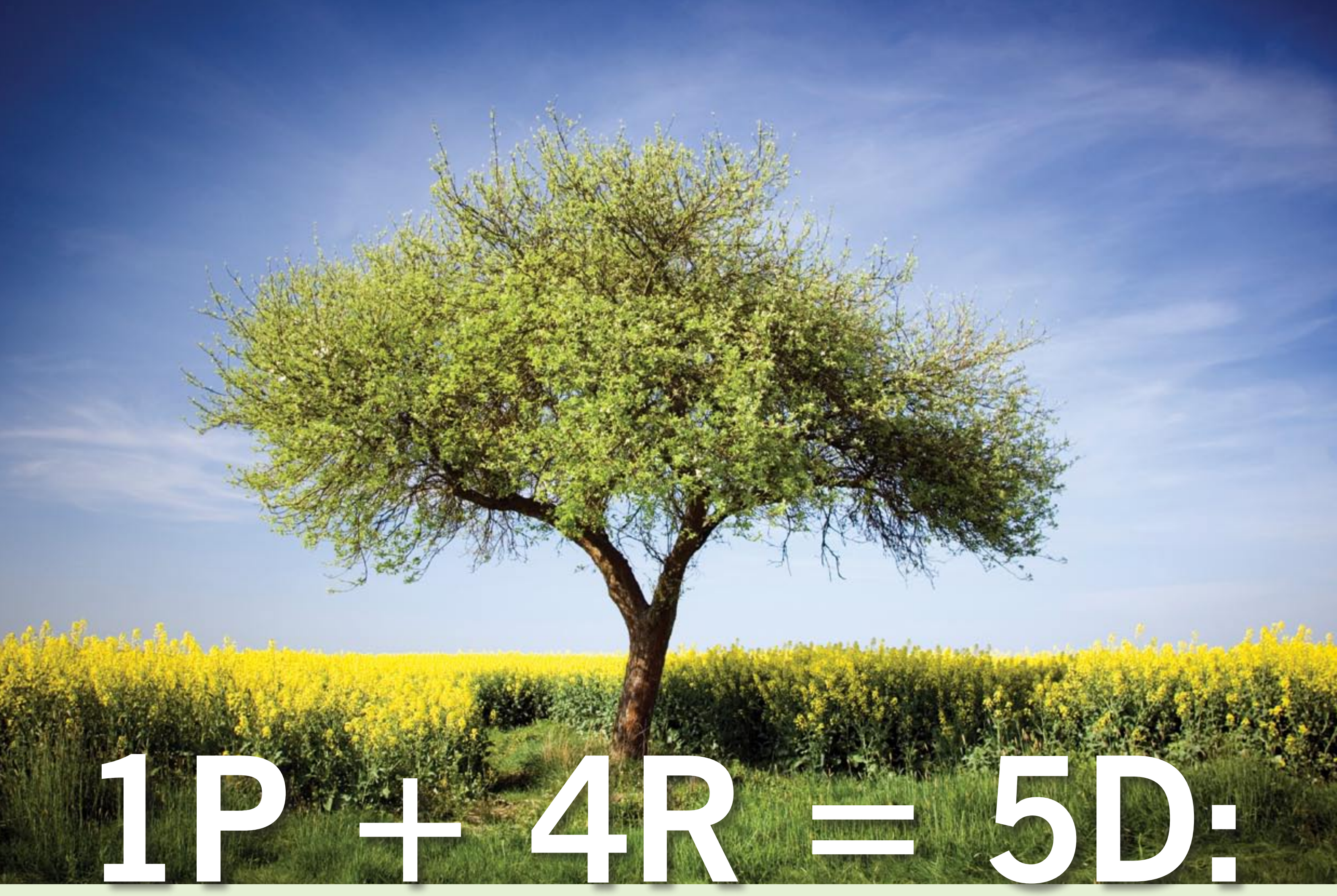

1P + 4R = 5D: An Equation for Deepening Mathematical Understanding

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No one will argue with the NCTM's assertion that students' implementation of and flexibility with varied representations are central to studying mathematics for deeper understanding (NCTM 2000). In our work with grades 6–8 mathematics teachers, however, we found reasons to be cautious when teaching activities that use multiple representations to develop a deeper understanding of procedures and concepts. In postactivity reflections, our teacher-learners consistently expressed a newfound appreciation for the variety of valid representations and strategies that can be used to solve a problem. This is a critical point at which to capture the learners' attention by asking this important question: "*Why* do the various representations and strategies lead to the same valid solution?"

