6 Paths, Loops, Cylinders, Suspensions, . . .

In certain situations (such as descent theorems for fundamental groups à la van Kampen) it is much more elegant, even indispensable for understanding something, to work with fundamental groupoids with respect to a suitable packet of base points....
—Alexander Grothendieck (1997)

Introduction. Chapter 0 introduced the categorical theme that *objects are completely determined by their relationships with other objects*. It has origins in theorem 0.1, a corollary of the Yoneda lemma stating that objects X and Y in a category are isomorphic if and only if the corresponding sets $\mathsf{Top}(Z, X)$ and $\mathsf{Top}(Z, Y)$ are isomorphic for all objects Z. This notion of gleaning information by "probing" one object with another is used extensively throughout algebraic topology where a famously useful probing object for topological spaces is the circle S^1 (and the sphere S^n, more generally).

A continuous map $S^1 \to X$ is a loop within the space X, so comparing $\mathsf{Top}(S^1, X)$ and $\mathsf{Top}(S^1, Y)$ amounts to comparing the set of all loops within X and Y. In practice, however, these are massively complicated sets. To declutter the situation, it's better to consider homotopy classes of loops, where loops aren't distinguished if one can be reshaped continuously to another. The question arises: "What are the most 'fundamental' loops in X, and do they differ from those in Y?" To simplify things further, it helps to consider only those loops that start and end at a fixed point in X. The set of such homotopy classes of loops forms a group called the *fundamental group* of X at the chosen point, which defines the object assignment of a functor $\mathsf{Top} \to \mathsf{Grp}$. And with this, the meaning of "algebraic topology" begins to come to life.

These ideas motivate a more general categorical study of pointed topological spaces and homotopy classes of maps between them. That is the goal of this chapter. Along the way, we'll encounter an interesting zoo of examples of such spaces; natural maps between them; and various adjunctions involving paths, loops, cylinders, cones, suspensions, wedges, and smashes. We open with a brief refresher in section 6.1 on homotopies and alternate ways of viewing them. In section 6.2 we motivate homotopy classes of based loops as a special case of a general construction called the fundamental groupoid. Focusing on "pointed" topological spaces in section 6.3 results in the fundamental group. We then analyze the pointed version of the product-hom adjunction, called the smash-hom adjunction, in section 6.4, and specializing yet further we obtain the suspension-loop adjunction in section 6.5. This adjunction and accompanying results on fibrations in section 6.6 provide a wonderfully short proof that the fundamental group of the circle is $\mathbb{Z}$. In section 6.6.3 we will showcase four applications of this result, and in section 6.7 we'll introduce the Seifert van Kampen theorem and use it to compute the fundamental groups of other familiar topological spaces.

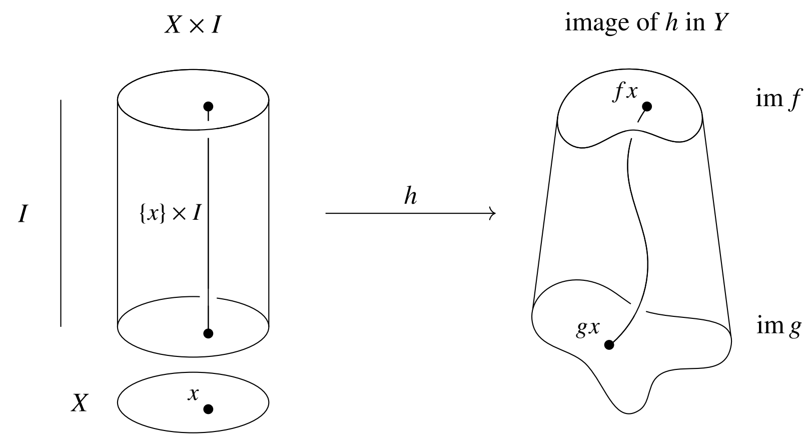
6.1 Cylinder-Free Path Adjunction

Adjunctions provide us with several different but equivalent pictures of homotopies. As usual, let $I = [0, 1]$. Recall from section 1.6 that a homotopy $h\colon I \times X \to Y$ between maps f and g is a map from $I \times X \to Y$ that on one end is the map $f = h(0, -)\colon X \to Y$ and the other end is the map $g = h(1, -)\colon X \to Y$. When such a homotopy exists, f and g are said to be homotopic, and we'll write $f \simeq g$. In this setup, one usually thinks of continuously reshaping f into g over time, which is parametrized by the unit interval. Because I is locally compact and Hausdorff, there is an adjunction between the functors $I \times -$ and $(-)^I$. We call the setup

$$I \times -\colon \mathsf{Top} \rightleftarrows \mathsf{Top}\colon (-)^I$$

the *cylinder-free path adjunction* since for a space X the space $I \times X$ is the cylinder on X and X^I is the space of paths in X. These paths are described as "free" to contrast them with a *based* path space—the space of all paths beginning at a given point—which we'll consider later on.

Under the cylinder-free path adjunction, there is a bijection $\mathsf{Top}(I \times X, Y) \cong \mathsf{Top}(X, Y^I)$ for any space Y. On the left-hand side are homotopies between maps $X \to Y$; on the right-hand side are maps that associate a point in X to a path in Y. This bijection is fairly intuitive. Suppose that h is a homotopy from f to g. Then for each $x \in X$ there is a path in Y from fx to gx. Simply fix x during the homotopy $h\colon I \times X \to Y$. This path is precisely the adjunct of h evaluated at x. So a homotopy can be viewed as a function from X to the paths in Y whose value at each x is a path from fx to gx.



Let's consider yet another way to view homotopies. For any spaces X and Y, the compact-open topology on Y^X is splitting, so we have an injection

$$\mathsf{Top}(X \times I, Y) \to \mathsf{Top}(I, Y^X)$$

So a homotopy may also be viewed as a map $I \to Y^X$. That is, a homotopy from f to g can be viewed as a path from the point f to the point g within the space of continuous functions Y^X. If X is locally compact Hausdorff, or if we're working in the category CGWH using the k-ified product and k-ified compact-open topologies, then $Y^{X\times I} \cong (Y^X)^I$ and so every path in the mapping space Y^X defines a homotopy.

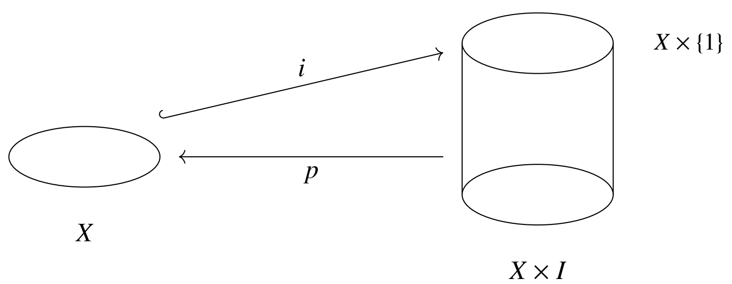
As the next theorems show, the space X, the cylinder $X \times I$, and the free path space X^I are indistinguishable in the eyes of homotopy theory. That is, these three spaces are homotopy equivalent. Homotopy equivalence was introduced in section 1.6; we'll restate it here for ease of reference.

Definition 6.1 Topological spaces X and Y are called *homotopy equivalent* if and only if there exist maps $f\colon X \to Y$ and $g\colon Y \to X$ with $fg \simeq \mathrm{id}_Y$ and $gf \simeq \mathrm{id}_X$. In this case, we write $X \simeq Y$ and f (or g) is called a *homotopy equivalence*. The category hTop is the category whose objects are spaces and whose morphisms are homotopy classes of continuous maps. So, $X \simeq Y$ if and only if $X \cong Y$ in hTop.

Theorem 6.1 The map $\pi\colon X^I \to X$ defined by $\gamma \mapsto \gamma 0$ and the map $i\colon X \to X^I$ defined by $x \mapsto c_x$, the constant path at x, are homotopy inverses.

Proof. Note that $i\pi\colon X^I \to X^I$ is the map that sends a path γ to $c_{\gamma 1}$, the constant path at $\gamma 1$. Let $h\colon X^I \times I \to X^I$ by $h(\gamma, t) = \gamma_t$ where $\gamma_t\colon I \to X$ is given by $\gamma_t s = \gamma(s + t - st)$. Then $h(\gamma, 0) = \gamma$ and $h(\gamma, 1) = c_{\gamma 1}$, and we see h is a homotopy between id_{X^I} and $i\pi$. Since $\pi i = \mathrm{id}_X$, there's nothing more to check. $\square$

Theorem 6.2 The map $i\colon X \to X\times I$ defined by $x \mapsto (x, 1)$ and the projection $p\colon X\times I \to X$ are homotopy inverses.



Proof. Exercise. $\square$

In any discussion on paths and homotopy, one is prompted to think of homotopies *between* paths. Category theory provides a good setting in which to explore this. To every space, one can associate to it a category whose objects are the points in that space and whose morphisms are homotopy classes of paths between those points. This category is called the fundamental groupoid.

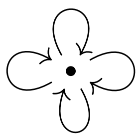
6.2 The Fundamental Groupoid and Fundamental Group

Let's first warm up with a few words about groupoids in general. As the name suggests, a groupoid is like a group. The connection is clear when one views groups from a categorical perspective.

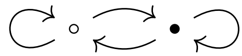
A group, as noted in chapter 0, is a category with a single object where every morphism is an isomorphism. A groupoid is a slight generalization of this.

Definition 6.2 A *groupoid* is a category in which every morphism is an isomorphism.

