1 Examples and Constructions

All of it was written by Sammy. . . . I wrote nothing.
—Henri Cartan (Jackson, 1999)

Introduction. Our goal in this chapter is to construct new topological spaces from given ones. We'll do so by focusing on four basic constructions: subspaces in section 1.2, quotients in section 1.3, products in section 1.4, and coproducts in section 1.5. To maintain a categorical perspective, the discussion of each construction will fit into the following template:

- **The classic definition**: an explicit construction of the topological space
- **The first characterization**: a description of the topology as either the coarsest or the finest topology for which maps into or out of the space are continuous, leading to a better definition
- **The second characterization**: a description of the topology in terms of a universal property as given in theorems 1.1, 1.2, 1.3, and 1.4

Before we construct topological spaces, it will be good to have some examples in mind. We'll begin then in section 1.1 with examples of topological spaces and continuous maps between them.

1.1 Examples and Terminology

Let's open with examples of spaces followed by examples of continuous functions.

1.1.1 Examples of Spaces

Example 1.1 Any set X may be endowed with the *cofinite* topology, where a set U is open if and only its complement $X \setminus U$ is finite (or if $U = \varnothing$). Similarly, any set may be equipped with the *cocountable* topology whose open sets are those whose complement is countable.

Example 1.2 The empty set $\varnothing$ and the one-point set $*$ are topological spaces in unique ways. For any space X, the unique functions $\varnothing \to X$ and $X \to *$ are continuous. The empty set is initial and the one-point set is terminal in Top, just as they are in Set.

Example 1.3 As we saw in section 0.1, $\mathbb{R}$ is a topological space with the usual metric topology, but it admits other topologies, too. For example, like any set, $\mathbb{R}$ has a cofinite