Bilinear forms and their matrices

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March 11, 2011

0.1 Definitions

A *bilinear form* on a vector space V over a field $\mathbb{F}$ is a map

$$H : V \times V \to \mathbb{F}$$

such that

(i) $H(v_1 + v_2, w) = H(v_1, w) + H(v_2, w),$ for all $v_1, v_2, w \in V$

(ii) $H(v, w_1 + w_2) = H(v, w_1) + H(v, w_2),$ for all $v, w_1, w_2 \in V$

(iii) $H(av, w) = aH(v, w),$ for all $v, w \in V, a \in \mathbb{F}$

(iv) $H(v, aw) = aH(v, w),$ for all $v, w \in V, a \in \mathbb{F}$

A bilinear form H is called *symmetric* if $H(v, w) = H(w, v)$ for all $v, w \in V$.

A bilinear form H is called *skew-symmetric* if $H(v, w) = -H(w, v)$ for all $v, w \in V$.

A bilinear form H is called *non-degenerate* if for all $v \in V$, there exists $w \in V$, such that $H(w, v) \neq 0$.

A bilinear form H defines a map $H^{\#} : V \to V^*$ which takes w to the linear map $v \mapsto H(v, w)$. In other words, $H^{\#}(w)(v) = H(v, w)$.

Note that H is non-degenerate if and only if the map $H^{\#} : V \to V^*$ is injective. Since V and V^* are finite-dimensional vector spaces of the same dimension, this map is injective if and only if it is invertible.

0.2 Matrices of bilinear forms

If we take $V = \mathbb{F}^n$, then every $n \times n$ matrix A gives rise to a bilinear form by the formula

$$H_A(v, w) = v^t A w$$

Example 0.1. Take $V = \mathbb{R}^2$. Some nice examples of bilinear forms are the ones coming from the matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$