1 Derive Triangle Matrix

What elementary matrix can be used to convert a singular matrix to upper triangle matrix with Left Multiplication

$$
\begin{aligned}
&\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}\\
&=\begin{bmatrix}r_1\\r_2\\r_3\end{bmatrix}\begin{bmatrix}u_1&u_2&u_3\end{bmatrix}\\
&=\begin{bmatrix}\langle r_1,u_1\rangle&\langle r_1,u_2\rangle&\langle r_1,u_3\rangle\\\langle r_2,u_1\rangle&\langle r_2,u_2\rangle&\langle r_2,u_3\rangle\\\langle r_3,u_1\rangle&\langle r_3,u_2\rangle&\langle r_3,u_3\rangle\end{bmatrix}\\
&=\begin{bmatrix}b\langle r_1,u_1\rangle&b\langle r_1,u_2\rangle&b\langle r_1,u_3\rangle\\a\langle r_2,u_1\rangle&a\langle r_2,u_2\rangle&a\langle r_2,u_3\rangle\\\langle r_3,u_1\rangle&\langle r_3,u_2\rangle&\langle r_3,u_3\rangle\end{bmatrix}\\
&=\begin{bmatrix}b&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&a&0\\0&0&1\end{bmatrix}\begin{bmatrix}\langle r_1,u_1\rangle&\langle r_1,u_2\rangle&\langle r_1,u_3\rangle\\\langle r_2,u_1\rangle&\langle r_2,u_2\rangle&\langle r_2,u_3\rangle\\\langle r_3,u_1\rangle&\langle r_3,u_2\rangle&\langle r_3,u_3\rangle\end{bmatrix}\\
&=\begin{bmatrix}1&0&0\\1&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}b&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&a&0\\0&0&1\end{bmatrix}\begin{bmatrix}\langle r_1,u_1\rangle&\langle r_1,u_2\rangle&\langle r_1,u_3\rangle\\\langle r_2,u_1\rangle&\langle r_2,u_2\rangle&\langle r_2,u_3\rangle\\\langle r_3,u_1\rangle&\langle r_3,u_2\rangle&\langle r_3,u_3\rangle\end{bmatrix}\\
&=\begin{bmatrix}\langle br_1,u_1\rangle&\langle br_1,u_2\rangle&\langle br_1,u_3\rangle\\\langle br_1,u_1\rangle-\langle ar_2,u_1\rangle&\langle br_1,u_2\rangle-\langle ar_2,u_2\rangle&\langle br_1,u_3\rangle-\langle ar_2,u_3\rangle\\\langle r_3,u_1\rangle&\langle r_3,u_2\rangle&\langle r_3,u_3\rangle\end{bmatrix}
\end{aligned}\tag{1}
$$

2 From intuition to proof

$$
\begin{aligned}
&L_0A_0=A_1\\
&L_1A_1=A_2\\
&L_k\ldots L_2L_1A_0=A_k=U\\
&L=L_k\ldots L_2L_1\\
&LA=U\\
&\text{Apply LU factorization on } L \text{ again}\\
&L'_k\ldots L'_1L'_0L=U'\\
&U' \text{ should be the identity}\\
&\Rightarrow L^{-1}=L'_k\ldots L'_1L'_0
\end{aligned}\tag{2}
$$

3 Let's prove it

$$L^{-1}=L'_k\ldots L'_1L'_0$$

$$
\begin{aligned}
&A_0=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}\\
&\Rightarrow\begin{bmatrix}1&0&0\\x_{21}&1&0\\y_{31}&0&1\end{bmatrix}\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\0&a_{22}&a_{23}\\0&a_{32}&a_{33}\end{bmatrix}\\
&\Rightarrow\langle r_2^*,v_1\rangle=\left\langle\begin{bmatrix}x_{21}\\1\\0\end{bmatrix},\begin{bmatrix}a_{11}\\a_{21}\\a_{31}\end{bmatrix}\right\rangle=0\\
&\Rightarrow a_{11}x_{21}+a_{21}=0\quad\Rightarrow x_{21}=\frac{-a_{21}}{a_{11}}\\
&\Rightarrow\langle r_3^*,v_1\rangle=\left\langle\begin{bmatrix}y_{31}\\0\\1\end{bmatrix},\begin{bmatrix}a_{11}\\a_{21}\\a_{31}\end{bmatrix}\right\rangle=0\\
&\Rightarrow a_{11}y_{31}+a_{31}=0\quad\Rightarrow y_{31}=\frac{-a_{31}}{a_{11}}\\
&\Rightarrow a_{kk}y_{3k}+a_{3k}=0\quad 1\rightarrow k\\
&\Rightarrow a_{kk}y_{ik}+a_{ik}=0\quad 3\rightarrow i\\
&\Rightarrow y_{ik}=\frac{-a_{ik}}{a_{kk}}\\
&\left\langle\begin{bmatrix}0\\\vdots\\y_{k,k}=1\\\vdots\\y_{i,k}\\\vdots\\0\end{bmatrix},\begin{bmatrix}a_{1k}\\\vdots\\a_{kk}\\\vdots\\a_{ik}\\\vdots\\a_{mk}\end{bmatrix}\right\rangle=0\\
&\Rightarrow y_{i,k}=\frac{-a_{i,k}}{a_{k,k}}\quad i\in[k+1,\ldots m]\\
&\text{Let } l_{i,k}=\frac{-a_{i,k}}{a_{k,k}}\quad i\in[k+1,\ldots m]\\
&v_k=\begin{bmatrix}0\\\vdots\\0\\l_{k+1,k}\\l_{k+2,k}\\\vdots\\l_{m,k}\end{bmatrix}\qquad\qquad l'_k=v_ke_k^*=v_k\otimes e_k\\
&l'_k=v_k\begin{bmatrix}0&\cdots&1&\cdots\end{bmatrix}\\
&l'_k=\begin{bmatrix}0v_k&\cdots&1v_k&\cdots\end{bmatrix}
\end{aligned}\tag{3}
$$

