Labor Market Dynamics with Sorting*

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Abstract

I propose a frictional search model of the labor market with worker and firm heterogeneity, sorting, and aggregate uncertainty. My main contribution is to show that labor market sorting, arising from a production complementarity between worker skills and firm productivities, can improve our understanding of the cyclical dynamics of the labor market. The model incorporates two new quantitatively important transmission channels. First, the model endogenously generates a wage rigidity of an empirically reasonable magnitude. Second, the firms form expectations about the endogenous distribution of unemployed worker types and the match surplus. The surplus function and distributions change with the aggregate state, which leads to the amplification of shocks. The model is calibrated to produce realistic degrees of wage dispersion and sorting. I propose a numerical procedure to keep track of the model's complex state space. Labor market sorting considerably propagates the model's response to shocks and brings it on par with empirical moments of labor market data, including vacancy dynamics.

Keywords: Sorting, Search and Matching, Mismatch, Aggregate Fluctuations, Wage Rigidity, Worker Heterogeneity, Firm Heterogeneity, Unemployment Dynamics

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1 Introduction

To make an appropriate labor market match, both workers and firms have to expend time and resources to find a suitable partner. This coordination friction is the essence of the Diamond-Mortensen-Pissarides (DMP) search and matching framework and explains the coexistence of unemployed workers and vacant jobs in the labor market.[1] Intuitively, frictions can be understood as an outcome of heterogeneity: workers and firms differ in terms of skills and productivities. This makes finding a suitable match costly.

The textbook DMP model is built around a representative agent. By using a Poisson process to model meetings between workers and firms, it elegantly incorporates the coordination friction without making heterogeneity across workers and firms explicit. The model has well-known shortcomings, however, in explaining empirical moments of labor market data: Shimer (2005), for example, points out that a simple dynamic DMP model fails to generate sufficient volatility in response to aggregate shocks, mainly because wages are fully flexible. Hornstein et al. (2011) add that standard search models do not generate as much wage dispersion as is observed in the data. In this paper, I argue that explicitly considering worker and firm heterogeneity and allowing for positive sorting in the labor market greatly improves the ability of search and matching models to match the data. Specifically, the model I propose overcomes the counterfactual implications of the standard model regarding the volatility of unemployment and vacancies in response to shocks and it has an empirically sound source of wage dispersion and rigidity built in to it.

I develop a dynamic DMP model with two-sided heterogeneity, labor market sorting, and aggregate shocks. The model features equilibrium wage dispersion, an endogenous wage rigidity, and sufficient amplification in response to shocks. At its core, the model is a Shimer and Smith (2000) economy. Sorting—generated by assuming a production complementarity between heterogeneous worker skills and firm productivities—implies that agents are not willing to match with every possible partner they meet when searching randomly. The standard coordination friction is thus enhanced because only a subset of all meetings generates new matches. Endogenous matching sets contain the acceptable partner types for all worker and firm types. They are determined by the match surplus, which depends on the underlying production complementarity. In response to shocks, the surplus, the matching sets, and the distribution of worker and firm types adapt, generating rich dynamics and amplification. Two new transmission channels are quantitatively important for the model's dynamics.

First, the model endogenously generates a wage rigidity of an empirically reasonable magnitude.[2] Shimer (2005) convincingly shows that the well-known lack of amplification

[1]The main references for this class of models are Diamond (1982), Mortensen (1982), Pissarides (1985), and Mortensen and Pissarides (1994). Pissarides (2000) provides an excellent textbook treatment.

[2]I find a moderate elasticity of wages with respect to labor productivity of 0.75. This elasticity is

in the standard model is due to the fully flexible wage with Nash bargaining, which reduces the incentive to create new jobs and limits the model's responsiveness to shocks. Adding labor market sorting to the picture leads to rigid wages, even with Nash bargaining. With sorting, wages depend on the *relative* labor market tightness, that is, the scarcity or abundance of other types in the bargaining worker-firm couple's matching sets. This property shields wages from fluctuations in aggregate labor market tightness and creates an endogenous rigidity.

Second, firms solve a type-specific dynamic job-creation problem: with free entry, they form an expectation about the future match surplus and the state-dependent distribution of unemployed worker types when deciding how many vacancies to post. I allow the strength of the production complementarity to be proportionate to labor productivity, so the option value of being in a good match, measured by the surplus function, adjusts over-proportionately in response to shocks. This increases the firms' incentive to post vacancies and leads to amplification. Together, the endogenous wage rigidity and the dynamic job-creation problem propagate the model's response to shocks to a degree that brings it on par with empirical moments of labor market data.[3]

The state space of my model includes endogenous distribution functions. This feature is the main difference between my model and that of Lise and Robin (2017), who show that search on the job with sequential auctions (building on Postel-Vinay and Robin (2002) and Robin (2011)) considerably simplifies the state space of a frictional search model with two-sided heterogeneity. They also find that a model with sorting and aggregate shocks fits many data moments well. The entry decision in their model, however, is static. Vacancy postings immediately adjust in response to shocks and, therefore, the model fails to match empirical vacancy dynamics. The dynamic entry decision in my model overcomes this problem and generates persistent vacancy dynamics. To focus on the computational complexity that arises from the endogenous distributions in the state space, in this paper I ignore search on the job. I believe that studying the dynamic entry problem is an important complement to Lise and Robin (2017). The logical next step in this literature will be to develop a model with both search on the job and a dynamic entry problem.

A technical contribution of this paper is a numerical procedure that allows keeping track of the model's complex state space. To analyze the response to aggregate shocks, it is necessary to compute the adjustment path of the match surplus, the matching sets, and the endogenous type distributions. The key idea to approach this challenge is to define

close to benchmark estimates for the U.S. labor market reported by Haefke et al. (2013), who find an elasticity of 0.8 with a standard error of 0.4.

[3]Pries (2008) also analyzes how worker heterogeneity impacts the dynamics of a search and matching model. He introduces heterogeneity in a simple way—two types of workers and homogeneous firms—and finds some amplification related to a changing composition of the pool of unemployed workers over the business cycle, which does not, however, sufficiently amplify the model.

auxiliary state variables for the integral terms that contain the additional endogenous objects of the model. This allows me to linearize the model around its steady state and use standard perturbation techniques. To judge the accuracy of my computational approach, I plug the simulated data back into the model's Bellman equations and find that the errors due to discretization and approximation are on average no bigger than those made when solving standard search and matching models via perturbation.[4]

This paper connects the large literature on cyclical dynamics of the labor market to recent empirical advances in identifying the extent of labor market sorting in the data. There is now an abundance of evidence that positive assortative matching (PAM), that is, the tendency of low (high) skill workers to work at low (high) productivity firms, is a prevalent feature of labor markets in many developed economies.[5] This evidence is my starting point and I argue that sorting provides both a micro-founded and empirically sound way of better aligning search and matching models with the data.

The most prominent existing approaches to solve the unemployment-volatility puzzle in the literature rely on making wages less responsive to shocks, either by simply assuming that wages are completely rigid (Hall, 2005), by modifying the calibration in order to increase the worker's outside option in the bargaining solution (Hagedorn and Manovskii, 2008), or by replacing Nash bargaining with an alternating offer bargaining game (Hall and Milgrom, 2008).[6] A counterpoint to these popular papers is Pissarides (2009), who modifies the firm entry problem, what is similar in spirit to this paper. The former mechanisms are widely used in DSGE macro models to generate wage stickiness, for instance, to study the transmission of monetary policy shocks. A recent example is Christiano et al. (2016), who use the alternating offer bargaining game of Hall and Milgrom (2008) to induce wage inertia in a New Keynesian model. My findings suggest that labor market sorting might provide an appealing alternative source of wage rigidity in a broad class of models. Incorporating the adjustment channels emphasized in this paper into models for monetary and fiscal policy analysis is a fascinating avenue for further research.

The remainder of this paper is structured as follows: Section 2 introduces the setup and derives stationary equilibria of both circular and hierarchical sorting models. Sec-

[4]Petrosky-Nadeau and Zhang (2016) show that solving the representative agent search and matching model in Hagedorn and Manovskii (2008) via log-linearization and perturbation creates a mean computational error of 3.75%. I find a mean error of 3.84% with slightly more dispersion.

[5]There is evidence of PAM in Germany, Sweden, Italy, Denmark, and the United States; see Andrews et al. (2008, 2012), Card et al. (2013), Hagedorn et al. (2017), Lopes de Melo (2018), Bonhomme et al. (2016), Bartolucci et al. (2018), Lise et al. (2016), Bagger and Lentz (2018), and Lochner and Schulz (2016).

[6]Hall (2005) shows that the volatility puzzle vanishes once wages are made fully inflexible, implying a counter-factual wage volatility of zero. Hagedorn and Manovskii (2008) show that the dynamics can be amplified by increasing the value of the workers' outside option of non-market activity. A higher calibrated value of the respective model parameter leads to lower wage outcomes in the Nash bargaining game, however, with the unrealistic consequence that a 15% increase in the value of non-market activity implies that the equilibrium unemployment rate doubles, see Hornstein et al. (2005) and Costain and Reiter (2008).

tion 3 analyzes the comparative statics of the model to shed some light on the sources of amplification. Section 4 adds aggregate uncertainty and analyzes wage determination and firm entry in the fully dynamic model. Section 5 discusses the computational strategy and presents results from numerical simulations of the sorting model in comparison to a baseline search and matching model and U.S. labor market data. Section 6 concludes by discussing my findings in light of the related theoretical and empirical literature.

2 Two Models of Assortative Matching

I construct a dynamic DMP model of the labor market with search frictions, sorting between heterogeneous workers and firms, and aggregate uncertainty. In this Section, I describe the model's general setup, two different production functions which induce sorting, and the stationary equilibrium of the economy for both cases. Section 4 introduces aggregate uncertainty in the form of a stochastic labor productivity process z. To simplify notation, this channel is neglected in deriving the stationary equilibrium.

2.1 General Setup

A continuum of workers is endowed with heterogeneous skills $x \in [0, 1]$ with a probability density function (pdf) $g_w(x)$. The measure of workers is exogenous and normalized to 1. A continuum of firms is heterogeneous in terms of productivity $y \in [0, 1]$ with pdf $g_f(y)$. Denote by $g_m(x, y)$ the two-dimensional joint distribution of active (i.e. producing) (x, y) matches. The distributions of employed workers, $g_e(x)$, and producing firms, $g_p(y)$, can be obtained by integrating out the respective dimension of $g_m(x, y)$. The distribution of unemployed worker types, $g_u(x)$, is obtained by subtracting the distribution of employed workers from $g_w(x)$, which is exogenous and fixed. The distribution of vacant firm types $g_v(y)$ is determined by free entry. The distributions of active matches, employed/unemployed workers, and producing/vacant firms are equilibrium objects. They integrate to the stocks of employed workers, E, producing firms, P, unemployed workers, U, and vacancies, V. M is the stock of active matches, which must equal E and P. Table 1 summarizes the relations between the underlying density functions, the distributions of matched and unmatched workers and firms, and the aggregate variables.

The heterogeneity of workers and firms is assumed to be one-dimensional.[7] The model does not allow for imperfect information with respect to worker and firm types. All agents know their own type and the types of all potential partners they meet.

Time is discrete. Agents are infinitely lived, risk neutral, and they maximize their future discounted income streams. The common discount factor is β. A production

[7]Both x and y can be viewed as a one-dimensional representation of a larger, multi-dimensional set of worker and firm characteristics. For a recent exploration of a multi-dimensional sorting model with random search, see Lindenlaub and Postel-Vinay (2016).

Table 1: Distribution functions of matched and unmatched worker and firm types

Distribution of	Relation	Aggregate Stock
Active matches	$g_m(x,y)$	$M = \iint g_m(x,y)\ \mathrm{d}x\mathrm{d}y$
Employed workers	$g_e(x) = \int g_m(x,y)\ \mathrm{d}y$	$E = \int g_e(x)\ \mathrm{d}x$
Unemployed workers	$g_u(x) = g_w(x) - g_e(x)$	$U = \int g_u(x)\ \mathrm{d}x$
Producing firms	$g_p(y) = \int g_m(x,y)\ \mathrm{d}x$	$P = \int g_p(y)\ \mathrm{d}y$
Vacant firms	$g_v(y) \to$ free entry	$V = \int g_v(y)\ \mathrm{d}y$

complementarity between worker skills and firm productivities (detailed below) induces labor market sorting, that is, for every worker (firm) an optimal firm (worker) exists and matching with this optimal partner would maximize output.[8] Search frictions, however, prevent the formation of these optimal matches, as in Shimer and Smith (2000). In a setting with random search, heterogeneity, and sorting, not all meetings necessarily result in an employment relationship. The agents optimally choose partners by forming a matching set that comprises all acceptable types on the other side of the market. The decision as to whether a partner is acceptable or not is solely determined by the match surplus, $\mathcal{S}(x,y)$, which can be negative when the value of joint production falls short of the workers' and firms' outside options.