For example, a category with two objects, each with identity morphisms, and an invertible morphism between the two is a groupoid.



group



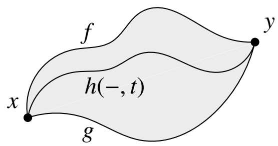
groupoid

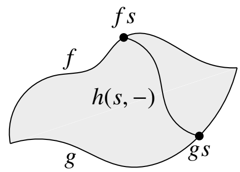
So groupoids are categories that are like groups but have possibly more than one object. For this reason, groupoids are a great way to gain intuition or inspiration when working with ideas in category theory. For example, ignoring the direction of arrows in a category results in a groupoid and hence in a structure that is very much like a group. In that sense, you might hope to "lift" theorems about groups to theorems about groupoids and then to theorems about categories. The Yoneda lemma is an example of this. It can be viewed as a lifting of *Cayley's theorem* in group theory—think about what the Yoneda lemma says when the category in question is a group.

Groupoids also appear naturally in topology when considering paths in a space. Think of a space as a category whose objects are the points in the space, and picture the morphisms to be paths between points. This doesn't quite work, though. The complication is that composition of paths isn't quite associative since the parametrization, and not just the image, is involved for paths defined as maps from the unit interval. But composition of paths is associative *up to path homotopy*. Path homotopy was defined in chapter 1. For convenience, we'll restate it here, using f and g for paths instead of α and β because we're about to think of paths as morphisms.

Definition 6.3 Two paths $f, g\colon I \to X$ from x to y in a space X are *path homotopic* if and only if there exists a homotopy $h\colon I \times I \to X$ from f to g that satisfies $h(0,t) = x$ and $h(1,t) = y$ for all t.

Under the product-hom adjunction, the homotopy may be viewed as a map $I \to X^I$ that lands in the subspace of X^I consisting of paths with fixed endpoints in X. A picture to have in mind is the cartoon below, which illustrates a simple homotopy between paths f and g. The picture can be interpreted in two equivalent ways. First, you can think of the homotopy as an extension of the paths to the shaded region. At each time t there is a path $h(-,t)$ from x to y. As t ranges from 0 to 1, one imagines f traversing the shaded region toward g. Alternatively, the homotopy can be viewed as a continuously varying family of paths from f to g. Indeed, for each point $s \in I$ there is a path from fs to gs.



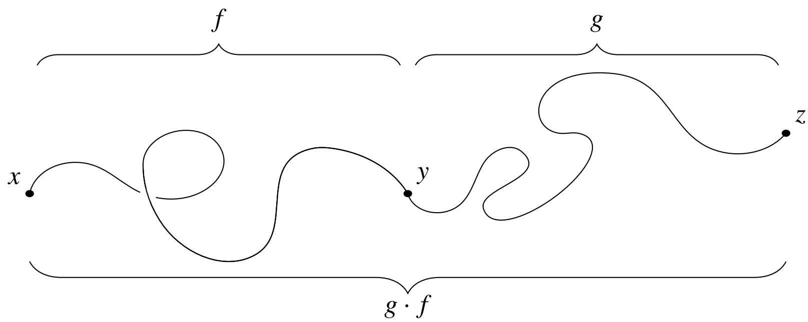


Given any two points $x, y \in X$ and a path f from one to the other, we'll be interested in the class $[f]$ of all paths from x to y that are path-homotopic to f. Points in X together with homotopy classes $[f]$ define a groupoid.

Definition 6.4 The *fundamental groupoid* $\pi_1 X$ of a space X is the category whose objects are points of X. A morphism $x \to y$ is a homotopy class of paths from x to y. Composition is given by *concatenation* of paths $[g] \circ [f] := [g \cdot f]$.

As introduced in section 2.1, if one has a path f and a path g, which begins where f ends, then their concatenation $g \cdot f$ is also a path, defined by

$$(g \cdot f)t = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$



This definition provides composition in $\pi_1 X$, making $\pi_1 X$ into a category. There are a few things to verify. First, check that composition is well defined: if $f' \simeq f$ and $g' \simeq g$ then $g' \cdot f' \simeq g \cdot f$. Also check that composition of homotopy classes of paths is associative, i.e., $(h \cdot g) \cdot f \simeq h \cdot (g \cdot f)$ for any three composable paths f, g, h. Finally, check that the homotopy class of the constant path at a point x is the identity morphism id_x. One further shows that $\pi_1 X$ is a groupoid by running paths in reverse. That is, for any path f from x to y, define a path g by $gt := f(1-t)$. This is a path from y to x and it satisfies $[g][f] = \mathrm{id}_x$ and $[f][g] = \mathrm{id}_y$. So every morphism in $\pi_1 X$ is an isomorphism.

Now consider the category Grpd whose objects are small groupoids and whose morphisms are functors between groupoids. It is not difficult to check that the fundamental groupoid defines a functor $\pi_1 \colon \mathsf{Top} \to \mathsf{Grpd}$. There's no question about what π_1 does on $\mathsf{Top}(X, Y)$. The pushforward of a morphism $\alpha \in \mathsf{Top}(X, Y)$ gives a map on paths $\alpha_* : X^I \to Y^I$. One only needs to check that α_* respects homotopy equivalence.

As a general remark, in any category C one can fix an object $X \in \mathsf{C}$ and consider the set of all isomorphisms from X to itself. Under composition, this set forms a group $\mathrm{Aut}\, X$. In particular, fixing an object $x_0 \in X$ in the fundamental groupoid $\pi_1 X$ yields a one-object category consisting of all isomorphisms from x_0 to itself. It is called *the fundamental group of X based at the point* x_0 and is denoted by $\pi_1(X, x_0)$.

Definition 6.5 The *fundamental group of X based at the point* x_0 is the group $\pi_1(X, x_0)$ of homotopy classes of loops based at $x_0 \in X$.

As another general remark, if G is a groupoid and x is any object in G, then $\mathrm{Aut}\, x$ thought of as a category with one object is a full subcategory of G. (Here we momentarily denote our object x with a lowercase letter, bearing in mind points in a topological space.) The inclusion of $\mathrm{Aut}\, x$ into the category G defines a fully faithful functor. If G is connected, meaning that there is a morphism between any two objects, then the inclusion of $\mathrm{Aut}\, x$ as a subcategory of G is also essentially surjective. Therefore, for any object x of a connected groupoid, there is an equivalence of categories between the group $\mathrm{Aut}\, x$ and the groupoid G. So, for any path connected space X, the fundamental group is equivalent as a category to the fundamental groupoid, and moreover, for any $x_0, x_1 \in X$, the fundamental groups are isomorphic $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.

Before we compute and make use of fundamental groups, we will spend the next section giving another interpretation of the fundamental group that is very good to know. Notice that a loop that begins and ends at a point x_0 is the same as a continuous function from the circle S^1 to X that sends the point $(1, 0) \in S^1$ to the point $x_0 \in X$. In this way, $\pi_1(X, x_0)$ may be viewed as homotopy classes of basepoint-preserving maps $S^1 \to X$; that is, maps that send a chosen point in S^1 to a chosen point in X. An effective, categorical way to treat basepoint-preserving maps is to consider a pair of spaces together with a point (X, x_0) as a single object and to think of basepoint-preserving maps as morphisms between these single objects.

6.3 The Categories of Pairs and Pointed Spaces

Define the *category of pairs of topological spaces* to be the category whose objects are pairs (X, A) where X is a topological space and A is a subspace of X. A morphism $f: (X, A) \to (Y, B)$ is a continuous function $f: X \to Y$ with $fA \subseteq B$. When we consider pairs for which the subset A consists of a single point, we obtain the category Top_* of pointed topological spaces. Objects in Top_* are pairs (X, x_0) where X is a topological space and x_0 is a designated point in X called the basepoint. A morphism is a continuous function $f: (X, x_0) \to (Y, y_0)$ with $fx_0 = y_0$. Such maps are said to *preserve* or *respect* basepoints.

Sometimes we'll write X for the pair (X, x_0) if it's understood that X has a basepoint $x_0 \in X$. In the case when X is locally compact and Hausdorff so that $\mathsf{Top}(X, Y)$ with the compact-open topology is exponential, then the set $\mathsf{Top}_*(X, Y)$ becomes a pointed space itself; it gets a topology as a subspace of $\mathsf{Top}(X, Y)$, and its basepoint is the constant map from X to the basepoint of Y.

It can be tempting to think that a choice of a basepoint in a space is only a matter of bookkeeping and that Top and Top_* may not be too different. But the categories do differ and in significant ways. For starters, colimits in Top_* are different than in Top. For example, the one-point space $*$ is both terminal and initial in Top_*, but it is not initial in Top. As we'll see in the next section, coproducts differ as well, though products do not.

Homotopies are more straightforward. A homotopy between spaces in Top_* is required to respect the basepoints: it's a map $h: I \times X \to Y$ such that $h(t, x_0) = y_0$ for all $t \in I$, which we'll call a *based homotopy*. And just as Top has a homotopy version hTop, so also does Top_* have a homotopy version hTop_*. This is the category whose objects are pointed topological spaces X and whose morphisms $X \to Y$ are homotopy classes of basepoint-preserving maps, the set of which is denoted by $\mathsf{hTop}_*(X, Y)$ or also by $[X, Y]_*$ or simply $[X, Y]$ if it's understood that we're working with basepoints. If X is locally compact and Hausdorff, then $[X, Y]$ is itself a pointed space. It gets a topology as a quotient of $\mathsf{Top}_*(X, Y)$, and its basepoint is the homotopy class of the constant map from X to the basepoint of Y.

