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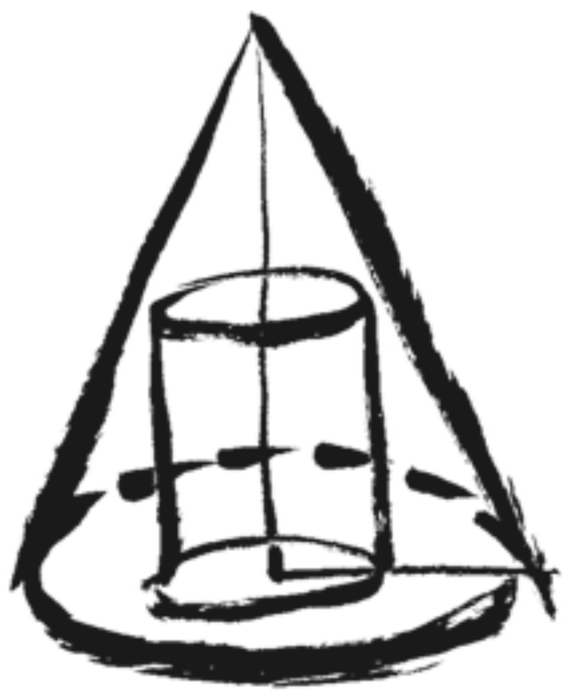

TEACHING GUIDE FOR SENIOR HIGH SCHOOL

Basic Calculus

CORE SUBJECT

This Teaching Guide was collaboratively developed and reviewed by educators from public and private schools, colleges, and universities. We encourage teachers and other education stakeholders to email their feedback, comments, and recommendations to the Commission on Higher Education, K to 12 Transition Program Management Unit - Senior High School Support Team at k12@ched.gov.ph. We value your feedback and recommendations.

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Introduction

As the Commission supports DepEd's implementation of Senior High School (SHS), it upholds the vision and mission of the K to 12 program, stated in Section 2 of Republic Act 10533, or the Enhanced Basic Education Act of 2013, that "every graduate of basic education be an empowered individual, through a program rooted on...the competence to engage in work and be productive, the ability to coexist in fruitful harmony with local and global communities, the capability to engage in creative and critical thinking, and the capacity and willingness to transform others and oneself."

To accomplish this, the Commission partnered with the Philippine Normal University (PNU), the National Center for Teacher Education, to develop Teaching Guides for Courses of SHS. Together with PNU, this Teaching Guide was studied and reviewed by education and pedagogy experts, and was enhanced with appropriate methodologies and strategies.

Furthermore, the Commission believes that teachers are the most important partners in attaining this goal. Incorporated in this Teaching Guide is a framework that will guide them in creating lessons and assessment tools, support them in facilitating activities and questions, and assist them towards deeper content areas and competencies. Thus, the introduction of the **SHS for SHS Framework.**

The SHS for SHS Framework

The SHS for SHS Framework, which stands for "Saysay-Husay-Sarili for Senior High School," is at the core of this book. The lessons, which combine high-quality content with flexible elements to accommodate diversity of teachers and environments, promote these three fundamental concepts:

SAYSAY: MEANING

Why is this important?

Through this Teaching Guide, teachers will be able to facilitate an understanding of the value of the lessons, for each learner to fully engage in the content on both the cognitive and affective levels.

HUSAY: MASTERY

How will I deeply understand this?

Given that developing mastery goes beyond memorization, teachers should also aim for deep understanding of the subject matter where they lead learners to analyze and synthesize knowledge.

SARILI: OWNERSHIP

What can I do with this?

When teachers empower learners to take ownership of their learning, they develop independence and self-direction, learning about both the subject matter and themselves.

The Parts of the Teaching Guide

This Teaching Guide is mapped and aligned to the DepEd SHS Curriculum, designed to be highly usable for teachers. It contains classroom activities and pedagogical notes, and integrated with innovative pedagogies. All of these elements are presented in the following parts:

1. **INTRODUCTION**
    - Highlight key concepts and identify the essential questions
    - Show the big picture
    - Connect and/or review prerequisite knowledge
    - Clearly communicate learning competencies and objectives
    - Motivate through applications and connections to real-life
2. **INSTRUCTION/DELIVERY**
    - Give a demonstration/lecture/simulation/ hands-on activity
    - Show step-by-step solutions to sample problems
    - Use multimedia and other creative tools
    - Give applications of the theory
    - Connect to a real-life problem if applicable
3. **PRACTICE**
    - Discuss worked-out examples
    - Provide easy-medium-hard questions
    - Give time for hands-on unguided classroom work and discovery
    - Use formative assessment to give feedback
4. **ENRICHMENT**
    - Provide additional examples and applications
    - Introduce extensions or generalisations of concepts
    - Engage in reflection questions
    - Encourage analysis through higher order thinking prompts
5. **EVALUATION**
    - Supply a diverse question bank for written work and exercises
    - Provide alternative formats for student work: written homework, journal, portfolio, group/individual projects, student-directed research project

Pedagogical Notes

The teacher should strive to keep a good balance between conceptual understanding and facility in skills and techniques. Teachers are advised to be conscious of the content and performance standards and of the suggested time frame for each lesson, but flexibility in the management of the lessons is possible. Interruptions in the class schedule, or students' poor reception or difficulty with a particular lesson, may require a teacher to extend a particular presentation or discussion.

Computations in some topics may be facilitated by the use of calculators. This is encour- aged; however, it is important that the student understands the concepts and processes involved in the calculation. Exams for the Basic Calculus course may be designed so that calculators are not necessary.

Because senior high school is a transition period for students, the latter must also be prepared for college-level academic rigor. Some topics in calculus require much more rigor and precision than topics encountered in previous mathematics courses, and treatment of the material may be different from teaching more elementary courses. The teacher is urged to be patient and careful in presenting and developing the topics. To avoid too much technical discussion, some ideas can be introduced intuitively and informally, without sacrificing rigor and correctness.

The teacher is encouraged to study the guide very well, work through the examples, and solve exercises, well in advance of the lesson. The development of calculus is one of humankind's greatest achievements. With patience, motivation and discipline, teaching and learning calculus effectively can be realized by anyone. The teaching guide aims to be a valuable resource in this objective.

On DepEd Functional Skills and CHED's College Readiness Standards

As Higher Education Institutions (HEIs) welcome the graduates of the Senior High School program, it is of paramount importance to align Functional Skills set by DepEd with the College Readiness Standards stated by CHED.

The DepEd articulated a set of 21st century skills that should be embedded in the SHS curriculum across various subjects and tracks. These skills are desired outcomes that K to 12 graduates should possess in order to proceed to either higher education, employment, entrepreneurship, or middle-level skills development.

On the other hand, the Commission declared the College Readiness Standards that consist of the combination of knowledge, skills, and reflective thinking necessary to participate and succeed - without remediation - in entry-level undergraduate courses in college.

The alignment of both standards, shown below, is also presented in this Teaching Guide - prepares Senior High School graduates to the revised college curriculum which will initially be implemented by AY 2018-2019.

College Readiness Standards Foundational Skills	DepEd Functional Skills
Produce all forms of texts (written, oral, visual, digital) based on: 1. Solid grounding on Philippine experience and culture; 2. An understanding of the self, community, and nation; 3. Application of critical and creative thinking and doing processes; 4. Competency in formulating ideas/arguments logically, scientifically, and creatively; and 5. Clear appreciation of one's responsibility as a citizen of a multicultural Philippines and a diverse world;	Visual and information literacies Media literacy Critical thinking and problem solving skills Creativity Initiative and self-direction
Systematically apply knowledge, understanding, theory, and skills for the development of the self, local, and global communities using prior learning, inquiry, and experimentation	Global awareness Scientific and economic literacy Curiosity Critical thinking and problem solving skills Risk taking Flexibility and adaptability Initiative and self-direction
Work comfortably with relevant technologies and develop adaptations and innovations for significant use in local and global communities;	Global awareness Media literacy Technological literacy Creativity Flexibility and adaptability Productivity and accountability
Communicate with local and global communities with proficiency, orally, in writing, and through new technologies of communication;	Global awareness Multicultural literacy Collaboration and interpersonal skills Social and cross-cultural skills Leadership and responsibility
Interact meaningfully in a social setting and contribute to the fulfilment of individual and shared goals, respecting the fundamental humanity of all persons and the diversity of groups and communities	Media literacy Multicultural literacy Global awareness Collaboration and interpersonal skills Social and cross-cultural skills Leadership and responsibility Ethical, moral, and spiritual values

K to 12 BASIC EDUCATION CURRICULUM
SENIOR HIGH SCHOOL – SCIENCE, TECHNOLOGY, ENGINEERING AND MATHEMATICS (STEM) SPECIALIZED SUBJECT

Correspondence between the Learning Competencies and the Topics in this Learning Guide

Course Title: Basic Calculus

Semester: Second Semester
No. of Hours/Semester: 80 hrs/sem
Prerequisite: Pre-Calculus

Subject Description: At the end of the course, the students must know how to determine the limit of a function, differentiate, and integrate algebraic, exponential, logarithmic, and trigonometric functions in one variable, and to formulate and solve problems involving continuity, extreme values, related rates, population models, and areas of plane regions.

CONTENT	CONTENT STANDARDS	PERFORMANCE STANDARDS	LEARNING COMPETENCIES	CODE	TOPIC NUMBER
Limits and Continuity	The learners demonstrate an understanding of... the basic concepts of limit and continuity of a function	The learners shall be able to... formulate and solve accurately real-life problems involving continuity of functions	The learners... 1. illustrate the limit of a function using a table of values and the graph of the function	**STEM_BC11LC-IIIa-1**	**1.1**
			2. distinguish between $\lim_{x \to c} f(x)$ and $f(c)$	**STEM_BC11LC-IIIa-2**	**1.2**
			3. illustrate the limit laws	**STEM_BC11LC-IIIa-3**	**1.3**
			4. apply the limit laws in evaluating the limit of algebraic functions (polynomial, rational, and radical)	**STEM_BC11LC-IIIa-4**	**1.4**
			5. compute the limits of exponential, logarithmic,and trigonometric functions using tables of values and graphs of the functions	**STEM_BC11LC-IIIb-1**	**2.1**
			6. evaluate limits involving the expressions $(\sin t)/t$, $(1-\cos t)/t$ and $(e^t - 1)/t$ using tables of values	**STEM_BC11LC-IIIb-2**	**2.2**
			7. illustrate continuity of a function at a number	**STEM_BC11LC-IIIc-1**	**3.1**
			8. determine whether a function is continuous at a number or not	**STEM_BC11LC-IIIc-2**	
			9. illustrate continuity of a function on an interval	**STEM_BC11LC-IIIc-3**	**3.2**
			10. determine whether a function is continuous on an interval or not.	**STEM_BC11LC-IIIc-4**	

CONTENT	CONTENT STANDARDS	PERFORMANCE STANDARDS	LEARNING COMPETENCIES	CODE	TOPIC NUMBER
			11. illustrate different types of discontinuity (hole/removable, jump/essential, asymptotic/infinite)	**STEM_BC11LC-IIId-1**	**4.1**
			12. illustrate the Intermediate Value and Extreme Value Theorems	**STEM_BC11LC-IIId-2**	**4.2**
			13. solves problems involving continuity of a function	**STEM_BC11LC-IIId-3**	**4.3**
Derivatives	basic concepts of derivatives	1. formulate and solve accurately situational problems involving extreme values	1. illustrate the tangent line to the graph of a function at a given point	**STEM_BC11D-IIIe-1**	**5.1**
			2. applies the definition of the derivative of a function at a given number	**STEM_BC11D-IIIe-2**	**5.3**
			3. relate the derivative of a function to the slope of the tangent line	**STEM_BC11D-IIIe-3**	**5.2**
			4. determine the relationship between differentiability and continuity of a function	**STEM_BC11D -IIIf-1**	**6.1**
			5. derive the differentiation rules	**STEM_BC11D-IIIf-2**	**6.2**
			6. apply the differentiation rules in computing the derivative of an algebraic, exponential, and trigonometric functions	**STEM_BC11D-IIIf-3**	
			7. solve optimization problems	**STEM_BC11D-IIIg-1**	**7.1**
		2. formulate and solve accurately situational problems involving related rates	8. compute higher-order derivatives of functions	**STEM_BC11D-IIIh-1**	**8.1**
			9. illustrate the Chain Rule of differentiation	**STEM_BC11D-IIIh-2**	**8.2**
			10. solve problems using the Chain Rule	**STEM_BC11D-IIIh-i-1**	
			11. illustrate implicit differentiation	**STEM_BC11D-IIIi-2**	**9.1**
			12. solve problems (including logarithmic, and inverse trigonometric functions) using implicit differentiation	**STEM_BC11D-IIIi-j-1**	
			13. solve situational problems involving related rates	**STEM_BC11D-IIIj-2**	**10.1**

CONTENT	CONTENT STANDARDS	PERFORMANCE STANDARDS	LEARNING COMPETENCIES	CODE	TOPIC NUMBER
Integration	antiderivatives and Riemann integral	1. formulate and solve accurately situational problems involving population models	1. illustrate an antiderivative of a function	**STEM_BC11I-IVa-1**	**11.1**
			2. compute the general antiderivative of polynomial, radical, exponential, and trigonometric functions	**STEM_BC11I-IVa-b-1**	**11.2-11.4**
			3. compute the antiderivative of a function using substitution rule and table of integrals (includingthose whose antiderivatives involve logarithmic and inverse trigonometric functions)	**STEM_BC11I-IVb-c-1**	**12.1**
			4. solve separable differential equations using antidifferentiation	**STEM_BC11I-IVd-1**	**13.1**
			5. solve situational problems involving exponential growth and decay, bounded growth, and logistic growth	**STEM_BC11I-IVe-f-1**	**14.1**
		2. formulate and solve accurately real-life problems involving areas of plane regions	6. approximate the area of a region under a curve using Riemann sums: (a) left, (b) right, and (c) midpoint	**STEM_BC11I-IVg-1**	**15.1**
			7. define the definite integral as the limit of the Riemann sums	**STEM_BC11I-IVg-2**	**15.2**
			8. illustrate the Fundamental Theorem of Calculus	**STEM_BC11I-IVh-1**	**16.1**
			9. compute the definite integral of a function using the Fundamental Theorem of Calculus	**STEM_BC11I-IVh-2**	**16.2**
			10. illustrates the substitution rule	**STEM_BC11I-IVi-1**	**17.1**
			11. compute the definite integral of a function using the substitution rule	**STEM_BC11I-IVi-2**	
			12. compute the area of a plane region using the definite integral	**STEM_BC11I-IVi-j-1**	**18.1**
			13. solve problems involving areas of plane regions	**STEM_BC11I-IVj-2**	**18.2**

Contents

Chapter 1

Limits and Continuity

LESSON 1: The Limit of a Function: Theorems and Examples

TIME FRAME: 4 hours

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Illustrate the limit of a function using a table of values and the graph of the function;
2. Distinguish between $\lim_{x \to c} f(x)$ and $f(c)$;
3. Illustrate the limit theorems; and
4. Apply the limit theorems in evaluating the limit of algebraic functions (polynomial, rational, and radical).

LESSON OUTLINE:

1. Evaluation of limits using a table of values
2. Illustrating the limit of a function using the graph of the function
3. Distinguishing between $\lim_{x \to c} f(x)$ and $f(c)$ using a table of values
4. Distinguishing between $\lim_{x \to c} f(x)$ and $f(c)$ using the graph of $y = f(x)$
5. Enumeration of the eight basic limit theorems
6. Application of the eight basic limit theorems on simple examples
7. Limits of polynomial functions
8. Limits of rational functions
9. Limits of radical functions
10. Intuitive notions of infinite limits

TOPIC 1.1: The Limit of a Function

DEVELOPMENT OF THE LESSON

(A) ACTIVITY

In order to find out what the students' idea of a limit is, ask them to bring cutouts of news items, articles, or drawings which for them illustrate the idea of a limit. These may be posted on a wall so that they may see each other's homework, and then have each one explain briefly why they think their particular cutout represents a limit.

(B) INTRODUCTION

Limits are the backbone of calculus, and calculus is called the Mathematics of Change. The study of limits is necessary in studying change in great detail. The evaluation of a particular limit is what underlies the formulation of the derivative and the integral of a function.

For starters, imagine that you are going to watch a basketball game. When you choose seats, you would want to be as close to the action as possible. You would want to be as close to the players as possible and have the best view of the game, as if you were in the basketball court yourself. Take note that you cannot actually be in the court and join the players, but you will be close enough to describe clearly what is happening in the game.

This is how it is with limits of functions. We will consider functions of a single variable and study the behavior of the function as its variable approaches a particular value (a constant). The variable can only take values very, very close to the constant, but it cannot equal the constant itself. However, the limit will be able to describe clearly what is happening to the function near that constant.

(C) LESSON PROPER

Consider a function f of a single variable x. Consider a constant c which the variable x will approach (c may or may not be in the domain of f). The limit, to be denoted by L, is the unique real value that $f(x)$ will approach as x approaches c. In symbols, we write this process as

$$\lim_{x \to c} f(x) = L.$$

This is read, **"The limit of $f(x)$ as x approaches c is L."**

LOOKING AT A TABLE OF VALUES

To illustrate, let us consider

$$\lim_{x \to 2}(1 + 3x).$$

Here, $f(x) = 1 + 3x$ and the constant c, which x will approach, is 2. To evaluate the given limit, we will make use of a table to help us keep track of the effect that the approach of x toward 2 will have on $f(x)$. Of course, on the number line, x may approach 2 in two ways: through values on its left and through values on its right. We first consider approaching 2 from its left or through values less than 2. Remember that the values to be chosen should be close to 2.

x	$f(x)$
1	4
1.4	5.2
1.7	6.1
1.9	6.7
1.95	6.85
1.997	6.991
1.9999	6.9997
1.9999999	6.9999997

Now we consider approaching 2 from its right or through values greater than but close to 2.

x	$f(x)$
3	10
2.5	8.5
2.2	7.6
2.1	7.3
2.03	7.09
2.009	7.027
2.0005	7.0015
2.0000001	7.0000003

Observe that as the values of x get closer and closer to 2, the values of $f(x)$ get closer and closer to 7. This behavior can be shown no matter what set of values, or what direction, is taken in approaching 2. In symbols,

$$\lim_{x \to 2}(1 + 3x) = 7.$$

EXAMPLE 1: Investigate

$$\lim_{x \to -1} (x^2 + 1)$$

by constructing tables of values. Here, $c = -1$ and $f(x) = x^2 + 1$.

We start again by approaching -1 from the left.

x	$f(x)$
-1.5	3.25
-1.2	2.44
-1.01	2.0201
-1.0001	2.00020001

Now approach -1 from the right.

x	$f(x)$
-0.5	1.25
-0.8	1.64
-0.99	1.9801
-0.9999	1.99980001

The tables show that as x approaches -1, $f(x)$ approaches 2. In symbols,

$$\lim_{x \to -1} (x^2 + 1) = 2.$$

EXAMPLE 2: Investigate $\lim_{x \to 0} |x|$ through a table of values.

Approaching 0 from the left and from the right, we get the following tables:

x	$\lvert x \rvert$
-0.3	0.3
-0.01	0.01
-0.00009	0.00009
-0.00000001	0.00000001

x	$\lvert x \rvert$
0.3	0.3
0.01	0.01
0.00009	0.00009
0.00000001	0.00000001

Hence,

$$\lim_{x \to 0} |x| = 0.$$

EXAMPLE 3: Investigate

$$\lim_{x \to 1} \frac{x^2 - 5x + 4}{x - 1}$$

by constructing tables of values. Here, $c = 1$ and $f(x) = \dfrac{x^2 - 5x + 4}{x - 1}$.
Take note that 1 is not in the domain of f, but this is not a problem. In evaluating a limit, remember that we only need to go very close to 1; we will not go to 1 itself.
We now approach 1 from the left.

x	$f(x)$
1.5	-2.5
1.17	-2.83
1.003	-2.997
1.0001	-2.9999

Approach 1 from the right.

x	$f(x)$
0.5	-3.5
0.88	-3.12
0.996	-3.004
0.9999	-3.0001

The tables show that as x approaches 1, $f(x)$ approaches -3. In symbols,

$$\lim_{x \to 1} \frac{x^2 - 5x + 4}{x - 1} = -3.$$

EXAMPLE 4: Investigate through a table of values

$$\lim_{x \to 4} f(x)$$

if

$$f(x) = \begin{cases} x + 1 & \text{if } x < 4 \\ (x-4)^2 + 3 & \text{if } x \geq 4. \end{cases}$$

This looks a bit different, but the logic and procedure are exactly the same. We still approach the constant 4 from the left and from the right, but note that we should evaluate the appropriate corresponding functional expression. In this case, when x approaches 4 from the left, the values taken should be substituted in $f(x) = x + 1$. Indeed, this is the part of the function which accepts values less than 4. So,

x	$f(x)$
3.7	4.7
3.85	4.85
3.995	4.995
3.99999	4.99999

On the other hand, when x approaches 4 from the right, the values taken should be substituted in $f(x) = (x-4)^2 + 3$. So,

x	$f(x)$
4.3	3.09
4.1	3.01
4.001	3.000001
4.00001	3.0000000001

Observe that the values that $f(x)$ approaches are not equal, namely, $f(x)$ approaches 5 from the left while it approaches 3 from the right. In such a case, we say that the limit of the given function *does not exist* **(DNE)**. In symbols,

$$\lim_{x \to 4} f(x) \text{ DNE.}$$

Remark 1: We need to emphasize an important fact. We do not say that $\lim_{x \to 4} f(x)$ "equals DNE", nor do we write "$\lim_{x \to 4} f(x) = \text{DNE}$", because "DNE" is not a value. In the previous example, "DNE" indicated that the function moves in different directions as its variable approaches c from the left and from the right. In other cases, the limit fails to exist because it is undefined, such as for $\lim_{x \to 0} \frac{1}{x}$ which leads to division of 1 by zero.

Remark 2: Have you noticed a pattern in the way we have been investigating a limit? We have been specifying whether x will approach a value c from the left, through values less than c, or from the right, through values greater than c. This direction may be specified in the limit notation, $\lim_{x \to c} f(x)$ by adding certain symbols.

- If x approaches c from the left, or through values less than c, then we write $\lim_{x \to c^-} f(x)$.
- If x approaches c from the right, or through values greater than c, then we write $\lim_{x \to c^+} f(x)$.

Furthermore, we say

$$\lim_{x \to c} f(x) = L$$

if and only if

$$\lim_{x \to c^-} f(x) = L \text{ and } \lim_{x \to c^+} f(x) = L.$$

In other words, for a limit L to exist, the limits from the left and from the right must both exist and be equal to L. Therefore,

$$\lim_{x\to c} f(x) \text{ DNE whenever } \lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x).$$

These limits, $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$, are also referred to as ***one-sided limits***, since you only consider values on one side of c.

Thus, we may say:

- in our very first illustration that $\lim_{x\to 2}(1+3x) = 7$ because $\lim_{x\to 2^-}(1+3x) = 7$ and $\lim_{x\to 2^+}(1+3x) = 7$.
- in Example 1, $\lim_{x\to -1}(x^2+1) = 2$ since $\lim_{x\to -1^-}(x^2+1) = 2$ and $\lim_{x\to -1^+}(x^2+1) = 2$.
- in Example 2, $\lim_{x\to 0}|x| = 0$ because $\lim_{x\to 0^-}|x| = 0$ and $\lim_{x\to 0^+}|x| = 0$.
- in Example 3, $\lim_{x\to 1}\dfrac{x^2-5x+4}{x-1} = -3$ because $\lim_{x\to 1^-}\dfrac{x^2-5x+4}{x-1} = -3$ and $\lim_{x\to 1^+}\dfrac{x^2-5x+4}{x-1} = -3$.
- in Example 4, $\lim_{x\to 4} f(x)$ DNE because $\lim_{x\to 4^-} f(x) \neq \lim_{x\to 4^+} f(x)$.

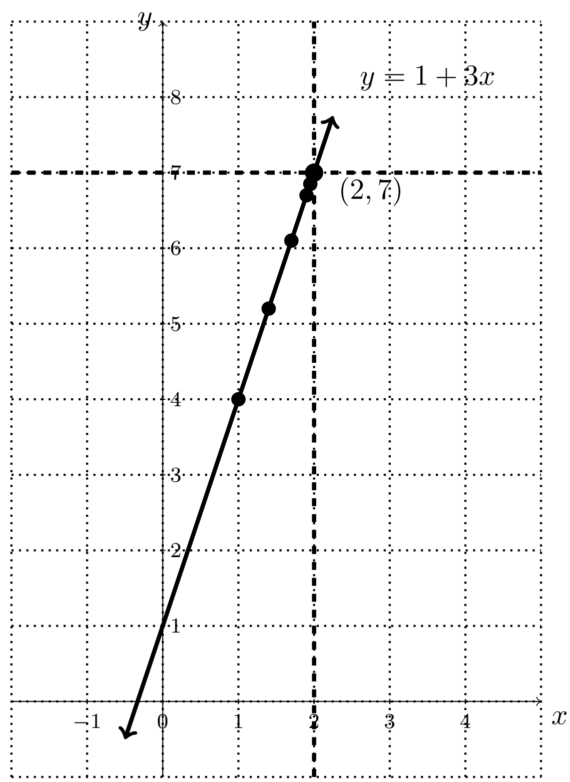
LOOKING AT THE GRAPH OF $y = f(x)$

If one knows the graph of $f(x)$, it will be easier to determine its limits as x approaches given values of c.

Consider again $f(x) = 1 + 3x$. Its graph is the straight line with slope 3 and intercepts $(0,1)$ and $(-1/3,0)$. Look at the graph in the vicinity of $x = 2$.

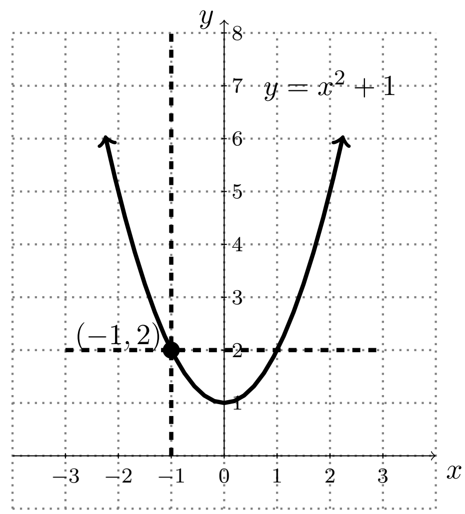
You can easily see the points (from the table of values in page 4) $(1,4)$, $(1.4,5.2)$, $(1.7,6.1)$, and so on, approaching the level where $y = 7$. The same can be seen from the right (from the table of values in page 4). Hence, the graph clearly confirms that

$$\lim_{x\to 2}(1+3x) = 7.$$



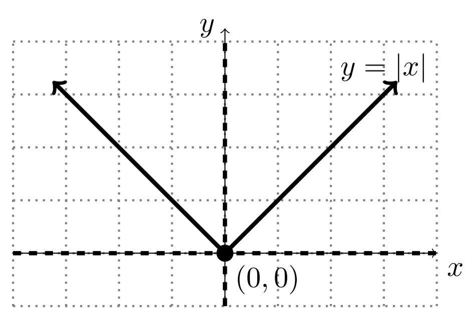
Let us look at the examples again, one by one.

Recall Example 1 where $f(x) = x^2 + 1$. Its graph is given by



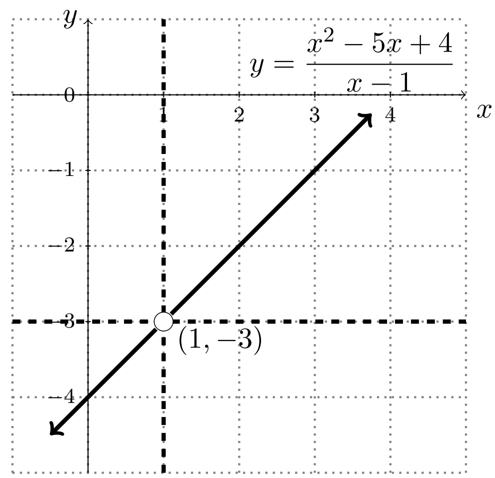
It can be seen from the graph that as values of x approach -1, the values of $f(x)$ approach 2.

Recall Example 2 where $f(x) = |x|$.



It is clear that $\lim_{x \to 0} |x| = 0$, that is, the two sides of the graph both move downward to the origin $(0, 0)$ as x approaches 0.

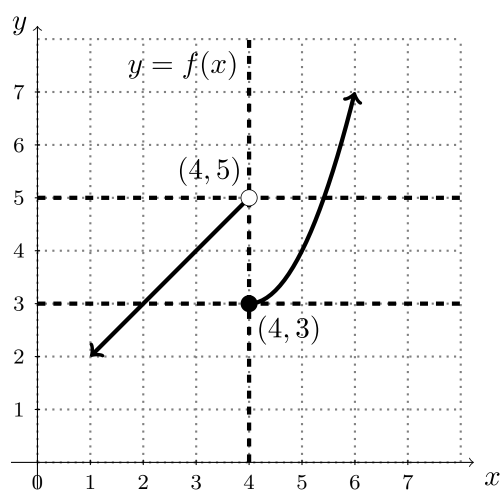
Recall Example 3 where $f(x) = \dfrac{x^2 - 5x + 4}{x - 1}$.



Take note that $f(x) = \dfrac{x^2 - 5x + 4}{x - 1} = \dfrac{(x-4)(x-1)}{x-1} = x - 4$, provided $x \neq 1$. Hence, the graph of $f(x)$ is also the graph of $y = x - 1$, excluding the point where $x = 1$.

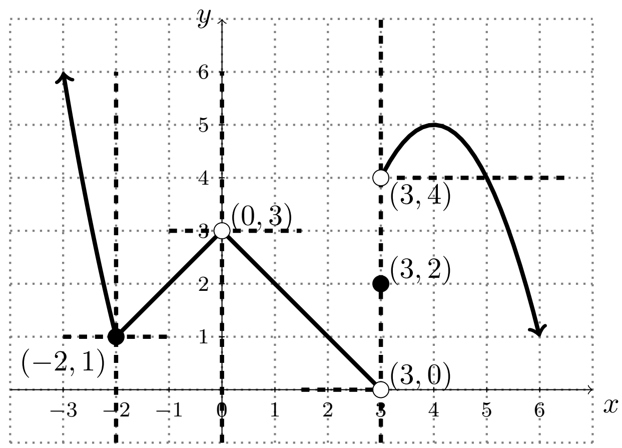
Recall Example 4 where

$$f(x) = \begin{cases} x + 1 & \text{if } x < 4 \\ (x-4)^2 + 3 & \text{if } x \geq 4. \end{cases}$$



Again, we can see from the graph that $f(x)$ has no limit as x approaches 4. The two separate parts of the function move toward different y-levels ($y = 5$ from the left, $y = 3$ from the right) in the vicinity of $c = 4$.

So, in general, if we have the graph of a function, such as below, determining limits can be done much faster and easier by inspection.



For instance, it can be seen from the graph of $y = f(x)$ that:

a. $\lim\limits_{x\to -2} f(x) = 1$.

b. $\lim\limits_{x\to 0} f(x) = 3$. Here, it does not matter that $f(0)$ does not exist (that is, it is undefined, or $x = 0$ is not in the domain of f). Always remember that what matters is the behavior of the function close to $c = 0$ and not precisely at $c = 0$. In fact, even if $f(0)$ were defined and equal to any other constant (not equal to 3), like 100 or -5000, this would still have no bearing on the limit. In cases like this, $\lim\limits_{x\to 0} f(x) = 3$ prevails regardless of the value of $f(0)$, if any.

c. $\lim\limits_{x\to 3} f(x)$ DNE. As can be seen in the figure, the two parts of the graph near $c = 3$ do not move toward a common y-level as x approaches $c = 3$.

(D) **EXERCISES** (Students may use calculators when applicable.)

Exercises marked with a star ($\star$) are challenging problems or may require a longer solution.

1. Complete the following tables of values to investigate $\lim\limits_{x\to 1}(x^2 - 2x + 4)$.

x	$f(x)$
0.5	
0.7	
0.95	
0.995	
0.9995	
0.99995	

x	$f(x)$
1.6	
1.35	
1.05	
1.005	
1.0005	
1.00005	

2. Complete the following tables of values to investigate $\lim\limits_{x \to 0} \dfrac{x-1}{x+1}$.

x	$f(x)$
-1	
-0.8	
-0.35	
-0.1	
-0.09	
-0.0003	
-0.000001	

x	$f(x)$
1	
0.75	
0.45	
0.2	
0.09	
0.0003	
0.000001	

3. Construct a table of values to investigate the following limits:

a. $\lim\limits_{x \to 3} \dfrac{10}{x-2}$

b. $\lim\limits_{x \to 7} \dfrac{10}{x-2}$

c. $\lim\limits_{x \to 2} \dfrac{2x+1}{x-3}$

d. $\lim\limits_{x \to 0} \dfrac{x^2+6}{x^2+2}$

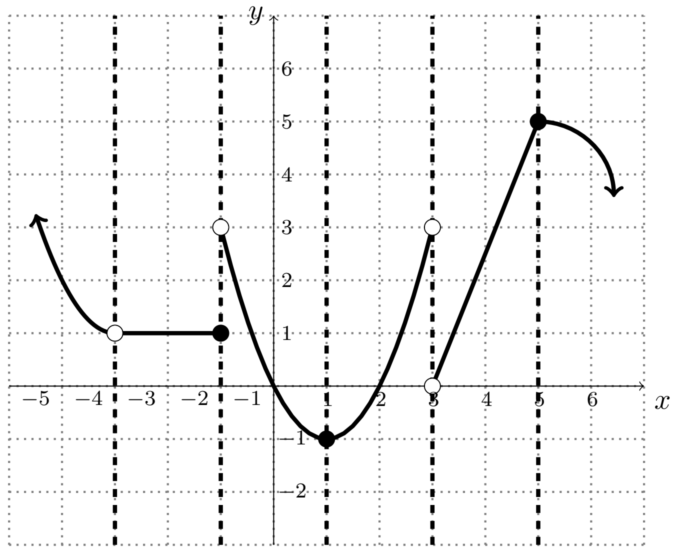
e. $\lim\limits_{x \to 1} \dfrac{1}{x+1}$

f. $\lim\limits_{x \to 0} f(x)$ if $f(x) = \begin{cases} 1/x & \text{if } x \leq -1 \\ x^2 - 2 & \text{if } x > -1 \end{cases}$

g. $\lim\limits_{x \to -1} f(x)$ if $f(x) = \begin{cases} 1/x & \text{if } x \leq -1 \\ x^2 - 2 & \text{if } x > -1 \end{cases}$

h. $\lim\limits_{x \to 1} f(x)$ if $f(x) = \begin{cases} x+3 & \text{if } x < 1 \\ 2x & \text{if } x = 1 \\ \sqrt{5x-1} & \text{if } x > 1 \end{cases}$

4. Consider the function $f(x)$ whose graph is shown below.



Determine the following:

a. $\lim_{x\to -3} f(x)$

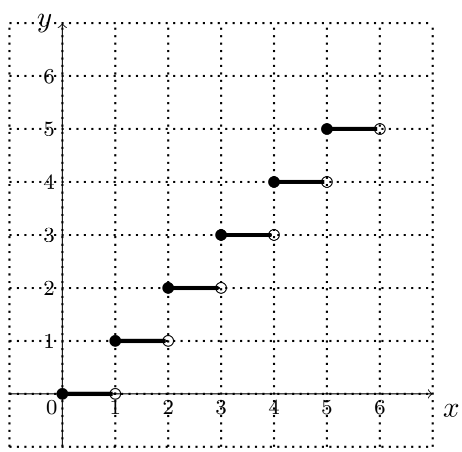
b. $\lim_{x\to -1} f(x)$

c. $\lim_{x\to 1} f(x)$

d. $\lim_{x\to 3} f(x)$

e. $\lim_{x\to 5} f(x)$

5. Consider the function $f(x)$ whose graph is shown below.



What can be said about the limit of $f(x)$

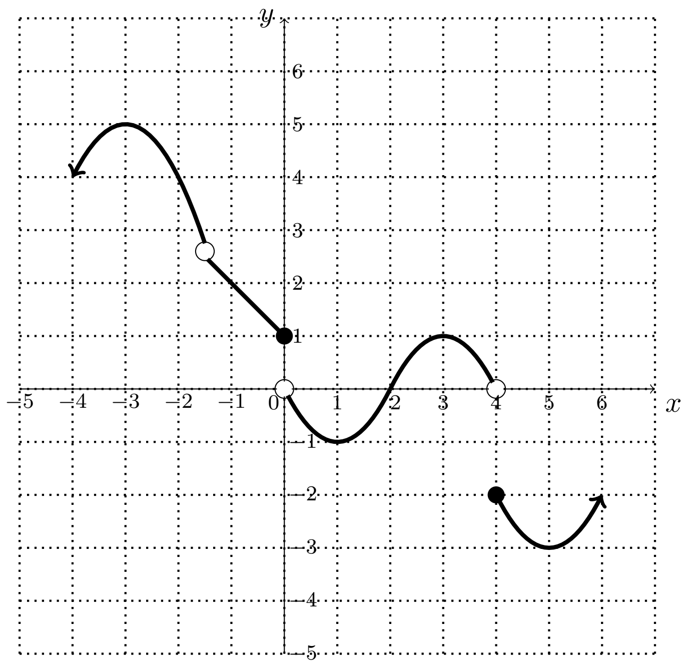
a. at $c = 1,\ 2,\ 3,$ and 4?

b. at integer values of c?

c. at $c = 0.4,\ 2.\,3, 4.7,$ and 5.5?

d. at non-integer values of c?

6. Consider the function $f(x)$ whose graph is shown below.



Determine the following:

a. $\lim_{x \to -1.5} f(x)$

b. $\lim_{x \to 0} f(x)$

c. $\lim_{x \to 2} f(x)$

d. $\lim_{x \to 4} f(x)$

Teaching Tip

Test how well the students have understood limit evaluation. It is hoped that by now they have observed that *for polynomial and rational functions* f, if c is in the domain of f, then to evaluate $\lim_{x \to c} f(x)$ they just need to substitute the value of c for every x in $f(x)$.

However, this is not true for general functions. Ask the students if they can give an example or point out an earlier example of a case where c is in the domain of f, but $\lim_{x \to c} f(x) \neq f(c)$.

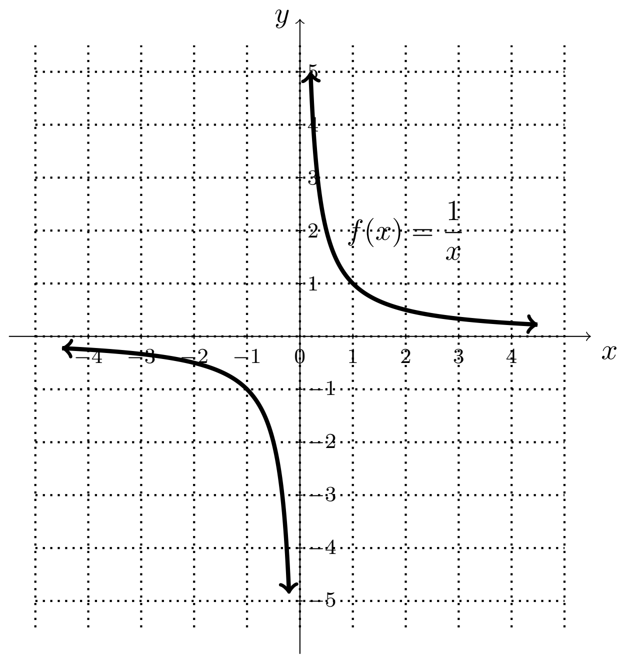
7. Without a table of values and without graphing $f(x)$, give the values of the following limits and explain how you arrived at your evaluation.

a. $\lim_{x \to -1} (3x - 5)$

b. $\lim_{x \to c} \dfrac{x^2 - 9}{x^2 - 4x + 3}$ where $c = 0, 1, 2$

$\star$c. $\lim_{x \to 3} \dfrac{x^2 - 9}{x^2 - 4x + 3}$

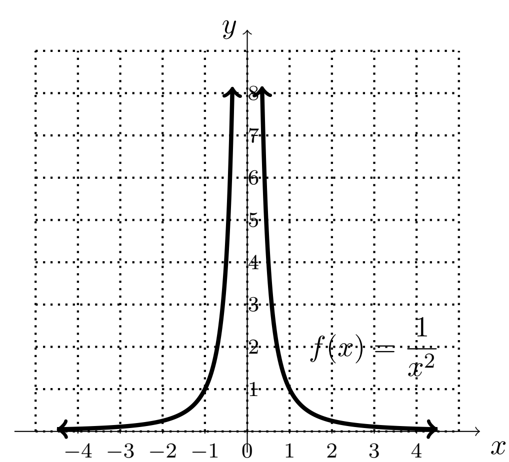
⋆8. Consider the function $f(x) = \dfrac{1}{x}$ whose graph is shown below.



What can be said about $\lim\limits_{x\to 0} f(x)$? Does it exist or not? Why?

Answer: The limit does not exist. From the graph itself, as x-values approach 0, the arrows move in opposite directions. If tables of values are constructed, one for x-values approaching 0 through negative values and another through positive values, it is easy to observe that the closer the x-values are to 0, the more negatively and positively large the corresponding $f(x)$-values become.

⋆9. Consider the function $f(x)$ whose graph is shown below. What can be said about $\lim\limits_{x\to 0} f(x)$? Does it exist or not? Why?



Answer: The limit does not exist. Although as x-values approach 0, the arrows seem to move in the same direction, they will not "stop" at a limiting value. In the absence of such a definite limiting value, we still say the limit does not exist. (We will revisit this function in the lesson about infinite limits where we will discuss more about its behavior near 0.)

⋆10. Sketch one possible graph of a function $f(x)$ defined on $\mathbb{R}$ that satisfies all the listed conditions.

a. $\lim_{x\to 0} f(x) = 1$

b. $\lim_{x\to 1} f(x)$ DNE

c. $\lim_{x\to 2} f(x) = 0$

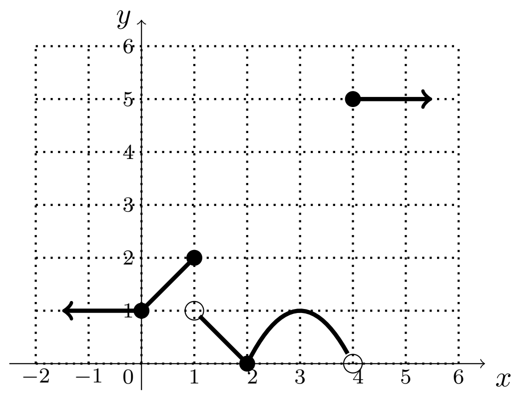
d. $f(1) = 2$

e. $f(2) = 0$

f. $f(4) = 5$

g. $\lim_{x\to c} f(x) = 5$ for all $c > 4$.

Possible answer (there are many other possibilities):



TOPIC 1.2: The Limit of a Function at c versus the Value of the Function at c

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

Critical to the study of limits is the understanding that the value of

$$\lim_{x \to c} f(x)$$

may be distinct from the value of the function at $x = c$, that is, $f(c)$. As seen in previous examples, the limit may be evaluated at values not included in the domain of f. Thus, it must be clear to a student of calculus that the exclusion of a value from the domain of a function does not prohibit the evaluation of the limit of that function at that excluded value, provided of course that f is defined at the points near c. In fact, these cases are actually the more interesting ones to investigate and evaluate.

Furthermore, the awareness of this distinction will help the student understand the concept of continuity, which will be tackled in Lessons 3 and 4.

(B) LESSON PROPER

We will mostly recall our discussions and examples in Lesson 1.
Let us again consider

$$\lim_{x \to 2} (1 + 3x).$$

Recall that its tables of values are:

x	$f(x)$
1	4
1.4	5.2
1.7	6.1
1.9	6.7
1.95	6.85
1.997	6.991
1.9999	6.9997
1.9999999	6.9999997

x	$f(x)$
3	10
2.5	8.5
2.2	7.6
2.1	7.3
2.03	7.09
2.009	7.027
2.0005	7.0015
2.0000001	7.0000003

and we had concluded that $\lim_{x \to 2} (1 + 3x) = 7$.

In comparison, $f(2) = 7$. So, in this example, $\lim_{x \to 2} f(x)$ and $f(2)$ are equal. Notice that the same holds for the next examples discussed:

$\lim_{x \to c} f(x)$	$f(c)$
$\lim_{x \to -1} (x^2 + 1) = 2$	$f(-1) = 2$
$\lim_{x \to 0} \lvert x \rvert = 0$	$f(0) = 0$

This, however, is not always the case. Let us consider the function

$$f(x) = \begin{cases} |x| & \text{if } x \neq 0 \\ 2 & \text{if } x = 0. \end{cases}$$

In contrast to the second example above, the entries are now unequal:

$\lim_{x \to c} f(x)$	$f(c)$
$\lim_{x \to 0} \lvert x \rvert = 0$	$f(0) = 2$

Does this in any way affect the existence of the limit? Not at all. This example shows that $\lim_{x \to c} f(x)$ and $f(c)$ may be distinct.

Furthermore, consider the third example in Lesson 1 where

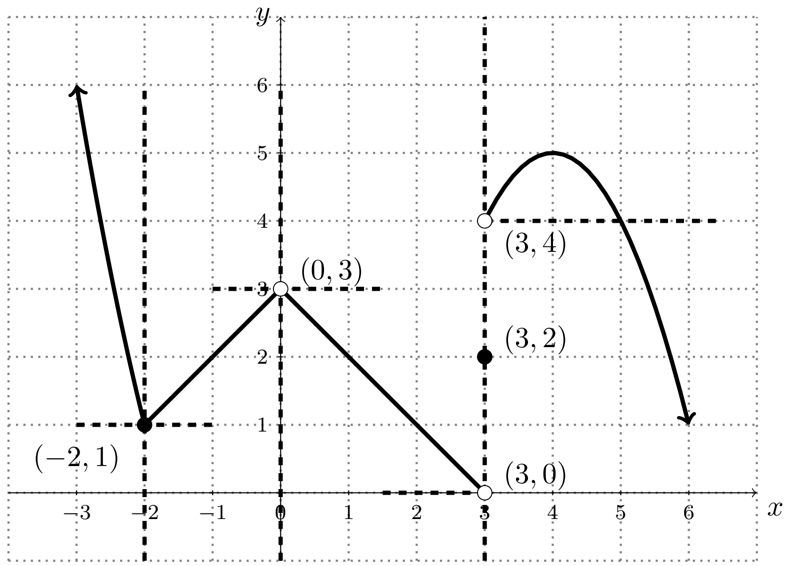
$$f(x) = \begin{cases} x + 1 & \text{if } x < 4 \\ (x - 4)^2 + 3 & \text{if } x \geq 4. \end{cases}$$

We have:

$\lim_{x \to c} f(x)$	$f(c)$
$\lim_{x \to 4} f(x)$ DNE	$f(4) = 2$

Once again we see that $\lim_{x \to c} f(x)$ and $f(c)$ are not the same.

A review of the graph given in Lesson 1 (redrawn below) will emphasize this fact.



We restate the conclusions, adding the respective values of $f(c)$:

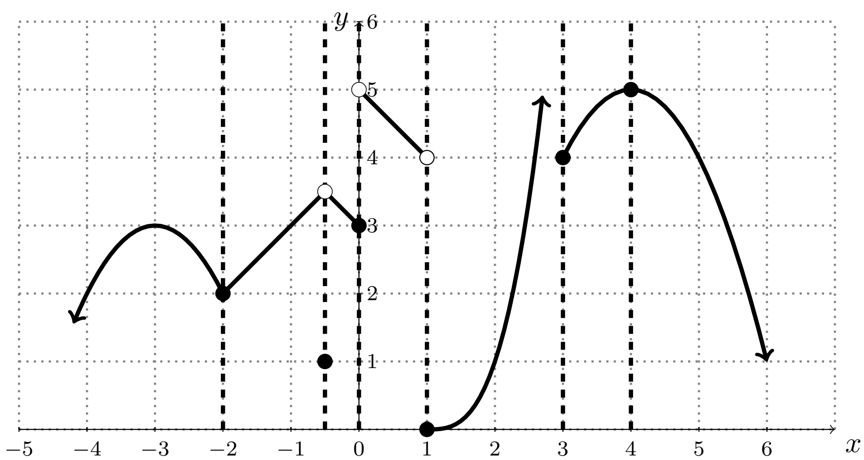
(a) $\lim_{x \to -2} f(x) = 1$ and $f(-2) = 1$.

(b) $\lim_{x \to 0} f(x) = 3$ and $f(0)$ does not exist (or is undefined).

(c) $\lim_{x \to 3} f(x)$ DNE and $f(3)$ also does not exist (or is undefined).

(C) **EXERCISES**

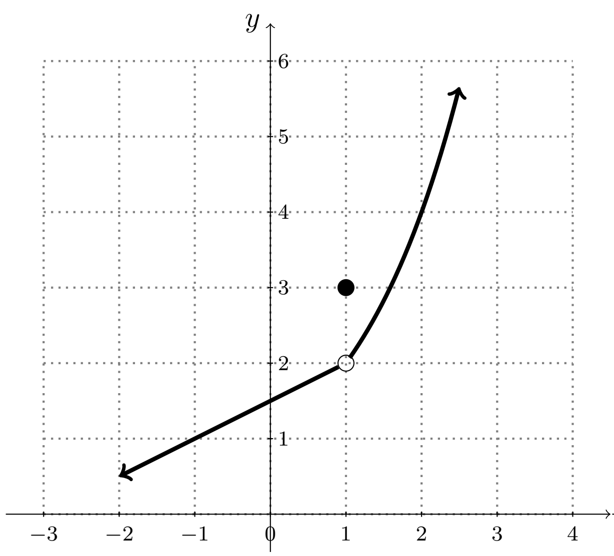
1. Consider the function $f(x)$ whose graph is given below.

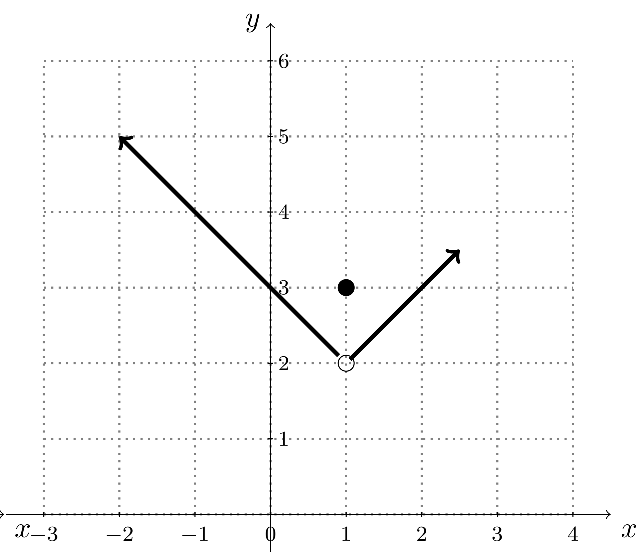


Based on the graph, fill in the table with the appropriate values.

c	$\lim_{x \to c} f(x)$	$f(c)$
-2		
$-1/2$		
0		
1		
3		
4		

2. For each given combination of values of $\lim_{x \to c} f(x)$ and $f(c)$, sketch the graph of a possible function that illustrates the combination. For example, if $\lim_{x \to 1} f(x) = 2$ and $f(1) = 3$, then a possible graph of $f(x)$ near $x = 1$ may be any of the two graphs below.





Do a similar rendition for each of the following combinations:

i. $\lim_{x \to 1} f(x) = 2$ and $f(1) = 2$

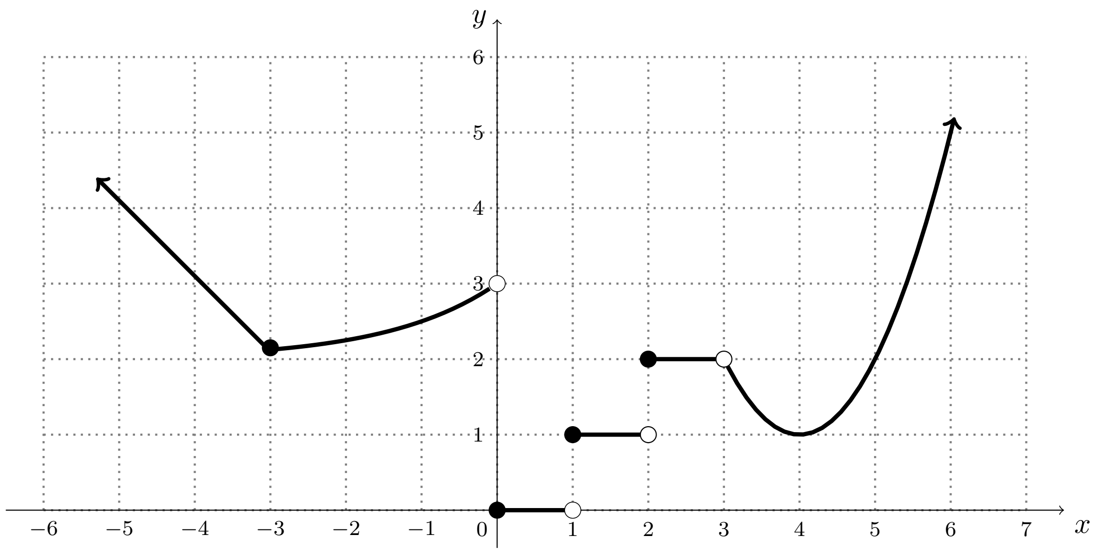
ii. $\lim_{x \to 1} g(x) = -1$ and $g(1) = 1$

iii. $\lim_{x \to 1} h(x)$ DNE and $h(1) = 0$

iv. $\lim_{x \to 1} j(x) = 2$ and $j(1)$ is undefined

v. $\lim_{x \to 1} p(x)$ DNE and $p(1)$ is undefined

3. Consider the function $f(x)$ whose graph is given below.



State whether $\lim_{x \to c} f(x)$ and $f(c)$ are equal or unequal at the given value of c. Also, state whether $\lim_{x \to c} f(x)$ or $f(c)$ does not exist.

i. $c = -3$

ii. $c = -2$

iii. $c = 0$

iv. $c = 0.5$

v. $c = 1$

vi. $c = 2$

vii. $c = 2.3$

viii. $c = 3$

ix. $c = 4$

x. $c = 6$

TOPIC 1.3: Illustration of Limit Theorems

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

Lesson 1 showed us how limits can be determined through either a table of values or the graph of a function. One might ask: Must one always construct a table or graph the function to determine a limit? Filling in a table of values sometimes requires very tedious calculations. Likewise, a graph may be difficult to sketch. However, these should not be reasons for a student to fail to determine a limit.

In this lesson, we will learn how to compute the limit of a function using Limit Theorems.

> **Teaching Tip**
>
> It would be good to recall the parts of Lesson 1 where the students were asked to give the value of a limit, without aid of a table or a graph. Those exercises were intended to lead to the Limit Theorems. These theorems are a formalization of what they had intuitively concluded then.

(B) LESSON PROPER

We are now ready to list down the basic theorems on limits. We will state eight theorems. These will enable us to directly evaluate limits, without need for a table or a graph.

In the following statements, c is a constant, and f and g are functions which may or may not have c in their domains.

1. The limit of a constant is itself. If k is any constant, then,

$$\lim_{x \to c} k = k.$$

   For example,

   i. $\lim_{x \to c} 2 = 2$
   ii. $\lim_{x \to c} -3.14 = -3.14$
   iii. $\lim_{x \to c} 789 = 789$

2. The limit of x as x approaches c is equal to c. This may be thought of as the substitution law, because x is simply substituted by c.

$$\lim_{x\to c} x = c.$$

For example,

i. $\lim\limits_{x\to 9} x = 9$

ii. $\lim\limits_{x\to 0.005} x = 0.005$

iii. $\lim\limits_{x\to -10} x = -10$

For the remaining theorems, we will assume that the limits of f and g both exist as x approaches c and that they are L and M, respectively. In other words,

$$\lim_{x\to c} f(x) = L, \quad \text{and} \quad \lim_{x\to c} g(x) = M.$$

3. <u>The Constant Multiple Theorem</u>: This says that the limit of a multiple of a function is simply that multiple of the limit of the function.

$$\lim_{x\to c} k \cdot f(x) = k \cdot \lim_{x\to c} f(x) = k \cdot L.$$

For example, if $\lim\limits_{x\to c} f(x) = 4$, then

i. $\lim\limits_{x\to c} 8 \cdot f(x) = 8 \cdot \lim\limits_{x\to c} f(x) = 8 \cdot 4 = 32.$

ii. $\lim\limits_{x\to c} -11 \cdot f(x) = -11 \cdot \lim\limits_{x\to c} f(x) = -11 \cdot 4 = -44.$

iii. $\lim\limits_{x\to c} \frac{3}{2} \cdot f(x) = \frac{3}{2} \cdot \lim\limits_{x\to c} f(x) = \frac{3}{2} \cdot 4 = 6.$

4. <u>The Addition Theorem</u>: This says that the limit of a sum of functions is the sum of the limits of the individual functions. Subtraction is also included in this law, that is, the limit of a difference of functions is the difference of their limits.

$$\lim_{x\to c}(\ f(x) + g(x)\ ) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L + M.$$

$$\lim_{x\to c}(\ f(x) - g(x)\ ) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = L - M.$$

For example, if $\lim\limits_{x\to c} f(x) = 4$ and $\lim\limits_{x\to c} g(x) = -5$, then

i. $\lim\limits_{x\to c} (f(x) + g(x)) = \lim\limits_{x\to c} f(x) + \lim\limits_{x\to c} g(x) = 4 + (-5) = -1.$

ii. $\lim\limits_{x\to c} (f(x) - g(x)) = \lim\limits_{x\to c} f(x) - \lim\limits_{x\to c} g(x) = 4 - (-5) = 9.$

5. <u>The Multiplication Theorem</u>: This is similar to the Addition Theorem, with multiplication replacing addition as the operation involved. Thus, the limit of a product of functions is equal to the product of their limits.

$$\lim_{x\to c}(f(x)\cdot g(x)) = \lim_{x\to c} f(x)\cdot \lim_{x\to c} g(x) = L\cdot M.$$

Again, let $\lim_{x\to c} f(x) = 4$ and $\lim_{x\to c} g(x) = -5$. Then

$$\lim_{x\to c} f(x)\cdot g(x) = \lim_{x\to c} f(x)\cdot \lim_{x\to c} g(x) = 4\cdot(-5) = -20.$$

Remark 1: The Addition and Multiplication Theorems may be applied to sums, differences, and products of more than two functions.

