5 Adjunctions and the Compact-Open Topology

Birds fly high in the air and survey broad vistas of mathematics out to the far horizon. They delight in concepts that unify our thinking and bring together diverse problems from different parts of the landscape. Frogs live in the mud below and see only the flowers that grow nearby. They delight in the details of particular objects, and they solve problems one at a time.
—Freeman Dyson (2009)

Introduction. Early in chapter 0 it was noted that category theory is an appropriate setting in which to discuss the concept of "sameness" between mathematical objects. This concept is captured by an isomorphism: a morphism from one object to another that is both left and right invertible. The discussion becomes especially interesting when those objects are categories. Two categories are isomorphic if there exists a pair of functors—one in each direction—whose compositions equal the identities. But equality is a lot of ask for! Isomorphisms of categories are too strict to be of much use. Relaxing the situation yields something better: categories C and D are *equivalent* if there exists a pair of functors $L\colon \mathsf{C} \rightleftarrows \mathsf{D} \colon R$ and natural isomorphisms $\mathrm{id}_{\mathsf{C}} \to RL$ and $LR \to \mathrm{id}_{\mathsf{D}}$.

Relaxing this a step further yields another gem of category theory: adjunctions. A pair of functors $L\colon \mathsf{C} \rightleftarrows \mathsf{D} \colon R$ forms an *adjunction*, and L and R are called *adjoint functors*, if there are natural transformations (not necessarily isomorphisms) $\eta\colon \mathrm{id}_{\mathsf{C}} \to RL$ and $\epsilon\colon LR \to \mathrm{id}_{\mathsf{D}}$ that, in addition, interact compatibly in a sense that can be made precise. Here the categories may not be equivalent, but don't think that adjunctions are mere second (or third) best. Quite often, relaxing a notion of equivalence results in a trove of rich mathematics. That is indeed the case here.

In this chapter, then, we introduce adjoint functors and use them to highlight several constructions in topology. We'll present the formal definition in section 5.1 and give some examples—free constructions in algebra, a forgetful functor from Top, and the Stone-Čech compactification—in sections 5.2, 5.3, and 5.5, respectively. Then we'll use a particularly nice adjunction—the product-hom adjunction—as motivation for putting an appropriate topology on function spaces. In section 5.6, we'll take an in-depth look at this topology, called the compact-open topology. Quite a few pages are devoted to this endeavor and some of the difficulties involved. Finally, section 5.7 closes with a discussion on the category of compactly generated weakly Hausdorff spaces—a "convenient" category of topological spaces. So in the pages to come, we'll be both birds and frogs. A categorical point of view oftentimes highlights and elevates the important properties that characterize an object or construction but fails to establish that such objects exist. Existence can require getting down in the mud.

5.1 Adjunctions

As mentioned above, an adjunction consists of a pair of functors L and R and a pair of natural transformations η and ϵ that interact in a certain way. But there is an equivalent definition, which is simpler to digest in a first introduction. We'll present this definition now and jump right into an example. The alternate definition will be given shortly after.

Definition 5.1 Let C and D be categories. An *adjunction* between C and D is a pair of functors $L\colon \mathsf{C} \to \mathsf{D}$ and $R\colon \mathsf{D} \to \mathsf{C}$ together with an isomorphism

$$\mathsf{D}(LX, Y) \overset{\cong}{\longleftrightarrow} \mathsf{C}(X, RY) \tag{5.1}$$

for each object X in C and each object Y in D that is natural in both components. The functor L is called the *left adjoint* and the functor R is called the *right adjoint*. We say the adjunction isomorphism—applied in either direction—sends a morphism to its *adjunct* (or *transpose*) and we write $\hat{f}$ for the adjunct of a map f. Together, all of this information is often denoted by

$$L\colon \mathsf{C} \rightleftarrows \mathsf{D}\colon R$$

or even more succinctly by $L \dashv R$.

To say that the isomorphism in (5.1) is "natural" in both components means that it arises via natural transformations. More precisely, notice that for each object X in C we get a pair of hom-functors

$$\mathsf{D}(LX, -) \qquad\qquad \mathsf{C}(X, R-)$$

from $\mathsf{D} \to \mathsf{Set}$. Similarly, for each object Y in D we have the hom-functors

$$\mathsf{D}(L-, Y) \qquad\qquad \mathsf{C}(-, RY)$$

from $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$. Saying the isomorphism $\mathsf{C}(LX, Y) \xrightarrow{\cong} \mathsf{D}(X, RY)$ is "natural in both components" means that there are natural transformations of functors

$$\mathsf{D}(LX, -) \xrightarrow{\quad\cong\quad} \mathsf{C}(X, R-)$$

$$\mathsf{D}(L-, Y) \xrightarrow{\quad\cong\quad} \mathsf{C}(-, RY)$$

Example 5.1 For any sets X, Y, and Z, the bijection $Y^{X\times Z} \xrightarrow{\cong} \left(Y^X\right)^Z$ arises from an adjunction. The functor $X\times -\colon \mathsf{Set} \to \mathsf{Set}$ is left adjoint to the functor $\mathsf{Set}(X, -)\colon \mathsf{Set} \to \mathsf{Set}$. To see it clearly, fix a set X, and define two functors

$$L := X \times -\colon \mathsf{Set} \to \mathsf{Set} \qquad\qquad R := \mathsf{Set}(X, -)\colon \mathsf{Set} \to \mathsf{Set}$$

Then

$$\mathsf{Set}(LZ, Y) = Y^{X\times Z} \cong \left(Y^X\right)^Z = \mathsf{Set}(Z, RY)$$

The setup $L\colon \mathsf{Set} \rightleftarrows \mathsf{Set}\colon R$ is called the *product-hom adjunction*. It will make several appearances in the pages to come. The "hom" in "product-hom" refers to "homomorphisms," which is how people used to, and sometimes still do, refer to "morphisms" in categories. We didn't want to try out a new name, like "product-mor," for this adjunction, however.

5.1.1 The Unit and Counit of an Adjunction

Suppose $L\colon \mathsf{C} \rightleftarrows \mathsf{D}\colon R$ is an adjunction with adjunction isomorphism

$$\varphi_{X,Y}\colon \mathsf{D}(LX, Y) \xrightarrow{\cong} \mathsf{C}(X, RY) \tag{5.2}$$

By setting $Y = LX$, we have an isomorphism

$$\varphi_{X,LX}\colon \mathsf{D}(LX, LX) \xrightarrow{\cong} \mathsf{C}(X, RLX)$$

Under this isomorphism, the morphism id_{LX} in the category D corresponds to a morphism $\eta_X := \varphi_{X,LX}(\mathrm{id}_{LX})\colon X \to RLX$ in the category C. As one can check, these maps assemble into a natural transformation

$$\eta\colon \mathrm{id}_{\mathsf{C}} \to RL$$

called the *unit* of the adjunction. In other words, the unit is comprised of the adjuncts of the identity maps: $\eta_X := \widehat{\mathrm{id}_{LX}}$. Similarly for $X = RY$, under the isomorphism

$$\varphi_{RY,Y}\colon \mathsf{D}(LRY, Y) \xrightarrow{\cong} \mathsf{C}(RY, RY)$$

the morphism id_{RY} in C corresponds to a morphism $\epsilon_Y := \widehat{\mathrm{id}_{RY}}\colon LRY \to Y$ that defines a natural transformation

$$\epsilon\colon LR \to \mathrm{id}_{\mathsf{D}}$$

called the *counit* of the adjunction.

Understanding the counit and unit of an adjunction helps to understand the isomorphisms (5.2) and their corresponding universal properties. For example, suppose $X \in \mathsf{C}$. For any $Y \in \mathsf{D}$ and $f\colon X \to RY$, there exists a unique $g\colon RLX \to RY$ so that $g\eta_X = f$. Here's the picture.

$$\begin{tikzcd} RLX \arrow[dr, dashed, "g"] & \\ X \arrow[u, "\eta_X"] \arrow[r, "f"] & RY \end{tikzcd}$$

Indeed, we can explicitly identify g as $R\hat{f}$. The equality $g\eta_X = f$ follows from the naturality of the adjunction isomorphism $\varphi\colon \mathsf{D}(LX, -) \to \mathsf{C}(X, R-)$.

Explicitly, fix X and Y, and let $f \in \mathsf{C}(X, RY)$. Then we have the adjunct map $\hat{f} \in \mathsf{D}(LX, Y)$ where $\varphi\hat{f} = f$. Moreover, this square commutes:

$$\begin{tikzcd} \mathsf{D}(LX, LX) \arrow[r, "\varphi"] \arrow[d, "\hat{f}_*"] & \mathsf{C}(X, RLX) \arrow[d, "(R\hat{f})_*"] \\ \mathsf{D}(LX, Y) \arrow[r, "\varphi"] & \mathsf{C}(X, RY) \end{tikzcd}$$

Choosing $\mathrm{id}_{LX} \in \mathsf{D}(LX, LX)$ and noting that $\varphi\,\mathrm{id}_{LX} = \eta_X$, commutativity implies $\varphi\hat{f} = R\hat{f}\eta$, that is $f = g\eta$ with $g = R\hat{f}$.

