Prove normal matrix is diagonalizable

$$\textbf{If } \mathbf{A}^*\mathbf{A} = \mathbf{A}\mathbf{A}^*$$

$$\textbf{Then } \mathbf{A} = \mathbf{P}^*\Lambda\mathbf{P}$$

Proof

$$\mathbf{A} = \mathbf{P}^*\Delta\mathbf{P}$$

Δ is upper triangle matrix according **Schur decomposition**

$\mathbf{P}$ is unitary matrix

First we need to show if $\mathbf{A}$ is **normal** then Δ is **normal**

$$\mathbf{A} = \mathbf{P}^*\Delta\mathbf{P}$$

$$\Rightarrow \mathbf{A}^* = (\mathbf{P}^*\Delta\mathbf{P})^*$$

$$\Rightarrow \mathbf{A}^* = \mathbf{P}^*(\mathbf{P}^*\Delta)^*$$

$$\Rightarrow \mathbf{A}^* = \mathbf{P}^*(\Delta^*\mathbf{P})$$

from (1) and (2)

$$\Rightarrow \mathbf{A}\mathbf{A}^* = (\mathbf{P}^*\Delta\mathbf{P})(\mathbf{P}^*\Delta^*\mathbf{P})$$

$$\Rightarrow \mathbf{A}\mathbf{A}^* = \mathbf{P}^*\Delta(\mathbf{P}\mathbf{P}^*)\Delta^*\mathbf{P}$$

$\mathbf{P}$ is unitary matrix

$$\Rightarrow \mathbf{P}^* = \mathbf{P}^{-1}$$

$$\Rightarrow \mathbf{A}\mathbf{A}^* = \mathbf{P}^*\Delta\Delta^*\mathbf{P}$$

$$\Rightarrow \mathbf{A}^*\mathbf{A} = \mathbf{P}^*\Delta^*\mathbf{P}\mathbf{P}^*\Delta\mathbf{P}$$

$$\Rightarrow \mathbf{A}^*\mathbf{A} = \mathbf{P}^*\Delta^*\Delta\mathbf{P}$$

$$\Rightarrow \mathbf{A}\mathbf{A}^* - \mathbf{A}^*\mathbf{A} = \mathbf{P}^*\Delta\Delta^*\mathbf{P} - \mathbf{P}^*\Delta^*\Delta\mathbf{P}$$

$$\Rightarrow \mathbf{A}\mathbf{A}^* - \mathbf{A}\mathbf{A}^* = \mathbf{P}^*(\Delta\Delta^* - \Delta^*\Delta)\mathbf{P} = \mathbf{0}$$

$$\Rightarrow \Delta\Delta^* - \Delta^*\Delta = \mathbf{0}$$

$$\Rightarrow \Delta\Delta^* = \Delta^*\Delta$$

Δ **is normal matrix**

$$\Rightarrow \vec{e_i}^*\Delta\Delta^*\vec{e_i} = \vec{e_i}^*\Delta^*\Delta\vec{e_i}$$

$$\Rightarrow \langle\Delta^*\vec{e_i}, \Delta^*\vec{e_i}\rangle = \langle\Delta\vec{e_i}, \Delta\vec{e_i}\rangle$$

$$\Rightarrow \|\Delta^*\vec{e_i}\|^2 = \|\Delta\vec{e_i}\|^2$$

$$\Rightarrow \|\Delta^*\vec{e_i}\| = \|\Delta\vec{e_i}\|$$

$\Rightarrow$ **The length of ith column and ith row in Δ are same**

Δ is upper triangle matrix

Let i to be the first row with nonzero off-diagonal element

$$A_{n,n} = \begin{pmatrix} a_{1,1} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & a_{2,2} & \cdots & 0 & \cdots & 0 \\ 0 & 0 & a_{i,i} & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ . & . & . & \vdots & \vdots & a_{n,n} \end{pmatrix}$$

If Δ is not diagonal matrix, then the ith column is $|a_{i,i}| \neq |a_{i,i}| + |*|$

This contracts our previous (3), therefore Δ must be diagonal matrix

Therefore $\mathbf{AA^*} = \mathbf{A^*A} \Rightarrow \mathbf{A} = \mathbf{P^*\Lambda P}$