We opened this chapter with a particular focus on the unit interval I and corresponding functor $\mathsf{Top}(I, -) : \mathsf{Top} \to \mathsf{Top}$. There we probed a space X with I and obtained the space of paths X^I. In the context of pointed spaces and homotopy classes of based maps, a fruitful choice of "probing space" is the sphere. By convention, the sphere S^n is a pointed space with basepoint $(1, 0, 0, \ldots, 0)$. Sometimes we'll refer to the basepoint of S^1 as 1 rather than $(1, 0)$, thinking of S^1 as the set of complex numbers z with $|z| = 1$. So one may consider the corresponding functor $\mathsf{Top}_*(S^n, -) = [S^n, -]$. When $n = 1$ this is precisely the fundamental group. That is, we have the interpretation of the fundamental group of a pointed space (X, x_0) as the set of homotopy classes of based maps from $(S^1, 1)$ to (X, x_0):

$$\pi_1(X, x_0) = [(S^1, 1), (X, x_0)]$$

This perspective has advantages. For one, it makes it clear that the fundamental group is a functor $\pi_1\colon \mathsf{Top}_* \to \mathsf{hTop}_* \to \mathsf{Set}$. It also makes it clear that it fits into a family of functors: for each $n = 0, 1, \ldots$ we have a homotopy functor $\pi_n := [S^n, -]$ from Top_* to Set called the nth *homotopy group*. What's not clear is that these functors land in Grp. The name will be justified in corollary 6.4.1 where it's shown that $\pi_n(X, x_0)$ is a group whenever $n \geq 1$.

Before moving on, we should look at the case when $n = 0$ since we've already defined a functor $\pi_0\colon \mathsf{Top} \to \mathsf{Set}$ in section 2.1.2. First, note that S^0 consists of the two points -1 and 1, and since any map $f\colon (S^0, 1) \to (X, x_0)$ must preserve basepoints, we have $1 \mapsto x_0$. So the map f is the same as a choice for $f(-1)$; that is, a point $* \to X$. Since homotopies between two such maps must also preserve basepoints, a homotopy is simply a path from one point to the other. This is consistent with our previous discussion of π_0 as the set of path components of X. If X has a basepoint, then $\pi_0(X)$ is a pointed set, the basepoint being the connected component of the basepoint of X.

So far, we've discussed some ideas in Top along with their based version in Top_*—objects, morphisms, homotopies, mapping spaces. Next, we turn to the product-hom adjuction $(X \times -) \dashv \mathsf{Top}(X, -)$ and its based version in Top_*. We already have the analogous mapping space: if X and Y have basepoints x_0 and y_0, then $\mathsf{Top}_*(X, Y)$ becomes a space as a subspace of $\mathsf{Top}(X, Y)$. The space $\mathsf{Top}_*(X, Y)$ is also based with the constant function $X \mapsto y_0$ as the basepoint. The Cartesian product $X \times Y$ also has a basepoint, (x_0, y_0). But as we'll see in the next section, the functor $X \times -$ is *not* left adjoint to $\mathsf{Top}_*(X, -)$. This motivates a new construction in Top_*, resulting in an adjunction that is the based version of the product-hom adjunction.

6.4 The Smash-Hom Adjunction

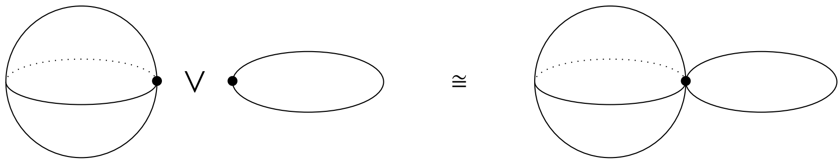
We'll begin with a more categorical discussion of the product in Top_*. To start, observe that there's a forgetful functor $U\colon \mathsf{Top}_* \to \mathsf{Top}$ that forgets basepoints. As usual, one should ask if it has a left adjoint. It does. The plus construction $+\colon \mathsf{Top} \to \mathsf{Top}_*$ is a left adjoint of U defined on objects by adding a point $X \mapsto X \coprod \{*\}$ and on morphisms $f\colon X \to Y$ by $f \mapsto \tilde{f}$ where $\tilde{f}$ extends f by sending the extra point of X to the extra point of Y. Notice that $\mathsf{Top}_*(+X, Y) \cong \mathsf{Top}(X, UY)$, so we have the adjunction

$$+\colon \mathsf{Top}_* \rightleftarrows \mathsf{Top}\colon U$$

This implies that U preserves limits. For example, the product of based spaces $(X, x_0) \times (Y, y_0)$ must be the product of the underlying topological spaces, and its basepoint is the point (x_0, y_0). Colimits, however, are different in Top_* compared with those in Top, as we hinted at earlier. In particular, the coproduct of pointed spaces X and Y is given a special name: the *wedge product*.

Definition 6.6 For pointed spaces (X, x_0) and (Y, y_0) the *wedge product* $X \vee Y$ is the quotient $X \sqcup Y/\sim$ where $x_0 \sim y_0$. The basepoint $*$ of $X \vee Y$ is the class $[x_0] = [y_0]$.

That is, the wedge product $X \vee Y$ is constructed by glueing X and Y together at their basepoints, and the basepoint of the new space is the single point where the two basepoints of X and Y have been identified. For example,



Note that there are inclusion maps

$$\begin{tikzcd}
(X, x_0) \arrow[dr, "i_X"] & & (Y, y_0) \arrow[dl, "i_Y"'] \\
& (X \vee Y, *) &
\end{tikzcd}$$

which together with the wedge product satisfy the following universal property. For any pointed space (Z, z_0) and any maps $f_X \colon (X, x_0) \to (Z, z_0)$ and $f_Y \colon (Y, y_0) \to (Z, z_0)$, there exists a unique map $f \colon (X \vee Y, *) \to (Z, z_0)$ so that $f_X = f i_X$ and $f_Y = f i_Y$.

$$\begin{tikzcd}
(X, x_0) \arrow[dr, "i_X"] \arrow[ddr, "f_X"'] & & (Y, y_0) \arrow[dl, "i_Y"'] \arrow[ddl, "f_Y"] \\
& (X \vee Y, *) \arrow[d, dashed, "f"] & \\
& (Z, z_0) &
\end{tikzcd}$$

We can establish this universal property for the wedge product in Top_* by using some of the universal properties already known for familiar coproducts in Top. First observe that a pointed space (X, x_0) is the same as a space X together with a map $* \to X$. So we can view the wedge product as the pushout:

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & X \vee Y \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

The universal property for the pushout implies that for any pointed space Z, a pair of basepoint-preserving maps $X \to Z$ and $Y \to Z$ is the same as a basepoint-preserving map $X \vee Y \to Z$. In other words, the wedge product is the coproduct in Top_*. We'll use this coproduct to give a refined version of the product-hom adjunction for based spaces.

To begin, consider the identification $\mathsf{Top}(X \times Y, Z) \cong \mathsf{Top}(Y, Z^X)$. In the context of pointed spaces, there are a couple of things to consider. First, if x_0 and z_0 are basepoints in Z and X respectively, then Z^X has a basepoint given by the constant function $f_0 \colon X \to z_0$. A map $f \colon Y \to Z^X$ on the right-hand side must preserve basepoints, meaning that it must satisfy $(fy_0)x = z_0$ for all $x \in X$. Additionally, for any $y \in Y$ the resulting map $fy \colon X \to Z$ must also preserve basepoints. That is, $(fy)x_0 = z_0$ for all $y \in Y$. Therefore, if the adjoint of a map $f \colon X \times Y \to Z$ is a basepoint preserving map $Y \to Z^X$, then f must be constant on $(\{x_0\} \times Y) \cup (X \times \{y_0\})$, sending it to z_0. This motivates the definition of the *smash product* of topological spaces.

Definition 6.7 The *smash product* of two pointed spaces (X, x_0) and (Y, y_0) is defined to be the quotient space

$$X \wedge Y \quad := \quad X \times Y / X \vee Y$$

where $X \vee Y$ is identified with the subspace $(\{x_0\} \times Y) \cup (X \times \{y_0\})$. It has (x_0, y_0) as a basepoint.

When X is locally compact and Hausdorff, the smash product is the quotient of the product by the minimal relation that ensures that there is a bijection of sets $\mathsf{Top}_*(X \wedge Y, Z) \cong \mathsf{Top}_*(Y, Z^X)$. So it is not surprising that the naturality of the product-hom adjunction in Top descends to yield the *smash-hom adjunction* among pointed spaces:

$$X \wedge - \colon \mathsf{Top}_* \rightleftarrows \mathsf{Top}_* \colon (-)^X$$

As we'll see next, an important case of the smash-hom adjuntion arises when X is taken to be the circle.

6.5 The Suspension-Loop Adjunction

There is a special name given to the smash product of the circle with a pointed space X—the *reduced suspension* of X. One can also smash X with the unit interval to obtain the *reduced cone* over X.

Definition 6.8 For a pointed space (X, x_0), the *reduced cone* CX and the *reduced suspension* are given by

$$CX := X \wedge I \quad \text{and} \quad \Sigma X := X \wedge S^1$$

More concretely,

$$
\begin{array}{rclcrcl}
CX & = & X \times I/\sim & \text{where} & (x,1) & \sim & (x',1) \\
 & & & & (x_0,t) & \sim & (x_0,s) \\
 & & & & & & \\
\Sigma X & = & X \times I/\sim & \text{where} & (x,0) & \sim & (x',0) \\
 & & & & (x,1) & \sim & (x',1) \\
 & & & & (x_0,t) & \sim & (x_0,s)
\end{array}
$$

for all $x, x' \in X$ and $s, t \in I$, and the basepoints of CX and ΣX are the classes $[x_0] = \{x_0\} \times I$. Simple sketches show that they look like quotients of (i) a cone drawn up from X to a point and (ii) a copy of X suspended between two points—one above, one below—as though by rigging lines. See figure 6.1.