Example 5.2 Let's look at the unit and counit of the product-hom adjunction $L\colon \mathsf{Set} \rightleftarrows \mathsf{Set}\colon R$ in Set where

$$L = X \times -\colon \mathsf{Set} \to \mathsf{Set} \qquad \text{and} \qquad R = \mathsf{Set}(X, -)\colon \mathsf{Set} \to \mathsf{Set}$$

The counit of this adjunction is the *evaluation map* $\mathrm{eval}\colon X \times Y^X \to Y$ defined by $\mathrm{eval}(x, f) = f(x)$. The unit is the map $Z \to (X \times Z)^X$ defined by $z \mapsto (-, z)$, where $(-, z)\colon X \to X \times Z$ is the function $x \mapsto (x, z)$.

The natural transformations η and ϵ provide another way to define an adjunction.

Definition 5.2 An *adjunction* between categories C and D is a pair of functors $L\colon \mathsf{C} \to \mathsf{D}$ and $R\colon \mathsf{D} \to \mathsf{C}$ together with natural transformations $\eta\colon \mathrm{id}_{\mathsf{C}} \to RL$ and $\epsilon\colon LR \to \mathrm{id}_{\mathsf{D}}$ such that for all objects $X \in \mathsf{C}$ and $Y \in \mathsf{D}$ the following triangles commute:

$$\begin{tikzcd} LX \arrow[r, "L\eta_X"] \arrow[dr, "\mathrm{id}_{LX}"'] & LRLX \arrow[d, "\epsilon_{LX}"] \\ & LX \end{tikzcd} \qquad \begin{tikzcd} RY \arrow[r, "\eta_{RY}"] \arrow[dr, "\mathrm{id}_{RY}"'] & RLRY \arrow[d, "R\epsilon_Y"] \\ & RY \end{tikzcd}$$

Verifying the equivalence of definitions 5.1 and 5.2 is a good exercise.

The next few sections contain additional examples of adjunctions. The first example arises in an algebraic context, while the remaining examples come from topology.

5.2 Free-Forgetful Adjunction in Algebra

Often, free constructions in algebra (free modules, free groups, free abelian groups, free monoids, etc.) are defined by universal properties. To be concrete, let's consider free groups, since modifying the discussion for other free constructions is usually easy. Here's the way a free group is commonly defined:

> *A free group on a set S is a group FS together with a map of sets $\eta\colon S \to FS$ satisfying the property that for any group G and any map of sets $f\colon S \to G$ there exists a unique group homomorphism $\hat{f}\colon FS \to G$ so that $\hat{f}\eta = f$.*

This diagram can help:

$$\begin{tikzcd} FS \arrow[dr, dashed, "\hat{f}"] & \\ S \arrow[u, "\eta"] \arrow[r, "f"'] & G \end{tikzcd}$$

and yet, the diagram is also confusing. After all, some objects in the diagrams are sets, some are groups, some of the arrows are set maps, and some are group homomorphisms. The situation becomes clearer when one observes that there is a *forgetful functor U* from groups to sets.

$$\mathsf{Grp} \xrightarrow{U} \mathsf{Set}$$

$$\begin{tikzcd} G \arrow[d, "\varphi"'] \arrow[r, phantom, "\longmapsto"] & UG \arrow[d, "U\varphi:=\varphi"] \\ G' & UG' \end{tikzcd}$$

It assigns to any group its underlying set (which explains the letter "U") and to any group homomorphism its underlying function. The adjective *forgetful* is often applied to a functor that "forgets" some or all of the structure of the objects in its codomain. It is a loose term that can be applied to lots of functors whose main job is to drop some data.

One may then define a *free group on a set S* to be a group FS and a map $\eta\colon S \to UFS$ with the property that, for all groups G and maps $f\colon S \to UG$, there exists a unique map $\hat{f}\colon FS \to G$ so that $f = U\hat{f}\eta$. So the right picture is in Set:

$$\begin{tikzcd} UFS \arrow[dr, dashed, "U\hat{f}"] & \\ S \arrow[u, "\eta"] \arrow[r, "f"'] & UG \end{tikzcd}$$

Notice that the "there exists" part of the definition of a free group says that for every group G, the map $\mathsf{Grp}(FS, G) \to \mathsf{Set}(S, UG)$ is surjective. The "unique" part of the definition says that $\mathsf{Grp}(FS, G) \to \mathsf{Set}(S, UG)$ is injective. The upshot is that "free" and "forgetful" form an adjoint pair $F\colon \mathsf{Set} \rightleftarrows \mathsf{Grp}\colon U$, providing the isomorphism,

$$\mathsf{Set}(S, UG) \quad \cong \quad \mathsf{Grp}(FS, G)$$

The unit of this adjunction defines the inclusion $\eta\colon S \to UFS$.

Remark 5.1 The universal property defining a free group can also be understood within the context of set maps $S \to UG$ for all groups G. So, one could make a category out of this context. Let's call this category U^S. An object in U^S is a group G and a set map $f\colon S \to UG$. A morphism between two objects $S \xrightarrow{f} UG$ and $S \xrightarrow{f'} UG'$ is a group homomorphism $\varphi\colon G \to G'$ so that $U\varphi f = f'$. That is,

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ UG \arrow[rr, "U\varphi"] & & UG' \end{tikzcd}$$

Then,

a free group on a set S is an initial object in the category U^S.

The context for the universal property is put into a category U^S that is built out of the undisguised material involved: the set S *and* the functor $U\colon \mathsf{Grp} \to \mathsf{Set}$. Then the universal object is a familiar notion (an initial object) from category theory.

The algebraic discussion here arose from considering a forgetful functor. One also has a forgetful functor in the topological setting, which gives rise to further adjunctions.

5.3 The Forgetful Functor $U\colon \mathsf{Top} \to \mathsf{Set}$ and Its Adjoints

There is a forgetful functor $U\colon \mathsf{Top} \to \mathsf{Set}$ that assigns to any topological space $(X, \mathcal{T}_X)$ the set X and to any continuous function $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ the function $f\colon X \to Y$. It is both a left and right adjoint in Top. Define a functor $D\colon \mathsf{Set} \to \mathsf{Top}$ that assigns to any set X the space $(X, \mathcal{T}_{\text{discrete}})$ with the discrete topology. To any function $f\colon X \to Y$, let $Df = f$, which is a continuous function. The setup $D\colon \mathsf{Set} \rightleftarrows \mathsf{Top}\colon U$ is an adjunction; for any set X and any space Y, we have

$$\mathsf{Top}(DX, Y) \quad \cong \quad \mathsf{Set}(X, UY)$$

On the right, we have arbitrary functions from the set X into the space Y, viewed as a set. On the left, we take continuous functions $DX \to Y$, which are *all* functions from $X \to Y$, since every function from a discrete space is continuous.

But here's another functor $I\colon \mathsf{Set} \to \mathsf{Top}$ that assigns to a set the same set with the indiscrete topology and to any function $f\colon X \to Y$, the same function, which will be continuous. Then $U\colon \mathsf{Top} \rightleftarrows \mathsf{Set}\colon I$ is an adjunction; for any space X and any set Y, we have

$$\mathsf{Set}(UX, Y) \quad \cong \quad \mathsf{Top}(X, IY)$$

On the left we have arbitrary functions from X, viewed as a set, to the set Y. On the right, we have continuous functions $X \to IY$, which are all functions $X \to Y$ since every function into an indiscrete space is continuous.

The universal properties arising from these adjunctions don't seem very interesting, but the fact that U is both a left and a right adjoint has important consequences. Notice this theorem in particular.

Theorem 5.1 If $L\colon \mathsf{C} \to \mathsf{D}$ has a right adjoint, then L is cocontinuous. If $R\colon \mathsf{D} \to \mathsf{C}$ has a left adjoint, then R is continuous.