Labor is assumed to be the only production input and capital is ignored. To focus on labor market sorting and the production complementarity between worker and firm types, I do not consider firm size and possible additional complementarities between workers within the same firm. This is equivalent to assuming that the firms' production function has constant returns to scale at the match level. Thus, firms' aggregate output is equal to the sum of what is produced by every individual worker at the firm. In other words, matches can be viewed as one-worker-one-machine relationships. The measure of active firms in the labor market is governed by free entry. Firms with a vacancy incur a per-period cost for keeping it open, representing expenses for posting the vacancy, screening applications, and the like. This cost is convex in the measure of type y vacancies posted. In the hierarchical sorting model considered below, the propensity to post vacancies will hence depend on firm type. The convex function c thus takes the measure of vacant firms $g_v(y)$ of type y as its argument. Firms enter the labor market by posting vacancies as long as the expected discounted value of production is at least as big as $c((g_v(y))$. In the hierarchical model, the convexity of $c(.)$ is critical to ensure a non-degenerate distribution of vacancies.

Only unemployed workers engage in random search.[9] I assume that meetings are

[8] The case where all workers and firms are matched to their optimal partner corresponds to the Walrasian first-best allocation in Becker (1973).

[9] I abstract from search on the job to study a dynamic firm entry problem. In Lise and Robin (2017), search on the job with sequential auctions (Postel-Vinay and Robin, 2002) leads to a static entry problem,

governed by a standard Cobb-Douglas type matching function with constant returns to scale, $M(U,V) = \vartheta U^{\xi} V^{1-\xi}$, where ξ $(1-\xi)$ is the elasticity of new matches with respect to unemployment (vacancies).[10] ϑ is a scaling parameter representing matching efficiency. The standard Poisson arrival rates are functions of aggregate labor market tightness $\theta = {}^{V}\!/_{U}$. $q_v(\theta) = {}^{M(U,V)}\!/_{V}$ is the rate at which vacant firms meet unemployed workers and, correspondingly, $q_u(\theta) = {}^{M(U,V)}\!/_{U}$ is the rate at which unemployed workers meet vacancies. $q_v(\theta)$ is decreasing and $q_u(\theta)$ is increasing in θ. Productive activity commences whenever a firm and a worker meet and find that they are jointly able to produce a non-negative surplus given their types. $\mathcal{S}(x,y)$ is then shared according to the standard Nash bargaining solution with worker bargaining power α.

Matches between firms and workers can be terminated for two reasons. They are subject to idiosyncratic separation shocks, which lead to immediate dissolution of the employment relationship. A match is subject to these shocks with an exogenous per-period probability δ. Additionally, in the presence of aggregate shocks (see Section 4), endogenous separations can occur at the margins of the agents' matching sets. When a negative productivity shock hits the economy, the surplus of previously marginally profitable matches may become negative. Since a negative surplus is always less than both parties' outside option, they prefer to separate.[11] In case of unemployment, workers receive flow benefits $b(x)$ every period, which represent the type-dependent value of home production or nonmarket activity.

The timing of the model is as follows: a period begins when the state of aggregate labor productivity z is revealed ($z = 1$ in steady state and thus muted in the following). Workers and firms form their optimal acceptance strategies based on the exogenous state z and the primitives of the model. Both endogenous and exogenous separations take place, following which new matches are formed. Workers and firms separated in the same period do not start their search until the next period. Finally, production commences and wages are paid.

2.2 Production Functions

Let the output of a producing (x,y) match be denoted $F(x,y)$. The production function is non-negative and twice continuously differentiable. Labor market sorting in the model economy is induced by a complementarity between worker and firm types in production. I

which has counter-factual implications for vacancy dynamics. Combining search on the job with dynamic firm entry in a sorting model would be an interesting future project.

[10]Note that using a linear search technology with heterogeneous workers and firms implies congestion effects between different worker and job types. I stick to the Cobb-Douglas matching function for simplicity and comparability to other studies. A quadratic search technology, as used in Shimer and Smith (2000), eliminates this congestion externality. Nöldeke and Tröger (2009) extend Shimer and Smith (2000) to models with linear search technologies.

[11]This mechanism is similar to Mortensen and Pissarides (1994) only that in my model aggregate rather than idiosyncratic productivity shocks trigger separations.

focus on positive assortative matching (PAM), which requires a supermodular production function, that is, positive cross-derivatives, $F_{xy} > 0$.[12]

More specifically, I rely on the equilibrium existence conditions in an optimal assignment economy with search frictions provided by Shimer and Smith (2000). Supermodularity of $F(x, y)$ alone is not sufficient in this setting. To ensure existence of a search equilibrium, $F(x, y)$, $F_x(x, y)$, and $F_{xy}(x, y)$ need to be log-supermodular.[13] These conditions also imply that the matching sets are nonempty, closed, and convex. Uniqueness cannot be established in this class of models for general parameter values. When solving the model numerically, I ensure that the mapping derived below is contracting in the parameter space I consider by repeatedly solving the model for different initial conditions.

Below, I introduce two types of production functions that induce sorting in equilibrium: the *circular* and the *hierarchical* model. Depending on the functional form of the production function, labor market sorting arises from a comparative advantage (circular model) or an absolute advantage (hierarchical model) argument. In sorting models with absolute advantage, for example Shimer and Smith (2000), high type workers (firms) will always produce more than low types, no matter what type of firm (worker) they are matched with. In other words, the production function implies an unambiguous hierarchy, or ranking, of workers and firms because it is strictly increasing in both dimensions, $F_x(x, y) > 0$, $F_y(x, y) > 0$.

In the comparative advantage sorting model, the worker/firm type does not matter by itself; only the interaction of x and y determines output. Examples of sorting models with comparative advantage include the circular models in Marimon and Zilibotti (1999), Gautier et al. (2010), and Gautier and Teulings (2015). The circular production function takes as input only the distance $d(x, y)$ between x and y, measured along the circumference of a circle, hence the name. Output is maximized for $d = 0$. As argued by Gautier and Teulings (2015), the circular model can be understood as a second-order Taylor approximation of a more general production technology.

In this paper, I consider both circular and hierarchical sorting models. The circular version is simpler because it reduces the dimensionality of the model: it can be solved in d rather than (x, y) and has a closed-form solution (explained below). Ultimately, however, the hierarchical sorting model is empirically more appealing: absolute advantage implies a ranking of workers and firms and most data used for constructing such rankings empirically and bring sorting models to the data, for instance, education, job tenure, firms size, or value added, are inherently hierarchical.

[12]This implies, for any $x' > x$ and $y' > y$, $F(x', y') + F(x, y) \geq F(x', y) + F(x, y')$. See Topkis (1998) for an excellent treatment of supermodularity and complementarity in the context of lattice theory.

[13]This implies, for any $x' > x$ and $y' > y$, $F(x', y')F(x, y) \geq F(x', y)F(x, y')$, $F_x(x', y')F_x(x, y) \geq F_x(x', y)F_x(x, y')$, and $F_{xy}(x', y')F_{xy}(x, y) \geq F_{xy}(x', y)F_{xy}(x, y')$.

2.3 The Circular Sorting Model

Define the distance between a worker type x and a firm type y match along the circle as

$$d(x, y) = \min\{x - y + 1, y - x\}.$$

Without loss of generality, I consider only the case $x > y$ because $y > x$ is completely symmetric. The maximum distance is $1/2$ since it starts decreasing again halfway around the circle. The circular production function $F(d)$ maps d onto output with an interior maximum at $F(d = 0)$:

$$F(d) = \bar{F} - \frac{1}{2}\gamma d. \tag{1}$$

The functional form I consider is comparable to that of Marimon and Zilibotti (1999), Gautier et al. (2010), and Gautier and Teulings (2015).[14] $\bar{F} > 0$ is the output of an optimal match with $d = 0$. The distance $d \in (0, \frac{1}{2})$ is a measure of mismatch between workers and firms. γ governs how quickly output is decreasing in distance. It represents the strength of the production complementarity and the cost of mismatch.[15] The smaller γ, the more substitutable are different types in production.[16] I consider only interior solutions in which matches are unacceptable beyond a certain distance, so γ has to be sufficiently high to ensure that, in equilibrium, workers do not accept jobs at every distance.

The circular sorting model has two properties that vastly simplify it compared to the hierarchical version considered below. First, it follows from Lemma 1 in Marimon and Zilibotti (1999) that if unemployed workers x are distributed uniformly along the circle initially, the distribution of vacancies y will also be uniform. This implies, second, that the values of unemployment, $\mathcal{U}$, and vacancies, $\mathcal{V}$, are the same for all worker and firm types. The value of nonmarket activity, b, and the vacancy posting costs, c, are constants and not type dependent. The values of employment and production, $\mathcal{E}(d)$ and $\mathcal{P}(d)$, depend on d only and not on the underlying worker and firm types. The same is true for the Nash wage bargain $W(d)$. These properties highlight why the circular version is a comparative advantage sorting model: the underlying worker and firm types do not matter for production by themselves; there is no hierarchy. I take these simplifications as my starting point and solve the circular model to obtain a closed-form expression for the interior cutoff distance, which I call d^*. In an interior solution ($d^* < \frac{1}{2}$), workers and firms will accept matches only up to a distance of d^* and reject matches beyond that

[14] Marimon and Zilibotti (1999) have a non-negative lower bound, which I do not need for my purposes. Gautier et al. (2010) and Gautier and Teulings (2015) let the output decrease quadratically in distance. This difference only affects the calibration of the model.

[15] γ is also known as the "complexity dispersion parameter" (Teulings and Gautier, 2004; Teulings, 2005).

[16] The limiting case $\gamma = 0$ would be a labor market without worker and firm heterogeneity. With $\gamma = \infty$, agents would match only with their optimal partner and nobody else.

point. This conforms with the matching set logic of the more general hierarchical model.

The following circular sorting model is a discrete time version of Marimon and Zilibotti (1999). As the focus of this paper is on the hierarchical version, I do not spend much time describing or solving the circular model.[17] The steady-state value functions are standard expect for the integral term in the value of unemployment and a vacant firm. Unmatched agents will accept matches up to d^* both to the left and to the right of their own position on the circle. For this reason, the integral terms are multiplied by 2.[18]

$$\mathcal{E}(d) = W(d) + \beta(\delta\mathcal{U} + (1-\delta)\mathcal{E}(d)) \tag{2}$$

$$\mathcal{U} = b + 2\beta q_u(\theta)\int_0^{d^*}(\mathcal{E}(c) - \mathcal{U})\mathrm{d}c \tag{3}$$

$$\mathcal{P}(d) = F(d) - W(d) + \beta(\delta\mathcal{V} + (1-\delta)\mathcal{P}(d)) \tag{4}$$

$$\mathcal{V} = -c + 2\beta q_v(\theta)\int_0^{d^*}(\mathcal{P}(c) - \mathcal{V})\mathrm{d}c \tag{5}$$

Due to the free entry assumption, $\mathcal{V}$ is 0 in equilibrium, so $c = 2\beta q_v(\theta)\int_0^{d^*}(\mathcal{P}(c))\mathrm{d}c$. The standard Nash bargaining solution then implies

$$\mathcal{E}(d) - \mathcal{U} = \frac{\alpha}{1-\alpha}\mathcal{P}(d), \tag{6}$$

where α is the workers' bargaining power. By plugging in (2), (3), and (4) and using the fact that $\frac{c}{q_v(\theta)}$ must be equal to $\mathcal{P}(d)$ with free entry, I obtain the following wage equation

$$W(d) = \alpha(F(d) + c\theta) + (1-\alpha)b, \tag{7}$$

which is just the Pissarides (2000) textbook wage equation with the additional twist that wage and output depend on the distance d in the circular sorting model.

