Notes on Tensor Products and the Exterior Algebra
For Math 245

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1 Tensor Products

1.1 Axiomatic definition of the tensor product

In linear algebra we have many types of products. For example,

- The scalar product: $V \times \mathbb{F} \to V$
- The dot product: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$
- The cross product: $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$
- Matrix products: $\mathsf{M}_{m\times k} \times \mathsf{M}_{k\times n} \to \mathsf{M}_{m\times n}$

Note that the three vector spaces involved aren't necessarily the same. What these examples have in common is that in each case, the product is a bilinear map. The tensor product is just another example of a product like this. If V_1 and V_2 are any two vector spaces over a field $\mathbb{F}$, the tensor product is a bilinear map:

$$V_1 \times V_2 \to V_1 \otimes V_2\,,$$

where $V_1 \otimes V_2$ is a vector space over $\mathbb{F}$. The tricky part is that in order to define this map, we first need to construct this vector space $V_1 \otimes V_2$.

We give two definitions. The first is an axiomatic definition, in which we specify the properties that $V_1 \otimes V_2$ and the bilinear map must have. In some sense, this is all we need to work with tensor products in a practical way. Later we'll show that such a space actually exists, by constructing it.

Definition 1.1. Let V_1, V_2 be vector spaces over a field $\mathbb{F}$. A pair (Y, μ), where Y is a vector space over $\mathbb{F}$ and $\mu : V_1 \times V_2 \to Y$ is a bilinear map, is called the **tensor product** of V_1 and V_2 if the following condition holds:

(*) Whenever β_1 is a basis for V_1 and β_2 is a basis for V_2, then

$$\mu(\beta_1 \times \beta_2) := \{\mu(x_1, x_2) \mid x_1 \in \beta_1,\ x_2 \in \beta_2\}$$

is a basis for Y.