3.9 Polynomial Rings

Let F be a field. By the *ring of polynomials* in the indeterminate, x, written as $F[x]$, we mean the set of all symbols $a_0 + a_1x + \ldots + a_nx^n$, where n can be any nonnegative integer and where the coefficients $a_1, a_2, \ldots, a_n$ are all in F. In order to make a ring out of $F[x]$ we must be able to recognize when two elements in it are equal, we must be able to add and multiply elements of $F[x]$ so that the axioms defining a ring hold true for $F[x]$. This will be our initial goal.

We could avoid the phrase "the set of all symbols" used above by introducing an appropriate apparatus of sequences but it seems more desirable to follow a path which is somewhat familiar to most readers.

DEFINITION: If $p(x) = a_0 + a_1x + \ldots + a_mx^m$ and $q(x) = b_0 + b_1x + \ldots + b_nx^n$ are in $F[x]$, then $p(x) = q(x)$ if and only if for every integer $i \geq 0$, $a_i = b_i$.

Thus two polynomials are declared to be equal if and only if their corresponding coefficients are equal.

DEFINITION: If $p(x) = a_0 + a_1x + \ldots + a_mx^m$ and $q(x) = b_0 + b_1x + \ldots + b_nx^n$ are both in $F[x]$, then

$$p(x) + q(x) = c_0 + c_1x + \ldots + c_tx^t$$

where for each i, $c_i = a_i + b_i$.

In other words, add two polynomials by adding their coefficients and collecting terms. To add $1 + x$ and $3 - 2x + x^2$ we consider $1 + x$ as $1 + x + 0x^2$ and add, according to the recipe given in the definition, to obtain as their sum $4 - x + x^2$.

The most complicated item, and the only one left for us to define for $F[x]$, is the multiplication.

DEFINITION: If $p(x) = a_0 + a_1x + \ldots + a_mx^m$ and $q(x) = b_0 + b_1x + \ldots + b_nx^n$, then

$$p(x)q(x) = c_0 + c_1x + \ldots + c_kx^k$$

where

$$c_t = a_tb_0 + a_{t-1}b_1 + a_{t-2}b_2 + \ldots + a_0b_t.$$

This definition says nothing more than: multiply the two polynomials by multiplying out the symbols formally, use the relation $x^\alpha x^\beta = x^{\alpha+\beta}$ and collect terms.

EXAMPLE: Let

$$p(x) = 1 + x - x^2, \quad q(x) = 2 + x^2 + x^3.$$

Here

$$a_0 = 1,\ a_1 = 1,\ a_2 = -1,\ a_3 = a_4 = \ldots = 0,$$

and

$$b_0 = 2,\ b_1 = 0,\ b_2 = 1,\ b_3 = 1,\ b_4 = b_5 = \ldots = 0.$$