$$
\begin{aligned}
&L_k=I+v_ke_k^*=\begin{bmatrix}1&&&&\\&\ddots&&&\\&&1&&\\&&l_{k+1,k}&&\\&&\vdots&\ddots&\\&&l_{m,k}&&1\end{bmatrix}\\
&L_k^{-1}=I-v_ke_k^*=\begin{bmatrix}1&&&&\\&\ddots&&&\\&&1&&\\&&-l_{k+1,k}&&\\&&\vdots&\ddots&\\&&-l_{m,k}&&1\end{bmatrix}\\
&L_kL_k^{-1}=(I+v_ke_k^*)(I-v_ke_k^*)\\
&L_kL_k^{-1}=I-v_ke_k^*v_ke_k^*\\
&L_kL_k^{-1}=I\quad\text{where }\langle e_k,v_k\rangle=0\\
&L_kL_{k+1}=(I+v_ke_k^*)(I+v_{k+1}e_{k+1}^*)\\
&L_kL_{k+1}=I+v_ke_k^*+v_{k+1}e_{k+1}^*+v_ke_k^*v_{k+1}e_{k+1}^*\\
&L_kL_{k+1}=I+v_ke_k^*+v_{k+1}e_{k+1}^*\quad\because\langle e_k^*,v_{k+1}\rangle=0\\
&L_kL_{k+1}=\begin{bmatrix}1&&&&\\&1&&&\\&-l_{k+1,k}&1&&\\&\vdots&-l_{k+2,k}&\ddots&\\&&\vdots&&\\&-l_{m,k}&-l_{m,k}&&1\end{bmatrix}\\
&\text{In general}\\
&L_1\ldots L_m=\begin{bmatrix}1&&&&&\\-l_{2,1}&1&&&&\\\vdots&-l_{k+1,k}&1&&&\\&\vdots&-l_{k+2,k+1}&\ddots&&\\&&\vdots&&&\\-l_{m,1}&-l_{m,k}&-l_{m,k+1}&&-l_{m,m-1}&1\end{bmatrix}\\
&\text{Try this}\\
&L_kL_{k-1}=I+v_ke_k^*+v_{k-1}e_{k-1}^*+v_ke_k^*v_{k-1}e_{k-1}^*\\
&\text{We can not go anywhere}\quad\because v_ke_k^*v_{k-1}e_{k-1}^*\neq 0\\
&\text{Let's try}\\
&L_{k-1}L_k=(I+v_{k-1}e_{k-1}^*)(I+v_ke_k^*)\\
&L_{k-1}L_k=I+v_{k-1}e_{k-1}^*+v_ke_k^*+v_{k-1}e_{k-1}^*v_ke_k^*\\
&L_{k-1}L_k=I+v_{k-1}e_{k-1}^*+v_ke_k^*\quad\because\langle e_{k-1},v_k\rangle=0\\
&L_{k-2}L_{k-1}L_k=(I+v_{k-2}e_{k-2}^*)(I+v_{k-1}e_{k-1}^*+v_ke_k^*)\\
&L_{k-2}L_{k-1}L_k=I+v_{k-2}e_{k-2}^*+v_{k-1}e_{k-1}^*+v_ke_k^*+v_{k-2}e_{k-2}^*v_{k-1}e_{k-1}^*+v_{k-2}e_{k-2}^*v_ke_k^*\\
&L_{k-2}L_{k-1}L_k=I+v_{k-2}e_{k-2}^*+v_{k-1}e_{k-1}^*+v_ke_k^*\quad\because\langle e_{k-2},v_{k-1}\rangle=0,\langle e_{k-2},v_k\rangle=0\\
&L_k^{-1}L_{k-1}^{-1}=(I-v_ke_k^*)(I-v_{k-1}e_{k-1}^*)\\
&L_k^{-1}L_{k-1}^{-1}=I-v_ke_k^*-v_{k-1}e_{k-1}^*+v_ke_k^*v_{k-1}e_{k-1}^*
\end{aligned}\tag{4}
$$

4 Code it up

`dog cat` α

```
mat vk(mat m, int k){
    mat id = identity(m.length);
    for(int kk=k + 1; kk<nrow; kk++){
        id.arr[kk][k] = m.arr[kk][k]/m[k][k];
    }
    return id;
}
```

5 Inverse of unit triangle matrix is also a triangle matrix