If we ended our inquiry here, without asking "why" different representations are valid, we could at least say that we have awakened a sense of possibility in our teachers. They might have returned to their classrooms with a more flexible attitude toward how

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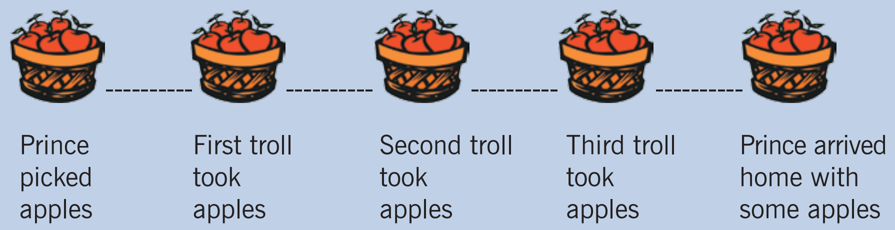
problems may be represented and solved and encourage flexibility in *their* students' representations and solutions. However, demonstrating that a problem can be represented and solved using manipulatives, tables, graphs, symbols, and so on does not mean that learners will understand that there are connections *between* and relationships *of* mathematical ideas embedded in the problem. On the contrary, students run the risk of developing understanding that is "a mile wide and an inch deep" (U.S. National Research Center 1996). Although students might be able to set up and solve a mathematical task in a variety of ways, they may have little or no idea *why* a particular representation works, *why* it may be preferred in a given situation, or *why* it is equivalent to a different approach. By asking, "*Why* do various representation and strategies lead to the same valid solution?" we open the door to exploring the connections and relationships within and between mathematical ideas that leads to a deeper understanding of mathematics. Using learners' representations to answer this question places them in a position to compare, question, or challenge their mathematical understanding and the understanding of others.

In this article, we will illustrate the equation 1P + 4R = 5D: From 1 problem (P), our teacher-learners generated 4 representations (R) that led to 5 discussions (D). These discussions focused on the relationships of events within the problem that led to each representation, the connections between the different ways that teachers chose to represent those relationships, and the equivalency of the representations. Although these discussions occurred within the context of a graduate course, we believe that they serve as realistic models for middle school teachers of the kinds of discussions they can and should facilitate in their own middle school mathematics classrooms.

Fig. 1 A general description of the Golden Apples problem



An important objective when discussing each solution strategy was to make explicit how a problem context was translated into a problem representation and how a representation led to a problem solution. A second important objective was to give those processes mathematical meaning by associating them with underlying mathematical ideas. We found it was helpful to discuss *procedures* as being the processes of creating a representation and solving a problem (e.g., converting words to mathematical symbols, symbolic manipulation) and *concepts* for talking about the underlying mathematical ideas (e.g., rate of change, equivalency, nesting).

ONE PROBLEM

Our teacher-learners solved the Golden Apples problem (Driscoll 1999, p. 22).

> A prince picked a basketful of golden apples in the enchanted orchard. On his way home, he was stopped by a troll who guarded the orchard. The troll demanded payment of one-half the apples, plus two more. The prince gave him the apples and set off again. A little farther on, he was stopped by a second troll. This troll demanded payment of one-half of the remaining apples, plus two more. The prince paid him and set off again. Just before the prince left the enchanted orchard, a third troll stopped him and demanded one-half of the remaining apples, plus two more. The prince paid him, and sadly went home. He had only two golden apples left. How many apples had the prince picked?

To identify the context of the problem, we asked, "What is going on in this situation?" The responses were (1) the prince picked some apples, (2) he was stopped by a troll who took some apples, (3) he was stopped by a second troll, (4) he was stopped by a third troll, and (5) he arrived home with two apples. **Figure 1** is a model that represents these five events; each box needs to be filled with a descriptive symbolic expression and related in some way. The teachers' knowledge of procedures helped them convert the written words to symbols for each event, and their knowledge of concepts helped them relate the events to one another and form a function rule. This was done in multiple ways, as illustrated below and in the ensuing discussions.