Proof. Recall from exercise 4.8 at the end of chapter 4 that there is a natural isomorphism:

$$\mathsf{C}(\operatorname{colim} F, Y) \quad \cong \quad \lim \mathsf{C}(F(-), Y)$$

for any functor $F\colon \mathsf{B} \to \mathsf{C}$. Therefore,

$$\begin{aligned} \mathsf{D}(L(\operatorname{colim} F), Y) \quad &\cong \quad \mathsf{C}(\operatorname{colim} F, RY) \\ &\cong \quad \lim \mathsf{C}(F-, RY) \\ &\cong \quad \lim \mathsf{D}(LF-, Y) \\ &\cong \quad \mathsf{D}(\operatorname{colim} LF, Y) \end{aligned}$$

Therefore, $L(\operatorname{colim} F)$ satisfies the universal property of $\operatorname{colim} LF$. And in particular, because colimits (if they exist) are unique up to unique isomorphism,

$$L(\operatorname{colim} F) \quad \cong \quad \operatorname{colim} LF$$

Thus, L is cocontinuous. By a similar argument—which we encourage the reader to verify—right adjoint functors are continuous. □

Corollary 5.1.1 Right adjoints preserve products.

Proof. Immediate. Products are limits. □

This explains why the constructions of products and coproducts, subspaces and quotients, equalizers and coequalizers, and pullbacks and pushforwards in Top must have, as an underlying set, the corresponding construction in Set: if the construction exists in Top then the forgetful functor $U\colon \mathsf{Top} \to \mathsf{Set}$ preserves it!

5.4 Adjoint Functor Theorems

What about a converse to theorem 5.1? Let $R : \mathsf{D} \to \mathsf{C}$ be any functor. Under what conditions will R have a left adjoint? Clearly, R must be continuous. Is the continuity of R sufficient? Not quite. Here's a nice way to think about it. For each object $X \in C$, look at the category R^X whose objects consist of an object Y in D together with a morphism $X \to RY$. A morphism between $f\colon X \to RY$ and $f'\colon X \to RY'$ is a morphism $g\colon Y \to Y'$ such that $(Rg)f = f'$. As in remark 5.1, given a functor $L\colon \mathsf{C} \to \mathsf{D}$, an object $LX \in \mathsf{D}$ satisfying $\mathsf{D}(LX, Y) \cong \mathsf{C}(X, RY)$ for all Y is an initial object in R^X. If there is an initial object LX in R^X for every object X, then they assemble functorially into a left adjoint $L\colon \mathsf{C} \to \mathsf{D}$. In any category, an initial object is the limit of the identity functor. So, R will have a left adjoint if and only if the identity functor on the category R^X has a limit for all objects X. If the category D is complete, then the functor R being continuous implies that R^X is complete. However, even if R^X is complete, the identity functor on R^X is usually not a *small* diagram. So, there are a suite of theorems known as *adjoint functor theorems* that assume the functor R is continuous and the category D is complete and that add some kind of hypothesis allowing one to use the fact that R^X has limits of all small diagrams to prove that the identity functor on R^X has a limit. We don't use any adjoint functor theorems in this book, but it's good to know they exist. Let's state one precisely with the *Solution Set Condition* (Mac Lane, 2013; Freyd, 1969) hypothesis.

The Solution Set Condition A functor $R\colon \mathsf{D} \to \mathsf{C}$ satisfies the *Solution Set Condition* if and only if for every object X in C, there exists a set of objects $\{Y_i\}$ in D and a set of morphisms

$$S = \{f_i\colon X \to RY_i\}$$

so that any $f\colon X \to RY$ factors through some $f_i \in S$ along a morphism $Y_i \to Y$ in D.

The Adjoint Functor Theorem Suppose D is complete and that $R\colon \mathsf{D} \to \mathsf{C}$ is a continuous functor satisfying the Solution Set Condition. Then R has a left adjoint $L\colon \mathsf{C} \to \mathsf{D}$.

For details beyond what we've already said, see the classic reference by Mac Lane (2013) or section 4.6 of the excellent book by Riehl (2016), and for an enlightening treatment of adjunctions including applications, see Spivak (2014). Before moving on, we should say that there are adjoint functor theorems for the existence of right adjoints as well. They suppose that $L\colon \mathsf{C} \to \mathsf{D}$ is a cocontinuous functor from a cocomplete category C, with some "co" version of the Solution Set Condition.

Our discussion of adjoint functor theorems arose from an observation about the forgetful functor on Top and its adjoints. Another notable adjunction in topology arises in a discussion on compactifications.

5.5 Compactifications

Definition 5.3 A *compactification* of a topological space is an embedding of the space as a dense subspace of a compact Hausdorff space.

So a compactification of X is a compact Hausdorff space Y and a continuous injection $i\colon X \to Y$ with $X \cong iX \subseteq Y$ and $\overline{X} = Y$. Note that only Hausdorff spaces have compactifications since every subspace of a Hausdorff space is Hausdorff.

Example 5.3 The inclusion $(0,1) \hookrightarrow [0,1]$ and the map $(0,1) \hookrightarrow S^1$ defined by $t \mapsto (\cos 2\pi t, \sin 2\pi t)$ are both compactifications. For the space $X = (0,1)$ with the discrete topology, the map $X \hookrightarrow [0,1]$ is not an embedding and is hence not a compactification.

5.5.1 The One-Point Compactification

If a compactification Y of a space X is obtained by adding a single point to X, then $X \hookrightarrow Y$ is called a *one-point compactification*—also sometimes called the *Alexandroff one-point* *compactification*. A space X has one-point compactification if and only if X is Hausdorff and locally compact. If a space has a one-point compactification, then it's unique.

To see this, suppose $X \hookrightarrow X^*$ is a compactification and $X^* \smallsetminus X = \{p\}$. The open neighborhoods of p are precisely the complements of compact subsets of X: the complement of an open set containing p is a closed subset of a compact space and so is compact. Conversely if K is a compact subset of $X \subset X^*$, then it is closed, so its complement in X^* is an open set containing p. Then, because points of X can be separated by open sets from the point $p \in X^* \smallsetminus X$, there's a neighborhood of every point of X contained in a compact set, so X is locally compact. The fact that X is a dense subset of X^* implies $\{p\}$ is not open, meaning that X is not already compact.

Conversely, beginning with any space X, one constructs a new space by adding a point p and defining the open neighborhoods of p to be complements of compact sets in X. If X is locally compact and Hausdorff and not compact, the result is a topology on $X^* := X \cup \{p\}$ that is compact and Hausdorff having X as a dense subset.

Now, what kind of property does the one-point compactification have?

Theorem 5.2 Suppose X is locally compact, Hausdorff, and not compact, and let $i\colon X \to X^*$ be the one-point compactification of X. If $e\colon X \to Y$ is any other compactification of X, then there exists a unique quotient map $q\colon Y \to X^*$ with $qe = i$.

$$\begin{tikzcd} & Y \arrow[d, dashed, "q"] \\ X \arrow[ur, "e"] \arrow[r, "i"'] & X^* \end{tikzcd}$$

Proof. The idea is that the quotient of Y obtained by identifying $Y \smallsetminus eX$ to one point is homeomorphic to X^*. The details are left as an exercise. □

This theorem can be useful, but it really doesn't say much more than "the one-point compactification of X is the smallest compactification of X," which you may find unsurprising. At the other extreme is the Stone-Čech compactification, which has good categorical properties.

5.5.2 The Stone-Čech Compactification

Let CH be the category whose objects are compact Hausdorff spaces and whose morphisms are continuous functions. There is a functor $U\colon \mathsf{CH} \to \mathsf{Top}$, which is just the inclusion of compact Hausdorff spaces as a subcategory of topological spaces and is the identity on objects and morphisms. The functor U has a left adjoint $\beta\colon \mathsf{Top} \to \mathsf{CH}$ called the *Stone-Čech compactification*. Constructions of β are outlined as construction 6.11 in May (2000) and in more detail in section 38 of Munkres (2000).

For now, let's just unwind this functorial description and see what it means. To say that β is a left adjoint of U means that for every topological space X and every compact Hausdorff space Y, we have a natural bijection

$$\mathsf{CH}(\beta X, Y) \cong \mathsf{Top}(X, UY) = \mathsf{Top}(X, Y)$$

This says continuous functions $f\colon X \to Y$ from a space X to a compact Hausdorff space Y correspond precisely to continuous functions $\hat{f}\colon \beta X \to Y$. Specifying the continuous functions from βX determines the space βX if it exists, but it doesn't *prove* it exists. For that, you need a construction as outlined in the references above, or you could invoke some version of the Adjoint Functor Theorem to prove that $U\colon \mathsf{CH} \to \mathsf{Top}$ has a left adjoint (Mac Lane, 2013).

