CE 604 Final Project

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1 Introduction

Obtaining an understanding of the bridge between theory and application of the Finite Element Method (FEM) can be difficult. Even after taking classes in both application and theory, understaning may still be limited. Currently, the gold standard for understanding FEM is [1]. Although it is known to be a well written book, the threshold for understanding is high. Dr. Kendrick Shepherd has taken steps to make this work more navigable for those early on in their understanding. He has produced a number of documents elaborating on the concepts of [1]. Yet, for many the theory behind FEM is vague.

This work has been compiled as a continuing effort to help make the theory behind FEM more accessible and will be applied to a beam. Much of what is presented is an elaboration on what is found in chapter 5.4 of [1]. This work also relies heavily on the derivation of the Timoshenko frame analysis [2].

A note regarding notation used in this work. Subscripts and indexing are important in the formulation of finite element analysis. Following the example of [1], the subscripts α and β will represent the integers 1 and 2. Other latin letters (e.g. i, j, k, l) will typically represent the integers 1,2,3, with the expection of e and n being any arbitarity integer.

The work will proceed in the following manner. First, the assumptions made in Timoshenko beam theory are explained. Particular attention will be placed on justifying these assumptions. Next, the theory of linear elasticity as provided in continuum mechanics is put forth. Finally, the variational equation (virtual work equation) will be stated, about which the previous sections provide background. A breif explanation of how to proceed toward the FEM will conclude.

2 Assumptions

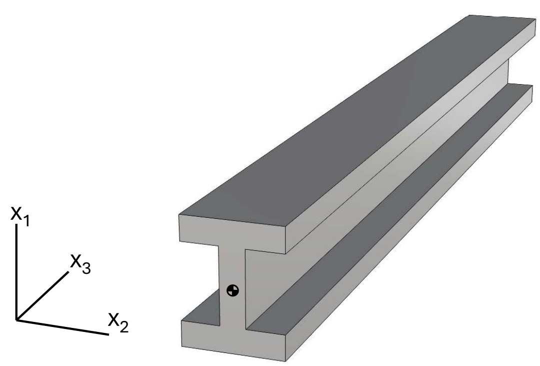


Figure 1: The definition of the beam, with the centroid marked.

2.1 Approximating the Beam First a beam is defined as the domain of interest. The beam is displayed in fig. 1. The domain of the beam is to first be divided into multiple sections, as shown in eq. (2.1) and fig. 2, where Ω is the beam, Ω^e is the eth element of the beam, and $\bigcup\limits_{e=1}^{n}$ is the union from element 1 to n.

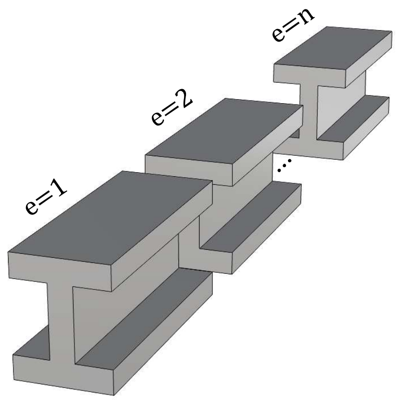


Figure 2: The beam subdivided into n elements

$$\Omega = \bigcup_{e=1}^{n} \Omega^e \tag{2.1}$$

Thus eq. (2.1) states that the domain of interest is to be composed of beam elements numbered from 1 to n. Furthermore, each element has local axes defined with respect to the principal axes as shown in eq. (2.2) and fig. 1, where h^e is the length of each element and A^e is the cross sectional area of each element.

$$\Omega^e = \{(x_1^e, x_2^e, x_3^e) | x_3^e \in [0, h^e], (x_1^e, x_2^2) \in A^e \subset \mathbb{R}^2\} \tag{2.2}$$

Furthermore, for the analysis the beam will be considered a one dimensional element. Thus, the beam will be approximated as a line with length h that will act at the centroid of the beam as shown by the center of gravity marker in fig. 1. In other texts, this is represented by eq. (2.3).

