CE 604 Final Project

Brian Shawcroft

1 Introduction

The purpose of this project is to make a clear connection between the fields of continuum mechanics and the finite element method. The material presented herein is based on Hughes chpt 5.4 and Dr. Shepherd's Timoshenko Frame Derivation. The motivation for undertaking this project is first and foremost to help myself gain a better understanding of these topics. I also hope that my work here will also help to accelerate other students learning as they try to understand the connection between continuum mechanics and the finite element method.

2 Assumptions

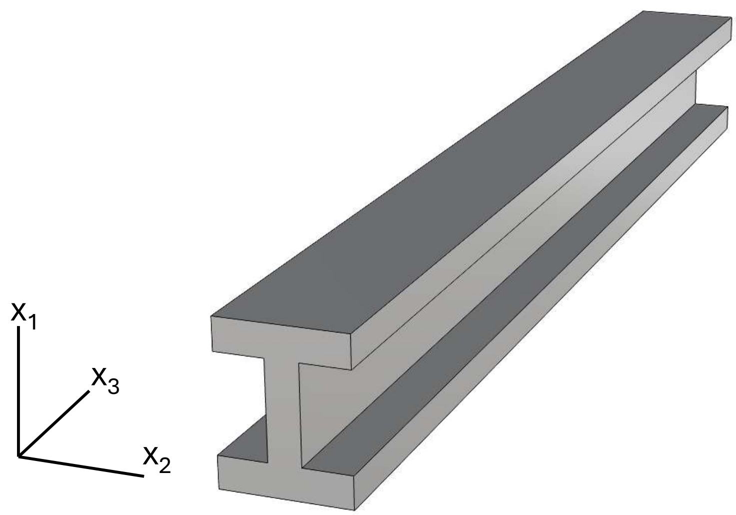


Figure 1: The definition of the beam

I will begin by outlining the assumptions made.

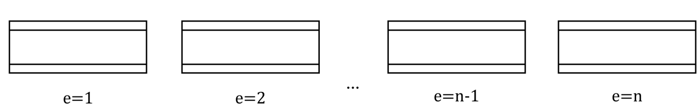
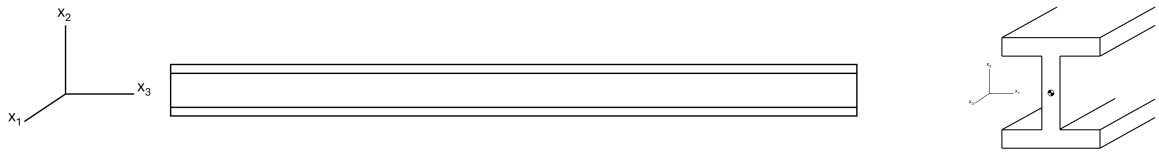


Figure 2: The beam subdivided into n elements



(a) side view

(b) isometric view

Figure 3: The definition of the beam

Approximating the Beam We will first define a beam as our domain of interest. The beam is displayed in fig. 1. The domain of our beam is to first be divided into multiple sections, as shown in eq. (2.1) and fig. 2, where Ω is the beam, Ω^e is the eth element of the beam, and $\overset{n}{\underset{e=1}{\cup}}$ is the union from element 1 to n.

$$\Omega = \overset{n}{\underset{e=1}{\cup}} \Omega^e \tag{2.1}$$

Thus eq. (2.1) states that the domain of interest is to be composed of beam elements numbered from 1 to n. Furthermore, each element has local axes defined with respect to the principal axes as shown in eq. (2.2) and fig. 3, where h^e is the length of each element and A^e is the cross sectional area of each element.

$$\Omega^e = \{(x_1^e, x_2^e, x_3^e) | x_3^e \in [0, h^e], (x_1^e, x_2^2) \in A^e \subset \mathbb{R}^2\} \tag{2.2}$$

