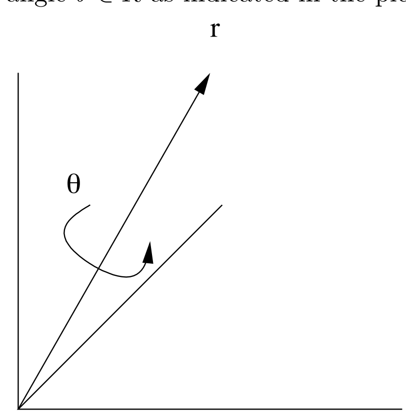
Chapter 13

The 3 dimensional rotation group

A rotation in space is a transformation $R : \mathbb{R}^3 \to \mathbb{R}^3$ determined by a unit vector $r \in \mathbb{R}^3$ and an angle $\theta \in R$ as indicated in the picture below.



A bit more precisely, the transformation $R = R(\theta, r)$ has the line through r as the axis, and the plane perpendicular to the line is rotated by the angle θ in the direction given by the right hand rule (the direction that the fingers of right hand point if the thumb points in the direction of r). R is a linear transformation, so it is represented by a matrix that we denote by the same symbol. It is invertible with inverse $R(-\theta, r)$. Therefore the set of rotations is a subset of $GL_3(\mathbb{R})$. We will show that it is a subgroup, and in particular that the product of two rotations is again a rotation. This is fairly obvious if the rotations share the same axis, but far from obvious in general. The trick is characterize the matrices that arise from rotations. Recall that a 3×3 matrix A is orthogonal if its columns are *orthonormal*, i.e. they unit vectors such that the dot product of any two is zero. This is equivalent to $A^T A = I$.

Lemma 13.1. *If A is orthogonal,* $\det A = \pm 1$.