Remark 2: The Constant Multiple Theorem is a special case of the Multiplication Theorem. Indeed, in the Multiplication Theorem, if the first function $f(x)$ is replaced by a constant k, the result is the Constant Multiple Theorem.

6. <u>The Division Theorem</u>: This says that the limit of a quotient of functions is equal to the quotient of the limits of the individual functions, provided the denominator limit is not equal to 0.

$$\begin{aligned}\lim_{x\to c}\frac{f(x)}{g(x)} &= \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}\\ &= \frac{L}{M}, \quad \text{provided } M \neq 0.\end{aligned}$$

For example,

i. If $\lim_{x\to c} f(x) = 4$ and $\lim_{x\to c} g(x) = -5$,

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{4}{-5} = -\frac{4}{5}.$$

ii. If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = -5$,

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{0}{-5} = 0.$$

iii. If $\lim\limits_{x \to c} f(x) = 4$ and $\lim\limits_{x \to c} g(x) = 0$, it is not possible to evaluate $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)}$, or we may say that the limit DNE.

7. The Power Theorem: This theorem states that the limit of an integer power p of a function is just that power of the limit of the function.

$$\lim_{x \to c}(f(x))^p = (\lim_{x \to c} f(x))^p = L^p.$$

For example,

i. If $\lim\limits_{x \to c} f(x) = 4$, then

$$\lim_{x \to c}(f(x))^3 = (\lim_{x \to c} f(x))^3 = 4^3 = 64.$$

ii. If $\lim\limits_{x \to c} f(x) = 4$, then

$$\lim_{x \to c}(f(x))^{-2} = (\lim_{x \to c} f(x))^{-2} = 4^{-2} = \frac{1}{4^2} = \frac{1}{16}.$$

8. The Radical/Root Theorem: This theorem states that if n is a positive integer, the limit of the nth root of a function is just the nth root of the limit of the function, provided the nth root of the limit is a real number. Thus, it is important to keep in mind that if n is even, the limit of the function must be positive.

$$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)} = \sqrt[n]{L}.$$

For example,

i. If $\lim\limits_{x \to c} f(x) = 4$, then

$$\lim_{x \to c} \sqrt{f(x)} = \sqrt{\lim_{x \to c} f(x)} = \sqrt{4} = 2.$$

ii. If $\lim\limits_{x \to c} f(x) = -4$, then it is not possible to evaluate $\lim\limits_{x \to c} \sqrt{f(x)}$ because then,

$$\sqrt{\lim_{x \to c} f(x)} = \sqrt{-4},$$

and this is not a real number.

(C) EXERCISES

1. Complete the following table.

c	$\lim_{x\to c} 2016$	$\lim_{x\to c} x$
-2		
$-1/2$		
0		
3.1416		
10		
$\sqrt{3}$		

2. Assume the following:

$$\lim_{x\to c} f(x) = \frac{3}{4}, \quad \lim_{x\to c} g(x) = 12, \quad \text{and} \quad \lim_{x\to c} h(x) = -3.$$

Compute the following limits:

a. $\lim_{x\to c}(-4 \cdot f(x))$

b. $\lim_{x\to c} \sqrt{12 \cdot f(x)}$

c. $\lim_{x\to c}(g(x) - h(x))$

d. $\lim_{x\to c}(f(x) \cdot g(x))$

e. $\lim_{x\to c} \dfrac{g(x) + h(x)}{f(x)}$

f. $\lim_{x\to c} \left(\dfrac{f(x)}{h(x)} \cdot g(x)\right)$

g. $\lim_{x\to c}(4 \cdot f(x) + h(x))$

h. $\lim_{x\to c}(8 \cdot f(x) - g(x) - 2 \cdot h(x))$

i. $\lim_{x\to c}(f(x) \cdot g(x) \cdot h(x))$

j. $\lim_{x\to c} \sqrt{-g(x) \cdot h(x)}$

k. $\lim_{x\to c} \dfrac{g(x)}{(h(x))^2}$

l. $\lim_{x\to c} \dfrac{g(x)}{(h(x))^2} \cdot f(x)$

3. Determine whether the statement is True or False. If it is false, explain what makes it false, or provide a counterexample.

a. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then $\lim_{x\to c}(f(x) \pm g(x))$ always exists.

b. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then $\lim_{x\to c}(f(x) \cdot g(x))$ always exists.

c. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then $\lim_{x\to c} \dfrac{f(x)}{g(x)}$ always exists.

d. If $\lim_{x\to c} f(x)$ exists and p is an integer, then $\lim_{x\to c}(f(x))^p$, where p is an integer, always exists.

e. If $\lim_{x\to c} f(x)$ exists and n is a natural number, then $\lim_{x\to c} \sqrt[n]{f(x)}$, always exists.

⋆f. If $\lim_{x\to c}(f(x) - g(x)) = 0$, then $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ are equal. Answer: False. (Take $f(x) = \dfrac{1}{x} = g(x)$ and $c = 0$.)

⋆g. If $\lim_{x\to c} \dfrac{f(x)}{g(x)} = 1$, then $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ are equal. Answer: False. (Take $f(x) = \dfrac{1}{x} = g(x)$ and $c = 0$.)

4. Assume the following:

$$\lim_{x\to c} f(x) = 1, \quad \lim_{x\to c} g(x) = -1, \quad \text{and} \quad \lim_{x\to c} h(x) = 2.$$

Compute the following limits:

a. $\lim_{x\to c}(f(x) + g(x))$

b. $\lim_{x\to c}(f(x) - g(x) - h(x))$

c. $\lim_{x\to c}(3 \cdot g(x) + 5 \cdot h(x))$

d. $\lim_{x\to c} \sqrt{f(x)}$

e. $\lim_{x\to c} \sqrt{g(x)}$

f. $\lim_{x\to c} \sqrt[3]{g(x)}$

g. $\lim_{x\to c}(h(x))^5$

h. $\lim_{x\to c} \dfrac{g(x) - f(x)}{h(x)}$

i. $\lim_{x\to c} \left(f(x) \cdot g(x) \cdot (h(x))^2\right)$

j. $\lim_{x\to c} \dfrac{1}{f(x)}$

k. $\lim_{x\to c} \dfrac{1}{g(x)}$

l. $\lim_{x\to c} \dfrac{1}{h(x)}$

m. $\lim_{x\to c} \dfrac{1}{f(x) - h(x)}$

n. $\lim_{x\to c} \dfrac{1}{f(x) + g(x)}$

5. Assume $f(x) = x$. Evaluate

a. $\lim_{x\to 4} f(x)$.

b. $\lim_{x\to 4} \dfrac{1}{f(x)}$.

c. $\lim_{x\to 4} \dfrac{1}{(f(x))^2}$.

d. $\lim_{x\to 4} -\sqrt{f(x)}$.

e. $\lim_{x\to 4} \sqrt{9 \cdot f(x)}$.

f. $\lim_{x\to 4}((f(x))^2 - f(x))$.

g. $\lim_{x\to 4}((f(x))^3 + (f(x))^2 + 2 \cdot f(x))$.

h. $\lim_{x\to 4} \sqrt[n]{3 \cdot (f(x))^2 + 4 \cdot f(x)}$.

i. $\lim_{x\to 4} \dfrac{(f(x))^2 - f(x)}{5 \cdot f(x)}$.

j. $\lim_{x\to 4} \dfrac{(f(x))^2 - 4f(x)}{(f(x))^2 + 4f(x)}$.

TOPIC 1.4: Limits of Polynomial, Rational, and Radical Functions

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

In the previous lesson, we presented and illustrated the limit theorems. We start by recalling these limit theorems.

Theorem 1. *Let c, k, L and M be real numbers, and let $f(x)$ and $g(x)$ be functions defined on some open interval containing c, except possibly at c.*

1. *If $\lim_{x\to c} f(x)$ exists, then it is unique. That is, if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} f(x) = M$, then $L = M$.*
2. $\lim_{x\to c} c = c$.
3. $\lim_{x\to c} x = c$
4. *Suppose $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$.*
   i. *(Constant Multiple)* $\lim_{x\to c}[k \cdot g(x)] = k \cdot M$.
   ii. *(Addition)* $\lim_{x\to c}[f(x) \pm g(x)] = L \pm M$.
   iii. *(Multiplication)* $\lim_{x\to c}[f(x)g(x)] = LM$.
   iv. *(Division)* $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, *provided* $M \neq 0$.
   v. *(Power)* $\lim_{x\to c}[f(x)]^p = L^p$ *for p, a positive integer.*
   vi. *(Root/Radical)* $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$ *for positive integers n, and provided that $L > 0$ when n is even.*

> **Teaching Tip**
> It would be helpful for the students if these limit theorems remain written on the board or on manila paper throughout the discussion of this lesson.

In this lesson, we will show how these limit theorems are used in evaluating algebraic functions. Particularly, we will illustrate how to use them to evaluate the limits of polynomial, rational and radical functions.

(B) LESSON PROPER

LIMITS OF ALGEBRAIC FUNCTIONS

We start with evaluating the limits of polynomial functions.

EXAMPLE 1: Determine $\lim\limits_{x\to 1}(2x+1)$.

Solution. From the theorems above,

$$\begin{aligned}
\lim_{x\to 1}(2x+1) &= \lim_{x\to 1} 2x + \lim_{x\to 1} 1 && \text{(Addition)} \\
&= \left(2\lim_{x\to 1} x\right) + 1 && \text{(Constant Multiple)} \\
&= 2(1)+1 && \left(\lim_{x\to c} x = c\right) \\
&= 2+1 \\
&= 3.
\end{aligned}$$

EXAMPLE 2: Determine $\lim\limits_{x\to -1}(2x^3-4x^2+1)$.

Solution. From the theorems above,

$$\begin{aligned}
\lim_{x\to -1}(2x^3-4x^2+1) &= \lim_{x\to -1} 2x^3 - \lim_{x\to -1} 4x^2 + \lim_{x\to -1} 1 && \text{(Addition)} \\
&= 2\lim_{x\to -1} x^3 - 4\lim_{x\to -1} x^2 + 1 && \text{(Constant Multiple)} \\
&= 2(-1)^3 - 4(-1)^2 + 1 && \text{(Power)} \\
&= -2-4+1 \\
&= -5.
\end{aligned}$$

EXAMPLE 3: Evaluate $\lim\limits_{x\to 0}(3x^4-2x-1)$.

Solution. From the theorems above,

$$\begin{aligned}
\lim_{x\to 0}(3x^4-2x-1) &= \lim_{x\to 0} 3x^4 - \lim_{x\to 0} 2x - \lim_{x\to 0} 1 && \text{(Addition)} \\
&= 3\lim_{x\to 0} x^4 - 2\lim_{x\to 0} x^2 - 1 && \text{(Constant Multiple)} \\
&= 3(0)^4 - 2(0) - 1 && \text{(Power)} \\
&= 0-0-1 \\
&= -1.
\end{aligned}$$

We will now apply the limit theorems in evaluating rational functions. In evaluating the limits of such functions, recall from Theorem 1 the Division Rule, and all the rules stated in Theorem 1 which have been useful in evaluating limits of polynomial functions, such as the Addition and Product Rules.

EXAMPLE 4: Evaluate $\lim\limits_{x\to 1}\dfrac{1}{x}$.

Solution. First, note that $\lim\limits_{x\to 1} x = 1$. Since the limit of the denominator is nonzero, we can apply the Division Rule. Thus,

$$\begin{aligned}\lim_{x\to 1}\frac{1}{x} &= \frac{\lim\limits_{x\to 1} 1}{\lim\limits_{x\to 1} x} \qquad \text{(Division)}\\ &= \frac{1}{1}\\ &= 1.\end{aligned}$$

EXAMPLE 5: Evaluate $\lim\limits_{x\to 2}\dfrac{x}{x+1}$.

Solution. We start by checking the limit of the polynomial function in the denominator.

$$\lim_{x\to 2}(x+1) = \lim_{x\to 2} x + \lim_{x\to 2} 1 = 2+1 = 3.$$

Since the limit of the denominator is not zero, it follows that

$$\lim_{x\to 2}\frac{x}{x+1} = \frac{\lim\limits_{x\to 2} x}{\lim\limits_{x\to 2}(x+1)} = \frac{2}{3} \qquad \text{(Division)}$$

EXAMPLE 6: Evaluate $\lim\limits_{x\to 1}\dfrac{(x-3)(x^2-2)}{x^2+1}$. First, note that

$$\lim_{x\to 1}(x^2+1) = \lim_{x\to 1} x^2 + \lim_{x\to 1} 1 = 1+1 = 2 \neq 0.$$

Thus, using the theorem,

$$\begin{aligned}\lim_{x\to 1}\frac{(x-3)(x^2-2)}{x^2+1} &= \frac{\lim\limits_{x\to 1}(x-3)(x^2-2)}{\lim\limits_{x\to 1}(x^2+1)} && \text{(Division)}\\ &= \frac{\lim\limits_{x\to 1}(x-3)\cdot\lim\limits_{x\to 1}(x^2-2)}{2} && \text{(Multipication)}\\ &= \frac{\left(\lim\limits_{x\to 1} x - \lim\limits_{x\to 1} 3\right)\left(\lim\limits_{x\to 1} x^2 - \lim\limits_{x\to 1} 2\right)}{2} && \text{(Addition)}\\ &= \frac{(1-3)(1^2-2)}{2}\\ &= 1.\end{aligned}$$

Theorem 2. *Let f be a polynomial of the form*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0.$$

If c is a real number, then

$$\lim_{x \to c} f(x) = f(c).$$

Proof. Let c be any real number. Remember that a polynomial is defined at any real number. So,

$$f(c) = a_n c^n + a_{n-1}c^{n-1} + a_{n-2}c^{n-2} + ... + a_1 c + a_0.$$

Now apply the limit theorems in evaluating $\lim\limits_{x \to c} f(x)$:

$$\begin{aligned}
\lim_{x \to c} f(x) &= \lim_{x \to c}(a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0) \\
&= \lim_{x \to c} a_n x^n + \lim_{x \to c} a_{n-1}x^{n-1} + \lim_{x \to c} a_{n-2}x^{n-2} + ... + \lim_{x \to c} a_1 x + \lim_{x \to c} a_0 \\
&= a_n \lim_{x \to c} x^n + a_{n-1} \lim_{x \to c} x^{n-1} + a_{n-2} \lim_{x \to c} x^{n-2} + ... + a_1 \lim_{x \to c} x + a_0 \\
&= a_n c^n + a_{n-1}c^{n-1} + a_{n-2}c^{n-2} + ... + a_1 c + a_0 \\
&= f(c).
\end{aligned}$$

Therefore, $\lim\limits_{x \to c} f(x) = f(c)$. □

EXAMPLE 7: Evaluate $\lim\limits_{x \to -1}(2x^3 - 4x^2 + 1)$.

Solution. Note first that our function

$$f(x) = 2x^3 - 4x^2 + 1,$$

is a polynomial. Computing for the value of f at $x = -1$, we get

$$f(-1) = 2(-1)^3 - 4(-1)^2 + 1 = 2(-1) - 4(1) + 1 = -5.$$

Therefore, from Theorem 2,

$$\lim_{x \to -1}(2x^3 - 4x^2 + 1) = f(-1) = -5.$$

Note that we get the same answer when we use limit theorems.

Theorem 3. *Let h be a rational function of the form $h(x) = \dfrac{f(x)}{g(x)}$ where f and g are polynomial functions. If c is a real number and $g(c) \neq 0$, then*

$$\lim_{x \to c} h(x) = \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}.$$

Proof. From Theorem 2, $\lim_{x\to c} g(x) = g(c)$, which is nonzero by assumption. Moreover, $\lim_{x\to c} f(x) = f(c)$. Therefore, by the Division Rule of Theorem 1,

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{f(c)}{g(c)}.$$

□

EXAMPLE 8: Evaluate $\lim_{x\to 1}\dfrac{1-5x}{1+3x^2+4x^4}$.

Solution. Since the denominator is not zero when evaluated at $x = 1$, we may apply Theorem 3:

$$\lim_{x\to 1}\frac{1-5x}{1+3x^2+4x^4} = \frac{1-5(1)}{1+3(1)^2+4(1)^4} = \frac{-4}{8} = -\frac{1}{2}.$$

We will now evaluate limits of radical functions using limit theorems.

EXAMPLE 9: Evaluate $\lim_{x\to 1}\sqrt{x}$.

Solution. Note that $\lim_{x\to 1} x = 1 > 0$. Therefore, by the Radical/Root Rule,

$$\lim_{x\to 1}\sqrt{x} = \sqrt{\lim_{x\to 1} x} = \sqrt{1} = 1.$$

EXAMPLE 10: Evaluate $\lim_{x\to 0}\sqrt{x+4}$.

Solution. Note that $\lim_{x\to 0}(x+4) = 4 > 0$. Hence, by the Radical/Root Rule,

$$\lim_{x\to 0}\sqrt{x+4} = \sqrt{\lim_{x\to 0}(x+4)} = \sqrt{4} = 2.$$

EXAMPLE 11: Evaluate $\lim_{x\to -2}\sqrt[3]{x^2+3x-6}$.

Solution. Since the index of the radical sign is odd, we do not have to worry that the limit of the radicand is negative. Therefore, the Radical/Root Rule implies that

$$\lim_{x\to -2}\sqrt[3]{x^2+3x-6} = \sqrt[3]{\lim_{x\to -2}(x^2+3x-6)} = \sqrt[3]{4-6-6} = \sqrt[3]{-8} = -2.$$

EXAMPLE 12: Evaluate $\lim_{x\to 2} \dfrac{\sqrt{2x+5}}{1-3x}$.

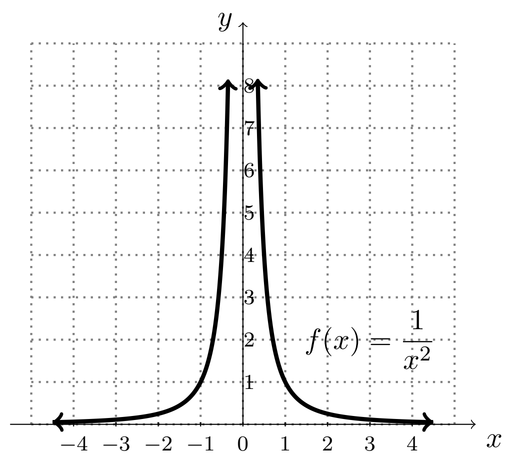
Solution. First, note that $\lim_{x\to 2}(1-3x) = -5 \neq 0$. Moreover, $\lim_{x\to 2}(2x+5) = 9 > 0$. Thus, using the Division and Radical Rules of Theorem 1, we obtain

$$\lim_{x\to 2} \frac{\sqrt{2x+5}}{1-3x} = \frac{\lim_{x\to 2} \sqrt{2x+5}}{\lim_{x\to 2} 1-3x} = \frac{\sqrt{\lim_{x\to 2}(2x+5)}}{-5} = \frac{\sqrt{9}}{-5} = -\frac{3}{5}.$$

INTUITIVE NOTIONS OF INFINITE LIMITS

We investigate the limit at a point c of a rational function of the form $\dfrac{f(x)}{g(x)}$ where f and g are polynomial functions with $f(c) \neq 0$ and $g(c) = 0$. Note that Theorem 3 does not cover this because it assumes that the denominator is nonzero at c.

Now, consider the function $f(x) = \dfrac{1}{x^2}$. Note that the function is not defined at $x = 0$ but we can check the behavior of the function as x approaches 0 intuitively. We first consider approaching 0 from the left.



x	$f(x)$
-0.9	1.2345679
-0.5	4
-0.1	100
-0.01	$10,000$
-0.001	$1,000,000$
-0.0001	$100,000,000$

Observe that as x approaches 0 from the left, the value of the function increases without bound. When this happens, we say that the limit of $f(x)$ as x approaches 0 from the left is *positive infinity*, that is,

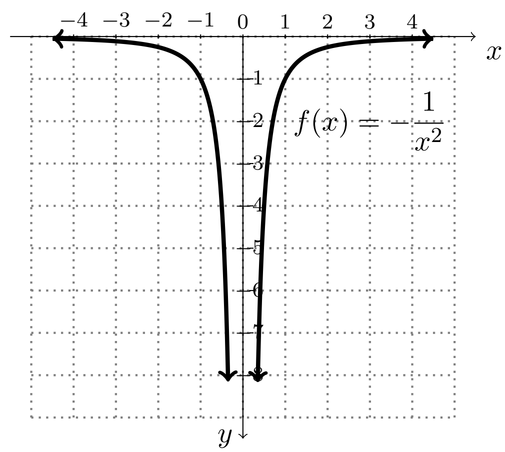
$$\lim_{x\to 0^-} f(x) = +\infty.$$

x	$f(x)$
0.9	1.2345679
0.5	4
0.1	100
0.01	$10,000$
0.001	$1,000,000$
0.0001	$100,000,000$

Again, as x approaches 0 from the right, the value of the function increases without bound, so, $\lim_{x\to 0^+} f(x) = +\infty$.

Since $\lim_{x\to 0^-} f(x) = +\infty$ and $\lim_{x\to 0^+} f(x) = +\infty$, we may conclude that $\lim_{x\to 0} f(x) = +\infty$.

Now, consider the function $f(x) = -\dfrac{1}{x^2}$. Note that the function is not defined at $x = 0$ but we can still check the behavior of the function as x approaches 0 intuitively. We first consider approaching 0 from the left.



x	$f(x)$
-0.9	-1.2345679
-0.5	-4
-0.1	-100
-0.01	$-10,000$
-0.001	$-1,000,000$
-0.0001	$-100,000,000$

This time, as x approaches 0 from the left, the value of the function decreases without bound. So, we say that the limit of $f(x)$ as x approaches 0 from the left is negative infinity, that is,

$$\lim_{x\to 0^-} f(x) = -\infty.$$

x	$f(x)$
0.9	-1.2345679
0.5	-4
0.1	-100
0.01	$-10,000$
0.001	$-1,000,000$
0.0001	$-100,000,000$

As x approaches 0 from the right, the value of the function also decreases without bound, that is, $\lim_{x\to 0^+} f(x) = -\infty$.

Since $\lim_{x\to 0^-} f(x) = -\infty$ and $\lim_{x\to 0^+} f(x) = -\infty$, we are able to conclude that $\lim_{x\to 0} f(x) = -\infty$.

We now state the intuitive definition of *infinite limits* of functions:

The limit of $f(x)$ as x approaches c is positive infinity, denoted by,

$$\lim_{x\to c} f(x) = +\infty$$

if the value of $f(x)$ increases without bound whenever the values of x get closer and closer to c. The limit of $f(x)$ as x approaches c is negative infinity, denoted by,

$$\lim_{x\to c} f(x) = -\infty$$

if the value of $f(x)$ decreases without bound whenever the values of x get closer and closer to c.

Let us consider $f(x) = \dfrac{1}{x}$. The graph on the right suggests that

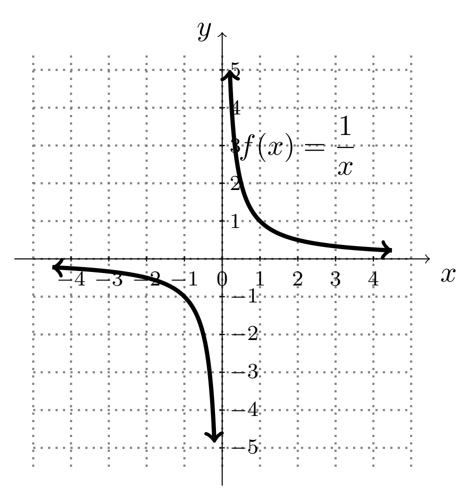
$$\lim_{x\to 0^-} f(x) = -\infty$$

while

$$\lim_{x\to 0^+} f(x) = +\infty.$$

Because the one-sided limits are not the same, we say that

$$\lim_{x\to 0} f(x) \text{ DNE}.$$



Remark 1: Remember that ∞ is NOT a number. It holds no specific value. So, $\lim_{x\to c} f(x) = +\infty$ or $\lim_{x\to c} f(x) = -\infty$ describes the behavior of the function near $x = c$, but it does not exist as a real number.

Remark 2: Whenever $\lim\limits_{x \to c^+} f(x) = \pm\infty$ or $\lim\limits_{x \to c^-} f(x) = \pm\infty$, we normally see the dashed vertical line $x = c$. This is to indicate that the graph of $y = f(x)$ is *asymptotic* to $x = c$, meaning, the graphs of $y = f(x)$ and $x = c$ are very close to each other near c. In this case, we call $x = c$ a ***vertical asymptote*** of the graph of $y = f(x)$.

Teaching Tip

Computing infinite limits is not a learning objective of this course, however, we will be needing this notion for the discussion on infinite essential discontinuity, which will be presented in Topic 4.1. It is enough that the student determines that the limit at the point c is $+\infty$ or $-\infty$ from the behavior of the graph, or the trend of the y-coordinates in a table of values.

(C) **EXERCISES**

I. Evaluate the following limits.

1. $\lim\limits_{w \to 1} (1 + \sqrt[3]{w})(2 - w^2 + 3w^3)$
2. $\lim\limits_{t \to -2} \dfrac{t^2 - 1}{t^2 + 3t - 1}$
3. $\lim\limits_{z \to 2} \left(\dfrac{2z + z^2}{z^2 + 4}\right)^3$
4. $\lim\limits_{x \to 0} \dfrac{x^2 - x - 2}{x^3 - 6x^2 - 7x + 1}$
5. $\lim\limits_{y \to -2} \dfrac{4 - 3y^2 - y^3}{6 - y - y^2}$
6. $\lim\limits_{x \to -1} \dfrac{x^3 - 7x^2 + 14x - 8}{2x^2 - 3x - 4}$
7. $\lim\limits_{x \to -1} \dfrac{\sqrt{x^2 + 3} - 2}{x^2 + 1}$
8. $\lim\limits_{x \to 2} \dfrac{\sqrt{2x} - \sqrt{6 - x}}{4 + x^2}$

II. Complete the following tables.

x	$\dfrac{x-5}{x-3}$	$\dfrac{x}{x^2-6x+9}$
2.5		
2.8		
2.9		
2.99		
2.999		
2.9999		

x	$\dfrac{x-5}{x-3}$	$\dfrac{x}{x^2-6x+9}$
3.5		
3.2		
3.1		
3.01		
3.001		
3.0001		

From the table, determine the following limits.

1. $\lim\limits_{x \to 3^-} \dfrac{x - 5}{x - 3}$
2. $\lim\limits_{x \to 3^+} \dfrac{x - 5}{x - 3}$
3. $\lim\limits_{x \to 3} \dfrac{x - 5}{x - 3}$
4. $\lim\limits_{x \to 3^-} \dfrac{x}{x^2 - 6x + 9}$
5. $\lim\limits_{x \to 3^+} \dfrac{x}{x^2 - 6x + 9}$
6. $\lim\limits_{x \to 3} \dfrac{x}{x^2 - 6x + 9}$

III. Recall the graph of $y = \csc x$. From the behavior of the graph of the cosecant function, determine if the following limits evaluate to $+\infty$ or to $-\infty$.

1. $\lim\limits_{x\to 0^-} \csc x$
2. $\lim\limits_{x\to 0^+} \csc x$
3. $\lim\limits_{x\to \pi^-} \csc x$
4. $\lim\limits_{x\to \pi^+} \csc x$

IV. Recall the graph of $y = \tan x$.

1. Find the value of $c \in (0, \pi)$ such that $\lim\limits_{x\to c^-} \tan x = +\infty$.
2. Find the value of $d \in (\pi, 2\pi)$ such that $\lim\limits_{x\to d^+} \tan x = -\infty$.

LESSON 2: Limits of Some Transcendental Functions and Some Indeterminate Forms

TIME FRAME: 4 hours

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Compute the limits of exponential, logarithmic, and trigonometric functions using tables of values and graphs of the functions;
2. Evaluate the limits of expressions involving $\frac{\sin t}{t}$, $\frac{1-\cos t}{t}$, and $\frac{e^t-1}{t}$ using tables of values; and
3. Evaluate the limits of expressions resulting in the indeterminate form $\text{“}\frac{0}{0}\text{”}$.

LESSON OUTLINE:

1. Exponential functions
2. Logarithmic functions
3. Trigonometric functions
4. Evaluating $\lim_{t\to 0} \frac{\sin t}{t}$
5. Evaluating $\lim_{t\to 0} \frac{1-\cos t}{t}$
6. Evaluating $\lim_{t\to 0} \frac{e^t-1}{t}$
7. Indeterminate form $\text{“}\frac{0}{0}\text{”}$

TOPIC 2.1: Limits of Exponential, Logarithmic, and Trigonometric Functions

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

Real-world situations can be expressed in terms of functional relationships. These functional relationships are called mathematical models. In applications of calculus, it is quite important that one can generate these mathematical models. They sometimes use functions that you encountered in precalculus, like the exponential, logarithmic, and trigonometric functions. Hence, we start this lesson by recalling these functions and their corresponding graphs.

(a) If $b > 0$, $b \neq 1$, the *exponential function with base* b is defined by

$$f(x) = b^x, \; x \in \mathbb{R}.$$

(b) Let $b > 0, b \neq 1$. If $b^y = x$ then y is called the *logarithm of* x *to the base* b, denoted $y = \log_b x$.

> **Teaching Tip**
> Allow students to use their calculators.

(B) LESSON PROPER

EVALUATING LIMITS OF EXPONENTIAL FUNCTIONS

First, we consider the natural exponential function $f(x) = e^x$, where e is called the *Euler number*, and has value 2.718281....

EXAMPLE 1: Evaluate the $\lim\limits_{x \to 0} e^x$.

Solution. We will construct the table of values for $f(x) = e^x$. We start by approaching the number 0 from the left or through the values less than but close to 0.

> **Teaching Tip**
> Some students may not be familiar with the natural number e on their scientific calculators. Demonstrate to them how to properly input powers of e on their calculators .

x	$f(x)$
-1	0.36787944117
-0.5	0.60653065971
-0.1	0.90483741803
-0.01	0.99004983374
-0.001	0.99900049983
-0.0001	0.999900049983
-0.00001	0.99999000005

Intuitively, from the table above, $\lim_{x\to 0^-} e^x = 1$. Now we consider approaching 0 from its right or through values greater than but close to 0.

x	$f(x)$
1	2.71828182846
0.5	1.6487212707
0.1	1.10517091808
0.01	1.01005016708
0.001	1.00100050017
0.0001	1.000100005
0.00001	1.00001000005

From the table, as the values of x get closer and closer to 0, the values of $f(x)$ get closer and closer to 1. So, $\lim_{x\to 0^+} e^x = 1$. Combining the two one-sided limits allows us to conclude that

$$\lim_{x\to 0} e^x = 1.$$

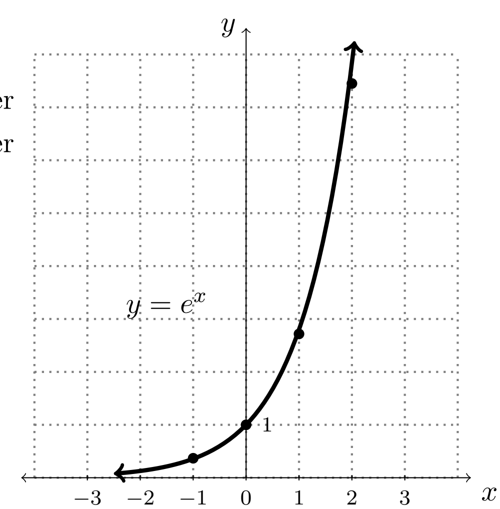
We can use the graph of $f(x) = e^x$ to determine its limit as x approaches 0. The figure below is the graph of $f(x) = e^x$.

Looking at Figure 1.1, as the values of x approach 0, either from the right or the left, the values of $f(x)$ will get closer and closer to 1. We also have the following:

(a) $\lim_{x\to 1} e^x = e = 2.718...$

(b) $\lim_{x\to 2} e^x = e^2 = 7.389...$

(c) $\lim_{x\to -1} e^x = e^{-1} = 0.367...$



EVALUATING LIMITS OF LOGARITHMIC FUNCTIONS

Now, consider the natural logarithmic function $f(x) = \ln x$. Recall that $\ln x = \log_e x$. Moreover, it is the inverse of the natural exponential function $y = e^x$.

EXAMPLE 2: Evaluate $\lim_{x \to 1} \ln x$.

Solution. We will construct the table of values for $f(x) = \ln x$. We first approach the number 1 from the left or through values less than but close to 1.

x	$f(x)$
0.1	-2.30258509299
0.5	-0.69314718056
0.9	-0.10536051565
0.99	-0.01005033585
0.999	-0.00100050033
0.9999	-0.000100005
0.99999	-0.00001000005

Intuitively, $\lim_{x \to 1^-} \ln x = 0$. Now we consider approaching 1 from its right or through values greater than but close to 1.

x	$f(x)$
2	0.69314718056
1.5	0.4054651081
1.1	0.0953101798
1.01	0.00995033085
1.001	0.00099950033
1.0001	0.000099995
1.00001	0.00000999995

Intuitively, $\lim_{x \to 1^+} \ln x = 0$. As the values of x get closer and closer to 1, the values of $f(x)$ get closer and closer to 0. In symbols,

$$\lim_{x \to 1} \ln x = 0.$$

We now consider the *common logarithmic function* $f(x) = \log_{10} x$. Recall that $f(x) = \log_{10} x = \log x$.

EXAMPLE 3: Evaluate $\lim_{x \to 1} \log x$.

Solution. We will construct the table of values for $f(x) = \log x$. We first approach the number 1 from the left or through the values less than but close to 1.

x	$f(x)$
0.1	-1
0.5	-0.30102999566
0.9	-0.04575749056
0.99	-0.0043648054
0.999	-0.00043451177
0.9999	-0.00004343161
0.99999	-0.00000434296

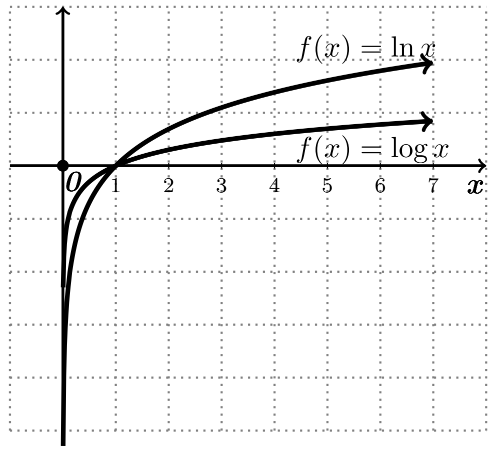
Now we consider approaching 1 from its right or through values greater than but close to 1.

x	$f(x)$
2	0.30102999566
1.5	0.17609125905
1.1	0.04139268515
1.01	0.00432137378
1.001	0.00043407747
1.0001	0.00004342727
1.00001	0.00000434292

As the values of x get closer and closer to 1, the values of $f(x)$ get closer and closer to 0. In symbols,

$$\lim_{x \to 1} \log x = 0.$$

Consider now the graphs of both the natural and common logarithmic functions. We can use the following graphs to determine their limits as x approaches 1..



The figure helps verify our observations that $\lim_{x\to 1} \ln x = 0$ and $\lim_{x\to 1} \log x = 0$. Also, based on the figure, we have

(a) $\lim_{x\to e} \ln x = 1$

(b) $\lim_{x\to 10} \log x = 1$

(c) $\lim_{x\to 3} \ln x = \ln 3 = 1.09...$

(d) $\lim_{x\to 3} \log x = \log 3 = 0.47...$

(e) $\lim_{x\to 0^+} \ln x = -\infty$

(f) $\lim_{x\to 0^+} \log x = -\infty$

TRIGONOMETRIC FUNCTIONS

EXAMPLE 4: Evaluate $\lim_{x\to 0} \sin x$.

Solution. We will construct the table of values for $f(x) = \sin x$. We first approach 0 from the left or through the values less than but close to 0.

x	$f(x)$
-1	-0.8414709848
-0.5	-0.4794255386
-0.1	-0.09983341664
-0.01	-0.00999983333
-0.001	-0.00099999983
-0.0001	-0.00009999999
-0.00001	-0.00000999999

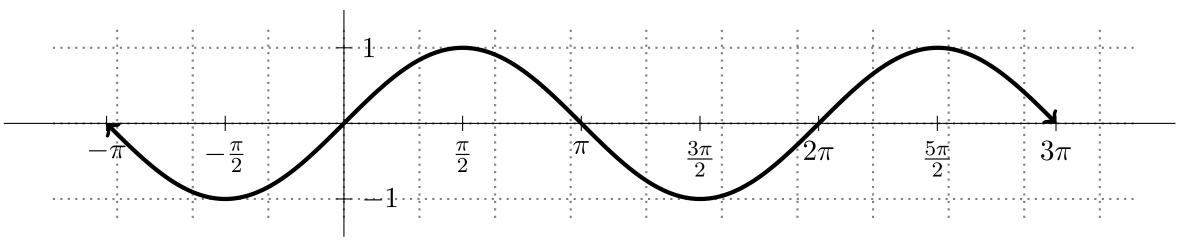
Now we consider approaching 0 from its right or through values greater than but close to 0.

x	$f(x)$
1	0.8414709848
0.5	0.4794255386
0.1	0.09983341664
0.01	0.00999983333
0.001	0.00099999983
0.0001	0.00009999999
0.00001	0.00000999999

As the values of x get closer and closer to 1, the values of $f(x)$ get closer and closer to 0. In symbols,

$$\lim_{x \to 0} \sin x = 0.$$

We can also find $\lim_{x \to 0} \sin x$ by using the graph of the sine function. Consider the graph of $f(x) = \sin x$.



The graph validates our observation in Example 4 that $\lim_{x \to 0} \sin x = 0$. Also, using the graph, we have the following:

(a) $\lim_{x \to \frac{\pi}{2}} \sin x = 1.$

(b) $\lim_{x \to \pi} \sin x = 0.$

(c) $\lim_{x \to -\frac{\pi}{2}} \sin x = -1.$

(d) $\lim_{x \to -\pi} \sin x = 0.$

Teaching Tip

Ask the students what they have observed about the limit of the functions above and their functional value at a point. Lead them to the fact that if f is either exponential, logarithmic or trigonometric, and if c is a real number which is in the domain of f, then

$$\lim_{x \to c} f(x) = f(c).$$

This property is also shared by polynomials and rational functions, as discussed in Topic 1.4.

(C) **EXERCISES**

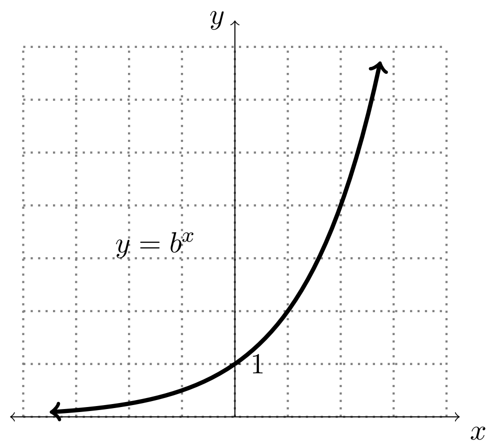
I. Evaluate the following limits by constructing the table of values.

1. $\lim_{x\to 1} 3^x$
2. $\lim_{x\to 2} 5^x$
3. $\lim_{x\to 4} \log x$
4. $\lim_{x\to 0} \cos x$
5. $\lim_{x\to 0} \tan x$

$\star$6. $\lim_{x\to \pi} \cos x$ Answer: -1

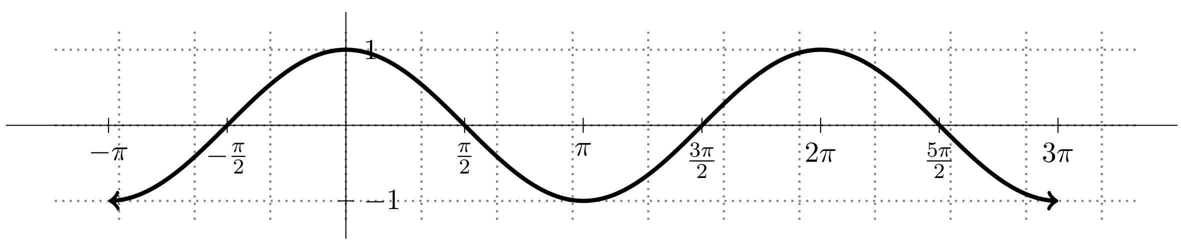
$\star$7. $\lim_{x\to \pi} \sin x$ Answer: 0

II. Given the graph below, evaluate the following limits:



1. $\lim_{x\to 0} b^x$
2. $\lim_{x\to 1.2} b^x$
3. $\lim_{x\to -1} b^x$

III. Given the graph of the cosine function $f(x) = \cos x$, evaluate the following limits:



1. $\lim_{x\to 0} \cos x$
2. $\lim_{x\to \pi} \cos x$
3. $\lim_{x\to \frac{\pi}{2}} \cos x$

TOPIC 2.2: Some Special Limits

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

We will determine the limits of three special functions; namely, $f(t) = \dfrac{\sin t}{t}$, $g(t) = \dfrac{1-\cos t}{t}$, and $h(t) = \dfrac{e^t - 1}{t}$. These functions will be vital to the computation of the derivatives of the sine, cosine, and natural exponential functions in Chapter 2.

(B) LESSON PROPER

THREE SPECIAL FUNCTIONS

We start by evaluating the function $f(t) = \dfrac{\sin t}{t}$.

EXAMPLE 1: Evaluate $\displaystyle\lim_{t\to 0} \frac{\sin t}{t}$.

Solution. We will construct the table of values for $f(t) = \dfrac{\sin t}{t}$. We first approach the number 0 from the left or through values less than but close to 0.

t	$f(t)$
-1	0.84147099848
-0.5	0.9588510772
-0.1	0.9983341665
-0.01	0.9999833334
-0.001	0.9999998333
-0.0001	0.99999999983

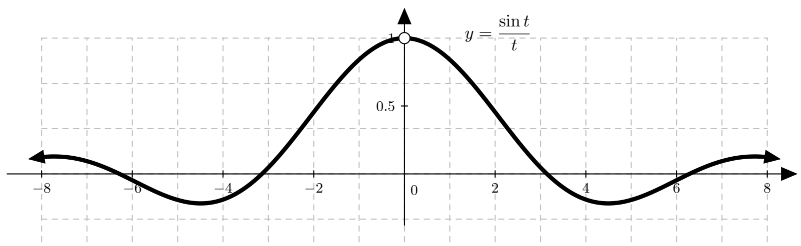
Now we consider approaching 0 from the right or through values greater than but close to 0.

t	$f(t)$
1	0.8414709848
0.5	0.9588510772
0.1	0.9983341665
0.01	0.9999833334
0.001	0.9999998333
0.0001	0.9999999983

Since $\lim_{t\to 0^-} \dfrac{\sin t}{t}$ and $\lim_{t\to 0^+} \dfrac{\sin t}{t}$ are both equal to 1, we conclude that

$$\lim_{t\to 0} \frac{\sin t}{t} = 1.$$

The graph of $f(t) = \dfrac{\sin t}{t}$ below confirms that the y-values approach 1 as t approaches 0.



Now, consider the function $g(t) = \dfrac{1-\cos t}{t}$.

EXAMPLE 2: Evaluate $\lim_{t\to 0} \dfrac{1-\cos t}{t}$.

Solution. We will construct the table of values for $g(t) = \dfrac{1-\cos t}{t}$. We first approach the number 1 from the left or through the values less than but close to 0.

t	$g(t)$
-1	-0.4596976941
-0.5	-0.2448348762
-0.1	-0.04995834722
-0.01	-0.0049999583
-0.001	-0.0004999999
-0.0001	-0.000005

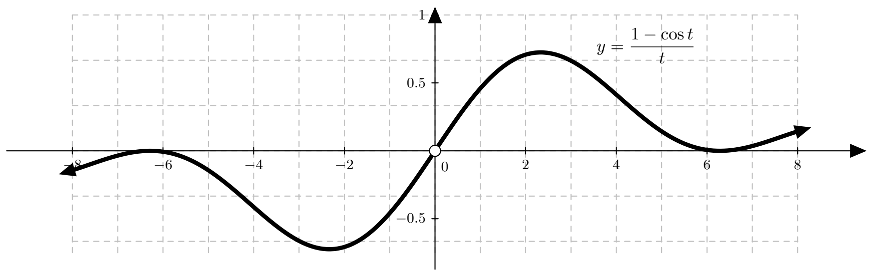
Now we consider approaching 0 from the right or through values greater than but close to 0.

t	$g(t)$
1	0.4596976941
0.5	0.2448348762
0.1	0.04995834722
0.01	0.0049999583
0.001	0.0004999999
0.0001	0.000005

Since $\lim_{t \to 0^-} \frac{1-\cos t}{t} = 0$ and $\lim_{t \to 0^+} \frac{1-\cos t}{t} = 0$, we conclude that

$$\lim_{t \to 0} \frac{1-\cos t}{t} = 0.$$

Below is the graph of $g(t) = \frac{1-\cos t}{t}$. We see that the y-values approach 0 as t tends to 0.



We now consider the special function $h(t) = \frac{e^t - 1}{t}$.

EXAMPLE 3: Evaluate $\lim_{t \to 0} \frac{e^t - 1}{t}$.

Solution. We will construct the table of values for $h(t) = \frac{e^t - 1}{t}$. We first approach the number 0 from the left or through the values less than but close to 0.

t	$h(t)$
-1	0.6321205588
-0.5	0.7869386806
-0.1	0.9516258196
-0.01	0.9950166251
-0.001	0.9995001666
-0.0001	0.9999500016

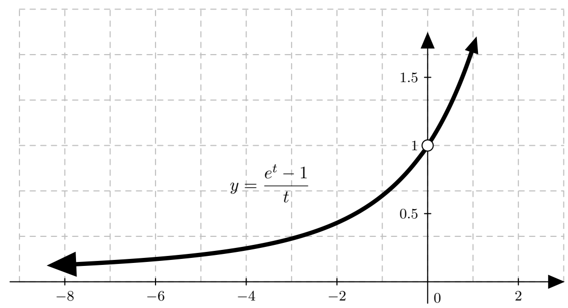
Now we consider approaching 0 from the right or through values greater than but close to 0.

t	$h(t)$
1	1.718281828
0.5	1.297442541
0.1	1.051709181
0.01	1.005016708
0.001	1.000500167
0.0001	1.000050002

Since $\lim_{x \to 0^-} \dfrac{e^t - 1}{t} = 1$ and $\lim_{x \to 0^+} \dfrac{e^t - 1}{t} = 1$, we conclude that

$$\lim_{x \to 0} \frac{e^t - 1}{t} = 1.$$

The graph of $h(t) = \dfrac{e^t - 1}{t}$ below confirms that $\lim_{t \to 0} h(t) = 1$.



INDETERMINATE FORM $\text{“}\dfrac{0}{0}\text{”}$

There are functions whose limits cannot be determined immediately using the Limit Theorems we have so far. In these cases, the functions must be manipulated so that the limit, if it exists, can be calculated. We call such limit expressions *indeterminate forms*.

In this lesson, we will define a particular indeterminate form, $\text{“}\dfrac{0}{0}\text{”}$, and discuss how to evaluate a limit which will initially result in this form.

Definition of Indeterminate Form of Type $\text{“}\dfrac{0}{0}\text{”}$

If $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$, then $\lim_{x \to c} \dfrac{f(x)}{g(x)}$ is called an **indeterminate form of type** $\text{“}\dfrac{0}{0}\text{”}$.

Remark 1: A limit that is indeterminate of type $\text{“}\dfrac{0}{0}\text{”}$ may exist. To find the actual value, one should find an expression equivalent to the original. This is commonly done by factoring or by rationalizing. Hopefully, the expression that will emerge after factoring or rationalizing will have a computable limit.

EXAMPLE 4: Evaluate $\lim_{x \to -1} \dfrac{x^2 + 2x + 1}{x + 1}$.

Solution. The limit of both the numerator and the denominator as x approaches -1 is 0. Thus, this limit as currently written is an indeterminate form of type $\dfrac{0}{0}$. However, observe that $(x + 1)$ is a factor common to the numerator and the denominator, and

$$\frac{x^2 + 2x + 1}{x + 1} = \frac{(x + 1)^2}{x + 1} = x + 1, \text{ when } x \neq -1.$$

Therefore,

$$\lim_{x \to -1} \frac{x^2 + 2x + 1}{x + 1} = \lim_{x \to -1} (x + 1) = 0.$$

EXAMPLE 5: Evaluate $\displaystyle\lim_{x \to 1} \frac{x^2 - 1}{\sqrt{x} - 1}$.

Solution. Since $\displaystyle\lim_{x \to 1} x^2 - 1 = 0$ and $\displaystyle\lim_{x \to 1} \sqrt{x} - 1 = 0$, then $\displaystyle\lim_{x \to 1} \frac{x^2 - 1}{\sqrt{x} - 1}$ is an indeterminate form of type $\frac{\text{“}0\text{”}}{0}$. To find the limit, observe that if $x \neq 1$, then

$$\frac{x^2 - 1}{\sqrt{x} - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} = \frac{(x - 1)(x + 1)(\sqrt{x} + 1)}{x - 1} = (x + 1)(\sqrt{x} + 1).$$

So, we have

$$\lim_{x \to 1} \frac{x^2 - 1}{\sqrt{x} + 1} = \lim_{x \to 1} (x + 1)(\sqrt{x} + 1) = 4.$$

> **Teaching Tip**
>
> In solutions of evaluating limits, it is a common mistake among students to forget to write the "*lim*" operator. They will write
>
> $$\lim_{x \to 1} \frac{x^2 - 1}{\sqrt{x} + 1} = (x + 1)(\sqrt{x} + 1) = 4,$$
>
> instead of always writing the limit operator until such time that they are already substituting the value $x = 1$. Of course, mathematically, the equation above does not make sense since $(x + 1)(\sqrt{x} + 1)$ is not always equal to 4. Please stress the importance of the "*lim*" operator.

Remark 2: We note here that the three limits discussed in Part 1 of this section,

$$\lim_{t \to 0} \frac{\sin t}{t}, \quad \lim_{t \to 0} \frac{1 - \cos t}{t}, \quad \text{and} \quad \lim_{x \to 0} \frac{e^t - 1}{t},$$

will result in $\frac{\text{“}0\text{”}}{0}$ upon direct substitution. However, they are not resolved by factoring or rationalization, but by a method which you will learn in college calculus.

(C) **EXERCISES**

I. Evaluate the following limits by constructing their respective tables of values.

1. $\lim_{t\to 0} \frac{t}{\sin t}$
2. $\lim_{t\to 0} \frac{t}{e^t - 1}$
3. $\lim_{t\to 0} \frac{\sin(2t)}{2t}$
4. $\lim_{t\to 0} \frac{1-\cos(3t)}{3t}$

⋆5. $\lim_{t\to 0} \frac{\sin t}{t} \cdot \frac{1-\cos t}{t}$ Answer: 0

⋆6. $\lim_{t\to 0} \frac{1-\cos t}{\sin t}$ Answer: 0

II. Evaluate the following limits:

1. $\lim_{w\to 1} (1+\sqrt[3]{w})(2-w^2+3w^3)$
2. $\lim_{t\to -1} \frac{t^2-1}{t^2+4t+3}$
3. $\lim_{z\to 2} \left(\frac{2z-z^2}{z^2-4}\right)^3$
4. $\lim_{x\to -1} \frac{x^2-x-2}{x^3-6x^2-7x}$
5. $\lim_{y\to -2} \frac{4-3y^2-y^3}{6-y-2y^2}$
6. $\lim_{x\to 4} \frac{x^3-7x^2+14x-8}{x^2-3x-4}$
7. $\lim_{x\to -1} \frac{\sqrt{x^2+3}-2}{x^2-1}$
8. $\lim_{x\to 2} \frac{\sqrt{2x}-\sqrt{6-x}}{4-x^2}$

⋆9. $\lim_{x\to 16} \frac{x^2-256}{4-\sqrt{x}}$ Answer: -256

⋆10. $\lim_{q\to -1} \frac{\sqrt{9q^2-4}-\sqrt{17+12q}}{q^2+3q+2}$ Answer: $-3\sqrt{5}$

LESSON 3: Continuity of Functions

TIME FRAME: 3-4 hours

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Illustrate continuity of a function at a point;
2. Determine whether a function is continuous at a point or not;
3. Illustrate continuity of a function on an interval; and
4. Determine whether a function is continuous on an interval or not.

LESSON OUTLINE:

1. Continuity at a point
2. Determining whether a function is continuous or not at a point
3. Continuity on an interval
4. Determining whether a function is continuous or not on an interval

---

TOPIC 3.1: Continuity at a Point

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

As we have observed in our discussion of limits in Topic (1.2), there are functions whose limits are not equal to the function value at $x = c$, meaning, $\lim_{x \to c} f(x) \neq f(c)$.

$\lim_{x \to c} f(x)$ is NOT NECESSARILY the same as $f(c)$.

This leads us to the study of continuity of functions. In this section, we will be focusing on the continuity of a function at a specific point.

Teaching Tip

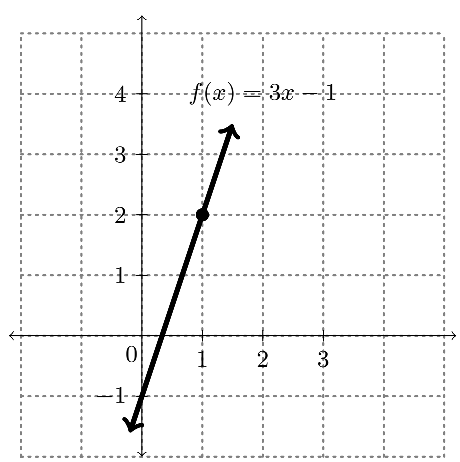
Ask the students to describe, in their own words, the term *continuous*. Ask them how the graph of a continuous function should look. Lead them towards the conclusion that a graph describes a continuous function if they can draw the entire graph without lifting their pen, or pencil, from their sheet of paper.

(B) LESSON PROPER

LIMITS AND CONTINUITY AT A POINT

What does "continuity at a point" mean? Intuitively, this means that in drawing the graph of a function, the point in question will be traversed. We start by graphically illustrating what it means to be continuity at a point.

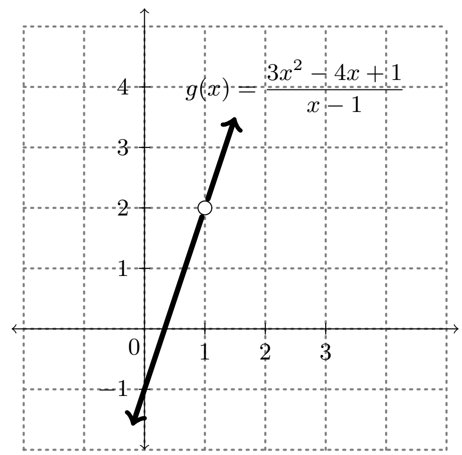
EXAMPLE 1: Consider the graph below.



Is the function continuous at $x = 1$?

Solution. To check if the function is continuous at $x = 1$, use the given graph. Note that one is able to trace the graph from the left side of the number $x = 1$ going to the right side of $x = 1$, without lifting one's pen. This is the case here. Hence, we can say that the function is continuous at $x = 1$.

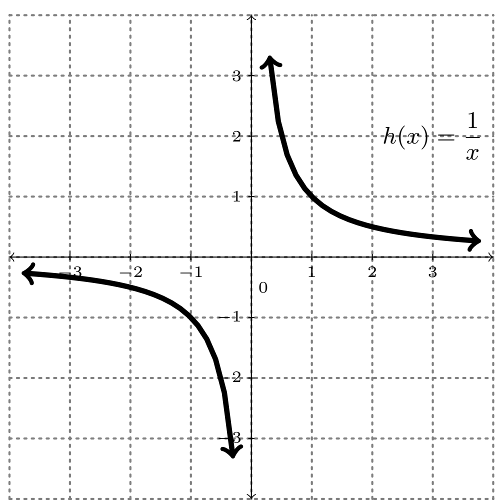
EXAMPLE 2: Consider the graph of the function $g(x)$ below.



Is the function continuous at $x = 1$?

Solution. We follow the process in the previous example. Tracing the graph from the left of $x = 1$ going to right of $x = 1$, one finds that s/he must lift her/his pen briefly upon reaching $x = 1$, creating a hole in the graph. Thus, the function is discontinuous at $x = 1$.

EXAMPLE 3: Consider the graph of the function $h(x) = \dfrac{1}{x}$.



Is the function continuous at $x = 0$?

Solution. If we trace the graph from the left of $x = 0$ going to right of $x = 0$, we have to lift our pen since at the left of $x = 0$, the function values will go downward indefinitely, while at the right of $x = 0$, the function values will go to upward indefinitely. In other words,

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty \text{ and } \lim_{x\to 0^+} \frac{1}{x} = \infty$$

Thus, the function is discontinuous at $x = 0$.

EXAMPLE 4: Consider again the graph of the function $h(x) = \frac{1}{x}$. Is the function continuous at $x = 2$?

Solution. If we trace the graph of the function $h(x) = \frac{1}{x}$ from the left of $x = 2$ to the right of $x = 2$, you will not lift your pen. Therefore, the function h is continuous at $x = 2$.

Suppose we are not given the graph of a function but just the function itself. How do we determine if the function is continuous at a given number? In this case, we have to check three conditions.

Three Conditions of Continuity

A function $f(x)$ is said to be **continuous** at $x = c$ if the following three conditions are satisfied:

(i) $f(c)$ exists;

(ii) $\lim_{x\to c} f(x)$ exists; and

(iii) $f(c) = \lim_{x\to c} f(x)$.

If at least one of these conditions is not met, f is said to be **discontinuous** at $x = c$.

EXAMPLE 5: Determine if $f(x) = x^3 + x^2 - 2$ is continuous or not at $x = 1$.