Furthermore, for the analysis the beam will be considered a one dimensional element. Thus, the beam will be approximated as a line with length h that will act at the centroid of the beam as shown by the center of gravity marker in fig. 3b. In other texts, this is represented by eq. (2.3).

$$0 = \int_{A^e} x_1^e \, dA \ = \int_{A^e} x_2^e \, dA \ = \int_{A^e} x_1^e x_2^e \, dA \tag{2.3}$$

Stress Tensor The second assummption is that $\sigma_{\alpha\beta} = 0$ for $\alpha, \beta \in \{1, 2\}$. The stress tensor σ is shown in eq. (2.4) and the stress element in fig. 4. By this assumption, the beam will not have normal stresses in the x_1 or x_2 directions, similar to an axial rod. The difference between the beam and an axial rod is that it can experience shear in the σ_{13} and σ_{23} directions). We know the beam will have loads applied in the x_1 or x_2 directions, which typically would mean a normal stress in those directions. However, given our assumptions the beam will still be able to support these loads by the shear. This assumption is made for consistency to ensure $\epsilon_{\alpha\beta} = 0$, since we are also neglecting the cross-sectional area of the beam as was shown in eq. (2.3). In section 3 it will be explained how we can still calculate $\epsilon_{\alpha\beta}$.

$$\sigma = \begin{bmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \tag{2.4}$$

If we transform σ into its principal stress form we get eq. (2.5). This is significant becasue it shows that the beam is in a plane stress condition. A general Mohr's circle for this state of stress is shown in section 2. This seems to justify the fact that in mechanics of materials almost all the stress tensors we worked with were simplified to the plane stress condition or similar.

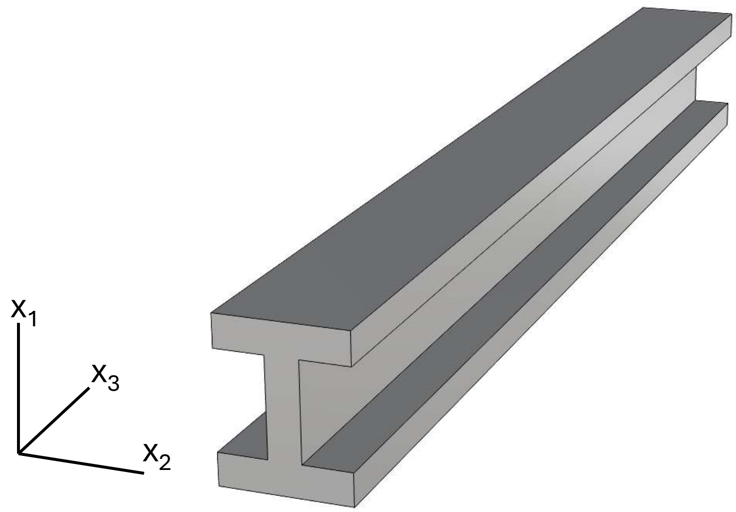


Figure 4: The stress element of the beam in its global coordinates. CHANGE PICTURE: make a stress element with coordinate axes shown.

(2.5)
$$\sigma * = \begin{bmatrix} \frac{\sigma_{33}+\sqrt{\sigma_{33}^2+4(\sigma_{13}^2+\sigma_{23}^2)}}{2} & 0 & 0 \\ 0 & \frac{\sigma_{33}-\sqrt{\sigma_{33}^2+4(\sigma_{13}^2+\sigma_{23}^2)}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Displacements The kinematics of the beam will be accounted for using eq. (2.6) and **????**. The translation of the beam is represented by w_i, where i is the direction of the translation. The rotation of the beam is represented by θ_i. As can be observed from the equations, w_i and θ_i are functions of only x_3, the location along the length of the beam. That is true because the beam is collapsed to a one-dimensional element. The overall displacement of a point on the beam is u_i, which acts as a function of x_1, x_2, and x_3. section 2 displays a positive displacement in each coordinate direction for both translations and rotations.