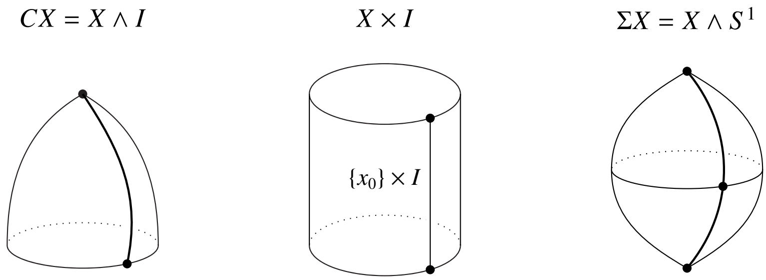


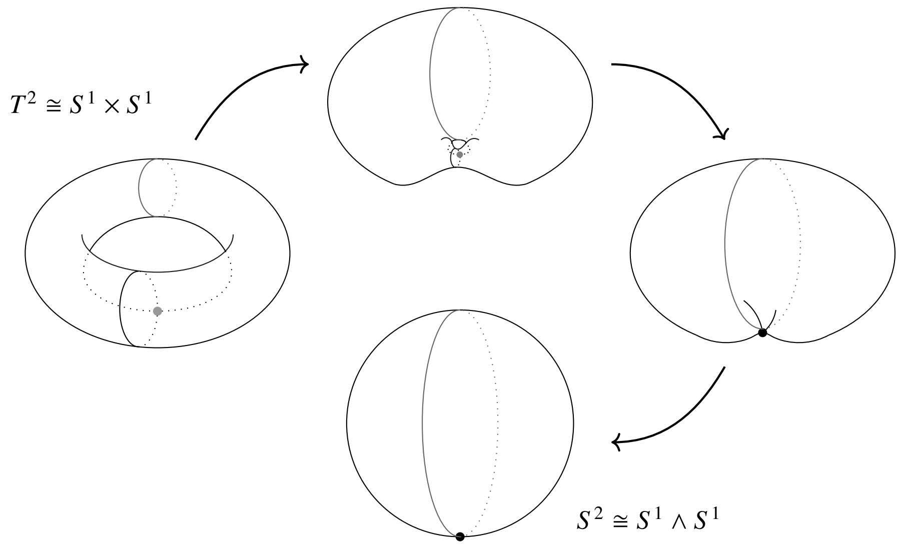
Figure 6.1 Sketches of the reduced cone (left) and reduced suspension (right), where points along the bold lines are identified.

Notice the identifications in the quotient of ΣX are consistent with those in definition 6.7—here we are using the fact that S^1 is obtained from I by identifying the endpoints 0 and 1. Now, if X does not have a basepoint, then we have the analogous "unreduced" constructions.

Definition 6.9 Let X be a topological space. The *cone* CX and *suspension* SX are defined to be

$$
\begin{array}{rclcrcl}
CX & := & X \times I/\sim & \text{where} & (x,1) & \sim & (x',1) \\
 & & & & & & \\
SX & := & X \times I/\sim & \text{where} & (x,0) & \sim & (x',0) \\
 & & & & (x,1) & \sim & (x',1)
\end{array}
$$

While the terminology may be new, suspensions arise in a familiar context: compactifications. The one-point compactification of $\mathbb{R}$, for example, is the circle S^1, and the one-point compactification of $\mathbb{R} \times \mathbb{R}$ is the sphere S^2. But another compactification of $\mathbb{R} \times \mathbb{R}$ is the torus $S^1 \times S^1$. How are these related? By the main property of the one-point compactification, S^2 is the quotient of $S^1 \times S^1$ by the extra points $(1 \times S^1) \cup (S^1 \times 1)$. That is, $S^1 \wedge S^1 \cong S^2$:



This argument works in general.

Theorem 6.3 Suppose X^* and Y^* are one-point compactifications of spaces X and Y. Then

$$X^* \wedge Y^* \quad \cong \quad (X \times Y)^*$$

where the extra points at infinity are the basepoints.

As a result, suspensions give a simple relationship between S^n and S^{n+1}.

Corollary 6.3.1 $\Sigma S^n \cong S^{n+1}$ for $n \geq 0$.

Notice that $S^1 \wedge -$ and $(-)^{S^1}$ are both functors from Top_* to Top_*, called the *reduced suspension* Σ and the *based loop functor* Ω, respectively. This gives rise to the *suspension-loop adjunction*.

$$\Sigma\colon \mathsf{Top}_* \rightleftarrows \mathsf{Top}_* : \Omega$$

The correspondence $\mathsf{Top}_*(\Sigma X, Y) \cong \mathsf{Top}_*(X, \Omega Y)$ is understood as follows. Suppose we have a map $f : \Sigma X \to Y$, and for any point $x \in X$, consider the subspace $\{x\} \times I$ of the cylinder $X \times I$. After forming the quotient ΣX, this space becomes $\{x\} \times S^1$, which is then mapped to Y via f. The assignment sending x to this loop is the adjunct of f. In particular, f must map the basepoint $*$ of ΣX to the basepoint y_0 of Y. This gives a map from $*$ to the constant loop at y_0. If we further pass to (basepoint-preserving) homotopy classes of morphisms, then we obtain for every pair of pointed spaces X and Y,

$$[\Sigma X, Y] \quad \cong \quad [X, \Omega Y]$$

which has important consequences.

Theorem 6.4 Let X be a pointed space. Then $\pi_n X \cong \pi_{n-1}\Omega X$ for each $n \geq 1$.

Proof. By corollary 6.3.1 and the suspension-loop adjunction,

$$\begin{aligned} \pi_n X &= [S^n, X] \\ &\cong [\Sigma S^{n-1}, X] \\ &\cong [S^{n-1}, \Omega X] \\ &= \pi_{n-1}\Omega X \end{aligned} \qquad \square$$

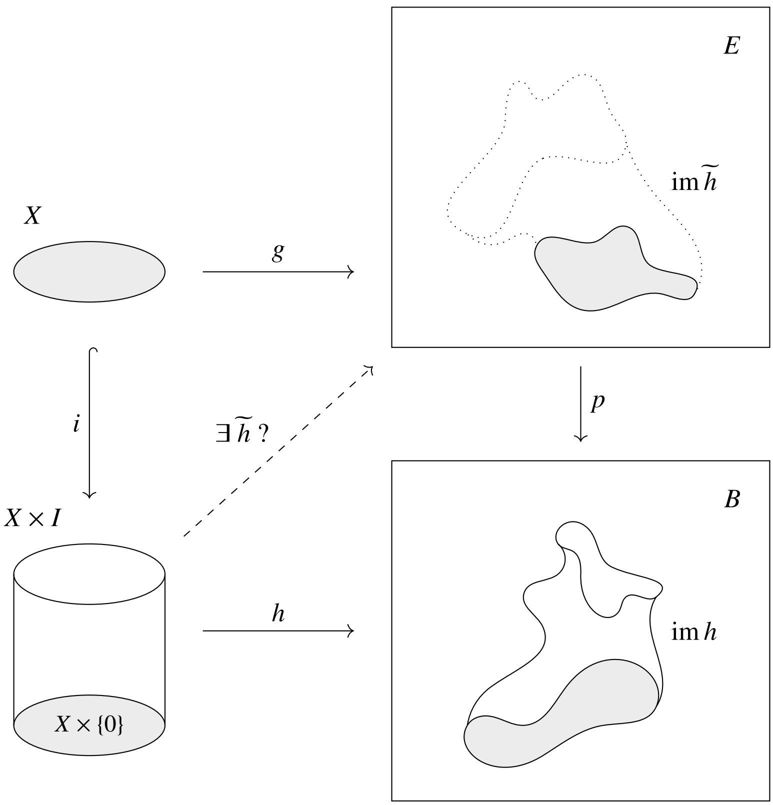
Since this construction is repeatable, we find that $\pi_n X \simeq \pi_1 \Omega^{n-1} X$. We already know that the fundamental group of any space is a group, and so we now know that the higher homotopy groups are also a group.

Corollary 6.4.1 Let (X, x_0) be a pointed space. Then $\pi_n(X, x_0)$ is a group for each $n \geq 1$.

In addition, it turns out that $\pi_n X$ is abelian if $n \geq 2$. We'll hint at the proof in theorem 6.6. Our next goal is to compute the fundamental group of some familiar spaces, starting with the circle. One would think the circle is a rather benign space, but to compute its fundamental group, we will use some new machinery—fibrations.

6.6 Fibrations and Based Path Spaces

Often in mathematics, one is interested in organizing similar objects into families. Usually, this is formalized as follows: one has a "total space" E that maps onto a "base" space B. The objects being organized into families are the fibers of the map $E \to B$. With this in mind, let us describe a situation that arises in homotopy theory. We have a map of topological spaces $p\colon E \to B$, and there is another space X that lies inside E in some way—say, $g\colon X \to E$—and which also lies in the base space B as $pg\colon X \to B$. Now suppose X lies within B as the initial part of a homotopy $h\colon X \times I \to B$. In this setup, one views the copy of X within E as being the first step in "lifting" the homotopy from B up to E. A natural question is: *Can we finish the task?* That is, can the rest of the homotopy in B be lifted to E? If the answer is "yes," then the map p is called a *fibration*.



This idea of lifting a homotopy has a dual notion: extending a homotopy. Start with a homotopy $A \times I \to X$, and suppose A sits inside another space Y by way of a map $i \colon A \to Y$. It's natural to wonder if the homotopy extends to Y. In particular, if a map $g \colon Y \cong Y \times \{0\} \to X$ already exists and is thought of as the first step of an extension, then one might hope that a full extension exists. If it does, then the map i is called a *cofibration*.