The unit of the Stone-Čech compactification adjunction

$$\beta\colon \mathsf{Top} \rightleftarrows \mathsf{CH}\colon U$$

defines a morphism $\eta\colon X \to U\beta X$. Since $U\beta X = \beta X$, the Stone-Čech compactification as a left adjoint of U doesn't just produce a compact Hausdorff space βX from any topological space X; it also produces a continuous function $\eta\colon X \to \beta X$ involved in a universal property. For every map $f\colon X \to Y$ between X and a compact Hausdorff space Y, there is a unique map $U\hat{f} = \hat{f}\colon \beta X \to Y$, the adjunct of f, so that $\hat{f}\eta = f$. Pictorally,

$$\begin{tikzcd}
\beta X \arrow[dr, dashed, "\hat{f}"] & \\
X \arrow[u, "\eta"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

In the case when X is locally compact and Hausdorff, the map $\eta\colon X \to \beta X$ is a compactification of X. That is, $\eta\colon X \to \beta X$ is an embedding and $\overline{X} = \beta X$. Then for any compact Hausdorff space Y, the map $\hat{f}\colon \beta X \to Y$ is the extension of the map $f\colon X \to Y$.

You might have noticed that the triangle above has the same flavor as the triangle defining free groups discussed earlier. This is no coincidence. In addition to the constructions already cited, one can use ultrafilters to construct the Stone-Čech compactification. For any space X, there is a natural topology on the set βX of ultrafilters on a X. The space βX with this topology is compact and Hausdorff, and the inclusion $\eta_X\colon X \to \beta X$ defined by sending a point to its principal ultrafilter is a realization of the Stone-Čech compactification. The fact that there exists an algebraic structure called a *monad* on the ultrafilter functor β sheds further light on resemblance between the Stone-Čech compactification and the free-forgetful adjunctions in algebraic categories mentioned earlier in this chapter. For details, we refer interested readers to the compactum article at the nLab (Stacey et al., 2019) as well as E. Manes's original paper (1969).

In closing, note that unlike the Stone-Čech compactification, the one-point compactification X^* of a locally compact Hausdorff space X is easy to define, but it doesn't have good properties with respect to morphisms. It definitely doesn't satisfy the condition that

$$\mathsf{CH}(X^*, Y) \cong \mathsf{Top}(X, Y)$$

For a simple example, consider $X = (0, 1)$ and its one point compactification $i\colon (0, 1) \to S^1$. Let $Y = [0, 1]$ and consider the inclusion $f\colon (0, 1) \to [0, 1]$. It cannot be extended to a continuous function from $S^1 \to [0, 1]$; there is no diagonal map that fits into the diagram below.

$$\begin{tikzcd} S^1 \\ (0,1) \arrow[u, "i"] \arrow[r, "f"] & {[0,1]} \end{tikzcd}$$

In the next section, we'll continue the discussion of adjunctions in topology. So far, we've discussed free constructions in algebra, the forgetful functor in Top and its adjoints, and compactifications. All of these are are united by the language of adjunctions. Next, we turn to the topic of *mapping spaces.* For any topological spaces X and Y, there is a set of continuous maps $\mathsf{Top}(X, Y)$ between them. Can that set be viewed as a *space* itself? That is, for any $X, Y \in \mathsf{Top}$, can $\mathsf{Top}(X, Y)$ also be regarded as an object in Top in a useful way? We'll see that finding a suitable topology for $\mathsf{Top}(X, Y)$ is more subtle than, say, finding a vector space structure on the set $\mathsf{Vect}_{\mathbf{k}}(V, W)$ of linear maps between vector spaces V and W. In chapter 1, universal properties in Set guided us as we constructed new spaces from old. We will also have categorical guidance on the journey to define topologies on mapping spaces. The guide this time is the product-hom adjunction in Set.

5.6 The Exponential Topology

Let X and Y be spaces. Consider the general problem of equipping the set of continuous functions $\mathsf{Top}(X, Y)$ with a topology making it a space of maps, or a *mapping space*. For the record, the product topology is usually not an appropriate topology for $\mathsf{Top}(X, Y)$ since it treats the space X only as an index set—it doesn't use the topology of X except to identify the continuous functions within the set of all functions $X \to Y$. But what properties should a topology on $\mathsf{Top}(X, Y)$ have? We take as guidance the following desired property.

Desired Property For a fixed space X, the functors

$$X \times - \colon \mathsf{Top} \to \mathsf{Top} \qquad \text{and} \qquad \mathsf{Top}(X, -)\colon \mathsf{Top} \to \mathsf{Top}$$

should form an adjoint pair. That is, for all spaces Y and Z, we should have an isomorphism of sets $\mathsf{Top}(X \times Z, Y) \cong \mathsf{Top}(Z, \mathsf{Top}(X, Y))$.

Let's begin to analyze this property. First think of three fixed spaces X, Y, and Z. One can obtain a function $X \to Y$ by starting with a function $g\colon X \times Z \to Y$ of two variables and by fixing one of the variables $z \in Z$, resulting in $g(-, z)\colon X \to Y$. We'd like a topology on

$\mathsf{Top}(X, Y)$ to have the property that, if the function g of two variables is continuous, then the assignment $z \mapsto g(-, z)$ will define a continuous map $Z \to \mathsf{Top}(X, Y)$. Going the other way, if we have a continuous map $Z \to \mathsf{Top}(X, Y)$, then we should be able to assemble the family of continuous maps from X to Y that are continuously parametrized by the space Z into a single continuous map $X \times Z \to Y$ of two variables.

Now, let's look more closely at the desired property. Let $g\colon X \times Z \to Y$ be continuous. Denote the adjunct by $\hat{g}\colon Z \to \mathsf{Top}(X, Y)$. For $\hat{g}$ to be continuous, the topology on $\mathsf{Top}(X, Y)$ should be rather coarse. However, if the topology on $\mathsf{Top}(X, Y)$ is too coarse (think of the indiscrete topology), then the set $\mathsf{Top}(Z, \mathsf{Top}(X, Y))$ will contain too many continuous functions—it will contain functions that are not the adjunct of any continuous map $g\colon X \times Z \to Y$. It's a balancing act that turns out neatly; if there exists a topology on $\mathsf{Top}(X, Y)$ so that for any space Z, the correspondence $g \mapsto \hat{g}$ defines a bijection of sets

$$\mathsf{Top}(X \times Z, Y) \cong \mathsf{Top}(Z, \mathsf{Top}(X, Y))$$

then that topology is unique (Arens and Dugundji, 1951; Escardó and Heckmann, 2002). Let's call it the *exponential topology* on $\mathsf{Top}(X, Y)$.

This balancing act is reminiscent of the constructions of new spaces from old in chapter 1. That's what we're doing here as well: given two spaces X and Y, we want to make a new topological space from the set of continuous maps from X to Y. In chapter 1, we had guidance from universal properties characterizing similar constructions in Set. Here, the product-hom adjunction in Set provides our categorical guidance. Let's go through some of the details.

Let's call a topology on $\mathsf{Top}(X, Y)$ *splitting* if the continuity of $g\colon Z \times X \to Y$ implies the continuity of $\hat{g}\colon Z \to \mathsf{Top}(X, Y)$. Let's call a topology on $\mathsf{Top}(X, Y)$ *conjoining* if the continuity of $\hat{g}\colon Z \to \mathsf{Top}(X, Y)$ implies the continuity of $g\colon Z \times X \to Y$. Then, to repeat the previous remarks using this terminology (Render, 1993), a topology on $\mathsf{Top}(X, Y)$ must be rather coarse to be splitting and must be rather fine to be conjoining. A topology on $\mathsf{Top}(X, Y)$ is exponential if and only if it is both splitting and conjoining.

Now keep in mind two things. First, the evaluation map is the counit of the product-hom adjunction in Set. Second, the adjunct of the evaluation map is the identity. Together, these give a very nice characterization of conjoining topologies.

Lemma 5.1 A topology on $\mathsf{Top}(X, Y)$ is conjoining if and only if the evaluation map $\mathrm{eval}\colon X \times \mathsf{Top}(X, Y) \to Y$ is continuous.

Proof. Assume we have a topology on $\mathsf{Top}(X, Y)$ for which the evaluation map is continuous. Consider a continuous map $\hat{g}\colon Z \to \mathsf{Top}(X, Y)$, and look at the following diagram

$$X \times Z \xrightarrow{\mathrm{id} \times \hat{g}} X \times \mathsf{Top}(X, Y) \xrightarrow{\mathrm{eval}} Y$$

The identity is continuous, $\hat{g}$ is continuous, and eval is continuous, so the composition $\mathrm{eval}(\mathrm{id} \times \hat{g})$ is continuous. The composition is precisely g, proving that the topology on $\mathsf{Top}(X, Y)$ is conjoining

For the other direction, assume we have a conjoining topology on $\mathsf{Top}(X, Y)$. Since the adjunct of the evaluation map $\widehat{\mathrm{eval}}\colon \mathsf{Top}(X, Y) \to \mathsf{Top}(X, Y)$ is the identity, which is always continuous, we conclude that the evaluation map is continuous. □

Lemma 5.2 Every splitting topology on $\mathsf{Top}(X, Y)$ is coarser than every conjoining topology.