Solution. We have to check the three conditions for continuity of a function.

(a) If $x = 1$, then $f(1) = 0$.

(b) $\lim_{x\to 1} f(x) = \lim_{x\to 1}(x^3 + x^2 - 2) = 1^3 + 1^2 - 2 = 0$.

(c) $f(1) = 0 = \lim_{x\to 1} f(x)$.

Therefore, f is continuous at $x = 1$.

EXAMPLE 6: Determine if $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ is continuous or not at $x = 0$.

Solution. We have to check the three conditions for continuity of a function.

(a) If $x = 0$, then $f(0) = 1$.

(b) $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{x^2 - x - 2}{x - 2} = \lim\limits_{x \to 0} \dfrac{(x-2)(x+1)}{x - 2} = \lim\limits_{x \to 0} (x + 1) = 1.$

(c) $f(0) = 1 = \lim\limits_{x \to 0} f(x)$.

Therefore, f is continuous at $x = 0$.

EXAMPLE 7: Determine if $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ is continuous or not at $x = 2$.

Solution. Note that f is not defined at $x = 2$ since 2 is not in the domain of f. Hence, the first condition in the definition of a continuous function is not satisfied. Therefore, f is discontinuous at $x = 2$.

EXAMPLE 8: Determine if

$$f(x) = \begin{cases} x + 1 & \text{if } x < 4, \\ (x - 4)^2 + 3 & \text{if } x \geq 4 \end{cases}$$

is continuous or not at $x = 4$. (This example was given in Topic 1.1.)

Solution. Note that f is defined at $x = 4$ since $f(4) = 3$. However, $\lim\limits_{x \to 4^-} f(x) = 5$ while $\lim\limits_{x \to 4^+} f(x) = 3$. Therefore $\lim\limits_{x \to 4^-} f(x)$ DNE, and f is discontinuous at $x = 4$.

Teaching Tip

The following seatwork is suggested at this point: Determine if $f(x) = \sqrt{x - 1}$ is continuous or not at $x = 4$.

Solution. We check the three conditions:

(a) $f(4) = \sqrt{4 - 1} = \sqrt{3} > 0$

(b) $\lim\limits_{x \to 4} \sqrt{x - 1} = \sqrt{4 - 1} = \sqrt{3}$

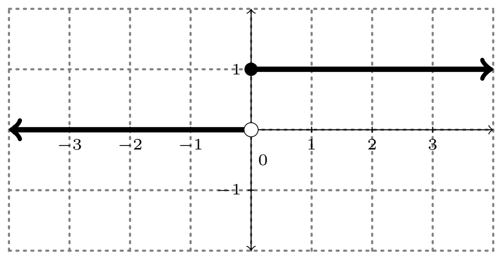
(c) $f(4) = \sqrt{3} = \lim\limits_{x \to 4} \sqrt{x - 1}$

Therefore, the function f is continuous at $x = 4$.

(C) **EXERCISES**

I. Given the graph below, determine if the function $H(x)$ is continuous at the following values of x:

1. $x = 2$
2. $x = -3$
3. $x = 0$



Heaviside function $H(x)$

II. Determine if the following functions are continuous at the given value of x.

1. $f(x) = 3x^2 + 2x + 1$ at $x = -2$
2. $f(x) = 9x^2 - 1$ at $x = 1$
3. $f(x) = \dfrac{1}{x-2}$ at $x = 2$
4. $h(x) = \dfrac{x-1}{x^2-1}$ at $x = 1$
5. $h(x) = \dfrac{x+1}{x^2-1}$ at $x = 1$
6. $g(x) = \sqrt{x-3}$ at $x = 4$
7. $g(x) = \dfrac{x}{\sqrt{4-x}}$ at $x = 8$
8. $g(x) = \dfrac{\sqrt{4-x}}{x}$ at $x = 0$

TOPIC 3.2: Continuity on an Interval

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

A function can be continuous on an interval. This simply means that it is continuous at every point on the interval. Equivalently, if we are able to draw the entire graph of the function on an interval without lifting our tracing pen, or without being interrupted by a hole in the middle of the graph, then we can conclude that the the function is continuous on that interval.

We begin our discussion with two concepts which are important in determining whether a function is continuous at the endpoints of closed intervals.

One-Sided Continuity

(a) A function f is said to be **continuous from the left at** $x = c$ if

$$f(c) = \lim_{x \to c^-} f(x).$$

(b) A function f is said to be **continuous from the right at** $x = c$ if

$$f(c) = \lim_{x \to c^+} f(x).$$

Here are known facts on continuities of functions on intervals:

Continuity of Polynomial, Absolute Value, Rational and Square Root Functions

(a) Polynomial functions are continuous everywhere.

(b) The absolute value function $f(x) = |x|$ is continuous everywhere.

(c) Rational functions are continuous on their respective domains.

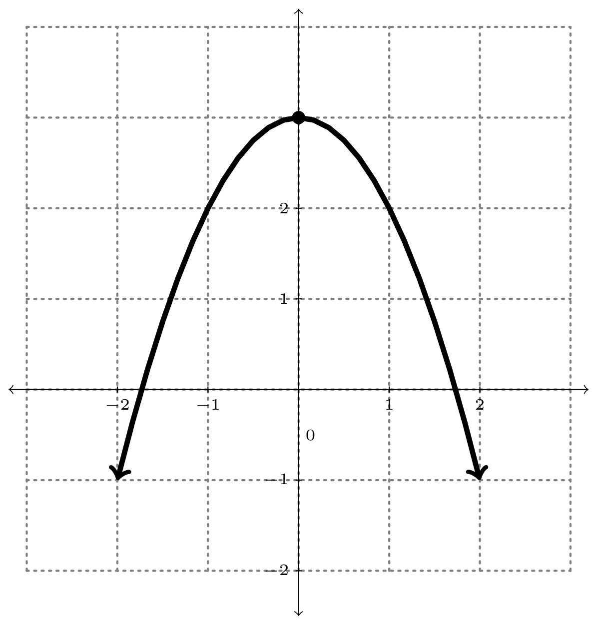
(d) The square root function $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

(B) **LESSON PROPER**

LIMITS AND CONTINUITY ON AN INTERVAL

We first look at graphs of functions to illustrate continuity on an interval.

EXAMPLE 1: Consider the graph of the function f given below.



Using the given graph, determine if the function f is continuous on the following intervals:

(a) $(-1, 1)$ (b) $(-\infty, 0)$ (c) $(0, +\infty)$

Solution. Remember that when we say "trace from the right side of $x = c$", we are tracing not from $x = c$ on the x-axis, but from the point $(c, f(c))$ *along the graph.*

(a) We can trace the graph from the right side of $x = -1$ to the left side of $x = 1$ without lifting the pen we are using. Hence, we can say that the function f is continuous on the interval $(-1, 1)$.

(b) If we trace the graph from any negatively large number up to the left side of 0, we will not lift our pen and so, f is continuous on $(-\infty, 0)$.

(c) For the interval $(0, +\infty)$, we trace the graph from the right side of 0 to any large number, and find that we will not lift our pen. Thus, the function f is continuous on $(0, +\infty)$.

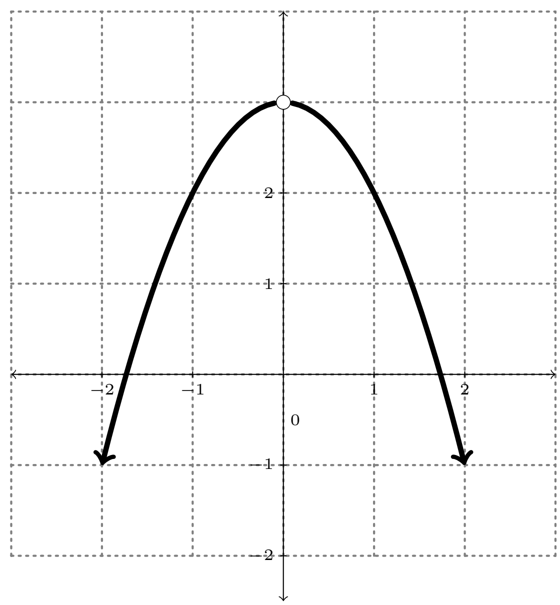
Teaching Tip

Please point these out after solving the previous example:

(a) The function is actually continuous on $[-1, 1]$, $[0, +\infty)$ and $(-\infty, 0]$ since the function f is defined at the endpoints of the intervals: $x = -1$, $x = 1$, and $x = 0$, and we are still able to trace the graph on these intervals without lifting our tracing pen.

(b) The function f is therefore continuous on the interval $(-\infty, +\infty)$ since if we trace the entire graph from left to right, we won't be lifting our pen. This is an example of a function which is continuous everywhere.

EXAMPLE 2: Consider the graph of the function h below.



Determine using the given graph if the function f is continuous on the following intervals:

a. $(-1, 1)$ b. $[0.5, 2]$

Solution. Because we are already given the graph of h, we characterize the continuity of h by the possibility of tracing the graph without lifting the pen.

(a) If we trace the graph of the function h from the right side of $x = -1$ to the left side of $x = 1$, we will be interrupted by a hole when we reach $x = 0$. We are forced to lift our pen just before we reach $x = 0$ to indicate that h is not defined at $x = 0$ and continue tracing again starting from the right of $x = 0$. Therefore, we are not able to trace the graph of h on $(-1, 1)$ without lifting our pen. Thus, the function h is not continuous on $(-1, 1)$.

(b) For the interval $[0.5, 2]$, if we trace the graph from $x = 0.5$ to $x = 2$, we do not have to lift the pen at all. Thus, the function h is continuous on $[0.5, 2]$.

Now, if a function is given without its corresponding graph, we must find other means to determine if the function is continuous or not on an interval. Here are definitions that will help us:

A function f is said to be **continuous**...

(a) everywhere if f is continuous at every real number. In this case, we also say f is continuous on $\mathbb{R}$.

(b) on (a, b) if f is continuous at every point x in (a, b).

(c) on $[a, b)$ if f is continuous on (a, b) and from the right at a.

(d) on $(a, b]$ if f is continuous on (a, b) and from the left at b.

(e) on $[a, b]$ if f is continuous on $(a, b]$ and on $[a, b)$.

(f) on (a, ∞) if f is continuous at all $x > a$.

(g) on $[a, \infty)$ if f is continuous on (a, ∞) and from the right at a.

(h) on $(-\infty, b)$ if f is continuous at all $x < b$.

(i) on $(-\infty, b]$ if f is continuous on $(-\infty, b)$ and from the left at b.

EXAMPLE 3: Determine the largest interval over which the function $f(x) = \sqrt{x+2}$ is continuous.

Solution. Observe that the function $f(x) = \sqrt{x+2}$ has function values only if $x + 2 \geq 0$, that is, if $x \in [-2, +\infty)$. For all $c \in (-2, +\infty)$,

$$f(c) = \sqrt{c+2} = \lim_{x \to c} \sqrt{x+2}.$$

Moreover, f is continuous from the right at -2 because

$$f(-2) = 0 = \lim_{x \to -2^+} \sqrt{x+2}.$$

Therefore, for all $x \in [-2, +\infty)$, the function $f(x) = \sqrt{x+2}$ is continuous.

EXAMPLE 4: Determine the largest interval over which $h(x) = \dfrac{x}{x^2-1}$ is continuous.

Solution. Observe that the given rational function $h(x) = \dfrac{x}{x^2-1}$ is not defined at $x = 1$ and $x = -1$. Hence, the domain of h is the set $\mathbb{R} \setminus \{-1, 1\}$. As mentioned at the start of this topic, a rational function is continuous on its domain. Hence, h is continuous over $\mathbb{R} \setminus \{-1, 1\}$.

EXAMPLE 5: Consider the function $g(x) = \begin{cases} x & \text{if } x \leq 0, \\ 3 & \text{if } 0 < x \leq 1, \\ 3 - x^2 & \text{if } 1 < x \leq 4, \\ x - 3 & \text{if } x > 4. \end{cases}$

Is g continuous on $(0, 1]$? on $(4, \infty)$?

Solution. Since g is a piecewise function, we just look at the 'piece' of the function corresponding to the interval specified.

(a) On the interval $(0, 1]$, $g(x)$ takes the constant value 3. Also, for all $c \in (0, 1]$,

$$\lim_{x \to c} g(x) = 3 = g(c).$$

Thus, g is continuous on $(0, 1]$.

(b) For all $x > 4$, the corresponding 'piece' of g is $g(x) = x - 3$, a polynomial function. Recall that a polynomial function is continuous everywhere in $\mathbb{R}$. Hence, $f(x) = x - 3$ is surely continuous for all $x \in (4, +\infty)$.

(C) EXERCISES

1. Is the function $g(x) = \begin{cases} x & \text{if } x \leq 0, \\ 3 & \text{if } 0 < x \leq 1, \\ 3 - x^2 & \text{if } 1 < x \leq 4, \\ x - 3 & \text{if } x > 4, \end{cases}$ continuous on $[1, 4]$? on $(-\infty, 0)$?

$\star$2. Do as indicated.

a. Find all values of m such that $g(x) = \begin{cases} x + 1 & \text{if } x \leq m, \\ x^2 & \text{if } x > m, \end{cases}$ is continuous everywhere.

b. Find all values of a and b that make

$$h(x) = \begin{cases} x + 2a & \text{if } x < -2, \\ 3ax + b & \text{if } -2 \leq x \leq 1, \\ 3x - 2b & \text{if } x > 1, \end{cases}$$

continuous everywhere.

LESSON 4: More on Continuity

TIME FRAME: 4 hours

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Illustrate different types of discontinuity (hole/removable, jump/essential, asymptotic/infinite);
2. Illustrate the Intermediate Value and Extreme Value Theorems; and
3. Solve problems involving the continuity of a function.

LESSON OUTLINE:

1. Review of continuity at a point
2. Illustration of a hole/removable discontinuity at a point
3. Illustration of a jump essential discontinuity at a point
4. Illustration of an infinite essential discontinuity at a point
5. Illustration of a consequence of continuity given by the Intermediate Value Theorem
6. Illustration of a consequence of continuity given by the Extreme Value Theorem
7. Situations which involve principles of continuity
8. Solutions to problems involving properties/consequences of continuity

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TOPIC 4.1: Different Types of Discontinuities

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

In Topic (1.2), it was emphasized that the value of $\lim_{x \to c} f(x)$ may be distinct from the value of the function itself at $x = c$. Recall that a limit may be evaluated at values which are not in the domain of $f(x)$.

In Topics (3.1) - (3.2), we learned that when $\lim_{x \to c} f(x)$ and $f(c)$ are equal, $f(x)$ is said to be continuous at c. Otherwise, it is said to be discontinuous at c. We will revisit the instances when $\lim_{x \to c} f(x)$ and $f(c)$ have unequal or different values. These instances of inequality and, therefore, discontinuity are very interesting to study. This section focuses on these instances.

(B) LESSON PROPER

Consider the functions $g(x)$, $h(x)$ and $j(x)$ where

$$g(x) = \begin{cases} \dfrac{3x^2 - 4x + 1}{x - 1} & \text{if } x \neq 1, \\ 1 & \text{if } x = 1. \end{cases}$$

$$h(x) = \begin{cases} x + 1 & \text{if } x < 4, \\ (x - 4)^2 + 3 & \text{if } x \geq 4. \end{cases}$$

and

$$j(x) = \frac{1}{x}, \quad x \neq 0.$$

We examine these for continuity at the respective values 1, 4, and 0.

(a) $\lim_{x \to 1} g(x) = 2$ but $g(1) = 1$.

(b) $\lim_{x \to 4} h(x)$ DNE but $h(4) = 3$.

(c) $\lim_{x \to 0} j(x)$ DNE and $f(0)$ DNE.

All of the functions are discontinuous at the given values. A closer study shows that they actually exhibit different types of discontinuity.

REMOVABLE DISCONTINUITY

A function $f(x)$ is said to have a ***removable discontinuity*** at $x = c$ if

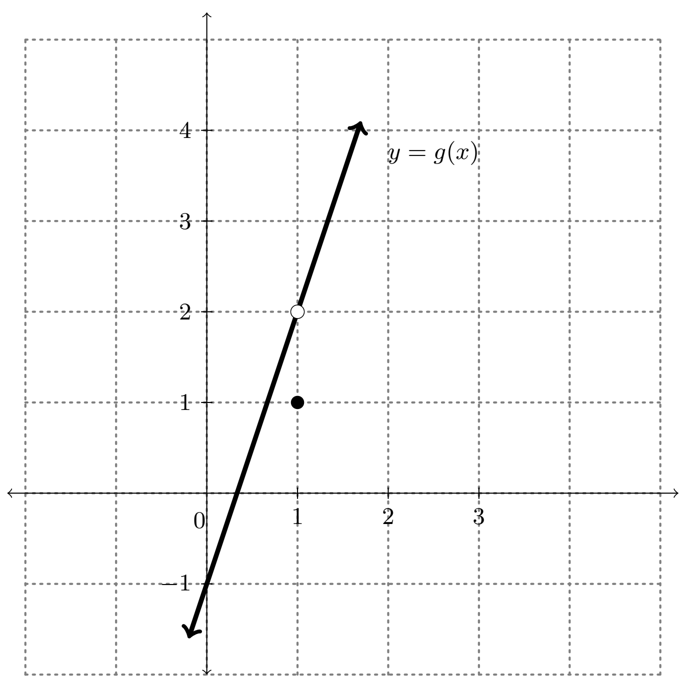
(a) $\lim_{x\to c} f(x)$ exists; and

(b) either $f(c)$ does not exist or $f(c) \neq \lim_{x\to c} f(x)$.

It is said to be *removable* because the discontinuity may be removed by *redefining* $f(c)$ so that it will equal $\lim_{x\to c} f(x)$. In other words, if $\lim_{x\to c} f(x) = L$, a removable discontinuity is remedied by the redefinition:

$$\text{Let } f(c) = L.$$

Recall $g(x)$ above and how it is discontinuous at 1. In this case, $g(1)$ exists. Its graph is as follows:



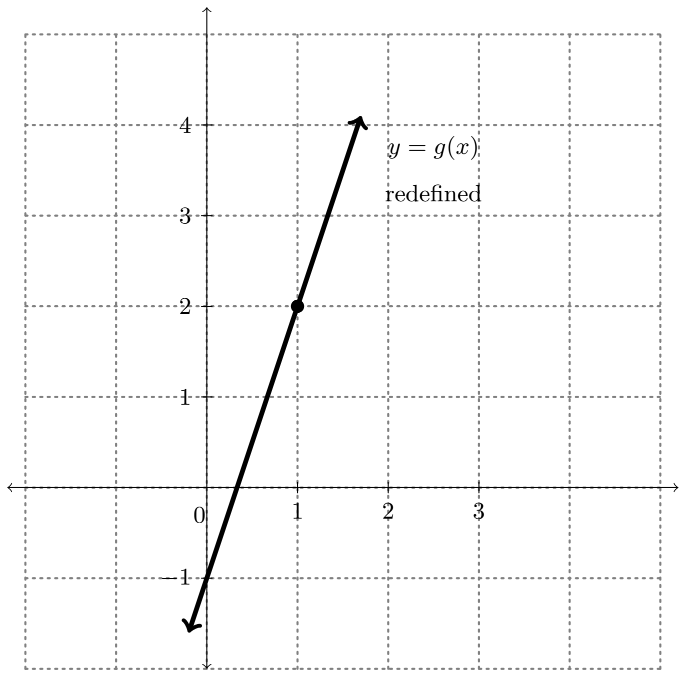
The discontinuity of g at the point $x = 1$ is manifested by the hole in the graph of $y = g(x)$ at the point $(1, 2)$. This is due to the fact that $f(1)$ is equal to 1 and not 2, while $\lim_{x\to 1} g(x) = 2$. We now demonstrate how this kind of a discontinuiy may be removed:

$$\text{Let } g(1) = 2.$$

This is called a *redefinition* of g at $x = 1$. The redefinition results in a "transfer" of the point $(1, 1)$ to the hole at $(1, 2)$. In effect, the hole is filled and the discontinuity is removed!

This is why the discontinuity is called a removable one. This is also why, sometimes, it is called a *hole* discontinuity.

We go back to the graph of $g(x)$ and see how redefining $f(1)$ to be 2 removes the discontinuity:



and revises the function to its continuous counterpart,

$$G(x) = \begin{cases} g(x) & \text{if } x \neq 1, \\ 2 & \text{if } x = 1. \end{cases}$$

ESSENTIAL DISCONTINUITY

A function $f(x)$ is said to have an **essential discontinuity** at $x = c$ if $\lim_{x \to c} f(x)$ DNE.

Case 1. If for a function $f(x)$, $\lim_{x \to c} f(x)$ DNE because the limits from the left and right of $x = c$ both exist but are not equal, that is,

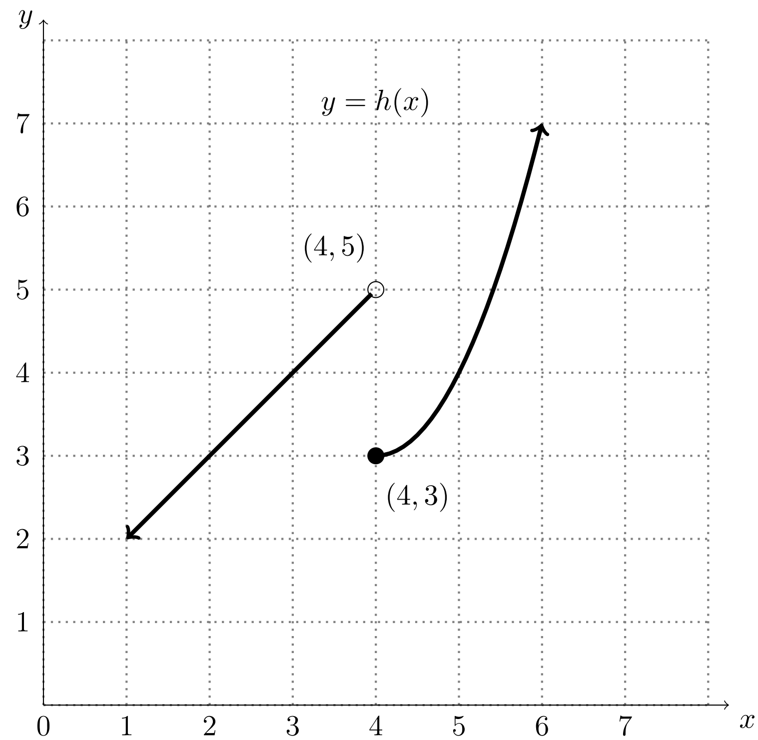
$$\lim_{x \to c^-} f(x) = L \text{ and } \lim_{x \to c^+} f(x) = M, \text{ where } L \neq M,$$

then f is said to have a *jump* essential discontinuity at $x = c$.

Recall the function $h(x)$ where

$$h(x) = \begin{cases} x+1 & \text{if } x < 4, \\ (x-4)^2+3 & \text{if } x \geq 4. \end{cases}$$

Its graph is as follows:



From Lesson 2, we know that $\lim_{x\to 4} h(x)$ DNE because

$$\lim_{x\to 4^-} h(x) = 5 \text{ and } \lim_{x\to 4^+} h(x) = 3.$$

The graph confirms that the discontinuity of $h(x)$ at $x = 4$ is certainly not removable. See, the discontinuity is not just a matter of having one point missing from the graph and putting it in; if ever, it is a matter of having a part of the graph entirely out of place. If we force to remove this kind of discontinuity, we need to connect the two parts by a vertical line from $(4, 5)$ to $(4, 3)$. However, the resulting graph will fail the Vertical Line Test and will not be a graph of a function anymore. Hence, this case has no remedy. From the graph, it is clear why this essential discontinuity is also called a *jump* discontinuity.

Case 2. If a function $f(x)$ is such that $\lim_{x\to c} f(x)$ DNE because either

(i) $\lim_{x \to c^-} f(x) = +\infty$, or

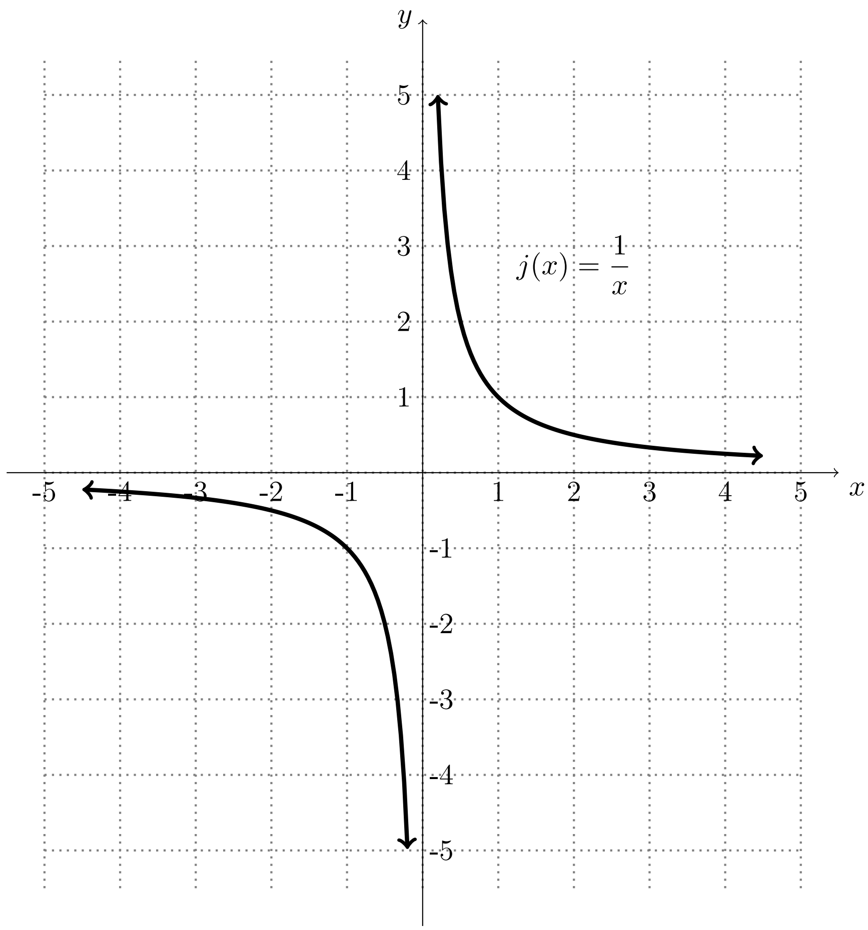
(ii) $\lim_{x \to c^-} f(x) = -\infty$, or

(iii) $\lim_{x \to c^+} f(x) = +\infty$, or

(iv) $\lim_{x \to c^+} f(x) = -\infty$,

then $f(x)$ is said to have an *infinite* discontinuity at $x = c$.

Recall $j(x) = \dfrac{1}{x}$, $x \neq 0$, as mentioned earlier. Its graph is as follows:

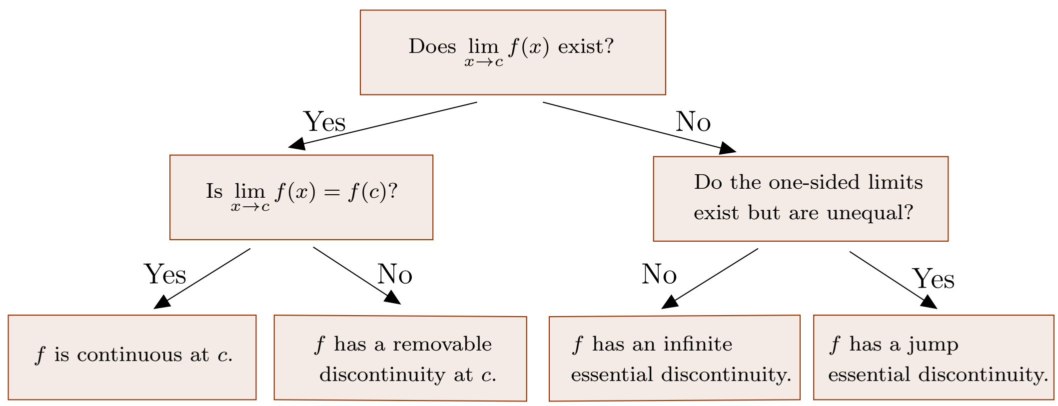


We have seen from Topic 1.4 that

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \quad \text{and} \quad \lim_{x \to 0^+} \frac{1}{x} = +\infty.$$

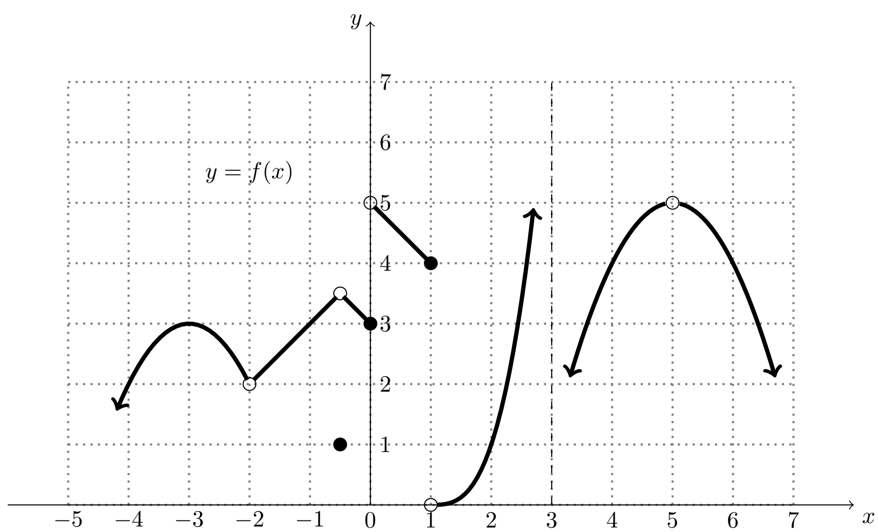
Because the limits are infinite, the limits from both the left and the right of $x = 0$ do not exist, and the discontinuity cannot be removed. Also, the absence of a left-hand (or right-hand) limit from which to "jump" to the other part of the graph means the discontinuity is permanent. As the graph indicates, the two ends of the function that approach $x = 0$ continuously move away from each other: one end goes upward without bound, the other end goes downward without bound. This translates to an asymptotic behavior as x-values approach 0; in fact, we say that $x = 0$ is a vertical asymptote of $f(x)$. Thus, this discontinuity is called an *infinite* essential discontinuity.

FLOWCHART. Here is a flowchart which can help evaluate whether a function is continuous or not at a point c. Before using this, make sure that the function is defined on an open interval containing c, except possibly at c.



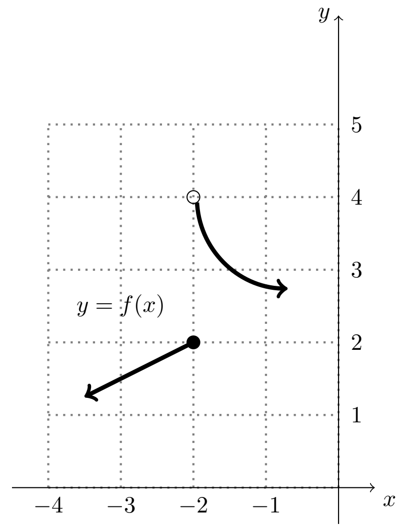
(C) **EXERCISES**

1. Consider the function $f(x)$ whose graph is given below.



Enumerate all discontinuities of $f(x)$ and identify their types. If a discontinuity is removable, state the redefiniton that will remove it. *Hint*: There are 6 discontinuities.

2. For each specified discontinuity, sketch the graph of a possible function $f(x)$ that illustrates the discontinuity. For example, if it has a jump discontinuity at $x = -2$, then a possible graph of f is



Do a similar rendition for f for each of the following discontinuities:

a. $\lim_{x\to 0} f(x) = 1$ and $f(0) = -3$

b. $\lim_{x\to 1} f(x) = -1$ and $f(1)$ DNE

c. $\lim_{x\to 2^-} f(x) = -2$ and $\lim_{x\to 2^+} f(x) = 2$

d. $\lim_{x\to 3^-} f(x) = -\infty$ and $\lim_{x\to 3^+} f(x) = +\infty$

e. $\lim_{x\to -1^-} f(x) = +\infty$, $\lim_{x\to -1^+} f(x) = 0$ and $f(-1) = 0$

f. $\lim_{x\to -1^-} f(x) = +\infty$, $\lim_{x\to -1^+} f(x) = 0$ and $f(-1) = -1$

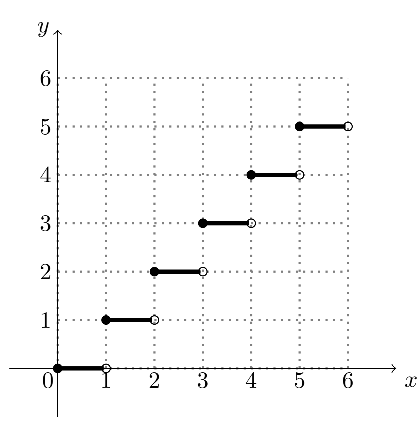
g. There is a removable discontinuity at $x = 1$ and $f(1) = 4$

h. There is a jump discontinuity at $x = 2$ and $f(2) = 3$

i. There is an infinite discontinuity at $x = 0$

j. There is an infinite discontinuity at $x = 0$ and $f(0) = -2$

3. Consider the function $f(x)$ whose graph is given below.



a. What kind of discontinuity is exhibited by the graph?

b. At what values of x does this type of discontinuity happen?

c. Can the discontinuities be removed? Why/Why not?

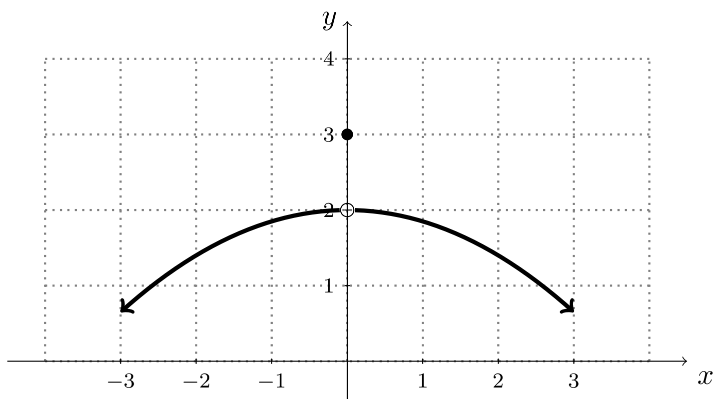
d. How many discontinuities do you see in the graph?

e. Based on the graph above, and assuming it is part of $f(x) = [\![x]\!]$, how many discontinuities will the graph of $f(x) = [\![x]\!]$ have?

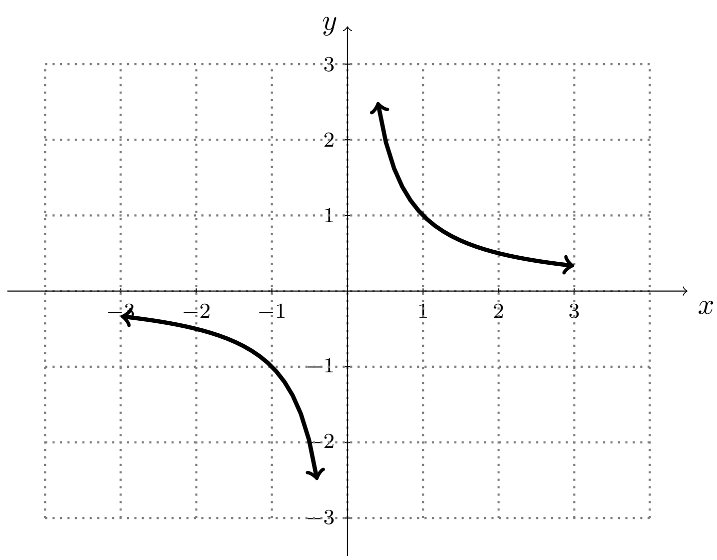
f. Assuming this is part of the graph of $f(x) = [\![x]\!]$, how would the discontinuities change if instead you have $f(x) = [\![2x]\!]$ or $f(x) = [\![3x]\!]$ or $f(x) = [\![0.5x]\!]$?

4. For each function whose graph is given below, identify the type(s) of discontinuity(ies) exhibited. Remedy any removable discontinuity with an appropriate redefinition.

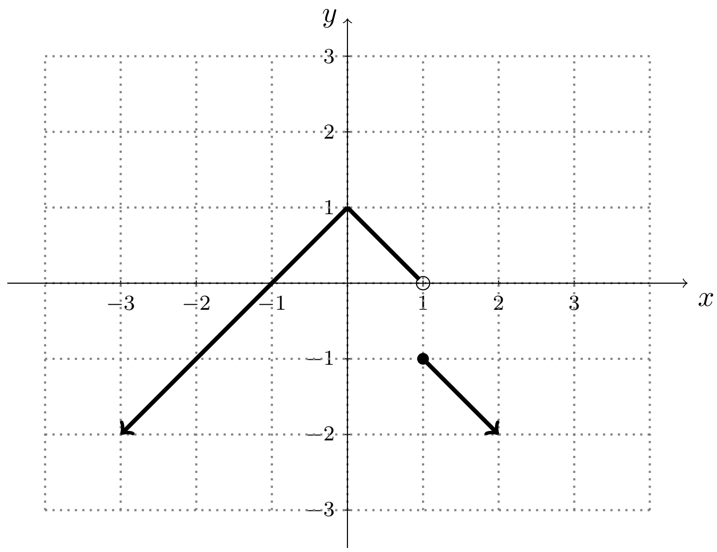
a. $y = f(x)$



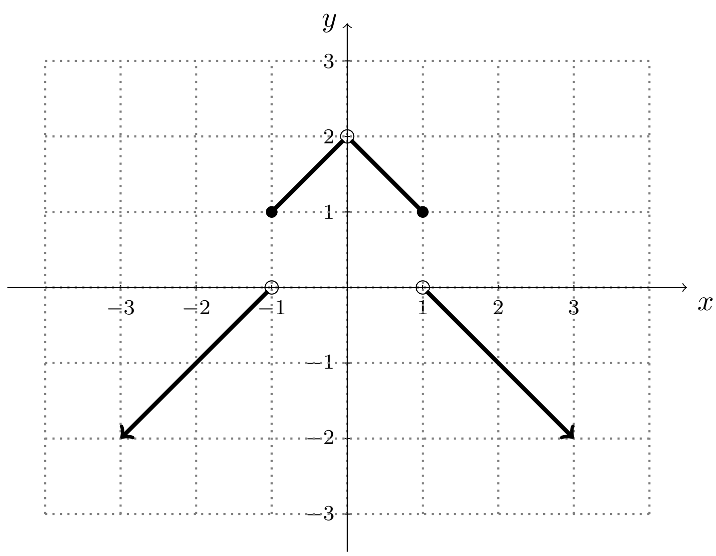
b. $y = g(x)$



c. $y = h(x)$



d. $y = j(x)$



5. Determine the possible points of discontinuity of the following functions and the type of discontinuity exhibited at that point. Remove any removable discontinuity. Sketch the graph of $f(x)$ to verify your answers.

a. $f(x) = \dfrac{1}{x^2}$

b. $f(x) = \dfrac{x^2 - 4}{x - 2}$

c. $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x \neq 2, \\ -4 & \text{if } x = 2. \end{cases}$

d. $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x < 2, \\ -4 & \text{if } x \geq 2. \end{cases}$

e. $f(x) = \dfrac{x - 2}{x^2 - 4}$

f. $f(x) = \dfrac{1}{x^2 - 9}$

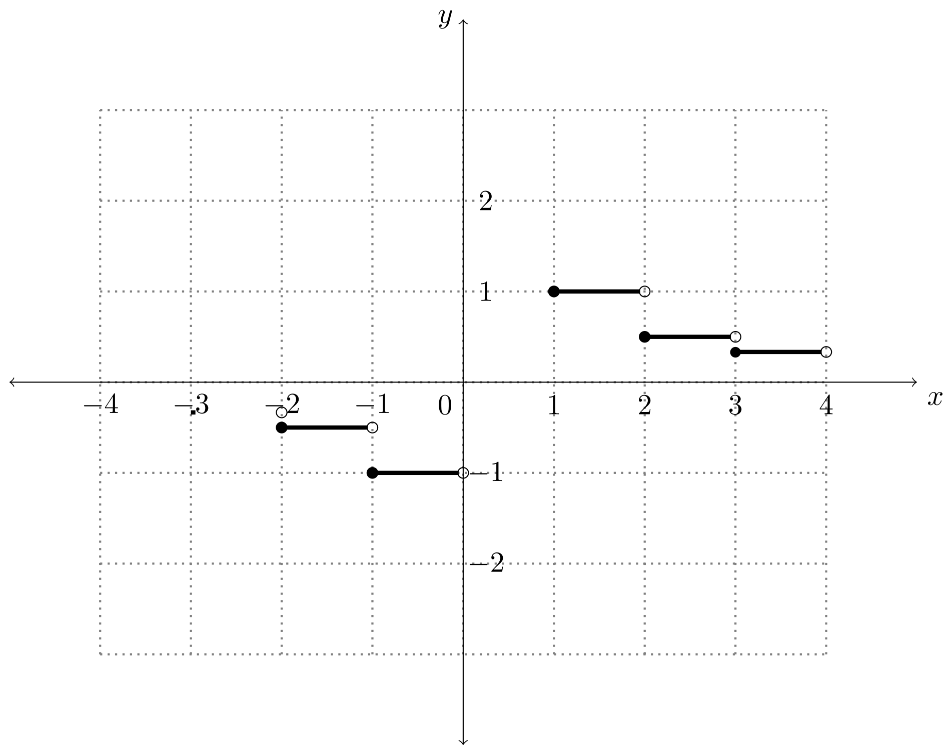
g. $f(x) = \tan x$

h. $f(x) = \cos x, \quad x \neq 2k\pi$, where k is an integer.

i. $f(x) = \csc x$

$\star$j. $f(x) = \dfrac{1}{[\![x]\!]}$

Answer to the starred exercise: First of all, $f(x)$ will be discontinuous at values where the denominator will equal 0. This means that x cannot take values in the interval $[0, 1)$. This will cause a big jump (or essential) discontinuity from where the graph stops right before (-1,-1) to where it resumes at (1,1).
Moreover, there will again be jump discontinuities at the integer values of x.



TOPIC 4.2: The Intermediate Value and the Extreme Value Theorems

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

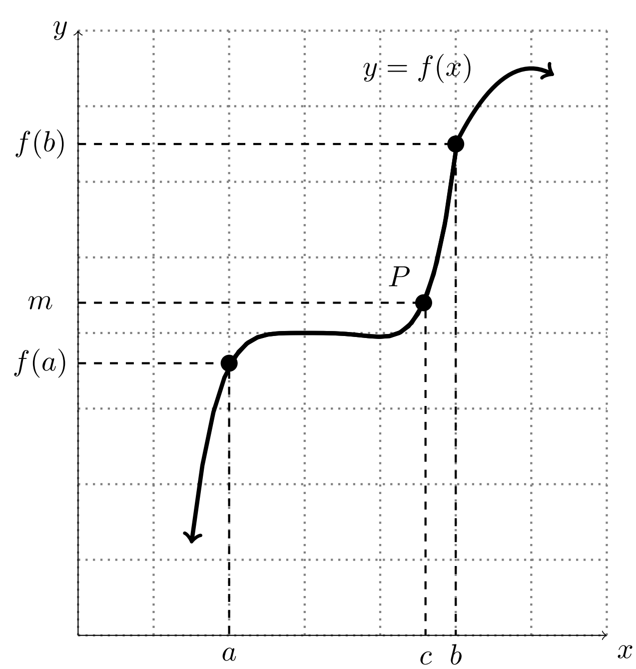
After discussing continuity at length, we will now learn two important consequences brought about by the continuity of a function over a closed interval. The first one is called the Intermediate Value Theorem or the IVT. The second one is called the Extreme Value Theorem or the EVT.

(B) LESSON PROPER

The Intermediate Value Theorem

The first theorem we will illustrate says that a function $f(x)$ which is found to be continuous over a closed interval $[a, b]$ will take any value between $f(a)$ and $f(b)$.

Theorem 4 (Intermediate Value Theorem (IVT))**.** *If a function $f(x)$ is continuous over a closed interval $[a, b]$, then for every value m between $f(a)$ and $f(b)$, there is a value $c \in [a, b]$ such that $f(c) = m$.*



Look at the graph as we consider values of m between $f(a)$ and $f(b)$. Imagine moving the dotted line for m up and down between the dotted lines for $f(a)$ and $f(b)$. Correspondingly, the dot P will move along the thickened curve between the two points, $(a, f(a))$ and $(b, f(b))$.

We make the following observations:

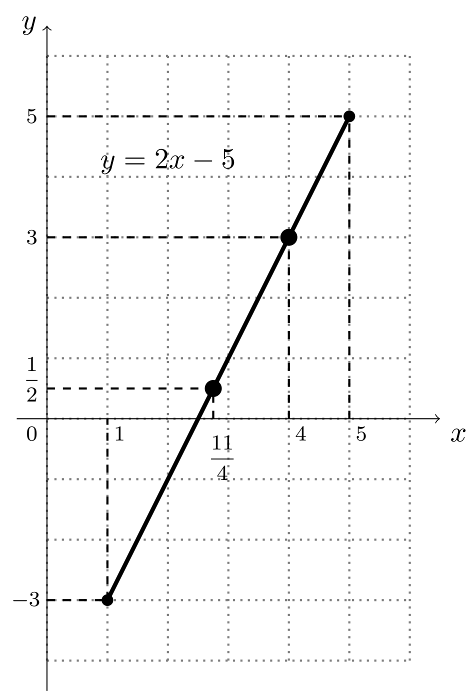
- As the dark dot moves, so will the vertical dotted line over $x = c$ move.
- In particular, the said line moves between the vertical dotted lines over $x = a$ and $x = b$.
- More in particular, for any value that we assign m in between $f(a)$ and $f(b)$, the consequent position of the dark dot assigns a corresponding value of c between a and b. This illustrates what the IVT says.

EXAMPLE 1: Consider the function $f(x) = 2x - 5$.

Since it is a linear function, we know it is continuous everywhere. Therefore, we can be sure that it will be continuous over any closed interval of our choice.

Take the interval $[1, 5]$. The IVT says that for any m intermediate to, or in between, $f(1)$ and $f(5)$, we can find a value intermediate to, or in between, 1 and 5.

Start with the fact that $f(1) = -3$ and $f(5) = 5$. Then, choose an $m \in [-3, 5]$, to exhibit a corresponding $c \in [1, 5]$ such that $f(c) = m$.



Choose $m = \dfrac{1}{2}$. By IVT, there is a $c \in [1, 5]$ such that $f(c) = \dfrac{1}{2}$. Therefore,

$$\frac{1}{2} = f(c) = 2c - 5 \quad \Longrightarrow \quad 2c = \frac{11}{2} \quad \Longrightarrow \quad c = \frac{11}{4}.$$

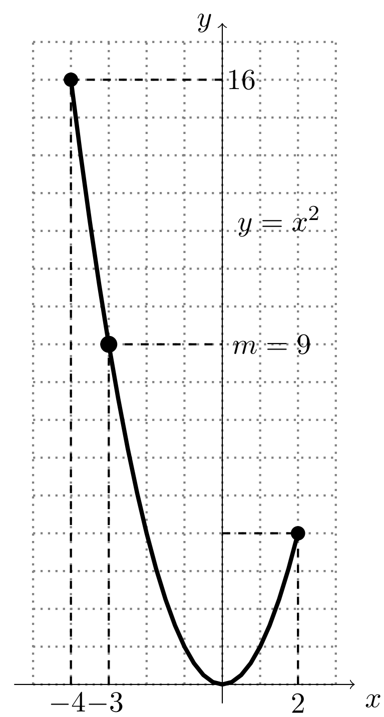
Indeed, $\dfrac{11}{4} \in (1, 5)$.

We can try another m-value in $(-3, 5)$. Choose $m = 3$. By IVT, there is a $c \in [1, 5]$ such that $f(c) = 3$. Therefore,

$$3 = f(c) = 2c - 5 \quad \Longrightarrow \quad 2c = 8 \quad \Longrightarrow \quad c = 4.$$

Again, the answer, 4, is in $[1, 5]$. The claim of IVT is clearly seen in the graph of $y = 2x - 5$.

EXAMPLE 2: Consider the simplest quadratic function $f(x) = x^2$.



Being a polynomial function, it is continuous everywhere. Thus, it is also continuous over any closed interval we may specify.

We choose the interval $[-4, 2]$. For any m in between $f(-4) = 16$ and $f(2) = 4$, there is a value c inside the interval $[-4, 2]$ such that $f(c) = m$.

Suppose we choose $m = 9 \in [4, 16]$. By IVT, there exists a number $c \in [-4, 2]$ such that $f(c) = 9$. Hence,

$$9 = f(c) = c^2 \quad \Longrightarrow \quad c = \pm 3.$$

However, we only choose $c = -3$ because the other solution $c = 3$ is not in the specified interval $[-4, 2]$.

Note: In the previous example, if the interval that was specified was [0,4], then the final answer would instead be $c = +3$.

Remark 1: The value of $c \in [a, b]$ in the conclusion of the Intermediate Value Theorem is not necessarily unique.

EXAMPLE 3: Consider the polynomial function

$$f(x) = x^3 - 4x^2 + x + 7$$

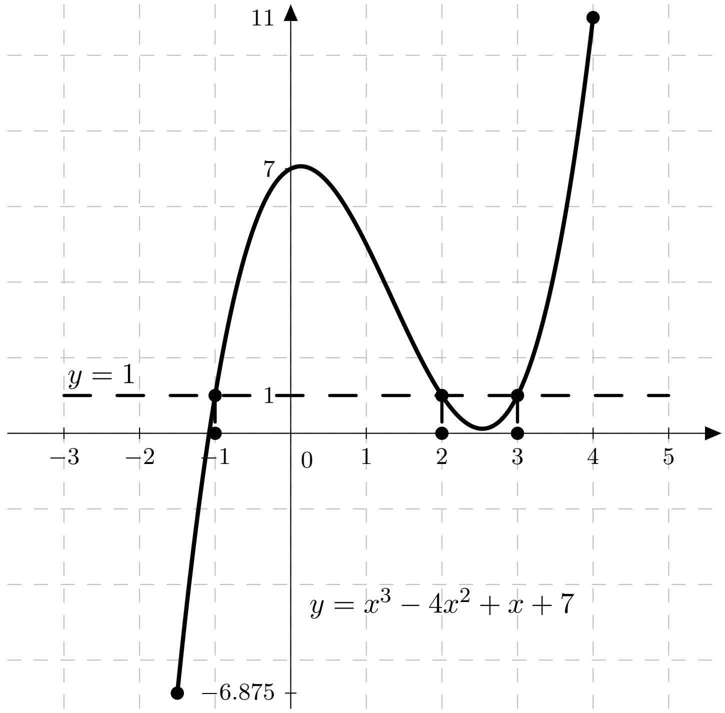
over the interval $[-1.5, 4]$ Note that

$$f(-1.5) = -6.875 \quad \text{and} \quad f(4) = 11.$$

We choose $m = 1$. By IVT, there exists $c \in [-1.5, 4]$ such that $f(c) = 1$. Thus,

$$\begin{aligned} & f(c) = c^3 - 4c^2 + c + 7 = 1 \\ \Longrightarrow \quad & c^3 - 4c^2 + c + 6 = 0 \\ \Longrightarrow \quad & (c+1)(c-2)(c-3) = 0 \\ \Longrightarrow \quad & c = -1 \text{ or } c = 2 \text{ or } c = 3. \end{aligned}$$

We see that there are three values of $c \in [-1.5, 4]$ which satisfy the conclusion of the Intermediate Value Theorem.



The Extreme Value Theorem

The second theorem we will illustrate says that a function $f(x)$ which is found to be continuous over a closed interval $[a, b]$ is guaranteed to have extreme values in that interval.

An extreme value of f, or *extremum*, is either a minimum or a maximum value of the function.

- A *minimum* value of f occurs at some $x = c$ if $f(c) \leq f(x)$ for all $x \neq c$ in the interval.
- A *maximum* value of f occurs at some $x = c$ if $f(c) \geq f(x)$ for all $x \neq c$ in the interval.

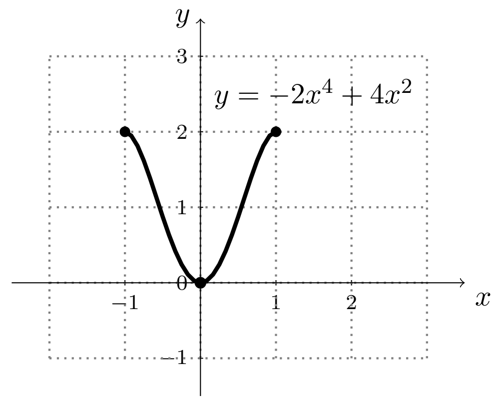
Theorem 5 (Extreme Value Theorem (EVT))**.** *If a function $f(x)$ is continuous over a closed interval $[a, b]$, then $f(x)$ is guaranteed to reach a maximum and a minimum on $[a, b]$.*

Note: In this section, we limit our illustration of extrema to graphical examples. More detailed and computational examples will follow once derivatives have been discussed.

EXAMPLE 4: Consider the function $f(x) = -2x^4 + 4x^2$ over $[-1, 1]$.

From the graph, it is clear that on the interval, f has

- The maximum value of 2, occurring at $x = \pm 1$; and
- The minimum value of 0, occurring at $x = 0$.



Remark 2: Similar to the IVT, the value $c \in [a, b]$ at which a minimum or a maximum occurs is not necessarily unique.

Here are more examples exhibiting the guaranteed existence of extrema of functions continuous over a closed interval.

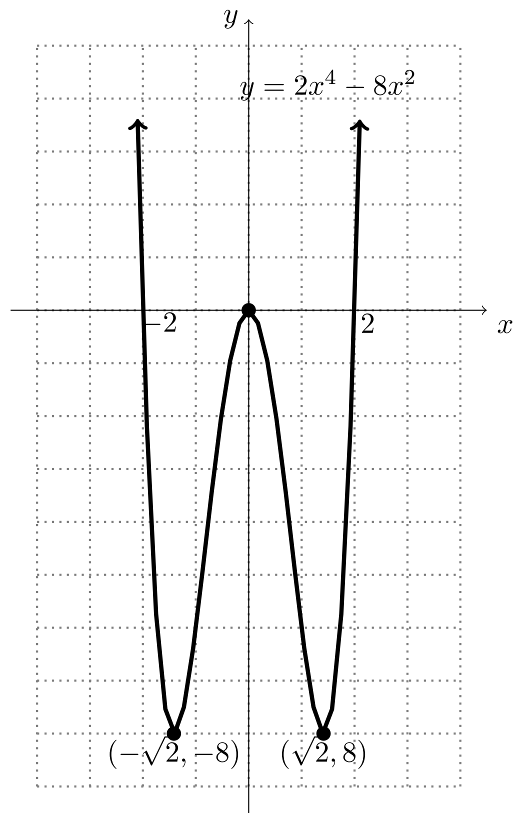
EXAMPLE 5: Consider Example 1. Observe that $f(x) = 2x - 5$ on $[1, 5]$ exhibits the extrema at the endpoints:

- The minimum occurs at $x = 1$, giving the minimum value $f(1) = -3$; and
- The maximum occurs at $x = 5$, giving the maximum value $f(5) = 5$.

EXAMPLE 6: Consider Example 2. $f(x) = x^2$ on $[-4, 2]$ exhibits an extremum at one endpoint and another at a point inside the interval (or, an *interior* point):

- The minimum occurs at $x = 0$, giving the minimum value $f(0) = 0$; and
- The maximum occurs at $x = -4$, giving the maximum value $f(-4) = 16$.

EXAMPLE 7: Consider $f(x) = 2x^4 - 8x^2$.



- On the interval $[-2, -\sqrt{2}]$, the extrema occur at the endpoints.
  - Endpoint $x = -2$ yields the maximum value $f(-2) = 0$.
  - Endpoint $x = -\sqrt{2}$ yields the minimum value $f(-\sqrt{2}) = -8$.
- On the interval $[-2, -1]$, one extremum occurs at an endpoint, another at an interior point.
  - Endpoint $x = -2$ yields the maximum value $f(-2) = 0$.
  - Interior point $x = -\sqrt{2}$ yields the minimum value $f(-\sqrt{2}) = -8$.
- On the interval $[-1.5, 1]$, the extrema occur at interior points.
  - Interior point $x = -\sqrt{2}$ yields the minimum value $f(-\sqrt{2}) = -8$.
  - Interior point $x = 0$ yields the maximum value $f(0) = 0$.
- On the interval $[-2, 2]$, the extrema occur at both the endpoints and several interior points.
  - Endpoints $x = \pm 2$ and interior point $x = 0$ yield the maximum value 0.
  - Interior points $x = \pm\sqrt{2}$ yield the minimum value -8.

Remark 3: Keep in mind that the IVT and the EVT are existence theorems ("*there is a value c ...*"), and their statements do not give a method for finding the values stated in their respective conclusions. It may be difficult or impossible to find these values algebraically especially if the function is complicated.

(C) **EXERCISES**

1. What value(s) of c, if any, will satisfy the IVT for the given function f and the given value m, on the given interval $[a, b]$. If there is (are) none, provide an explanation.

a. $f(x) = x^2 - 1,\ m = 2,\ [-1, 2]$

b. $f(x) = x^2 - 1,\ m = 2,\ [-1, 1]$

c. $f(x) = x^3 + 2,\ m = 3,\ [0, 3]$

d. $f(x) = \sin x,\ m = 1/2,\ [-\pi, \pi]$

e. $f(x) = x^3 - 3x^2 + 3x - 1,\ m = -1,\ [-1, 2]$

f. $f(x) = 4,\ m = 4,\ [-2, 2]$

g. $f(x) = x,\ m = 4,\ [-2, 2]$

h. $f(x) = x^2,\ m = 4,\ [-2, 2]$

2. Sketch the graph of each $f(x)$ in Item (a) to verify your answers.

3. Referring to your graphs in Item (b), where does each $f(x)$ attain its minimum and maximum values? Compute for the respective minimum and maximum values.