Stationary Equilibrium of the Circular Model

The main advantage of the circular model from a computational perspective is that it delivers a closed-form solution for the matching cutoff d^*. I find the cutoff by using the fact that the value of a producing firm must be 0 at the cutoff in an interior solution.

$$\mathcal{P}(d^*) = (1-\alpha)(F(d^*) - b) - \alpha c\theta = 0 \tag{8}$$

Equation (8) can easily be solved for d^*. To find the value of aggregate labor market tightness θ compatible with d^* and free entry, plug the value functions into Equation 6, substitute out the wage using $\mathcal{P}(d)$, integrate on both sides, and substitute the integral

[17] For more details, see Marimon and Zilibotti (1999) and the proofs therein.

[18] I use c as the variable of integration to avoid notational confusion.

term out of the entry condition. The equilibrium θ value is unique because I consider interior solutions only.

2.4 The Hierarchical Model

I now turn to the more general, hierarchical case. In the sorting model with absolute advantage, high type workers (firms) always produce more, no matter with which type of firm (worker) they are matched. Following Shimer and Smith (2000), I use a simple functional form assumption that features log-supermodularity of the function itself, its first derivatives, and cross-derivatives to ensure the existence of a search equilibrium.[19]

$$F(x, y) = \exp(x \times y). \tag{9}$$

Recall that x and y are bounded on $[0, 1]$, so $\min(F(x, y)) = 1$ and $\max(F(x, y)) = \mathrm{e}$.[20] I now define the surplus function

$$\mathcal{S}(x, y) = \mathcal{P}(x, y) - \mathcal{V}(y) + \mathcal{E}(x, y) - \mathcal{U}(x), \tag{10}$$

which depends on the four option value equations (see below) for a producing firm, a vacant job, an employed worker, and an unemployed worker for all (x, y) combinations. To capture the logic behind the matching sets in a way that is algebraically convenient, I define a simple match indicator function:

$$\mu(x, y) = \begin{cases} 1 & \text{if } \ \mathcal{S}(x, y) > 0 \\ 0 & \text{if } \ \mathcal{S}(x, y) < 0 \end{cases} \tag{11}$$

$\mu(x, y)$ equals 1 whenever a firm of type y is willing to match with a worker of type x and vice versa. When necessary for clarity, I indicate that $\mu(x, y) = 1$ ($\mu(x, y) = 0$) by writing $\mu^+(x, y)$ ($\mu^-(x, y)$). The choice of with whom to match is determined solely by the surplus value function $\mathcal{S}(x, y)$. In case the surplus is positive, both parties will agree to form a match, so the decision is mutually consistent.[21] When the surplus turns out to be negative after a meeting, both parties prefer to continue their search due to their higher outside options.

$$\mathcal{E}(x, y) = W(x, y) + \underbrace{\beta\delta\mathcal{U}(x)}_{\text{separation}} + \underbrace{\beta(1-\delta)\max\{\mathcal{E}(x, y), \mathcal{U}(x)\}}_{\text{continued employment}} \tag{12}$$

[19]That is, for any $x' > x$ and $y' > y$, $F(x', y')F(x, y) \geq F(x', y)F(x, y')$, $F_x(x', y')F_x(x, y) \geq F_x(x', y)F_x(x, y')$, and $F_{xy}(x', y')F_{xy}(x, y) \geq F_{xy}(x', y)F_{xy}(x, y')$.

[20]This functional form is also used in Teulings and Gautier (2004).

[21]In the model with continuous distributions, the surplus is never exactly 0. Due to discretization, I have to allow for some smoothing of $\mu(x, y)$ when the surplus is very small.

$\mathcal{E}(x, y)$ represents the value of a type x worker employed at firm type y and consists of the flow payment of the match-specific wage, $W(x, y)$, plus the value of the two possible outcomes in the next period, discounted by β. With probability δ this match is subject to an idiosyncratic separation shock and the worker receives the value of unemployment, $\mathcal{U}(x)$. Accordingly, with probability $(1 - \delta)$, the worker continues to receive the value of employment at firm x. The max operator allows for the possibility that the value of employment falls below the value of unemployment in the next period, for instance, due to a productivity shock. The asset value of an unemployed worker of type x is defined as follows, with the limits of integration being equal to the boundaries of x and y, $[0, 1]$:

$$
\begin{aligned}
\mathcal{U}(x) = b(x) + \underbrace{\beta(1 - q_u(\theta))\mathcal{U}(x)}_{\text{no meeting}} + \underbrace{\beta q_u(\theta) \int_0^1 \frac{g_v(y)}{V} \mu^+(x, y)\mathcal{E}(x, y)\mathrm{d}y}_{\text{successful match}} \\
+ \underbrace{\beta q_u(\theta)\mathcal{U}(x) \int_0^1 \frac{g_v(y)}{V} \mu^-(x, y)\mathrm{d}y}_{\text{meet unacceptable firm}}
\end{aligned}
\tag{13}
$$

In case of unemployment, all workers receive a type-specific value of home production $b(x)$ with $\frac{\partial b(x)}{\partial x} > 0$. In the following period, there is a probability $(1 - q_u(\theta))$ that the unemployed worker will not meet any firm and remains unemployed. With probability $q_u(\theta)$, the job finding rate, a meeting with a firm will occur. Whenever the type of firm y is an element of this worker's matching set (and vice versa), that is, $\mu(x, y) = 1$, a match is formed and production starts. Note that $g_v(y)/V$ under the integral sign represents the probability of meeting every specific firm type. This probability acts like a weight on the surplus. In case the firm turns out to be an unsuitable match ($\mu(x, y) = 0$), both parties continue to search. The firms' asset value equations are symmetrical:

$$
\mathcal{P}(x, y) = F(x, y) - W(x, y) + \underbrace{\beta\delta\mathcal{V}(y)}_{\text{separation}} + \underbrace{\beta(1 - \delta) \max\{\mathcal{P}(x, y), \mathcal{V}(y)\}}_{\text{continued production}}
\tag{14}
$$

The flow payment received by the firm in a productive employment relationship is the match-specific output $F(x, y)$ minus the wage, $W(x, y)$. In the next period, the match breaks up with probability δ, leading to the option value of a vacancy, or continues with probability $(1 - \delta)$. Finally, the asset value of a vacancy is as follows:

$$
\begin{aligned}
\mathcal{V}(y) = -c(g_v(y)) + \underbrace{\beta(1 - q_v(\theta))\mathcal{V}(y)}_{\text{no meeting}} + \underbrace{\beta q_v(\theta) \int_0^1 \frac{g_u(x)}{U} \mu^+(x, y)\mathcal{P}(x, y)\mathrm{d}x}_{\text{successful match}} \\
+ \underbrace{\beta q_v(\theta)\mathcal{V}(y) \int_0^1 \frac{g_u(x)}{U} \mu^-(x, y)\mathrm{d}x}_{\text{meet unacceptable worker}}
\end{aligned}
\tag{15}
$$

The cost of maintaining an open vacancy is determined by the convex function, $c(g_v(y))$, which must be paid every period. In the next period, there is the possibility of not meeting a worker, of meeting a suitable worker and filling the job, or of meeting an unsuitable worker and continuing to search.

The Nash bargaining solution determines how the surplus is divided in the event of a suitable match. The workers' bargaining power parameter is $\alpha \in (0,1)$. For both the worker and the firm, the respective share of surplus equals the additional value of being matched compared to the value of continued search, which serves as threat point in the bargaining game.

$$\alpha\mathcal{S}(x,y) = \mathcal{E}(x,y) - \mathcal{U}(x) \tag{16}$$

$$(1-\alpha)\mathcal{S}(x,y) = \mathcal{P}(x,y) - \mathcal{V}(y) \tag{17}$$

Note that the values of employment (production) and unemployment (a vacancy) are equalized when $\mathcal{S}(x,y) = 0$, which is exactly the definition of the matching sets. It is possible to simplify the four Bellman equation (Equations (12) to (15)) by adding and subtracting the value of unemployment and a vacancy respectively. The surplus sharing rules of Equations (16) and (17) can then be plugged in and the surplus function shows up under the integral signs.[22]

$$\mathcal{E}(x,y) = W(x,y) + \beta\left(\mathcal{U}(x) + \alpha(1-\delta)\max\{\mathcal{S}(x,y),0\}\right), \tag{18}$$

$$\mathcal{U}(x) = b(x) + \beta\left(\mathcal{U}(x) + \alpha q_u(\theta)\int_0^1 \frac{g_v(y)}{V}\max\{\mathcal{S}(x,y),0\}\mathrm{d}y\right), \tag{19}$$

$$\mathcal{P}(x,y) = F(x,y) - W(x,y) + \beta\left(\mathcal{V}(y) + (1-\alpha)(1-\delta)\max\{\mathcal{S}(x,y),0\}\right), \tag{20}$$

$$\mathcal{V}(y) = -c(g_v(y)) + \beta\left(\mathcal{V}(y) + (1-\alpha)q_v(\theta)\int_0^1 \frac{g_u(x)}{U}\max\{\mathcal{S}(x,y),0\}\mathrm{d}x\right). \tag{21}$$

Stationary Equilibrium of the Hierarchical Model

Due the integral terms in the values of unemployment (Equation (19)) and a vacancy (Equation (21)), the hierarchical sorting model has no closed-form solution. Value function iteration on a discrete grid can be applied in this context to numerically approximate the model's stationary equilibrium. Technically, this procedure relies on the conjecture that in a given parameter space, the surplus function is a contraction mapping.[23]

The first step is to compute the fixed point of the surplus value function $\mathcal{S}(x,y)$, which is obtained by plugging Equations (18) to (20) into Equation (10). Note that

[22] Note that $\int_0^1 \frac{g_v(y)}{V}\max\{\mathcal{S}(x,y),0\}\mathrm{d}y$ is equivalent to $\int_0^1 \frac{g_v(y)}{V}\mu(x,y)\mathcal{S}(x,y)\mathrm{d}y$.

[23] This approach to solve the model is similar in spirit to the procedures described in Shimer and Smith (2000), Hagedorn et al. (2017), and Lise and Robin (2017). I thank Robert Shimer for sharing the code used to produce the numerical results in Shimer and Smith (2000).

$\mathcal{V}(y) = 0 \;\forall\; y$ due to free entry, so the value of a vacancy drops out.

$$\begin{aligned} \mathcal{S}(x,y) = F(x,y) &+ \beta(1-\delta)\max\{\mathcal{S}(x,y),0\} \\ &- \left(b(x) + \beta\alpha q_u(\theta)\int_0^1 \frac{g_v(y)}{V}\max\{\mathcal{S}(x,y),0\}\;\mathrm{d}y\right) \end{aligned} \tag{22}$$

The surplus function includes the workers' outside option in the large brackets due to the Nash bargaining assumption. This is an important difference from the model of Lise and Robin (2017), who show that incorporating search on the job with sequential auctions (Postel-Vinay and Robin, 2002) removes the distributional terms from the surplus value and, hence, from the state space of the model. By solving the fixed point problem for every (x,y) combination, the equilibrium matching sets of all worker and firm types summarized in $\mu(x,y)$ are pinned down by $\mathcal{S}(x,y) \geq 0$. The agents' acceptance strategy is as follows: for every (x,y) combination with a weakly positive surplus, $\mu(x,y)$ contains a 1 and a 0 otherwise.[24]

In the second step, knowing $\mathcal{S}(x,y)$ and $\mu(x,y)$, an equilibrium flow condition can be used to solve for the endogenous distribution of unemployed workers.

$$\delta g_m(x,y) = g_u(x)\; q_u(\theta)\; \frac{g_v(y)}{V}\; \mu(x,y). \tag{23}$$

The left-hand side of Equation (23) represents the number of dissolved active matches for every (x,y) combination in equilibrium. Without stochastic labor productivity, all match dissolutions occur exogenously with probability δ. On the right-hand side of the equation, the flow out of unemployment for the measure of all unemployed workers of type x, given by $g_u(x)$, is determined by the job-finding rate, $q_u(\theta)$, times the probability of meeting a specific firm type y, ${g_v(y)}/{V}$, times the match indicator function that takes the value 1 if the respective (x,y) combination produces a positive surplus. New matches are created solely for (x,y) combinations that produce a (weakly) positive surplus in equilibrium. Integrating the firm dimension out of Equation (23) and substituting in $g_w(x) - g_u(x)$ on the left-hand side yields the following expression for $g_u(x)$:

$$g_u(x) = \frac{\delta g_w(x)}{q_u(\theta)\int_0^1 \frac{g_v(y)}{V}\mu(x,y)\mathrm{d}y + \delta} \tag{24}$$

Note that integrating Equation (24) over x yields an expression comparable to the textbook equation pinning down equilibrium unemployment, that is, the Beveridge Curve:

$$U = \frac{\delta}{\delta + q_u(\theta)\iint \frac{g_v(y)}{V}\mu(x,y)\mathrm{d}y\mathrm{d}x}$$

[24]In practice, however, it is necessary to apply some smoothing to the acceptance strategy to ensure convergence. I allow for mixed strategy solutions close to the cutoff, following Hagedorn et al. (2017).