Proof. Let $\mathcal{T}, \mathcal{T}'$ be topologies on $\mathsf{Top}(X, Y)$. If $\mathcal{T}'$ is conjoining, then the evaluation map $X \times (\mathsf{Top}(X, Y), \mathcal{T}') \to Y$ is continuous. If in addition $\mathcal{T}$ is splitting, then the adjunct of the evaluation map $X \times (\mathsf{Top}(X, Y), \mathcal{T}') \to Y$ is continuous. Since the adjunct of the evaluation map is the identity $(\mathsf{Top}(X, Y), \mathcal{T}') \to (\mathsf{Top}(X, Y), \mathcal{T})$, we conclude that $\mathcal{T} \subseteq \mathcal{T}'$. □

The balancing act follows directly from Lemma 5.2.

Theorem 5.3 If there exists an exponential topology on $\mathsf{Top}(X, Y)$, then it is unique.

Proof. Suppose $\mathcal{T}$ and $\mathcal{T}'$ are exponential topologies on $\mathsf{Top}(X, Y)$. Since $\mathcal{T}$ is splitting and $\mathcal{T}'$ is conjoining, we have $\mathcal{T} \subseteq \mathcal{T}'$. And vice versa: we have $\mathcal{T}' \subseteq \mathcal{T}$ since $\mathcal{T}'$ is splitting and $\mathcal{T}$ is conjoining. □

The catch, as you might have guessed, is that there might not exist an exponential topology on $\mathsf{Top}(X, Y)$—there may be a gap between the splitting topologies and the conjoining topologies on $\mathsf{Top}(X, Y)$. So at this point, you might wonder about taking an "adjoint functor theorem" approach to finding a right adjoint to the functor $X \times -\colon \mathsf{Top} \to \mathsf{Top}$. To do so, we would have to explore the extent to which $X \times -$ preserves colimits. Whether $X \times -$ preserves colimits depends on X. The spaces X for which the functor $X \times -$ preserves colimits were characterized in 1970 as those spaces that are *core-compact* (Day and Kelly, 1970). We'll skip the definition of core-compact here, except to say that a Hausdorff space is core-compact if and only if it is locally compact. It follows that for a locally compact Hausdorff space X, there does exist an exponential topology on $\mathsf{Top}(X, Y)$ for any space Y. Even better, this exponential topology, when X is locally compact Hausdorff, coincides with what is classically called the *compact-open topology* (Fox, 1945). So, at least when X is locally compact and Hausdorff, we can get started thinking about the desired categorical properties of mapping spaces using ideas in classical topology. This is what we do in the next few sections.

Before going on to the compact-open topology, we should give a general categorical definition. In any category C that has finite products, one can ask if for all $X, Y \in \mathsf{C}$ the set $\mathsf{C}(X, Y)$ can be considered as an object in C and if so, whether it supports a product-hom adjunction $X \times -\colon \mathsf{C} \rightleftarrows \mathsf{C}\colon \mathsf{C}(X, -)$. If the answer is yes, then the category is referred to as *Cartesian closed*. The fact that exponential topologies do not always exist implies that the category Top is *not* Cartesian closed. One might try to find a "convenient" category of topological spaces in which the product-hom adjunction holds that is also rich enough to contain the spaces we care about (Brown, 2006; Isbell, 1975; Steenrod, 1967; Stacey et al., 2019). One might guess that locally compact Hausdorff spaces are such a category. But no. Nevermind that we might want some non-Hausdorff spaces. If X and Y are both locally compact and Hausdorff, then $\mathsf{Top}(X, Y)$ may not be. In the final section 5.7 of this chapter, we discuss the search for a convenient Cartesian closed category of topological spaces. That discussion will involve a shift in perspective that, once again, is illuminated with adjunctions.

5.6.1 The Compact-Open Topology

Let's now define the compact-open topology, and try to give you a feel for it.

Definition 5.4 Let X and Y be topological spaces. For each compact set $K \subseteq X$ and each open set $U \subseteq Y$, define $S(K, U) := \{f \in \mathsf{Top}(X, Y) \mid fK \subseteq U\}$. The sets $S(K, U)$ form a subbasis for a topology on $\mathsf{Top}(X, Y)$ called the *compact-open topology*.

Notice that a subbasis for the product topology on $\mathsf{Top}(X, Y)$ consists of sets

$$S(F, U) = \{(f : X \to Y) \mid fF \subseteq U\}$$

where $F \subseteq X$ is finite and $U \subseteq Y$ is open. That is, the product topology is what one might call the "finite-open" topology. In the case when X has the discrete topology, all functions $X \to Y$ are continuous and the compact-open topology on $\mathsf{Top}(X, Y)$ coincides with the product topology on $\mathsf{Top}(X, Y)$. More generally, every finite set is compact, so the compact-open topology is finer than the product-topology. As a consequence, fewer filters converge in the compact-open topology than in the product topology.

In fact, a closer look at convergence can give you a good feel for the difference between the product topology and the compact-open topology. Let's start by looking at sequences. A sequence of functions $\{f_n : [0, 1] \to [0, 1]\}_{n \in \mathbb{N}}$ converges to a limiting function f in the product topology if and only if the sequence converges pointwise. On the other hand, a sequence of functions $\{f_n : [0, 1] \to [0, 1]\}_{n \in \mathbb{N}}$ converges to f in the compact-open topology if and only if the sequence converges uniformly. To see this, consider a more general situation. Suppose that X is compact and Y is a metric space. Then $\mathsf{Top}(X, Y)$ becomes a metric space with the metric defined by

$$d(f, g) := \sup_{x \in X} d(fx, gx)$$

Two functions $f, g \in \mathsf{Top}(X, Y)$ are close in this metric if their values fx and gx are close for all points $x \in X$. A sequence $\{f_n\}$ in $\mathsf{Top}(X, Y)$ converges to f in this metric topology if and only if for all $\varepsilon > 0$ there exists an $n \in N$ so that for all $k > n$ and for all $x \in X$,

$d(f_k x, g_k x) < \varepsilon$. The fact that when X is compact and Y is a metric space the compact-open topology coincides with this metric topology is the content of the next theorem. First, a lemma.

Lemma 5.3 Let X be a metric space and let U be open. For every compact set $K \subseteq U$, there is an $\varepsilon > 0$ so that for any $x \in K$ and any $y \in X \smallsetminus U$, $d(x, y) > \varepsilon$.

Proof. This is a straightforward argument using the definition of compactness. □

Theorem 5.4 Let X be compact and Y be a metric space. The compact-open topology on $\mathsf{Top}(X, Y)$ is the same as the metric topology.

Proof. Let $f \in \mathsf{Top}(X, Y)$ and $\varepsilon > 0$ be given. Consider $B(f, \varepsilon)$. We'll find a set O that is open in the compact-open topology, with $f \in O \subseteq B(f, \varepsilon)$. Hence, compact-open neighborhoods of f refine the metric neighborhoods of f, proving that the compact-open topology is finer than the metric topology. Now, since X is compact, its image fX is compact. Since the collection $\left\{B\left(fx, \frac{\varepsilon}{3}\right)\right\}_{x\in X}$ is an open cover of fX it has a finite subcover

$$\left\{B\left(fx_1, \tfrac{\varepsilon}{3}\right), \ldots, B\left(fx_n, \tfrac{\varepsilon}{3}\right)\right\}$$

Define compact subsets $\{K_1, \ldots, K_n\}$ of X and open subsets $\{U_1, \ldots, U_n\}$ of Y by

$$K_i := \overline{f^{-1}\left(B\left(fx_i, \tfrac{\varepsilon}{3}\right)\right)} \qquad \text{and} \qquad U_i := B\left(fx_i, \tfrac{\varepsilon}{2}\right)$$

Since f is continuous, $f\overline{A} \subseteq \overline{fA}$ for any set A. In particular,

$$fK_i \subseteq \overline{B\left(fx_i, \tfrac{\varepsilon}{3}\right)} \subseteq B\left(fx_i, \tfrac{\varepsilon}{2}\right) = U_i$$

for each $i = 1, \ldots, n$. Therefore, f is in the open set $O := \cap_{i=1}^{n} S(K_i, U_i)$. To see that $O \subseteq B(f, \varepsilon)$, let $g \in O$. If $x \in K_i$ for some i, we have $fx, gx \in U_i$ since $f, g \in S(K_i, U_i)$. Therefore,

$$d(fx, gx) \leq d(fx, fx_i) + d(fx_i, gx) = \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon$$

Since the balls $\left\{B\left(fx_i, \frac{\varepsilon}{3}\right)\right\}$ cover fX, the compact sets $\{K_i\}$ cover X and every point x lies in K_i for some i. Therefore, $d(fx, gx) < \varepsilon$ for every $x \in X$, and so $d(f, g) < \varepsilon$ in $\mathsf{Top}(X, Y)$.