4. Determine whether the given function will have extrema (both a maximum and a minimum) on the interval indicated. If not, provide an explanation.

a. $f(x) = x^2 - 1,\ (-1, 2)$

b. $f(x) = |x|,\ [0, 1]$

c. $f(x) = |x|,\ (0, 1)$

d. $f(x) = \sin x,\ (-\pi, \pi)$

e. $f(x) = \sin x,\ [-\pi/2, \pi/2]$

f. $f(x) = x^3 - 3x^2 + 3x - 1,\ (-1, 1)$

g. $f(x) = 1/x,\ [-2, 2]$

h. $f(x) = [\![x]\!],\ [0, 1]$

$\star$5. The next items will show that the hypothesis of the Intermediate Value Theorem – that f must be continuous on a closed and bounded interval – is indispensable.

a. Find an example of a function f defined on $[0, 1]$ such that $f(0) \neq f(1)$ and there exists no $c \in [0, 1]$ such that

$$f(c) = \frac{f(0) + f(1)}{2}.$$

(*Hint*: the function must be discontinuous on $[0, 1]$.)
Possible answer: Piecewise function defined by $f(x) = 1$ on $[0, 1)$ and $f(1) = 0$.

b. Find an example of a function f defined on $[0, 1]$ but is only continuous on $(0, 1)$ and such that there exists no value of $c \in [0, 1]$ such that

$$f(c) = \frac{f(0) + f(1)}{2}$$

Possible answer: Piecewise function defined by $f(x) = 1$ on $[0, 1)$ and $f(1) = 0$.

$\star$6. The next items will show that the hypothesis of the Extreme Value Theorem – that f must be continuous on a closed and bounded interval – is indispensable.

a. Find an example of a function f defined on $[0,1]$ such that f does not attain its absolute extrema on $[0,1]$. (*Hint*: the function must be discontinuous on $[0,1]$.)
Possible answer: Piecewise function defined by $f(x) = x$ on $(0,1)$ and $f(0) = f(1) = \dfrac{1}{2}$.

b. Find an example of a function f that is continuous on $(0,1)$ but does not attain its absolute extrema on $[0,1]$
Possible answer: Piecewise function defined by $f(x) = x$ on $(0,1)$ and $f(0) = f(1) = \dfrac{1}{2}$.

7. Determine whether the statement is *true* or *false*. If you claim that it is false, provide a counterexample.

a. If a function is continuous on a closed interval $[a,b]$, then it has a maximum and a minimum on that interval. Answer: True

b. If a function is discontinuous on a closed interval, then it has no extreme value on that interval. Answer: False, for example the piecewise function $f(x) = 0$ on $[0,1/2]$ and $f(x) = 1$ on $(1/2,1]$ achieve its extrema but it is discontinuous on $[0,1]$.

c. If a function has a maximum and a minimum over a closed interval, then it is continuous on that interval. Answer: False, same counterexample as above

d. If a function has no extreme values on $[a,b]$, then it is discontinuous on that interval. Answer: True

e. If a function has either a maximum only or a minimum only over a closed interval, then it is discontinuous on that interval. Answer: True

8. Determine whether the given function will have extrema (both a maximum and a minimum) on the interval indicated. If not, provide an explanation.

a. $f(x) = |x+1|,\ [-2,3]$
b. $f(x) = -|x+1| + 3,\ (-2,2)$
c. $f(x) = [\![x]\!],\ [1,2)$
d. $f(x) = [\![x]\!],\ [1,2]$
e. $f(x) = \cos x, [0,2\pi]$
f. $f(x) = \cos x, [0,2\pi)$
g. $f(x) = x^4 - 2x^2 + 1,\ [-1,1]$
h. $f(x) = x^4 - 2x^2 + 1,\ (-3/2,3/2)$

$\star$9. Sketch a graph each of a random f over the interval $[-3,3]$ showing, respectively,

a. f with more than 2 values c in the interval satisfying the IVT for $m = 1/2$.
b. f with only one value c in the interval satisfying the IVT for $m = -1$.
c. f with exactly three values c in the interval satisfying $m = 0$.
d. f with a unique maximum at $x = -3$ and a unique minimum at $x = 3$.
e. f with a unique minimum and a unique maximum at interior points of the interval.

f. f with two maxima, one at each endpoint, and a unique minimum at an interior point.

g. f with two maxima, one at each endpoint, and two minima occurring at interior points.

h. f with three maxima, one at each endpoint and another at an interior point, and a unique minimum at an interior point.

i. f with three zeros, one at each endpoint and another at an interior point, a positive maximum, and a negative minimum.

j. f with four maxima and a unique minimum, all occurring at interior points.

$\star$10. State whether the given situation is *possible* or *impossible*. When applicable, support your answer with a graph. Consider the interval to be $[-a, a]$, $a > 0$, for all items and that $c \in [-a, a]$. Suppose also that each function f is continuous over $[-a, a]$.

a. $f(-a) < 0$, $f(a) > 0$ and there is a c such that $f(c) = 0$.

b. $f(-a) < 0$, $f(a) < 0$ and there is a c such that $f(c) = 0$.

c. $f(-a) > 0$, $f(a) > 0$ and there is a c such that $f(c) = 0$.

d. f has exactly three values c such that $f(c) = 0$.

e. f has exactly three values c such that $f(c) = 0$, its minimum is negative, its maximum is positive.

f. f has exactly three values c such that $f(c) = 0$, its minimum is positive, its maximum is negative.

g. f has a unique positive maximum, a unique positive minimum, and a unique value c such that $f(c) = 0$.

h. f has a unique positive maximum, a unique negative minimum, and a unique value c such that $f(c) = 0$.

i. f has a unique positive maximum, a unique negative minimum, and two values c such that $f(c) = 0$.

j. f has a unique positive maximum, a unique positive minimum, and five values c such that $f(c) = 0$, two of which are $c = \pm a$.

k. f has two positive maxima, two negative minima, and no value c such that $f(c) = 0$.

l. f has two positive maxima, one negative minimum, and a unique value c such that $f(c) = 0$.

m. f has two positive maxima, one negative minimum found between the two maxima, and a unique value c such that $f(c) = 0$.

n. f has two maxima, two minima, and no value c such that $f(c) = 0$.

o. f has two maxima, two minima, and a unique value c such that $f(c) = 0$.

⋆11. Determine whether the given function will have extrema (both a maximum and a minimum) on the interval indicated. If not, provide an explanation.

a. $f(x) = \sin x,\ (-\pi/2, \pi/2)$

b. $f(x) = \sin x,\ [-\pi/2, \pi/2)$

c. $f(x) = \dfrac{1}{x-1},\ [2, 4]$

d. $f(x) = \dfrac{1}{x-1},\ [-4, 4]$

e. $f(x) = \begin{cases} 2 - \sqrt{-x} & \text{if } x < 0, \\ 2 - \sqrt{x} & \text{if } x \geq 0, \end{cases}\ [-3, 3]$

f. $f(x) = \begin{cases} (x-2)^2 + 2 & \text{if } x < -1, \\ (x-2)^2 - 1 & \text{if } x \geq -1, \end{cases}\ [-3, 3]$

g. $f(x) = -x^4 + 2x^2 - 1,\ [-1, 1]$

h. $f(x) = -x^4 + 2x^2 - 1,\ \left(-\dfrac{3}{2}, \dfrac{3}{2}\right)$

TOPIC 4.3: Problems Involving Continuity

This is an OPTIONAL topic. It is intended for the enrichment of the students, to enhance their understanding of continuity and the properties it makes possible, such as stated in the Intermediate Value Theorem.

DEVELOPMENT OF THE LESSON

(A) **INTRODUCTION**

Continuity is a very powerful property for a function to possess. Before we even move on to its possibilities with respect to differentiation and integration, let us take a look at some types of problems which may be solved if one has knowledge of the continuity of the function(s) involved.

(B) **LESSON PROPER**

For every problem that will be presented, we will provide a solution that makes use of continuity and takes advantage of its consequences, such as the Intermediate Value Theorem (IVT).

APPROXIMATING ROOTS (Method of Bisection)

Finding the roots of polynomials is easy if they are special products and thus easy to factor. Sometimes, with a little added effort, roots can be found through synthetic division. However, for most polynomials, roots, can at best, just be approximated.

Since polynomials are continuous everywhere, the IVT is applicable and very useful in approximating roots which are otherwise difficult to find. In what follows, we will always choose a closed interval $[a, b]$ such that $f(a)$ and $f(b)$ differ in sign, meaning, $f(a) > 0$ and $f(b) < 0$, or $f(a) < 0$ and $f(b) > 0$.

In invoking the IVT, we take $m = 0$. This is clearly an intermediate value of $f(a)$ and $f(b)$ since $f(a)$ and $f(b)$ differ in sign. The conclusion of the IVT now guarantees the existence of $c \in [a, b]$ such that $f(c) = 0$. This is tantamount to looking for the roots of polynomial $f(x)$.

EXAMPLE 1: Consider $f(x) = x^3 - x + 1$. Its roots cannot be found using factoring and synthetic division. We apply the IVT.

- Choose any initial pair of numbers, say -3 and 3.
- Evaluate f at these values.

$$f(-3) = -23 < 0 \text{ and } f(3) = 25 > 0.$$

  Since $f(-3)$ and $f(3)$ differ in sign, a root must lie between -3 and 3.
- To approach the root, we trim the interval.
  - Try $[0, 3]$. However, $f(0) = 1 > 0$ like $f(3)$ so no conclusion can be made about a root existing in $[0, 3]$.
  - Try $[-3, 0]$. In this case, $f(0)$ and $f(-3)$ differ in sign so we improve the search space for the root from $[-3, 3]$ to $[-3, 0]$.
- We trim further.
  - $f(-1) = 1 > 0$ so the root is in $[-3, -1]$.
  - $f(-2) = -5 < 0$ so the root is in $[-2, -1]$.
  - $f(-\frac{3}{2}) = -\frac{7}{8} < 0$ so the root is in $[-\frac{3}{2}, -1]$.
  - $f(-\frac{5}{4}) = \frac{19}{64} > 0$ so the root is in $[-\frac{3}{2}, -\frac{5}{4}]$.
- Further trimming and application of the IVT will yield the approximate root $x = -\frac{53}{40} = -1.325$. This gives $f(x) \approx -0.0012$.

The just-concluded procedure gave one root, a negative one. There are two more possible real roots.

FINDING INTERVALS FOR ROOTS

When finding an exact root of a polynomial, or even an approximate root, proves too tedious, some problem-solvers are content with finding a small interval containing that root

EXAMPLE 2: Consider again $f(x) = x^3 - x + 1$. If we just need an interval of length 1, we can already stop at $[-2, -1]$. If we need an interval of length $1/2$, we can already stop at $[-\frac{3}{2}, -1]$. If we want an interval of length $1/4$, we stop at $[-\frac{3}{2}, -\frac{5}{4}]$.

EXAMPLE 3: Consider $f(x) = x^3 - x^2 + 4$. Find three distinct intervals of length 1, or less, containing a root of $f(x)$.

When approximating, we may choose as sharp an estimate as we want. The same goes for an interval. While some problem-solvers will make do with an interval of length 1, some may want a finer interval, say, of length $1/4$. We should not forget that this type of search is possible because we are dealing with polynomials, and the continuity of polynomials everywhere allows us repeated use of the IVT.

SOME CONSEQUENCES OF THE IVT

Some interesting applications arise out of the logic used in the IVT.

EXAMPLE 4: We already know from our first lessons on polynomials that the degree of a polynomial is an indicator of the number of roots it has. Furthermore, did you know that a polynomial of odd degree has at least one real root?

Recall that a polynomial takes the form,

$$f(x) = a_0 x^n + a_1 x^{n-1} + ... + a_{n-1} x + a_n$$

where $a_0, a_1, ..., a_n$ are real numbers and n is an odd integer.

Take for example $a_0 = 1$. So,

$$f(x) = x^n + a_1 x^{n-1} + ... + a_{n-1} x + a_n.$$

Imagine x taking bigger and bigger values, like ten thousand or a million. For such values, the first term will far outweigh the total of all the other terms. See, if x is positive, for big n the value of $f(x)$ will be positive. If x is negative, for big n the value of $f(x)$ will be negative.

We now invoke the IVT. Remember, n is odd.

- Let a be a large-enough negative number. Then, $f(a) < 0$.
- Let b be a large-enough positive number. Then, $f(b) > 0$.

By the IVT, there is a number $c \in (a, b)$ such that $f(c) = 0$. In other words, $f(x)$ does have a real root!

Teaching Tip

Ask the class why the claim may not hold for polynomials of even degree.

Answer: It is possible that the graphs of polynomials of even degree only stay above the x-axis, or only below the x-axis. For example, the graph of $f(x) = x^2 + 1$ stays only above the x-axis and therefore does not intersect x-axis, that is, $f(x)$ has no roots.

CHAPTER 1 EXAM

I. Complete the following tables of values to investigate $\lim_{x\to 1}(2x+1)$.

x	$f(x)$
0.5	
0.7	
0.95	
0.995	
0.9995	

x	$f(x)$
1.6	
1.35	
1.05	
1.005	
1.0005	

II. Using the tables of values above, determine the following:

1. $\lim_{x\to 1^-}(2x+1)$
2. $\lim_{x\to 1^+}(2x+1)$
3. $\lim_{x\to 1}(2x+1)$

III. Evaluate the following using Limit Theorems.

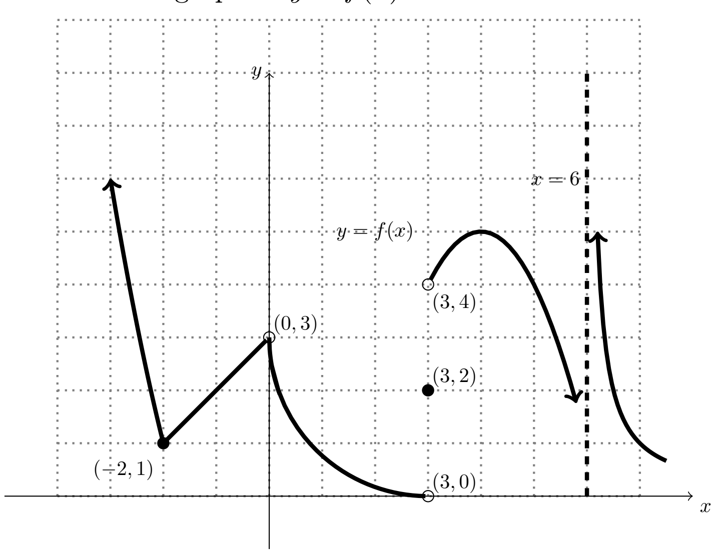
1. $\lim_{x\to 1}\dfrac{x^2-4x+3}{x^2-1}$
2. $\lim_{t\to 0}\dfrac{3t-2\sin t+(e^t-1)}{t}$

IV. Let f be the function defined below.

$$f(x)=\begin{cases}\left|\dfrac{x^2+3x}{x+3}\right|, & \text{if} \quad x\le 0, x\ne -3\\ x+1, & \text{if} \quad 0<x<1\\ \sqrt{x}, & \text{if} \quad x\ge 1.\end{cases}$$

Discuss the continuity of f at $x=-3$, $x=0$ and $x=1$. If discontinuous, give the type of discontinuity.

V. Consider the graph of $y=f(x)$ below.



At the following x-coordinates, write whether (A) f is continuous, (B) f has a removable discontinuity, (C) f has an essential jump discontinuity, or (D) f has an essential infinite discontinuity.

1. $x=-2$
2. $x=0$
3. $x=3$
4. $x=6$

Chapter 2

Derivatives

LESSON 5: The Derivative as the Slope of the Tangent Line

TIME FRAME: 4 hours

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Illustrate the tangent line to the graph of a function at a given point;
2. Apply the definition of the derivative of a function at a given number; and
3. Relate the derivative of a function to the slope of the tangent line.

LESSON OUTLINE:

1. Tangent and secant lines to a circle
2. Tangent line to the graph of an arbitrary function
3. Cases where the tangent line does not exist
4. The slope of the tangent line of an arbitrary function
5. The equation of the tangent line
6. The definition and evaluation of the derivative

TOPIC 5.1: The Tangent Line to the Graph of a Function at a Point

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

You may start by writing the word "TANGENT LINE" on the board and asking the class what they know about a tangent line or where they first heard the word "tangent".

Possible answers:

(a) line which touches the graph at only one point
(b) line touching a circle at one point
(c) trigonometric function tangent (and cotangent)
(d) (from some dictionaries) touching but not intersecting (cutting through)
(e) right beside
(f) limiting position of a secant line
(g) tangent comes from the Latin word *tángere* meaning "to touch" (cf. Noli me Tángere)

Teaching Tip

Accept all of these and acknowledge that all of these are correct in almost all cases but remark that these coud not be very precise in general. In fact, the formal definition of a tangent line is stated using limits. You may list down the following contentions/elaborations to the list above:

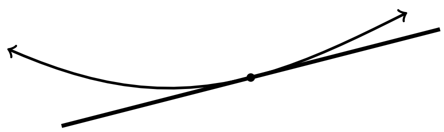
Point-by-point deliberations of the above list.

(a) Which of the following is a tangent line?

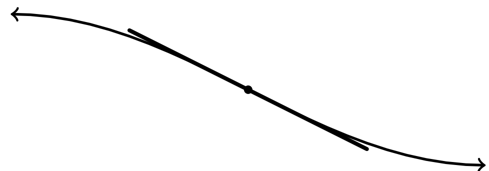
at least two points of intersection — only one point of intersection

(b) Correct; but what if the graph is not a circle?
(c) In fact, the tangent and contangent of an angle are measures of line segments which are tangent to a circle (see enrichment).

(d) Touching and intersecting are very similar concepts in mathematics. The distinction may arise from the interpretation that a line touching a curve stays entirely on one side (below or above) of the curve, while intersecting means that the tangent line "cuts through" the curve (the tangent line is above the curve to the left of the point of tangency while it is below the curve to the right of the point of tangency, or vice versa).



tangent line "touches" the graph

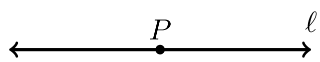


tangent line "cuts through" the graph

Is the line on the second graph a tangent line or not?

(e) English usage: adjacent with no space in between

Teacher: Also, ask them what they think is the tangent line to a line.



What is the tangent line to ℓ at P?

Answer: The tangent line to a line is itself. See Example 3 below.

(B) **ACTIVITY**

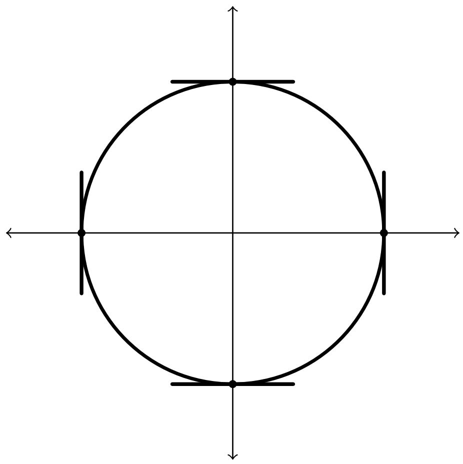
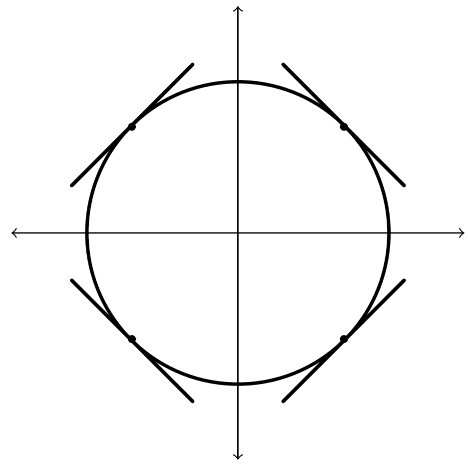
TANGENT LINES TO CIRCLES

- Recall from geometry class that a **tangent line to a circle** centered at O is a line intersecting the circle at exactly one point. It is found by constructing the line, through a point A on the circle, that is perpendicular to the segment (radius) $\overline{OA}$.
- A **secant line to a circle** is a line intersecting the circle at two points.

Remark 1: The difficulty in defining the concept of the tangent line is due to an axiom in Euclidean geometry that states that **a line is uniquely determined by two distinct points.**

Thus, the definition of a tangent line is more delicate because it is determined by only one point, and infinitely many lines pass through a point.

Draw the unit circle and mark several points (including $(0, \pm 1)$ and $(\pm 1, 0)$) on it. Ask the class how they would draw tangent lines at these points. Elicit from them the following facts about the tangent lines to the circle at different points:

(a) At $(\pm 1, 0)$, the tangent lines are vertical;

(b) At $(0, \pm 1)$, the tangent lines are horizontal;

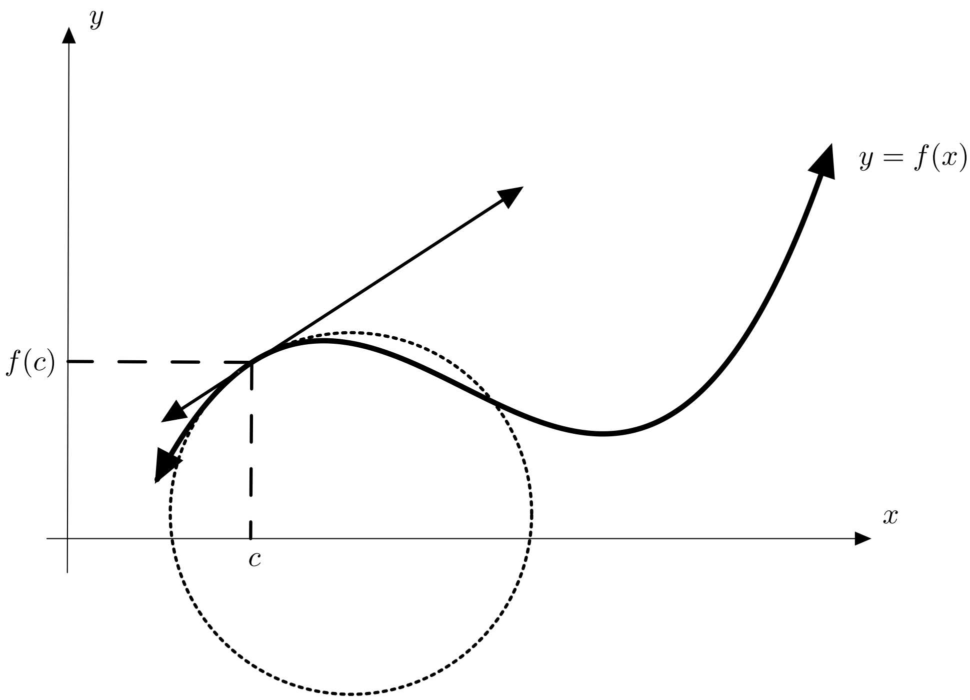
(c) At points in the first and third quadrants, the tangent lines are slanting to the left; and

(d) At points in the second and fourth quadrants, the tangent lines are slanting to the right.

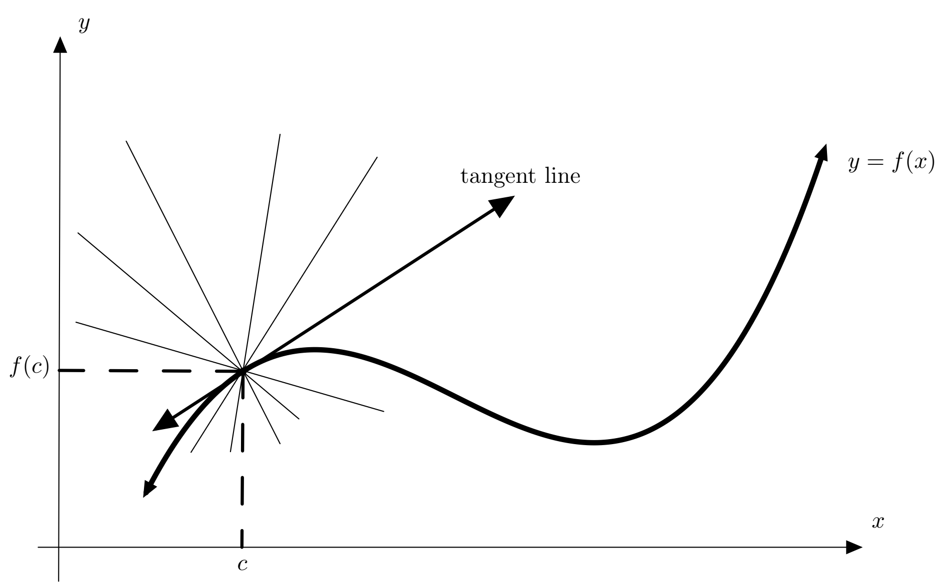
(C) **LESSON PROPER**

HOW TO DRAW TANGENT LINES TO CURVES AT A POINT

The definition of a tangent line is not very easy to explain without involving limits. Students can imagine that locally, the curve looks like an arc of a circle. Hence, they can draw the tangent line to the curve as they would to a circle.

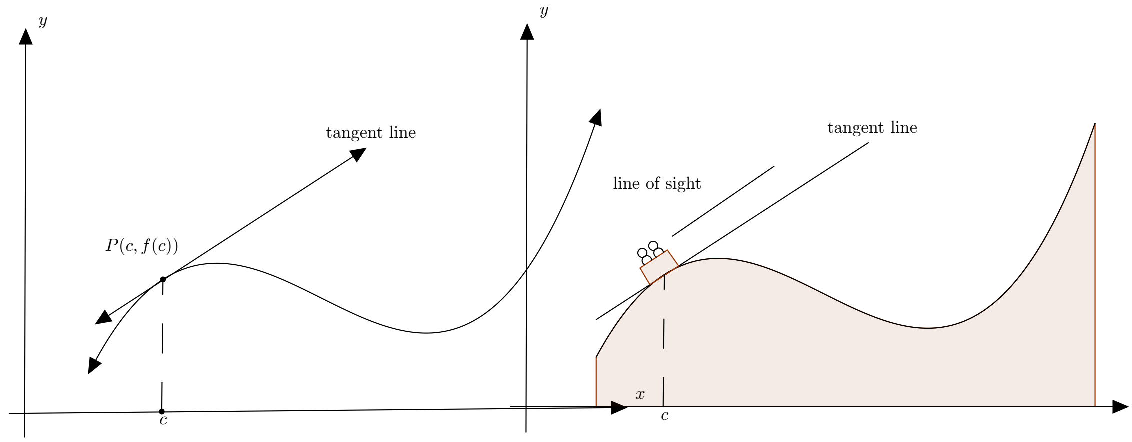


One more way to see this is to choose the line through a point that locally looks most like the curve. Among all the lines through a point $(c, f(c))$, the one which best approximates the curve $y = f(x)$ near the point $(c, f(c))$ is the **tangent line** to the curve at that point.



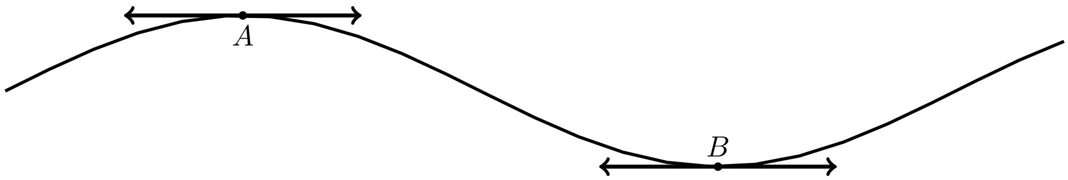
Among all lines passing through $(c, f(c))$, the tangent line is the closest to the curve locally.

Another way of qualitatively understanding the tangent line is to visualize the curve as a roller coaster (see [7], p. 103). The tangent line to the curve at a point is parallel to the line of sight of the passengers looking straight ahead and sitting erect in one of the wagons of the roller coaster.



y
tangent line
$P(c, f(c))$
c
y
x
tangent line
line of sight
x
c

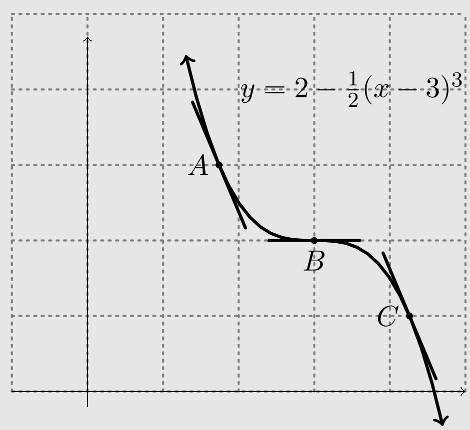
EXAMPLE 1: Ask the class what they think are the tangent lines at the "peaks" and "troughs" of a smooth curve.



Notice that on the unit circle, these points correspond to the points $(0, 1)$ and $(0, -1)$, so whenever the graph is smooth (meaning, there are no sharp corners), the tangent lines at the "peaks" and "troughs" are always **horizontal**.

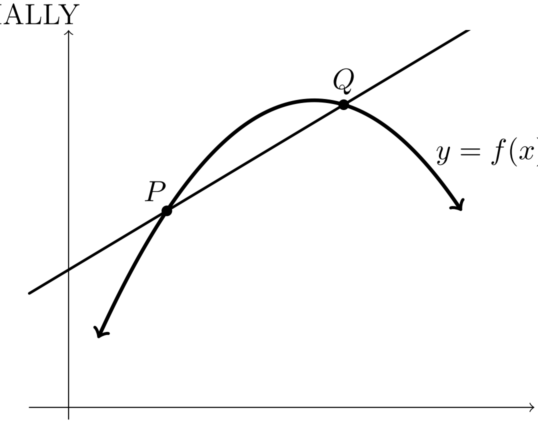
Boardwork

EXAMPLE 2: The following is the graph of $y = 2 - \frac{1}{2}(x-3)^3$. Ask the class to draw the tangent lines at each of the given points A, B, and C.



THE TANGENT LINE DEFINED MORE FORMALLY

The precise definition of a tangent line relies on the notion of a *secant line.* Let C be the graph of a continuous function $y = f(x)$ and let P be a point on C. A secant line to $y = f(x)$ through P is any line connecting P and another point Q on C. In the figure on the right, the line $\overline{PQ}$ is a secant line of $y = f(x)$ through P.

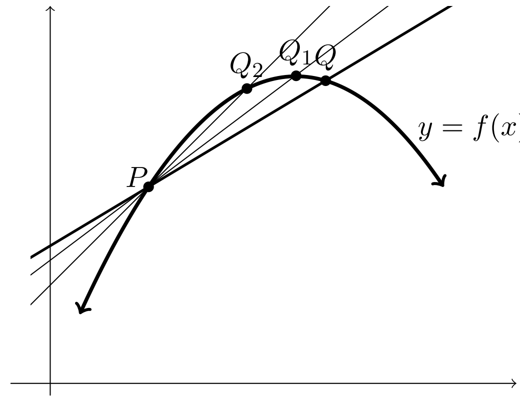


We now construct the tangent line to $y = f(x)$ at P.

Choose a point Q on the right side of P, and connect the two points to construct the secant line $\overline{PQ}$.

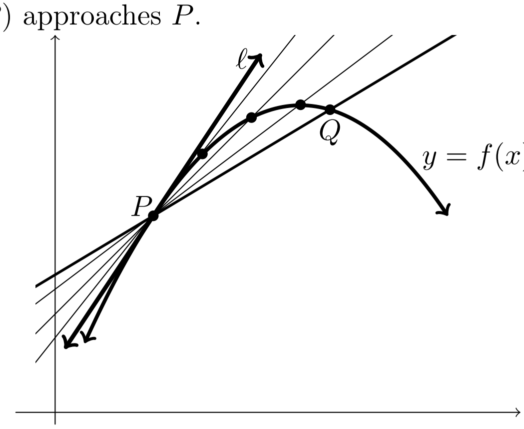
Choose another point Q_1 in between P and Q. Connect the two points P and Q_1 to construct the secant line $\overline{PQ_1}$.

Choose another point Q_2 in between P and Q_1. Construct the secant line $\overline{PQ_2}$.



Consider also the case when Q is to the left of P and perform the same process. Intuitively, we can define the tangent line through P to be the limiting position of the secant lines $\overline{PQ}$ as the point Q (whether to the left or right of P) approaches P.

If the sequence of secant lines to the graph of $y = f(x)$ through P approaches one limiting position (in consideration of points Q to the left and from the right of P), then we define this line to be the tangent line to $y = f(x)$ at P.



We summarize below the definitions of the secant line through a point, and the tangent line at a point of the graph of $y = f(x)$.

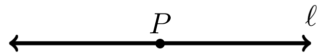
Definition

Let C be the graph of a continuous function $y = f(x)$ and let P be a point on C.

1. A **secant line** to $y = f(x)$ through P is any line connecting P and another point Q on C.
2. The **tangent line** to $y = f(x)$ at P is the limiting position of all secant lines $\overline{PQ}$ as $Q \to P$.

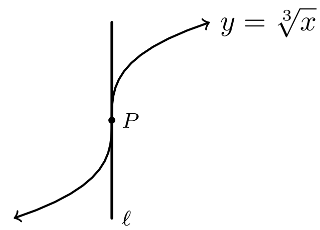
Remark 2: Notice the use of the articles *a* and *the* in the definition above. It should be emphasized that if a tangent line exists, then it must be unique, much the same as in limits.

EXAMPLE 3: The tangent line to another line at any point is the line itself. (This debunks the idea that a tangent line touches the graph at only one point!) Indeed, let ℓ be a line and let P be on ℓ. Observe that no matter what point Q on ℓ we take, the secant line $\overline{PQ}$ is ℓ itself. Hence, the limiting position of a line ℓ is ℓ itself.



The tangent line to ℓ at any point P is itself!

EXAMPLE 4: Our definition of the tangent line allows for a vertical tangent line. We have seen this on the unit circle at points $(1, 0)$ and $(-1, 0)$. A vertical tangent line may also exist even for continuous functions. Draw the curve $y = \sqrt[3]{x}$ and mark the point $P(0, 0)$. Allow the class to determine the tangent line to the graph at P using the formal definition. Consider the two cases: when Q is to the right of P and when Q is to the left of P.

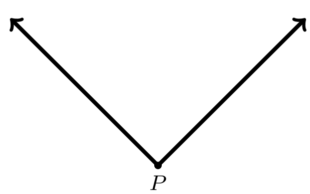


CURVES THAT DO NOT HAVE TANGENT LINES

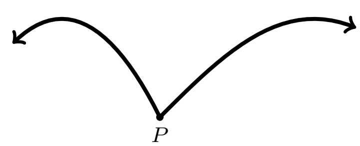
It is possible that the tangent line to a graph of a function at a point $P(x_0, f(x_0))$ does not exist. There are only two cases when this happens:

1. *The case when the function is not continuous at* x_0: It is clear from the definition of the tangent line that the function must be continuous.
2. *The case when the function has a sharp corner/cusp at* P: This case produces different limiting positions of the secant lines $\overline{PQ}$ depending on whether Q is to the left or to the right of P.

Remark 3: The word "sharp corner" is more commonly used for joints where only lines are involved. For example, the absolute value function $y = |x|$ has a sharp corner at the origin. In contrast, the term "cusp" is often used when at least one graph involved represents a nonlinear function. See the graphs below.



corner at P



cusp at P

In the above examples, each has a sharp corner/cusp at P. Choosing Q to be points to the left of P produces a different limiting position than from choosing Q to the right of P. Since the two limiting positions do not coincide, then the tangent line at P does not exist. (This is the same thing that happens when the limit from the left of c differs from the limit from the right of c, where we then conclude that the limit does not exist.)

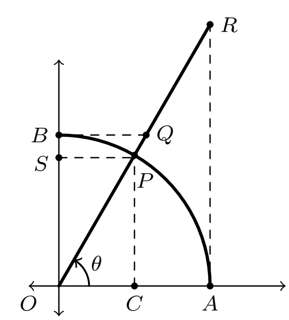
Teaching Tip: Suggestions for learning assessment

1. Assess learning on this topic by drawing graphs on the board, marking points and asking volunteers to draw the tangent line at each point. Make sure that you mark all stationary points (where the tangent line is horizontal) and inflection points (where the tangent line "cuts through" the graph).

2. To assess the students' understanding of the formal definition, get a straight edge and try to animate on the board what happens as the secant lines approach the tangent line. Of course, the pivot must be on the point where the tangent line is wanted. This can also be done using a string that is fixed at the point where we want the tangent line to be. Let the class try this as boardwork. Consider points which are to the left and to the right of the fixed point.

3. Notice that the formal definition of the tangent line involves a limit of a sequence of lines. This poses a problem because the limits that have been discussed before are for functions which involve numbers, and not for geometric objects. Ask the class what number is associated with a line so that instead of looking for a limiting position, we would be looking for a limit of an algebraic expression. Answer: slope of a line

ENRICHMENT



1. Show using similar triangles that $\tan\theta$ and $\cot\theta$ are the respective measures of line segments $\overline{AR}$ and $\overline{BQ}$, which are tangent to the unit circle. (*Hint: It should be clear, beforehand, that* $|\overline{AO}| = |\overline{OB}| = 1$, $\cos\theta = |\overline{OC}| = |\overline{SP}|$ *and* $\sin\theta = |\overline{PC}| = |\overline{SO}|$.)

2. We have seen that for a smooth graph, the tangent lines at the stationary points are always horizontal. Ask the class if the converse is true, that is, if the tangent line at a point P is horizontal, does it follow that P is a local extremum point? Answer: No, see point B in Example 2.

TOPIC 5.2: The Equation of the Tangent Line

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

In the previous lesson, we defined the tangent line at a point P as the limiting position of the secant lines $\overline{PQ}$, where Q is another point on the curve, as Q approaches P. There is a slight problem with this definition because we have no means of computing the limit of lines. Hence, we need to work on the numbers that characterize the lines.

> **Teaching Tip**
>
> Ask the class what things determine lines; in particular, what is the minimum that you should know so that you can draw one and only one line. Expect correct answers such as *two points, slope and y-intercept, the two intercepts, a point and the slope.* Give prominence to the last of the list, emphasizing that there are infinitely many lines passing thru a point; there are infinitely many (parallel) lines with the same slope, but there is only one line passing through a point with a given slope.

- Recall the slope of a line passing through two points (x_0, y_0) and (x, y).

> **Recall: Slope of a Line**
>
> A line ℓ passing through distinct points (x_0, y_0) and (x, y) has slope
>
> $$m_\ell = \frac{y - y_0}{x - x_0}.$$

EXAMPLE 1: Given $A(1, -3)$, $B(3, -2)$, and $C(-1, 0)$, what are the slopes of the lines $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$?

Solution. The slope of $\overline{AB}$ is

$$m_{\overline{AB}} = \frac{-2 - (-3)}{3 - 1} = \frac{1}{2}.$$

The slope of $\overline{AC}$ is

$$m_{\overline{AC}} = \frac{0 - (-3)}{-1 - 1} = -\frac{3}{2}.$$

The slope of $\overline{BC}$ is

$$m_{\overline{BC}} = \frac{0 - (-2)}{-1 - 3} = -\frac{2}{4} = -\frac{1}{2}.$$

- Recall the point-slope form of the equation of the line with slope m and passing through the point $P(x_0, y_0)$.

Recall: Point-Slope Form

The line passing through (x_0, y_0) with slope m has the equation

$$y - y_0 = m(x - x_0).$$

EXAMPLE 2: From Example 1 above, since $m_{\overline{AB}} = \frac{1}{2}$, then using $A(1, -3)$ as our point, then the point-slope form of the equation of $\overline{AB}$ is

$$y - (-3) = \frac{1}{2}(x - 1) \quad \text{or} \quad y + 3 = \frac{1}{2}(x - 1).$$

Teaching Tip

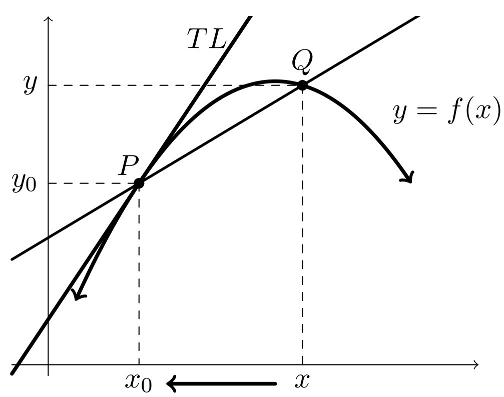
Ask the class what would happen if we had chosen $B(3, -2)$ as our point instead of A. Answer: We would get an equivalent equation.

(B) **LESSON PROPER**

THE EQUATION OF THE TANGENT LINE

Given a function $y = f(x)$, how do we find the equation of the tangent line at a point $P(x_0, y_0)$?

Consider the graph of a function $y = f(x)$ whose graph is given below. Let $P(x_0, y_0)$ be a point on the graph of $y = f(x)$. Our objective is to find the equation of the tangent line (TL) to the graph at the point $P(x_0, y_0)$.



- Find any point $Q(x, y)$ on the curve.
- Get the slope of this secant line $\overline{PQ}$.

$$m_{\overline{PQ}} = \frac{y - y_0}{x - x_0}.$$

- Observe that **letting Q approach P is equivalent to letting x approach x_0.**

> **Teaching Tip**
>
> Illustrate this by choosing points x_1 and x_2 in between x and x_0 and projecting it vertically to the corresponding points Q_1 and Q_2 on the graph. The class should be able to see this equivalence.

We use the formal definition of the tangent line:

- Since the tangent line is the limiting position of the secant lines as Q approaches P, it follows that the slope of the tangent line (TL) at the point P is the limit of the slopes of the secant lines $\overline{PQ}$ as x approaches x_0. In symbols,

$$m_{TL} = \lim_{x \to x_0} \frac{y - y_0}{x - x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

- Finally, since the tangent line passes through $P(x_0, y_0)$, then its equation is given by

$$y - y_0 = m_{TL}(x - x_0).$$

> **Teaching Tip**
>
> It is up to you if you want your students to put this into standard (slope-intercept) form. This is of course not an objective but it helps for easy checking of final answers.

SUMMARY AND EXAMPLES

Equation of the Tangent Line

To find the equation of the tangent line to the graph of $y = f(x)$ at the point $P(x_0, y_0)$, follow this 2-step process:

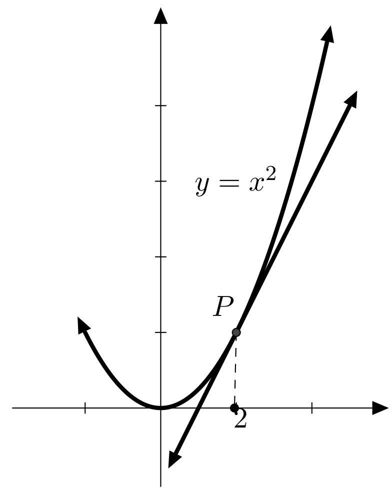
- Get the slope of the tangent line by computing

$$m = \lim_{x \to x_0} \frac{y - y_0}{x - x_0} \quad \text{or} \quad m = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

- Substitute this value of m and the coordinates of the known point $P(x_0, y_0)$ into the point-slope form to get

$$y - y_0 = m(x - x_0).$$

EXAMPLE 3: Find the equation of the tangent line to $y = x^2$ at $x = 1$.



Solution. To get the equation of the line, we need the point $P(x_0, y_0)$ and the slope m. We are only given $x_0 = 2$. However, the y-coordinate of x_0 is easy to find by substituting $x_0 = 2$ into $y = x^2$. This gives us $y_0 = 4$. Hence, P has the coordinates $(2, 4)$. Now, we look for the slope:

$$\lim_{x \to x_0} \frac{y - y_0}{x - x_0} = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4.$$

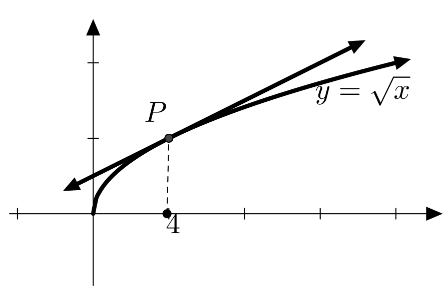
Finally, the equation of the tangent line with slope $m = 4$ and passing through $P(2, 4)$ is

$$y - 4 = 4(x - 2) \quad \text{or} \quad y = 4x - 4.$$

Teaching Tip

If you want to make your own examples, make sure that P is a point on the curve! For example, it does not make sense to find the equation of the tangent line to $y = x^2$ at the point $P(2, 3)$ since P is not on the parabola ($3 \neq 2^2$). You could modify this to $P(2, 4)$ or better yet, for later examples, you can just ask for the tangent line at a specific x-coordinate.

EXAMPLE 4: Find the slope-intercept form of the tangent line to $f(x) = \sqrt{x}$ at $x = 4$.



Solution. Again, we find the y-coordinate of $x_0 = 4$: $y_0 = f(x_0) = \sqrt{x_0} = \sqrt{4} = 2$. Hence, P has coordinates $(4, 2)$. Now, we look for the slope of the tangent line. Notice that we have to rationalize the numerator to evaluate the limit.

$$
\begin{aligned}
m = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} &= \lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2} \\
&= \lim_{x\to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)} \\
&= \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}.
\end{aligned}
$$

Finally, with point $P(4,2)$ and slope $m=\frac{1}{4}$, the equation of the tangent line is

$$y-2=\frac{1}{4}(x-4) \quad \text{or} \quad y=\frac{x}{4}+1.$$

The next example shows that our process of finding the tangent line works even for horizontal lines.

EXAMPLE 5: Show that the tangent line to $y=3x^2-12x+1$ at the point $(2,-11)$ is horizontal.

Solution. Recall that a horizontal line has zero slope. Now, computing for the slope, we get:

$$
\begin{aligned}
m = \lim_{x\to x_0} \frac{y-y_0}{x-x_0} &= \lim_{x\to 2} \frac{(3x^2-12x+1)-(-11)}{x-2} \\
&= \lim_{x\to 2} \frac{3(x^2-4x+4)}{x-2} \\
&= \lim_{x\to 2} (3(x-2)) = 0.
\end{aligned}
$$

Since the slope of the tangent line is 0, it must be horizontal. Its equation is

$$y-(-11)=0(x-2) \quad \text{or} \quad y=-11.$$

EXAMPLE 6: Verify that the tangent line to the line $y=2x+3$ at $(1,5)$ is the line itself.

Solution. We first compute for the slope of the tangent line. Note that $x_0=1$ and $y_0=5$.

$$m=\lim_{x\to x_0}\frac{y-y_0}{x-x_0}=\lim_{x\to 1}\frac{(2x+3)-5}{x-1}=\lim_{x\to 1}\frac{2x-2}{x-1}=2.$$

Therefore, substituting this into the point-slope form with $P(1,5)$ and $m=2$, we get

$$y-5=2(x-1) \quad \text{i.e.,} \quad y=2x+3.$$

This is the same equation as that of the given line.

(C) **EXERCISES**

Find the standard (slope-intercept form) equation of the tangent line to the following functions at the specified points:

1. $f(x) = 3x^2 - 12x + 1$ at the point $(0, 1)$ Answer: $y = -12x + 1$
2. $f(x) = 2x^2 - 4x + 5$ at the point $(-1, 11)$ Answer: $y = -8x - 9$
3. $f(x) = \sqrt{x+9}$ at the point where $x = 0$ Answer: $y = \dfrac{x}{6} + 3$

$\star$4. $f(x) = \sqrt{25 - x^2}$ at the point where $x = 4$ Answer: $y = -\frac{4x}{3} + \frac{25}{3}$

$\star$5. $f(x) = x^2 + \sqrt{x}$ at the point where $x = 1$ Answer: $y = \frac{5x}{2} - \frac{1}{2}$

Teaching Tip

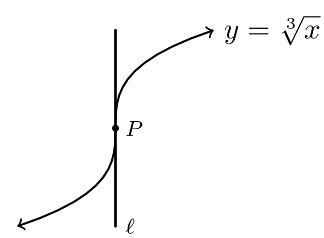
It is not productive to dwell on finding the slope of the tangent line to very complicated functions. A more efficient way of finding this will be given in the next sections when we discuss derivatives.

(D) **ENRICHMENT**

This section explores the equation of vertical tangent lines. In the last topic, we remarked that vertical tangent lines may exist. However, we know that the slope of a vertical line does not exist or is undefined. How do we reconcile these seemingly contradicting ideas?

For example, consider the vertical tangent line to the graph of $y = \sqrt[3]{x}$ at $P(0, 0)$. If we compute its slope, we have

$$m = \lim_{x \to x_0} \frac{y - y_0}{x - x_0} = \lim_{x \to 0} \frac{\sqrt[3]{x} - 0}{x - 0} = \lim_{x \to 0} \frac{1}{\sqrt[3]{x^2}}.$$



Observe that the last expression is undefined $\left(\dfrac{1}{0}\right)$ if we substitute x with 0. Hence, the slope of this tangent line is undefined.

Therefore, our computation for the slope of the tangent line to this curve is actually consistent with our idea of the slope of a vertical line. The next question to ask is: Does this tangent line have an equation? The answer is yes. Recall that a vertical line passing through the point (x_0, y_0) possesses the equation $x = x_0$.

Equation of Vertical Tangent Lines

Let f be a function that is continuous at x_0. Assuming that the tangent line to the graph of $y = f(x)$ at the point $P(x_0, y_0)$ is vertical, then its equation is

$$x = x_0.$$

TOPIC 5.3: The Definition of the Derivative

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

The expression

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

computes more than just the slope of the tangent line. The fraction in the limit also gives us the relative change of the function on the time or space interval $[x_0, x]$. Thus, we may interpret this limit as the instantaneous rate of change of f with respect to x. (See Enrichment) It is therefore fitting to make an abstraction out of this. Afterwards, we can regard this as a tool that can be used in finding the slope of the tangent line, or in finding the instantaneous rate of change. In what follows, we define this expression as the *derivative of $f(x)$ at x_0*.

(B) LESSON PROPER

We present the formal definition of the derivative.

Definition of the Derivative
Let f be a function defined on an open interval $I \subseteq \mathbb{R}$, and let $x_0 \in I$. The derivative of f at x_0 is defined to be

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

if this limit exists. That is, the derivative of f at x_0 is the slope of the tangent line at $(x_0, f(x_0))$, if it exists.

Notations: If $y = f(x)$, the derivative of f is commonly denoted by

$$f'(x),\ D_x[f(x)],\ \frac{d}{dx}[f(x)],\ \frac{d}{dx}[y],\ \frac{dy}{dx}.$$

Remark 1: Note that the limit definition of the derivative is inherently *indeterminate*! Hence, the usual techniques for evaluating limits which are indeterminate of type $\frac{0}{0}$ are applied, e.g., **factoring**, **rationalization**, or **using one of the following established limits**:

(i) $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = 1$ (ii) $\displaystyle\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$ (iii) $\displaystyle\lim_{x \to 0} \frac{e^x - 1}{x} = 1.$

EXAMPLE 1: Compute $f'(1)$ for each of the following functions:

1. $f(x) = 3x - 1$
2. $f(x) = 2x^2 + 4$
3. $f(x) = \dfrac{2x}{x+1}$
4. $f(x) = \sqrt{x+8}$

***Solution*.** Here, x_0 is fixed to be equal to 1. Using the definition above,

$$f'(1) = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1}.$$

Remember that what we are computing, $f'(1)$, is just the slope of the tangent line to $y = f(x)$ at $x = 1$.

1. Note that $f(1) = 2$, so by factoring

$$\begin{aligned} f'(1) &= \lim_{x\to 1} \frac{(3x-1) - 2}{x - 1} \\ &= \lim_{x\to 1} \frac{3(x-1)}{x-1} \\ &= \lim_{x\to 1} 3 \\ &= 3. \end{aligned}$$

2. Here, $f(1) = 6$ so again, by factoring,

$$\begin{aligned} f'(1) &= \lim_{x\to 1} \frac{(2x^2+4) - 6}{x - 1} \\ &= \lim_{x\to 1} \frac{2(x+1)(x-1)}{x-1} \\ &= \lim_{x\to 1} 2(x+1) \\ &= 4. \end{aligned}$$

3. We see that $f(1) = 1$. So, from the definition,

$$f'(1) = \lim_{x\to 1} \frac{\frac{2x}{x+1} - 1}{x - 1}.$$

We multiply both the numerator and the denominator by $x + 1$ to simplify the complex fraction:

$$\begin{aligned} f'(1) &= \lim_{x\to 1} \frac{\frac{2x}{x+1} - 1}{x - 1} \cdot \frac{x+1}{x+1} \\ &= \lim_{x\to 1} \frac{2x - (x+1)}{(x-1)(x+1)} \\ &= \lim_{x\to 1} \frac{x-1}{(x-1)(x+1)} \\ &= \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}. \end{aligned}$$

4. Note that $f(1) = 3$. Therefore, by rationalizing the numerator (meaning, multiplying by $\sqrt{x+8} + 3$),

$$\begin{aligned} f'(1) &= \lim_{x \to 1} \frac{\sqrt{x+8}-3}{x-1} \cdot \frac{\sqrt{x+8}+3}{\sqrt{x+8}+3} \\ &= \lim_{x \to 1} \frac{(x+8)-9}{(x-1)(\sqrt{x+8}+3)} \\ &= \lim_{x \to 1} \frac{1}{\sqrt{x+8}+3} \\ &= \frac{1}{6}. \end{aligned}$$

AN EQUIVALENT DEFINITION OF THE DERIVATIVE

Recall that we have defined the derivative of a function f at x_0 as follows:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

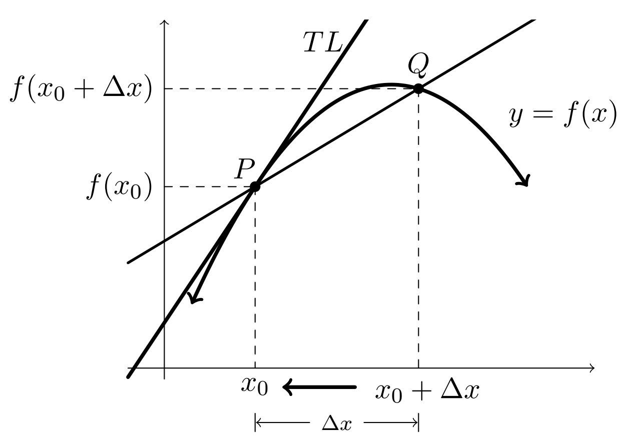
There is another definition of the derivative which is derived by using the substitution

$$x = x_0 + \Delta x \tag{2.1}$$

into the above limit definition of the derivative.

Observe that Δx measures the displacement as we move from x to x_0. Thus, in the figure below, the point to the right of x_0 becomes $x_0 + \Delta x$, if $\Delta x > 0$. It should be clear algebraically from (2.1), and from the figure, that **letting x approach x_0 is equivalent to letting Δx approach 0.** Applying the substitution, we now have

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{(x_0 + \Delta x) - x_0}.$$



We present this fact formally below:

Alternative Definition of the Derivative

Let f be a function defined on an open interval $I \subseteq \mathbb{R}$, and let $x \in I$. The derivative of f at x is defined to be

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{2.2}$$

if this limit exists.

Teaching Tip

Confusion may arise when students think of Δx as $\Delta \times x$ or Δ as an operator that affects x. Here, Δx should be treated like any other variable. This is why many textbooks use h instead of Δx. In this case, equation (2.2) becomes

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

Usually, this is the definition used to obtain the general expression of the derivative of a function at any point $x \in I$.

Remark 2: Please remind the students that $f(x_0 + h)$ is basically a composition of two functions. Therefore, it is determined by replacing all instances of x in the definition of f by $x_0 + h$. For example, if

$$f(x) = x^2 + 3x \qquad \text{and} \qquad g(x) = \cos(3x) - e^x,$$

then

$$f(x_0 + h) = (x_0 + h)^2 + 3(x_0 + h) \quad \text{and} \quad g(x_0 + h) = \cos(3(x_0 + h)) - e^{x_0 + h}.$$

EXAMPLE 2: Let $f(x) = \sin x$, $g(x) = \cos x$, and $s(x) = e^x$. Find $f'(2\pi)$, $g'(\pi)$, and $s'(3)$.

Solution. We use the alternative definition of the derivative.

(a) Here, we substitute $x_0 = 2\pi$.

$$\begin{aligned} f'(2\pi) &= \lim_{h \to 0} \frac{f(2\pi + h) - f(2\pi)}{h} \\ &= \lim_{h \to 0} \frac{\sin(2\pi + h) - 0}{h}. \end{aligned}$$

Using the sum identity of the sine function: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, and noting that $\sin(2\pi) = 0$ and $\cos(2\pi) = 1$, we get

$$\begin{aligned} f'(2\pi) &= \lim_{h\to 0} \frac{\sin(2\pi)\cos h + \cos(2\pi)\sin h}{h} \\ &= \lim_{h\to 0} \frac{\sin h}{h} \\ &= 1. \qquad\qquad \text{(Why?)} \end{aligned}$$

(b)

$$\begin{aligned} g'(\pi) &= \lim_{h\to 0} \frac{g(\pi+h) - g(\pi)}{h} \\ &= \lim_{h\to 0} \frac{\cos(\pi+h) - (-1)}{h}. \end{aligned}$$

Using the sum identity of the cosine function: $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, and noting that $\cos\pi = -1$ and $\sin\pi = 0$, we get

$$\begin{aligned} g'(\pi) &= \lim_{h\to 0} \frac{\cos\pi\cos h - \sin\pi\sin h + 1}{h} \\ &= \lim_{h\to 0} \frac{-\cos h + 1}{h} \\ &= 0. \qquad\qquad \text{(Why?)} \end{aligned}$$

(c)

$$\begin{aligned} s'(3) &= \lim_{h\to 0} \frac{s(3+h) - s(3)}{h} \\ &= \lim_{h\to 0} \frac{e^{3+h} - e^3}{h}. \end{aligned}$$

Using the exponent laws, $e^{3+h} = e^3 e^h$. Moreover, since e^3 is just a constant, we can factor it out of the limit operator. So,

$$\begin{aligned} s'(3) &= \lim_{h\to 0} \frac{e^3 e^h - e^3}{h} \\ &= e^3 \lim_{h\to 0} \frac{e^h - 1}{h} \\ &= e^3. \qquad\qquad \text{(Why?)} \end{aligned}$$

Seatwork: Let $f(x) = 2x^2 + 3x - 1$. Use the definition of the derivative to find $f'(-1)$.