FOUR REPRESENTATIONS

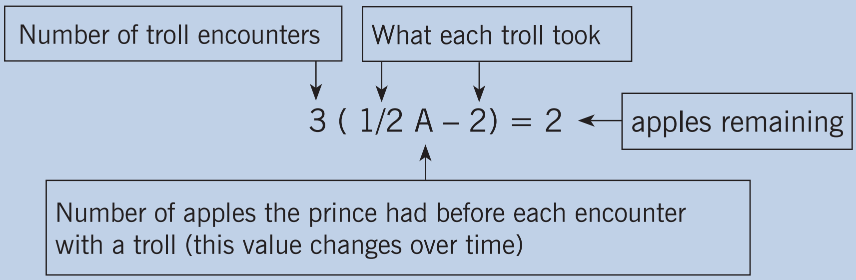
Figure 2 shows four representations of the problem that typified the work of the ten teachers in the class. One representation is incorrect, two include solutions, and one is extended to a generalized form. These features will be highlighted in the discussions presented in the next section.

FIVE DISCUSSIONS

1. Erroneous Representation

In most classrooms in which teachers use multiple representations to build

Fig. 2 Four representations of the Golden Apples problem



(a)

Let n = apples remaining	$n = 2$
Double $n + 4 = n^1$	$2 \times 2 + 4 = 8$
Double $n^1 + 4 = n^2$	$2 \times 8 + 4 = 20$
Double $n^2 + 4 = n^3$	$2 \times 20 + 4 = 44$
n^3 is the number of apples picked	$n^3 = 44$

(b)

Apples remaining

2 + 2 = 4 × 2 = 8

8 + 2 = 10 × 2 = 20

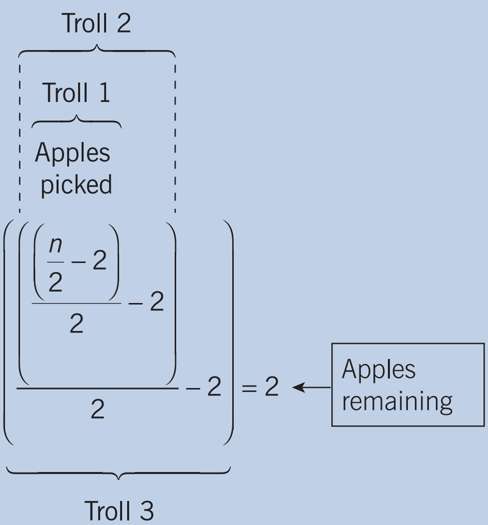
20 + 2 = 22 × 2 = 44

Apples taken home	0	1	2	3	4	5
Number of apples picked	28	36	44	52	60	68

8

$8x + 28$, where x is the number of apples taken home

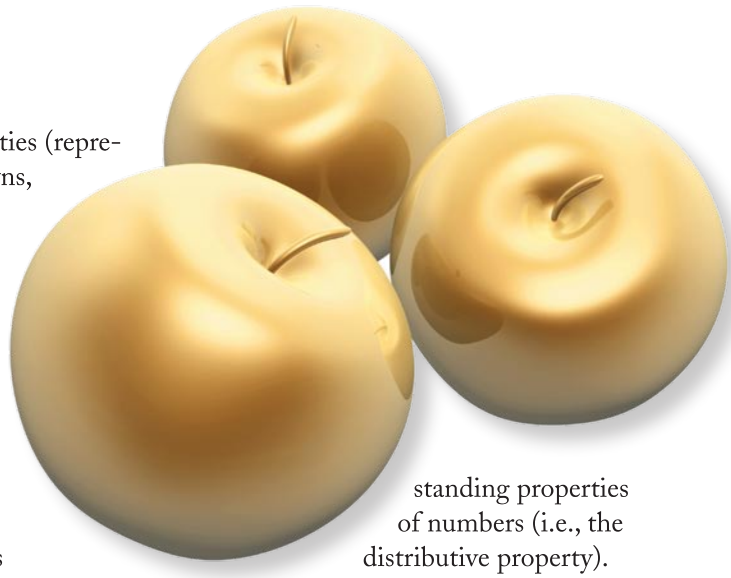
(c)