To show that the metric topology is finer than the compact-open topology, let $K \subseteq X$ be compact, $U \subseteq Y$ be open, and consider $f \in S(K, U)$. From Lemma 5.3, we know there exists a fixed $\varepsilon > 0$ so that for any $y \in fK$ and any $y' \in Y \smallsetminus fU$, $d(y, y') \geq \varepsilon$. Then if $g \in B(f, \varepsilon)$, we have $d(fx, gx) < \varepsilon$ for every $x \in X$. Therefore, if $x \in K$, then $gx \in U$, and we see that $gK \subseteq U$. This proves $B(f, \varepsilon) \subseteq S(K, U)$. If $O = S(K_1, U_1) \cap \cdots \cap S(K_n, U_n)$ is any basic open set in the compact-open topology, we have the open metric ball $B(f, \varepsilon) \subseteq O$ where $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$. This proves that every basic open set in the compact-open topology is open in the metric topology and hence the metric-topology is finer than the compact-open topology. □

Now that we've given a feel for the compact-open topology, let's look at it in the context of mapping spaces. The first thing to notice is that the compact-open topology is coarse enough to be splitting.

Theorem 5.5 For any spaces X and Y, the compact-open topology on $\mathsf{Top}(X,Y)$ is splitting.

Proof. Let Z be any space, and suppose $g\colon X \times Z \to Y$ is continuous. To show that the adjunct $\hat{g}\colon Z \to \mathsf{Top}(X,Y)$ is continuous, consider a subbasic open set $S(K,U)$ in $\mathsf{Top}(X,Y)$. We need to show that $(\hat{g})^{-1}S(K,U) = \{z \in Z \mid g(K,z) \subseteq U\}$ is open in Z. Let $z \in (\hat{g})^{-1}S(K,U)$. So, $z \in Z$ and $g(K,z) \subseteq U$. Since g is continuous, we know that $g^{-1}U = \{(x,z) \mid g(x,z) \subseteq U\}$ is open in $X \times Z$ and contains $K \times \{z\}$. Therefore, the Tube Lemma says there are open sets V and W with $K \subseteq V$ and $z \in W$ with $K \times \{z\} \subseteq V \times W \subseteq g^{-1}U$. Then, $z \in W \subseteq (\hat{g})^{-1}S(K,U)$ as needed. □

Theorem 5.6 If X is locally compact and Hausdorff and Y is any space, then the compact-open topology on $\mathsf{Top}(X,Y)$ is exponential.

Proof. We only need to check that the compact-open topology is conjoining, and this is equivalent to showing that the evaluation map $\mathrm{eval}\colon X \times \mathsf{Top}(X,Y) \to Y$ is continuous at every point (x,f). Let $(x,f) \in X \times \mathsf{Top}(X,Y)$, and let $U \subseteq Y$ be an open set containing the evaluation $\mathrm{eval}(x,f) = fx$. Because f is continuous, $f^{-1}U$ is an open set in X containing x. Since X is locally compact and Hausdorff, there exists an open set $V \subseteq X$ with $K := \overline{V}$ compact and $x \in V \subseteq K \subseteq f^{-1}U$. This implies that $fx \in fK \subseteq U$. Then $V \times S(K,U)$ is an open set in $X \times \mathsf{Top}(X,Y)$ with $(x,f) \in V \times S(K,U)$ and $\mathrm{eval}(V \times S(K,U)) \subseteq U$. □

And thus, when X is locally compact and Hausdorff and $\mathsf{Top}(X,Y)$ is equipped with the compact-open topology, we have the desired property listed at the opening of section 5.6. As an application, let's prove theorem 2.20 from chapter 2. We begin with a lemma.

Lemma 5.4 If $f\colon X \to Y$ is a quotient map and Z is locally compact and Hausdorff, then $f \times \mathrm{id}_Z\colon X \times Z \to Y \times Z$ is a quotient map.

Proof. Let $f\colon X \to Y$ be a quotient map. We want to prove that the product $Y \times Z$ has the quotient topology inherited from the map $f \times \mathrm{id}_Z$. So consider $Y \times Z$ with two possibly distinct topologies: $(Y \times Z)_p$ will denote the product topology, and $(Y \times Z)_q$ will denote the quotient topology inherited from the map $f \times \mathrm{id}_Z\colon X \times Z \to Y \times Z$.

The universal property of the quotient topology tells us which maps out of $(Y \times Z)_q$ are continuous. In particular, $\mathrm{id}\colon (Y \times Z)_q \to (Y \times Z)_p$ is continuous since $f \times \mathrm{id}_Z\colon X \times Z \to (Y \times Z)_p$ is continuous. That is, in the diagram below, the dashed map is continuous because the solid diagonal is continuous.

$$\begin{tikzcd}
 & X \times Z \arrow[d, "\pi"] \arrow[ddl, bend right, "f\times \mathrm{id}_Z"'] \\
 & (Y \times Z)_q \arrow[dl, dashed, "\mathrm{id}"] \\
(Y \times Z)_p &
\end{tikzcd}$$

So we only have to prove that the identity in the other direction $\mathrm{id}\colon (Y \times Z)_p \to (Y \times Z)_q$ is continuous. Since Z is locally compact Hausdorff, it suffices to prove that the adjunct $\widehat{\mathrm{id}}\colon Y \to \mathsf{Top}(Z, (Y \times Z)_q)$ is continuous. As a map out of Y, $\widehat{\mathrm{id}}$ will be continuous if its precomposition with the quotient map f is continuous. That is, in the diagram below, the dashed map will be continuous if the solid diagonal map is continuous.

$$\begin{tikzcd}
 & X \arrow[d, "f"] \arrow[ddl, bend right] \\
 & Y \arrow[dl, dashed, "\widehat{\mathrm{id}}"] \\
\mathsf{Top}(Z, (Y \times Z)_q) &
\end{tikzcd}$$

The solid diagonal map in the picture is $\widehat{\pi}$, the adjunct of the continuous quotient map $\pi\colon X \times Z \to (Y \times Z)_q$, and so it is indeed continuous. □

Theorem 5.7 If $X_1 \twoheadrightarrow Y_1$ and $X_2 \twoheadrightarrow Y_2$ are quotient maps and Y_1 and X_2 are locally compact and Hausdorff, then $X_1 \times X_2 \twoheadrightarrow Y_1 \times Y_2$ is a quotient map.

Proof. Suppose Y_1 and X_2 are locally compact and Hausdorff and that $f_1\colon X_1 \twoheadrightarrow Y_1$ and $f_2\colon X_2 \twoheadrightarrow Y_2$ are quotient maps. By the lemma, the two maps $f_1 \times \mathrm{id}_{X_2}\colon X_1 \times X_2 \twoheadrightarrow Y_1 \times X_2$ and $\mathrm{id}_{Y_1} \times f_2\colon Y_1 \times X_2 \twoheadrightarrow Y_1 \times Y_2$ are quotient maps. Therefore the composition

$$(\mathrm{id}_{Y_1} \times f_2) \circ (f_1 \times \mathrm{id}_{X_2}) : X_1 \times X_2 \twoheadrightarrow Y_1 \times Y_2$$

is a quotient map. □

In the next section, we'll give an application to analysis. Unlike $\mathbb{R}^n$, whose compact subsets are nicely characterized by the *Heine-Borel theorem* as the sets that are closed and bounded, it's not always easy to decide when a subset of $\mathsf{Top}(X, Y)$ is compact. Ascoli's theorem in the next section provides a criterion.

5.6.2 The Theorems of Ascoli and Arzela

It's not difficult to decide when a family of functions is compact using the product topology.

Theorem 5.8 If X is any space and Y is Hausdorff, then a subset $A \subseteq \mathsf{Top}(X, Y)$ has compact closure in the product topology if and only if for each $x \in X$, the set $A_x = \{fx \in Y \mid f \in A\}$ has compact closure in Y.

Proof. This was exercise 2.19 at the end of chapter 2. □

If we can identify families of functions for which the product topology and the compact-open topology coincide, then we have necessary and sufficient conditions for such families to be compact in the compact-open topology. The following definition provides a common way to make such an identification.

Definition 5.5 Let X be a topological space and (Y, d) be a metric space. A family $A \subseteq \mathsf{Top}(X, Y)$ is called *equicontinuous at* $x \in X$ if and only if for every $\varepsilon > 0$, there exists an open neighborhood U of x so that for every $u \in U$ and for every $f \in A$, $d(fx, fu) < \varepsilon$. If $\mathcal{F}$ is equicontinuous for every $x \in X$, the family A is simply called *equicontinuous*.