Definition 6.10 A map $p \colon E \to B$ is called a *fibration* if and only if for any maps h and g making the outer square commute, there exists $\widetilde{h} \colon Y \to X^I$ so that the whole diagram commutes:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"] \arrow[ur, dashed, "\widetilde{h}"] & B \end{tikzcd}$$

Here i is the map sending each $x \in X$ to $(x, 0)$. Often, E is referred to as the *total space* of the fibration while B is called the *base space* of the fibration.

Dually, a map $i\colon A \to Y$ is a *cofibration* if for any maps h and g making the outer square commute, there exists $\widetilde{h}\colon Y \to X^I$ so that the whole diagram commutes:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"'] & X^I \arrow[d, "e"] \\ Y \arrow[r, "g"] \arrow[ur, dashed, "\widetilde{h}"] & X \end{tikzcd}$$

Here the map e is evaluation at 0. It sends a path γ to its starting point $\gamma(0)$.

These are challenging definitions. Here's the takeaway. Fibrations are maps into B with the property that, *if a homotopy in B lifts to the total space at a point, then it lifts completely.* Cofibrations are the maps out of space A where, *if a homotopy in A extends at a point, then it extends completely.* This follows quickly from a consideration of what commutativity means in these diagrams:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"] & B \end{tikzcd} \qquad \Bigg| \qquad \begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"'] & X^I \arrow[d, "e"] \\ Y \arrow[r, "g"] & X \end{tikzcd}$$

For the left diagram to commute, the map $h(-,0)\colon X \times \{0\} \to B$, which is hi up to the isomorphism $X \cong X \times \{0\}$ and must be equal to pg, which is to say g is lift of the homotopy h at the point 0. Now let's turn to the homotopy extension diagram at the right. Recall the adjunction $\mathsf{Top}(A \times I, X) \cong \mathsf{Top}(A, X^I)$. So h in the diagram corresponds to a homotopy $A \times I \to X$. And, up to the isomorphism $X \cong X \times \{0\}$, the map g between $Y \to X^{\{0\}}$ extends the homotopy h along i at the point 0.

We've referred to fibrations and cofibrations as "dual" notions, but know that some care must be taken to properly dualize concepts in mathematics—there's usually more going on under the hood. But here's the simple idea: (co)fibrations are maps for which (extensions) lifts exist as soon as we have them for a single point. These ideas respectively are called the *homotopy lifting property* and *homotopy extension property*, and you can generally think of a (co)fibration as a map with the homotopy (extension) lifting property for all spaces.

We'll give some examples below, but first, it's good to know that the lifting/extension properties enjoyed by (co)fibrations are especially potent in a discussion of homotopy theory. From the homotopical viewpoint, any continuous function *is* either a fibration or a cofibration. Let's say a few brief words about this.

6.6.1 Mapping Path Space and Mapping Cylinder

Any continuous function factors as a homotopy equivalance followed by a fibration. Dually, any continuous function factors as a cofibration followed by a homotopy equivalence. In other words, any map can be replaced by a fibration or a cofibration at will, "up to homotopy." Even better, this a constructive statement. Given any map $f\colon X \to Y$, we can explicitly construct a homotopy equivalance and fibration that realizes this factorization, and it is similar for cofibrations. The homotopy equivalences appearing in these statements involve two topological spaces associated to f: its *mapping path space* and its *mapping cylinder.*

Definition 6.11 The *mapping path space* P_f of a map f is the pullback:

$$\begin{tikzcd} P_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ \mathcal{P}Y \arrow[r] & Y \end{tikzcd}$$

In other words, P_f consists of pairs $(x, \gamma) \in X \times \mathcal{P}Y$ where $fx = \gamma 1$. Sometimes P_f is also called the *mapping cocylinder*.

A nice consequence is that the assignment $x \mapsto (x, c_{fx})$ defines a homotopy equivalence from X to P_f. (Its homotopy inverse is simply projection onto the first factor.) And the map $P_f \to Y$ sending the pair (x, γ) to $\gamma 1$ is a fibration. The idea behind the proof is similar to that in example 6.1. Then, as claimed, we have a factorization of f as a homotopy equivalence followed by a fibration:

$$\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[rr, bend right, "f"'] & P_f \arrow[r, two heads] & Y \end{tikzcd}$$

Definition 6.12 The *mapping cylinder M_f of f* is the pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X \times I \arrow[d] \\ Y \arrow[r] & M_f \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where $X \to X \times I$ is the map $x \mapsto (x, 0)$. In other words, M_f is the quotient of the disjoint union of Y and $X \times I$ obtained by identifying fx and $(x, 0)$ for each $x \in X$. One imagines M_f as a "cylinder" whose top is a copy of X and whose base is fX, which sits inside Y.

It can be shown that the map $X \to M_f$ defined by $x \mapsto [(x, 1)]$ is a cofibration, that Y is homotopy equivalent to M_f, and that any map $f\colon X \to Y$ factors through this cofibration and equivalence,

$$\begin{tikzcd} X \arrow[r, hook] \arrow[rr, bend right, "f"'] & M_f \arrow[r, "\simeq"] & Y \end{tikzcd}$$

The fact that any map is, up to homotopy, either a fibration or a cofibration composed with a homotopy equivalence suggests that the triad—fibration, cofibration, and homotopy equivalence—is intrinsically useful in an exploration of homotopy theory. Indeed, it propels one to the study of *model category theory*. A *model category* is a complete and cocomplete category with three classes of morphisms called fibrations, cofibrations, and weak equivalences that satisfy certain conditions. Or, more succinctly, it's a category in which one can "do homotopy theory." As you might hope, Top with homotopy equivalences and (co)fibrations as given in definition 6.10 provides a prime example of a model category (see Strøm (1972)), but it's not the only model structure on Top, nor is Top the only category with a model structure. We mention these ideas simply to whet the appetite. A more categorical discussion of (co)fibrations, equivalences, model categories, and general categorical homotopy theory may be found in Riehl (2014).

Let's now return to the task at hand—examples.

6.6.2 Examples and Results

To give a first example of a fibration, we'll introduce a new mapping space associated to a pointed space X.

Definition 6.13 The mapping space $\mathcal{P}X = \mathsf{Top}_*((I,0),(X,x_0))$ is called the *based path space* of X.

So points in $\mathcal{P}X$ are paths that start at x_0 and end at some $x \in X$. This space of based paths is itself a pointed space with the constant path $c_{x_0}\colon I \mapsto x_0$ serving as the basepoint.

You'll have noticed we view the interval I as a pointed space in two ways: either with basepoint 0 or with basepoint 1. When constructing the reduced cone CX on a pointed space X as $CX = X \wedge I$, we regard the basepoint of I to be 1. We like our cones to be right-side up, not upside down. On the other hand, when constructing the based path space $\mathcal{P}X$ on a pointed space X as $\mathsf{Top}_*(I,X)$, we regard the basepoint of I to be 0. We like the basepoint of X to be the beginning—rather than the end—of a path.

But the endpoints of paths are of interest. There is a map $p\colon \mathcal{P}X \to X$ which sends a path γ to its endpoint $\gamma 1 \in X$. This map provides a nice connection between $\mathcal{P}X$ and another important mapping space: the fiber $p^{-1}x_0$ consisting of all loops at x_0. That is, $p^{-1}x_0 = \Omega X$, a situation typically illustrated as a diagram:

$$\begin{tikzcd} \Omega X \arrow[r] & \mathcal{P}X \arrow[d, "p"] \\ & X \end{tikzcd}$$

There are other ways in which p is a particularly nice kind of map. It sends c_{x_0} to x_0, so it is basepoint preserving. Moreover, it is a fibration.

Example 6.1 For any based space X, the map $p\colon \mathcal{P}X \to X$ sending a path γ to its endpoint $\gamma(1)$ is a fibration. Suppose we have the commuting square

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d] & \mathcal{P}X \arrow[d, "p"] \\ Z \times I \arrow[r, "h"] & X \end{tikzcd}$$

where Z is any pointed space. Notice that for a fixed $z \in Z$, gz is a path in X ending at the point $h(z, 0)$, which, with z still fixed, is the starting point of the path h_z. So define $\widetilde{h}\colon Z \times I \to \mathcal{P}X$ to be the parameterization of the concatenation $h_z \cdot gz$ given by:

$$\widetilde{h}(z,t)s = \begin{cases} gz(s(1+t)) & \text{if } 0 \le s \le \frac{1}{1+t} \\ h(z, s(1+t)-1) & \text{if } \frac{1}{1+t} \le s \le 1. \end{cases}$$

One can check that $\widetilde{h}$ preserves basepoints and commutes with the diagram.

While on the topic of based path spaces, here is a good property to know about.

Proposition 6.1 $\mathcal{P}X$ is contractible.

Proof. Let $*$ denote the one-point space. The composition $* \to \mathcal{P}X \to *$ is equal to id_*, so to prove $\mathcal{P}X$ is homotopy equivalent to $*$, we need only show the composition $\mathcal{P}X \to * \to \mathcal{P}X$ that sends a path γ to c_{x_0} is homotopic to $\mathrm{id}_{\mathcal{P}X}$.

Define $h\colon \mathcal{P}X \times I \to \mathcal{P}X$ by $h(\gamma, t) = \gamma_t$, where $\gamma_t\colon I \to X$ is the path $\gamma_t(s) = \gamma(s+(1-s)t)$. Then h is the identity on $\mathcal{P}X$ at $t = 0$, and h is the constant map $\gamma \mapsto c_{\gamma 1}$ at $t = 1$. Further, h is basepoint-preserving since $h(c_{x_0}, t) = c_{x_0}$ for all t. $\square$

Here is another important example of a fibration.

