \frac{\varepsilon}{a} = 1$, we obtain that the function $\gamma(t)$ is bounded in the interval $(-\infty, \infty)$.

It is obvious that every function $\varphi(x)$ defined by formula (12), where the function γ is bounded and peridic with period 1, satisfies equation (2) and belongs to the class K^s for every $s < r$.

From Theorem 2 the following corollary results immediately:

COROLLARY 3. If a function $\varphi(x)$ satisfies equation (2) and $\varphi \in K^s$ for a certain s, $0 < s < r$, then $\varphi \in K^{s'}$ for each s' such that $0 < s' < r$.

R e m a r k 4. If $r < 0$, then we define the class K^r as follows: $\varphi(x)$ belongs to K^r iff $\varphi\left(\frac{1}{x}\right)$ belongs to K^{-r}. It follows from this definition that Theorems 1 and 2 remain valid for $r < 0$ and $s < 0$.

§ 2. Now we are going to give an application of Corollary 2 to the evaluation of an improper integral.

We consider the well-known improper integral (t real)

$$f(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} exp\,(-x^2) \cos tx \, dx. \tag{13}$$

Now we shall show that $f(t) = exp\left(\frac{-t^2}{4}\right)$ using the theory of functional equations.

Note that the function $f(a)$ defined by integral (13) satisfies functional equation (11). In fact, we have by (13)

$$[f(t)]^2 = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} exp(-x^2-y^2)\cos tx \cos ty \, dxdy =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} exp(-x^2-y^2)\cos t(x+y)\, dxdy +$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} exp(-x^2-y^2)\cos t(x-y)\, dxdy = \frac{1}{2\pi}[I_1 + I_2]$$

Putting in I_1

$$x = \frac{1}{\sqrt{2}} u - \frac{1}{\sqrt{2}} v, \quad y = \frac{1}{\sqrt{2}} u + \frac{1}{\sqrt{2}} v$$

we have (Jacobian $= 1$, $x^2 + y^2 = u^2 + v^2$)

$$I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} exp(-u^2-v^2)\cos 2tu \, dudv =$$

$$= \int_{-\infty}^{\infty} exp(-v^2)\, dv \int_{-\infty}^{\infty} exp(-u^2)\cos 2tu \, du = \pi f(\sqrt{2}\, t),$$

since

$$\int_{-\infty}^{\infty} exp(-x^2)\, dx = \sqrt{\pi}.$$

Analogously we have $I_2 = \pi f(\sqrt{2}\, t)$. Thus the function $f(t)$ satisfies equation (11) and moreover

$$f(0) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} exp(-x^2)\, dx = 1.$$

It follows from well-known criteria of the differentiation of improper integrals that the function $f(t)$ is twice differentiable for each t (cf. [1], p. 610, theorem 3) as well as that

$$f''(t) = -\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} x^2 exp(-x^2)\cos tx \, dx,$$

whence

$$f''(0) = -\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} x^2 exp(-x^2)\, dx = -\frac{1}{2}.$$

By Corollary 2 we have $f(t) = exp\left(\frac{-t^2}{4}\right)$

It is also possible to evaluate integral (13) using functional equations in several variables (cf. [2]).

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MAREK CEZARY ZDUN

O RÓWNANIU FUNKCYJNYM BÖTTCHERA I JEGO ZASTOSOWANIU DO OBLICZANIA ZNANEJ CAŁKI NIEWŁAŚCIWEJ

Streszczenie

W pierwszej części rozpatruje się równanie funkcyjne (1) w klasie funkcji K^s postaci $\varphi(x) = a_0 + a_1 x + \ldots + a_{[s]} x^{[s]} + x^s g(x)$, gdzie $g(x)$ jest pewną funkcją ciągłą w zerze. Oznaczamy $r = \log_a p$.

Zostały udowodnione następujące twierdzenia.

TWIERDZENIE 1. Jedynymi rozwiązaniami równania (1) w klasie funkcji K^r spełniającymi warunek $\varphi(0) = 1$ są funkcje $\varphi(x) = \exp cx^r$.

TWIERDZENIE 2. Jeżeli $0 < s < r$, to rozwiązanie równania (1) w klasie funkcji K^s zależy od dowolnej funkcji. Wyraża się ono wzorem

$$\varphi(x) = \exp\{x^r \gamma(\log_a x)\},$$

gdzie γ jest dowolną funkcją okresową o okresie 1 i ograniczoną lub $\varphi(x) = 0$.

Okazuje się, że jeżeli funkcja φ spełnia równanie (1) oraz $\varphi \in K^s$ dla pewnego s, $0 < s < r$, to $\varphi \in K^{s'}$ takiego, że $0 < s' < r$.

W drugiej części pokazane jest zastosowanie równań funkcyjnych o jednej zmiennej do obliczania całek niewłaściwych na przykładzie całki danej wzorem (13).

Okazuje się, że funkcja określona wzorem (13) spełnia równanie funkcyjne (10) oraz należy do klasy K^1. Na podstawie Twierdzenia 1 możemy obliczyć wartość całki (13).

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