Solution. Note that $f(-1) = -2$. Thus,

$$\begin{aligned} f'(-1) &= \lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)} \\ &= \lim_{x \to -1} \frac{2x^2 + 3x - 1 + 2}{x + 1} \\ &= \lim_{x \to -1} \frac{(2x + 1)(x + 1)}{x + 1} \\ &= \lim_{x \to -1} (2x + 1) = -1. \end{aligned}$$

INSTANTANEOUS VELOCITY OF A PARTICLE IN RECTILINEAR MOTION

The derivative of a function is also interpreted as the instantaneous rate of change. We discuss here a particular quantity which is important in physics – the instantaneous velocity of a moving particle. The general setting of the difference between the average and instantaneous rates of change will be presented in the Enrichment section.

Suppose that an object or a particle starts from a fixed point A and moves along a straight line towards a point B. Suppose also that its position along line $\overline{AB}$ at time t is s. Then the motion of the particle is completely described by the ***position function***

$$s = s(t), \quad t \geq 0$$

and since the particle moves along a line, it is said to be in ***rectilinear motion***.

EXAMPLE 3: Suppose that a particle moves along a line with position function $s(t) = 2t^2 + 3t + 1$ where s is in meters and t is in seconds.

a. What is its initial position?

b. Where is it located after $t = 2$ seconds?

c. At what time is the particle at position $s = 6$?

Solution.

a. The initial position corresponds to the particle's location when $t = 0$. Thus,

$$s(0) = 2(0)^2 + 3(0) + 1.$$

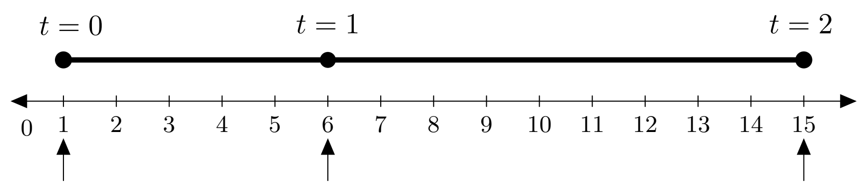
This means that the particle can be found 1 meter to the right of the origin.

b. After 2 seconds, it can now be found at position $s(2) = 2(2)^2 + 3(2) + 1 = 15$ meters.

c. We equate $s(t) = 2t^2 + 3t + 1 = 6$. So,

$$2t^2 + 3t - 5 = 0 \iff (2t+5)(t-1) = 0 \iff t = -\frac{5}{2} \text{ or } t = 1.$$

Since time cannot be negative, we choose $t = 1$ second.



Now, we ask: What is the particle's velocity at the instant when time $t = 1$?

Recall that the formula for the ***average velocity*** of a particle is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time elapsed}} = \frac{s(t_{\text{final}}) - s(t_{\text{initial}})}{t_{\text{final}} - t_{\text{initial}}}.$$

This poses a problem because at the instant when $t = 1$, there is no elapsed time. We remedy this by computing the velocity at short time intervals with an endpoint at $t = 1$.

For example, on the time interval $[1, 2]$, the velocity of the particle is

$$v = \frac{s(2) - s(1)}{2 - 1} = \frac{15 - 6}{2 - 1} = 9 \text{ m/s}.$$

We compute the particle's velocity on shorter intervals:

Time Interval	Average Velocity
$[1, 1.5]$	8
$[1, 1.1]$	7.2
$[1, 1.01]$	7.02
$[1, 1.001]$	7.002

Time Interval	Average Velocity
$[0.5, 1]$	6
$[0.9, 1]$	6.8
$[0.99, 1]$	6.98
$[0.999, 1]$	6.998

We see from the tables above that the velocities of the particle on short intervals ending or starting at $t = 1$ approach 7 m/s as the lengths of the time intervals approach 0. This limit

$$\lim_{t \to 1} \frac{s(t) - s(1)}{t - 1}$$

is what we refer to as the ***instantaneous velocity*** of the particle at $t = 1$. However, the limit expression above is precisely the definition of the derivative of s at $t = 1$, and the instantaneous velocity is actually the slope of the tangent line at the point $t = t_0$ if the function in consideration is the position function. We make the connection below.

Instantaneous velocity

Let $s(t)$ denote the position of a particle that moves along a straight line at each time $t \geq 0$. The instantaneous velocity of the particle at time $t = t_0$ is

$$s'(t_0) = \lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0},$$

if this limit exists.

EXAMPLE 4: A ball is shot straight up from a building. Its height (in meters) from the ground at any time t (in seconds) is given by $s(t) = 40 + 35t - 5t^2$. Find

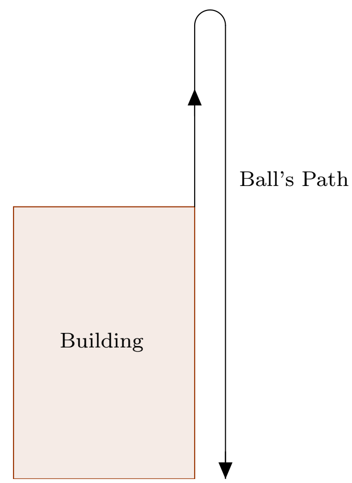
a. the height of the building.

b. the time when the ball hits the ground.

c. the average velocity on the interval $[1, 2]$.

d. the instantaneous velocity at $t = 1$ and 2.

e. the instantaneous velocity at any time t_0.



Solution.

a. The height of the building is the initial position of the ball. So, the building is $s(0) = 40$ meters tall.

b. The ball is on the ground when the height s of the ball from the ground is zero. Thus we solve the time t when $s(t) = 0$:

$$30 + 40t - 5t^2 = 0 \iff 5(8 - t)(1 + t) = 0 \iff t = 8 \text{ or } t = -1.$$

Since time is positive, we choose $t = 8$ seconds.

c. The average velocity of the ball on $[1, 2]$ is $\dfrac{s(2) - s(1)}{2 - 1} = \dfrac{90 - 70}{2 - 1} = 20$ m/s.

d. Then instantaneous velocity at time $t = 1$ is

$$\lim_{t \to 1} \frac{s(t) - s(1)}{t - 1} = \lim_{t \to 1} \frac{(40 + 35t - 5t^2) - 70}{t - 1} = \lim_{t \to 1} \frac{-5(t - 6)(t - 1)}{t - 1} = 25 \text{ m/s}.$$

At time $t = 2$,

$$\lim_{t \to 2} \frac{s(t) - s(2)}{t - 2} = \lim_{t \to 2} \frac{(40 + 35t - 5t^2) - 90}{t - 2} = \lim_{t \to 2} \frac{-5(t - 5)(t - 2)}{t - 2} = 15 \text{ m/s}.$$

e. The instantaneous velocity function at any time t_0 is

$$\begin{aligned}\lim_{t\to t_0}\frac{s(t)-s(t_0)}{t-t_0} &= \lim_{t\to 1}\frac{(40+35t-5t^2)-(40+35t_0-5t_0^2)}{t-t_0}\\ &= \lim_{t\to 1}\frac{5(t-t_0)(7-(t+t_0))}{t-t_0}\\ &= (35-10t_0)\text{ m/s.}\end{aligned}$$

Remark 3: Note that in (e) of the previous example, if we set $t_0 = 4$, then the instantaneous velocity is -5 m/s. The negative sign gives the direction of the velocity (since it is a vector quantity), and means that the ball is moving towards the negative direction of the line, in this case, moving downwards. If the sign of the velocity is positive, then the particle is moving towards the positive direction (that is, in the direction from the origin towards the positive numbers) of the s-axis.

(C) EXERCISES

1. For each of the following functions, find the indicated derivative using the definition.

a. $f(x) = x^2 - 4x + 1$; $f'(2)$ — Answer: 0

b. $f(x) = x^3 + 2$; $f'(-2)$ — Answer: 12

c. $f(x) = 2x^4 + 3x^3 - 2x + 7$; $f'(0)$ — Answer: -2

d. $f(x) = \sqrt{2x+7}$; $f'(1)$ — Answer: $1/3$

e. $f(x) = 1 + \sqrt{x^2+3x+6}$; $f'(2)$ — Answer: $7/8$

f. $f(x) = \dfrac{x}{x+4}$; $f'(-3)$ — Answer: 4

g. $f(x) = \dfrac{x^2+3}{4-x^2}$; $f'(-1)$ — Answer: $-14/9$

h. $f(x) = \dfrac{\sqrt{x+3}}{x-4}$; $f'(1)$ — Answer: $-11/36$

i. $f(x) = \dfrac{3-\sqrt{5x-9}}{2x-1}$; $f'(2)$ — Answer: $-23/18$

j. $f(x) = 2\sin(\pi x)$; $f'(-3)$ — Answer: -2π

k. $f(x) = x^2\cos x$; $f'(0)$ — Answer: 0

l. $f(x) = e^{x+1}$; $f'(-1)$ — Answer: 1

⋆m. $f(x) = \sqrt[3]{x-1}$; $f'(9)$ — Answer: $1/12$

⋆n. $f(x) = \sqrt[3]{7x+6}$; $f'(3)$ — Answer: $7/27$

⋆o. $f(x) = \tan(3x)$; $f'(\pi/4)$ — Answer: 6

⋆p. $f(x) = 2\cos x - 3\sin(2x)$; $f'(\pi/6)$ — Answer: -4

⋆q. $f(x) = 1 - 3\sin^2 x$; $f'(\pi/3)$ — Answer: $-3\sqrt{3}/2$

$\star$r. $f(x) = xe^x$; $f'(1)$ Answer: $2e$

$\star$s. $f(x) = \dfrac{x}{1+\sin(\pi x)}$; $f'(1/2)$ Answer: $1/2$

$\star$t. $f(x) = \dfrac{x+\cos(3\pi x)}{x^2}$; $f'(1)$ Answer: 1

2. A billiard ball is hit and travels in a straight line. If s centimeters is the distance of the ball from its initial position at t seconds, then $s = 100t2 + 100t$. If the ball hits the cushion at 39 cm from its initial position, at what velocity does it hit the cushion at that particular instant? (*Hint*: Determine first the time when $s = 39$ and find the instantaneous velocity at this time.)

3. A particle is moving along a straight line and its position at any time $t \geq 0$ (in seconds) is given by $s(t) = t^3 - 6t^2 + 9t$ meters.

   a. Find the average velocity of the particle on the time interval $[0, 2]$?

   b. Find the velocity of the particle at the instant when $t = 2$?

4. A stone is thrown vertically upward from the top of a building. If the equation of the motion of the stone is $s = -5t2 + 30t + 200$, where s is the directed distance from the ground in meters and t is in seconds,

   a. what is the height of the building?

   b. what is the average velocity on the time interval $[1, 3]$?

   c. what is the instantaneous velocity at time $t = 1$?

   d. at what time will the stone hit the ground?

   e. what is the instantaneous velocity of the stone upon impact?

(D) ENRICHMENT

In this section, we discuss average and instantaneous rate of change.

AVERAGE RATE OF CHANGE

There are several instances where the rate of change of a certain quantity is of interest.

(a) A statistician may be interested in how the population of a certain city changes with time.

(b) A civil engineer may want to know how the length of metal bars changes with temperature.

(c) A company manager can study how the production cost of a certain product increases as the number of manufactured products also increases.

(d) A chemist may be interested in how the volume of a certain compound changes with increased pressure.

To compute the average rate of change of a quantity y with respect to another quantity x, we need first to establish a function f that describes the relationship of x and y. Suppose that this relationship is given by $y = f(x)$. Now, fix x_0 in the domain of f. Then, the **average rate of change of f over the interval** $[x_0, x]$ is

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0}.$$

EXAMPLE 5: Suppose the cost (in pesos) of manufacturing x liters of a certain solution is given by $C(x) = 20 + 5x - 2x^2 + x^3$. Compute the average rate of change of the cost y of producing x liters over (i) $[1, 4]$, (ii) $[1, 2]$, (iii) $[1, 1.5]$.

Solution. To solve this, we need the values of $C(1), C(1.5), C(2)$, and $C(4)$. Computing, we get $C(1) = 20 + 5(1) - 2(1)^2 + 1^3 = ₱24$. Similarly, $C(1.5) = ₱26.375$, $C(2) = ₱30$, and $C(4) = ₱72$.

(i) $\dfrac{C(4) - C(1)}{4 - 1} = \dfrac{72 - 24}{4 - 1} = \dfrac{48}{3} = ₱16/\text{liter}$

(ii) $\dfrac{C(2) - C(1)}{2 - 1} = \dfrac{30 - 24}{2 - 1} = \dfrac{6}{1} = ₱6/\text{liter}$

(iii) $\dfrac{C(1.5) - C(1)}{1.5 - 1} = \dfrac{26.375 - 24}{1.5 - 1} = \dfrac{2.375}{0.5} = ₱4.75/\text{liter}$

Further Discussion: Try to ask the class how these values are interpreted. So, for instance, the values above mean that you need ₱24 to produce the first liter. Treating this as our baseline, we see that we need $C(4) - C(1) = ₱48$ to produce three more liters, while $C(2) - C(1) = ₱6$ more to produce one more liter. However, you need only $C(1.5) - C(1) = ₱2.375$ to produce half a liter more.

Teaching Tip

Ask the class why there are cases wherein the production cost of the jth product is different from that of the kth product. In the above example, the first product costs ₱24 while the second one costs only ₱6 given that the first liter is already produced.

A possible answer to this is that most of the time, the cost of producing the first product is significantly higher because it involves the overhead costs of running the manufacturing process. This is also the reason why products that are manufactured in bulk are cheaper compared to those that are "made-to-order."

Hence, on the average, given that the company has already manufactured one liter, the production of the next three liters will cost ₱16/liter while it only costs ₱4.75/liter for the next half-liter. You can then ask the class how much the cost per liter is if the company wants to produce only a small amount after the first liter.

This table gives the average rate of change of the cost over $[1,a]$ where a is a number very close to 1.

a	$\frac{C(a)-C(1)}{a-1}$
1.3	4.39
1.2	4.24
1.1	4.11
1.01	4.0101

We see that the average rate of change over $[1,a]$ approaches 4 as a approaches 1. We say that this limit is the **instantaneous rate of change** of the cost function at $x=1$.

Average and Instantaneous Rate of Change

Suppose f is a function and $y=f(x)$.

(a) The **average rate of change of y with respect to x on** $[x_0,x]$ is

$$\frac{\Delta f}{\Delta x}=\frac{f(x)-f(x_0)}{x-x_0}.$$

(b) The **instantaneous rate of change of y with respect to x at** $x=x_0$ is

$$\lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x}=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0).$$

The instantaneous rate of change is what we will define to be the derivative of y with respect to x.

EXAMPLE 6: Verify that the instantaneous rate of change of the cost function above at $x=1$ is equal to ₱4/liter.

Solution. Recall that the instantaneous rate of change of a function C at $x=1$ is precisely the derivative of C at $x=1$. Since $C(1)=24$, then

$$C'(1)=\lim_{x\to 1}\frac{C(x)-C(1)}{x-1}=\lim_{x\to 1}\frac{(20+5x-2x^2+x^3)-24}{x-1}.$$

Using synthetic or long division, we see that $\dfrac{(20+5x-2x^2+x^3)-24}{x-1}=x^2-x+4$. Therefore, it follows that

$$C'(1)=\lim_{x\to 1}(x^2-x+4)=4.$$

LESSON 6: Rules of Differentiation

TIME FRAME: 3 hours

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Determine the relationship between differentiability and continuity;
2. Derive the differentiation rules; and
3. Apply the differentiation rules in computing the derivatives of algebraic, exponential, and trigonometric functions.

LESSON OUTLINE:

1. Definitions on continuity and differentiability
2. Relationship between differentiability and continuity
3. Sample exercises on the relationship between differentiability and continuity
4. Derivation of the different differentiation rules
5. Examples on the different differentiation rules
6. Applications

TOPIC 6.1: Differentiability Implies Continuity

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

The difference between continuity and differentiability is a critical issue. Most, but not all, of the functions we encounter in calculus will be differentiable over their entire domain. Before we can confidently apply the rules regarding derivatives, we need to be able to recognize the exceptions to the rule.

Recall the following definitions:

Definition 1 (Continuity at a Number)**.** *A function f is* ***continuous at a number*** *c if all of the following conditions are satisfied:*

(*i*) *$f(c)$ is defined;*

(*ii*) *$\lim_{x\to c} f(x)$ exists; and*

(*iii*) *$\lim_{x\to c} f(x) = f(c)$.*

If at least one of the these conditions is not satisfied, the function is said to be ***discontinuous*** *at c.*

Definition 2 (Continuity on $\mathbb{R}$)**.** *A function f is said to be* ***continuous everywhere*** *if f is continuous at every real number.*

Definition 3. *A function f is* ***differentiable at the number*** *c if*

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$

exists.

(B) LESSON PROPER

We now present several examples of determining whether a function is continuous or differentiable at a number.

EXAMPLE 1:

1. The piecewise function defined by
$$f(x) = \begin{cases} \dfrac{x^2+2x-3}{x-1} & \text{if } x \neq 1, \\ 4 & \text{if } x = 1, \end{cases}$$
is continuous at $c = 1$. This is because $f(1) = 4$,
$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{(x-1)(x+3)}{x-1} = 4,$$
and $f(1) = \lim\limits_{x\to 1} f(x)$.

2. The function defined by
$$f(x) = \begin{cases} -x^2 & \text{if } x < 2, \\ 3-x & \text{if } x \geq 2. \end{cases}$$
is not continuous at $c = 2$ since $\lim\limits_{x\to 2^-} f(x) = -4 \neq 1 = \lim\limits_{x\to 2^+} f(x)$, hence the $\lim\limits_{x\to 2} f(x)$ does not exist.

3. Consider the function $f(x) = \sqrt[3]{x}$. By definition, its derivative is
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h} \cdot \frac{\sqrt[3]{(x+h)^2}+\sqrt[3]{(x+h)(x)}+\sqrt[3]{x^2}}{\sqrt[3]{(x+h)^2}+\sqrt[3]{(x+h)(x)}+\sqrt[3]{x^2}} \\ &= \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt[3]{(x+h)^2}+\sqrt[3]{(x+h)(x)}+\sqrt[3]{x^2})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt[3]{(x+h)^2}+\sqrt[3]{(x+h)(x)}+\sqrt[3]{x^2}} \\ &= \frac{1}{3\sqrt[3]{x^2}}. \end{aligned}$$
Since $f'(1) = \dfrac{1}{3\sqrt[3]{1^2}} = \dfrac{1}{3}$, then f is differentiable at $x = 1$. On the other hand, $f'(0)$ does not exist. Hence f is not differentiable at $x = 0$.

4. The function defined by
$$f(x) = \begin{cases} 5x & \text{if } \; x < 1 \\ 2x+3 & \text{if } \; x \geq 1 \end{cases}$$
is continuous at $x = 1$ but is not differentiable at $x = 1$. Indeed, $f(1) = 2(1) + 3 = 5$. Now,
   - If $x < 1$, then $f(x) = 5x$ and so $\lim\limits_{x\to 1^-} 5x = 5$.
   - If $x > 1$, then $f(x) = 2x + 3$ and so $\lim\limits_{x\to 1^+} (2x+3) = 5$.

Since the one-sided limits exist and are equal to each other, the limit exists and equals 5. So,

$$\lim_{x \to 1} f(x) = 5 = f(1).$$

This shows that f is continuous at $x = 1$. On the other hand, computing for the derivative,

- For $x < 1$, $f(x) = 5x$ and $\displaystyle\lim_{h \to 0^-} \frac{5(x+h) - (5x)}{h} = 5.$
- For $x > 1$, $f(x) = 2x + 3$ and $\displaystyle\lim_{h \to 0^+} \frac{(2(x+h)+3) - (2x+3)}{h} = 2.$

Since the one-sided limits at $x = 1$ do not coincide, the limit at $x = 1$ does not exist. Since this limit is the definition of the derivative at $x = 1$, we conclude that f is not differentiable at $x = 1$.

5. Another classic example of a function that is continuous at a point but not differentiable at that point is the absolute value function $f(x) = |x|$ at $x = 0$. Clearly, $f(0) = 0 = \lim_{x \to 0} |x|$. However, if we look at the limit definition of the derivative,

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}.$$

Note that the absolute value function is defined differently to the left and right of 0 so we need to compute one-sided limits. Note that if h approaches 0 from the left, then it approaches 0 through negative values. Since $h < 0 \implies |h| = -h$, it follows that

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1.$$

Similarly, if h approaches 0 from the right, then h approaches 0 through positive values. Since $h > 0 \implies |h| = h$, we obtain

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1.$$

Hence, the derivative does not exist at $x = 0$ since the one-sided limits do not coincide.

The previous two examples prove that continuity does not necessarily imply differentiability. That is, there are functions which are continuous at a point, but is not differentiable at that point. The next theorem however says that the converse is always TRUE.

Theorem 6. *If a function f is differentiable at a, then f is continuous at a.*

Proof. That function f is differentiable at a implies that $f'(a)$ exists. To prove that f is continuous at a, we must show that

$$\lim_{x \to a} f(x) = f(a),$$

or equivalently,

$$\lim_{h\to 0} f(a+h) = f(a).$$

If $h \neq 0$, then

$$\begin{aligned} f(a+h) &= f(a) + f(a+h) - f(a) \\ &= f(a) + \frac{f(a+h)-f(a)}{h} \cdot h. \end{aligned}$$

Taking the limit as $h \to 0$, we get

$$\begin{aligned} \lim_{h\to 0} f(a+h) &= \lim_{h\to 0} f(a) + \lim_{h\to 0} \left[\frac{f(a+h)-f(a)}{h} \cdot h\right] \\ &= f(a) + f'(a)\cdot 0 \\ &= f(a). \end{aligned}$$

□

Remark 1:

(a) If f is continuous at $x = a$, it does not mean that f is differentiable at $x = a$.

(b) If f is not continuous at $x = a$, then f is not differentiable at $x = a$.

(c) If f is not differentiable at $x = a$, it does not mean that f is not continuous at $x = a$.

(d) A function f is not differentiable at $x = a$ if one of the following is true:

  i. f is not continuous at $x = a$.

  ii. the graph of f has a *vertical* tangent line at $x = a$.

  iii. the graph of f has a corner or cusp at $x = a$.

Teaching Tip

A lot of students *erroneously* deduce that the verb for "getting the derivative" is "to derive". Please correct this. The right verb is "to differentiate". Moreover, the process of getting the derivative is "differentiation" — not "derivation".

(C) **EXERCISES:**

1. Suppose f is a function such that $f'(5)$ is undefined. Which of the following statements is always true?

   a. f must be continuous at $x = 5$.

   b. f is definitely not continuous at $x = 5$.

c. There is not enough information to determine whether or not f is continuous at $x = 5$. Answer: (a) False. Counterexample: any function that is not continuous at 5; (b) False. Counterexample $f(x) = |x - 5|$; (c) True.

2. Which of the following statements is/are always true?

   I. A function that is continuous at $x = a$ is differentiable at $x = a$.

   II. A function that is differentiable at $x = a$ is continuous at $x = a$.

   III. A function that is NOT continuous at $x = a$ is NOT differentiable at $x = a$.

   IV. A function that is NOT differentiable at $x = a$ is NOT continuous at $x = a$.

   a. none of them

   b. I and III

   c. I and IV

   d. II and IV

   e. II and III

   f. all of them

   Answer: only (e) is always true

3. Suppose that f is a function that is continuous at $x = -3$. Which of the following statements are true?

   a. f must be differentiable at $x = -3$. Answer: False, e.g. $f(x) = |x + 3|$.

   b. f is definitely not differentiable at $x = -3$. Answer: False, e.g. $f(x) = x$.

4. Consider the function defined by

$$f(x) = \begin{cases} x^2 & \text{if} \quad x < 3, \\ 6x - 9 & \text{if} \quad x \geq 3. \end{cases}$$

   For each statement below, write *True* if the statement is correct and *False*, otherwise. At $x = 3$, the function is

   a. undefined. Answer: False

   b. differentiable but not continuous. Answer: False

   c. continuous but not differentiable. Answer: False

   d. both continuous and differentiable. Answer: True

   e. neither continuous nor differentiable. Answer: False

5. Determine the values of x for which the function is continuous.

   a. $f(x) = \dfrac{x+5}{x^2 - 5x + 4}$

   b. $f(x) = \dfrac{x}{4 - x^2}$

   c. $f(x) = \dfrac{3x+1}{2x^2 - 8x}$

   d. $f(x) = \dfrac{4x}{x^2 - 9}$

   e. $f(x) = \sqrt{x - 3}$

   f. $f(x) = \sqrt{\dfrac{x-2}{x+3}}$

6. Determine the largest subset of $\mathbb{R}$ where $f(x) = \sqrt{25 - x^2}$ is continuous.

7. Is the function defined by $g(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?

8. Is the function defined by $f(x) = |x - 1|$ differentiable at $x = 1$?

9. Is the function defined by

$$f(x) = \begin{cases} x^3 - 3 & \text{if} \quad x \leq 2, \\ x^2 + 1 & \text{if} \quad x > 2. \end{cases}$$

continuous at $x = 2$? differentiable at $x = 2$?

10. Consider the function defined by $f(x) = \sqrt{x}$. Is f differentiable at $x = 1$? at $x = 0$? at $x = -1$?

TOPIC 6.2: The Differentiation Rules and Examples Involving Algebraic, Exponential, and Trigonometric Functions

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

How do we find derivatives?

The procedure for finding the exact derivative directly from a formula of the function without having to use graphical methods is called differential calculus. In practice, we use some rules that tell us how to find the derivative of almost any function. In this lesson, we will introduce these rules.

You may start by asking the students to compute the derivatives of the following functions using the limit definition (formal definition):

(a) $f(x) = 3x^2 + 4$

(b) $g(x) = \dfrac{-4x^2 + 3}{x - 1}$

Give them a few minutes to solve.

After a few minutes, (using their answers) tell the students that computing the derivative of a given function from the definition is usually time consuming. Thus, this lesson will help them compute the derivative of a given function more easily.

(B) LESSON PROPER

We first recall the definition of the derivative of a function.

The derivative of the function f the function f' whose value at a number x in the domain of f is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2.3}$$

if the limit exists.

For example, let us compute the derivative of the first function of the seatwork above: $f(x) = 3x^2 + 4$. Let us first compute the numerator of the quotient in (2.3):

$$\begin{aligned} f(x+h) - f(x) &= (3(x+h)^2 + 4) - (3x^2 + 4) \\ &= (3x^2 + 6xh + 3h^2 + 4) - (3x^2 + 4) \\ &= 6xh + 3h^2 \\ &= h(6x + 3h). \end{aligned}$$

Therefore,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{3(x+h)^2+4-(3x^2+4)}{h}\\
&= \lim_{h\to 0}\frac{h(6x+3h)}{h}\\
&= \lim_{h\to 0}(6x+3h)\\
&= 6x.
\end{aligned}$$

We see that computing the derivative using the definition of even a simple polynomial is a lengthy process. What follows next are rules that will enable us to find derivatives easily. We call them **DIFFERENTIATION RULES**.

Teaching Tip

You may prove some of the rules and let the class derive the other rules.

DIFFERENTIATING CONSTANT FUNCTIONS

The graph of a constant function is a horizontal line and a horizontal line has zero slope. The derivative measures the slope of the tangent, and so the derivative is zero.

RULE 1: <u>The Constant Rule</u>

If $f(x) = c$ where c is a constant, then $f'(x) = 0$. The derivative of a constant is equal to zero.

Proof:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$

EXAMPLE 1:

(a) If $f(x) = 10$, then $f'(x) = 0$.

(b) If $h(x) = -\sqrt{3}$, then $h'(x) = 0$.

(c) If $g(x) = 5\pi$, then $g'(x) = 0$.

DIFFERENTIATING POWER FUNCTIONS

A function of the form $f(x) = x^k$, where k is a real number, is called a **power function**. Below are some examples of power functions.

(a) $f(x) = x$

(b) $g(x) = x^2$

(c) $h(x) = \sqrt{x}$

(d) $p(x) = \sqrt[4]{x}$

(e) $\ell(x) = x^{-5}$

(f) $s(x) = 1/x^8$

The definition of the derivative discussed in the previous lesson can be used to find the derivatives of many power functions. For example, it can be shown that

$$\begin{gathered}\text{If } f(x) = x^2, \text{ then } f'(x) = 2x.\\ \text{If } f(x) = x^3, \text{ then } f'(x) = 3x^2.\\ \text{If } f(x) = x^4, \text{ then } f'(x) = 4x^3.\\ \text{If } f(x) = x^5, \text{ then } f'(x) = 5x^4.\\ \text{If } f(x) = x^6, \text{ then } f'(x) = 6x^5.\end{gathered}$$

Notice the pattern in these derivatives. In each case, the new power of f becomes the coefficient in f' and the power of f' is one less than the original power of f. In general, we have the following rule:

RULE 2: The Power Rule

If $f(x) = x^n$ where $n \in \mathbb{N}$, then $f'(x) = nx^{n-1}$.

Teaching Tip

At the least, prove the case when $n = 1$ and $n = 2$.

Proof. (*The cases $n = 1$ and $n = 2$*) Using the limit definition,

$$f(x) = x \implies f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h) - x}{h} = \lim_{h\to 0} 1 = 1 = 1x^{1-1}.$$

On the other hand, if $f(x) = x^2$, then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}\\ &= \lim_{h\to 0} \frac{h(2x+h)}{h} = 2x = 2x^{2-1}.\end{aligned}$$

□

If you wish to demonstrate the general case, the proof is stated below.

For the function $f(x) = x^n$, in computing $f(x+h) = (x+h)^n$, we need to invoke the Binomial Theorem:

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + \frac{n!}{r!(n-r)!}x^{n-r}h^r + \cdots + nxh^{n-1} + h^n.$$

Note that in the binomial expansion,

(i) there are $n+1$ terms;

(ii) the coefficient of the rth term is the binomial coefficient $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$; and

(iii) the sum of the exponent of x and the exponent of h is always equal to n.

Proof. (*The general case*)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{\left(x^n + nx^{n-1}h + + \cdots + \frac{n!}{r!(n-r)!}x^{n-r}h^r + \cdots + nxh^{n-1} + h^n\right) - x^n}{h} \\
&= \lim_{h\to 0} \left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \cdots + \frac{n!}{r!(n-r)!}x^{n-r}h^{r-1} + \cdots + h^{n-1}\right] \\
&= nx^{n-1}
\end{aligned}$$

□

Remark 1: Observe that the statement of the power rule restricts the exponent to be a natural number (since the Binomial Theorem is invoked). However, this formula holds true even for exponents $r \in \mathbb{R}$:

$$f(x) = x^r \quad \Longrightarrow \quad f'(x) = rx^{r-1} \quad \text{for all } r \in \mathbb{R}.$$

For example, if $f(x) = x^{-\pi}$, then $f'(x) = (-\pi)x^{(-\pi-1)}$.

EXAMPLE 2:

1. If $f(x) = x^3$, then $f'(x) = 3x^{3-1} = 3x^2$.
2. Find $g'(x)$ where $g(x) = \dfrac{1}{x^2}$.

Solution. In some cases, the laws of exponents must be used to rewrite an expression before applying the power rule. Thus, we first write $g(x) = \dfrac{1}{x^2} = x^{-2}$ before we apply the Power Rule. We have:

$$g'(x) = (-2)x^{-2-1} = -2x^{-3} \quad \text{or} \quad \frac{-2}{x^3}.$$

3. If $h(x) = \sqrt{x}$, then we can write $h(x) = x^{\frac{1}{2}}$. So we have,

$$\begin{aligned} h'(x) &= \frac{1}{2}x^{\frac{1}{2}-1} \\ &= \frac{1}{2}x^{-\frac{1}{2}} \qquad \text{or} \qquad \frac{1}{2\sqrt{x}} \end{aligned}$$

> **Teaching Tip**
>
> This rule is very basic and should be mastered. You can give some more examples until the students are confidently able to apply the power rule.

DIFFERENTIATING A CONSTANT TIMES A FUNCTION

RULE 3: The Constant Multiple Rule

If $f(x) = k\,h(x)$ where k is a constant, then $f'(x) = k\,h'(x)$.

> **Teaching Tip**
>
> Rule 3 states that the derivative of a constant times a differentiable function is the constant times the derivative of the function. Its proof is a direct consequence of the constant multiple theorem for limits.

EXAMPLE 3:

Find the derivatives of the following functions.

(a) $f(x) = 5x^{\frac{3}{4}}$ (b) $g(x) = \frac{1}{3}\sqrt[3]{x}$ (c) $h(x) = -\sqrt{3}\,x$

Solution. We use Rule 3 in conjunction with Rule 2.

(a) $f'(x) = 5 \cdot \dfrac{3}{4}x^{\frac{3}{4}-1} = \dfrac{15}{4}x^{-\frac{1}{4}}$.

(b) $g(x) = \dfrac{1}{3}x^{\frac{1}{3}} \implies g'(x) = \dfrac{1}{3} \cdot \dfrac{1}{3}x^{\frac{1}{3}-1} = \dfrac{1}{9}x^{-\frac{2}{3}}$.

(c) $h'(x) = -\sqrt{3}x^{1-1} = -\sqrt{3}$.

DIFFERENTIATING SUMS AND DIFFERENCES OF FUNCTIONS

RULE 4: The Sum Rule

If $f(x) = g(x) + h(x)$ where g and h are differentiable functions, then $f'(x) = g'(x) + h'(x)$.

Teaching Tip

Rule 4 states that the derivative of the sum of two differentiable functions is the sum of the derivatives of the functions. Its proof relies on the Addition Theorem for limits.

EXAMPLE 4: Refer to Example 3 above and ask the students to perform the following:

(a) Differentiate the following:

(i) $f(x) + g(x)$ (ii) $g(x) + h(x)$ (iii) $f(x) + h(x)$

(b) Use Rules 3 and 4 to differentiate the following: (Hint: $f(x) - g(x) = f(x) + (-1)g(x)$.)

(i) $f(x) - g(x)$ (ii) $g(x) - h(x)$ (iii) $f(x) - h(x)$

Solution.

(a) Copying the derivatives in the solution of Example (3), and substituting them into the formula of the Sum Rule, we obtain

(i) $\frac{15}{4}x^{-\frac{1}{4}} + \frac{1}{9}x^{-\frac{2}{3}}$. (ii) $\frac{1}{9}x^{-\frac{2}{3}} + (-\sqrt{3})$. (iii) $\frac{15}{4}x^{-\frac{1}{4}} + (-\sqrt{3})$.

(b) Using Rules 3 and 4, we deduce that the derivative of $f(x) - g(x)$ is equal to the difference of their derivatives: $f'(x) - g'(x)$. Therefore we obtain

(i) $\frac{15}{4}x^{-\frac{1}{4}} - \frac{1}{9}x^{-\frac{2}{3}}$. (ii) $\frac{1}{9}x^{-\frac{2}{3}} - (-\sqrt{3})$. (iii) $\frac{15}{4}x^{-\frac{1}{4}} - (-\sqrt{3})$.

Remark 2:

(a) The Sum Rule can also be extended to a sum of a finite number of functions. If

$$f(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$$

where $f_1, f_2, \ldots, f_n$ are differentiable functions, then

$$f'(x) = f_1'(x) \ + \ f_2'(x) \ + \cdots + \ f_n'(x).$$

(b) The same is true for the difference of a finite number of functions. That is,

$$f'(x) = f_1'(x) \ - \ f_2'(x) \ - \cdots - \ f_n'(x).$$

RULE 5: The Product Rule

If f and g are differentiable functions, then

$$D_x[f(x)\,g(x)] = f(x)\,g'(x) + g(x)\,f'(x).$$

Rule 5 states that the derivative of the product of two differentiable functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

The derivative of the product is NOT the product of their derivatives! Indeed, if $f(x) = x$ and $g(x) = x^2$, then

$$D_x[f(x) \cdot g(x)] = D_x[x \cdot x^2] = D_x(x^3) = 3x^2.$$

However,

$$D_x[f(x)] \cdot D_x[g(x)] = D_x(x) \cdot D_x(x^2) = 1 \cdot 2x = 2x.$$

Clearly, $3x^2 \neq 2x$, and therefore

$$D_x[f(x) \cdot g(x)] \neq D_x[f(x)] \cdot D_x[g(x)].$$

Teaching Tip

Presentation of this proof is *optional*, but is encouraged to be given to advanced classes.

Proof. Suppose f and g are both differentiable functions and let $H(x) = f(x)\,g(x)$. Then

$$H'(x) = \lim_{h\to 0} \frac{H(x+h) - H(x)}{h} = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Adding and subtracting $f(x+h)g(x)$ in the numerator (*the mathematical trick of adding* 0 *in a useful manner*) will help simplify this limit. That is,

$$\begin{aligned}
H'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h)[g(x+h) - g(x)]}{h} + \frac{g(x)[f(x+h) - f(x)]}{h}\right] \\
&= \lim_{h\to 0} \left[\frac{f(x+h)[g(x+h) - g(x)]}{h}\right] + \lim_{h\to 0} \left[\frac{g(x)[f(x+h) - f(x)]}{h}\right] \\
&= \lim_{h\to 0} f(x+h) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x) \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= f(x)g'(x) + f'(x)g(x)
\end{aligned}$$

$\square$

EXAMPLE 5:

(a) Find $f'(x)$ if $f(x) = (3x^2 - 4)(x^2 - 3x)$

Solution.

$$\begin{aligned} f'(x) &= (3x^2-4)D_x(x^2-3x) + (x^2-3x)D_x(3x^2-4) \\ &= (3x^2-4)(2x-3) + (x^2-3x)(6x) \\ &= 6x^3 - 9x^2 - 8x + 12 + 6x^3 - 18x^2 \\ &= 12x^3 - 27x^2 - 8x + 12. \end{aligned}$$

Remark 3: In the above example, we could have also multiplied the two factors and get

$$f(x) = 3x^4 - 9x^3 - 4x^2 + 12x.$$

Then, by the Rules 2,3 and 4, the derivative of f is

$$f'(x) = 12x^2 - 27x^2 - 8x + 12$$

which is consistent with the one derived from using the product rule.

(b) Find $f'(x)$ if $f(x) = \sqrt{x}\,(6x^3 + 2x - 4)$.

Solution. Using product rule,

$$\begin{aligned} f'(x) &= x^{1/2}\,D_x(6x^3+2x-4) + D_x(x^{1/2})\,(6x^3+2x-4) \\ &= x^{1/2}\,(18x^2+2) + \frac{1}{2}x^{-1/2}\,(6x^3+2x-4) \\ &= 18x^{5/2} + 2x^{1/2} + 3x^{5/2} + x^{1/2} - 2x^{-1/2} \\ &= 21x^{5/2} + 3x^{1/2} - 2x^{-1/2}. \end{aligned}$$

DIFFERENTIATING QUOTIENTS OF TWO FUNCTIONS

What is the derivative of $\dfrac{x^2+2}{x-3}$? More generally, we would like to have a formula to compute the derivative of $\dfrac{f(x)}{g(x)}$ if we already know $f'(x)$ and $g'(x)$.

Teaching Tip

The derivation/proof below is *optional* but is encouraged to be presented to advanced classes.

We derive the Quotient Rule using the Product Rule.

Notice that $\frac{f(x)}{g(x)}$ can be written as a product of two functions: $f(x) \cdot \frac{1}{g(x)}$. Hence, we can compute the derivative if we know $f'(x)$ and $\left[\frac{1}{g(x)}\right]'$. Now,

$$\begin{aligned}
D_x\left[\frac{1}{g(x)}\right] &= \lim_{h\to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} \\
&= \lim_{h\to 0} \left[\frac{g(x) - g(x+h)}{g(x+h)g(x)} \cdot \frac{1}{h}\right] \\
&= \lim_{h\to 0} \left[-\frac{g(x+h) - g(x)}{h} \cdot \frac{1}{g(x+h)g(x)}\right] \\
&= -g'(x) \cdot \frac{1}{[g(x)]^2}
\end{aligned}$$

Thus, using the product rule

$$\begin{aligned}
D_x\left[\frac{f(x)}{g(x)}\right] &= D_x\left[f(x) \cdot \frac{1}{g(x)}\right] \\
&= f(x)\left[\frac{1}{g(x)}\right]' + \left[\frac{1}{g(x)}\right] f'(x) \\
&= f(x) \cdot \frac{-g'(x)}{[g(x)]^2} + \left[\frac{f'(x)}{g(x)}\right] \\
&= \frac{-g'(x)f(x) + f'(x)g(x)}{[g(x)]^2}.
\end{aligned}$$

Putting everything together, we have the following rule:

RULE 6: <u>The Quotient Rule</u>

Let $f(x)$ and $g(x)$ be two differentiable functions with $g(x) \neq 0$. Then

$$D_x\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

The rule above states that the derivative of the quotient of two functions is the fraction having as its ***denominator*** the *square of the original denominator*, and as its ***numerator*** the *denominator times the derivative of the numerator minus the numerator times the derivative of the denominator.*

Teaching Tip

- Remind again the students that the derivative of a quotient is **NOT** equal to the quotient of their derivatives, that is,

$$D_x\left[\frac{f(x)}{g(x)}\right] \neq \frac{D_x[f(x)]}{D_x[g(x)]}.$$

- Since subtraction is not commutative, it matters which function you first copy and which one you first differentiate. A very common mnemonic for the quotient rule is

$$D_x\left[\frac{\text{high}}{\text{low}}\right] = \frac{\text{low } D(\text{high}) - \text{high } D(\text{low})}{\text{low squared}}.$$

EXAMPLE 6:

(a) Let $h(x) = \dfrac{3x+5}{x^2+4}$. Compute $h'(x)$.

Solution. If $h(x) = \dfrac{3x+5}{x^2+4}$, then $f(x) = 3x+5$ and $g(x) = x^2+4$ and therefore $f'(x) = 3$ and $g'(x) = 2x$. Thus,

$$\begin{aligned} h(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \\ &= \frac{(x^2+4)(3) - (3x+5)(2x)}{(x^2+4)^2} \\ &= \frac{3x^2+12-6x^2-10x}{(x^2+4)^2} \\ &= \frac{12-10x-3x^2}{(x^2+4)^2}. \end{aligned}$$

(b) Find $g'(x)$ if $g(x) = \dfrac{2x^4+7x^2-4}{3x^5+x^4-x+1}$.

Solution.

$$\begin{aligned} g'(x) &= \frac{(3x^5+x^4-x+1)\,D_x(2x^4+7x^2-4) - (2x^4+7x^2-4)\,D_x(3x^5+x^4-x+1)}{(3x^5+x^4-x+1)^2} \\ &= \frac{(3x^5+x^4-x+1)(8x^3+14x) - (2x^4+7x^2-4)(15x^4+4x^3-1)}{(3x^5+x^4-x+1)^2}. \end{aligned}$$

DIFFERENTIATING TRIGONOMETRIC FUNCTIONS

This time we will look at the derivatives of the trigonometric functions:

$$\sin x, \quad \cos x, \quad \tan x, \quad \sec x, \quad \csc x, \quad \cot x.$$

You may start this topic by recalling/reviewing the trigonometric functions involving some identities, that is,

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

and some important limits previously discussed such as,

$$\lim_{t\to 0} \frac{\sin t}{t} = 1 \qquad \text{and} \qquad \lim_{t\to 0} \frac{\cos t - 1}{t} = 0.$$

RULE 7: <u>Derivatives of trigonometric functions</u>

(a) $D_x(\sin x) = \cos x$

(b) $D_x(\cos x) = -\sin x$

(c) $D_x(\tan x) = \sec^2 x$

(d) $D_x(\cot x) = -\csc^2 x$

(e) $D_x(\sec x) = \sec x \tan x$

(f) $D_x(\csc x) = -\csc x \cot x$

The proof of (a) will be presented below. Statement (b) can be proven similarly.

Proof. Expanding $\sin(x+h)$ using the trigonometric sum identity, we obtain

$$\begin{aligned}
D_x(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} \\
&= \lim_{h\to 0} \left[\cos x \cdot \frac{\sin h}{h} + (-\sin x) \cdot \frac{1-\cos h}{h}\right] \\
&= \cos x \cdot \lim_{h\to 0} \left(\frac{\sin h}{h}\right) - \sin x \cdot \lim_{h\to 0} \left(\frac{1-\cos h}{h}\right) \\
&= (\cos x)(1) - (\sin x)(0) \\
&= \cos x.
\end{aligned}$$

$\square$

Statements (c) - (f) can be proved using Statements (a) and (b) and the Quotient Rule. We only present the proof of (c) below.

Proof. Using Quotient Rule and Statements (a) and (b) above, we have

$$
\begin{aligned}
D_x(\tan x) = D_x\left[\frac{\sin x}{\cos x}\right] &= \frac{\cos x\, D_x(\sin x) - \sin x\, D_x(\cos x)}{(\cos x)^2} \\
&= \frac{\cos x(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}.
\end{aligned}
$$

Using the identity $\cos^2 x + \sin^2 x = 1$, we get $D_x(\tan x) = \dfrac{1}{\cos^2 x} = \sec^2 x$. □

EXAMPLE 7: Differentiate the following functions:

(a) $f(x) = \sec x + 3\csc x$ (b) $g(x) = x^2 \sin x - 3x\cos x + 5\sin x$

Solution. Applying the formulas above, we get

(a) If $f(x) = \sec x + 3\csc x$, then

$$f'(x) = \sec x \tan x + 3(-\csc x \cot x) = \sec x \tan x - 3\csc x \cot x.$$

(b) If $g(x) = x^2 \sin x - 3x\cos x + 5\sin x$, then

$$
\begin{aligned}
g'(x) &= [(x^2)(\cos x) + (\sin x)(2x)] - 3[(x)(-\sin x) + (\cos x(1)] + 5(\cos)x)] \\
&= x^2\cos x + 2x\sin x + 3x\sin x - 3\cos x + 5\cos x \\
&= x^2\cos x + 5x\sin x + 2\cos x.
\end{aligned}
$$

Remark 4:

(a) Whenever Rule 7 is applied to problems where the trigonometric functions are viewed as functions of angles, the unit measure must be in radians.

(b) Every trigonometric function is differentiable on its domain. In particular, the sine and cosine functions are everywhere differentiable.

DIFFERENTIATING AN EXPONENTIAL FUNCTION

RULE 8: <u>Derivative of an exponential function</u>

$$\text{If } f(x) = e^x, \text{ then } f'(x) = e^x.$$

Proof. (*Optional*) Using the definition of the derivative,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}.$$

Using a law of exponent, $e^{x+h} = e^x \cdot e^h$. Therefore, since $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$,

$$f'(x) = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} = e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h} = e^x.$$

□

EXAMPLE 8:

(a) Find $f'(x)$ if $f(x) = 3e^x$.

Solution. Applying Rules 3 and 7, we have

$$f'(x) = 3D_x[e^x] = 3e^x.$$

(b) Find $g'(x)$ if $g(x) = -4x^2e^x + 5xe^x - 10e^x$.

Solution. Applying Rule 5 to the first two terms and Rule 3 to the third term, we have

$$\begin{aligned} g'(x) &= [(-4x^2)(e^x) + (e^x)(-8x)] + [(5x)(e^x) + (e^x)(5) - 10 \cdot e^x] \\ &= -4x^2e^x - 3xe^x - 5e^x. \end{aligned}$$

(c) Find $h'(x)$ if $h(x) = e^x \sin x - 3e^x \cos x$.

Solution. We apply the Product Rule to each term.

$$\begin{aligned} h'(x) &= [(e^x)(\cos x) + (\sin x)(e^x)] - 3 \cdot [(e^x)(-\sin x) + (\cos x)(e^x)] \\ &= e^x \cos x + e^x \sin x + 3e^x \sin x - 3e^x \cos x \\ &= e^x(4\sin x - 2\cos x). \end{aligned}$$

(d) Find $\dfrac{dy}{dx}$ where $y = \dfrac{17}{e^x x^e + 2x - 3\sqrt{x}}$.

Solution. Using Quotient Rule (also Product Rule when differentiating $x^e e^x$), we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{(e^x x^e + 2x - 3\sqrt{x})(0) - (17)([(e^x)(ex^{e-1}) + (x^e)(e^x)] + 2 - \frac{3}{2\sqrt{x}})}{(e^x x^e + 2x - 3\sqrt{x})^2} \\ &= \frac{-17e^{x+1}x^{e-1} - 17e^x x^e - 34 + \frac{51}{2\sqrt{x}}}{(e^x x^e + 2x - 3\sqrt{x})^2}.\end{aligned}$$

Remark 5: Since the domain of the exponential function is the set of real numbers, and $\dfrac{d}{dx}[e^x] = e^x$, it follows that the exponential function is differentiable everywhere.

(C) EXERCISES

1. Let $f(x) = mx+b$, the line with slope m and y-intercept b. Use the rules of differentiation to show that $f'(x) = m$. This again proves that the derivative of a line is just the slope of the line.

2. Find the derivatives of the following functions. Locate the derivative in the table below and encircle the answer. Keep working until you have five encircled answers in a line horizontally, vertically or diagonally. (BINGO!)

a. $y = x^2 - x + 1$

b. $f(x) = \dfrac{1}{2x+1}$

c. $y = (3x-1)(2x+5)$

d. $g(x) = x^3 - 3x^2 + 2$

e. $y = \dfrac{2x+5}{3x-2}$

f. $y = (2x^2+2)(x^2+3)$

g. $f(x) = \dfrac{x^2+1}{x^2-1}$

h. $y = x^3 - 4x^2 - 3x$

i. $f(x) = 5x^3 - 3x^5$

j. $y = (2x+3)^2$

k. $f(x) = (x + \dfrac{1}{x})^2$

l. $y = x^2(x^3-1)$

m. $f(x) = \dfrac{2x}{3x^2+1}$

n. $y = \dfrac{x}{x^2+1}$

o. $f(x) = (x-2)(x+3)$

$2x - 2x^{-3}$	$2x + \frac{1}{x^2}$	$2x - 1$	$\frac{19}{3x-2}$	$-(2x^3 + 2)$
$\frac{-4x}{(x^2-1)^2}$	$12x - 7$	$4x$	$(x+1)^{-2}$	$3(x^2+3x)^2$
$5x^4 - 2x$	$6x^3$	$8x + 12$	$\frac{2-6x^2}{(3x^2+1)^2}$	$8x^3 + 16x$
$15(x^2 - x^4)$	$3x^2 - 6x$	$3x^2 - 8x - 3$	$\frac{2x}{(1-x^2)^2}$	$\frac{-19}{(3x-2)^2}$
$12x + 13$	$2x + 1$	$\frac{-2}{(2x+1)^2}$	$\frac{1-x^2}{(x^2+1)^2}$	$2\left(x + \frac{1}{x}\right)$

3. Use the Rules of Differentiation to differentiate the following functions:

   a. $f(x) = 2x^3 + 6x$
   b. $g(x) = 7x^4 - 3x^2$
   c. $y(x) = 4x^3 - 18x^2 + 6x$
   d. $h(x) = (3x + 4)^2$
   e. $h(x) = 9x^{2/3} + \frac{2}{\sqrt[4]{x}}$

4. Find the derivative of each of the following functions:

   a. $f(u) = (4u + 5)(7u^3 - 2u)$
   b. $h(t) = \frac{4}{t^{\frac{2}{3}}}$
   c. $f(z) = -6z^3 - \frac{1}{2z}$
   d. $g(s) = \frac{3s - s^3}{s^2 + 1}$
   e. $f(a) = \frac{17}{2a^3} + \frac{1}{8a^2} - 11a$
   f. $f(r) = (r^2 + 2r - 3)(3r + 4)$
   g. $g(b) = \frac{3b-4}{b+5} \cdot (2b + 5)$
   h. $h(v) = \frac{3v^2 - 4v + 1}{(3v^2 - 2v + 1)(7v^3 - v^2 + 3v - 5)}$

5. Find $\frac{dy}{dx}$ and simplify the result, if possible.

   a. $y = \sqrt{x} - \frac{1}{\sqrt{x}}$
   b. $y = x^2 + \pi^2 + x^\pi$
   c. $y = x^2 \sec x$
   d. $y = \frac{\sin x - 1}{\cos x}$
   e. $y = \frac{1}{e^x + 2}$
   f. $y = e^x + x^e + e^e$
   g. $y = x^2 \sin x \cos x$
   h. $y = \frac{8}{x} - \tan x \cot x$

6. Find the derivative of $f(x) = x^2 - 3x$. Use the result to find the slope of the tangent line to the curve $f(x) = x^2 - 3x$ at the point where $x = 2$.

7. If $y = \frac{1}{x}$, find y' and use this result to find the points on the curve $y = \frac{1}{x}$ where the tangent line has the slope -4.

8. Find all points on the graph of $y = (x + 3)^2$ at which the tangent line is parallel to the line with equation $y - 4x + 2 = 0$.

LESSON 7: Optimization

TIME FRAME: 3 hours lecture + 1 hour exercises

LEARNING OUTCOME: At the end of the lesson, the learner shall be able to solve optimization problems.

LESSON OUTLINE:

1. Mathematical modeling
2. Critical and extreme points
3. Fermat's theorem and the extreme value theorem
4. Word Problems

TOPIC 7.1: Optimization using Calculus

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

This topic presents one of the most important applications of calculus – optimization.

We first review mathematical modelling. Next, we define the notions of a critical point and an extremum of a function f. Then, we proceed with the discussion of the two important theorems – Fermat's and the Extreme Value Theorem. Finally, we discuss the step-by-step solutions of word problems involving optimization.

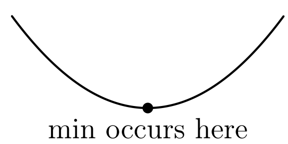
(B) ACTIVITY

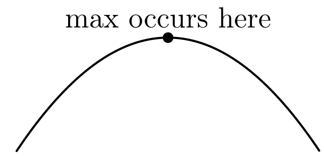
- Discuss with the class the importance of optimization in every decision process. Ask them how they maximize or minimize something when they go to school, shopping, church, etc: For example, they look at the greatest advantage and the least disadvantage when they choose:
  - Routes they take (minimize time travel, traffic congestion)
  - What to wear (maximize appearance/personality)
  - What food to eat at the canteen (minimize costs, maximize hunger satisfaction), etc.

- You could also expound on the adage "Everything in moderation." Ask which quantities they know become bad if they are increased or decreased too much. For example:
  - Sleep/rest (fatigue/weakness vs. conditions linked to depression/anxiety, obesity and heart problems, etc.: The rule of thumb is 8 hours.)
  - Study hours (delinquent/failure in course vs. lacking time to do other aspects of life e.g., family time/social life: The rule of thumb is $1-3$ hours every night after class)
  - Food intake (being underweight vs. overweight with optimal values specified by RDA (recommended dietary allowance)), etc.

- Now, wrap up the discussion by reinforcing the idea that logical decisions in real life are made by optimizing some quantities (objectives) that depend on things you can control (variables). Of course, in real life, there are several objectives and variables involved. In this section, we will only be concerned with problems with one objective that depends on controlling only one variable.

- You may also want to let them recall that they have done optimization before using parabolas. (*Refer to the figures that follow.*) The idea was that in a parabola opening upwards, minimization is possible and the minimum is attained at the vertex (analogous idea for parabolas opening downwards). This method, however, is very limited because the objective function must be a quadratic function.





(C) **LESSON PROPER**

REVIEW OF MATHEMATICAL MODELING (This part can be skipped and integrated in problem solving proper)

Before we start with problem solving, we recall key concepts in mathematical modeling (It was your first topic in General Mathematics). Functions are used to describe physical phenomena. For example:

- The number of people y in a certain area that is infected by an epidemic after some time t;
- The concentration c of a drug in a person's bloodstream t hours after it was taken;
- The mice population y as the snake population x changes, etc.

We model physical phenomena to help us predict what will happen in the future. We do this by finding or constructing a function that exhibits the behavior that has already been observed. In the first example above, we want to find the function $y(t)$. For example, if $y(t) = 1000 \cdot 2^{-t}$, then we know that initially, there are $y(0) = 1000$ affected patients. After one hour, there are $y(1) = 1000 \cdot 2^{-1} = 500$ affected patients.

Observe that the independent variable here is time t and that the quantity y depends on t. Since y is dependent on t, it now becomes possible to optimize the value of y by controlling at which time t you will measure y.

We now look at some examples.

EXAMPLE 1: For each of the following, determine the dependent quantity Q and the independent quantity x on which it depends. Then, find the function $Q(x)$ that accurately models the given scenario.

(a) The product P of a given number x and the number which is one unit bigger.
Answer: $P(x) = x(x+1) = x^2 + x$

(b) The volume V of a sphere of a given radius r Answer: $V(r) = 4/3\,\pi r^3$

(c) The volume V of a right circular cone with radius 3cm and a given height h
Answer: $V(h) = 3\pi h$

(d) The length ℓ of a rectangle with width 2cm and a given area A
Answer: $\ell(A) = A/2$

(e) The total manufacturing cost C of producing x pencils if there is an overhead cost of ₱100 and producing one pencil costs ₱2
Answer: $C(x) = 100 + 2x$

(f) The volume of the resulting open-top box when identical squares with side x are cut off from the four corners of a 3 meter by 5 meter aluminum sheet and the sides are then turned up Answer: $V(x) = x(3-2x)(5-2x)$.

CRITICAL POINTS AND POINTS WHERE EXTREMA OCCUR

Here, we define several concepts. We will see later that extreme points occur at critical points.

Definition

Let f be a function that is continuous on an open interval I containing x_0.

- We say that x_0 is a **critical point** of f if $f'(x_0) = 0$ or $f'(x_0)$ does not exist (that is, f has a corner or a cusp at $(x_0, f(x_0))$).
- We say that the **maximum** occurs at x_0 if the value $f(x_0)$ is the largest among all other functional values on I, that is,

$$f(x_0) \geq f(x) \quad \text{for all } x \in I.$$

- We say that the **minimum** of f occurs at x_0 if the value $f(x_0)$ is the smallest among all the other functional values on I, that is,

$$f(x_0) \leq f(x) \quad \text{for all } x \in I.$$

- We say that an **extremum** of f occurs at x_0 if either the maximum or the minimum occurs at x_0.