Compared to the textbook version, the above expression is augmented with the double integral term in the denominator, which has to be smaller than 1 if $\exists(x,y).\mu(x,y)=0$. Thus, equilibrium unemployment must be higher in the sorting model.

To determine the distribution of vacant firm types in the stationary equilibrium, I apply the free entry condition for all y. Any firm type will post vacancies as long as the discounted value of the job is at least as high as the cost implied by $c(g_v(y))$.

$$c(g_v(y)) = \beta(1-\alpha)q_v(\theta)\int_0^1 \frac{g_u(x)}{U}\mu(x,y)\mathcal{S}(x,y)\mathrm{d}x. \tag{25}$$

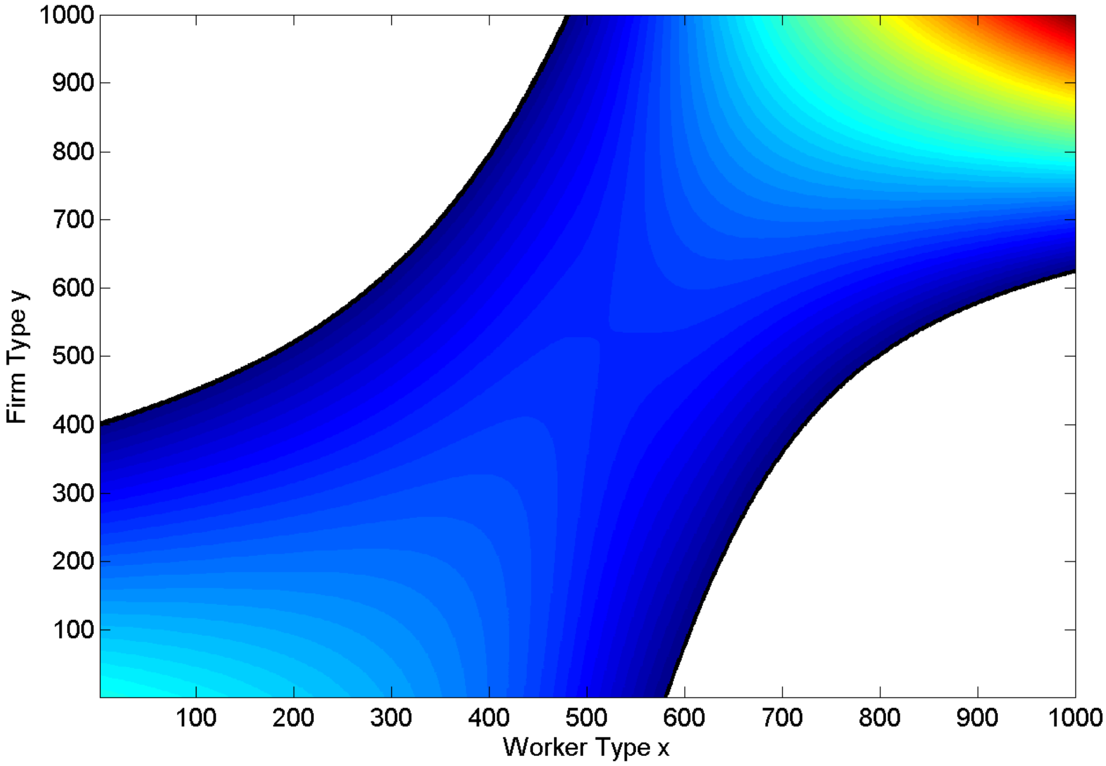
Given $g_u(x)$, $\mu(x,y)$, and $\mathcal{S}(x,y)$, Equation 25 can be solved for the measure of vacancies posted by every firm type y in equilibrium. Integrating over $g_u(x)$ and $g_v(y)$ yields the aggregate stocks of unemployed workers and vacant firms, which in turn determine aggregate labor market tightness, arrival rates, and the flow of new meetings via the matching function.

The stationary equilibrium of the hierarchical sorting model consists of the objects $\{\mathcal{S}(x,y), \mu(x,y), g_u(x), g_v(y)\}$, which are jointly determined by the surplus value function (Equation (22)), the steady state flow condition (Equation (23)), and free entry of vacancies. The solution algorithm has two steps: it alternates between computing the fixed point of the surplus function for all (x,y) combinations and updating the distributions of unemployed workers and vacant firm types using steady-state flows and the free entry condition, respectively. This procedure is repeated until the decision rule converges.[25] The equilibrium can be computed with high precision in a relatively short amount of time.

To help visualize the properties of the stationary equilibrium of the hierarchical sorting model, Figure 1 presents a projection of the surplus function on the type space, along with the matching cutoffs. The model is solved on a discrete grid with 1,000 worker and firm types. The hierarchy implied by the sorting model is immediately apparent, as the surplus increases quickly toward the upper-right corner. Interestingly, the surplus also increases toward the lower-left corner, that is, for matches between low type workers and low type firms. This property is a direct expression of the production complementarity in the sorting model. The surplus shrinks toward the cutoffs of the matching sets because the relatively higher ranked partner always needs to be compensated for his higher outside option. With absolute advantage and the production function of Equation (9), it is relatively more important to be optimally matched for low type workers and firms than for medium types. In other words, this type of worker has a higher incentive to be sorted and this is reflected in the surplus function.

[25] Convergence is achieved once the absolute difference of the surplus between two iterations is less than 10^{-12}.

Figure 1: Surplus and Matching Set Cutoffs in Equilibrium



3 Comparative Statics

I now introduce aggregate labor productivity into the hierarchical sorting model. Labor productivity z can be imagined as an underlying technology that enables labor to be used productively. Thus, it influences the output of every match in equal measure:

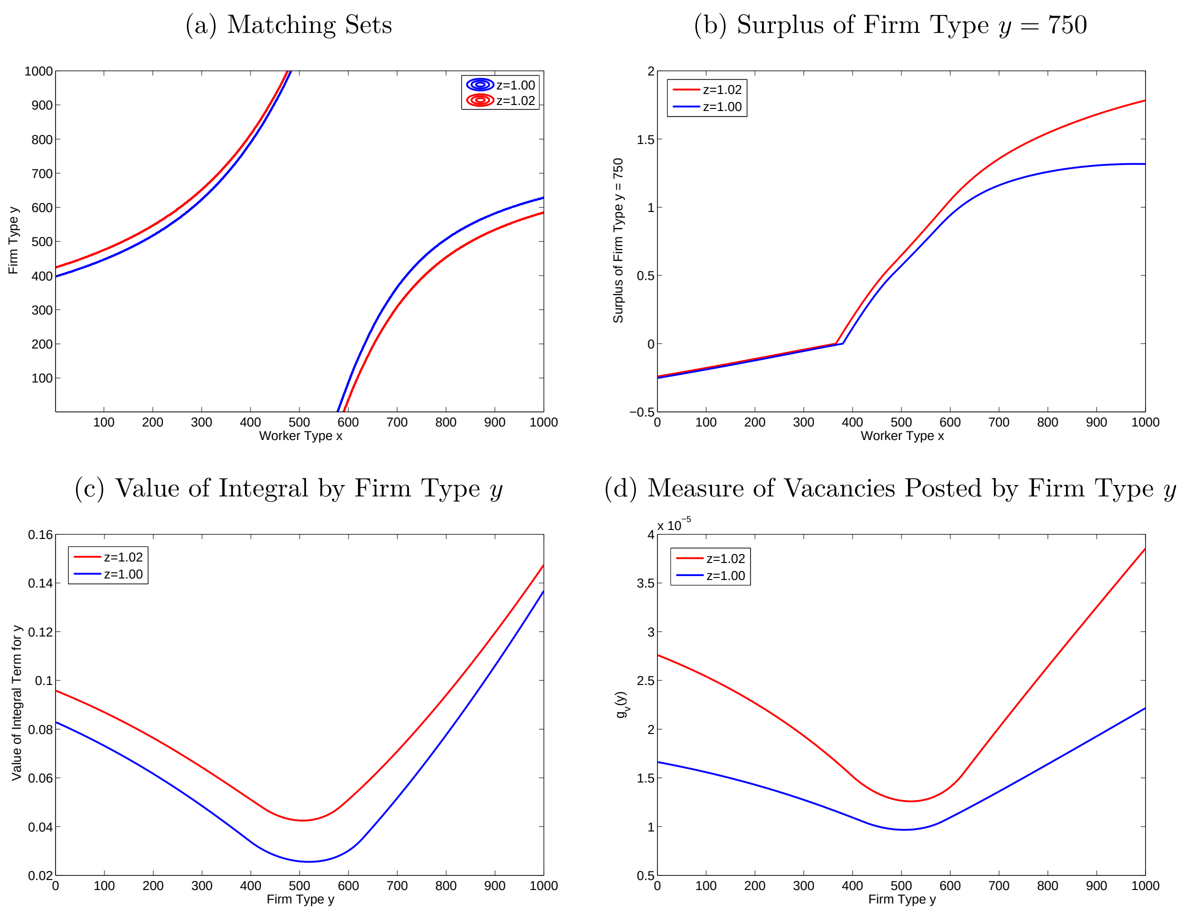
$$F(x, y, z) = F(x, y) \times z, \tag{26}$$

so the output of a match between worker x and firm y with labor productivity z is simply the product of the match-specific part, $F(x, y)$, and the potentially time-varying but type-independent aggregate labor productivity, z. To better understand the properties that lead to an amplification of shocks in the dynamic model, I study the comparative statics of the model, shown in Figure 2. z changes from its steady-state value of 1 to 1.02. A key property of the production function $F(x, y, z) = z \times e^{xy}$ as introduced above is that the strength of the production complementarity is positively correlated with z.[26] This implies procyclical sorting, that is, the incentive to be optimally matched increases with labor productivity.[27]

[26]This is a result of the (log-)supermodularity of the first derivative in the production function: $\frac{\partial F(x,y,z)}{\partial x} = F_x(x, y, z) = x\ z\ e^{xy}$.

[27]To date, there is no conclusive empirical evidence about the cyclicality of labor market sorting. Lochner and Schulz (2016) document that the degree of labor market sorting in Germany seems to be

Figure 2: Comparative Statics of a Change in z



In response to a change in z from 1.00 (blue curves) to 1.02 (red curves), the equilibrium of the hierarchical sorting model changes, as shown in Figure 2. First, the matching sets become wider; see Panel 2a. The set of acceptable matches increases for all worker and firm types. The boundaries do not shift in parallel but become (more) asymmetric in response to an increase in z. This property of the model turns out to be important for the wage adjustments in the dynamic sorting model. Panel 2b shows how the surplus function shifts upward for a firm of type $y = 750$. Note that even though the matching set becomes only slightly wider, the overall surplus of this firm increases more than is proportional in response to a change in z, which illustrates that being optimally sorted becomes more desirable as labor productivity increases. This property is key to the amplification I find in the dynamic sorting model because it implies that the integral terms also increase more than proportionally in response to shocks in z. Panel 2c shows how the integral term of the firm in the job creation condition (Equation (25)) increases with z for all firm types y. Hence, all firms will have an incentive to post more vacancies as labor becomes more productive, as is observed what in Panel 2d, which shows the measure of vacancies posted $g_v(y)$ by every firm type y. Note that vacancy posting increases particularly for low and high type firms, in line with the properties of the surplus function shown in Figure 1. Quantitatively, the depicted change of z from 1.00 to 1.02 leads to an overall increase of the vacancy rate by more than 50%. Unemployment falls by about 6%. These comparative statics results make a strong case for finding a significant amplification effect in the dynamic sorting model.

