1 LU Factorization

Factor the matrix A to lower L and upper U triangle matrices

$$\begin{aligned} Ax &= b \\ LUx &= b \\ \text{let} \quad Ux &= y \\ x &= U^{-1}y \\ Ly &= b \\ y &= L^{-1}b \\ x &= U^{-1}y = U^{-1}L^{-1}b \end{aligned} \tag{1}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{aligned} a_{33}x_3 &= b_3 \\ x_3 &= \frac{b_3}{a_{33}} \\ a_{22}x_2 + a_{13}x_3 &= b_2 \\ a_{22}x_2 &= b_2 - a_{13}x_3 \\ a_{22}x_2 &= b_2 - a_{13}\left(\frac{b_3}{a_{33}}\right) \\ x_2 &= \frac{b_2 - a_{13}\left(\frac{b_3}{a_{33}}\right)}{a_{22}} \end{aligned} \tag{2}$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{11}x_1 &= b_1 - (a_{11}x_2 + a_{12}x_3) \\ x_1 &= \frac{b_1 - (a_{11}x_2 + a_{12}x_3)}{a_{11}} \\ x_1 &= \frac{b_1 - \left(a_{11}\frac{b_2 - a_{13}\frac{b_3}{a_{33}}}{a_{22}} + a_{12}\frac{b_3}{a_{33}}\right)}{a_{11}} \end{aligned}$$

2 Example

$$\begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \tag{3}$$
$$\Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 7 \end{bmatrix} \tag{4}$$
$$\Rightarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Proof. coming soon □