(d)

understanding, errors will occur. In discussions of solutions, errors can reveal a lot about the student's mathematical thinking, such as how the solver was using procedures and concepts to represent and solve a problem situation. Errors in representations can occur in notation, in structure (or form), or as an inaccurate translation of the context. For example, in **figure 2b**, a notational error occurred; exponents were used instead of subscripts to denote different instantiations of the unknown n. **Figure 2c** contained an example (sometimes referred to as run-on equal signs) of a structural error; the representations conveyed the idea that 2 + 2 was equivalent to 4 × 2 and 8, which was incorrect. Errors provide opportunities to emphasize the importance of the standardized use of notation and form to accurately convey one's ideas to others; however, a teacher must balance the identification and correction of procedural errors with the recognition of developing conceptual understanding. In both **figure 2b** and **2c**, for example, the teachers demonstrated that they understood the basic concept of undoing a process of events, in spite of their errors with notation and problem structure.

Figure 2a contained a translation error. The teacher used the letter A to represent the number of apples the prince was holding when he met a troll. Contextually, she knew that when the prince met a troll, he had to give away half the apples and two more. Procedurally, she wrote this as 1/2 A – 2. Contextually, she knew this transaction occurred three times and that at the conclusion the prince had two apples. Procedurally, she wrote this as 3(1/2 A – 2) = 2. In an ensuing discussion, she revealed that she knew that the value of A was not constant; the number of apples the prince was holding changed each time he encountered a troll. Consequently, she



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realized that she should not represent this phenomenon with a single static variable but did not know how to represent the changing number as a nested quantity. Also in contention in this representation was the meaning of the 3. A transaction did not occur three times as indicated but occurred in successive stages, which is represented by "nesting" each previous representation of a transaction into the next. The way that this teacher *conceptually* thought about representing the problem was similar to the thinking represented in **figure 2d.** Because her representation was included in a larger discussion of the solution strategies, she had the opportunity to see that similarity and increase her understanding of how to correctly represent the situation.

2. Properties of Numbers and the Form of a Representation

Differences in representations sometimes occurred as a result of the way problem solvers defined and expressed the quantities. A discussion of this phenomenon occurred in response to a subtle difference in the representations in **figures 2b** and **2c.** In both representations, the teachers began with the end result (the 2 apples the prince had when he returned home), then multiplied and added the apples back in through a series of three encounters with trolls. This is sometimes called an "unwinding" strategy (Koedinger and MacLaren 1997). The difference between these representations arose from how each teacher used or did not use a letter to represent a quantity. In **figure 2b**, the teacher assigned the letter n to represent the number of apples the prince had when he arrived home. She stated that you double n and add 4 to represent the number of apples the prince had before his encounter with the third troll (which she represented as n^1, instead of the correctly subscripted n_1). She repeated this calculation two more times with new quantities (represented as new unknowns, once again with incorrect superscripts) to find the number of apples picked. In contrast, in the top half of **figure 2c**, the teacher did not use a letter to make a general representation but began with the 2 remaining apples the prince had when he reached home, added 2, and then multiplied the sum by 2 (represented erroneously with run-on equal signs). She repeated this procedure two more times to find the number of apples the prince picked. We asked, "What does the 4 represent in **figure 2b**?" to highlight the procedural differences in the representations. In the ensuing discussion, teachers noted that in **figure 2b**, because the teacher used n to represent the apples the prince had when he got home, she could not complete the step of adding 2 apples to the number of apples the prince had, as shown in **figure 2c.** Instead, she represented the outcome of adding back in the 2 apples as $n + 2$ (which she did not show). Next, to "unwind" the process of a troll taking half the apples, she doubled this sum. Because the sum was represented as an expression, she used the distributive property (which she also did not show); $2(n + 2) = 2n + 4$, or "double n then add 4." In **figure 2c**, the teacher did not use a letter to represent a known quantity (the number of apples the prince brought home), so she did not have to apply any properties beyond addition and multiplication. Thus, the choice of assigning a letter to a known quantity (the number of apples the prince had), which allowed us to generalize to a function rule and quickly find outcomes for any given quantity, increased the value of understanding properties of numbers (i.e., the distributive property).