$$0 = \int_{A^e} x_1^e \, dA = \int_{A^e} x_2^e \, dA = \int_{A^e} x_1^e x_2^e \, dA \tag{2.3}$$

2.2 Stress Tensor The second assummption is that $\sigma_{\alpha\beta} = 0$. The stress tensor σ is shown in eq. (2.4) and the stress element in fig. 3.

$$\sigma = \begin{bmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \tag{2.4}$$

If we transform σ into its principal stress form we get eq. (2.5). This is significant becasue it shows that the beam is in a plane stress condition. A general Mohr's circle for this state of stress is shown in section 2.2. This seems to justify the fact that in previous classes almost all the stress tensors we worked with were simplified to the plane stress condition or similar.

$$\sigma* = \begin{bmatrix} \frac{\sigma_{33} + \sqrt{\sigma_{33}^2 + 4(\sigma_{13}^2 + \sigma_{23}^2)}}{2} & 0 & 0 \\ 0 & \frac{\sigma_{33} - \sqrt{\sigma_{33}^2 + 4(\sigma_{13}^2 + \sigma_{23}^2)}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.5}$$

This assumption is less intuitive and familiar than the other assumptions made in this section. For this reason, two approaches will be put forth to verify it's validity. The first approach relies heavily on the one-dimensionality of the approximated beam coupled with the material properties. The second apporach considers the details of how forces are supported using of Timoshenko Beam theory.

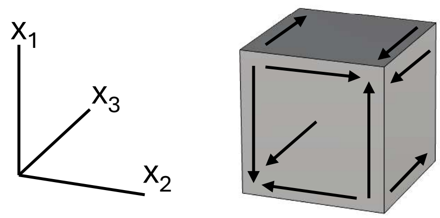


Figure 3: The stress element of the beam in its global coordinates.

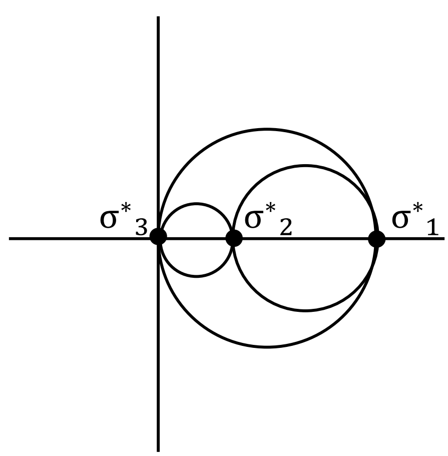


Figure 4: The generalized Mohr's circle for the plane stress condition.

Approach 1 To begin, let $\epsilon_{\alpha\beta} = 0$, which is verified by eq. (3.21). Now, eq. (2.6) can be derived from eq. (2.12). Note that $\sigma_{\alpha\alpha}$ are not neccesarily equal to zero as previously assumed.

$$\begin{aligned}
\sigma_{11} &= \lambda\epsilon_{33} \\
\sigma_{22} &= \lambda\epsilon_{33} \\
\sigma_{33} &= (\lambda + 2\mu)\epsilon_{33} \\
\sigma_{12} &= 0 \\
\sigma_{13} &= 2\mu\epsilon_{13} \\
\sigma_{23} &= 2\mu\epsilon_{23}
\end{aligned} \tag{2.6}$$

Next, consider the equations for λ and μ with respect to modulus of elasticity (E) and Poisson's ratio (ν), see eqs. (2.7) and (2.8).

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{2.7}$$

$$\mu = \frac{2E}{2(1+\nu)} \tag{2.8}$$

The physical representation of ν deals with the compresibility of a material. For a material in axial compression, ν represents the materials tendency to expand in the transverse direction. The values of ν may range from 0 to 0.5. If $\nu = 0$ the material will experience no exapnsion when compressed. On the other hand, when $\nu = 0.5$ it will expand to maintain a constant volume.