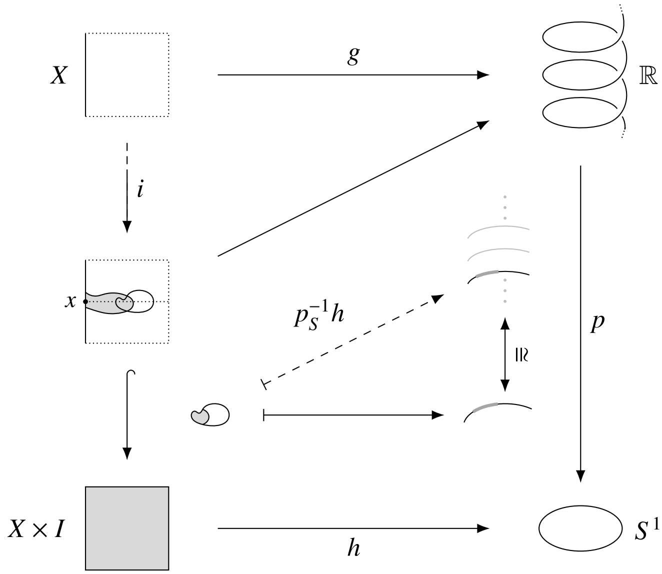
Example 6.2 The map p from $\mathbb{R}$ to S^1 given by $y \mapsto e^{2\pi i y}$ is a fibration with fiber $\mathbb{Z}$. That is, if the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & \mathbb{R} \arrow[d, "p"] \\ X \times I \arrow[r, "h"] & S^1 \end{tikzcd}$$

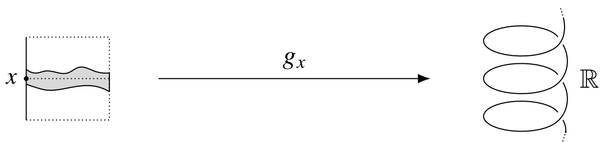
then there is a lift of the homotopy h through p.

The key here is that p is a local homeomorphism: for every $y \in \mathbb{R}$ there is an open neighborhood of y that maps homeomorphically onto its image—for instance, the interval $(y - \frac{1}{2}, y + \frac{1}{2})$ would work. So there is an open cover $\mathcal{U}$ of S^1 so that for each $U \in \mathcal{U}$, the inverse image $p^{-1}U$ is a collection of disjoint open sets in $\mathbb{R}$, each of which is homeomorphic to U. Because h is continuous, its inverse image $h^{-1}\mathcal{U}$ is an open cover of $X \times I$. Further, on each of the compact subspaces $\{x\} \times I$ there is a finite subcover, say $\mathcal{V}_x \subseteq h^{-1}\mathcal{U}$.

Lifting h inductively along any one of the $\mathcal{V}_x$ isn't hard. We need two observations. First, if a lift $g_x \colon S \to \mathbb{R}$ is defined for a nonempty subset $S \subseteq h^{-1}U \in \mathcal{V}_x$, then $g_x S$ is contained entirely in one of the disjoint homeomorphic copies of U in $p^{-1}U$—call it U_S. Second, p restricts to a homeomorphism $p_S \colon U_S \to U$. We can thus extend the domain of such a g_x to include all of $h^{-1}U$ by declaring $g_x := p_S^{-1}h$ on $h^{-1}U$. Diagrammatically, we have:



Inducting through the finite subcover (using the given g as the base step) we define for each $x \in X$ a lift g_x whose domain includes the open cover $\mathcal{V}_x$ of $\{x\} \times I$:



It's possible that three or more sets in the cover $\mathcal{V}_x$ overlap, in which case there is a choice of inductions extending g to a g_x. However, all such inductions must agree on any overlaps precisely because p is a local homeomorphism, so the g_x are well defined.

This same observation guarantees that any two lifts g_x and $g_{x'}$ must be equal on the intersection of their domains. Therefore the g_x assemble to uniquely determine a map

$\widetilde{h}\colon X \times I \to \mathbb{R}$ such that

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "i"'] & \mathbb{R} \arrow[d, "p"] \\ X \times I \arrow[r, "h"] \arrow[ur, dashed, "\widetilde{h}"] & S^1 \end{tikzcd}$$

commutes. In other words, the exponential map p is a fibration.

These examples give two different fibrations whose base space is the circle.

$$\begin{tikzcd} \mathcal{P}S^1 \arrow[dr, "\text{Example 6.1}"'] & & \mathbb{R} \arrow[dl, "\text{Example 6.2}"] \\ & S^1 & \end{tikzcd}$$

Naturally one wonders if this triangle can be completed. Are $\mathbb{R}$ and $\mathcal{P}S^1$ related? Yes, both spaces are contractible, so there is a homotopy equivalence between them. As the next theorem shows, these fibrations must therefore have homotopy equivalent fibers.

Theorem 6.5 Suppose p and q are fibrations with base space B and f is a map of total spaces causing the diagram to commute:

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & D \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

If f is a homotopy equivalence, then f induces a homotopy equivalence between fibers.

Commutativity of the triangle implies $fp^{-1}b \subseteq q^{-1}b$ for all $b \in B$, which is to say that f is a *fiber-preserving map*. But its homotopy inverse $f'\colon D \to E$ may not be fiber preserving, and the homotopies witnessing $ff' \simeq \mathrm{id}_D$ and $f'f \simeq \mathrm{id}_E$ may not be comprised of fiber-preserving maps. However, if one can replace f' with a fiber-preserving homotopy equivalent map g that satisfies $fg \simeq \mathrm{id}_D$ and $gf \simeq \mathrm{id}_E$, where each homotopy is comprised of fiber-preserving maps, then for each $b \in B$ the map f can restrict to a homotopy equivalence between fibers, $p^{-1}b \simeq q^{-1}b$. The theorem thus rests on producing such a map g.

Proof. By assumption, there is a homotopy h' from ff' to id_D. Postcomposing it with the fibration q gives homotopy $h\colon D \times I \to B$ from pf' to q, and the outer square commutes by commutativity of the triangle:

$$\begin{tikzcd} D \arrow[r, "f'"] \arrow[d] & E \arrow[d, "p"] \\ D \times I \arrow[r, "h"] \arrow[ur, dashed, "\widetilde{h}"] & B \end{tikzcd}$$

Since p is a fibration, there is a lift $\widetilde{h}$ with $\widetilde{h}(-,0) = f'$. We claim that $g := \widetilde{h}(-,1)$ is the desired map. First note that g is fiber preserving by commutativity of the previous diagram. Moreover, fg and id_D are homotopic by $k\colon D \times I \to D$ defined by

$$k(d,t) = \begin{cases} f\widetilde{h}(d, 1-2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ h'(d, 2t-1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

The map $k(-,t)$, however, may not be fiber preserving for each $t \in I$. To work around this, one can define a homotopy $M\colon D \times I \times I \to D$ from the homotopy qk to a homotopy between q and itself so that the square commutes:

$$\begin{tikzcd} D \times I \arrow[r, "k"] \arrow[d] & D \arrow[d, "q"] \\ D \times I \times I \arrow[r, "M"] \arrow[ur, dashed, "L"] & B \end{tikzcd}$$

Since q is a fibration, M lifts to a homotopy L. Since L fits into the diagram, it provides the desired fiber-preserving homotopy from fg to id_D:

$$fg = k(-,0) = L(-,0,0) \simeq L(-,1,0) = k(-,1) = \mathrm{id}_D$$

A similar story shows $gf \simeq \mathrm{id}_E$. Here's the sketch. First use the fibration p together with the homotopy witnessing $f'f \simeq \mathrm{id}_E$ to get a homotopy $E \times I \to B$ that fits into the square

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & D \arrow[d, "q"] \\ E \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

The lift of the homotopy defines a map $\bar{g}\colon E \to D$. One can then show $g\bar{g} \simeq \mathrm{id}_E$ through a fiber-preserving homotopy, from which it follows that $\bar{g} = \mathrm{id}_D\,\bar{g} \simeq fg\bar{g} \simeq f$ and so $gf \simeq g\bar{g} \simeq \mathrm{id}_E$. $\square$

Right away, we obtain several important corollaries.

Corollary 6.5.1 The loop space of the circle ΩS^1 is homotopy equivalent to $\mathbb{Z}$

Proof. Both $\mathbb{R} \to S^1$ and $\mathcal{P}S^1 \to S^1$ are fibrations by earlier examples. Moreover, both $\mathcal{P}S^1$ and $\mathbb{R}$ are contractible (proposition 6.1 and example 1.21). There is thus a homotopy equivalence between them that commutes with the fibrations

$$\begin{tikzcd} \mathcal{P}S^1 \arrow[rr] \arrow[dr] & & \mathbb{R} \arrow[dl] \\ & S^1 & \end{tikzcd}$$

By theorem 6.5 it induces a homotopy equivalence between the fibers ΩS^1 and $\mathbb{Z}$ of $\mathcal{P}S^1 \to S^1$ and $\mathbb{R} \to S^1$, respectively. $\square$

In fact, this idea holds in greater generality.

Corollary 6.5.2 Let $p\colon E \to B$ be a fibration with fiber F. If E is contractible, then F is homotopy equivalent to the loop space ΩB.

Immediately, we obtain the next result.

Corollary 6.5.3 The fundamental group of S^1 is isomorphic to $\mathbb{Z}$.

