Cross Ratios

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1 Projective Geometry and Cross ratios

Definition 1. The *projective plane* $\mathbb{P}^2$ is the set of lines through an observation point O in three dimensional space. A *projective line* l is a plane passing through O, and a *projective point* P is a line passing through O. If the line defining P is contained in the plane defining l, we say that $P \in l$.

If $\mathbb{A}^2$ is an ordinary plane which does not pass through O, then we can identify most projective points of $\mathbb{P}^2$ with ordinary points on $\mathbb{A}^2$ by taking the intersection of the line defining the projective point with $\mathbb{A}^2$. The projective line which is defined by a plane passing through O and parallel to $\mathbb{A}^2$ is called the *line at infinity*, or the *horizon line*. Projective points contained in the line at infinity are called *infinite points*.

If we take $O = (0,0,0)$, then we can put coordinates on the projective plane as follows. Every projective point P is a line through O and some other point (p,q,r). Then every point on the line defining P is of the form $(\lambda p, \lambda q, \lambda r)$ for some λ. We write $P = [p : q : r]$, where the colons indicate that we only care about the ratios of the coordinates. If $\mathbb{A}^2$ is the plane $z = 1$, then the ordinary point on $\mathbb{A}^2$ corresponding to P is $(\frac{p}{r}, \frac{q}{r}, 1)$, or if we ignore the z-coordinate it is just $(\frac{p}{r}, \frac{q}{r})$. If $r = 0$, then P is an infinite point with *slope* $\frac{q}{p}$.