4 The Dynamic Sorting Model

In the dynamic sorting model, z is stochastic, so I study the properties of the model under aggregate uncertainty. Let z_t denote the realization of aggregate labor productivity z in period t. I assume that z_t follows an AR(1) process (in logs):

$$z_t = \rho \, z_{t-1} + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2), \tag{27}$$

with the autocorrelation ρ and innovation parameter σ_ϵ calibrated below. To simplify notation, let z' be next period's realization $(t+1)$. Time subscripts are omitted in the following to avoid notational clutter.

I define the state of the system to be $\Omega(g_m(x, y, z), z)$. This state contains the exogenous state variable z and the endogenous state $g_m(x, y, z)$, which is the distribution of active matches. All endogenous objects depending on the state have Ω as an argument in the following.

roughly aligned with the business cycle.

With this notation, the dynamic surplus value function becomes

$$
\begin{aligned}
\mathcal{S}(x, y, \Omega) &= F(x, y, z) + \beta \mathbb{E}\left[(1-\delta)\max\{\mathcal{S}(x, y, \Omega'), 0\}\right] \\
&\quad - \beta \mathbb{E}\left[b(x) - \alpha q_u(\theta(\Omega')) \int_0^1 \frac{g_v(y, \Omega')}{V(\Omega')} \mu(x, y, \Omega') \mathcal{S}(x, y, \Omega') \, \mathrm{d}y\right].
\end{aligned} \tag{28}
$$

$F(x, y, z)$ depends on the exogenous state only. Home production $b(x)$ does not depend on z. $\mathbb{E}$ is the expectation operator regarding the future aggregate state. It includes all information available in the period the expectation is formed.

4.1 Job Creation

The key condition for the analysis of sorting models out of steady state in this paper is the dynamic entry problem of the firm. I assume free entry for all firm types y, so $\mathcal{V}(y) = 0 \; \forall \; y$. A firm of type y will post vacancies as long as the expected discounted value of production is at least as high as the cost implied by $c(.)$:

$$
c(g_v(y, \Omega)) = \beta(1-\alpha)\mathbb{E}\left[q_v(\theta(\Omega')) \int_0^1 \frac{g_u(x, \Omega')}{U(\Omega')} \mu(x, y, \Omega') \mathcal{S}(x, y, \Omega') \, \mathrm{d}x\right]. \tag{29}
$$

In the hierarchical sorting model, firm entry depends on the expectation of the integral over the firm-specific matching set, summarized in $\mu(x, y, \Omega')$, the surplus function $\mathcal{S}(x, y, \Omega')$, and the probability measure of meeting every specific worker type within the firms matching set, $\frac{g_u(x,\Omega')}{U(\Omega')}$.[28] Due to the state dependence, the job-creation condition in the hierarchical sorting model is much richer than in standard search and matching models: $c = \beta(1-\alpha)\mathbb{E}[q_v(\theta(z'))\mathcal{S}(z')]$. Note that vacancy posting costs are constant in the standard problem and the expectation does not involve an integral term. I show in the quantitative section of the paper that the additional objects on the right-hand side of Equation (29) are critical for the model's response to aggregate shocks: all three endogenous variables of the augmented problem depend on the state and adjust in response to shocks. For a given firm, the surplus function shifts upward more than proportionally after a positive shock; recall Figure 2. Additionally, the cardinality of the matching sets increases since more potential matches now yield a positive surplus. There is an additional surplus to be realized both with workers who have been in the firm's matching set before *and* with new workers on the margins of the matching set. This channel is quantitatively important for a large share of the amplification result documented in Section 5.

[28] The dynamic entry problem is the main difference between this paper and Lise and Robin (2017). In their model with search on the job, entry is a static problem because the surplus function does not depend on the integral terms contained in Equations (19) and (21) of my model.

4.2 Wage Formation

I now turn to the derivation of match-specific wages in the hierarchical sorting model using the Nash bargaining solution and free entry:

$$\mathcal{E}(x,y,\Omega)-\mathcal{U}(x,\Omega)=\frac{\alpha}{1-\alpha}\mathcal{P}(x,y,\Omega). \tag{30}$$

Plugging in the value functions, maximizing the Nash product, and engaging in some algebra yields an expression that determines the match-specific wage in the dynamic model, $W(x,y,\Omega)$:

$$\begin{aligned} W(x,y,\Omega) &= \alpha F(x,y,z)+(1-\alpha)b(x) \\ &\quad + (1-\alpha)\beta\alpha\mathbb{E}\left[q_u(\theta(\Omega'))\int_0^1 \frac{g_v(y,\Omega')}{V(\Omega')}\mu(x,y,\Omega')\mathcal{S}(x,y,\Omega')\ \mathrm{d}y\right]. \end{aligned} \tag{31}$$

The wage of a given worker is a convex combination of the match-specific output, $F(x,y,z)$, and the worker's outside option, his value of being unemployed. With respect to the integral term, the same logic as in the firms' job-creation problem applies: the outside option depends on the expected value of the surplus with all other potential employers in the matching set, weighted by the distribution. The higher the surplus and the higher the probability of meeting every specific firm type in the matching set, the higher the bargained wage because the worker has more valuable outside options for which he needs to be compensated. After factoring out α, Equation (29) can be plugged into the wage equation to arrive at

$$\begin{aligned} W(x,y,\Omega) = \alpha &\left(F(x,y,z)+c(g_v(y,\Omega))\mathbb{E}\left[\theta(\Omega')\frac{\int_0^1 \frac{g_v(y,\Omega')}{V(\Omega')}\mu(x,y,\Omega')\mathcal{S}(x,y,\Omega')\ \mathrm{d}y}{\int_0^1 \frac{g_u(x,\Omega')}{U(\Omega')}\mu(x,y,\Omega')\mathcal{S}(x,y,\Omega')\ \mathrm{d}x}\right]\right) \\ &+(1-\alpha)b(x). \end{aligned} \tag{32}$$

Now, the same logic regarding the integral term over the matching set also applies for the firm in the bargaining game: the expected value of the surplus, the matching set, and the distribution of other unemployed worker types influence the negotiated wage negatively through the denominator: the more workers who are available in the firm's matching set, the better the firm's outside option of continued search, and the lower the match-specific bargained wage with worker type x because the firm needs to be compensated. Note that aggregate labor market tightness $\theta(\Omega')$ in front of the quotient cancels out with $1/V(\Omega')$ in the numerator and ${}^{1}/_{U(\Omega')}$ in the denominator:

$$W(x,y,\Omega) = \alpha\left(F(x,y,z) + c(y)\mathbb{E}\left[\frac{\int_0^1 g_v(y,\Omega')\mu(x,y,\Omega')\mathcal{S}(x,y,\Omega')\ \mathrm{d}y}{\int_0^1 g_u(x,\Omega')\mu(x,y,\Omega')\mathcal{S}(x,y,\Omega')\ \mathrm{d}x}\right]\right) + (1-\alpha)b(x). \tag{33}$$

The match-specific wage $W(x,y,\Omega)$ does not depend on aggregate labor market tightness, in contrast to the standard model. This is an important feature of the model, because it disconnects wages from fluctuations in aggregate tightness. Instead, the quotient, which I call *relative* labor market tightness, determines wages. Relative labor market tightness is the expected ratio of the two integral terms, which include the distributions of vacancies and unemployed workers, and the surpluses with all types within the respective matching sets. (33) provides a natural generalization of the textbook wage equation to the framework with heterogeneous workers and firms. The key difference between the sorting model and the baseline DMP model is now obvious. Compare Equation (33) with its textbook counterpart in a dynamic DMP model with state z and homogeneous firms and workers:[29]

$$W(z) = \alpha(F(z) + \kappa\mathbb{E}\theta(z')) + (1-\alpha)b. \tag{34}$$

In the textbook model, the wage is positively impacted by aggregate labor market tightness. The higher $\theta = {}^{V}/_{U}$, the more difficult it is for firms to hire a worker. If there is fierce competition between many firms for relatively few unemployed workers, wages are higher. In this setting, the fully flexible Nash-bargained wage absorbs all the fluctuations in θ, which is the essence of the lack of amplification in the standard model (Shimer, 2005). In the sorting model, the impact of aggregate labor market tightness on wages is partially muted because θ is replaced by the relative labor market tightness, which I now define as follows:

$$\Theta(x,y,\Omega) = \frac{\int_0^1 g_v(y,\Omega)\mu(x,y,\Omega)\mathcal{S}(x,y,\Omega)\ \mathrm{d}y}{\int_0^1 g_u(x,\Omega)\mu(x,y,\Omega)\mathcal{S}(x,y,\Omega)\ \mathrm{d}x}.$$

The integral term in the numerator (denominator) represents a specific worker (firm) type's option value of search, that is, the value of the surplus function over the respective matching sets, properly weighted by the distribution of suitable types. Note that in the hierarchical sorting model, the endogenous distributions of unemployed workers and vacant firm types are typically not uniform, even with underlying uniform type densities. Relative labor market tightness can have an impact on the wage bargain that is very different from aggregate tightness in the standard model: aggregate tightness may be high (and unemployment low), but if the measure of unemployed workers within a firm's matching set is high, the firm has no incentive to pay a high wage and workers

[29]Note that this equation corresponds to Shimer's Equation (7) (Shimer (2005), p. 41), which is a slightly generalized version of Equation (1.20) in Pissarides (2000), p. 17. Equation (33) collapses to Equation (34) in the case of homogeneous workers and firms and constant output.

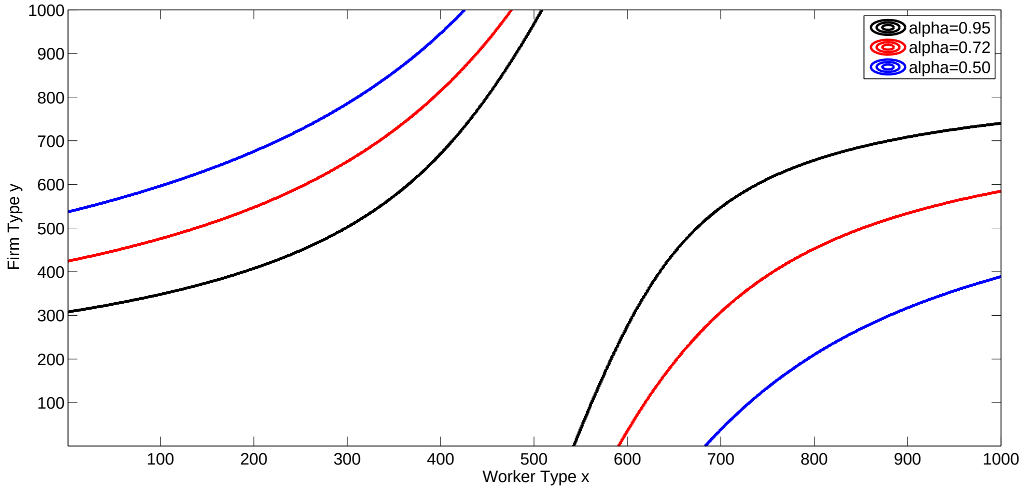
extract less, even if unemployment outside the firm's matching set is very low. Thus, the worker's bargaining position does not depend on scarcity or abundance of other unemployed workers outside the matching set of his potential employer. This mechanism de-links match-specific wages from the influence of aggregate labor market conditions.

Wage Rigidity with Sorting

In the dynamic setting, aggregate fluctuations lead to an under-proportional wage adjustment whenever $\Theta(x, y)$ is smaller than aggregate labor market tightness θ. In the calibrated model, aggregate labor market tightness θ is normalized to 1. So whenever the denominator in $\Theta(x, y)$, the firm's integral term, is bigger than the worker's integral term in the numerator, $\Theta(x, y) < \theta$ will hold. This weakens the link between aggregate fluctuations and the wage bargain: wages no longer respond to shocks in a fully flexible manner. It is well known that mechanisms which disconnect wages and aggregate fluctuations are key to enable search and matching models to generate amplification. The finding that such a disconnect can endogenously arise in a setting with heterogeneity, sorting, and Nash bargaining is novel and a central contribution of this paper. In the following, I argue that the relative labor market tightness term is smaller than 1 in equilibrium due to the fact that the surplus function exhibits an asymmetry that is positively related to the workers' bargaining parameter.