Proof. By the previous corollary, $\Omega S^1 \simeq \mathbb{Z}$ which implies

$$\pi_0 \Omega S^1 \cong \pi_0 \mathbb{Z}$$

The left-hand side is $\pi_1 S^1$ by theorem 6.4. The right-hand side is the set of path components of $\mathbb{Z}$, which is simply $\mathbb{Z}$. $\square$

Now, you might be concerned that $\pi_0\mathbb{Z}$ is merely a set with no additional structure since, after all, the corollary to theorem 6.4 held only in the case when $n \geq 1$. But by the homotopy invariance of π_0, it follows that $\pi_0\mathbb{Z}$ is isomorphic to $\pi_0\Omega S^1 = [S^0, \Omega S^1]$, and the latter, being a set of (homotopy classes of) maps into a group, is itself a group. So we are assured that $\pi_0\mathbb{Z} \cong \mathbb{Z}$ is indeed a group.

Another important consequence is the following.

Corollary 6.5.4 The nth homotopy group of the circle is trivial for $n \geq 2$.

Proof. If $n \geq 2$, then

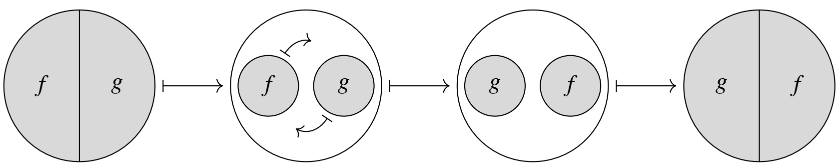
$$\pi_n S^1 = \pi_{n-1}\Omega S^1 = \pi_{n-1}\mathbb{Z} = [S^{n-1}, \mathbb{Z}]$$

The result follows since S^{n-1} is connected for $n > 1$ so any basepoint-preserving map from it to $\mathbb{Z}$ must be constant to the basepoint of $\mathbb{Z}$. $\square$

While on the topic of nth homotopy groups, here is the result promised earlier.

Theorem 6.6 For any space X the nth homotopy group $\pi_n X$ is abelian for $n \geq 2$.

Proof. The proof is left to exercise 6.4 at the end of the chapter. The picture below gives a hint when $n = 2$.



$\square$

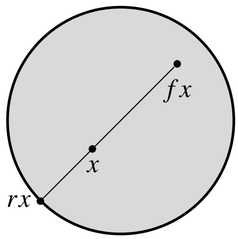
Now that we've computed the fundamental group of the circle, we'll present a few applications of the result. Afterward, we'll turn our attention to the fundamental groups of other familiar spaces. As we'll see then, a theorem of Seifert and van Kampen provides a methodical way of doing so.

6.6.3 Applications of $\pi_1 S^1$

The isomorphism $\pi_1 S^1 \cong \mathbb{Z}$ leads to some nice results, some of which we showcase here. To start, recall that in section 2.1 we proved that every map from the closed interval $[-1, 1]$ to itself must have a fixed point. Here is the two-dimensional analog.

Brouwer's Fixed Point Theorem Every map $D^2 \to D^2$ has a fixed point.

Proof. Suppose $f\colon D^2 \to D^2$ is a map with no fixed points. Then for any $x \in D^2$, there is a unique ray starting at fx and passing through x.



Let rx denote the point where this ray intersects the boundary of the disc $S^1 = \partial D^2$, and observe that r is a continuous map which satisfies $r(x) = x$ for all $x \in S^1$. Thus $ri = \mathrm{id}_{S^1}$ where $i\colon S^1 \hookrightarrow D^2$ is the inclusion. Choosing a basepoint in S^1 and applying π_1, we obtain

$$\begin{tikzcd} \pi_1(S^1, 1) \arrow[r, "\pi_1 i"] \arrow[rr, bend right, "\pi_1(id_{S^1}) = id_{\pi_1(S^1,1)}"'] & \pi_1(D^2, 1) \arrow[r, "\pi_1 r"] & \pi_1(S^1, 1) \end{tikzcd}$$

But this is impossible since $\pi_1 D^2 \cong 0$ while $\pi_1 S^1 \cong \mathbb{Z}$. Emphatically, the identity does not factor through the constant map at 0. $\square$

The following result from linear algebra is a corollary.

Perron-Frobenius Theorem Every 3×3 matrix with positive entries has a positive eigenvalue.

Proof. Let Δ^2 denote the subset of $\mathbb{R}^3$ consisting of all points (x, y, z) satisfying $x+y+z = 1$, where each coordinate lies in the interval $[0, 1]$. That is, Δ^2 is the face opposite the origin of the unit tetrahedron in the first quadrant of $\mathbb{R}^3$. Now if A is any 3×3 matrix with real positive entries, define a linear map B by

$$Bv = \tfrac{1}{\lambda_v} Av$$

where λ_v is the sum of the coordinates of the vector Av. Then B is a linear transformation from Δ^2 to itself. And since Δ^2 and the disc D^2 are homeomorphic, B must have a fixed point. Thus there is a vector w so that $w = Bw$ and so $Aw = \lambda_w w$ where, by assumption, λ_w must be positive. $\square$

We have two more applications of $\pi_1 S^1$ to share. Both require the following definition.

Definition 6.14 Let $f\colon (S^1, 1) \to (S^1, f1)$. Choose a path $f1 \to 1$, which defines an isomorphism $\pi_1(S^1, f1) \simeq \pi_1(S^1, 1)$. Then $\pi_1 f$ sends a generator $[\gamma] \in \pi_1(S^1, 1)$ to an integer multiple of $[\gamma]$. This integer, denoted $\deg f$, is called the *degree of f*.

A check is required to make sure that $\deg f$ does not depend on the choice of path, which follows from the fact that $\pi_1(S^1, 1)$ is abelian. Also, notice that $\deg f$ only depends on the homotopy class of f since $\pi_1 f$ and $\pi_1 g$ are equal as group homomorphisms whenever f and g are homotopy equivalent maps.

Example 6.3 The degree of the identity map on S^1 is 1. The degree of the map sending z to iz is also 1 since rotation by 90° is homotopic to the identity map. And for any $n \geq 1$, the degree of the map $z \mapsto z^n$ is n.

Theorem 6.7 If $f\colon S^1 \to S^1$ has degree $n \neq 1$, then f has a fixed point.

Proof. If f does not have a fixed point, define $h\colon S^1 \times I \to S^1$ by

$$h(x, t) = \frac{(1-t)fx + tx}{|(1-t)fx + tx|}$$

Then h gives a homotopy between f and id_{S^1} and so $\deg f = 1$. $\square$

The notion of degree provides yet another application of $\pi_1 S^1 \cong \mathbb{Z}$.

The Fundamental Theorem of Algebra Every polynomial

$$pz = z^n + c_{n-1}z^{n-1} + \cdots + c_0$$

with $c_i \in \mathbb{C}$ and $n \neq 0$ has a root in $\mathbb{C}$.

Proof. Let $n \neq 0$, and suppose f does not have a root. Then

$$h(z, t) = \frac{p(tz)}{|p(tz)|}$$

defines a homotopy between $\frac{p}{|p|}$ and $\frac{c}{|c|}$, the latter being the constant map at c_0. Thus $\frac{p}{|p|}$ must have degree 0. On the other hand,

$$i(z, t) = \frac{t^n p(\frac{z}{t})}{|p(\frac{z}{t})|}$$

defines a homotopy between $\frac{p}{|p|}$ and the map $z \mapsto z^n$ which has degree n. Thus $0 = \deg \frac{p}{|p|} = n$, which is a contradiction. $\square$

With $\pi_1 S^1$ in hand, let's now turn to compute the fundamental group of other spaces. A result of Seifert and van Kampen gives us a tool for doing so.

6.7 The Seifert van Kampen Theorem

There is a common strategy employed in mathematics, which we mentioned early on in section 2.1: information about parts and how they interact is often used to obtain information about a whole. This is especially true in topology, where (as we've seen) spaces are oftentimes decomposed into open sets and information about those sets and their intersection is used to obtain information about the space. This approach is particularly valuable when computing fundamental groupoids (and fundamental groups). If a space X can be decomposed as the union of open sets U and V and if the fundamental groupoids of U, V and $U \cap V$ are known, then we can expect to understand something about the fundamental groupoid of X. As the next theorem shows, it can be understood completely—it is a colimit involving the fundamental groupoids of the spaces comprising X.

Seifert van Kampen Theorem Suppose U and V are open subsets of a topological space $X = U \cup V$. Then one has the following diagram of spaces and continuous functions:

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & X \end{tikzcd}$$

Applying π_1 yields a pushout diagram in the category of groupoids.

$$\begin{tikzcd} \pi_1(U \cap V) \arrow[r] \arrow[d] & \pi_1 U \arrow[d] \\ \pi_1 V \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \pi_1 X \end{tikzcd}$$

Proof. Here is the idea of the proof. If G is any groupoid fitting into a diagram as on the left,

$$\begin{tikzcd} \pi_1(U \cap V) \arrow[r] \arrow[d] & \pi_1 U \arrow[dd, bend left, "f"] \\ \pi_1 V \arrow[dr, bend right, "g"'] & \\ & G \end{tikzcd} \qquad \begin{tikzcd} \pi_1(U \cap V) \arrow[r] \arrow[d] & \pi_1 U \arrow[d] \arrow[dr, bend left, "f"] & \\ \pi_1 V \arrow[r] \arrow[drr, bend right, "g"'] & \pi_1 X \arrow[dr, dashed, "\Phi"] & \\ & & G \end{tikzcd}$$

then we must construct a functor Φ completing the diagram as on the right. On objects, this is straightforward. Let $x \in X$. If $x \in U$, define $\Phi x = fx$, and if $x \in V$, define $\Phi x = gx$. If $x \in U \cap V$, these two assignments agree.