So a family of functions is equicontinuous if, within a neighborhood, one can bound the variation of every function in the family by a single epsilon. For this section, the most important facts about equicontinuous families are that the compact-open topology agrees with the product topology on them and that their closures are also equicontinuous. We leave the proofs of these two facts as exercises for you to solve.

Lemma 5.5 Let X be a topological space and (Y, d) be a metric space. If $A \subseteq \mathsf{Top}(X, Y)$ is an equicontinuous family, then the subspace topology on A of $\mathsf{Top}(X, Y)$ with compact-open topology is the same as the subspace topology on A of $\mathsf{Top}(X, Y)$ with the product topology.

Lemma 5.6 If $A \subseteq \mathsf{Top}(X, Y)$ is equicontinuous, then the closure of A in $\mathsf{Top}(X, Y)$ using the product topology is also equicontinuous.

Putting these ideas together gives us the famous theorems of Ascoli and Arzela.

Ascoli's Theorem Let X be locally compact Hausdorff, and let (Y, d) be a metric space. A family $\mathcal{F} \subseteq \mathsf{Top}(X, Y)$ has compact closure if and only if $\mathcal{F}$ is equicontinuous and for every $x \in X$, the set $F_x := \{fx \mid f \in \mathcal{F}\}$ has compact closure.

Arzela's Theorem Let X be compact, (Y, d) be a metric space and $\{f_n\}$ be a sequence of functions in $\mathsf{Top}(X, Y)$. If $\{f_n\}$ is equicontinuous and if for each $x \in X$ the set $\{f_n x\}$ is bounded, then $\{f_n\}$ has a subsequence that converges uniformly.

5.6.3 Enrich the Product-Hom Adjunction in Top

In the case that $\mathsf{Top}(X, Y)$ has an exponential topology, we know it's unique. Let's introduce some notation for it.

Definition 5.6 Define the space Y^X to be the set $\mathsf{Top}(X, Y)$ with its exponential topology, provided it exists.

Suppose we have an exponential topology on $\mathsf{Top}(X, Y)$ and use Y^X to denote it. We have a bijection of sets

$$\mathsf{Top}(Z \times X, Y) \cong \mathsf{Top}(Z, Y^X) \tag{5.3}$$

Now, we can put the compact-open topology on these mapping spaces. One can then ask the question: under what conditions is the bijection of sets in (5.3) a homeomorphism? One answer is: when X is locally compact and Hausdorff and when Z is Hausdorff. Instead of proving this (as done in Hatcher, 2002, 529–532), let's prove a slightly weaker version by assuming that Z is, additionally, locally compact. Then the compact open topology $\mathsf{Top}(Z, Y^X)$ will be exponential. Also, since the product of locally compact Hausdorff spaces is locally compact Hausdorff (exercise 2.9 at the end of chapter 2), the compact open topology on $\mathsf{Top}(Z \times X, Y)$ will be exponential. So, we're looking at proving that

$$Y^{Z \times X} \cong \left(Y^X\right)^Z$$

The payoff in proving the weaker version is that we have a lot more adjunctions available and we can give a clean, categorical argument, as in Strickland (2009), without diving in to the topologies themselves.

Theorem 5.9 If X and Z are locally compact Hausdorff, then for any space Y, the isomorphism of sets $\mathsf{Top}(Z \times X, Y) \to \mathsf{Top}(Z, \mathsf{Top}(X, Y))$ is a homeomorphsim of spaces.

Proof. Throughout this proof, remember that for any spaces A and C, the compact open topology on $\mathsf{Top}(A, C)$ is splitting. This means that the adjunct of a continuous map $A \times B \to C$, which is a function $B \to \mathsf{Top}(A, C)$, is also continuous. Also, if B is locally compact and Hausdorff, then for any space C, the compact open topology on $\mathsf{Top}(B, C)$ is conjoining. This is equivalent to the statement that for any space C, the evaluation map $B \times \mathsf{Top}(B, C) \to \mathsf{Top}(C)$ is continuous.

Now suppose X and Z are locally compact Hausdorff. Because X is locally compact Hausdorff, the evaluation map $X \times (Y^Z)^X \to Y^Z$ is continuous. Also, because Z is locally compact Hausdorff, the evaluation map $Z \times Y^Z \to Y$ is continuous. Therefore, the composition

$$Z \times X \times (Y^Z)^X \to Z \times Y^Z \to Y$$

is continuous. Now, to keep things clear, set $A = Z \times X$ and set $B = (Y^Z)^X$. We have a continuous function $g\colon A \times B \to Y$. Since the compact-open topology is always splitting, the adjunct $\hat{g}\colon B \to \mathsf{Top}(A, Y)$ of the map g is continuous. Restoring $A = Z \times X$ and

$B = (Y^Z)^X$ reveals that the map $\hat{g}\colon (Y^Z)^X \to \mathsf{Top}(Z \times X, Y)$ is continuous. This map $\hat{g}$ is precisely the map that we want to show is a homeomorphism. We've shown it's continuous, so we're halfway done.

Because $Z \times X$ is locally compact Hausdorff, the evaluation map

$$Z \times X \times Y^{Z \times X} \to Y$$

is continuous. Now, thinking of this as a continuous map from $Z \times -$, its adjunct $X \times Y^{Z \times X} \to Y^Z$ is continuous. And again. Finally, thinking of this as a continuous map from $X \times -$, we conclude its adjunct $Y^{Z \times X} \to (Y^Z)^X$ is continuous. This is the inverse of the continuous map $\hat{g}$ in the previous paragraph, completing the proof. □

Now, let's make some concluding remarks about the compact-open topology on $\mathsf{Top}(X, Y)$ when X is locally compact Hausdorff. From a certain point of view, it's insufficient to work with locally compact Hausdorff spaces. For example, these spaces are not closed under many common constructions. The colimit of the following diagram of locally compact Hausdorff spaces

$$\mathbb{R} \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \hookrightarrow \cdots$$

is not locally compact. Moreover, even if X is locally compact and Hausdorff, Y^X with the compact-open topology may not be locally compact and Hausdorff, so the construction of a topology on a mapping space is not repeatable. One solution to all these issues involves more adjunctions—"*k*-ification" and "weak-Hausdorfication." This is the next topic.

5.7 Compactly Generated Weakly Hausdorff Spaces

Here, we present a bird's eye view of constructing a topology on $\mathsf{Top}(X, Y)$ in a more general setting. The main idea is to find a "convenient category" of topological spaces which has limits and colimits, has exponential objects with the desirable property listed in section 5.6, and is large enough to contain the spaces we care about. The category of compactly generated weakly Hausdorff spaces is such a category. The importance of compactly generated spaces for topologies on function spaces was recognized early on by Brown (1964). A categorical perspective along the lines we present here, including the behavior of compactly generated spaces under limits, occurred later (Steenrod, 1967). Finally, Hausdorff was replaced with weakly Hausdorff yielding what is often considered the most convenient category of topological spaces (McCord, 1969).

Be aware that the terminology is not used consistently in the literature. For example, "compactly generated" in May (1999) means what we call "compactly generated and weak Hausdorff." Here we err on over-adjectivized terminology in an effort to avoid any confusion, especially since we omit most of the proofs. We'll start with a few definitions.

Definition 5.7 A space X is *compactly generated* if and only if for all compact spaces K and continuous maps $f\colon K \to X$, the set $f^{-1}A$ being closed (or open) implies A is closed (or open).

Definition 5.8 A space X is *weakly Hausdorff* if and only if for all compact spaces K and continuous maps $f\colon K \to X$, the image fK is closed in X.

Definition 5.9 Let CG, WH, and CGWH denote the full subcategories of compactly generated, weakly Hausdorff, and compactly generated weakly Hausdorff spaces, respectively.

Example 5.4 The category CG includes all locally compact spaces and all first countable spaces. The category CGWH includes locally compact Hausdorff spaces and metric spaces. Notice that weakly Hausdorff lies between T_1 and T_2. The interested reader will find these properties are all distinct by searching for examples in Steen and Seebach (1995).

The reader might recall from chapter 1 that we introduced constructions of topologies in three ways: first, with a classic definition, which described the open sets; second, with a better definition that characterized the topology among a set of possible topologies; and third, with a still better definition, which characterized which functions into or out of the construction are continuous. Let's describe k-ification in these three ways. Let X be any space. First, one can define a topology on X to consist of all sets U so that $f^{-1}U$ is open for some continuous $f\colon K \to X$ from a compact space K. This topology is compactly generated. In fact, it is the smallest compactly generated topology containing the original topology on X. It is further characterized by the property that a function $g\colon X \to Y$ out of X with this topology is continuous if and only for all compact K and all functions $f : K \to X$, $gf\colon K \to Y$ is continuous. The set X with this compactly generated topology is called the *k-ification* of X and is denoted by kX. The k-ification of a map $f\colon X \to Y$ is the map f viewed as a map $kX \to kY$, which will be continuous. Thus, k-ification defines a functor $k\colon \mathsf{Top} \to \mathsf{CG}$.