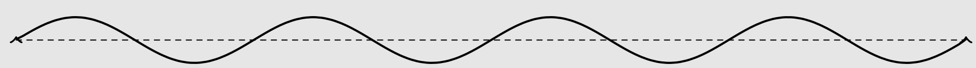
Teaching Tip

Sometimes, there is an abuse of terminology when we say x_0 is an extremum or an extreme point of f. The proper terms are the following:

- An extremum of f occurs at x_0.
- f has an extremum at $x = x_0$
- $(x_0, f(x_0))$ is an extreme point of f.
- $f(x_0)$ is an extreme value of f.

You are encouraged to use the above terms but are gently reminded to avoid penalizing the students for being imprecise (e.g., saying "x_0 is an extremum of f" instead of "f has an extremum at x_0").

Also, note that extremum points may not be unique, as illustrated by the sine curve:



Here, there are infinitely many points where the maximum and minimum occur.

EXAMPLE 2: Find all critical points of the given function f.

(a) $f(x) = 3x^2 - 3x + 4$

(b) $f(x) = x^3 - 9x^2 + 15x - 20$

(c) $f(x) = x^3 - x^2 - x - 10$

(d) $f(x) = x - 3x^{1/3}$

(e) $f(x) = x^{5/4} + 10x^{1/4}$

Solution. We differentiate f and find all values of x such that $f'(x)$ becomes zero or undefined.

(a) Note that f is differentiable everywhere, so critical points will only occur when f' is zero. Differentiating, we get $f'(x) = 6x - 3$. Therefore, $f'(x) = 0 \Leftrightarrow 6x - 3 = 0 \Leftrightarrow x = 1/2$. So $x = 1/2$ is a critical point.

(b) $f'(x) = 3x^2 - 18x + 15 = 3(x^2 - 6x + 5) = 3(x - 5)(x - 1)$. Hence the critical points are 1 and 5.

(c) $f'(x) = 3x^2 - 2x - 1 = (3x + 1)(x - 1)$. So, the critical points are $-1/3$ and 1.

(d) $f'(x) = 1 - x^{-2/3} = \dfrac{x^{2/3} - 1}{x^{2/3}}$. Observe that f' is zero when the numerator is zero, or when $x = 1$. Moreover, f' is undefined when the denominator is zero, i.e., when $x = 0$. So, the critical points are 0 and 1.

(e) $f'(x) = \frac{5}{4}x^{1/4} + \frac{10}{4}x^{-3/4} = \frac{5(x+2)}{4x^{3/4}}$. Note that the domain of f is $[0, \infty)$; therefore -2 cannot be a critical point. The only critical point is 0.

FERMAT'S THEOREM AND THE EXTREME VALUE THEOREM

We now discuss the theory behind optimization using calculus. The first gives a relationship between the critical points and extremum points. The second is a guarantee that a problem has extreme points. We will illustrate the theorems by graphs.

Theorem 7 (Fermat's Theorem)**.** *Let f be continuous on an open interval I containing x_0. If f has an extremum at x_0, then x_0 must be a critical point of f.*

To illustrate this, recall that the derivative of f at x_0 is the slope of the tangent line of f at x_0.

Teaching Tip

Draw several graphs of smooth curves and let the class deduce that the extreme points are where there is a horizontal tangent line (the slope of the tangent line is 0). Then, cross-reference this to the definition of a critical point when f is differentiable ($f'(x_0) = 0$).

Also, there are points where f may not be differentiable. So, we also have to check those points where f' does not exist. This is still part of the definition of a critical point.

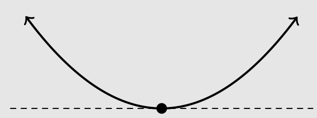
Theorem 8 (Extreme Value Theorem)**.** *Let f be a function which is continuous on a closed and bounded interval $[a, b]$. Then the extreme values (maximum and minimum) of f always exist, and they occur either at the endpoints or at the critical points of f.*

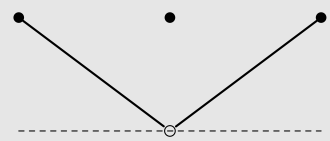
Teaching Tip

It is important to stress the assumptions of the Extreme Value Theorem. The function f must be **continuous** on an interval that is **closed and bounded**. To illustrate this, consider

- The parabola $f(x) = x^2$ defined on $\mathbb{R}$. It indeed has a minimum point at the origin, but it does not have a maximum (since the values go to infinity as x approaches $\pm\infty$. The conclusion of the Extreme Value Theorem fails because even if f is continuous, the interval on which it is defined is not bounded.

- The function $f(x) = |x|$ if $x \in [-1, 0) \cup (0, 1]$ and 1 if $x = 0$. Even if f is defined on the closed and bounded interval $[-1, 1]$, the function is **not** continuous there. So, the conclusion fails. As we see from the graph, f has a maximum occurring at $x = -1, 0, 1$ but does not possess a minimum because the value 0 is not attained.





EXAMPLE 3: Find the extrema of the given functions on the interval $[-1, 1]$. (These functions are the same as in the previous exercise.)

(a) $f(x) = 3x^2 - 3x + 4$

(b) $f(x) = x^3 - 9x^2 + 15x - 20$

(c) $f(x) = x^3 - x^2 - x - 10$

(d) $f(x) = x - 3x^{1/3}$

Solution. We remember that we have solved all critical points of f in the previous exercise. However, we will only consider those critical points on the interval $[-1, 1]$. Moreover, by the Extreme Value Theorem, we also have to consider the endpoints. So, what remains to be done is the following:

- Get the functional values of all these critical points inside $[-1, 1]$;
- Get the functional values at the endpoints; and
- Compare the values. The highest one is the maximum value while the lowest one is the minimum value.

(a) There's only one critical point, $x = 1/2$, and the endpoints are $x = \pm 1$. We present the functional values in a table.

x	-1	$1/2$	1
$f(x)$	10	$13/4$	4

Clearly, the maximum of f occurs at $x = -1$ and has value 10. The minimum of f occurs at $x = 1/2$ and has value $13/4$.

(b) The critical points of f are 1 and 5, but since we limited our domain to $[-1, 1]$, we are only interested with $x = 1$. Below is the table of functional values at this critical point, as well as those at the endpoints.

x	-1	1
$f(x)$	-45	-13

Therefore, the maximum value -13 occurs at $x = 1$ while the minimum value -45 occurs at $x = -1$.

(c) Considering the critical points and the endpoints, we consider the functional values at $-1/3, -1$ and 1:

x	-1	$-1/3$	1
$f(x)$	-11	$-25/27$	-11

Thus, the maximum point is $(-1/3, -25/27)$ while the minimum points are $(1, -11)$ and $(-1, -11)$.

(d) $f(0) = 0$, $f(1) = -2$ and $f(-1) = 2$. So, the maximum point is $(-1, 2)$ while the minimum point is $(1, -2)$.

OPTIMIZATION: APPLICATION OF EXTREMA TO WORD PROBLEMS

Many real-life situations require us to find a value that best suits our needs. If we are given several options for the value of a variable x, how do we choose the "best value?" Such a problem is classified as an ***optimization problem***. We now apply our previous discussion to finding extremum values of a function to solve some optimization problems.

Suggested Steps in Solving Optimization Problems

1. If possible, draw a diagram of the problem.

2. Assign variables to all unknown quantities involved.

3. Specify the objective function. This function must be continuous.

    i. Identify the quantity, say q, to be maximized or minimized.

    ii. Formulate an equation involving q and other quantities. Express q in terms of a single variable, say x. If necessary, use the information given and relationships between quantities to eliminate some variables.

    iii. The objective function is

$$\begin{aligned} &\textbf{maximize} \quad q = f(x) \\ \text{or } &\textbf{minimize} \quad q = f(x). \end{aligned}$$

4. Determine the domain or constraints of q from the physical restrictions of the problem. The domain must be a closed and bounded interval.

5. Use appropriate theorems involving extrema to solve the problem. Make sure to give the exact answer (with appropriate units) to the question.

EXAMPLE 4: Find the number in the interval $[-2, 2]$ so that the difference of the number from its square is maximized.

Solution. Let x be the desired number. We want to maximize

$$f(x) = x^2 - x$$

where $x \in [-2, 2]$. Note that f is continuous on $[-2, 2]$ and thus, we can apply the Extreme Value Theorem.

We first find the critical numbers of f in the interval $(-2, 2)$. We have

$$f'(x) = 2x - 1,$$

which means that we only have one critical number in $(-2, 2)$: $x = \dfrac{1}{2}$.

Then we compare the function value at the critical number and the endpoints. We see that

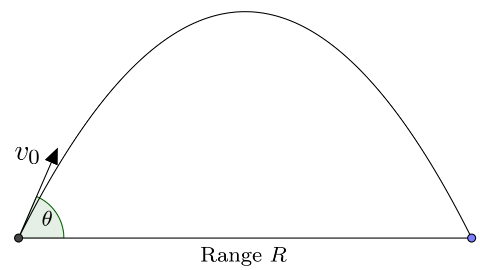
$$f(-2) = 6, \qquad f(2) = 2, \qquad f\left(\frac{1}{2}\right) = -\frac{1}{4}.$$

From this, we conclude that f attains a maximum on $[-2, 2]$ at the left endpoint $x = -2$. Hence, the number we are looking for is -2.

EXAMPLE 5: The range R (distance of launch site to point of impact) of a projectile that is launched at an angle $\theta \in [0°, 90°]$ from the horizontal, and with a fixed initial speed of v_0 is given by

$$R(\theta) = \frac{v_0^2}{g} \sin 2\theta,$$

where g is the acceleration due to gravity. Show that this range is maximized when $\theta = 45°$.



Solution. Let $R(\theta)$ denote the range of the projectile that is launched at an angle θ, measured from the horizontal. We need to maximize

$$R(\theta) = \frac{v_0^2}{g} \sin 2\theta$$

where $\theta \in [0, \pi/2]$. Note that R is continuous on $[0, \pi/2]$ and therefore the Extreme Value Theorem is applicable.

We now differentiate R to find the critical numbers on $[0, \pi/2]$:

$$R'(\theta) = \frac{v_0^2}{g} 2 \cos 2\theta = 0 \quad \Longleftrightarrow \quad 2\theta = \pi/2.$$

Hence $\theta = \pi/4 = 45°$ is a critical number.

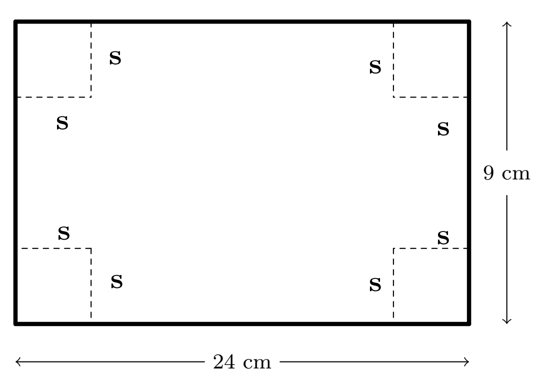
Finally, we compare the functional values:

$$f(0) = 0, \quad f(\pi/4) = \frac{v_0^2}{g}, \quad f(\pi/2) = 0.$$

Thus, we conclude that f attains its maximum at $\theta = \pi/4$, with value v_0^2/g.

EXAMPLE 6:

A rectangular box is to be made from a piece of cardboard 24 cm long and 9 cm wide by cutting out identical squares from the four corners and turning up the sides. Find the volume of the largest rectangular box that can be formed.



Solution. Let s be the length of the side of the squares to be cut out, and imagine the "flaps" being turned up to form the box. The length, width and height of the box would then be $24-2s,\ 9-2s$, and s, respectively. Therefore, the volume of the box is

$$V(s) = (24-2s)(9-2s)s = 2(108s - 33s^2 + 2s^3).$$

We wish to maximize $V(s)$ but note that s should be nonnegative and should not be more than half the width of the cardboard. That is, $s \in [0, 4.5]$. (The case $s = 0$ or $s = 4.5$ does not produce any box because one of the dimensions would become zero; but to make the interval closed and bounded, we can think of those cases as *degenerate* boxes with zero volume). Since V is just a polynomial, it is continuous on the closed and bounded interval $[0, 4.5]$. Thus, the Extreme Value Theorem applies. Now

$$V'(s) = 216 - 132s + 12s^2 = 4(54 - 33s + 3s^2) = 4(3s-6)(s-9)$$

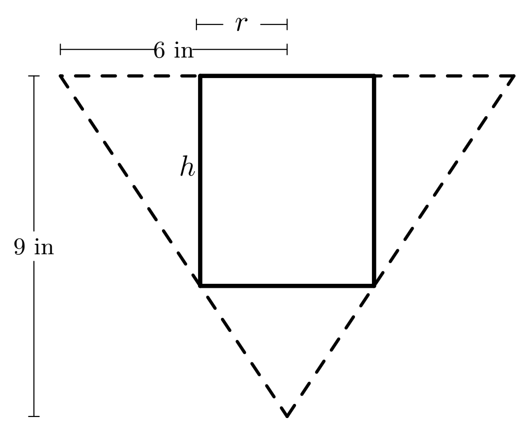
and hence the only critical number of V in $(0, 4.5)$ is 2 ($s = 9$ is outside the interval).

We now compare the functional values at the endpoints and at the critical points:

s	0	2	4.5
$V(s)$	0	200	0

Therefore, from the table, we see that V attains its maximum at $s = 2$, and the maximum volume is equal to $V(2) = 200$ cm^3.

EXAMPLE 7: Determine the dimensions of the right circular cylinder of greatest volume that can be inscribed in a right circular cone of radius 6 cm and height 9 cm.



Solution. Let h and r respectively denote the height and radius of the cylinder. The volume of the cylinder is $\pi r^2 h$.

Looking at the central cross-section of the cylinder and the cone, we can see similar triangles, and so

$$\frac{6}{6-r} = \frac{9}{h}. \tag{2.4}$$

We can now write our objective function as

$$V(r) = 9\pi r^2 - \frac{3}{2}\pi r^3 = 3\pi r^2\left(3 - \frac{r}{2}\right),$$

and it is to be maximized. Clearly, $r \in [0, 6]$. Since V is continuous on $[0, 6]$, the Extreme Value Theorem is applicable.

Now,

$$V'(r) = 18\pi r - \frac{9}{2}\pi r^2 = 9\pi r\left(2 - \frac{r}{2}\right)$$

and hence, our only critical number is 4 on $(0, 6)$. We now compare the functional values at the endpoints and at the critical points:

r	0	4	6
$V(r)$	0	48π	0

We see that the volume is maximized when $r = 4$, with value $V(4) = 48\pi$. To find the dimensions, we solve for h from (2.4).

If $r = 4$,

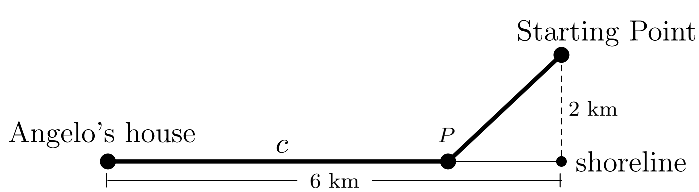
$$\frac{6}{6-2} = \frac{9}{h} \quad \Longrightarrow \quad h = 3.$$

Therefore, the largest circular cylinder that can be inscribed in the given cone has dimensions $r = 4$ cm and height $h = 3$ cm.

Teaching Tip

The next example is optional because in finding the derivative of the function, you usually use a technique – Chain Rule – that is yet to be discussed. One way to remedy this is to use the definition to find the derivative. This will need a few more steps which your time frame may not allow.

EXAMPLE 8: Angelo, who is in a rowboat 2 kilometers from a straight shoreline wishes to go back to his house, which is on the shoreline and 6 kilometers from the point on the shoreline nearest Angelo. If he can row at 6 kph and run at 10 kph, how should he proceed in order to get to his house in the least amount of time?



Solution. Let c be the distance between the house and the point P on the shore from which Angelo will start to run. Using the Pythagorean Theorem, we see that the distance he will travel by boat is $\sqrt{4+(6-c)^2}$.

Note that speed$=\frac{\text{distance}}{\text{time}}$. Thus, he will sail for $\frac{\sqrt{4+(6-c)^2}}{6}$ hours and run for $\frac{c}{10}$ hours. We wish to minimize

$$T(c) = \frac{\sqrt{4+(6-c)^2}}{6} + \frac{c}{10}.$$

We can assume that $c \in [0,6]$. We now differentiate T. Observe that our previously discussed rules of differentiation are not applicable to this function because we have not yet discussed how to differentiate compositions of functions. We use the definition of the derivative instead.

$$\begin{aligned}
T'(c) &= \lim_{x\to c} \frac{T(x)-T(c)}{x-c} \\
&= \lim_{x\to c} \frac{\left(\frac{\sqrt{4+(6-x)^2}}{6} + \frac{x}{10}\right) - \left(\frac{\sqrt{4+(6-c)^2}}{6} + \frac{c}{10}\right)}{x-c} \\
&= \frac{1}{10}\lim_{x\to c}\frac{x-c}{x-c} + \frac{1}{6}\lim_{x\to c}\left(\frac{\sqrt{4+(6-x)^2}-\sqrt{4+(6-c)^2}}{x-c}\right).
\end{aligned}$$

We now rationalize the expression in the first limit by multiplying the numerator and denominator by $\sqrt{4+(6-x)^2}+\sqrt{4+(6-c)^2}$. This yields:

$$\begin{aligned}
T'(c) &= \frac{1}{10} + \frac{1}{6}\lim_{x\to c}\frac{(4+(6-x)^2)-(4+(6-c)^2)}{(x-c)(\sqrt{4+(6-x)^2}+\sqrt{4+(6-c)^2}} \\
&= \frac{1}{10} + \frac{1}{6}\lim_{x\to c}\frac{-12(x-c)+(x-c)(x+c)}{(x-c)(\sqrt{4+(6-x)^2}+\sqrt{4+(6-c)^2})} \\
&= \frac{1}{10} + \frac{1}{6}\lim_{x\to c}\frac{-12+x+c}{\sqrt{4+(6-x)^2}+\sqrt{4+(6-c)^2}} \\
&= \frac{1}{10} + \frac{1}{6}\cdot\frac{-12+2c}{2\sqrt{4+(6-c)^2}}.
\end{aligned}$$

Solving for the critical numbers of f on $(0,6)$, we solve

$$T'(c) = \frac{1}{10} + \frac{1}{6}\cdot\frac{-12+2c}{2\sqrt{4+(6-c)^2}} = \frac{3\sqrt{4+(6-c)^2}-30+5c}{30\sqrt{4+(6-c)^2}} = 0,$$

and we get $c = \frac{9}{2}$.

Comparing function values at the endpoints and the critical number,

$$T(0) = \frac{\sqrt{40}}{6}, \qquad T\left(\frac{9}{2}\right) = \frac{13}{15}, \qquad T(6) = \frac{14}{15},$$

we see that the minimum of T is attained at $c = \frac{9}{2}$. Thus, Angelo must row up to the point P on the shore $\frac{9}{2}$ kilometers from his house and $\frac{3}{2}$ kilometers from the point on the shore nearest him. Then he must run straight to his house.

(D) **EXERCISES**

1. Find the extrema of the following function on the given interval, if there are any, and determine the values of x at which the extrema occur.

   a. $f(x) = \dfrac{x}{x^2+2}$ on $[-1, 4]$

   b. $f(x) = x\sqrt{4-x^2}$ on $[-1, 2]$

   c. $f(x) = 2\sec x - \tan x$ on $\left[0, \frac{\pi}{4}\right]$

   d. $f(x) = \dfrac{1}{x}$ on $[-2, 3]$

   e. $f(x) = 4x^3 - 3x^2 - 6x + 3$ on $[0, 10]$

   f. $f(x) = 4x^3 - 3x^2 - 6x + 3$ on $[-1, 10]$

   g. $f(x) = 2\cos x$ on $[\frac{-2\pi}{3}, \frac{\pi}{3}]$

   h. $f(x) = \sqrt{4-x^2}$ on $[-2, 2]$

   i. $f(x) = \dfrac{4}{(x-1)^2}$ on $[2, 5]$

   $\star$j. $f(x) = \dfrac{\sin x}{\cos x - \sqrt{2}}$ on $[0, 2\pi]$

   $\star$k. $f(x) = x^2 + \cos(x^2)$ on $[-\sqrt{\pi}, \sqrt{\pi}]$

2. Answer the following optimization problems systematically:

   a. A closed box with a square base is to have a volume of 2000 cm^3. The material for the top and bottom of the box costs ₱30 per square centimeter and the material for the sides cost ₱15 per square centimeter. Find the dimensions of the box so that the total cost of material is the least possible and all its dimensions do not exceed 20 cm.

   b. An open box is to be made from a 16 cm by 30 cm piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides. How long should the sides of the squares be to obtain a box with the largest volume?

   c. An offshore oil well located at a point W that is 5 km from the closest point A on a straight shoreline. Oil is to be piped from W to a shore point B that is 8 km from A by piping it on a straight line underwater from W to some shore point P between A and B and then on to B via pipe along the shoreline. If the cost of laying pipe is ₱10,000,000/km underwater and ₱5,000,000/km over land, where should the point P be located to minimize the cost of laying the pipe?

   d. Find the radius and height of the right circular cylinder of largest volume that can be inscribed in a right circular cone with radius 4 cm and height 3 cm.

   e. Find the dimensions of the largest rectangle that can be inscribed in the right triangle with sides 3, 4 and 5 if (a) two sides of the rectangle are on the legs of the triangle, and if (b) a side of the rectangle is on the hypotenuse of the triangle.

f. Find the area of the largest rectangle having two vertices on the x-axis and two vertices above the x-axis and on the parabola with equation $y = 9 - x^2$.

g. Find an equation of the tangent line to the curve $y = x^3 - 3x^2 + 5x,\ 0 \leq x \leq 3$ that has the least slope.

h. A closed cylindrical can is to hold 1 cubic meter of liquid. Assuming there is no waste or overlap, how should we choose the height and radius to minimize the amount of material needed to manufacture the can? (Assume that both dimensions do not exceed 1 meter.)

SOLUTION TO STARRED EXERCISES

1.j. $f'(x) = \dfrac{(\cos x - \sqrt{2})\cos x - \sin x(-\sin x)}{(\cos x - \sqrt{2})^2} = 0 \Leftrightarrow \cos^2 x + \sin^2 x - \sqrt{2}\cos x = 0$. Since $\cos^2 x + \sin^2 x = 1$, we see that $f'(x) = 0$ whenever $\cos x = \sqrt{2}/2$, meaning, when $x = \pi/4$ or $x = 7\pi/4$. Along with the endpoints, we check their functional values:

x	$f(x) = \sin x/(\cos x - \sqrt{2})$
$\pi/4$	$\sqrt{2}/2$
$7\pi/4$	$-1/3$
0	0
2π	0

Therefore, f attains its maximum $\sqrt{2}/2$ at $x = \pi/4$ and its minimum $-1/3$ at $x = 7\pi/4$.

1.k. $f'(x) = 2x - 2x\sin(x^2) = 0 \Leftrightarrow 2x(1 - \sin(x^2)) = 0 \Leftrightarrow x = 0$ or $x^2 = \pi/2$. Hence, the critical points of f are $x = 0$ and $x = \pm\sqrt{\pi/2}$. Along with the endpoints, we now check the functional values.

x	$f(x) = x^2 + \cos(x^2)$
0	1
$\pm\sqrt{\pi/2}$	$\pi/2$
$\pm\sqrt{\pi}$	$\pi - 1$

Hence, the maximum $(\pi - 1)$ occurs at $x = \pm\sqrt{\pi}$ and the minimum 1 occurs at $x = 0$.

LESSON 8: Higher-Order Derivatives and the Chain Rule

TIME FRAME: 4 hours

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Compute higher-order derivatives of functions;
2. Illustrate the Chain Rule of differentiation; and
3. Solve problems using the Chain Rule.

LESSON OUTLINE:

1. Definition of higher order derivatives
2. Examples on computing higher order derivatives
3. The Chain Rule
4. Examples and problems

TOPIC 8.1: Higher-Order Derivatives of Functions

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

Consider the polynomial function $f(x) = x^4 + 3x^3 - 4x + 2$. Its derivative,

$$f'(x) = 4x^3 + 9x^2 - 4,$$

is again differentiable. Therefore, we may still differentiate f' to get

$$(f')'(x) = f''(x) = 12x^2 + 18x.$$

This is again another differentiable function. We call f'' and its subsequent derivatives the higher-order derivatives of f. Like f', which we interpret as the slope of the tangent line, the function f'' will have an interpretation on the graph of f (This will be a lesson in college). It also has an important interpretation in physics if f denotes the displacement of a particle. For now, this lesson will reinforce our skills in differentiating functions.

(B) LESSON PROPER

Consider the function $y = f(x)$. The derivative $y', f'(x), D_x y$ or $\dfrac{dy}{dx}$ is called the **first derivative** of f with respect to x. The derivative of the first derivative is called the *second derivative* of f with respect to x and is denoted by any of the following symbols:

$$y'' \ , \ f''(x) \ , \ D_x^2 \ y \ , \ \frac{d^2y}{dx^2}$$

The **third derivative** of f with respect to x is the derivative of the second derivative and is denoted by any of the following symbols:

$$y''' \ , \ f'''(x) \ , \ D_x^3 \ y \ , \ \frac{d^3y}{dx^3}$$

In general, the n^{th} ***derivative*** of f with respect to x is the derivative of the $(n-1)^{st}$ derivative and is denoted by any of the following symbols:

$$y^{(n)} \ , \ f^{(n)}(x) \ , \ D_x^n \ y \ , \ \frac{d^ny}{dx^n}$$

Formally, we have the following definition.

Definition 4. *The n^{th} **derivative of the function** f is defined recursively by*

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \qquad \textit{for } n = 1, \textit{ and}$$

$$f^{(n)}(x) = \lim_{\Delta x \to 0} \frac{f^{(n-1)}(x+\Delta x) - f^{(n-1)}(x)}{\Delta x} \quad \textit{for } n > 1,$$

provided that these limits exist. Thus, the nth derivative of f is just the derivative of the $(n-1)^{st}$ derivative of f.

Remark 1:

(a) The function f can be written as $f^{(0)}(x)$.

(b) In the notation $f^{(n)}(x)$, n is called the ***order*** of the derivative.

EXAMPLE 1:

1. Find the fourth derivative of the function $f(x) = x^5 - 3x^4 + 2x^3 - x^2 + 4x - 10$.

   Solution. We differentiate the function repeatedly and obtain

$$\begin{aligned} f'(x) &= 5x^4 - 12x^3 + 6x^2 - 2x + 4 \\ f''(x) &= 20x^3 - 36x^2 + 12x - 2 \\ f'''(x) &= 60x^2 - 72x + 12 \\ f^{(4)}(x) &= 120x - 72. \end{aligned}$$

2. Find the first and second derivatives of the function defined by

$$y = (3x^2 - 4)(x^2 - 3x).$$

   Solution. Using the Product Rule, we compute the first derivative:

$$\begin{aligned} y' &= (3x^2-4)D_x(x^2-3x) + (x^2-3x)D_x(3x^2-4) \\ &= (3x^2-4)(2x-3) + (x^2-3x)(6x) \\ &= 6x^3 - 9x^2 - 8x + 12 + 6x^3 - 18x^2 \\ &= 12x^3 - 27x^2 - 8x + 12. \end{aligned}$$

   Similarly, we obtain the second derivative:

$$\begin{aligned} y'' &= D_x(12x^3 - 27x^2 - 8x + 12) \\ &= 36x^2 - 54x - 8. \end{aligned}$$

3. Let $y = \dfrac{3x+5}{x^2+4}$. Find $\dfrac{d^2y}{dx^2}$.

Solution. Using the Quotient Rule twice, we have

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(x^2+4)D_x(3x+5) - (3x+5)D_x(x^2+4)}{(x^2+4)^2} \\
&= \frac{(x^2+4)(3) - (3x+5)(2x)}{(x^2+4)^2} \\
&= \frac{3x^2+12-6x^2-10x}{(x^2+4)^2} \\
&= \frac{12-10x-3x^2}{(x^2+4)^2},
\end{aligned}$$

and

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{12-10x-3x^2}{(x^2+4)^2}\right) \\
&= \frac{d}{dx}\left(\frac{12-10x-3x^2}{x^4+8x^2+16}\right) \\
&= \frac{(x^4+8x^2+16)\dfrac{d}{dx}(12-10x-3x^2) - (12-10x-3x^2)\dfrac{d}{dx}(x^4+8x^2+16)}{(x^4+8x^2+16)^2} \\
&= \frac{(x^4+8x^2+16)(-10-6x) - (12-10x-3x^2)(4x^3+16x)}{(x^4+8x^2+16)^2} \\
&= \frac{6x^5+30x^4-48x^3+80x^2-288x-160}{(x^4+8x^2+16)^2}.
\end{aligned}$$

4. Find the third derivative of the function defined by $g(x) = -4x^2e^x + 5xe^x - 10e^x$.

Solution. We differentiate repeatedly (applying the Product Rule) and obtain

$$\begin{aligned}
g^{(1)}(x) &= [(-4x^2)(e^x) + (e^x)(-8x)] + [(5x)(e^x) + (e^x)(5) - 10\cdot e^x] \\
&= -4x^2e^x - 3xe^x - 5e^x \\
&= e^x(-4x^2-3x-5).
\end{aligned}$$

$$\begin{aligned}
g^{(2)}(x) &= (e^x)(-8x-3) + (-4x^2-3x-5)(e^x) \\
&= e^x(-4x^2-11x-8).
\end{aligned}$$

$$\begin{aligned}
g^{(3)}(x) &= (e^x)(-8x-11) + (-4x^2-11x-8)(e^x) \\
&= e^x(-4x^2-19x-19).
\end{aligned}$$

5. If $f(x) = e^x \sin x - 3e^x \cos x$, find $f^{(5)}(x)$.

Solution. We differentiate repeatedly (applying the Product Rule) and obtain

$$\begin{aligned} f^{(1)}(x) &= [(e^x)(\cos x) + (\sin x)(e^x)] - 3 \cdot [(e^x)(-\sin x) + (\cos x)(e^x)] \\ &= e^x \cos x + e^x \sin x + 3e^x \sin x - 3e^x \cos x \\ &= e^x(4 \sin x - 2 \cos x). \end{aligned}$$

$$\begin{aligned} f^{(2)}(x) &= e^x[4(\cos x) - 2(-\sin x)] + (4 \sin x - 2 \cos x)(e^x) \\ &= e^x(2 \cos x + 6 \sin x). \end{aligned}$$

$$\begin{aligned} f^{(3)}(x) &= e^x[2(-\sin x) + 6(\cos x)] + (2 \cos x + 6 \sin x)(e^x) \\ &= e^x(8 \cos x + 4 \sin x). \end{aligned}$$

$$\begin{aligned} f^{(4)}(x) &= e^x[8(-\sin x) + 4(\cos x)] + (8 \cos x + 4 \sin x)(e^x) \\ &= e^x(12 \cos x - 4 \sin x). \end{aligned}$$

$$\begin{aligned} f^{(5)}(x) &= e^x[12(-\sin x) - 4(\cos x)] + (12 \cos x - 4 \sin x)(e^x) \\ &= e^x(8 \cos x - 16 \sin x). \end{aligned}$$

(C) EXERCISES

1. Find y', y'', y''', and y'''' for the following expressions:

   a. $y = x^4$
   b. $y = x^{-7}$
   c. $y = x^{\frac{22}{7}}$
   d. $y = x^{-\frac{1}{3}}$
   e. $y = x^{3.2}$
   f. $y = x^{-3.5}$

2. Find the first and second derivatives of the following:

   a. $y = x\sqrt{x}$
   b. $f(u) = \dfrac{1}{u^3}$
   c. $f(t) = \sqrt[3]{t}$
   d. $g(x) = \dfrac{1}{x^2\sqrt{x}}$
   e. $k(t) = \dfrac{t}{t^2\sqrt{t}}$
   f. $h(s) = \dfrac{s^3\sqrt{s}}{\sqrt[3]{s}}$

3. If $f(x) = 6x^5 - 5x^4 + 3x^3 - 7x^2 + 9x - 14$, then find $f^{(n)}(x)$ for all $n \in \mathbb{N}$.
4. Find $f'\left(\frac{\pi}{6}\right)$, $f''\left(\frac{\pi}{6}\right)$, and $f'''\left(\frac{\pi}{6}\right)$, given $f(x) = -\sin x$.
5. If $g(x) = \sin x$, then find $g^{(n)}(x)$ for all $n \in \mathbb{N}$.
6. Find $f^{(5)}(x)$ if $f(x) = \sqrt[3]{3x + 2}$.

$\star$7. Find $D_x^{55}(\cos x)$ and $D_x^{56}(\cos x)$.

Answer: $D_x^{55}(\cos x) = \sin x$ and $D_x^{56}(\cos x) = \cos x$

$\star$8. Find $f^{(n)}(x)$ for all $n \in \mathbb{N}$ where $f(x) = e^{-x}$. (Hint: Write $f(x)$ as a quotient.)

Answer: $f^{(n)}(x) = -e^{-x}$ if n is odd while $f^{(n)}(x) = e^{-x}$ when n is even.

$\star$9. Let $p(x)$ be a polynomial of degree n with leading coefficient 1. What is $p^{(k)}(x)$ if (a) $k = n$; and if (b) $k > n$.

Answer: (a) $n!$; (b) 0

TOPIC 8.2: The Chain Rule

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

Consider the following functions:

(a) $f(x) = (3x^2 - 2x + 4)^2$

(b) $y = \sin 2x$

Teaching Tip

Ask the students to find the derivatives $f'(x)$ and y' of the functions above, before continuing with your lecture.

Expect some students to use the Power Rule (even when it is not applicable):

$$D_x[(3x^2 - 2x + 4)^2] = 2(3x^2 - 2x + 4). \quad \text{(This is incorrect!)}$$

Some may expand the expression first to get

$$f(x) = 9x^4 + 4x^2 + 16 - 12x^3 + 24x^2 - 16x = 9x^4 - 12x^3 + 28x^2 - 16x + 16$$

before differentiating:

$$f'(x) = 36x^3 - 36x^2 + 56x - 16. \quad \text{(This is correct!)}$$

Some may even use product rule and first write the function

$$f(x) = (3x^2 - 2x + 4)^2 = (3x^2 - 2x + 4)(3x^2 - 2x + 4).$$

Hence,

$$\begin{aligned} f'(x) &= (3x^2 - 2x + 4)D_x(3x^2 - 2x + 4) + (3x^2 - 2x + 4)D_x(3x^2 - 2x + 4) \\ &= 2 \cdot (3x^2 - 2x + 4)(6x - 2) \\ &= (36x^3 - 24x^2 + 48x) - (12x^2 - 8x + 16) \\ &= 36x^3 - 36x^2 + 56x - 16. \quad \text{(This is correct!)} \end{aligned}$$

Ask them if the last two (correct) methods will be doable if

$$f(x) = (3x^2 - 2x + 4)^{200}.$$

On the other hand, for the function $y = \sin 2x$, some of them may have used a trigonometric identity to first rewrite y into

$$y = \sin 2x = 2 \sin x \cos x.$$

In this case

$$\begin{aligned} y' &= 2[(\sin x)(-\sin x) + (\cos x)(\cos x)] \\ &= 2(\cos^2 x - \sin^2 x) \\ &= 2\cos(2x). \end{aligned}$$

In this lesson students will learn a rule that will allow them to differentiate a given function without having to perform any preliminary multiplication, or apply any special formula.

(B) LESSON PROPER

The Chain Rule below provides for a formula for the derivative of a composition of functions.

Theorem 9 (Chain Rule)**.** *Let f be a function differentiable at c and let g be a function differentiable at $f(c)$. Then the composition $g \circ f$ is differentiable at c and*

$$D_x(g \circ f)(c) = g'(f(c)) \cdot f'(c).$$

Remark 1: Another way to state the Chain Rule is the following: If y is a differentiable function of u defined by $y = f(u)$ and u is a differentiable function of x defined by $u = g(x)$, then y is a differentiable function of x, and the derivative of y with respect to x is given by

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

In words, the derivative of a composition of functions is the derivative of the *outer* function evaluated at the inner function, times the derivative of the inner function.

EXAMPLE 1:

(a) Recall our first illustration $f(x) = (3x^2 - 2x + 4)^2$. Find $f'(x)$ using the Chain Rule.

Solution. We can rewrite $y = f(x) = (3x^2 - 2x + 4)^2$ as $y = f(u) = u^2$ where $u = 3x^2 - 2x + 4$, a differentiable function of x. Using the Chain Rule, we have

$$\begin{aligned} f'(x) = y' &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= (2u)(6x - 2) \\ &= 2(3x^2 - 2x + 4)(6x - 2) \\ &= 36x^3 - 36x^2 + 56x - 16. \end{aligned}$$

(b) For the second illustration, we have $y = \sin(2x)$. Find y' using the Chain Rule.

Solution. We can rewrite $y = \sin(2x)$ as $y = f(u)$ where $f(u) = \sin u$ and $u = 2x$. Hence,

$$\begin{aligned} y' &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \cos u \cdot 2 \\ &= 2\cos(2x). \end{aligned}$$

(c) Find $\dfrac{dz}{dy}$ if $z = \dfrac{4}{(a^2 - y^2)^2}$, where a is a real number.

Solution. Notice that we can write $z = \dfrac{4}{(a^2 - y^2)^2}$ as $z = 4(a^2 - y^2)^{-2}$. Applying the Chain Rule, we have

$$\begin{aligned} \frac{dz}{dy} &= 4 \cdot -2(a^2 - y^2)^{-2-1} \cdot \frac{d}{dy}(a^2 - y^2) \\ &= -8(a^2 - y^2)^{-3} \cdot -2y \\ &= 16y(a^2 - y^2)^{-3} \\ &= \frac{16y}{(a^2 - y^2)^3}. \end{aligned}$$

Now, suppose we want to find the derivative of a power function of x. That is, we are interested in $D_x[f(x)^n]$. To derive a formula for this, we let $y = u^n$ where u is a differentiable function of x given by $u = f(x)$. Then by the Chain Rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= nu^{n-1} \cdot f'(x) \\ &= n[f(x)]^{n-1} \cdot f'(x) \end{aligned}$$

Thus, $D_x[f(x)]^n = n[f(x)]^{n-1} \cdot f'(x)$. This is called the GENERALIZED POWER RULE.

EXAMPLE 2:

(a) What is the derivative of $y = (3x^2 + 4x - 5)^5$?

Solution.

$$\begin{aligned} D_x[(3x^2+4x-5)^5] &= 5\cdot(3x^2+4x-5)^{5-1}\cdot D_x(3x^2+4x-5) \\ &= 5(3x^2+4x-5)^4(6x+4). \end{aligned}$$

(b) Find $\dfrac{dy}{dx}$ where $y=\sqrt{3x^3+4x+1}$.

Solution.

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}(3x^3+4x+1)^{\frac{1}{2}-1}D_x(3x^3+4x+1) \\ &= \frac{1}{2}(3x^3 4x+1)^{-\frac{1}{2}}(9x^2+4) \\ &= \frac{9x^2+4}{2\sqrt{3x^3 4x+1}}. \end{aligned}$$

(c) Find $\dfrac{dy}{dx}$ where $y=(\sin 3x)^2$.

Solution.

$$\begin{aligned} \frac{dy}{dx} &= 2\cdot(\sin 3x)^{2-1}\cdot\frac{d}{dx}(\sin 3x) \\ &= 2(\sin 3x)\cdot\cos 3x\cdot\frac{d}{dx}(3x) \\ &= 2(\sin 3x)(\cos 3x)\cdot 3 \\ &= 6\sin 3x\cos 3x. \end{aligned}$$

(d) Differentiate $(3x^2-5)^3$.

Solution.

$$\begin{aligned} \frac{d(3x^2-5)^3}{dx} &= 3(3x^2-5)^{3-1}\cdot D_x(3x^2-5) \\ &= 3(3x^2-5)^2\cdot 6x \\ &= 18x(3x^2-5)^2. \end{aligned}$$

(e) Consider the functions $y = 3u^2 + 4u$ and $u = x^2 + 5$. Find $\dfrac{dy}{dx}$.

Solution. By the Chain Rule, we have $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ where $\dfrac{dy}{du} = 6u+4$ and $\dfrac{du}{dx} = 2x$. Thus,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= (6u+4)(2x) \\ &= \left[6(x^2+5)+4\right](2x) \\ &= (6x^2+34)(2x) \\ &= 12x^3 + 68x. \end{aligned}$$

(C) **EXERCISES**

1. Use the Chain Rule to find $\dfrac{dy}{dx}$ in terms of x.

   a. $y = (u-2)^3 \quad \text{and} \quad u = \dfrac{1}{2x+1}$

   b. $y = \sqrt{u} \quad \text{and} \quad u = 5x^2 - 3$

   c. $y = \dfrac{1}{u+1} \quad \text{and} \quad u = \dfrac{1}{x+1}$

2. Solve for $\dfrac{dy}{dx}$ and simplify the result.

   a. $y = (3x-2)^5(2x^2+5)^6$

   b. $y = \left(\dfrac{5x-1}{2x+3}\right)^3$

   c. $y = x\sqrt{4-x^2}$

3. Find the derivatives of the following functions as specified:

   a. $y = \sin x^2, \quad y''$

   b. $h(z) = (9z+4)^{\frac{3}{2}}, \quad h^{(3)}(z)$

   c. $f(x) = \sin(\cos x), \quad f''(x)$

   d. $g(x) = (x^2+1)^{17}, \quad g^{(2)}(x)$

   e. $y = (t^3 - \sqrt{t})^{-3.2}, \quad y''$

   f. $y = x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}, \quad \dfrac{d^2y}{dx^2}$

4. Find the first and second derivatives of the following:

   a. $y = e^{\cos x}$

   b. $f(x) = (ax+b)^7$

   c. $h(t) = \cot^3(e^t)$

   d. $g(x) = 3xe^{3x}$

   e. $y = e^x - e^{-x}$

   f. $s = \dfrac{5}{3+e^t}$

5. Find $\dfrac{dy}{dx}$ if $y = \sin(\sqrt{x^3+1})$.

6. Find $h'\left(\dfrac{\pi}{3}\right), h''\left(\frac{\pi}{3}\right)$, and $h'''\left(\dfrac{\pi}{3}\right)$, given $h(x) = \sin x \cos 3x$.
7. If $f(x) = 6x^5 - 5x^4 + 3x^3 - 7x^2 + 9x - 14$, then find $f^{(n)}(x)$ for all $x \in \mathbb{N}$.

For problems (8) and (9), please refer to the table below:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$	$(f \circ g)(x)$	$(f \circ g)'(x)$
-2	2	-1	1	1		
-1	1	2	0	2		
0	-2	1	2	-1		
1	0	-2	-1	2		
2	1	0	1	-1		

8. Use the table of values to determine $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = 1$ and $x = 2$.
9. Use the table of values to determine $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = -2, -1$ and 0.
10. Show that $z'' + 4z' + 8z = 0$ if $z = e^{-2x}(\sin 2x + \cos 2x)$.
11. If $f(x) = x^4$ and $g(x) = x^3$, then $f'(x) = 4x^3$ and $g'(x) = 3x^2$. The Chain Rule multiplies derivatives to get $12x^5$. But $f(g(x)) = \left(x^3\right)^4 = x^{12}$ and its derivative is NOT $12x^5$. Where is the flaw?

LESSON 9: Implicit Differentiation

TIME FRAME: 2 hours

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Illustrate implicit differentiation;
2. Apply the derivatives of the natural logarithmic and inverse tangent functions; and
3. Use implicit differentiation to solve problems.

LESSON OUTLINE:

1. What is implicit differentiation?
2. The derivative of the natural logarithmic and inverse tangent functions
3. Solving problems using implicit differentiation
4. Enrichment: Derivatives of other transcendental functions

TOPIC 9.1: What is Implicit Differentiation?

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

The majority of differentiation problems in basic calculus involves functions y written *explicitly* as functions of the independent variable x. This means that we can write the function in the form $y = f(x)$. For such a function, we can find the derivative directly. For example, if

$$y = 4x^5 + \cos(2x - 7),$$

then the derivative of y with respect to x is

$$\frac{dy}{dx} = 20x^4 - 2\sin(2x - 7).$$

However, some functions y are written *implicitly* as functions of x. This means that the expression is not given directly in the form $y = f(x)$. A familiar example of this is the equation

$$x^2 + y^2 = 5,$$

which represents a circle of radius $\sqrt{5}$ with its center at the origin $(0, 0)$. We can think of the circle as the union of the graphs of two functions, namely the function represented by the upper semi-circle and the function represented by the lower semi-circle. Suppose that we wish to find the slope of the line tangent to the circle at the point $(-2, 1)$.

The solution is to find the the derivative at the point $(-2, 1)$. Since the equation of the circle is not complicated, one way to do this is to write y explicitly in terms of x. Thus, from $x^2 + y^2 = 5$, we obtain $y = \pm\sqrt{5 - x^2}$. The positive square root represents the upper semi-circle while the negative square root represents the bottom semi-circle. Since the point $(-2, 1)$ is on the upper semi-circle, the slope of the tangent line is now obtained by differentiating the function

$$y = \sqrt{5 - x^2} = (5 - x^2)^{1/2}$$

and evaluating the derivative at $x = -2$. Thus, using the Chain Rule,

$$\frac{dy}{dx} = \frac{1}{2}(5 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{5 - x^2}}.$$

Therefore, the slope of the tangent line at the point $(-2, 1)$ is the value of the above derivative evaluated at $x = -2$, namely

$$\frac{dy}{dx} = \frac{2}{\sqrt{5 - (-2)^2}} = \frac{2}{\sqrt{1}} = 2.$$

In the above example, we obtained the required derivative because we were able to write y explicitly in terms of the variable x. That is, we were able to transform the original equation into an equation of the form $y = f(x)$, with the variable y on one side of the equation, and the other side consisting of an expression in terms of x.

However, there are many equations where it is difficult to express y in terms of x. Some examples are:

(a) $y^3 + 4y^2 + 3x^2 + 10 = 0$

(b) $\sin^2 x = 4\cos^3(2y + 5)$

(c) $x + y^3 = \ln(xy^4)$

In this lesson, we will learn another method to obtain derivatives. The method is called *implicit differentiation.*

(B) LESSON PROPER

We have seen that functions are not always given in the form $y = f(x)$ but in a more complicated form that makes it difficult or impossible to express y explicitly in terms of x. Such functions are called implicit functions, and y is said to be defined implicitly. In this lesson, we explain how these can be differentiated using a method called *implicit differentiation.*

Differentiating quantities involving only the variable x with respect to x is not a problem; for instance, the derivative of x is just 1. But if a function y is defined implicitly, then we need to apply the Chain Rule in getting its derivative. So, while the derivative of x^2 is $2x$, the derivative of y^2 is

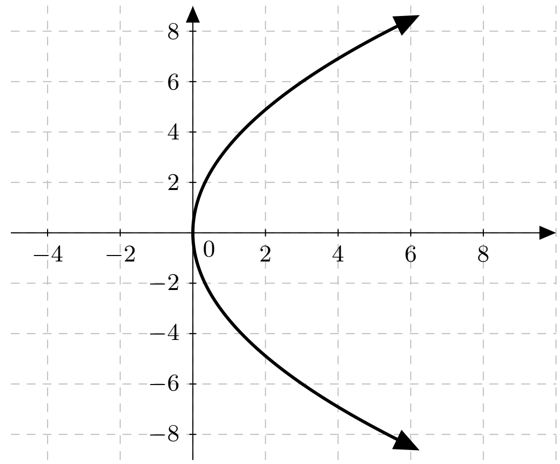
$$2y\frac{dy}{dx}.$$

More generally, if we have the expression $f(y)$, where y is a function of x, then

$$\frac{d}{dx}\left(f(y)\right) = \frac{d}{dy}\left(f(y)\right) \cdot \frac{dy}{dx}.$$

In order to master implicit differentiation, students need to review and master the application of the Chain Rule.

Consider a simple expression such as $y^2 = 4x$. Its graph is a parabola with vertex at the origin and opening to the right.



If we consider only the upper branch of the parabola, then y becomes a function of x. We can obtain the derivative dy/dx by applying the Chain Rule. When differentiating terms involving y, we are actually applying the Chain Rule, that is, we first differentiate with respect to y, and then multiply by dy/dx. Differentiating both sides with respect to x, we have

$$\begin{aligned} & y^2 &=& \quad 4x, \\ \Longrightarrow & \frac{d}{dx}(y^2) &=& \frac{d}{dx}(4x) \\ \Longrightarrow & 2y\frac{dy}{dx} &=& \quad 4. \end{aligned}$$

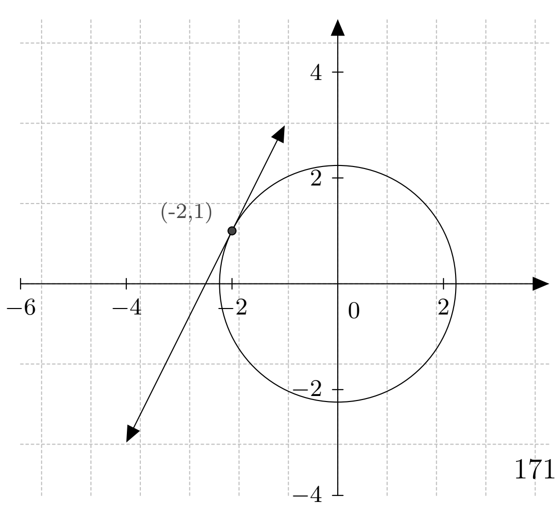
Solving for dy/dx, we obtain

$$\frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}.$$

Notice that the derivative contains y. This is typical in implicit differentiation.

Let us now use implicit differentiation to find the derivatives dy/dx in the following examples. Let us start with our original problem involving the circle.

EXAMPLE 1: Find the slope of the tangent line to the circle $x^2 + y^2 = 5$ at the point $(-2, 1)$.



$$\begin{aligned} & x^2 + y^2 &=& \quad 5 \\ \Longrightarrow & \frac{d}{dx}(x^2 + y^2) &=& \frac{d}{dx}(5) \\ \Longrightarrow & \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &=& \quad 0 \\ \Longrightarrow & 2x + 2y\frac{dy}{dx} &=& \quad 0. \end{aligned}$$

Solution. Solving for dy/dx, we obtain

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}.$$

Substituting $x = -2$ and $y = 1$, we find that the slope is

$$\frac{dy}{dx} = 2.$$

Notice that this is a faster and easier way to obtain the derivative.

EXAMPLE 2: Find $\dfrac{dy}{dx}$ for $y^3 + 4y^2 + 3x^2y + 10 = 0$.

Solution. Differentiating both sides of the equation gives

$$\begin{aligned} & \frac{d}{dx}(y^3 + 4y^2 + 3x^2y + 10) = \frac{d}{dx}(0) \\ \Longrightarrow \qquad & \frac{d}{dx}(y^3) + \frac{d}{dx}(4y^2) + \frac{d}{dx}(3x^2y) + \frac{d}{dx}(10) = 0 \\ \Longrightarrow \qquad & 3y^2\frac{dy}{dx} + 8y\frac{dy}{dx} + 3x^2\frac{dy}{dx} + 6xy + 0 = 0. \end{aligned}$$

We collect the terms involving dy/dx and rearrange to get

$$\frac{dy}{dx}(3y^2 + 8y + 3x^2) + 6xy = 0.$$

Thus,

$$\frac{dy}{dx} = \frac{-6xy}{3y^2 + 8y + 3x^2}.$$

Note that the derivative of the term $3x^2y$ is obtained by applying the Product Rule. We consider $3x^2$ as one function and y as another function.

Implicit differentiation can be applied to any kind of function, whether they are polynomial functions, or functions that involve trigonometric and exponential, quantitites.

EXAMPLE 3: Find $\dfrac{dy}{dx}$ for $x + y^3 = e^{xy^4}$.

Solution. Differentiating both sides with respect to x gives

$$\begin{aligned} & \frac{d}{dx}(x + y^3) = \frac{d}{dx}(e^{xy^4}) \\ \Longrightarrow \qquad & 1 + 3y^2\frac{dy}{dx} = e^{xy^4}\frac{d}{dx}(xy^4) \\ \Longrightarrow \qquad & 1 + 3y^2\frac{dy}{dx} = e^{xy^4}\left(4xy^3\frac{dy}{dx} + y^4\right) \end{aligned}$$

Collecting all terms with dy/dx gives

$$\begin{aligned} & \frac{dy}{dx}\left(3y^2 - e^{xy^4}4xy^3\right) = -1 + e^{xy^4}y^4 \\ \Longrightarrow \qquad & \frac{dy}{dx} = \frac{-1 + e^{xy^4}y^4}{3y^2 - e^{xy^4}4xy^3}. \end{aligned}$$

DERIVATIVES OF THE NATURAL LOGARITHMIC AND INVERSE TANGENT FUNCTIONS

The derivatives of some inverse functions can be found by implicit differentiation. Take, for example, the natural logarithmic function

$$y = \ln x.$$

Note that it is the inverse function of the exponential function. To find dy/dx, we first rewrite this into

$$e^y = x \tag{2.5}$$

and then differentiate implicitly:

$$\begin{aligned} & \frac{d}{dx}(e^y) = \frac{d}{dx}(x) \\ \Longrightarrow \quad & e^y \frac{dy}{dx} = 1 \\ \Longrightarrow \quad & \frac{dy}{dx} = \frac{1}{e^y}. \end{aligned} \tag{2.6}$$

However, from (2.5), $e^y = x$. Hence after substituting this to (2.6), we see that

$$y = \ln x \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{1}{x}.$$

Now, we do a similar process as above to find the derivative of the inverse tangent function. Let's consider

$$y = \tan^{-1} x.$$

Applying the tangent function to both sides gives

$$\tan y = x. \tag{2.7}$$

Now, we apply implicit differentiation. So,

$$\begin{aligned} & \frac{d}{dx}(\tan y) = \frac{d}{dx}(x) \\ \Longrightarrow \quad & \sec^2 y \frac{dy}{dx} = 1 \\ \Longrightarrow \quad & \frac{dy}{dx} = \frac{1}{\sec^2 y}. \end{aligned} \tag{2.8}$$

Next, we have to find a way so that by the use of (2.7), we may be able to write (2.8) as a function of x. The relationship between $\tan y$ and $\sec^2 y$ is straightforward from one of the trigonometric identitites:

$$\sec^2 y - \tan^2 y = 1.$$

Therefore, from (2.7), $\sec^2 y = 1 + \tan^2 y = 1 + x^2$. Finally, substituting this into (2.8), we get the following result:

$$y = \tan^{-1} x \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{1}{1 + x^2}.$$

We summarize these two derivatives with consideration to Chain Rule.

Derivatives of the Natural Logarithmic and Inverse Tangent Functions

Suppose u is a function of x. Then

- $\dfrac{d}{dx}(\ln u) = \dfrac{1}{u} \cdot \dfrac{du}{dx}$
- $\dfrac{d}{dx}(\tan^{-1} u) = \dfrac{1}{1+u^2} \cdot \dfrac{du}{dx}$

EXAMPLE 4: Find dy/dx.

(a) $y = \ln(7x^2 - 3x + 1)$

(b) $y = \tan^{-1}(2x - 3\cos x)$

(c) $y = \ln(4x + \tan^{-1}(\ln x))$

Solution.

(a) $\dfrac{dy}{dx} = \dfrac{1}{7x^2 - 3x + 1} \cdot (14x - 3)$

(b) $\dfrac{dy}{dx} = \dfrac{1}{1 + (2x - 3\cos x)^2} \cdot (2 + 3\sin x)$

(c) $\dfrac{dy}{dx} = \dfrac{1}{4x + \tan^{-1}(\ln x)} \cdot \left(4 + \dfrac{1}{1 + (\ln x)^2} \cdot \dfrac{1}{x}\right).$

EXAMPLE 5: Find $\dfrac{dy}{dx}$ for $\cos(y^2 - 3) = \tan^{-1}(x^3) + \ln y$.

Solution. Differentiating both sides gives

$$\begin{aligned} & \frac{d}{dx}(\cos(y^2-3)) = \frac{d}{dx}(\tan^{-1}(x^3) + \ln y) \\ \Longrightarrow \qquad & -\sin(y^2-3) \cdot 2y \cdot \frac{dy}{dx} = \frac{1}{1+(x^3)^2} \cdot 3x^2 + \frac{1}{y} \cdot \frac{dy}{dx}. \end{aligned}$$

Collecting terms with dy/dx:

$$\begin{aligned} & \frac{dy}{dx}\left(-2y\sin(y^2-3) - \frac{1}{y}\right) = \frac{3x^2}{1+x^6} \\ \Longrightarrow \qquad & \frac{dy}{dx} = \frac{\dfrac{3x^2}{1+x^6}}{-2y\sin(y^2-3) - \dfrac{1}{y}}. \end{aligned}$$

EXAMPLE 6: Find $\dfrac{dy}{dx}$ for $\tan^{-1} y = 3x^2 y - \sqrt{\ln(x - y^2)}$.

Solution. Differentiating both sides with respect to x gives

$$
\begin{aligned}
& \frac{d}{dx}\left(\tan^{-1} y\right) = \frac{d}{dx}\left(3x^2y - (\ln(x-y^2))^{1/2}\right) \\
\Longrightarrow \quad & \frac{1}{1+y^2} \cdot \frac{dy}{dx} = 3x^2\frac{dy}{dx} + 6xy - \frac{1}{2}\left(\ln(x-y^2)\right)^{-1/2}\frac{1}{x-y^2}\left(1-2y\frac{dy}{dx}\right) \\
\Longrightarrow \quad & \frac{1}{1+y^2} \cdot \frac{dy}{dx} = 3x^2\frac{dy}{dx} + 6xy - \frac{(\ln(x-y^2))^{-1/2}}{2(x-y^2)} + \frac{y(\ln(x-y^2))^{-1/2}}{x-y^2}\frac{dy}{dx}.
\end{aligned}
$$

We now isolate dy/dx:

$$
\begin{aligned}
& \frac{dy}{dx}\left(\frac{1}{1+y^2} - 3x^2 - \frac{y(\ln(x-y^2))^{-1/2}}{x-y^2}\right) = 6xy - \frac{(\ln(x-y^2))^{-1/2}}{2(x-y^2)} \\
\Longrightarrow \quad & \frac{dy}{dx} = \frac{6xy - \dfrac{(\ln(x-y^2))^{-1/2}}{2(x-y^2)}}{\dfrac{1}{1+y^2} - 3x^2 - \dfrac{y(\ln(x-y^2))^{-1/2}}{x-y^2}}.
\end{aligned}
$$

Seatwork: Find the derivative with respect to x of

$$\tan^{-1}(x^2) - \ln(2x+3) = 4\cos^3(2y+5).$$

Note that the equation defines y implicitly as a function of x.