(2.6)
$$u_1(x_1, x_2, x_3) = w_1(x_3) - x_2\theta_3(x_3)$$

(2.7)
$$u_2(x_1, x_2, x_3) = w_2(x_3) + x_1\theta_3(x_3)$$

(2.8)
$$u_3(x_1, x_2, x_3) = w_3(x_3) - x_1\theta_2(x_3) + x_2\theta_1(x_3)$$

Note that in eq. (2.6) the $x_2\theta_3(x_3)$ term is subtracted $w_1(x_3)$, while in eq. (2.7) $x_1\theta_3(x_3)$ is added to $w_2(x_3)$. This is an artifact of the positive/negative convection for rotation. For a point P at some positive valued (x_1, x_2), the u_1 displacement is decreased by a x_3 rotation. Similarly for the same point P, the u_2 displacement is increased by the θ_3 rotation, see fig. 7. The same sort of scenario occurs in eq. (2.8), see fig. 8

It is important to note that the anlysis will be limited or enhanced by the accuracy of these equations. In this case, warping in the beam is not accounted for.

3 Clasical Linear Elastic Theory

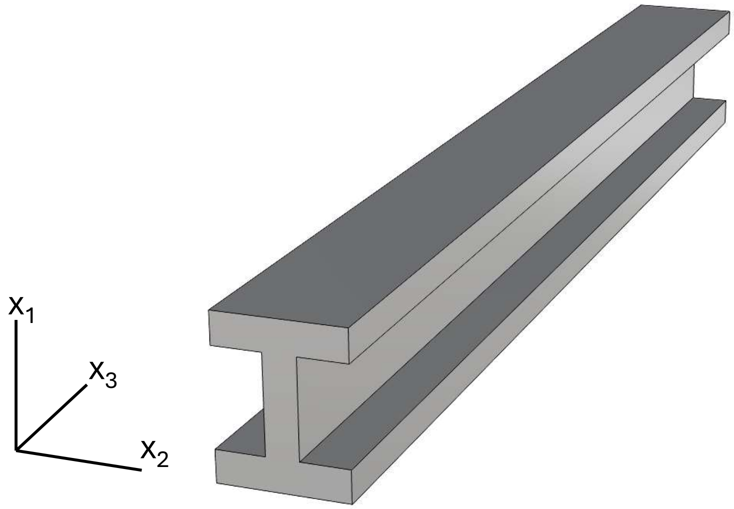
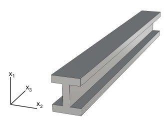
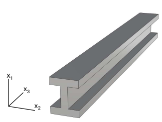


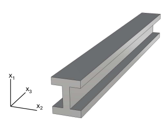
Figure 5: The generalized Mohr's circle for the plane stress condition. CHANGE PICTURE: make a Mohr's circle like the one on pg 85 (pg 25 of 61-80) in Continuum Mechanics Text



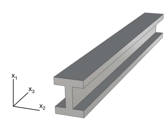
(a) No displacement



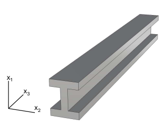
(b) w_1



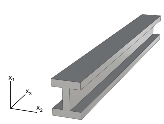
(c) w_2



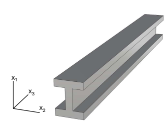
(d) w_3



(e) θ_1



(f) θ_2



(g) θ_3

Figure 6: Positive displacements in each coordinate direction for both translation and rotation are shown. CHANGE PICTURES

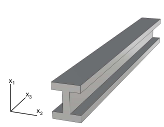


Figure 7: A pictoral representation of sign convetion for θ_3 rotations affecting u_α displacements. CHANGE PICTURE

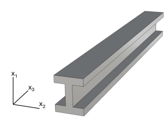


Figure 8: A pictoral representation of sign convetion for θ_α rotations affecting u_3 displacements. CHANGE PICTURE