Figure 3 plots the cutoffs of $\mu(x, y)$ in steady state for three values of the bargaining parameter (i.e., $\alpha = 0.5, 0.72, 0.95$). When $\alpha = 0.5$ (blue), the matching set cutoffs are slightly asymmetric. To see this, note that the lowest worker type is acceptable for all firm types below, roughly, 520, whereas the least productive firm is acceptable to all workers up to almost type 700. The asymmetry increases with α. The extreme case of $\alpha = 0.95$ (black) underlines this. The best worker type is extremely picky; he will only match with firms better than type 750. The best firm accepts workers down to rank 420 and is thus much more tolerant with respect to mismatch. Typically, calibrated search and matching models assign the worker a higher bargaining parameter. In the following, I use $\alpha = 0.72$ (red), as in Shimer (2005). This value implies a significant asymmetry, which is sufficient to ensure that $\Theta(x, y) < 1 \ \forall \ (x, y)$, implying an under-proportional wage adjustment in response to shocks. Intuitively, greater bargaining power on the worker's side translates to narrower matching sets because the worker can afford to be more picky when choosing the firm types with which to match. The worker will always extract a higher fraction of the surplus in a match, so the matching set can be smaller in order to be indifferent between the value of employment and unemployment. Firms, in turn, optimally choose wider matching sets because they command only a small share of the match-specific surplus. With small bargaining power, a wider matching set is necessary to equalize the values of production and a vacancy.

Figure 3: Equilibrium Matching Sets and Worker Bargaining Power



The hierarchical sorting model exhibits an asymmetry of the surplus function that translates into wider matching sets for the firms, a relative labor market tightness that is smaller than aggregate labor market tightness for all (x, y) combinations, and, accordingly, wages that do not fully adjust to shocks, and thus an endogenous wage rigidity arises. The numerical simulations in the following quantify to what degree this endogenous rigidity influences the model's dynamics in conjunction with the augmented job-creation condition.

5 Numerical Simulations and Empirical Analysis

I now test the quantitative importance of the augmented job creation and wage determination mechanisms in the dynamic hierarchical sorting model. It is common in this class of dynamic macro models to judge the model's empirical performance by its ability to match specific data moments. The Shimer Puzzle revolves around the search and matching model's ability (or lack thereof) to explain the volatility of the unemployment rate, the vacancy rate, aggregate labor market tightness, and the job-finding rate over the business cycle. I run numerical simulations in a stochastic environment where aggregate shocks to labor productivity drive the business cycle. The stochastic labor productivity process z is defined in Section 4 and calibrated below.

I find that second moments of simulated data from the sorting model are very close to moments from U.S. data, due to both the augmented wage formation mechanism and the additional margins in the job-creation condition. Both channels depend on the complex state space of the model, which determines the additional endogenous objects: the match surplus, matching sets, and type distributions. They all change with the state, so the computations need to handle forward-looking expectations of the match surplus, the matching sets, and the distributions of unmatched worker and firm types.

5.1 Computation

The dynamic sorting model has a complex state space Ω, which consists of the exogenous state z (labor productivity) and the endogenous state $g_m(x, y, z)$ (joint distribution of active matches). All other endogenous objects—the surplus, the matching sets, the distributions of unmatched worker and firm types, the stocks U and V, the arrival rates $q_u(\theta)$ and $q_v(\theta)$, and aggregate labor market tightness θ—follow from the state of the system Ω.

Keeping track of Ω's evolution is computationally challenging because it contains an endogenous distribution function and is hence infinite-dimensional. Discretization is inevitable, but the dimensionality of the state space is still huge for the case of 100 distinct worker and firm types, which is the number I use in my simulations.

I approach this computational complexity by defining two auxiliary state variables for the integral terms in the job-creation condition and in the wage equation, which contain all the high-dimensional endogenous objects. This trick allows me to log-linearize the model around its steady state. I can then apply standard techniques, in this case second-order perturbation, to solve and simulate the dynamic model. To keep track of the deviations of the auxiliary state variables during the simulation, I need to resolve the model conditional on every draw of the exogenous state z. I compute the numerical differentials of the auxiliary state variables with respect to z in a "brute-force" fashion for all (x, y) and z combinations. The key to success for this method is a very fast algorithm that solves the model using the procedure outlined in Section 2.4. Once I know how the integral terms change in response to shocks, I simply plug the deviations from the auxiliary state variables' steady state back into the log-linearized system, thus obtaining policy functions of the integral terms.[30]

Since I make heavy use of discretization and approximations to simulate the model, I need to check the reliability of my computational approach. It is well known that log-linearization around the steady state is error prone in nonlinear systems.[31] The approximation becomes increasingly worse with large shocks that push the dynamic model far away from its steady state. Fortunately, the calibrated labor productivity process used in the context of U.S. labor market dynamics is not very volatile. The calibrated standard deviation of z is only 2%, so the model always remains in relatively close vicinity to the

[30]For convenience, I use Dynare (Adjemian et al., 2014) for the underlying standard computations, e.g., checking stability of the system and computing policy functions. The steady state is computed using an external routine and the same routine is called during simulations to account for the fluctuations of the high-dimensional endogenous objects (surplus, matching sets, probability distributions of types).

[31]Using more accurate projection methods to simulate the dynamic sorting model with its complex state space is beyond the scope of this paper.

steady state. I check my computations by plugging simulated data back into the Bellman equations to discover whether the data solve them. The mean computational error I make is quite small at 3.8%. This value is very close to the error one makes when solving simple dynamic search and matching models using log-linearization and perturbation, so my method of dealing with the additional complexity of the sorting model's state space does not appear to significantly increase computational errors.[32] Figures A.1 and A.2 in the Appendix show the distribution of computational errors and positive correlation of the errors with z.

5.2 Calibration Based on Business Cycle Properties

As in Shimer (2005), a time period is set to be one quarter. Table 2 shows the calibration of the model based on the U.S. labor market data used for the simulation exercise. To ensure comparability of the dynamics of the augmented model with the results in Shimer (2005), identical parameter values are used whenever possible. A value of 0.1 for the separation rate translates into an average employment spell of about 2.5 years during the United States in the relevant period (1951–2003). The quarterly discount rate is set to 0.012, representing an annual interest rate of roughly 5%. The discount factor (as it appears in the model equations) is thus $\beta = {}^{1}/{}_{1.012} \approx 0.99$. The matching function elasticity is set to 0.72, in line with Shimer (2005), which is within the empirically supported range from matching function estimations reported by Petrongolo and Pissarides (2001). I set the bargaining parameter equal to the matching function elasticity, that is, I follow the Hosios (1990) condition for socially efficient vacancy posting in the decentralized equilibrium.[33]

Several parameters need to be recalibrated in the hierarchical sorting model. The value of nonmarket activity $b(x)$ is type dependent in my model. I calibrate it to be $0.223 \times \underset{x}{\operatorname{argmax}}\ F(x, y)$. This implies that home production $b(x)$ has a mean of 40% of the output a worker of type x can produce in his optimal match. This assumption is a natural extension of Shimer (2005), who assumes a constant b of 0.4 when output is normalized to 1. The efficiency parameter of the aggregate matching function, ϑ, needs to be increased in the sorting model to take into account that not all meetings result in matches. A value of 2 implies, along with the other parameter values, that the *net* job finding rate, that is, the rate of matches that are formed after a meeting, is close to the value Shimer (2005) constructs from the data, which is 1.355 (quarterly).

[32] Petrosky-Nadeau and Zhang (2016) show that solving the representative agent search and matching model in Hagedorn and Manovskii (2008) via log-linearization and perturbation creates a mean computational error of 3.75%. I find a mean error of 3.84% with slightly more dispersion.

[33] It is unclear whether the Hosios condition holds in the sorting model. I do not claim that vacancy posting is socially efficient in my model.

Table 2: Parameter values for the quarterly calibration of the search and matching model for the U.S. labor market (1951–2003)

Parameter	Symbol	Value	Source
Discount factor	β	0.99	Shimer (2005)
Separation rate	δ	0.1	Shimer (2005)
Workers' bargaining power	α	0.72	Shimer (2005)
Matching function elasticity	ξ	0.72	Shimer (2005)
Matching function constant	ϑ	2	Recalibrated
Value of nonmarket activity	$b(x)$	$0.223 \times \underset{x}{\operatorname{argmax}}\ F(x,y)$	Recalibrated
Vacancy posting costs	c_0	0.03	New Parameter
	c_1	0.4	New Parameter
First order autocorrelation	ρ	0.765	Hagedorn
Standard deviation	σ_ϵ	0.013	& Manovskii (2008)

The convex vacancy posting cost function takes the following form:

$$c(g_v(y)) = \frac{c_0}{1+c_1} g_v(y)^{1+c_1}$$

c_0 and c_1 are set to the values shown in Table 2 to target a steady-state aggregate labor market tightness of 1.[34] The calibrated economy has a steady-state unemployment rate of about 7.8%.[35]

The stochastic labor productivity process z can be imagined as an underlying technology that enables labor to be used productively. It is type independent and affects all matches in equal measure. As in Shimer (2005), it is normalized to 1 in steady state and calibrated to resemble empirical labor productivity in the United States over the relevant period of time. Regarding the functional form, I follow Hagedorn and Manovskii (2008) and set up stochastic labor productivity as a first-order autoregressive process:[36]

$$z_t = \rho\ z_{t-1} + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2). \tag{35}$$

$\rho \in (0,1)$ captures the degree of first-order autocorrelation of the AR(1) process. In-

[34] I need slightly more curvature in my model compared to the values estimated in Lise and Robin (2017), mainly to ensure quick convergence.

[35] This is slightly higher than typically targeted values of steady-state unemployment. The reason lies in the adapted Beveridge Curve (see Equation (2.4)), which implies higher equilibrium unemployment when the matching sets do not cover the whole type space. This property of the model is inconsequential for the quantitative exercise in this paper.

[36] Many closely related papers use more general Markov chains to add a stochastic dimension to the model. An AR(1) process is a homogeneous Markov process iff the error terms are i.i.d. I prefer to use the AR(1) process under this assumption for computational reasons.

novations are drawn from a Gaussian distribution with mean 0 and standard deviation σ_ϵ. Both parameters are set to match quarterly U.S. labor productivity.[37] All values in Table 2 are based on quarterly data. Shimer's (2005) simulation results as well as my own results are reported as deviations from a HP trend, which is conventional in the literature.[38]

Using the calibration in Table 2, the model produces realistic amounts of wage dispersion and labor market sorting. The standard deviation of log wages in equilibrium is about 0.418. Spearman's rank correlation coefficient, which is a measure for the degree of labor market sorting, is 0.095. This is only a moderate degree of positive sorting, so the strong complementarity of worker and firm types assumed via the form of the production function does not translate into a strongly sorted distribution of matches, in line with what is observed in U.S. data.[39]

5.3 The Amplification Effect of Sorting

I find that the hierarchical sorting model produces a large amount of amplification in response to shocks. Second moments of simulated time series data are of the same order of magnitude as the volatility observed in U.S. labor market data for the relevant period of time. In particular, the simulated standard deviations of unemployment, vacancies, labor market tightness, and the job-finding rate are much closer to empirical second moments than simulated data from standard search and matching models. Table 3 compares my results to those of Shimer (2005) and the empirical data moments.

The first two rows of Table 3 show the well-known unemployment volatility problem emphasized by Shimer (2005). The standard deviations of unemployment, U, vacancies, V, labor market tightness, θ, and the job-finding rate, $q_u(\theta)$, in simulated time series data miss the standard deviations in the data by a factor of about 10 to 20. As a first exercise, I run simulations of a model from which I remove worker and firm heterogeneity, set output to 1, and use the exact same calibration as in Shimer (2005), so vacancy posting costs c and the value of home production b are constants. The results are set out in the

[37] Shimer (2005), Hornstein et al. (2005), and Hagedorn and Manovskii (2008) report the parameter values necessary to represent U.S. labor productivity "as seasonally adjusted quarterly real average output per person in the non-farm business sector constructed by the BLS" (Hagedorn and Manovskii (2008), p. 1694).