Solution.

$$
\begin{aligned}
& \frac{d}{dx}(\tan^{-1}(x^2) - \ln(2x+3)) = \frac{d}{dx}\left(4\cos^3(2y+5)\right) \\
\Longrightarrow \quad & \frac{1}{1+x^4} \cdot 2x - \frac{2}{2x+3} = 12\cos^2(2y+5) \cdot -\sin(2y+5) \cdot 2\frac{dy}{dx} \\
\Longrightarrow \quad & \frac{dy}{dx} = \frac{\dfrac{x}{1+x^4} - \dfrac{1}{2x+3}}{-12\cos^2(2y+5)\sin(2y+5)}.
\end{aligned}
$$

(C) **EXERCISES.**

1. Find $\dfrac{dy}{dx}$ for the following:
   a. $x^3 + y^3 = 8$
   b. $y \sin y = xy$
   c. $\tan^{-1}(x+3y) = x^4$
   d. $e^{2y} + x^3 = y$
   $\star$e. $\ln(3xy) = x + x^5$
2. Find the slope of the tangent line to $x\sqrt[3]{y} + y\sqrt[3]{x} = 10$ at (1,8).

3. Find the slope of the tangent line to $x^2e^y + y^2e^x = 2e$ at (1,1).

$\star$4. Find the equation of the tangent line to $x^2 - 3xy + y^2 = -1$ at (2,1).

$\star$5. Consider $xy^2 + x^2y = 6$.

   a. Find $\dfrac{dy}{dx}$.

   b. Find the slope of the tangent at the point (1,2).

   c. Find the point where the tangent line is horizontal.

   d. Find the point where the tangent line is vertical.

ANSWERS TO STARRED EXERCISES

1.e.

$$\frac{1}{3xy}\cdot 3\left(x\frac{dy}{dx}+y\right) = 1+5x^4$$
$$\frac{3x\cdot\frac{dy}{dx}}{3xy}+\frac{3y}{3xy} = 1+5x^4$$
$$\frac{1}{y}\cdot\frac{dy}{dx}+\frac{1}{x} = 1+5x^4$$
$$\frac{dy}{dx} = y\left(1+5x^4-\frac{1}{x}\right) = \frac{y(5x^5+x-1)}{x}$$

4. Implicitly differentiating $x^2 - 3xy + y^2 = -1$ yields

$$2x - 3\left(x\cdot\frac{dy}{dx}+y\cdot 1\right)+2y\cdot\frac{dy}{dx} = 0.$$

Substituting (x, y) with $(2, 1)$ gives

$$2(2) - 3\left(2\cdot\frac{dy}{dx}+1\right)+2\cdot\frac{dy}{dx} = 0.$$

Solving this for dy/dx will give $dy/dx = 1/4$. Hence, the equation of the tangent line is $(y-1) = \dfrac{1}{4}(x-2)$ or, in general form, $x - 4y + 2 = 0$.

5. a. Differentiating implicitly,

$$\left(x\cdot 2y\frac{dy}{dx}+y^2+x^2\cdot\frac{dy}{dx}+2xy\right) = 0$$
$$\frac{dy}{dx}\left(2xy+x^2\right) = -(y^2+2xy)$$
$$\frac{dy}{dx} = -\frac{y^2+2xy}{2xy+x^2}$$

b. We just substitute $(x, y) = (1, 2)$ into the above equation:

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,2)} = -\frac{2^2+2(1)(2)}{2(1)(2)+1^2} = -\frac{8}{5}.$$

c. The tangent line is horizontal when $dy/dx = 0$. Thus,

$$\begin{aligned} \frac{dy}{dx} &= -\frac{(y^2+2xy)}{(2xy+x^2)} = 0 \\ \implies \quad y^2 + 2xy &= 0 \\ \implies \quad y(y+2x) &= 0 \\ \implies \quad y = 0 &\text{ or } y = -2x. \end{aligned}$$

We now go back to the original equation: $xy^2+x^2y = 6$. Note that if $y = 0$, we get $0 = 6$, which is a contradiction. On the other hand, if $y = -2x$, then $4x^3-2x^3 = 6$ or $x = \sqrt[3]{3}$. So, the tangent line is horizontal at the point $(\sqrt[3]{3}, -2\sqrt[3]{3})$.

d. Similarly, vertical tangent lines occur when the derivative dy/dx is undefined, i.e. when $x = 0$ or $x = -2y$. Again $x = 0$ produces a contradiction while $x = -2y$ yields $y = \sqrt[3]{3}$. So, vertical tangent lines occur at the point $(-2\sqrt[3]{3}, \sqrt[3]{3})$.

(D) **ENRICHMENT**

DERIVATIVES OF OTHER TRANSCENDENTAL FUNCTIONS

We have so far learned the derivatives of the following transcendental functions:

(a) $\dfrac{d}{dx}(e^x) = e^x$ (b) $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$ (c) $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$

We now explore the derivatives of b^x, $\log_b x$, and other inverse trigonometric functions.

I. Let $b > 0$, $b \neq 1$. To find the derivative of $y = b^x$, we first rewrite it in another form:

$$y = b^x = e^{\ln b^x} = e^{x \ln b}. \tag{2.9}$$

Hence, using Chain Rule, $\dfrac{dy}{dx} = e^{x\ln b} \cdot \ln b$. Therefore, using (2.9),

$$y = b^x \quad \implies \quad \frac{dy}{dx} = b^x \cdot \ln b.$$

II. Let $b > 0$, $b \neq 1$. To find the derivative of

$$y = \log_b x, \tag{2.10}$$

we may now use implicit differentiation. We first rewrite (2.10) into

$$b^y = x. \tag{2.11}$$

Therefore,

$$\begin{aligned} \frac{d}{dx}(b^y) &= \frac{d}{dx}(x) \\ \implies \quad b^y \cdot \ln b \frac{dy}{dx} &= 1 \\ \implies \quad \frac{dy}{dx} &= \frac{1}{b^y \cdot \ln b}. \end{aligned} \tag{2.12}$$

Finally substituting (2.11) into (2.12) gives

$$y = \log_b x \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{1}{x \ln b}.$$

III. We only show how to find the derivative of $\sin^{-1} x$, and leave the analogous proofs of the other derivatives to the teacher. Let $x \in [-1, 1]$, and consider $y = \sin^{-1} x$. Applying the sine function to both sides of the equation gives

$$\sin y = x. \tag{2.13}$$

Implicitly differentiating (2.13), we obtain

$$\begin{aligned} & \frac{d}{dx}(\sin y) = \frac{d}{dx}(x) \\ \Longrightarrow \quad & \cos y \frac{dy}{dx} = 1 \\ \Longrightarrow \quad & \frac{dy}{dx} = \frac{1}{\cos y}. \end{aligned} \tag{2.14}$$

We now find a way to express $\cos y$ in terms of x using equation (2.13). However, we know from trigonometry that

$$\begin{aligned} & \cos^2 y + \sin^2 y = 1 \\ \Longrightarrow \quad & \cos y = \pm\sqrt{1 - \sin^2 y}. \end{aligned}$$

Recall that in Precalculus, the range of $y = \sin^{-1} x$ has been restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (the fourth and first quadrants). Therefore, $\cos y > 0$ and so we only choose $\cos y = +\sqrt{1 - \sin^2 y}$. Finally, substituting this into (2.14) gives

$$y = \sin^{-1} x \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

We summarize these three derivatives of the other inverse trigonometric functions:

Summary of Derivatives of Transcendental Functions

Let u be a differentiable function of x.

(a) $\dfrac{d}{dx}(e^u) = e^u \dfrac{du}{dx}$

(b) $\dfrac{d}{dx}(b^u) = b^u \cdot \ln b \dfrac{du}{dx}$

(c) $\dfrac{d}{dx}(\ln u) = \dfrac{1}{u} \dfrac{du}{dx}$

(d) $\dfrac{d}{dx}(\log_b u) = \dfrac{1}{u \ln b} \dfrac{du}{dx}$

(e) $\dfrac{d}{dx}(\sin^{-1}(x)) = \dfrac{1}{\sqrt{1 - x^2}} \dfrac{du}{dx}$

(f) $\dfrac{d}{dx}(\cos^{-1}(x)) = -\dfrac{1}{\sqrt{1 - x^2}} \dfrac{du}{dx}$

(g) $\dfrac{d}{dx}(\tan^{-1}(x)) = \dfrac{1}{1 + x^2} \dfrac{du}{dx}$

(h) $\dfrac{d}{dx}(\cot^{-1}(x)) = -\dfrac{1}{1 + x^2} \dfrac{du}{dx}$

(i) $\dfrac{d}{dx}(\sec^{-1}(x)) = \dfrac{1}{x\sqrt{x^2 - 1}} \dfrac{du}{dx}$

(j) $\dfrac{d}{dx}(\csc^{-1}(x)) = -\dfrac{1}{x\sqrt{x^2 - 1}} \dfrac{du}{dx}$

Teaching Tip

A curious student will realize that the derivative of a cofunction of a trigonometric function is the negative of the derivative of the original trigonometric function. This is because of the identities

- $\cos^{-1} x = \dfrac{\pi}{2} - \sin^{-1}(x)$
- $\cot^{-1} x = \dfrac{\pi}{2} - \tan^{-1}(x)$
- $\csc^{-1} x = \dfrac{\pi}{2} - \sec^{-1}(x)$.

EXAMPLE 7: Find $\dfrac{dy}{dx}$ for the following.

(a) $y = \log_2 x$

(b) $y = 5^x$

(c) $3^y = \sec^{-1} x - \cos^{-1}(y^2+1)$

Solution. (a) $\dfrac{dy}{dx} = \dfrac{1}{x \ln 2}$

(b) $\dfrac{dy}{dx} = 5^x \cdot \ln 5$

(c) We use implicit differentiation here.

$$\begin{aligned}
3^y \ln 3 \frac{dy}{dx} &= \frac{1}{x\sqrt{x^2-1}} - \left(-\frac{1}{\sqrt{1-(y^2+1)^2}} \cdot 2y \frac{dy}{dx} \right) \\
\left(3^y \ln 3 - \frac{1}{\sqrt{1-(y^2+1)^2}} \cdot 2y \right) \frac{dy}{dx} &= \frac{1}{x\sqrt{x^2-1}} \\
\frac{dy}{dx} &= \frac{\dfrac{1}{x\sqrt{x^2-1}}}{3^y \ln 3 - \dfrac{1}{\sqrt{1-(y^2+1)^2}} \cdot 2y}.
\end{aligned}$$

LESSON 10: Related Rates

TIME FRAME: 2 hours lecture

LEARNING OUTCOME: At the end of the lesson, the learner shall be able to solve situational problems involving related rates.

LESSON OUTLINE:

1. Process of solving related rates problems
2. Word problems

TOPIC 10.1: Solutions to Problems Involving Related Rates

DEVELOPMENT OF THE LESSON

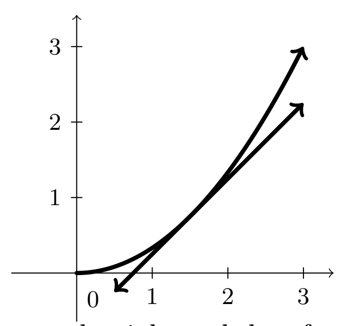
(A) INTRODUCTION

This section culminates the chapter on derivatives. The discussion on related rates concerns quantities which change (increase/decrease) with time, and which are related by an equation. Differentiating this equation with respect to time gives an equation of relationship between the rates of change of the quantities involved. Therefore, if we know the rates of change of all but one quantity, we are able to solve this using the aforementioned relationship between the rates of change.

As motivation, imagine a water droplet falling into a still pond, producing ripples that propagate away from the center. Ideally, the ripples are concentric circles which increase in radius (and also in area) as time goes on. Thus, the radius and area of a single ripple are changing at rates that are related to each other. This means that if we know how fast the radius is changing, we should be able to determine how fast the area is changing at any point in time, and vice versa.

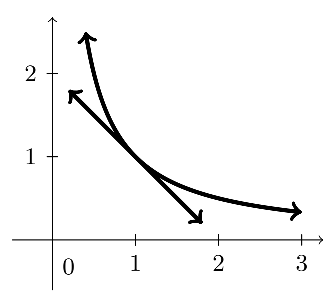
We first need to recall that aside from being the slope of the tangent line to a function at a point, the derivative is also interpreted as a rate of change. The sign of the derivative indicates whether the function is increasing or decreasing.

Suppose the graph of a differentiable function is increasing. This means that as x increases, the y-value also increases. Hence, its graph would typically start from the bottom left and increase to the top right of the frame. Refer to the figure on the right.



Observe that the tangent line to the graph at any point slants to the right and therefore, has a positive slope. This, in fact, describes increasing differentiable functions: A differentiable function is increasing on an interval if its derivative is positive on that interval.

Similarly, a differentiable function is decreasing on an interval if and only if its derivative is negative on that interval.



Remark

Let x be a differentiable function which represents a quantity that changes with time t, then

- $\frac{dx}{dt}$ is the rate of change of x with respect to t;
- $\frac{dx}{dt}$ is positive if and only if x increases with time; and
- $\frac{dx}{dt}$ is negative if and only if x decreases with time.

The unit of measurement of $\frac{dx}{dt}$ is $\frac{\text{unit of measurement of } x}{\text{unit of measurement of } t}$.

(B) **LESSON PROPER**

A ***related rates*** problem concerns the relationship among the rates of change of several variables with respect to time, given that each variable is also dependent on the others. In particular, if y is dependent on x, then the rate of change of y with respect to t is dependent on the rate of change of x with respect to t, that is, $\frac{dy}{dt}$ is dependent on $\frac{dx}{dt}$.

To be systematic in our solutions, we present the following steps:

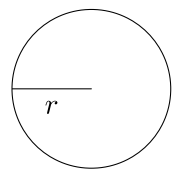
Suggestions in solving problems involving related rates:

1. If possible, provide an illustration for the problem that is valid for any time t.
2. Identify those quantities that change with respect to time, and represent them with variables. (Avoid assigning variables to quantities which are constant, that is, which do not change with respect to time. Label them right away with the values provided in the problem.)
3. Write down any numerical facts known about the variables. Interpret each rate of change as the derivative of a variable with respect to time. Remember that if a quantity decreases over time, then its rate of change is negative.
4. Identify which rate of change is being asked, and under what particular conditions this rate is being computed.
5. Write an equation showing the relationship of all the variables by an equation that is valid for any time t.
6. Differentiate the equation in (5) implicitly with respect to t.
7. Substitute into the equation, obtained in (6), all values that are valid at the particular time of interest. Sometimes, some quantities still need to be solved by substituting the particular conditions written in (4) to the equation in (6). Then, solve for what is being asked in the problem.

8. Write a conclusion that answers the question of the problem. Do not forget to include the correct units of measurement.

EXAMPLE 1: A water droplet falls onto a still pond and creates concentric circular ripples that propagate away from the center. Assuming that the area of a ripple is increasing at the rate of 2π cm^2/s, find the rate at which the radius is increasing at the instant when the radius is 10 cm.

Solution. We solve this step-by-step using the above guidelines.
(1) Illustration



(2) Let r and A be the radius and area, respectively, of a circular ripple at any time t.
(3) The given rate of change is $\dfrac{dA}{dt} = 2\pi$.
(4) We are asked to find $\dfrac{dr}{dt}$ at the instant when $r = 10$.
(5) The relationship between A and r is given by the formula for the area of a circle:

$$A = \pi r^2.$$

(6) We now differentiate implicitly with respect to time. (Be mindful that all quantities here depend on time, so we should always apply Chain Rule.)

$$\frac{dA}{dt} = \pi(2r)\frac{dr}{dt}.$$

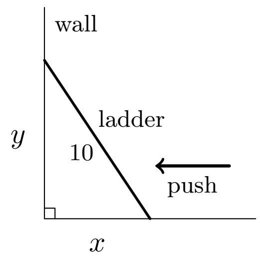
(7) Substituting $\dfrac{dA}{dt} = 2\pi$ and $r = 10$ gives

$$\begin{aligned} & 2\pi = \pi \cdot 2(10)\frac{dr}{dt} \\ \Longrightarrow \qquad & \frac{dr}{dt} = \frac{1}{10}. \end{aligned}$$

(8) Conclusion: The radius of a circular ripple is increasing at the rate of $\dfrac{1}{10}$ cm/s.

EXAMPLE 2: A ladder 10 meters long is leaning against a wall. If the bottom of the ladder is being pushed horizontally towards the wall at 2 m/s, how fast is the top of the ladder moving when the bottom is 6 meters from the wall?

Solution. We first illustrate the problem.



Let x be the distance between the bottom of the ladder and the wall. Let y be the distance between the top of the ladder and the ground (as shown). Note that the length of the ladder is not represented by a variable as it is constant.

We are given that $\dfrac{dx}{dt} = -2$. (Observe that this rate is negative since the quantity x decreases with time.)

We want to find $\dfrac{dy}{dt}$ at the instant when $x = 6$.

Observe that the wall, the ground and the ladder determine a right triangle. Hence, the relationship between x and y is given by the the Pythagorean Theorem:

$$x^2 + y^2 = 100. \tag{2.15}$$

Differentiating both sides with respect to time t gives

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0. \tag{2.16}$$

Before we proceed to the next step, we ask ourselves if we already have everything we need. So, dx/dt is given, dy/dt is the quantity required, x is given, BUT, we still do not have y.

This is easy to solve by substituting the given condition $x = 6$ into the equation in (2.15). So,

$$6^2 + y^2 = 100 \quad \Longrightarrow \quad y = \sqrt{100 - 36} = \sqrt{64} = 8.$$

Finally, we substitute all the given values into equation (2.16):

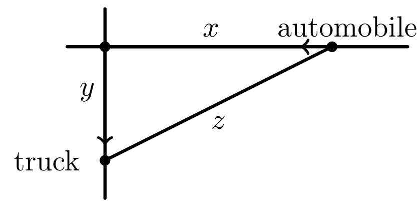
$$2(6)(-2) + 2(8)\frac{dy}{dt} = 0 \quad \Longrightarrow \quad \frac{dy}{dt} = \frac{24}{16} = \frac{3}{2}.$$

Thus, the distance between the top of the ladder and the ground is increasing at the rate of 1.5 m/s. Equivalently, we can also say that the top of the ladder is moving at the rate of 1.5 m/s.

EXAMPLE 3: An automobile traveling at the rate of 20 m/s is approaching an intersection. When the automobile is 100 meters from the intersection, a truck traveling at the rate of 40 m/s crosses the intersection. The automobile and the truck are on perpendicular roads. How fast is the distance between the truck and the automobile changing two seconds after the truck leaves the intersection?

Solution. Let us assume that the automobile is travelling west while the truck is travelling south as illustrated below.

Let x denote the distance of the automobile from the intersection, y denote the distance of the truck from the intersection, and z denote the distance between the truck and the automobile.



Then we have $\dfrac{dx}{dt} = -20$ (the negative rate is due to the fact that x decreases with time) and $\dfrac{dy}{dt} = 40$. We want to find $\dfrac{dz}{dt}$ when $t = 2$.

The equation relating x, y and z is given by the Pythagorean Theorem. We have

$$x^2 + y^2 = z^2. \tag{2.17}$$

Differentiating both sides with respect to t,

$$\begin{aligned} & \frac{d}{dt}\left[x^2 + y^2\right] &=& \frac{d}{dt}\left[z^2\right] \\ \Longrightarrow \quad & 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &=& 2z\frac{dz}{dt} \\ \Longrightarrow \quad & x\frac{dx}{dt} + y\frac{dy}{dt} &=& z\frac{dz}{dt}. \end{aligned}$$

Before substituting the given values, we still need to find the values of x, y and z when $t = 2$. This is found by the distance-rate-time relationship:

$$\textit{distance} \;=\; \textit{rate} \times \textit{time}.$$

For the automobile, after 2 seconds, it has travelled a distance equal to $(rate)(time) = 20(2) = 40$ from the 100 ft mark. Therefore, $x = 100 - 40 = 60$. On the other hand, for the truck, it has travelled $y = (rate)(time) = 40(2) = 80$. The value of z is found from (2.17):

$$z = \sqrt{x^2 + y^2} = \sqrt{60^2 + 80^2} = \sqrt{10^2(36 + 64)} = 100.$$

Finally,

$$40(-20) + 80(40) = 100\frac{dz}{dt} \quad \Longrightarrow \quad \frac{dz}{dt} = 20.$$

Thus, the distance between the automobile and the truck is increasing at the rate of 20 meters per second.

The next example is peculiar in the sense that in the (main) equation relating all variables, some variables may be related to each other by an equation independent from the main one. In this case, it is best to minimize the number of variables of the main equation by incorporating the other equation into the main equation.

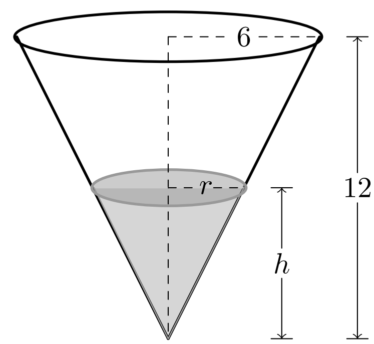
For instance, consider the area of a rectangle problem. We know that $A = \ell w$ (main equation). If we also know that $\ell = 2w$, then our main equation equation can now be rewritten as

$$A = (2w)w = 2w^2.$$

The need to write one variable in terms of another will be apparent when only one of them has a given rate of change.

EXAMPLE 4: Water is pouring into an inverted cone at the rate of 8 cubic meters per minute. If the height of the cone is 12 meters and the radius of its base is 6 meters, how fast is the water level rising when the water is 4-meter deep?

Solution. We first illustrate the problem.



Let V be the volume of the water inside the cone at any time t. Let h, r be the height and radius, respectively, of the cone formed by the volume of water at any time t.

We are given $\dfrac{dV}{dt} = 8$ and we wish to find $\dfrac{dh}{dt}$ when $h = 4$.

Now, the relationship between the three defined variables is given by the volume of the cone:

$$V = \frac{\pi}{3}r^2h.$$

Observe that the rate of change of r is neither given nor asked. This prompts us to find a relationship between r and h. From the illustration, we see that by the proportionality relations in similar triangles, we obtain

$$\frac{r}{h} = \frac{6}{12}$$

or $r = \dfrac{h}{2}$. Thus,

$$V = \frac{\pi}{3}r^2h = \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3.$$

Differentiating both sides with respect to t,

$$\begin{array}{rrcl} & \dfrac{d}{dt}(V) & = & \dfrac{d}{dt}\left(\dfrac{\pi}{12}h^3\right) \\ \Longrightarrow & \dfrac{dV}{dt} & = & \dfrac{\pi}{4}h^2\dfrac{dh}{dt}. \end{array}$$

Thus, after substituting all given values, we obtain

$$8 = \frac{\pi}{4}(4)^2\frac{dh}{dt} \quad \Longrightarrow \quad \frac{dh}{dt} = \frac{32}{16\pi} = \frac{2}{\pi}.$$

Finally, we conclude that the water level inside the cone is rising at the rate of $\dfrac{2}{\pi}$ meters/minute.

(C) **EXERCISES**

1. Starting from the same point, Reden starts walking eastward at 60 cm/s while Neil starts running towards the south at 80 cm/s. How fast is the distance between Reden and Neil increasing after 2 seconds?

2. A woman standing on a cliff is watching a motor boat through a telescope as the boat approaches the shoreline directly below her. If the telescope is 25 meters above the water level and if the boat is approaching the cliff at 20 m/s, at what rate is the acute angle made by the telescope with the vertical changing when the boat is 250 meters from the shore?

3. A balloon, in the shape of a right circular cylinder, is being inflated in such a way that the radius and height are both increasing at the rate of 3 cm/s and 8 cm/s, respectively. What is the rate of change of its total surface area when its radius and height are 60 cm and 140 cm, respectively?

4. If two resistors with resistance R_1 and R_2 are connected in parallel, the total resistance R in ohms is given by $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$. If R_1 and R_2 are increasing at 0.4 ohms/s and 0.25 ohms/s, respectively, how fast is R changing when $R_1 = 600$ ohms and $R_2 = 400$ ohms?

5. A baseball diamond has the shape of a square with sides 90 ft long. A player 60 ft from second base is running towards third base at a speed of 28 ft/min. At what rate is the player's distance from the home plate changing?

6. Shan, who is 2 meters tall, is approaching a post that holds a lamp 6 meters above the ground. If he is walking at a speed of 1.5 m/s, how fast is the end of his shadow moving (with respect to the lamp post) when he is 6 meters away from the base of the lamp post?

$\star$7. Water is being poured at the rate of 2π m^3/min. into an inverted conical tank that is 12-meter deep with a radius of 6 meters at the top. If the water level is rising at the rate of $\frac{1}{6}$ m/min and there is a leak at the bottom of the tank, how fast is the water leaking when the water is 6-meter deep?

(D) **ENRICHMENT**

1. A ladder 20-meter long is leaning against an embankment inclined 60° with the horizontal. If the bottom of the ladder is being moved horizontally towards the embankment at 1 m/s, how fast is the top of the ladder moving when the bottom is 4 meters from the embankment?

2. A boat is pulled in by means of a winch on the dock 4 meters above the deck of the boat. The winch pulls in rope at the rate of 1 m/s. Determine the speed of the boat when there is 3 meters of rope out.

3. A ladder, inclined at 60° with the horizontal is leaning against a vertical wall. The foot of the ladder is 3 m away from the foot of the wall. A boy climbs the ladder such that his distance z m with respect to the foot of the ladder is given by $z = 6t$, where t is the time in seconds. Find the rate at which his vertical distance from the ground changes with respect to t. Find the rate at which his distance from the foot of the wall is changing with respect to t when he is 3 m away from the foot of the ladder.

⋆4. A particle is moving along the curve $y = \sqrt{x}$. As the particle passes through the point (4,2), its x-coordinate changes at a rate of 3 cm/s. How fast is the distance of the particle from the origin changing at this instant?

ANSWERS TO STARRED EXERCISES

(C) 7. Let V, h and r be the volume, height and radius respectively, of the water collected in the cone at any point t. Then $V = V_{\text{in}} - V_{\text{out}}$. Moreover, it is given that

$$\frac{dV_{\text{in}}}{dt} = +2\pi, \quad \text{and} \quad \frac{dh}{dt} = +\frac{1}{6}.$$

Now, the formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$. We wish to express V in terms of a single variable only. Since we have dh/dt in the given, it is better to express r in terms of h.

Using similar triangles and the fact that the height and radius of the conical container are 12 and 6 respectively, we have

$$\frac{r}{h} = \frac{6}{12} \quad \text{which means} \quad r = \frac{h}{2}.$$

Substituting this expression r, we get $V = \frac{1}{3}\pi h \left(\frac{h}{2}\right)^2 = \frac{\pi h^3}{12}$. Differentiating with respect to t (don't forget to apply Chain Rule) gives

$$\frac{dV}{dt} = \frac{\pi}{12}(3h^2)\frac{dh}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt}.$$

Hence, at the instant when $h = 6$,

$$\frac{dV}{dt} = \frac{\pi(6)^2}{4} \cdot \frac{1}{6} = \frac{3\pi}{2}.$$

Finally, $\frac{dV}{dt} = \frac{dV_{\text{in}}}{dt} - \frac{dV_{\text{out}}}{dt}$. This implies that

$$\frac{dV_{\text{out}}}{dt} = \frac{dV_{\text{in}}}{dt} - \frac{dV}{dt} = 2\pi - \frac{3\pi}{2} = \frac{\pi}{2}\frac{\text{m}^3}{\text{min}}.$$

(D) 4. Let h, x and y denote the distance from the origin, x-coordinate and y-coordinate respectively, of a point P on the curve $y = \sqrt{x}$ at any time t. We are given that $\frac{dx}{dt} = +3$ and we want to find $\frac{dh}{dt}$ at the instant when $x = 4$ and $y = 2$.
Using the distance formula,

$$h = x^2 + y^2.$$

Since the curve has equation $y = \sqrt{x}$, the formula above simplifies to

$$h = x^2 + x.$$

Differentiating with respect to t gives

$$\frac{dh}{dt} = (2x + 1)\frac{dx}{dt}.$$

Substituting $x = 4$ and $\frac{dx}{dt} = 3$ gives $\frac{dh}{dt} = 9(3) = 27$ cm/s.

CHAPTER 2 EXAM

I. Write TRUE if the statement is true, otherwise, write FALSE.

(a) The sixth derivative of $y = \sin x$ is itself.

(b) The linear function $y = mx + b$, where $m \neq 0$, has no maximum value on any open interval.

(c) If the function f is continuous at the real number $x = a$, then f is differentiable at the real number $x = a$.

(d) If $f(c)$ is an extremum, then $f'(c)$ does not exist.

(e) A differentiable function $f(x)$ is decreasing on (a, b) whenever $f'(x) > 0$ for all $x \in (a, b)$.

II. Use the definition of derivative to find $f'(x)$ if $f(x) = \dfrac{1}{x}$.

III. Find $\dfrac{dy}{dx}$ of the following functions. (Do not simplify.)

1. $y = \dfrac{1 - \cos x}{1 + x^3}$
2. $y = \sqrt{\tan(x^2 - 1)}$
3. $y = e^{\sin x}$
4. $y = \ln(2^x + 1)$

IV. Find the equation of the tangent line to the curve $x^2 + 5xy + y^2 + 3 = 0$ at the point $(1, -1)$.

V. An open box is to be made from a 8 inches by 18 inches piece of cardboard by cutting squares of equal size from the four corners and bending up the sides. How long should the sides of the squares be to obtain a box with the largest volume?

VI. If $f(x) = x^4 + x^3 + x^2 + x - 1$, then find $f^{(n)}(x)$ for all $n \in \mathbb{N}$.

VII. A ladder 8 meters long is leaning against a wall. If the bottom of the ladder is being pulled horizontally away from the wall at 2 meters per second, how fast is the top of the ladder moving when the bottom is 3 meters from the wall?

Chapter 3

Integration

LESSON 11: Integration

TIME FRAME: 5 hours

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Illustrate the antiderivative of a function;
2. Compute the general antiderivative of polynomial functions;
3. Compute the general antiderivative of root functions;
4. Compute the general antiderivative of exponential functions; and,
5. Compute the general antiderivative of trigonometric functions.

LESSON OUTLINE:

1. Antiderivatives of functions
2. Antiderivatives of polynomial and root functions
3. Antiderivatives of integrals yielding exponential and logarithmic functions
4. Antiderivatives of trigonometric functions

TOPIC 11.1: Illustration of an Antiderivative of a Function

DEVELOPMENT OF THE LESSON

(A) ACTIVITY

Matching Type. Match the functions in Column A with their corresponding derivatives in Column B.

Column A	Column B
1. $F(x) = x^3 + 2x^2 + x$	a. $f(x) = 4x^3 + 3x^2$
2. $F(x) = x^3 + x^2 + x$	b. $f(x) = 9x^2 + 3$
3. $F(x) = x^4 + 3x^3 + 1$	c. $f(x) = 3x^2 + 2x + 1$
4. $F(x) = x^4 + x^3 + 2$	d. $f(x) = 9x^2 - 3$
5. $F(x) = x^2 + 2x + 1$	e. $f(x) = 2x + 2$
6. $F(x) = x^2 - 2x + 1$	f. $f(x) = 2x - 2$
7. $F(x) = 3x^3 + 3x + 1$	g. $f(x) = 3x^2 + 4x + 1$
8. $F(x) = 3x^3 - 3x + 1$	h. $f(x) = 4x^3 + 9x^2$

(B) INTRODUCTION

In the previous discussions, we learned how to find the derivatives of different functions. Now, we will introduce the inverse of differentiation. We shall call this process antidifferentiation. A natural question then arises:

> Given a function f, can we find a function F whose derivative is f?

Definition 5. *A function F is an* ***antiderivative*** *of the function f on an interval I if $F'(x) = f(x)$ for every value of x in I.*

> **Teaching Tip**
>
> The teacher must go back to the activity and explain to the students that the functions in Column A are indeed antiderivatives of the their corresponding derivatives in Column B.

(C) **LESSON PROPER**

ANTIDERIVATIVES OR INDEFINITE INTEGRALS

We will now give examples of antiderivatives of functions.

EXAMPLE 1:

(a) An antiderivative of $f(x) = 12x^2+2x$ is $F(x) = 4x^3+x^2$. As we can see, the derivative of F is given by $F'(x) = 12x^2 + 2x = f(x)$.

(b) An antiderivative of $g(x) = \cos x$ is $G(x) = \sin x$ because $G'(x) = \cos x = g(x)$.

Remark 1: The antiderivative F of a function f is **not** unique.

EXAMPLE 2:

(a) Other antiderivatives of $f(x) = 12x^2 + 2x$ are $F_1(x) = 4x^3 + x^2 - 1$ and $F_2(x) = 4x^3 + x^2 + 1$. In fact, any function of the form $F(x) = 4x^3 + x^2 + C$, where $C \in \mathbb{R}$ is an antiderivative of $f(x)$. Observe that $F'(x) = 12x^2 + 2x + 0 = 12x^2 + 2x = f(x)$.

(b) Other antiderivatives of $g(x) = \cos x$ are $G_1(x) = \sin x + \pi$ and $G_2(x) = \sin x - 1$. In fact, any function $G(x) = \sin x + C$, where $C \in \mathbb{R}$ is an antiderivative of $g(x)$.

Theorem 10. *If F is an antiderivative of f on an interval I, then every antiderivative of f on I is given by $F(x) + C$, where C is an arbitrary constant.*

Remark 2: Using the theorem above, we can conclude that if F_1 and F_2 are antiderivatives of f, then $F_2(x) = F_1(x) + C$. That is, F_1 and F_2 differ only by a constant.

Terminologies and Notations:

- ***Antidifferentation*** is the process of finding the antiderivative.
- The symbol $\displaystyle\int$, also called the ***integral sign***, denotes the operation of antidifferentiation.
- The function f is called the ***integrand***.
- If F is an antiderivative of f, we write $\displaystyle\int f(x)\,dx = F(x) + C$.
- The symbols $\displaystyle\int$ and dx go hand-in-hand and dx helps us identify the variable of integration.
- The expression $F(x)+C$ is called the ***general antiderivative*** of f. Meanwhile, each antiderivative of f is called a ***particular antiderivative*** of f.

(D) **EXERCISES**

1. Determine the general antiderivatives of the following functions:

   a. $f(x) = 6x^5 - x^4 + 2x^2 - 3$

   b. $f(x) = -3$

   c. $g(x) = x^3 - 3x^2 3x - 1$

   ⋆d. $h(x) = \sin x + \cos x - \sec^2 x$ Answer: $H(x) = -\cos x + \sin x - \tan x + C$

2. Matching type: Match the functions in Column A with their corresponding antiderivatives in Column B.

Column A	Column B
a. $f(x) = 3x^2 + 2x + 1$	a. $F(x) = 3x^3 - x$
b. $f(x) = 9x^2 - 1$	b. $F(x) = x^3 + x^2 + x$
c. $f(x) = x^2 - 2$	c. $F(x) = 2x^2 - \dfrac{1}{3}x^3$
d. $f(x) = (x+1)(x-1)$	d. $F(x) = -2x^2 + \dfrac{1}{3}x^3$
e. $f(x) = x(4-x)$	e. $F(x) = \dfrac{1}{3}x^3 - 2x + 1$
f. $f(x) = x(x-4)$	f. $F(x) = \dfrac{1}{3}x^3 - x + 1$

TOPIC 11.2: Antiderivatives of Algebraic Functions

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

As previously discussed, the process of antidifferentiation is just the inverse process of finding the derivatives of functions. We have shown in the previous lesson that a function can have a family of antiderivatives.

We will look at antiderivatives of different types of functions. Particularly, we will find the antiderivatives of polynomial functions, rational functions and radical functions.

(B) LESSON PROPER

Let us recall first the following differentiation formulas:

(a) $D_x(x) = 1$.

(b) $D_x(x^n) = nx^{n-1}$, where n is any real number.

(c) $D_x[a(f(x))] = aD_x[f(x)]$.

(d) $D_x[f(x) \pm g(x)] = D_x[f(x)] \pm D_x[g(x)]$.

The above formulas lead to the following theorem which are used in obtaining the antiderivatives of functions. We apply them to integrate polynomials, rational functions and radical functions.

Theorem 11. (Theorems on Antidifferentiation)

(a) $\int dx = x + C$.

(b) If n is any real number and $n \neq -1$, then

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C.$$

(c) If a is any constant and f is a function, then

$$\int af(x) \, dx = a \int f(x) \, dx.$$

(d) If f and g are functions defined on the same interval,

$$\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx.$$

Teaching Tip

- Remind the students that expressions of the form $\dfrac{1}{x^p}$ and $\sqrt[q]{x^p}$ are integrated using the formula (b) of Theorem 11, since they can be rewritten into x^{-p} and $x^{p/q}$ respectively.

- In case a student asks, please reiterate in class that the case when $n=-1$ for $\displaystyle\int x^n\,dx$ will be discussed later.

EXAMPLE 1: Determine the following antiderivatives:

1. $\displaystyle\int 3\,dx$
2. $\displaystyle\int x^6\,dx$
3. $\displaystyle\int \frac{1}{x^6}\,dx$
4. $\displaystyle\int 4\sqrt{u}\,du$
5. $\displaystyle\int \left(12x^2+2x\right)\,dx$
6. $\displaystyle\int t\left(2t-3\sqrt{t}\right)\,dt$
7. $\displaystyle\int \frac{x^2+1}{x^2}\,dx$

Solution. We will use the Theorems on Antidifferentiation to determine the antiderivatives.

1. Using (a) and (c) of the theorem, we have $\displaystyle\int 3\,dx = 3\int dx = 3x + C$.
2. Using (b) of the theorem, we have $\displaystyle\int x^6\,dx = \frac{x^{6+1}}{6+1}+C=\frac{x^7}{7}+C$.
3. Using (b) of the theorem, we have
$$\int \frac{1}{x^6}\,dx=\int x^{-6}\,dx=\frac{x^{-6+1}}{-6+1}+C=\frac{x^{-5}}{-5}+C=-\frac{1}{5x^5}+C.$$
4. Using (b) and (c) of the theorem, we have
$$\int 4\sqrt{u}\,du=4\int u^{\frac{1}{2}}\,du=\frac{4u^{\frac{1}{2}+1}}{\frac{1}{2}+1}+C=\frac{4u^{\frac{3}{2}}}{\frac{3}{2}}+C=\frac{8u^{\frac{3}{2}}}{3}+C.$$
5. Using (b), (c) and (d) of the theorem, we have
$$\int\left(12x^2+2x\right)\,dx=12\int x^2\,dx+2\int x\,dx=\frac{12x^3}{3}+\frac{2x^2}{2}+C=4x^3+x^2+C.$$
6. Using (b), (c) and (d), we have
$$\int t\left(2t-3\sqrt{t}\right)\,dt=\int\left(2t^2-3t^{\frac{3}{2}}\right)\,dt=2\int t^2\,dt-3\int t^{\frac{3}{2}}\,dt=\frac{2t^3}{3}-\frac{6t^{\frac{5}{2}}}{5}+C.$$
7. Using (a), (b) and (d), we have
$$\int\frac{x^2+1}{x^2}\,dx=\int\left(1+x^{-2}\right)\,dx=x+\frac{x^{-1}}{-1}+C=x-\frac{1}{x}+C.$$

Teaching Tip

- A common mistake in antidifferentiation is distributing the integral sign over a product or a quotient. Please reiterate that

$$\int f(x)g(x)\,dx \neq \int f(x)\,dx \cdot \int g(x)\,dx \quad \text{and} \quad \int \frac{f(x)}{g(x)}\,dx \neq \frac{\int f(x)\,dx}{\int g(x)\,dx}.$$

- In antidifferentiation, it is better to rewrite a product or a quotient into a sum or difference. This technique was done in the last two items of the previous example.

(C) **EXERCISES**

1. $\int x^2\,dx$
2. $\int (x^3 + 2x^2)\,dx$
3. $\int \left(\frac{3}{4}x^4 + 3x^2 + 1\right)\,dx$
4. $\int x^{-2}\,dx$
5. $\int (x^{-3} + x^{-2} + x^{-1})\,dx$
6. $\int x^{-100}\,dx$
7. $\int (3x^{-2} + x + 2)\,dx$
8. $\int 1,000\,dx$
9. $\int \sqrt[3]{x}\,dx$
10. $\int \sqrt[3]{x^2}\,dx$
11. $\int \sqrt[4]{x^4}\,dx$
12. $\int \sqrt[5]{x^{10}}\,dx$
13. $\int (u^2 + u + 1)\,du$
14. $\int \left(\frac{1}{4}v^4 + v^2 + v\right)\,dv$
15. $\int \frac{w^3 + w^2 + w}{w^3}\,dw$
16. $\int \sqrt[4]{y}\,dy$
17. $\int t^2\left(\sqrt[3]{t} + \sqrt{t} - t^{-\frac{1}{2}}\right)\,dt$

TOPIC 11.3: Antiderivatives of Functions Yielding Exponential Functions and Logarithmic Functions

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

We will introduce the antiderivatives of two important functions namely, the exponential function

$$f(x) = e^x$$

and the function yielding the logarithmic function

$$f(x) = \ln x.$$

We will present first the basic formulas, then their examples once we have already discussed integration by substitution. The technique of integration by substitution will help us integrate complicated functions yielding exponential and logarithmic functions.

(B) LESSON PROPER

In this lesson, we will present the basic formulas for integrating functions that yield exponential and logarithmic functions. Let us first recall the following differentiation formulas:

(a) $D_x(e^x) = e^x$.

(b) $D_x(a^x) = a^x \ln a$.

(c) $D_x(\ln x) = \dfrac{1}{x}$.

Because antidifferentiation is the inverse operation of differentiation, the following theorem should be immediate.

Theorem 12. (Theorems on integrals yielding the exponential and logarithmic functions)

(a) $\displaystyle\int e^x \, dx = e^x + C.$

(b) $\displaystyle\int a^x \, dx = \frac{a^x}{\ln a} + C.$ *Here,* $a > 0$ *with* $a \neq 1$.

(c) $\displaystyle\int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln |x| + C.$

These statements in the theorem make sense since, as we had discussed in differentiation, the derivative of

$$F(x) = e^x + C$$

is

$$f(x) = F'(x) = e^x + 0 = e^x.$$

Also, the derivative of

$$G(x) = a^x$$

is

$$g(x) = G'(x) = a^x \ln a.$$

For the next function, we first recall that $|x| = \sqrt{x^2}$. So, using Chain Rule, the derivative of

$$H(x) = \ln|x| + C = \ln\sqrt{x^2} + C$$

is

$$h(x) = H'(x) = \frac{1}{\sqrt{x^2}}\frac{1}{2\sqrt{x^2}}(2x+0) = \frac{1}{x}.$$

Note that we used $y = \ln|x|$ because it has a larger domain than $y = \ln x$, as $\ln x$ is defined only on positive real numbers.

EXAMPLE 1: Find the integrals of the following functions.

(a) $\displaystyle\int (e^x + 2^x)\,dx$

(b) $\displaystyle\int 3^x\,dx$

(c) $\displaystyle\int 3^{x+1}\,dx$

(d) $\displaystyle\int \frac{2}{x}\,dx$

Solution. We will use the theorem to determine the integrals.

(a) Using (a) and (b) of theorem, we have

$$\int (e^x + 2^x)\,dx = \int (e^x)\,dx + \int (2^x)\,dx = e^x + \frac{2^x}{\ln 2} + C.$$

(b) Using (b) of the theorem, we have

$$\int 3^x\,dx = \frac{3^x}{\ln 3} + C.$$

(c) Using (b) of the theorem, we have

$$\int 3^{x+1}\,dx = \int (3^x)(3^1)\,dx = 3\int (3^x)\,dx = 3\frac{3^x}{\ln 3} + C.$$

(d) Using (c) of the theorem, we get

$$\int \frac{2}{x}\,dx = 2\int \frac{1}{x}\,dx = 2\ln|x| + C.$$

We will take more integrals yielding the exponential and logarithmic functions once we discuss integration by substitution. For instance, we will try to see how we can evaluate the following:

(a) $\displaystyle\int e^{3x}\,dx$ (b) $\displaystyle\int 2^{4x}\,dx$ (c) $\displaystyle\int \frac{1}{2x-1}\,dx.$

(C) **EXERCISES**

Solve the following integrals:

(1) $\displaystyle\int 2e^{x}\,dx$

(2) $\displaystyle\int (2e^{x}+4^{x})\,dx$

(3) $\displaystyle\int 2(5^{x})\,dx$

(4) $\displaystyle\int 5^{x+2}\,dx$

(5) $\displaystyle\int \frac{2}{x}\,dx$

(6) $\displaystyle\int 3x^{-1}\,dx$

TOPIC 11.4: Antiderivatives of Trigonometric Functions

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

We already know how to differentiate the trigonometric functions. As we said, the process of finding the antiderivatives of functions is the inverse of such a process. At this point, we will find the antiderivatives of trigonometric functions.

(B) LESSON PROPER

Let us first recall the following differentiation formulas of the different trigonometric functions:

(a) $D_x(\sin x) = \cos x$

(b) $D_x(\cos x) = -\sin x$

(c) $D_x(\tan x) = \sec^2 x$

(d) $D_x(\sec x) = \sec x \tan x$

(e) $D_x(\cot x) = -\csc^2 x$

(f) $D_x(\csc x) = -\cot x \csc x$.

We now present the formulas for the antiderivatives of trigonometric functions.

Theorem 13. (Antiderivatives of Trigonometric Functions)

(a) $\displaystyle\int \sin x\, dx = -\cos x + C$

(b) $\displaystyle\int \cos x\, dx = \sin x + C$

(c) $\displaystyle\int \sec^2 x\, dx = \tan x + C$

(d) $\displaystyle\int \csc^2 x\, dx = -\cot x + C$

(e) $\displaystyle\int \sec x \tan x\, dx = \sec x + C$

(f) $\displaystyle\int \csc x \cot x\, dx = -\csc x + C$

EXAMPLE 1: Determine the antiderivatives of the following:

(a) $\displaystyle\int (\cos x - \sin x)\ dx$

(b) $\displaystyle\int \cot^2 x\, dx$

(c) $\displaystyle\int \tan^2 v\, dv$

(d) $\displaystyle\int \frac{\sin x}{\cos^2 x}\, dx$

Solution. We will use the theorem on antiderivatives of trigonometric functions.

(a) Using (a) and (b) of the theorem, we have

$$\begin{aligned}\int (\cos x - \sin x)\ dx &= \int \cos x\, dx - \int \sin x\, dx \\ &= \sin x - (-\cos x) + C = \sin x + \cos x + C.\end{aligned}$$

(b) Since we know that $\cot^2 x = \csc^2 - 1$, then

$$\int \cot^2 x\, dx = \int \left(\csc^2 x - 1\right)\, dx = \int \csc^2 x - \int dx = -\cot^2 - x + C.$$

(c) Since $\tan^2 v = \sec^2 v - 1$, we have

$$\int \tan^2 v\, dv = \int \left(\sec^2 v - 1\right)\, dv = \int \sec^2 v\, dv - \int dv = \tan v - v + C.$$

(d) $\displaystyle\int \frac{\sin x}{\cos^2 x}\, dx = \int \frac{\sin x}{\cos x}\frac{1}{\cos x}\, dx = \int \tan x \sec x\, dx = \sec x + C.$

Note that we have just presented $\displaystyle\int \sin x\, dx$ and $\displaystyle\int \cos x\, dx$. One can then ask the following:

(a) $\displaystyle\int \tan x\, dx = ?$

(b) $\displaystyle\int \cot x\, dx = ?$

(c) $\displaystyle\int \sec x\, dx = ?$

(d) $\displaystyle\int \csc x\, dx = ?$

We will derive these integrals once we discuss the technique of integration by substitution.

(C) **EXERCISES**

Evaluate the following integrals:

1. $\displaystyle\int \tan^2 x\, dx$
2. $\displaystyle\int \cot^2 x\, dx$
3. $\displaystyle\int \frac{\cos x}{\sin^2 x}\, dx$
4. $\displaystyle\int \frac{\cos x}{\sin^2 x}\, dx$
5. $\displaystyle\int \left[\frac{\sin x}{\cos^2 x} + x\right]\, dx$
6. $\displaystyle\int (\sin u + u)\, du$
7. $\displaystyle\int (1 - \cos v)\, dv$

⋆8. $\displaystyle\int \frac{1}{\sec y \tan y}\, dy$

Answer: $-\cos y + C$

⋆9. $\displaystyle\int \frac{1}{\sin^2 x \cos^2 x}\, dx$

Answer: $-\cot x + \tan x + C$

LESSON 12: Techniques of Antidifferentiation

TIME FRAME: 7 hours

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Compute the antiderivative of a function using the substitution rule; and
2. Compute the antiderivative of a function using a table of integrals (including those whose antiderivatives involve logarithmic and inverse trigonometric functions).

LESSON OUTLINE:

1. Antidifferentiation by substitution
2. Table of Integrals

TOPIC 12.1: Antidifferentiation by Substitution and by Table of Integrals

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

Antidifferentiation is more challenging than differentiation. To find the derivative of a given function, there are well-established rules that are always applicable to differentiable functions. For antidifferentiation, the antiderivatives given in the previous lesson may not suffice to integrate a given function.

A prerequisite is knowledge of the basic antidifferentiation formulas. Some formulas are easily derived, but most of them need to be memorized.

No hard and fast rules can be given as to which method applies in a given situation. In college, several techniques such as *integration by parts*, *partial fractions*, *trigonometric subtitution* will be introduced. This lesson focuses on the most basic technique - antidifferentiation by substitution - which is the inverse of the Chain Rule in differentiation.

There are occasions when it is possible to perform a difficult piece of integration by first making a ***substitution***. This has the effect of changing the variable and the integrand. The ability to carry out integration by substitution is a skill that develops with practice and experience, but sometimes a sensible substitution may not lead to an integral that can be evaluated. We must then be prepared to try out alternative substitutions.

Suppose we are given an integral of the form $\displaystyle\int f(g(x))\cdot g'(x)\,dx$. We can transform this into another form by changing the independent variable x to u using the substitution $u = g(x)$. In this case, $\dfrac{du}{dx} = g'(x)\,dx$. Therefore,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

This change of variable is one of the most important tools available to us. This technique is called ***integration by substitution***. It is often important to guess what will be the useful substitution.

Usually, we make a substitution for a function whose derivative also occurs in the integrand.

(B) **LESSON PROPER**

EXAMPLE 1: Evaluate $\int (x+4)^5\,dx$.

Solution. Notice that the integrand is in the fifth power of the expression $(x+4)$. To tackle this problem, we make a ***substitution***. We let $u = x+4$. The point of doing this is to change the integrand into a much simpler u^5. However, we must take care to substitute appropriately for the term dx too.

Now, since $u = x+4$ it follows that $\frac{du}{dx} = 1$ and so $du = dx$. So, substituting $(x+4)$ and dx, we have

$$\int (x+4)^5\,dx = \int u^5\,du.$$

The resulting integral can be evaluated immediately to give $\frac{u^6}{6} + C$. Recalling that $u = x+4$, we have

$$\begin{aligned}\int (x+4)^5\,dx &= \int u^5\,du \\ &= \frac{u^6}{6} + C \\ &= \frac{(x+4)^6}{6} + C.\end{aligned}$$

An alternative way of finding the antiderivative above is to expand the expression in the integrand and antidifferentiate the resulting polynomial (of degree 5) term by term. We will NOT do this. Obviously, the solution above is simpler than the mentioned alternative.

EXAMPLE 2: Evaluate $\int (x^5+2)^9\; 5x^4\,dx$.

Solution. If we let $u = x^5+2$, then $du = 5x^4\,dx$, which is precisely the other factor in the integrand. Thus, in terms of the variable u, this is essentially just a power rule integration.

That is,

$$\begin{aligned}\int (x^5+2)^9\; 5x^4\,dx &= \int u^9\,du, \quad \text{where} \quad u = x^5+2 \\ &= \frac{u^{10}}{10} + C \\ &= \frac{(x^5+2)^{10}}{10} + C.\end{aligned}$$

Again, the alternative way is to expand out the expression in the integrand, and integrate the resulting polynomial (of degree 49) term by term. Again, we would rather NOT do this.

EXAMPLE 3: Evaluate $\displaystyle\int \frac{z^2}{\sqrt{1+z^3}}\,dz$.

Solution. In this example, we let $u = 1+z^3$ so that $\dfrac{du}{dz} = 3z^2$. If $u = 1+z^3$, then we need to express $z^2\,dz$ in the integrand in terms of du or a constant multiple of du.

From $\dfrac{du}{dz} = 3z^2$ it follows that $du = 3z^2\,dz$ and $z^2\,dz = \dfrac{1}{3}\,du$. Thus,

$$
\begin{aligned}
\int \frac{z^2}{\sqrt{1+z^3}}\,dz &= \int \frac{1}{\sqrt{1+z^3}} \cdot z^2\,dz \\
&= \int \frac{1}{\sqrt{u}} \cdot \frac{1}{3}\,du \\
&= \frac{1}{3}\int u^{-\frac{1}{2}}\,du \\
&= \frac{1}{3}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C_1\right) \\
&= \frac{2}{3}u^{\frac{1}{2}} + \frac{C_1}{3} \\
&= \frac{2}{3}(1+z^3)^{\frac{1}{2}} + C \quad \text{where} \quad C = \frac{C_1}{3}.
\end{aligned}
$$

Teaching Tip

To avoid unnecessary arithmetic on the constant of integration, we will henceforth write C as a separate summand, and add it only after integrating.

EXAMPLE 4: Evaluate $\displaystyle\int \frac{x}{\sqrt{x^2-1}}\,dx$.

Solution. Notice that if $u = x^2-1$, then $du = 2x\,dx$. This implies that $x\,dx = \dfrac{1}{2}\,du$, so we have

$$
\begin{aligned}
\int \frac{x}{\sqrt{x^2-1}}\,dx &= \int \frac{1}{u^{\frac{1}{2}}} \cdot \frac{1}{2}\,du \\
&= \frac{1}{2}\int u^{-\frac{1}{2}}\,du \\
&= \frac{1}{2}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + C \\
&= (x^2-1)^{\frac{1}{2}} + C.
\end{aligned}
$$

EXAMPLE 5: Evaluate $\displaystyle\int \frac{x^3}{\sqrt{x^3+5}}\,dx$.

Solution. Let $u = x^3+5$. Then $x^3 = u-5$ and $du = 3x^2\,dx$. Thus,

$$\begin{aligned}
\int \frac{x^3}{\sqrt{x^3+5}}\,dx &= \int \frac{u-5}{u^{\frac{1}{2}}}\,dx \\
&= \int \left(\frac{u}{u^{\frac{1}{2}}} - \frac{5}{u^{\frac{1}{2}}}\right)dx \\
&= \int (u^{\frac{1}{2}} - 5u^{-\frac{1}{2}})\,dx \\
&= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{5u^{\frac{1}{2}}}{\frac{1}{2}} + C \\
&= \frac{2}{3}u^{\frac{3}{2}} - 10u^{\frac{1}{2}} + C \\
&= \frac{2}{3}(x^3+5)^{\frac{3}{2}} - 10(x^3+5)^{\frac{1}{2}} + C.
\end{aligned}$$

EXAMPLE 6: Evaluate $\displaystyle\int \frac{dx}{x\sqrt[3]{\ln x}}$.

Solution. We substitute $u = \ln x$ so that $du = \dfrac{1}{x}dx$, which occurs in the integrand. Thus,

$$\begin{aligned}
\int \frac{dx}{x\sqrt[3]{\ln x}} &= \int \frac{1}{\sqrt[3]{u}}\,du \\
&= \int u^{-\frac{1}{3}}\,du \\
&= \frac{u^{\frac{2}{3}}}{\frac{2}{3}} + C \\
&= \frac{3}{2}(\ln x)^{\frac{2}{3}} + C
\end{aligned}$$

We recall the theorem we stated in the previous lesson.

Theorem 14. (Theorems on integrals yielding the exponential and logarithmic functions)

1. $\displaystyle\int e^x\,dx = e^x + C.$

2. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C.$ *Here,* $a > 0$ *with* $a \neq 1$.

3. $\displaystyle\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C.$

As we said before, we will have more integrals yielding the exponential and logarithmic functions once we have discussed integration by substitution.

EXAMPLE 7: Evaluate the following integrals.

(a) $\displaystyle\int e^{3x}\,dx$

(b) $\displaystyle\int 2^{4x}\,dx$

(c) $\displaystyle\int \frac{1}{2x-1}\,dx$

Solution.

(a) We let $u = 3x$. Then $du = 3dx$. Hence, $dx = \dfrac{du}{3}$. So,

$$\int e^{3x}\,dx = \int e^u \frac{du}{3} = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{3x} + C.$$

(b) Here, we let $u = 4x$ and so $du = 4dx$. Thus, $dx = \dfrac{du}{4}$. Hence, we have

$$\int 2^{4x}\,dx = \int 2^u \frac{du}{4} = \frac{1}{4}\int 2^u\,du = \frac{1}{4}\frac{2^u}{\ln 2} + C = \frac{1}{4\ln 2}2^{4x} + C.$$

(c) Suppose we let $u = 2x-1$. Then $du = 2dx$. Hence, $dx = \dfrac{du}{2}$. We have

$$\int \frac{1}{2x-1}\,dx = \int \frac{1}{u}\frac{du}{2} = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|2x-1| + C.$$

EXAMPLE 8: Evaluate $\displaystyle\int \cos(4x+3)\,dx$.