3. Equivalency and Inverse Strategies

Consider the context of—and the five events that occurred in—this problem (**fig. 1**). We described **figures 2b** and **2c** as representations of the concept of "unwinding"; the teachers reversed the order of the occurrence of the events. **Figure 2d**, on the other hand, represented the events in the order they occurred, in what we described as a "nesting" configuration. In **figure 2d** the unknown n represented the number of apples initially picked: The number of apples the prince had when he encountered the first troll. The first troll took half, represented by $n/2$, and 2 more, represented by $(n/2 - 2)$. This quantity represented the number of apples the prince had when he encountered the second troll. This nested quantity was acted on with division and subtraction in the ensuing negotiation with the second troll and likewise when the prince encountered the third troll; the representation of his remaining apples is nested in division by 2 and subtraction of 2.

Comparing the representation in **figure 2d** with **2b** or **2c** initiated a discussion of the mathematical ideas of reciprocity and equivalency. While comparing their representations, our teachers did not voice *explicit* understandings of such concepts as inverse, identity, or reciprocity. However, they were quick to notice that the representations in **figures 2c** and **2d** used "opposite" operations; for example, addition in **figure 2c** was used versus that of subtraction in **figure 2d** and multiplication in **2c** versus division in **2d**. Using the contextual representations in **figure 1** as a guide, they were able to easily describe how each representation started from "opposite" points—the orchard or the palace—and in exactly "opposite" order. Therefore, it made sense to them conceptually that these representations were equivalent; they used "opposite" operations to describe the "opposite" order of events. We point out, again, the need for caution and clarity when discussing a representation in relation to its underlying mathematical idea: "Opposite" did not have the same mathematical meaning as "inverse." For the sake of developing a deeper understanding of mathematical concepts, it was important to emphasize correct word use.

4. Equivalency and Symbolic Representation and Manipulation

The table in **figure 2c** was generated when the teacher manipulated the value for the number of apples the prince took home (n), then worked through her algorithm to find corresponding values of apples picked. She identified the constant rate of change and y-intercept and wrote an expression to represent the general case for apples picked, given the number of apples remaining ($8x + 28$). When one teacher asked how to show that the representations in **2c** and **2d** were equivalent, the group discussed setting the equation in **figure 2d** equal to x rather than 2 and "simplifying." By repeatedly adding 2 to both sides of the equation and multiplying both sides by 2, they reduced the equation to $n = 8x + 28$: the number of apples picked (n), given the number of apples remaining (x). Hence, using multiple representations of the problem situation allowed a discussion of equivalency that was grounded in the procedures for simplifying expressions. Coupled with the opportunity to develop a *contextual* understanding of equivalency in the previous discussion, our teachers used a rich environment to make connections and see relationships of mathematical ideas.

A deep understanding of mathematics does not necessarily result from being able to represent a mathematical situation in multiple ways

5. Mathematical Trajectories

Finally, these four representations (including the erroneous representation in **fig. 2a)** allowed our teachers to have frank discussions about their mathematical understanding and the implications for their own teaching. These discussions included the fact that some solutions were more sophisticated or further along in terms of mathematical thinking, which might have negative implications for some students in their classroom. Also discussed was how the nested representation in **figure 2d** was related to, and possibly evolved from, the "not quite" representation in **figure 2a**. Another point involved how the representation in **figure 2b**, although not as elegant as **2d**, may have been the most efficient if all one wanted to do was answer the question. The definition of mathematical proficiency (Kilpatrick, Swafford, and Findell 2001) includes strands such as "strategic competence" and "adaptive reasoning." When teachers and students realize that their understanding of procedures and concepts at any given moment lies on a continuum of mathematical understanding, they also realize the flexibility required to exercise strategy and adaptability.

CONCLUSION

A deep understanding of mathematics does not necessarily result from being able to represent a mathematical situation in multiple ways. We present a case, representative of the NCTM's Communication Standard (2000), which illustrates that the power of multiple representations to develop deep understanding of mathematical procedures and concepts resides in the classroom discussions that result when reconciling the representations to each other. When learners contribute their representations to the pool of possibilities, even when they are syntactically similar (i.e., all symbolic) or contain errors, they enter into the discussions and position themselves for deep mathematical learning.

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