Although ν is a material property, it's siginificance may be influenced by geometry. For example, an axial rod of length l and diameter d is composed of an isotropic material with a maximal Poisson's ration $\nu = 0.5$. When this rod is tensioned to stretch some Δl, the diameter will also change by some Δd so that the volume V does not change. If the rod is slender, then $\Delta d << \Delta l$. Therefore, Δd may be negligible for a sufficiently slender rod.

The approximated beam is the most extreme case of a slender element because it is one-dimensional. The cross-section will not expand or contract under compression or tension. Therefore, ν must be equal to zero. Consequently, λ will also be zero. If we substitute zero for λ in eq. (2.6), we obtain that $\sigma_{\alpha\alpha} = 0$.

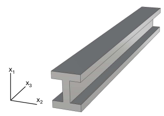
Approach 2 The controlling internal forces of a beam are moment and shear. By the above assumptions, the beam is capable of supporting shear and axial force. By Timoshenko beam theory the moment is transfered into an axial force in the beam ($\sigma_{bending} = \frac{My}{I}$). In this manner, the controlling forces are accounted for.

Locally, the beam may experience nonzero $\sigma_{\alpha\beta}$ stresses. However, these stresses will be much smaller than the controling shear and moment-induced axial forces. This will be to the degree that the transverse normal forces will be negligible. Thus, it is reasonable to make the global approximation that $\sigma_{\alpha\beta} = 0$

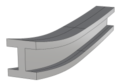
2.3 Kinematics The kinematics of the beam will be accounted for using eqs. (2.9) to (2.11). The translation of the beam is represented by w_i, where i is the direction of the translation. The rotation of the beam is represented by θ_i. As can be observed from the equations, w_i and θ_i are functions of only x_3, the location along the length of the beam. That is true because the beam is collapsed to a one-dimensional element. The overall displacement of a point on the beam is u_i, which acts as a function of x_1, x_2, and x_3. section 2.3 displays a positive displacement in each coordinate direction for both translations and rotations.

$$u_1(x_1, x_2, x_3) = w_1(x_3) - x_2\theta_3(x_3) \tag{2.9}$$

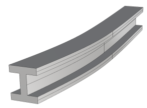
$$u_2(x_1, x_2, x_3) = w_2(x_3) + x_1\theta_3(x_3) \tag{2.10}$$



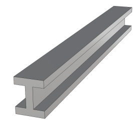
(a) No displacement



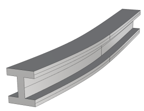
(b) w_1



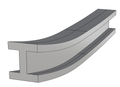
(c) w_2



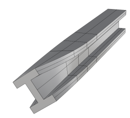
(d) w_3



(e) θ_1



(f) θ_2



(g) θ_3

Figure 5: Displacements in each coordinate direction for both translation and rotation are shown.

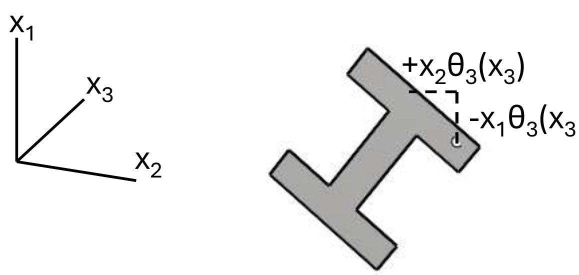


Figure 6: A pictoral representation of sign convetion for θ_3 rotations affecting u_α displacements.

$$u_3(x_1, x_2, x_3) = w_3(x_3) - x_1\theta_2(x_3) + x_2\theta_1(x_3) \tag{2.11}$$

