Chapter 2
Linear Spaces

Abstract This chapter is a rigorous introduction to linear spaces, but with a strong emphasis on infinite dimensional linear spaces. No prior knowledge of linear spaces is assumed, so that all definitions and proofs are included, but some mathematical maturity is assumed, dictating the level of detail given. In particular, all results (e.g., existence of dimension) are given in full generality using Zorn's Lemma.

Keywords Linear space · Vector space · Linear transformation · Operator · Dimension · Quotient linear space · Product linear space · Inner product space · Normed space · Cauchy-Schwarz inequality

This chapter assumes a rudimentary understanding of the linear structure of $\mathbb{R}^n$, a very rich structure, both algebraically and geometrically. Elements in $\mathbb{R}^n$, when thought of as vectors, that is as entities representing direction and magnitude, can be used to form parallelograms, can be scaled, the angle between two vectors can be computed, and the length of a vector can be found. These geometric features are given algebraically by means of, respectively, vector addition, scalar multiplication, the inner product of two vectors, and the norm of a vector. In this chapter these notions are abstracted to give rise to the concepts of linear space, inner product space, and normed space.

The chapter gives a detailed presentation of all of the relevant notions of linear spaces, provides examples, and contains rigorous proofs of all of the results therein. Exploiting the assumption of a rudimentary understanding of the linear structure of $\mathbb{R}^n$, and thus of finite dimensional linear spaces, the chapter has a clear secondary goal, namely to explore the subtleties of infinite dimensional linear spaces. This is an absolute necessity with Hilbert spaces in mind, since virtually all interesting examples of Hilbert spaces are infinite dimensional.

A consequence of this infinite dimensional theme of the chapter is that some proofs are considerably more involved than their finite dimensional counterparts. In particular, the theorems establishing the existence of bases and the concept of dimension are sophisticated and resort to an application of Zorn's Lemma. Also, cardinality considerations are important, since one needs to be able to compute, at least a little bit, with infinite quantities. To facilitate an easier reading of this chapter,

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