Theorem 5.10 The following setup is an adjunction: $U\colon \mathsf{CG} \rightleftarrows \mathsf{Top}\colon k$, where U is the inclusion of $\mathsf{CG} \to \mathsf{Top}$ and k is the k-ification functor.

Proof. Proved as theorem 3.2 in Steenrod (1967). □

Theorem 5.1 then implies that k preserves limits and U preserves colimits. The statement that k preserves limits implies that the limit of a diagram in Top is sent by k to the limit of the k-ification of the diagram. To clarify what this means, consider two compactly generated spaces X and Y. The product $X \times Y$ in Top may not be compactly generated, but $k(X \times Y)$ is the product of X and Y in CG. This means that $k(X \times Y)$ satisfies the universal property to be a product for compactly generated spaces. The quantifier in the universal property for the product $k(X \times Y)$ involves fewer spaces, and so in general, the topology on $k(X \times Y)$ is finer than the topology on $X \times Y$.

Now, the consequences of U being a left adjoint means that U preserves colimits. This means that *if* a diagram in CG has a colimit in CG, *then* it must agree with the colimit of the diagram in Top. While there's no obvious reason that colimits in CG need exist, it is nevertheless true.

Theorem 5.11 CG is a cocomplete category.

Proof. See appendix A of Lewis (1978). □

Now let's add weakly Hausdorff to the picture with an analogous "weak-Hausdorffification" functor q. Let X be any space. We define the weakly Hausdorff space qX to be the quotient of X by the smallest closed equivalence relation in $X \times X$. You can think about what the open sets of qX are and how to characterize which functions into or out of (which will it be?) qX are continuous.

Theorem 5.12 There is an adjunction $q\colon \mathsf{CG} \rightleftarrows \mathsf{CGWH}\colon U$ where U is the inclusion of $\mathsf{CGWH} \to \mathsf{CG}$.

Proof. See appendix A of Lewis (1978). □

As a consequence, take a diagram in CGWH. The colimit of this diagram may not be weakly Hausdorff, but it is compactly generated by theorem 5.11. Then apply the functor q, which as a left adjoint preserves colimits and hence yields a space in CGWH that must be the colimit of the diagram in CGWH.

As for limits, *if* a diagram in CGWH has a limit in CGWH, then it must agree with the limit of the diagram in CG. While there's no obvious reason that limits in CGWH need exist, it is nevertheless true.

Theorem 5.13 CGWH is a complete category.

Proof. See proposition 2.22 in Strickland (2009). □

The upshot of this back and forth game between adjoint functors produces a category CGWH that is closed under limits and colimits. But be careful of multiple interpretations. For example, for two compactly generated weakly Hausdorff spaces X and Y, we have the "old" product, denoted by $X \times_o Y$, which is the product in Top. We also have the new product, denoted by $X \times Y$, which is the product in CGWH.

Now, for compactly generated weakly Hausdorff spaces X and Y, let $Y^X = k\mathsf{Top}(X, Y)$. That is, the topology we put on the space of maps from X to Y is the k-ification of the compact-open topology.

Theorem 5.14 If X and Y are CGWH, then Y^X is in CGWH. For a fixed X, the assignment $Y \mapsto Y^X$ defines a functor $-^X\colon \mathsf{CGWH} \to \mathsf{CGWH}$ that fits into the adjunction

$$X \times -\colon \mathsf{CGWH} \rightleftarrows \mathsf{CGWH}\colon -^X$$

inducing homeomorphisms of spaces $Y^{X\times Z} \cong (Y^X)^Z$.

Proof. See Lewis (1978). $\square$

Corollary 5.14.1 For $X, Y, Z \in \mathsf{CGWH}$,

(i) the functor $- \times X$ preserves colimits.

(ii) the functor $-^X$ preserves limits.

(iii) the functor Y^- takes colimits to limits.

(iv) composition $Z^Y \times Y^X \to Z^X$ is continuous.

(v) evaluation $\mathrm{eval}\colon X \times Y^X \to Y$ is continuous.

Thus, the category CGWH is Cartesian closed. The category of compactly generated weakly Hausdorff spaces has other good properties, too. We recommend the excellent notes in Strickland (2009) for statements and proofs.

We opened this chapter with the idea that relaxing a notion of equivalence can result in rich mathematics. As we've seen, the data of an adjunction $L \dashv R$ is similar to, but not quite the same as, the data of an equivalence between categories C and D. This relaxed version provides a characterization about relationships between objects in these categories: if you know all morphisms $LX \to Y$ in D, then you know all morphisms $X \to RY$ in C, and vice versa. When the category of interest is Top and X is a locally compact Hausdorff space, the compact-open topology provides a bijection of mapping spaces $\mathsf{Top}(X \times Z, Y) \cong \mathsf{Top}(Z, \mathsf{Top}(X, Y))$ for all spaces Y and Z. More generally, the functors $X \times -$ and $\mathsf{C}(X, -)$ form an adjoint pair when the category C is a convenient one, such as CGWH.

In the next chapter, we'll again use a loosened-up version of equivalence—*homotopy equivalence*—to motivate other rich constructions and adjunctions in topology. As introduced in chapter 1, a homotopy between functions $f, g\colon X \to Y$ is a map $h\colon I \times X \to Y$ with $h(0, -) = f$ and $h(1, -) = g$. Since the unit interval is locally compact and Hausdorff, there is a bijection $\mathsf{Top}(I \times X, Y) \cong \mathsf{Top}(X, \mathsf{Top}(I, Y))$ for all spaces X and Y. On the left are homotopies and on the right are maps into $\mathsf{Top}(I, Y)$, the space of all paths in Y. This dual perspective, together with the ideas in this chapter, naturally leads to a discussion on homotopy, path spaces, and more adjunctions that arise in topology.

Exercises

1. Prove that definitions 5.1 and 5.2 are equivalent.

2. Let $L\colon \mathsf{C} \rightleftarrows \mathsf{D}\colon R$ be an adjunction with unit η and counit ϵ. Let C' be the full subcategory of C whose objects are those $X \in \mathsf{C}$ for which η_X is an isomorphism. Define D' similarly. Show that C' and D' are equivalent categories.

3. Give examples and justify your answers.

a) Find a space X and a space Y for which the evaluation map $\mathsf{Top}(X, Y) \times X \to Y$ is not continuous.

b) Find a space X and a space Y for which the evaluation map $\mathsf{Top}(X, Y) \times X \to Y$ is not continuous.

4. Prove Lemmas 5.5 and 5.6

5. Let $A = \{f \in \mathsf{Top}([0, 1], [0, 1]) \mid f \text{ is differentiable and } |f'| \leq 1\}$. Prove that $\overline{A}$ is compact.

6. Define a family of functions $\mathcal{F} \subset \mathsf{Top}([0, 1], \mathbb{R})$ by $\mathcal{F} = \{f_a\}_{0<a\leq 1}$ where $f_a(x) = 1 - \frac{x}{a}$. Prove or disprove: $\mathcal{F}$ is compact in the compact-open topology.

7. If Y is a subspace of a space X, then a map $r\colon X \to Y$ is called a *retract* of X onto Y, and Y is said to be a retract of X if $ri = \mathrm{id}_Y$ where $i\colon Y \to X$ is the inclusion map. Let $\beta\colon \mathsf{Top} \to \mathsf{CH}$ be the Stone-Čech compactification functor. Prove that any compact Hausdorff space X is a retract of $\beta U X$ where $U\colon \mathsf{CH} \to \mathsf{Top}$ includes CH as a subcategory of Top.

8. Suppose that Y is locally compact Hausdorff. Prove that for any spaces X and Z, composition $\mathsf{Top}(X, Y) \times \mathsf{Top}(Y, Z) \to \mathsf{Top}(X, Z)$ is continuous.

9. Show that a space X is Hausdorff if and only if the diagonal D is closed in $X \times_o X$. Here $X \times_o X$ means the product topology. Show that a space X is Hausdorff if and only if the diagonal D is closed in $X \times X$, where $X \times X$ means $k(X \times_o X)$.

10. Complete the proof of theorem 5.1. That is, mimic the proof that left adjoints preserve colimits, substituting a right adjoint and limit. This essentially boils down to finding an analogous natural isomorphism, allowing you to "pull a limit in the second argument of a hom out to a limit of homs."