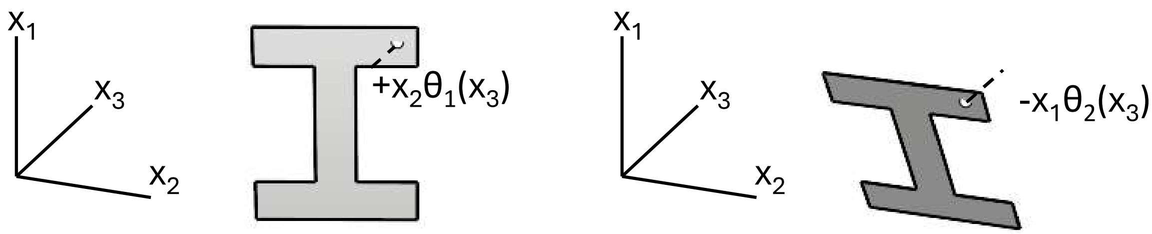
Note that in eq. (2.9) the $x_2\theta_3(x_3)$ term is subtracted $w_1(x_3)$, while in eq. (2.10) $x_1\theta_3(x_3)$ is added to $w_2(x_3)$. This is an artifact of the positive/negative convention for rotation. For a point P at some positive valued (x_1, x_2), the u_1 displacement is decreased by a x_3 rotation. Similarly for the same point P, the u_2 displacement is increased by the θ_3 rotation, see fig. 6. The same sort of scenario occurs in eq. (2.11), see fig. 7

It is important to note that the anlysis will be limited or enhanced by the accuracy of these equations. In this case, warping in the beam is not accounted for.

2.4 Elastic Deformation All deformation in the beam is assumed to be elastic, as opposed to plastic. Furthermore, we assume that the beam is homogeneous and isotropic. This leads us to the special case for defining the stress tensor as eq. (2.12), where λ and μ are Lameé constants describing material properties.

$$\sigma_{ij} = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij} \tag{2.12}$$

2.5 Small Displacements As is customary in elastic methods, it is assumed that only small displacements occur.



(a) θ_1 (b) θ_2

Figure 7: A pictoral representation of sign convetion for θ_α rotations affecting u_3 displacements.

3 Linear Elasticity

Now that the beam and it's behavior are defined by the above assumptions, it is time to introduce linear elasticity.

Recall that eq. (3.13) represents the balance of linear momentum.

$$\sigma_{ij,j} + f_i = \rho u_{,tt} \tag{3.13}$$

For static cases, it is accurate to assume that $u_{,tt} = 0$. This assumption allows for the simplification to eq. (3.14)

$$\sigma_{ij,j} + f_i = 0 \tag{3.14}$$

Equation (3.15) is commonly known as hooke's law. It relates stress and strain using the fourth order tensor C_{ijkl}.

$$\sigma_{ij} = C_{ijkl}\epsilon_{kk} \tag{3.15}$$

As mentioned ealier, in this work we make the assumption that the beam consists of a homogenous and isotropic material. Therefore, Hooke's law takes the special form of eq. (2.12).

The infinitesimal strain tensor is described in eq. (3.16).

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{3.16}$$

This is often written as eq. (3.17) for simplifyed notation.

$$\epsilon_{ij} = u_{(i,j)} \tag{3.17}$$

When performing a linear elastic analysis, the quantities of interest are typically forces and displacemnts. To solve for the unkonw forces and displacements, we begin with those that are known. Equation (3.18) describe the known displacements, typically at the boundary conditions. Equation (3.19) are the known forces acting on the beam.

$$u_i = g_i \; on \; \Gamma_g \tag{3.18}$$

$$\sigma_{ij} n_j = h_i \ on \ \Gamma_h \tag{3.19}$$

With the theory of linear elasticity defined, it is now possible to address the calculation of $\epsilon_{\alpha\beta}$. By the assumptions of section 2.3, we find that $\epsilon_{\alpha\beta} = 0$. This is shown by recalling th displacement vector u_i and eq. (3.17). Solving for $u_{\alpha,\beta}$ yeilds eq. (3.20).