[38] The Hodrick-Prescott (HP) filter is a technique for decomposing the trend and the cyclical component of a time series (Hodrick and Prescott, 1997). Shimer (2005) sets the smoothing parameter of the filter to $\lambda = 10^5$ instead of to the more common value of $\lambda = 1600$ for quarterly data. This makes the cyclical component less volatile and more persistent. I use the same value as Shimer to generate comparable moments. Hornstein et al. (2005) point out that a more volatile trend, using the common smoothing parameter $\lambda = 1600$ for quarterly data, "has almost no effect on the relative volatilities" (p. 23).

[39] Some evidence for the degree of labor market sorting in the United States is presented by Lise et al. (2016), who make a parametric assumption about the production function (CES) and directly estimate the elasticity of substitution. They find evidence for a relatively small degree of positive sorting, but the magnitude of the estimated substitution elasticity is not readily comparable to a rank correlation coefficient.

Table 3: Actual and simulated standard deviations of labor market variables

	Standard deviations	U	V	θ	$q_u(\theta)$	z	$F(x,y,z)$
1.	U.S. data	0.190	0.202	0.382	0.118	0.02	-
2.	Results of Shimer (2005)	0.009	0.027	0.035	0.010	0.02	-
3.	No sorting, no heterogeneity	0.009	0.026	0.035	0.010	0.02	-
4.	Sorting, hierarchical model	0.102	0.277	0.380	0.168	0.02	0.06

Note: Rows 1 & 2: Based on Tables 1 and 3 in Shimer (2005), pp. 28, 39. Calculated based on quarterly U.S. data, 1951–2003. Rows 3 & 4: Standard deviations of simulated data from my model with and without sorting. All moments come from HP-filtered data with $\lambda = 10^5$.

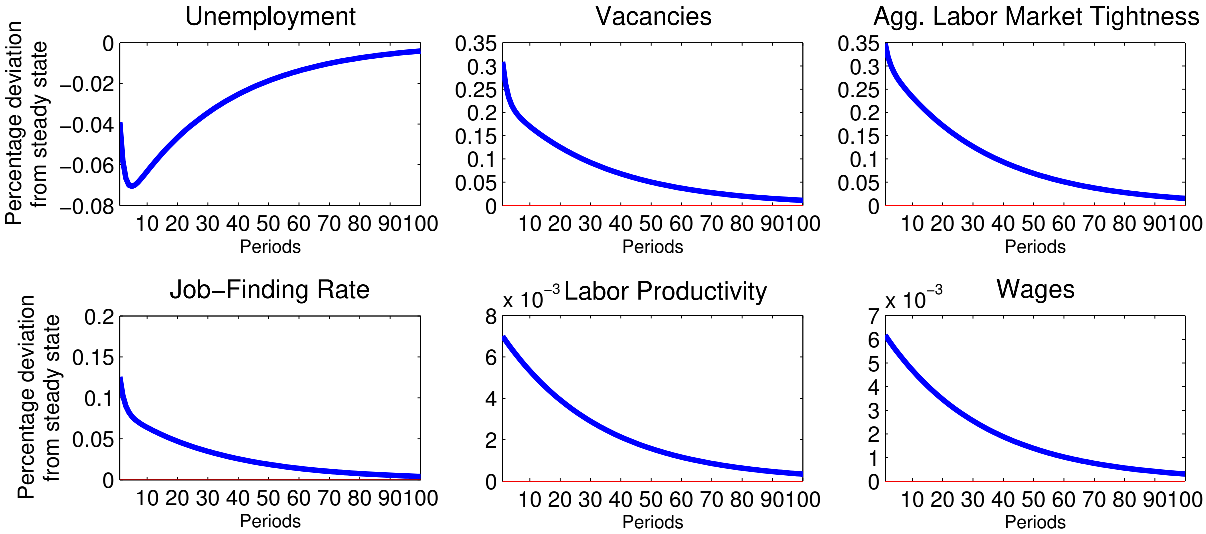
third row of Table 3. They are nearly exactly the same as Shimer's when sorting and heterogeneity are switched off.

The main results are reported in the fourth row of Table 3. Note that the volatility of z, the calibrated underlying labor productivity process, is the same in all models. In the hierarchical sorting model, however, overall match-specific output ($F(x, y, z)$) fluctuates somewhat more than z. This is because, as explained in the comparative statics exercise in Section 3, the chosen production function implies that sorting itself is procyclical. It becomes relatively more valuable to be optimally matched as z increases, so $F(x, y, z)$ is more volatile in response to shocks than z alone.

The second moments of simulated time series data from the hierarchical sorting model are much closer to the data than are those from the standard model without sorting. The standard deviation of the HP-filtered time series of labor market tightness (0.380) is very close to the data (0.382). The sorting model generates realistic dynamics of labor market tightness via two transmission channels: the additional margins of adjustment in the firms' job-creation problem (Section 4.1) and the endogenous wage rigidity (Section 4.2). The simulated standard deviations of vacancies (V) and the job-finding rate (q^u_t)) are also much higher than in the baseline model; they even overshoot their empirical counterparts to some extent. The standard deviation of unemployment (U) is amplified as well but remains somewhat lower than the empirical value.

To illustrate the dynamics of the hierarchical sorting model, Figure 4 shows impulse response functions of six key variables: unemployment, vacancies, aggregate labor market tightness, and the job-finding rate, as well as the autoregressive labor productivity process and wages. In response to a positive shock to labor productivity, unemployment falls by about 6 percentage points initially and shows a hump-shaped return to steady state. Vacancy posting, job finding, and aggregate tightness of the labor market show strong positive reactions directly after the shock. All impulse responses show a high degree of persistence as well as realistic correlations and cyclical properties. For instance, unemployment and vacancies move in opposite directions in response to the shock and are

Figure 4: Impulse Response Functions of Key Variables in the Search and Matching Model with Sorting



thus highly negatively correlated (Beveridge Curve). Note also the under-proportional adjustment of wages to the shock in labor productivity: the initial adjustment of wages is roughly 85% of the jump in labor productivity in the depicted example, so the endogenously generated wage rigidity becomes apparent.

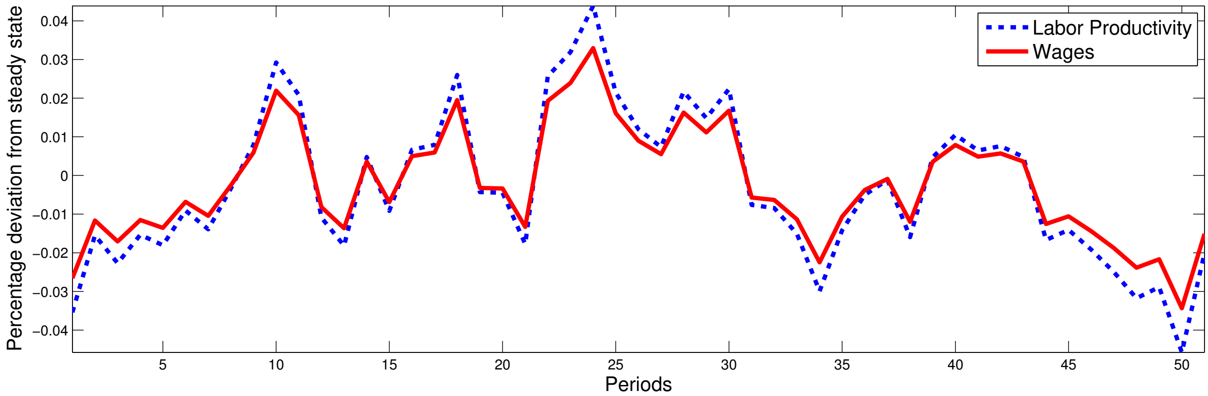
I now take a closer look at the simulation results. The augmented search and matching model with sorting entails two additional channels that lead to amplification of the dynamic model: the additional margins in job creation and the endogenous wage rigidity. How does each channel contribute quantitatively to the improved empirical performance and how large is the implied degree of wage rigidity?

5.4 The Degree of Wage Rigidity

That rigid wages can amplify search and matching models' response to shocks is well known in the literature and typically based on altering the assumptions underlying wage determination. (Hall, 2005; Hagedorn and Manovskii, 2008; Hall and Milgrom, 2008). The argument in favor of rigid wages in the context of the sorting model is novel and different. It was shown in Section 4.2 that rigid wages originate from an equilibrium property of the model: with sorting, the values of workers' and firms' outside options are asymmetric, leading to a skewed surplus function. Workers optimally choose narrower matching sets because they have greater bargaining power. This is based on the calibration of the bargaining parameter and the fact that workers receive a positive value of home production in the event of unemployment, whereas a vacant firm earns nothing. This generates a wage rigidity of an empirically reasonable magnitude without disregarding Nash bargaining or abandoning Shimer's calibration strategy.

Perfectly flexible wages, as in the baseline model, always fully respond to changes of

Figure 5: Simulated Time Series of Wages and Labor Productivity



Note: Fluctuations around trend of simulated data for labor productivity and wages, HP-filtered with $\lambda = 10^5$. Hierarchical model. The 50-period time frame was randomly chosen.

labor productivity, leading to a one-to-one co-movement. The elasticity of wages with respect to labor productivity is thus simply 1. Totally rigid wages, on the other hand, would not react at all to changes in labor productivity, implying an elasticity of 0 (as in Hall (2005)). Figure 5 shows how simulated wages (red, solid) and labor productivity (blue, dashed) deviate from trend in the model with sorting. The data stem from a simulation of the hierarchical model and the 50-period timeframe is randomly chosen. In the standard search and matching model, these two time series are congruent. In the model with sorting, however, wages turn out to be less volatile and do not fully adjust, as can be seen in Figure 5. Wages follow labor productivity closely but adjust underproportionally. The amplitude of labor productivity deviations is larger. The elasticity of wages with respect to labor productivity, a measure for the rigidity of wages, hence must be somewhat smaller than 1. This is exactly the result one would expect from comparing the sorting model to the baseline model. In the baseline model, wages are too responsive. After a favorable shock, workers soak up most of the extra productivity via immediately adjusting wages. This leads to an insufficient responsiveness of other model variables, particularly vacancies, due to a lack of incentives for the firm. In the model with sorting, however, the one-to-one link between wages and labor productivity is weakened. The endogenous wage rigidity, which is visible in Figure 5, limits the extent to which wages adjust in response to shocks. This increases firms' incentives to create new jobs and leads to amplification.

I refer to the empirical literature to investigate whether the model-generated rigidity is of a reasonable magnitude. I rely on Haefke et al. (2013), who focus primarily on the different degrees of wage rigidity for newly hired workers as compared to established employment relationships. I can compare the reported wage elasticity with respect to labor productivity for new hires to my results. Haefke et al. (2013) note that this value "is an appropriate and informative calibration target for search and matching models."[40] Haefke et al. (2013) find an elasticity of wages with respect to labor productivity of 0.8 with a relatively large standard error of 0.4. Using simulated data from the model with sorting, this elasticity can be simply computed as the coefficient η_1 of a simple linear regression of wages on labor productivity in logs and first differences:

$$\Delta \log W_t(x, y, z) = \eta_0 + \eta_1 \Delta \log z_t + \varepsilon_t \tag{36}$$

Running this regression yields a wage elasticity of $\eta_1 = 0.751$ for the hierarchical model, which is well within the empirically supported range. Hagedorn and Manovskii (2008) also compute an elasticity from U.S. wage and productivity data and report a coefficient of 0.449, which also falls within the supported range, albeit at the lower end. The elasticity/derivative implied by the alternating-offer bargaining game proposed by Hall and Milgrom (2008) is about 0.7, which is close to what I find from the model with sorting. It is reassuring that the rigidity lies close to these benchmarks.

Recall that the endogenous wage rigidity is, as described above, not the only modification of the model with sorting. Given the strong amplification effect, the degree of wage rigidity I find appears to be small at first sight. However, the value is well in line with empirical evidence on wage rigidities; a more extreme rigidity would not be justifiable empirically. It is thus instructive to decompose the effect of sorting in the dynamic model based on the two sources of amplification: the endogenous wage rigidity and the additional margins of adjustment in job creation.