To define Φ on a homotopy class of paths from $x \to y$, choose a representative path $\gamma\colon I \to X$, and use compactness of I to subdivide the path γ into a composition of paths $\gamma_n \cdots \gamma_1$, each component of which lies in either U or V. Then, define $\Phi([\gamma])$ to be the composition of $(f \text{ or } g)[\gamma_n] \cdots (f \text{ or } g)[\gamma_2](f \text{ or } g)[\gamma_1]$ as the case may be. To see that $\Phi[\gamma]$ is well defined, let $\gamma' \simeq \gamma$, and choose a homotopy h between γ and γ'. Use compactness of $I \times I$ to subdivide the image of h into rectangles that lie entirely in U or V. The details are left as an exercise. $\square$

The next result is an important consequence.

Proposition 6.2 Suppose U and V are open subsets of a topological space $X = U \cup V$, and suppose $x_0 \in U \cap V$. Then one has the following diagram of spaces and continuous functions:

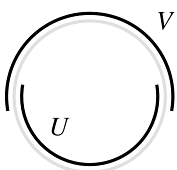
$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & X \end{tikzcd}$$

If $U \cap V$ is path connected, then the following diagram is a pushout in the category of groups.

$$\begin{tikzcd} \pi_1(U \cap V, x_0) \arrow[r] \arrow[d] & \pi_1(U, x_0) \arrow[d] \\ \pi_1(V, x_0) \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \pi_1(X, x_0) \end{tikzcd}$$

Proof. By the general remark following the definition of the fundamental group, the fundamental groups are equivalent as categories to the fundamental groupoids since $U \cap V$ is path connected. $\square$

To see why $U \cap V$ must be path connected, consider the circle $X = S^1$, and let U and V be the subsets as indicated.



Then $U \cap V$ is homotopy equivalent to a two-point space and U and V are both contractible. Assuming Seifert van Kampen gives a supposed pushout of groups,

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 1 \arrow[d] \\ 1 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \pi_1(S^1, 1) \end{tikzcd}$$

But the pushout of the diagram $1 \leftarrow 1 \rightarrow 1$ is the trivial group 1, and thus $1 \cong \pi_1(S^1, 1) \cong \mathbb{Z}$, which is of course a contradiction.

Now you might be wondering, "What exactly *are* pushouts in the category Grp?" The coproduct of two groups G and H is their free product $G * H$, the group generated by the generators of both G and H with relations (equations satisfied by the generators) coming from the relations in G and H. The pushout, then, of a diagram of groups $H \leftarrow K \rightarrow G$ is a quotient of the free product such that the diagram commutes:

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "g"'] & G \arrow[d] \\ H \arrow[r] & (G * H)/N \end{tikzcd}$$

One concludes that N must be the (smallest) normal subgroup of $G * H$ generated by the relations $fk = gk$ for each $k \in K$. This construction is sometimes called the *amalgamated free product*.

6.7.1 Examples

Let's close by using the Seifert van Kampen theorem to compute the fundamental group of some familiar spaces.

Example 6.4 Suppose $X = S^2$ is the sphere. Let U be all of S^2 except for the point $(0, 0, 1)$, and let V be all of S^2 except for $(0, 0, -1)$. Then $U \cap V$ is homotopy equivalent to a circle, and thus its fundamental group is isomorphic to $\mathbb{Z}$. And since both U and V are contractible we obtain the diagram of groups

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & 1 \arrow[d] \\ 1 \arrow[r] & \pi_1(S^2, 1) \end{tikzcd}$$

The fundamental group of S^2 is then trivial.

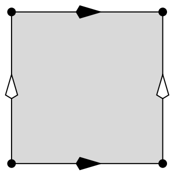
Example 6.5 Consider the wedge product of two circles $S^1 \vee S^1$, a "figure eight." Let U and V be the indicated subspaces.

Notice that $U \cap V$ is contractible while U and V are each homotopy equivalent to a circle. Thus, the fundamental group of each is isomorphic to $\mathbb{Z}$. This is isomorphic to the free group on a single generator, so let's write $\pi_1 U \cong F\alpha$ and $\pi_1 V \cong F\beta$ where α and β are the loops generating $\pi_1 U$ and $\pi_1 V$, respectively. Then we have the diagram:

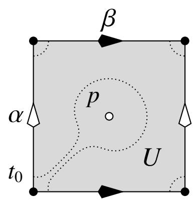
$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & F\alpha \arrow[d] \\ F\beta \arrow[r] & \pi_1(S^1 \vee S^1, s_0) \end{tikzcd}$$

The fundamental group of $S^1 \vee S^1$ is therefore the free group on two generators, $F\alpha * F\beta \cong F(\alpha, \beta)$.

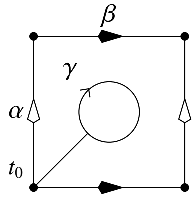
Example 6.6 Recall from example 1.17 that we can view the torus T as the quotient of a square with opposite sides identified:



All four corners of the square are identified to a single point, say, t_0. So let's consider the pointed torus (T, t_0). Now suppose p is any other point in T. Set $V = T \setminus \{p\}$, and let U be a "small" disc containing both p and t_0. Naming the two edges α and β, we have a situation like this:



Then U is contractible. Further, V retracts onto the wedge of the loops α and β. To see this, think of removing the point p and retract the remaining gray area onto the boundary of the square. Keeping in mind that opposite sides are identified, one obtains the wedge product of the loops (α, t_0) and (β, t_0), namely, $(S_1 \vee S_1, t_0)$. Hence, by example 6.5, $\pi_1(V, t_0)$ is given by the free product on two generators which we may as well call α and β. Finally, the intersection of U and V is a punctured disc: $U \setminus \{p\}$ which retracts onto the inner "lollipop" below



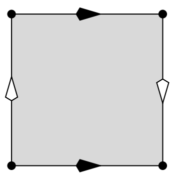
whose fundamental group is given by the free group generated by the loop γ moving from t_0 out along the diagonal, counterclockwise around the circle, and back down the diagonal to t_0. Therefore, by Seifert van Kampen,

$$\begin{tikzcd} F\gamma \arrow[r] \arrow[d] & 1 \arrow[d] \\ F(\alpha,\beta) \arrow[r] & \pi_1(T, t_0) \end{tikzcd}$$

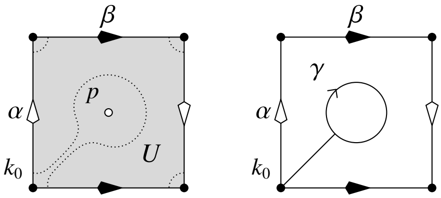
But under the inclusion $U \cap V \to V$, the loop γ is sent to itself. And under the retraction of V onto the boundary of the square, γ maps to $\alpha\beta\alpha^{-1}\beta^{-1}$; that is, the group homomorphism $F\gamma \to F(\alpha,\beta)$ sends γ to $\alpha\beta\alpha^{-1}\beta^{-1}$. Therefore $\pi_1(T, t_0)$ is the quotient of $F(\alpha,\beta)$ generated by the relation $\alpha\beta\alpha^{-1}\beta^{-1} = 1$. This group is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

This example shows that $\pi_1(S^1 \times S^1) \cong \pi_1 S^1 \times \pi_1 S^1$, which isn't too surprising. As the categorically minded reader can check, the functor π_1 takes products to products.

Example 6.7 Referring again to example 1.17, recall that the Klein bottle K is obtained by identifying opposite sides of a square as shown.



We can compute its fundamental group in the same way as for the torus. Let k_0 be the single vertex of the square, and let p be another point in K. If U is an open disc containing p and $V = K \setminus \{p\}$, then by the same arguments as in the previous example, we have the following setup



implies

$$\begin{tikzcd} F\gamma \arrow[rr] \arrow[dd] & & 1 \arrow[dd] \\ & \gamma \arrow[r, mapsto] \arrow[d, mapsto] & 1 \\ & \alpha\beta\alpha\beta^{-1} & \\ F(\alpha,\beta) \arrow[rr] & & \pi_1(K,k_0) \end{tikzcd}$$

by Seifert van Kampen. The conclusion is that $\pi_1(K,k_0)$ is isomorphic to a group with a *presentation* given in two generators and one relation, namely, $\langle \alpha,\beta \mid \alpha\beta\alpha\beta^{-1} \rangle$.

But before too heartily congratulating ourselves on the calculation, recall that deciding whether a given group presentation describes the trivial group or not can't algorithmically be determined (Wikipedia, 2019). This is a variant of the word problem and is indeed equivalent to the halting problem in computing. Some wariness in working with group presentations is in order.

Exercises

1. Prove that the maps $d_0, d_1 \colon X \to X \times I$ defined by $d_0 x = (x, 0)$ and $d_1 x = (x, 1)$ and the projection $s \colon X \times I \to X$ are homotopy equivalences.

2. Prove that if $f \colon S^1 \to S^1$ satisfies $\|f(x) - x\| < 1$ for all x, then f is surjective.

3. The n-dimensional projective space is naturally pointed. Its basepoint is the class of the basepoint of S^n in the quotient $\mathbb{RP}^n \simeq S^n/\sim$, where antipodal points have been identified.

a) Prove that $\pi_1 \mathbb{RP}^2 \cong \mathbb{Z}/2\mathbb{Z}$.

b) Compute $\pi_1(\mathbb{RP}^2 \vee \mathbb{RP}^2)$.

c) Prove or disprove: $\mathbb{RP}^2 \vee \mathbb{RP}^2$ is a retract of $\mathbb{RP}^2 \times \mathbb{RP}^2$.

4. Learn what the *Eckmann-Hilton argument* is and how to use it to prove that the higher homotopy groups of a space are abelian.

5. Suppose A and Y are locally compact and Hausdorff and $f \colon A \to Y$ is a cofibration. Prove that for any space Z, the map $f^* \colon Z^Y \to Z^X$ is a fibration.