$$u_{\alpha,\beta} = \begin{bmatrix} 0 \ -\theta_3(x_3) \\ \theta_3(x_3) \ 0 \end{bmatrix} \tag{3.20}$$

Expanding $\epsilon_{\alpha\beta}$ yeilds eq. (3.21).

$$\begin{aligned} \epsilon_{\alpha\beta} &= \frac{1}{2}(u_{1,2} + u_{2,1}) \\ &= \frac{1}{2}(-\theta_3(x_3) + \theta_3(x_3)) \\ &= 0 \end{aligned} \tag{3.21}$$

Thus we verify that according to our assumptions of section 2.3 $\epsilon_{\alpha\beta} = 0$. However, we will use eq. (3.22) to calculate the strain for α and β directions.

$$\epsilon_{\alpha\beta} = -\frac{\lambda\epsilon_{33}}{2(\lambda + \mu)}\delta_{\alpha\beta} \tag{3.22}$$

This equation is derived from the assumptions in section 2.2, which can be used to redifine $\sigma_{\alpha\beta}$ as eq. (3.23). This in turn expands to eq. (3.24).

$$0 = \sigma_{\alpha\beta} = \lambda\delta_{\alpha\beta}\epsilon_{kk} + 2\mu\epsilon_{\alpha\beta} \tag{3.23}$$

$$\begin{aligned} 0 = \sigma_{\alpha\alpha} = {} & \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33})\delta_{11} + 2\mu\epsilon_{11} \\ & + \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33})\delta_{22} + 2\mu\epsilon_{22} \end{aligned} \tag{3.24}$$

Finally, simplifying we obtain eq. (3.25).

$$\begin{aligned} 0 &= \lambda(2\epsilon_{\alpha\alpha} + 2\epsilon_{33}) + 2\mu\epsilon_{\alpha\alpha} \\ \epsilon_{\alpha\alpha} &= -\frac{\lambda}{\lambda + \mu}\epsilon_{33} \end{aligned} \tag{3.25}$$

From here, eq. (3.22) can be found by substituting the previous equation into eq. (3.23).

It has been shown that the assumptions of sections 2.2 and 2.3 are inconsistent with repsect to strain, in particular $\epsilon_{\alpha\beta}$. Typically, eq. (3.22) is prefered for strain calculations. Also, this inconsistenty does not void the analysis method.

4 Variational Equation

The variational equation (a.k.a virtual work) is the next step towards FEM. It is shown in eq. (4.26).

$$\int_\Omega \sigma_{ij,j}\, \hat{u}_i \, d\Omega + \int_\Omega f_i \, \hat{u}_i \, d\Omega = 0 \tag{4.26}$$

By applying the divergence theorem, boundary conditions, and changes of variables, we arrive at eq. (4.27).

$$0 = \sum_{e=1}^{n_{el}} \int_0^{l^e} \left(\hat{u}_1 \bar{u}_1 \ddot{u}_1 + \hat{u}_2 \bar{u}_2 \ddot{u}_2 + \hat{u}_3 \bar{u}_3 \ddot{u}_3 + \hat{\theta}_x \bar{I}_x \ddot{\theta}_x + \hat{\theta}_y \bar{I}_y \ddot{\theta}_y + \hat{\theta}_z \bar{I}_z \ddot{\theta}_z \right) d\xi \tag{4.27}$$

This is the discretized weak form of the virtual work equation, which is used for FEM formulation. Essentially, this is the boundary between continuum mechanics and FEM. To continue into FEM basis functions are applied, and the equations are implemented in a matrix formulation. To continue to derive FEM for a Timoshenko beam refer to [1].

5 Time Spent

I spent 37 hours on my final project.

References

[1] T. J. R. Hughes, *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*, Dover Publications, Mineola, New York, reprint of the 1987 prentice-hall edition ed., 2000.

[2] K. M. Shepherd, *Timoshenko Frame Derivation*.