5.5 Rigid Wages vs. Job Creation

Sorting influences firms' forward-looking vacancy posting decisions. Recall the firms' job creation condition:

$$c(g_f(y)) = \beta(1-\alpha)\mathbb{E}\left[q_v(\theta(\Omega'))\int_0^1 \frac{g_u(x,\Omega')}{U(\Omega')}\max\{\mathcal{S}(x,y,\Omega'),0\}\ \mathrm{d}x\right]. \tag{37}$$

There are a number of additional margins of adjustment in the entry problem of the hierarchical sorting model: after a positive productivity shock, the inversely U-shaped surplus function shifts upward as shown in the comparative statics exercise. In my parametrization, firms choose wider matching sets in response to positive shocks because the surplus from an increased number of potential matches is larger than zero. This leads to a higher

[40] Haefke et al. (2013), p. 898. Since the baseline search and matching model with Nash bargaining is essentially a model of new hires, the elasticity of wages with respect to labor productivity for this group is a reasonable target to match. Note that wages do not play any allocational role in a random search model. The Nash bargaining solution simply determines how the surplus is shared in every time period given the state of the model in the same period. Thus, the length of an employment spell does not influence wages and there is no meaningful distinction between new hires and existing employment relationships.

Table 4: Amplification effect of an imposed wage rigidity

	Standard deviations	U_t	V_t	θ_t	q_t^u
1.	U.S. data	0.190	0.202	0.382	0.118
2.	Results of Shimer (2005)	0.009	0.027	0.035	0.010
3.	Sorting, hierarchical model	0.102	0.277	0.380	0.168
4.	Rigidity imposed, $\eta_1 = 0.751$	0.031	0.084	0.114	0.051

Note: Rows 1 & 2: Based on Tables 1 and 3 in Shimer (2005), pp. 28/39. Calculated based on quarterly U.S. data, 1951–2003. Rows 3 & 4: Standard deviations of simulated data from my model. All moments come from HP-filtered data with $\lambda = 10^5$.

expectation of the value of matches with all workers in the firms' matching sets. An opposing force is exerted by the changing distribution of unemployed worker types. Since unemployment falls in response to a positive shock, it will be harder for firms to meet workers within their matching sets.[41]

The combined effect of the additional channels in the sorting model is that firms post more vacancies in response to shocks by increasing $g_v(y)$ subject to the convex cost function. Relative to the standard model, this creates an amplification of shocks because the value of being optimally sorted increases and the matching set and surplus function adjust accordingly. For a productivity shock of equal magnitude, the right-hand side of Equation (37) increases more than the expected future value of a job in the standard model without sorting. Additionally, this implies that wages do not need to be extremely rigid in the sorting model to generate sufficient volatility.

To quantify the contributions of the endogenous wage rigidity and the larger number of vacancy postings to overall amplification, I take the elasticity of wages with respect to labor productivity calculated above and impose it on a model without sorting and heterogeneity. The gap in volatility, which cannot be accounted for by the effect of rigid wages alone, must then be the effect of sorting on job creation. The last row of Table 4 shows the results of this exercise.

The volatility of labor market variables with an imposed wage rigidity of $\eta_1 = 0.751$ is too small compared to the data. As expected, the rigidity amplifies the model's response, but only by a factor of about 2–3 (depending on the variable). Thus, the relatively moderate model-generated wage rigidity does not suffice to generate sufficient amplification. This is not surprising. Additionally, the effect of endogenous separations, which is present in the dynamic sorting model, is quantitatively small. This is in line with the comparative statics result that the matching sets do not change much in response to changes in labor productivity. I thus conclude that the large amplification effect of sorting must be

[41] The described adjustments are not unambiguous and depend on the model's parameterization. For instance, as the value of being optimally matched increases after a positive shock, the matching sets could also become smaller.

primarily driven by additional job creation, which is, in turn, driven by the more than proportional adjustment of the surplus function.

5.6 The Dynamics of the Circular Model

As a final computational test, I run simulations of the simple circular sorting model derived in Section 2.3. The model has a closed-form solution for the matching cutoff $d^* \in (0, \frac{1}{2})$, the maximum distance along the circle workers and firms are willing to accept when forming a match. Due to this closed-form solution, solving and simulating the circular sorting model is much simpler than the hierarchical version. The matching cutoff is procyclical in the circular model, so the acceptable distance from the optimal match increases in response to a positive shock. This channel amplifies job creation in a manner similar to that found with the hierarchical model. The question is whether the simpler circular model also produces sufficient amplification or if the additional complexity of the hierarchical sorting model is necessary to match empirical labor market dynamics

I simulate the model using the Shimer (2005) calibration, the value of home production b and vacancy posting costs c are constants. I calibrate the parameters of the circular production function (Equation (1)) in order to match the output dispersion of the hierarchical production function. I find that the circular model produces a small amount of amplification. The standard deviation of simulated labor market tightness is about 0.052. This is a small increase in comparison to Shimer (2005) (0.035) but still far from the empirical moment to be matched (0.382). I conclude from this exercise that the additional features of the more complex hierarchical sorting model, particularly the endogenously adjusting distributions in the state space, make an important contribution to explaining the observed cyclical dynamics of the labor market.

6 Discussion and Conclusions

In this paper I construct a search and matching model with two-sided heterogeneity, sorting, and aggregate shocks. The model's relationship to previous literature is best understood from two reference points. The first reference point is the optimal assignment model of heterogeneous agents in a frictionless market following Becker (1973), the classical sorting model. In this frictionless environment, a production complementarity makes it optimal for all types to match with only their unique optimal counterpart. Shimer and Smith (2000) take this model out of its Walrasian equilibrium by adding frictions and making search costly, thus introducing the concept of matching sets that do not cover the whole type space.

The second reference point is the baseline DMP search and matching model of the labor market and its dynamic version.[42] The DMP model has been very successful in explaining equilibrium unemployment and a number of important stylized facts of labor market data (e.g., the Beveridge Curve). In the DMP model, every agent is willing to match with every other agent, but frictions limit the number of encounters in the labor market and, therefore, equilibrium unemployment exists. Wages are totally flexible in this model. In the dynamic version with aggregate shocks, wages adjust instantaneously and follow labor productivity one-to-one. This feature is central to the volatility puzzle discussion started by Shimer (2005) and Hall (2005).[43]

I construct a model in between these two reference points: in a labor market with two-sided heterogeneity, search frictions, sorting, and aggregate shocks, suboptimal matches between heterogeneous jobs and workers arise in equilibrium and persist through time. I show that this setup has the potential to generate sufficient volatility in response to shocks. The model endogenously generates a wage rigidity and amplification via the firms' dynamic entry problem. These channels amplify the model to a degree that brings it on par with empirical moments of labor market data. I do not rely on any additional assumptions besides heterogeneity and sorting. Recent advances in empirically identifying the extent of sorting in labor markets (Andrews et al., 2008, 2012; Card et al., 2013; Hagedorn et al., 2017; Lopes de Melo, 2018; Bonhomme et al., 2016; Bartolucci et al., 2018; Lise et al., 2016; Bagger and Lentz, 2018; Lochner and Schulz, 2016) make adding sorting to a search and matching model a both micro-founded and empirically supported complement to existing approaches to better align search and matching models with the data.

Rigidities of prices and wages play a key role in modern macroeconomics. Blanchard and Galí (2010) survey a number of papers that explore the implications of real and nominal rigidities in different variants of real business cycle (RBC) or New-Keynesian macro (NKM) models and also link them to the Shimer puzzle literature. The models developed by Merz (1995), Andolfatto (1996), Christoffel and Linzert (2005), Krause and Lubik (2007), Faia (2008), and Gertler and Trigari (2009) are all examples of DSGE models that integrate variants of price and wage rigidities to create persistence and sufficient amplification for the analysis of different kinds of shocks. Gertler and Trigari (2009), for example, adapt the well-known Calvo (1983) pricing structure for wage formation in the labor market. Rigid wages are generated by allowing only a fraction of matches to

[42] The main references for this class of models are Diamond (1982), Mortensen (1982), Pissarides (1985), and Mortensen and Pissarides (1994). Pissarides (2000) provides an excellent textbook treatment.

[43] Hall (2005) shows that the volatility puzzle vanishes once wages are made completely inflexible. However, this implies a counter-factual wage volatility of zero. Hagedorn and Manovskii (2008) show that the dynamics can be amplified by increasing the value of the workers' outside option of nonmarket activity. A higher calibrated value of the respective bargaining parameter leads to lower wages in the Nash bargaining game. Hornstein et al. (2005) and Costain and Reiter (2008) show that the calibration strategy proposed by Hagedorn and Manovskii (2008) is implausible because a 15% increase in the value of non-market activity implies that the equilibrium unemployment rate doubles.

renegotiate wages in every period. Another recent example is Christiano et al. (2016), who use the alternating-offer bargaining game of Hall and Milgrom (2008) to induce wage inertia in a New Keynesian model.

In the light of this literature, this paper's key contribution is to show that a search and matching model with two-sided heterogeneity and sorting is able to jointly generate rigid wages and large unemployment fluctuations as observed in the data. This result demonstrates that sufficient amplification of shocks and the simple Nash bargaining solution are not mutually exclusive. Incorporating a frictional labor market with sorting into a large macroeconomic model with heterogeneous agents will be a fascinating topic for future research.

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A Computational Appendix

A.1 Computational Errors

To check the accuracy of the computational method described in Section 5.1, I plug simulated data from the dynamic sorting model back into the Bellman equations of the model. Ideally, the simulated data would solve these equations exactly. However, I make heavy use of discretization and approximation techniques, so it is reasonable to expect some imprecision. For convenience, I use the wage equation for this test because it contains both the firms' and the workers' integral terms:

$$W(x,y,\Omega) - \alpha\left(F(x,y,z) + c(g_v(y,\Omega))\mathbb{E}\left[\frac{\int_0^1 g_v(y,\Omega')\max\{\mathcal{S}(x,y,\Omega'),0\}\,\mathrm{d}y}{\int_0^1 g_u(x,\Omega')\max\{\mathcal{S}(x,y,\Omega'),0\}\,\mathrm{d}x}\right]\right) \\ - (1-\alpha)b(x) \stackrel{?}{=} 0.$$

Solving the dynamic sorting model by log-linearization and perturbation results in a mean computational error of 3.84%. The 2.5th, 50th, and 97.5th percentiles of the distribution are −15.0%, 4.67%, and 17.6%, respectively. This distribution is slightly left skewed due to the fact that the model's response to shocks is not symmetric around the steady state, for example, because of endogenous separations, which only happen after negative shock. Figures A.1 and A.2 show a histogram of the computational errors and a scatter plot that shows the positive correlation of the errors with z.

A recent paper by Petrosky-Nadeau and Zhang (2016) can serve as a benchmark for the size and distributions of the errors. The authors show that solving a representative agent search and matching model via log-linearization and perturbation—they use Hagedorn and Manovskii (2008) as an example—creates a mean computational error of 3.75% with the 2.5th, 50th, and 97.5th percentiles of the distribution being −11.1%, −3.66%, and 8.76%, respectively. I conclude from this that the errors resulting from the computational approach in this paper lead to errors of an expectable magnitude, even though the errors I find are slightly more dispersed than what Petrosky-Nadeau and Zhang (2016) find for a representative agent search and matching model.

Figure A.1: Histogram of Computational Errors

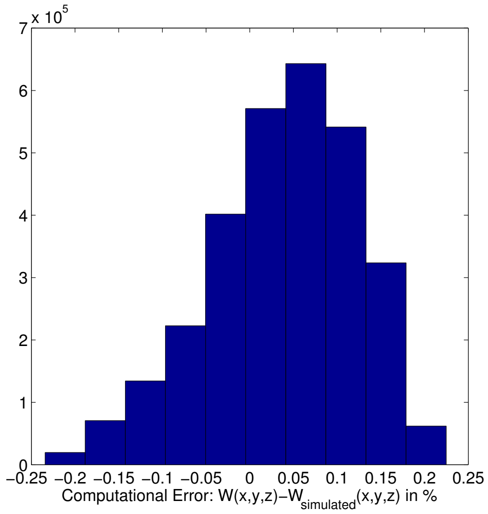

Figure A.2: Scatter Plot of Computational Errors and Stochastic Labor Productivity z