Solution. Observe that if we make the substitution $u = 4x+3$, the integrand will contain a much simpler form, $\cos u$, which we can easily integrate. So, if $u = 4x+3$, then $du = 4\,dx$ and $dx = \dfrac{1}{4}\,du$. So,

$$\begin{aligned}\int \cos(4x+3)\,dx &= \int \cos u \cdot \frac{1}{4}\,du \\ &= \frac{1}{4}\int \cos u\,du \\ &= \frac{1}{4}\sin u + C \\ &= \frac{1}{4}\sin(4x+3) + C.\end{aligned}$$

EXAMPLE 9: Evaluate the integral $\int \sin x \cos x \; dx$.

Solution. Note that if we let $u = \sin x$, its derivative is $du = \cos x \; dx$ which is the other factor in the integrand and our integral becomes

$$\begin{aligned} \int \sin x \cos x \; dx &= \int u \; du \\ &= \frac{u^2}{2} + C_1 \\ &= \frac{\sin^2 x}{2} + C_1. \end{aligned}$$

Alternative solution to the problem: If we let $u = \cos x$, then $du = -\sin x \; dx$ which is also the other factor in the integrand. Even if the integral

$$\int \sin x \cos x \; dx = -\frac{\cos^2 x}{2} + C_2$$

looks different from the above answer, we can easily show that the two answers are indeed equal with by using the trigonometric identity $\sin^2 x + \cos^2 x = 1$. In this case, $C_2 = C_1 + \frac{1}{2}$.

EXAMPLE 10: Evaluate the integral $\int e^{\sin x} \; \cos x \; dx$.

Solution. We let $u = \sin x$ so that the other factor in the integrand $\cos x \; dx = du$. Thus, the integral becomes

$$\begin{aligned} \int e^{\sin x} \; \cos x \; dx &= \int e^u \; du \\ &= e^u + C \\ &= e^{\sin x} + C. \end{aligned}$$

Recall that we had earlier presented the integrals $\int \sin x \, dx$ and $\int \cos x \, dx$. Now that we already know integration by substitution, we can now present the integrals of other trigonometric functions: $\tan x$, $\cot x$, $\sec x$, and $\csc x$.

First, let us use substitution technique to find $\int \tan x \, dx$. Note that $\tan x = \dfrac{\sin x}{\cos x}$. Hence,

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx.$$

Now, if we let $u = \cos x$, then $du = -\sin x\,dx$. Hence, we have

$$\begin{aligned}
\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx \\
&= \int \frac{1}{u}\,(-du) \\
&= -\int \frac{1}{u}\,du \\
&= -\ln|u| + C \\
&= -\ln|\cos x| + C.
\end{aligned}$$

Equivalently, $-\ln|\cos x| + C = \ln|\cos x|^{-1} = \ln|\sec x| + C$.

Similarly, we can use substitution technique to show $\int \cot x\,dx = \ln|\sin x| + C$. Here, we use $\cot x = \dfrac{\cos x}{\sin x}$ and choose $u = \sin x$.

Let us now find $\int \sec x\,dx$. The usual trick is to multiply the numerator and the denominator by $\sec x + \tan x$.

$$\int \sec x\,dx = \int (\sec x)\frac{\sec x + \tan x}{\sec x + \tan x}\,dx = \int \frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x)\,dx.$$

Now, if we let $u = \sec x + \tan x$, then $du = \sec x \tan x + \sec^2 x$. Thus, we have

$$\begin{aligned}
\int \sec x\,dx &= \int \frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x)\,dx. \\
&= \int \frac{1}{u}\,du \\
&= \ln|u| + C \\
&= \ln|\sec x + \tan x| + C.
\end{aligned}$$

Similarly, we can show $\int \csc x\,dx = \ln|\csc x - \cot x| + C$.
Hence, we have the following formulas:

1. $\int \tan x\,dx = -\ln|\cos x| + C = \ln|\sec u| + C$.
2. $\int \cot x\,dx = \ln|\sin x| + C$.
3. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$.
4. $\int \csc x\,dx = \ln|\csc x - \cot x| + C$.

Teaching Tip

It would help the students if the formulas above are written either on a sheet of manila paper or on the board while the following examples are discussed.

EXAMPLE 11: Evaluate $\int x^4 \sec(x^5)\,dx$.

Solution. We let $u = x^5$. Then $du = 5x^4 dx$. Thus, $x^4 dx = \dfrac{du}{5}$. We have

$$\begin{aligned}
\int x^4 \sec(x^5)\,dx &= \int \sec u \frac{du}{5} \\
&= \frac{1}{5}\int \sec u\,du \\
&= \frac{1}{5}\ln|\sec u + \tan u| + C \\
&= \frac{1}{5}\ln|\sec x^5 + \tan x^5| + C.
\end{aligned}$$

EXAMPLE 12: Evaluate $\displaystyle\int \frac{4+\cos\frac{x}{4}}{\sin\frac{x}{4}}\,dx$.

Solution. Let $u = \frac{x}{4}$. Then $du = \frac{1}{4}\,dx$ and so $4\,du = dx$. Thus, we have

$$\begin{aligned}
\int \frac{4+\cos\frac{x}{4}}{\sin\frac{x}{4}}\,dx &= \int \frac{4+\cos u}{\sin u}\,4du \\
&= 4\int [\frac{4}{\sin u} + \frac{\cos u}{\sin u}]\,du \\
&= 4\int \Big[4\csc u + \cot u\Big]\,du \\
&= 4\int \cot u\,du + 16\int \csc u\,du \\
&= 4\ln|\sin u| + 16\ln|\csc u - \cot u| + C \\
&= 4\ln\left|\sin\frac{x}{4}\right| + 16\ln\left|\csc\frac{x}{4} - \cot\frac{x}{4}\right| + C.
\end{aligned}$$

INTEGRALS OF INVERSE CIRCULAR FUNCTIONS

We now present the formulas for integrals yielding the inverse circular functions.

1. $\displaystyle\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$
2. $\displaystyle\int \frac{du}{1+u^2} = \tan^{-1} u + C$
3. $\displaystyle\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u + C$

If the constant 1 in these integrals is replaced by some other positive number, one can use the following generalizations:

Let $a > 0$. Then

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C \tag{3.1}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \tag{3.2}$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C \tag{3.3}$$

Teaching Tip

Again, it would help the students if the formulas above are written either on a sheet of manila paper or on the board while the following examples are discussed.

EXAMPLE 13: Evaluate $\displaystyle\int \frac{1}{\sqrt{9 - x^2}}\, dx$.

Solution. From Formula (3.1) with $a = 3$, we write this into

$$\int \frac{1}{\sqrt{9 - x^2}}\, dx = \int \frac{1}{\sqrt{3^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{3}\right) + C.$$

EXAMPLE 14: $\displaystyle\int \frac{dx}{9x^2 + 36}$

Solution. Let $u = 3x$, $du = 3\,dx$. Then from Formula (3.2),

$$\begin{aligned}
\int \frac{dx}{9x^2 + 36} &= \frac{1}{3}\int \frac{du}{u^2 + 36} \\
&= \frac{1}{3} \cdot \frac{1}{6}\tan^{-1}\left(\frac{u}{6}\right) + C \\
&= \frac{1}{18}\tan^{-1}\left(\frac{3x}{6}\right) + C \\
&= \frac{1}{18}\tan^{-1}\left(\frac{x}{2}\right) + C.
\end{aligned}$$

EXAMPLE 15: Evaluate $\displaystyle\int \frac{dx}{x\ln x\sqrt{(\ln x)^2-9}}$.

Solution. Let $u = \ln x$, $du = \dfrac{1}{x}\,dx$. Then from Formula (3.3),

$$\begin{aligned}
\int \frac{dx}{x\ln x\sqrt{(\ln x)^2-9}} &= \int \frac{du}{u\sqrt{u^2-9}} \\
&= \frac{1}{3}\sec^{-1}\left(\frac{u}{3}\right) + C \\
&= \frac{1}{3}\sec^{-1}\left(\frac{\ln x}{3}\right) + C.
\end{aligned}$$

EXAMPLE 16: $\displaystyle\int \frac{dx}{\sqrt{9+8x-x^2}}$.

Solution. Observe that by completing the squares, and Formula (3.1),

$$\begin{aligned}
\int \frac{dx}{\sqrt{9+8x-x^2}} &= \int \frac{dx}{\sqrt{9-(x^2-8x)}} \\
&= \int \frac{dx}{\sqrt{9-(x^2-8x+16-16)}} \\
&= \int \frac{dx}{\sqrt{25-(x-4)^2}}.
\end{aligned}$$

Let $u = x-4$, $du = dx$. Then

$$\begin{aligned}
\int \frac{dx}{\sqrt{9+8x-x^2}} &= \int \frac{du}{\sqrt{25-u^2}} \\
&= \sin^{-1}\frac{u}{5} + C \\
&= \sin^{-1}\left(\frac{x-4}{5}\right) + C.
\end{aligned}$$

EXAMPLE 17: $\displaystyle\int \frac{18x+3}{9x^2+6x+2}\,dx$.

Solution. Let $u = 9x^2+6x+2$, $du = (18x+6)\,dx$. Then

$$\begin{aligned}
\int \frac{18x+3}{9x^2+6x+2}\,dx &= \int \frac{18x+6}{9x^2+6x+2}\,dx - \int \frac{3}{9x^2+6x+2}\,dx \\
&= \int \frac{du}{u} - 3\int \frac{dx}{9x^2+6x+1+1} \\
&= \ln|u| - 3\int \frac{dx}{(3x+1)^2+1}.
\end{aligned}$$

Let $v = 3x + 1$, $dv = 3\,dx$. Then by Formula (3.2),

$$\begin{aligned}\int \frac{18x+3}{9x^2+6x+2}\,dx &= \ln|u| - \frac{3}{3}\int \frac{1}{v^2+1}\,dv \\ &= \ln|u| - \tan^{-1} v + C \\ &= \ln|9x^2+6x+2| - \tan^{-1}(3x+1) + C.\end{aligned}$$

Teaching Tip

End the entire lesson by giving the students a table of all integral formulas introduced.

(C) SUMMARY/TABLE OF INTEGRALS

1. $\displaystyle\int dx = x + C$
2. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, if $n \neq -1$
3. $\displaystyle\int af(x)\,dx = a\int f(x)\,dx$
4. $\displaystyle\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
5. $\displaystyle\int e^x\,dx = e^x + C$
6. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C$
7. $\displaystyle\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$
8. $\displaystyle\int \sin x\,dx = -\cos x + C$
9. $\displaystyle\int \cos x\,dx = \sin x + C$
10. $\displaystyle\int \sec^2 x\,dx = \tan x + C$
11. $\displaystyle\int \csc^2 x\,dx = -\cot x + C$
12. $\displaystyle\int \sec x \tan x\,dx = \sec x + C$
13. $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$
14. $\displaystyle\int \tan x\,dx = -\ln|\cos x| + C$
15. $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$
16. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$
17. $\displaystyle\int \csc x\,dx = \ln|\csc x - \cot x| + C$
18. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$

(D) EXERCISES

I. Determine the antiderivatives of the following functions:

1. $f(x) = (x+1)^{100}$
2. $f(x) = (x^2+x)^{10}(2x+1)$
3. $g(x) = \dfrac{x^3}{\sqrt{1-x^4}}$
4. $h(x) = \dfrac{6x^2+2}{\sqrt{x^3+x+1}}$

II. Evaluate the following integrals:

1. $\displaystyle\int \frac{x^2 - x}{1 + 3x^2 - 2x^3}\, dx$
2. $\displaystyle\int \frac{\cos x}{1 + 2\sin x}\, dx$
3. $\displaystyle\int \frac{x^2 + 2}{x + 1}\, dx$
4. $\displaystyle\int \frac{3x^5 - 2x^3 + 5x^2 - 2}{x^3 + 1}\, dx$
5. $\displaystyle\int \frac{\log_4(x^2)}{x}\, dx$
6. $\displaystyle\int \frac{(\log_4 x)^2}{x}\, dx$
7. $\displaystyle\int \frac{1}{x + \sqrt{x}}\, dx$
8. $\displaystyle\int (\sec 4x - \cot 4x)\, dx$
9. $\displaystyle\int \sec x \csc x\, dx$
10. $\displaystyle\int 6^x e^x\, dx$
11. $\displaystyle\int \frac{1}{x^2 e^{\frac{1}{x}}}\, dx$
12. $\displaystyle\int \frac{5^{\sqrt{x}} e^{5^{\sqrt{x}}}}{\sqrt{x}}\, dx$
13. $\displaystyle\int 7^x \tan(7^x)\, dx$
14. $\displaystyle\int \frac{3^x\left[\cos\left(3^x\right) - \sin\left(3^x\right)\right]}{\sin^2\left(3^x\right)}\, dx$
15. $\displaystyle\int \frac{e^{4x}}{e^{4x} + 1}\, dx$
16. $\displaystyle\int \frac{e^{3x}}{e^x - 5}\, dx$
17. $\displaystyle\int e^{x + e^x}\, dx$
18. $\displaystyle\int \frac{3^{\log_2(\sin x)} + \sec x}{\tan x}\, dx$
19. $\displaystyle\int \frac{1}{\sqrt{1 - 4x^2}}\, dx$
20. $\displaystyle\int \frac{1}{x\sqrt{25x^2 - 4}}\, dx$
21. $\displaystyle\int \frac{dx}{(1 + x^2)\sqrt{16 - (\tan^{-1} x)^2}}$
22. $\displaystyle\int \frac{e^x}{e^{2x+2} + 2}\, dx$
23. $\displaystyle\int \frac{8}{x^2 - 6x + 25}\, dx$
24. $\displaystyle\int \frac{6}{(2 - x)\sqrt{x^2 - 4x + 3}}\, dx$
25. $\displaystyle\int \frac{2x}{\sqrt{4x - x^2}}\, dx$
26. $\displaystyle\int \frac{x}{x^2 + x + 1}\, dx$
27. $\displaystyle\int \frac{1}{9x^2 + 25}\, dx$
28. $\displaystyle\int \frac{1}{x\sqrt{16x^2 - 9}}\, dx$
29. $\displaystyle\int \frac{1}{e^x + e^{-x}}\, dx$
30. $\displaystyle\int \frac{1}{(2x - 3)\sqrt{4x(x - 3)}}\, dx$

LESSON 13: Application of Antidifferentiation to Differential Equations

TIME FRAME: 3 hours

LEARNING OUTCOME: At the end of the lesson, the learner shall be able to solve separable differential equations using antidifferentiation.

LESSON OUTLINE:

1. Separable differential equations

TOPIC 13.1: Separable Differential Equations

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

Imagine you are behind a microscope in a biology laboratory and you are trying to count the growing number of bacteria in a petri dish. You observe that at any point in time, one-fourth of the population is reproduced. You may eventually give up on counting because the bacteria are multiplying very fast, but you can use your calculus skills to find the population of bacteria at any given time.

Let y denote the population at any time t. According to your observation, the change in the population at any time is $\frac{1}{4}y$. Since the derivative quantifies change, the above observation can be expressed mathematically as

$$\frac{dy}{dt} = \frac{1}{4}y.$$

This is an example of a differential equation, and our objective is to recover the population y from the above equation using integration.

(B) LESSON PROPER

A ***differential equation (DE)*** is an equation that involves x, y and the derivatives of y. The following are examples of differential equations:

(a) $\frac{dy}{dx} = 2x + 5$ (b) $\frac{dy}{dx} = -\frac{x}{y}$ (c) $y'' + y = 0$

The ***order*** of a differential equation pertains to the highest order of the derivative that appears in the differential equation.

The first two examples above are first-order DEs because they involve only the first derivative, while the last example is a second-order DE because y'' appears in the equation.

A ***solution*** to a differential equation is a function $y = f(x)$ or a relation $f(x, y) = 0$ that satisfies the equation.

Teaching Tip

In the answers below, we will always try to express y in terms of x. However, recalling our discussion on implicit differentiation, it may be difficult or impossible to express y explicitly in terms of x. Therefore, a solution to a differential equation may as well be only a relation.

For example, $y = x^2 + 5x + 1$ is a solution to $\dfrac{dy}{dx} = 2x + 5$ since

$$\frac{d}{dx}(y) = \frac{d}{dx}(x^2 + 5x + 1) = 2x + 5.$$

The relation $x^2 + y^2 = 1$ is a solution to $\dfrac{dy}{dx} = -\dfrac{x}{y}$ because if we differentiate the relation implicitly, we get

$$2x + 2y\frac{dy}{dx} = 0 \qquad \Longrightarrow \qquad \frac{dy}{dx} = -\frac{x}{y}.$$

Finally, $y = \sin x$ solves the differential equation $y'' + y = 0$ since $y' = \cos x$ and $y'' = -\sin x$, and therefore

$$y'' + y = (-\sin x) + \sin x = 0.$$

Solving a differential equation means finding *all* possible solutions to the DE.

A differential equation is said to be ***separable*** if it can be expressed as

$$f(x)\,dx = g(y)\,dy,$$

where f and g are functions of x and y, respectively. Observe that we have separated the variables in the sense that the left-hand side only involves x while the right-hand side is purely in terms of y.

If it is possible to separate the variables, then we can find the solution of the differential equation by simply integrating:

$$\int f(x)\,dx = \int g(y)\,dy$$

and applying appropriate techniques of integration. Note that the left-hand side yields a function of x, say $F(x)+C_1$, while the right-hand side yields a function of y, say $G(y)+C_2$. We thus obtain

$$F(x) = G(y) + C \qquad \text{(Here, } C = C_2 - C_1\text{)}$$

which we can then express into a solution of the form $y = H(x) + C$, if possible.

We will now look at some examples of how to solve separable differential equations.

EXAMPLE 1: Solve the differential equation $\dfrac{dy}{dt} = \dfrac{1}{4}\,y$.

Solution. Observe that $y = 0$ is a solution to the differential equation. Suppose that $y \neq 0$. We divide both sides of the equation by y to separate the variables:

$$\frac{dy}{y} = \frac{1}{4}\,dt.$$

Integrating the left-hand side with respect to y and integrating the right-hand side with respect to t yield

$$\int \frac{dy}{y} = \int \frac{1}{4}\,dt \qquad \Longrightarrow \qquad \ln|y| = \frac{1}{4}t + C.$$

Taking the exponential of both sides, we obtain

$$|y| = e^{\frac{1}{4}t+C} = e^{C}\,e^{\frac{1}{4}t}.$$

Therefore, $y = \pm A e^{\frac{1}{4}t}$, where $A = e^{C}$ is any positive constant. Therefore, along with the solution $y = 0$ that we have found at the start, the solution to the given differential equation is $y = Ae^{\frac{1}{4}t}$ where A is any real number.

EXAMPLE 2: Solve the differential equation $2y\,dx - 3x\,dy = 0$.

Solution. If $y = 0$, then $\dfrac{dy}{dx} = 0$ which means that the function $y = 0$ is a solution to the differential equation $2y - 3x\dfrac{dy}{dx} = 0$. Assume that $y \neq 0$. Separating the variables, we get

$$2y\,dx = 3x\,dy$$

$$\frac{2}{x}\,dx = \frac{3}{y}\,dy.$$

Integrating both sides, we obtain

$$\begin{aligned}
\int \frac{2}{x}\,dx &= \int \frac{3}{y}\,dy \\
2\ln|x| &= 3\ln|y| + C \\
\ln|x|^2 - C &= \ln|y|^3 \\
|y|^3 &= e^{-C}\,e^{\ln|x|^2} = e^{-C}\,|x|^2.
\end{aligned}$$

Therefore, the solutions are $y = \pm\sqrt[3]{A}\,x^{2/3}$ where $A = e^{-C}$ is any positive constant. Along with the trivial solution $y = 0$, the set of solutions to the differential equation is $y = Bx^{2/3}$ where B is any real number. *Note: The absolute value bars are dropped since x^2 is already nonnegative.*

EXAMPLE 3: Solve the equation $xy^3\,dx + e^{x^2}\,dy = 0$.

Solution. First, we separate the variables. We have

$$\begin{aligned} -xy^3\,dx &= e^{x^2}\,dy \\ -xe^{-x^2}\,dx &= \frac{1}{y^3}\,dy. \end{aligned}$$

Integrating both sides of the equation with respect to their variables, we have

$$-\int xe^{-x^2}\,dx = \int y^{-3}\,dy.$$

Meanwhile, if $u = -x^2$, then $du = -2x\,dx$ so that $-\dfrac{du}{2} = x\,dx$. Hence,

$$-\int xe^{-x^2}\,dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{-x^2} + C.$$

Therefore,

$$\begin{aligned} -\int xe^{-x^2}\,dx &= \int y^{-3}\,dy \\ \frac{1}{2}e^{-x^2} &= -\frac{y^{-2}}{2} + C. \end{aligned}$$

If we solve for y in terms of x, we get $y = \pm\big(A - e^{-x^2}\big)^{-1/2}$, where $A = 2C$ is any constant.

EXAMPLE 4: Solve the equation $3(y+2)\;dx - xy\;dy = 0$.

Solution. Separating the variables of the differential equation gives us

$$\begin{aligned} 3\,(y+2)\;dx &= xy\;dy \\ \frac{3}{x}\;dx &= \frac{y}{y+2}\;dy \\ \frac{3}{x}\;dx &= \frac{(y+2)-2}{y+2}\;dy \\ \frac{3}{x}\;dx &= \left(1 - \frac{2}{y+2}\right)\;dy. \end{aligned}$$

Now that we have separated the variables, we now integrate the equation term by term:

$$\begin{aligned} \int \frac{3}{x}\;dx &= \int \left(1 - \frac{2}{y+2}\right)\;dy \\ 3\ln|x| &= y - 2\ln|y+2| + C. \end{aligned}$$

Teaching Tip

The solution to the above example is impossible to express in the form $y = H(x)+C$. So, we are contented with expressing this as a relation.

Note that in the previous examples, a constant of integration is always present. If there are initial conditions, or if we know that the solution passes through a point, we can solve this constant and get a ***particular solution*** to the differential equation.

EXAMPLE 5: Find the particular solutions of the following given their corresponding initial conditions:

(a) $\dfrac{dy}{dt} = \dfrac{1}{4}y$ when $y = 100$ and $t = 0$

(b) $2y\,dx - 3x\,dy = 0$ when $x = 1$ and $y = 1$

(c) $xy^3\,dx + e^{x^2}\,dy = 0$ when $x = 0$ and $y = 1$

(d) $3(y+2)\,dx - xy\,dy = 0$ when $x = 1$ and $y = -1$

Solution. We will use the general solutions from the previous examples.

(a) The solution to Example 1 is $y = A\,e^{1/4t}$. Using the conditions $y = 100$ and $t = 0$, we get $100 = Ae^0$. Hence, $A = 100$ and therefore the particular solution is $y = 100e^{\frac{1}{4}t}$.

(b) The solution to Example 2 is $y = B\,x^{2/3}$. Substituting $(x, y) = (1, 1)$ gives $1 = B\,1^{2/3} = B$. Hence, the particular solution is $y = \sqrt[3]{x^2}$.

(c) From the Example 3, the general solution is $y = \pm\dfrac{1}{\sqrt{A - e^{-x^2}}}$. Substituting $(x, y) = (1, 0)$ yields $1 = \pm\dfrac{1}{\sqrt{A-1}}$. Since the square root of a real number is never negative, $\sqrt{A-1} = +1$ and so $A = 2$. Finally, the particular solution is $y = \pm\dfrac{1}{\sqrt{2 - e^{-x^2}}}$.

(d) From Example 4, the general solution is $3\ln|x| = y - 2\ln|y+2| + C$. Substituting the given values $(x, y) = (1, -1)$, we obtain $3\ln|1| = -1 - 2\ln|-1+2| + C$. Simplifying this gives $C = 1$. Hence, the particular solution is the relation $3\ln|x| = y - 2\ln|y+2| + 1$.

(C) EXERCISES

1. Determine whether each of the following differential equations is separable or not, if it is separable, rewrite the equation in the form $g(y)\,dy = f(x)\,dx$.

a. $\dfrac{dy}{dx} + 4y = 8$

b. $\dfrac{dy}{dx} = \sqrt{1+x^2}$

c. $\dfrac{dy}{dx} = 3y^2 - y^2 \sin x$

d. $x\dfrac{dy}{dx} = (x-y)^2$

e. $\dfrac{dy}{dx} + 4y = x^2$

f. $\dfrac{dy}{dx} = \sin(x+y)$

g. $\dfrac{dy}{dx} + xy = 4x$

h. $y\dfrac{dy}{dx} = e^{x-3y^2}$

2. Find the general solution of the following differential equations.

a. $\dfrac{dy}{dx} = \dfrac{x}{y}$

b. $\dfrac{dy}{dx} = y(1+e^x)$

c. $\dfrac{dy}{dx} = 9x^2y$

d. $\dfrac{4}{y^3}\dfrac{dy}{dx} = \dfrac{1}{x}$

e. $\dfrac{dy}{dx} = \dfrac{1}{xy^3}$

f. $\dfrac{dy}{dx} = \dfrac{2y}{x}$

g. $\dfrac{dy}{dx} = x^3 - 3x^2 + x$

h. $\dfrac{dy}{dx} = -\dfrac{x}{2y}$

i. $\dfrac{dy}{dx} = \dfrac{\cos^2 y}{\sin^2 x}$

j. $\cos y\dfrac{dy}{dx} = \sin x$

k. $xy\dfrac{dy}{dx} = y^2 + 9$

3. Solve the following initial-value problems.

a. $\dfrac{dy}{dx} = x^2y^3 \quad \text{and} \quad y(3) = 1$

b. $\dfrac{dy}{dx} = (1+y^2)\tan x \quad \text{and} \quad y(0) = \sqrt{3}$

c. $\dfrac{dy}{dx} = 1 + y^2 \quad \text{and} \quad y(\pi) = 0$

d. $\dfrac{dy}{dx} = \dfrac{x}{y} \quad \text{and} \quad y(-3) = -2$

e. $\dfrac{dy}{dx} = \dfrac{y+1}{x}, x \neq 0 \text{ and } y(-1) = 1$

f. $\dfrac{dy}{dx} = \dfrac{x^3}{y^2} \quad \text{and} \quad y(2) = 3$

g. $\dfrac{dy}{dx} = \dfrac{3x}{y} \quad \text{and} \quad y(6) = -4$

h. $\dfrac{dy}{dx} = \cos(2x) \quad \text{and} \quad y(0) = 1$

i. $\dfrac{dy}{dx} = 4 - y \quad \text{and} \quad y(0) = 2$

4. Find all constant solutions, if possible, to each of the following differential equations.

a. $\dfrac{dy}{dx} = xy - 4x$

b. $\dfrac{dy}{dx} - 6y = 3$

c. $y\dfrac{dy}{dx} = xy^2 - 9x$

d. $\dfrac{dy}{dx} = \sin y$

5. Find all solutions to the differential equation $\dfrac{dy}{dx} = -\dfrac{x}{y-3}$ and a particular solution satisfying $y(0) = 1$.

6. Find the particular solution of the differential equation $\dfrac{d^2y}{dx^2} = -\dfrac{3}{x^4}$ determine by the initial conditions $y = \dfrac{1}{2}$ and $\dfrac{dy}{dx} = -1$ when $x = 1$.

LESSON 14: Application of Differential Equations in Life Sciences

TIME FRAME: 8 hours

LEARNING OUTCOME: At the end of the lesson, the learner shall be able to solve situational problems involving: exponential growth and decay, bounded growth, and logistic growth.

LESSON OUTLINE:

1. Understanding which parts of a problem play roles in a differential or integral expression
2. Recognizing whether a situation is indicative of exponential, bounded, or logistic growth
3. Setting up a differential or integral given situations involving exponential, bounded, and logistic growth
4. Solving these differential equations and integrals

TOPIC 14.1: Situational Problems Involving Growth and Decay Problems

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

When studying a real-world problem, the ability to recognize the type of mathematical situation it may adhere to is an advantage. It is an added advantage if the problem describes certain patterns with already known solution approaches. Such is the case for so-called exponential, bounded and logistic growth. The following discussion focuses on situations falling under these categories, where growth (or decay) is expressed as a rate of change. Hence, the solution is obtained via integration.

Teaching Tip

This is a good opportunity for the students to practice the use of the calculator. Some computations will require rational powers of e and natural logarithms.

(B) LESSON PROPER

EXPONENTIAL GROWTH AND DECAY

The simplest growth model for a population depends only on the occurrence of births and deaths. Births and deaths, in turn, depend on the current size of a population. In particular, they are fractions or percentages of the population. Thus, if $y = f(t)$ is the size of a certain population at time t, and the birth and death rates are given by positive constants b and d, respectively, the rate of change in the population at time t is given by

$$\frac{dy}{dt} = by - dy .$$

Before we continue, we will henceforth replace $b - d$ with the constant k.

$$\frac{dy}{dt} = ky .$$

To depict rates of change, it follows that k may be positive or negative. If it is positive, meaning $b > d$ or there are more births than deaths, it denotes *growth*. If it is negative, meaning $b < d$ or there are more deaths than births, it denotes *decay*.

How does integration come into play?

Did you recognize the differential equation above as a separable one? If so, then you know that solving it will yield $y = f(t)$ or the size of the population at any time t. Indeed,

$$\frac{dy}{dt} = ky$$

and so,

$$\frac{dy}{y} = k\,dt.$$

Integrating both sides of the equation

$$\int \frac{dy}{y} = \int k\,dt,$$

we get

$$\ln y = kt + C_1.$$

Thus,

$$e^{\ln y} = e^{kt+C_1}.$$

Therefore,

$$y = e^{kt} \cdot e^{C_1},$$

and so it follows that

$$y = C \cdot e^{kt},$$

where $C = e^{C_1}$.

The foregoing result explains why this pattern of growth is called *exponential* or *unbounded growth*.

Teaching Tip

Point out immediately the several "automatic" assumptions in the solution, namely,

- Why is it possible to divide by y?
- Why is "$\ln y$" in the third line and not "$\ln |y|$"?
- Why does "$\ln y$" in the third line not have any constant of integration?

Ask the students to figure out the reasons why these assumptions are allowed and "automatic."

We improve the resulting equation by solving for C. To find a particular solution, the value of C must be determined. This can be done if the value of y is given at a particular time t. For instance, if at $t = 0$, we know that $y = y_0$, then

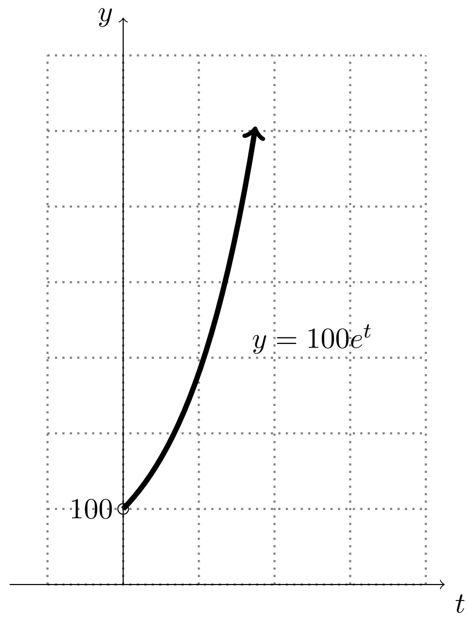
$$\begin{aligned} y_0 &= C \cdot e^{k \cdot 0} \\ &= C \cdot e^0 \\ &= C \end{aligned}$$

This makes $y = y_0 e^{kt}$.

A sample graph will illustrate why exponential growth is also called *unbounded growth*. Assume a certain population begins with 100 individuals with a growth rate $k = 1$. The population at any time $t > 0$ is given by

$$y = 100e^t$$

whose graph is as follows:



EXAMPLE 1: Suppose that a colony of lice grows exponentially. After 1 day, 50 lice are counted. After 3 days, 200 were counted. How many are there originally? What is the exponential growth equation for the colony?

Solution. Recall the exponential growth equation and identify information given in the problem that will help answer the question.

- $y_1 = 50$ means that $50 = y_0 e^{k \cdot 1}$.
- $y_3 = 200$ means that $200 = y_0 e^{k \cdot 3}$.

Note that these two equations will give us the values for the two unknowns, y_0 and e^k.

$$\begin{aligned} 50 &= y_0 e^k \\ 200 &= y_0 e^{3k}. \end{aligned}$$

From the first equation, $y_0 = 50e^{-k}$. Using this in the second equation,

$$\begin{aligned}
200 &= (50\, e^{-k})e^{3k} \\
200 &= 50\, e^{2k} \\
4 &= e^{2k} \\
4 &= (e^k)^2 \\
\text{or} \quad 2 &= e^k.
\end{aligned}$$

Substituting this in the first equation,

$$\begin{aligned}
50 &= y_0 \cdot 2 \\
\text{or} \quad y_0 &= 25.
\end{aligned}$$

We now have the answers to the two questions given. First, there were originally 25 lice in the colony. Second, the exponential growth equation for the given word problem is

$$y = 25 \cdot 2^t.$$

Now, let us take a decay problem.

EXAMPLE 2: The rate of decay of radium is said to be proportional to the amount of radium present. If the half-life of radium is 1690 years and there are 200 grams on hand now, how much radium will be present in 845 years?

Solution. The exponential decay equation again starts off as $y = Ce^{kt}$.

Since there are 200 grams present at the start, the equation immediately evolves to

$$y = 200e^{kt}.$$

A half-life of 1690 years means that the initial amount of 200 grams of radium will reduce to half, or just 100 grams, in 100 years. Thus,

$$100 = 200e^{k \cdot 1690}.$$

This gives

$$e^k = \left(\frac{1}{2}\right)^{1/1690},$$

and consequently,

$$y = 200\left(\frac{1}{2}\right)^{t/1690}.$$

To answer the problem,

$$
\begin{aligned}
y &= 200\left(\frac{1}{2}\right)^{845/1690} \\
&= 200\left(\frac{1}{2}\right)^{1/2} \\
&= 200\left(\frac{1}{\sqrt{2}}\right) \\
&\approx 200\left(\frac{1}{0.707}\right) \\
&= 141.4.
\end{aligned}
$$

Therefore, after 845 years, there will be approximately 141.4 grams of radium left.

BOUNDED GROWTH

Thinking back, populations cannot really grow without bound. In many cases, the population is limited by some resource, such as food or space. This limiting quantity or upper bound is sometimes referred to as the *carrying capacity*, and researchers measure the difference between this limiting quantity and the actual population. If the carrying capacity is given by a positive constant, K, the rate of change of y with respect to time t is proportional to the difference $(K - y)$. That is,

$$\frac{dy}{dt} = k(K - y)\ .$$

This type of growth is called *bounded growth*.

Now, $\dfrac{dy}{dt} = k(K - y)$ implies that

$$\frac{dy}{(K - y)} = k\ dt.$$

Integrating both sides of the equation, we get

$$-\ln|K - y| = kt + C_1$$

and so

$$|K - y| = e^{-kt + C_1}.$$

Finally, we have

$$|K - y| = C \cdot e^{-kt}.$$

Two cases emerge from the absolute value expression on the left: $K - y > 0$ and $K - y < 0$. The former means that population is lower than the carrying capacity, while the latter has the opposite meaning. It is more usual that the former happens, thus for the succeeding computation we consider the former, i.e., $y < K$.

$$
\begin{aligned}
y < K \quad &\Longrightarrow \quad |K - y| \;=\; K - y \\
&\Longrightarrow \quad K - y \;=\; C \cdot e^{-kt} \\
&\Longrightarrow \quad y \;=\; K - C \cdot e^{-kt}.
\end{aligned}
$$

For future problem-solving, it is useful to note that C is equal to $K - y_0$, where again y_0 is the initial population, or the population at time t.

> **Teaching Tip**
>
> It would be good to ask the students to explore the consequence on the resulting equation if $y > K$, and the possible scenarios when this may occur.

EXAMPLE 3: A certain *pawikan* breeding site is said to be able to sustain 5000 *pawikans*. One thousand *pawikans* are brought there initially. After a year, this increased to 1100 *pawikans*. How many *pawikans* will there be after 5 years? Assume that *pawikans* follow the limited growth model.

Solution. We recall the bounded growth equation and identify parts given in the word problem.

- $K = 5000$.
- $y_0 = 1000$. This means that $C = 5000 - 1000 = 4000$ and the equation becomes

$$y \;=\; 5000 - 4000 \cdot e^{-kt}.$$

The population after 1 year, $y_1 = 1100$, means we can substitute y with 1100 and t with 1 to obtain e^{-k}.

$$
\begin{aligned}
1100 &= 5000 - 4000 \cdot e^{-k} \\
4000 \cdot e^{-k} &= 5000 - 1100 \\
&= 3900 \\
e^{-k} &= \frac{3900}{4000} \\
&= 0.975.
\end{aligned}
$$

With the values we have enumerated and solved, the bounded equation is now of the form

$$y \;=\; 5000 - 4000 \cdot (0.975)^t.$$

We can now find the required population, y_5.

$$\begin{aligned} y &= 5000 - 4000 \cdot (0.975)^5 \\ &\approx 5000 - 4000 \cdot (0.881) \\ &= 5000 - 3524 \\ &= 1476. \end{aligned}$$

Therefore, there will be approximately 1476 *pawikans* in the breeding site.

The next example illustrates a sort of "decay." Remember we said earlier that there are occasions when $y > K$? This is one instance.

Teaching Tip

At this point, it would be good to relate this to the remarks at the start of bounded growth that sometimes $y > K$. Ideally, cooling would have been one of the scenarios mentioned when you asked for cases that $y > K$. (See next example.)

EXAMPLE 4: Suppose that newly-baked cupcakes are taken out of the oven which is set at 100 degrees. Room temperature is found to be 25 degrees, and in 15 minutes the cupcakes are found to have a temperature of 50 degrees. Determine the approximate temperature of the cupcakes after 30 minutes.

Solution. Newton's Law of Cooling states that the rate of change of the temperature of an object is equal to the difference between the object's temperature and that of the surrounding air. This gives the differential equation

$$\frac{dy}{dt} = -k(y - 25).$$

Since the situation anticipates that the temperature of an object, y, will decrease towards that of the surrounding air, y_a. Thus, y is assumed to be greater than y_a. Furthermore, to denote the decrease, the constant of proportionality is written as $-k$, with $k > 0$. t in this problem is measured in minutes.

By separation of variables, this becomes

$$y = 25 + Ce^{-kt}.$$

- $y_0 = 100$, we get $C = 75$ and the equation becomes

$$y = 25 + 75e^{-kt}.$$

- The 50-degree temperature after 15 minutes gives

$$e^{-k} = \left(\frac{1}{3}\right)^{1/15},$$

and the equation changes further to

$$y = 25 + 75\left(\frac{1}{3}\right)^{t/15}.$$

We can now proceed to approximate the temperature after 30 minutes:

$$\begin{aligned} y &= 25 + 75\left(\frac{1}{3}\right)^{30/15} \\ &= 25 + 75\left(\frac{1}{3}\right)^{2} \\ &= 25 + 75\left(\frac{1}{9}\right) \\ &\approx 25 + 8.33 \\ &= 33.33. \end{aligned}$$

Hence, after 30 minutes, the cupcakes' temperature will be approximately 33 degrees.

LOGISTIC GROWTH

Further studies say that it is more appropriate for the rate of change of a population to be expressed as proportional to both the size of the population, y, and the difference between a limiting quantity, K, and the size of the population. Hence,

$$\frac{dy}{dt} = ky(K-y).$$

This is called *logistic growth*.

In preparation for integration, we write the above equation as

$$\frac{dy}{y(K-y)} = k\,dt,$$

where the left side of the equation may be written as

$$\frac{1}{K}\left(\frac{1}{y} + \frac{1}{K-y}\right).$$

We solve this differential equation.

$$
\begin{aligned}
\int \frac{1}{y(K-y)}\,dy &= \int k\,dt \\
\int \frac{1}{K}\left(\frac{1}{y}+\frac{1}{K-y}\right)dy &= \int k\,dt \\
\frac{1}{K}\int \left(\frac{1}{y}+\frac{1}{K-y}\right)dy &= \int k\,dt \\
\int \left(\frac{1}{y}+\frac{1}{K-y}\right)dy &= K\int k\,dt \\
\ln|y| - \ln|K-y| &= Kkt + C \\
\ln\left|\frac{y}{K-y}\right| &= Kkt + C \\
\left|\frac{y}{K-y}\right| &= e^{Kkt+C}
\end{aligned}
$$

We follow the same assumption for bounded growth that $y < K$. Thus, $\left|\frac{y}{K-y}\right| = \frac{y}{K-y}$, and

$$\frac{y}{K-y} = e^{Kkt+c}.$$

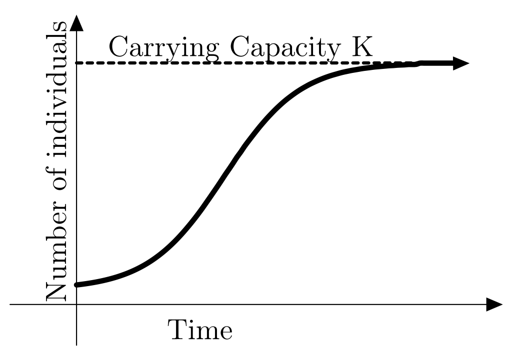
Finally, we isolate y.

$$
\begin{aligned}
y &= (K-y)e^{Kkt+c} \\
y + ye^{Kkt+c} &= Ke^{Kkt+c} \\
y &= \frac{Ke^{Kkt+c}}{1+e^{Kkt+c}} \\
&= \frac{K}{e^{-(Kkt+c)}+1} \\
&= \frac{K}{C\cdot e^{-Kkt}+1}, \text{ where } C = e^{-c}
\end{aligned}
$$

The logistic equation is thus,

$$y = \frac{K}{1+C\cdot e^{-Kkt}},$$

where C and K are positive constants. Its graph is shown on the right.



Teaching Tip

In case the students ask, be ready to show them

- *the solution to* $\dfrac{1}{K}\left(\dfrac{1}{y}+\dfrac{1}{K-y}\right)=\dfrac{1}{y(K-y)}$;
- *the interim steps in the isolation of* y*; and*
- *why* C *is guaranteed to be positive.*

The last bullet may be a recitation question.

EXAMPLE 5: Ten Philippine eagles were introduced to a national park 10 years ago. There are now 23 eagles in the park. The park can support a maximum of 100 eagles. Assuming a logistic growth model, when will the eagle population reach 50?

Solution. To solve the problem, we first recognize how the given information will fit into and improve our equation.

- Since $K = 100$, we have $y = \dfrac{100}{1 + C \cdot e^{-100kt}}$
- Since $y_0 = 23$, we can solve for C.

$$\begin{aligned} 10 &= \frac{100}{1 + C \cdot e^0} \\ 10 &= \frac{100}{1 + C} \\ 10 + 10C &= 100 \\ 10C &= 90 \\ \text{or } C &= 9. \end{aligned}$$

Hence, the equation becomes

$$y = \frac{100}{1 + 9 \cdot e^{-100kt}}.$$

The current population of 23 eagles is equal to the population after 10 years, or $y_{10} = 23$.

This piece of information allows us to solve for the exponential term.

$$\begin{aligned}
23 &= \frac{100}{1+9e^{-100k\cdot 10}} \\
23 &= \frac{100}{1+9e^{-1000k}} \\
1+9e^{-1000k} &= \frac{100}{23} \\
9e^{-1000k} &= \frac{100}{23}-1 \\
9e^{-1000k} &= \frac{77}{23} \\
e^{-1000k} &= \frac{77}{23\cdot 9} \\
\text{or} \quad e^{-1000k} &\approx 0.37.
\end{aligned}$$

Instead of solving for k, it will suffice to find a substitute for $e^{-Kk} = e^{-100k}$. Clearly, if $e^{-1000k} \approx 0.37$, then $e^{-100k} \approx (0.37)^{1/10}$. So,

$$y = \frac{100}{1+9\cdot(0.37)^{t/10}}.$$

We are now ready to answer the question, "When will the eagle population reach 50?" Given the most recent version of our logistic equation, we just substitute y with 50 and solve for t, the time required to have 50 eagles in the population.

$$\begin{aligned}
50 &= \frac{100}{1+9\cdot(0.37)^{t/10}} \\
50(1+9\cdot(0.37)^{t/10}) &= 100 \\
50+450\cdot(0.37)^{t/10} &= 100 \\
(0.37)^{t/10} &= \frac{100-50}{450} \\
(0.37)^{t/10} &= \frac{1}{9} \\
(0.37)^{t} &= \left(\frac{1}{9}\right)^{10} \\
\ln(0.37)^{t} &= \ln\left(\frac{1}{9}\right)^{10} \\
t\cdot\ln(0.37) &= 10\cdot\ln\left(\frac{1}{9}\right) \\
t &= 10\cdot\frac{\ln(1/9)}{\ln(0.37)} \\
t &\approx 10(2.2) = 22.
\end{aligned}$$

The eagle population in the said national park will reach 50 in approximately 22 years.

(C) **EXERCISES**

(a) The population of Barangay Siksikan is increasing at a rate proportional to its current population. In the year 2000, the population was 10,000. In 2003, it became 15,000. What was its population in 2009? In approximately what year will its population be 100,000?

(b) Certain bacteria cells are being observed in an experiment. The population triples in 1 hour. If at the end of 3 hours, the population is 27,000, how many bacteria cells were present at the start of the experiment? After how many hours, approximately, will the number of cells reach 1 million?

(c) The half-life of a radioactive substance refers to the amount of time it will take for the quantity to decay to half as much as it was originally. Substance Q has a half-life of 20 years. If in 2015, 100g of Q was at hand, how much will be at hand in 2055? How much will be at hand in 2060?

(d) Your parents bought a car in 2012 at the price of P1.2 million. The value of your car will depreciate over the years due to use. Thus, in 2015 your car is valued at P900,000. What will be your car?s worth in 2018? In what year, approximately, will your car just be worth P200,000?

(e) Marimar comes from an island-town of 2000 people. She goes to the mainland with a friend and there they catch a highly contagious virus. A week after their return to their island-town, 8 people are infected. How many will be affected after another week? Their public health center decides that once 30

(f) As the biologist for a certain crocodile farm, you know that its carrying capacity is 20,000 crocodiles. You initially release 5000 crocodiles into the farm. After 6 weeks, the crocodile population has increased to 7500. In how many weeks will the population reach 10,000? In how many weeks will the population reach 20,000?

LESSON 15: Riemann Sums and the Definite Integral

TIME FRAME: 4 hours

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Approximate the area of a region under a curve using Riemann sums: (a) left, (b) right, and (c) midpoint; and
2. Define the definite integral as the limit of the Riemann sums.

LESSON OUTLINE:

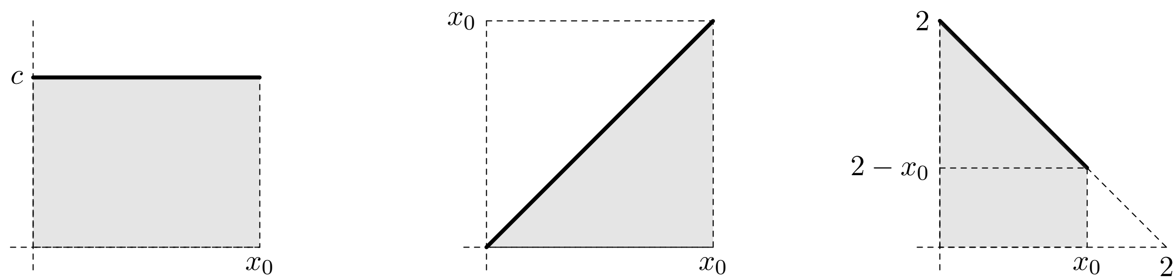
1. Method of exhaustion
2. Riemann sums and Partition points
3. Left, right, and midpoint Riemann sums
4. Refinement
5. Irregular partition
6. Formal definition of a definite integral
7. Geometric interpretation of a definite integral
8. Computing definite integrals
9. Properties of the definite integral
10. The definite integral as a net signed area
11. Exercises and enrichment

TOPIC 15.1: Approximation of Area using Riemann Sums

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

We know that the derivative of a function represents the slope of the tangent line or its instantaneous rate of change. We now ask, what does an antiderivative represent? To answer this, we draw the following on the board:



$f(x) = c$ $\qquad$ $g(x) = x$ $\qquad$ $h(x) = 2 - x$

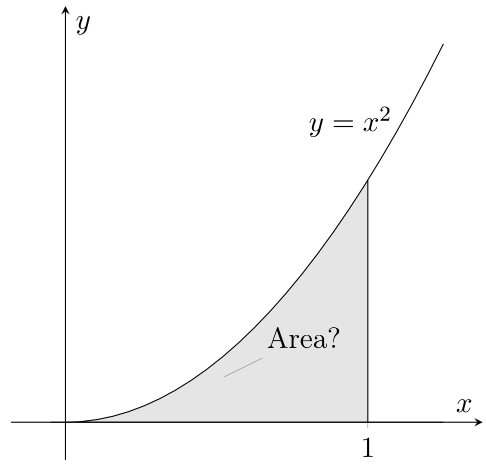
We can then ask some of our students to fill in the following table:

Function	Antiderivative	Area of shaded region
$f(x) = c$	cx	cx_0
$g(x) = x$	$\frac{1}{2}x^2$	$\frac{1}{2}x_0 \cdot x_0$
$h(x) = 2 - x$	$2x - \frac{1}{2}x^2$	$\frac{1}{2}(2 - x_0 + 2)(x_0)$

We see a striking relationship between the area of the region below the graph of a function and the antiderivative of the function. We now suspect that antidifferentiation has something to do with the computation of areas below curves. In this section, we first investigate how to approximate the area of general regions.

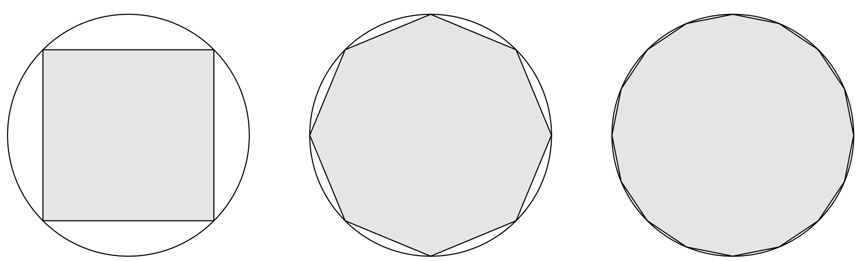
(B) LESSON PROPER

Notice that geometry provides formulas for the area of a region bounded by straight lines, like those above. However, it does not provide formulas to compute the area of a general region. For example, it is quite impossible to compute for the area of the region below the parabola $y = x^2$ using geometry alone.



Even the formula for the area of the circle $A = \pi r^2$ uses a limiting process. Before, since people only knew how to find the area of polygons, they tried to cover the area of a circle by inscribing n-gons until the error was very small. This is called the *Method of Exhaustion.*

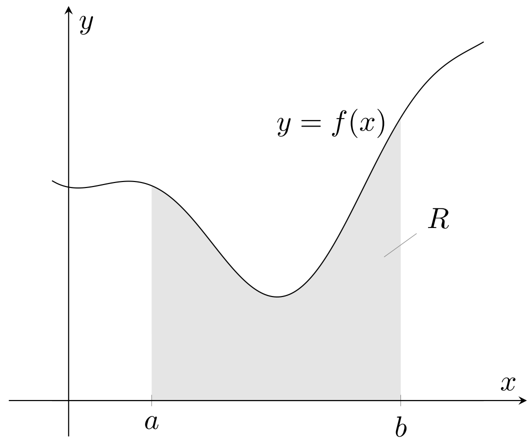
The method of exhaustion is attributed to the ancient Greek mathematician Antiphon of Athens (ca. 5th century BCE), who thought of inscribing a sequence of regular polygons, each with double the number of sides than the previous one, to approximate the area of a circle.



Our method for approximating the area of a region uses the same technique. However, instead of inscribing regular n-gons, we use the simplest polygon – rectangles.

RIEMANN SUMS

Throughout this lesson, we will assume that function f is positive (that is, the graph is above the x-axis), and continuous on the closed and bounded interval $[a, b]$. The **goal** of this lesson is to approximate the area of the region R bounded by $y = f(x)$, $x = a$, $x = b$, and the x-axis.



We first partition $[a, b]$ regularly, that is, into congruent subintervals. Similar to the method of exhaustion, we fill R with rectangles of equal widths. The *Riemann sum of f* refers to the number equal to the combined area of these rectangles. Notice that as the number of rectangles increases, the Riemann sum approximation of the exact area of R becomes better and better.

Of course, the Riemann sum depends on how we construct the rectangles and with how many rectangles we fill the region. We will discuss three basic types of Riemann sums: Left, Right, and Midpoint.

PARTITION POINTS

First, we discuss how to divide equally the interval $[a, b]$ into n subintervals. To do this, we compute the *step size* Δx, the length of each subinterval:

$$\Delta x = \frac{b-a}{n}.$$

Next, we let $x_0 = a$, and for each $i = 1, 2, \ldots, n$, we set the ith intermediate point to be $x_i = a + i\Delta x$. Clearly, the last point is $x_n = a + n\Delta x = a + n\left(\frac{b-a}{n}\right) = b$. Please refer to the following table:

x_0	x_1	x_2	x_3	$\cdots$	x_i	$\cdots$	x_{n-1}	x_n
a	$a + \Delta x$	$a + 2\Delta x$	$a + 3\Delta x$	$\cdots$	$a + i\Delta x$	$\cdots$	$a + (n-1)\Delta x$	b

We call the collection of points $\mathcal{P}_n = \{x_0, x_1, \ldots, x_n\}$ a set of *partition points* of $[a, b]$. Note that to divide an interval into n subintervals, we need $n + 1$ partition points.

Teaching Tip

The above process is the same as inserting $n - 1$ arithmetic means between a and b.

EXAMPLE 1: Find the step size Δx and the partition points needed to divide the given interval into the given number of subintervals.

Example: $[0,1]$; $6 \;\Rightarrow\; \Delta x = \frac{1}{6}, \quad \mathcal{P}_6 = \left\{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\right\}$.

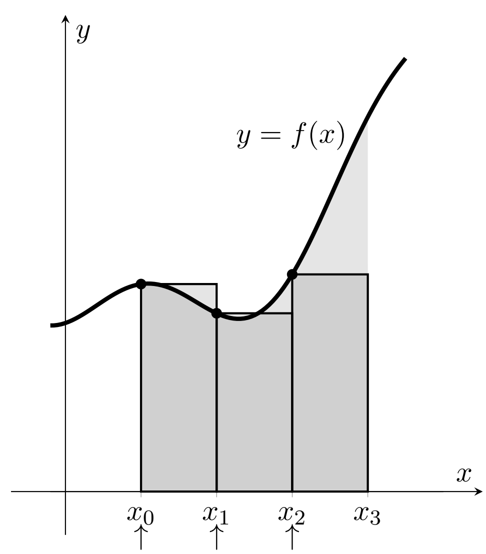
- $[0,1]$; 7 Answer: $\Delta x = \frac{1}{7}, \quad \mathcal{P}_7 = \{0, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, 1\}$
- $[2,5]$; 6 Answer: $\Delta x = \frac{1}{2}, \quad \mathcal{P}_6 = \{2, \frac{5}{2}, 3, \frac{7}{2}, 4\frac{9}{2}, 5\}$
- $[-3,4]$; 4 Answer: $\Delta x = \frac{7}{4}, \quad \mathcal{P}_4 = \{-3, -\frac{5}{4}, \frac{1}{2}, \frac{9}{4}, 4\}$
- $[-5,-1]$; 5 Answer: $\Delta x = \frac{4}{5}, \quad \mathcal{P}_5 = \{-5, -\frac{21}{5}, -\frac{17}{5}, -\frac{13}{5}, -\frac{9}{5}, -1\}$
- $[-10,-3]$; 8 Answer: $\Delta x = \frac{7}{8}, \quad \mathcal{P}_8 = \{-10, -\frac{73}{8}, -\frac{33}{4}, -\frac{59}{8}, -\frac{13}{2}, -\frac{45}{8}, -\frac{19}{4}, -\frac{31}{8}, -3\}$

Assume that the interval $[a,b]$ is already divided into n subintervals. We then cover the region with rectangles whose bases correspond to a subinterval. The three types of Riemann sum depend on the heights of the rectangles we are covering the region with.

LEFT RIEMANN SUM

The nth left Riemann sum L_n is the sum of the areas of the rectangles whose heights are the functional values of the left endpoints of each subinterval.

For example, we consider the following illustration. We subdivide the interval into three subintervals corresponding to three rectangles. Since we are considering left endpoints, the height of the first rectangle is $f(x_0)$, the height of the second rectangle is $f(x_1)$, and the height of the third rectangle is $f(x_2)$.



Therefore, in this example, the 3rd left Riemann sum equals

$$L_3 = f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + f(x_2)(x_3 - x_2) = \left(f(x_0) + f(x_1) + f(x_2)\right)\Delta x.$$

In general, if $[a, b]$ is subdivided into n intervals with partition points $\{x_0, x_1, \ldots, x_n\}$, then the nth left Riemann sum equals

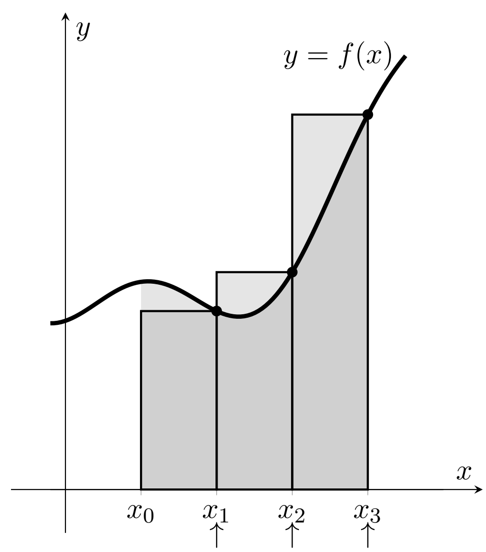
$$L_n = (f(x_0) + f(x_1) + \ldots + f(x_{n-1}))\,\Delta x = \sum_{k=1}^{n} f(x_{k-1})\Delta x.$$

We define the right and midpoint Riemann sums in a similar manner.

RIGHT RIEMANN SUM

The nth right Riemann sum R_n is the sum of the areas of the rectangles whose heights are the functional values of the right endpoints of each subinterval.

For example, we consider the following illustration. We subdivide the interval into three subintervals corresponding to three rectangles. Since we are considering right endpoints, the height of the first rectangle is $f(x_1)$, the height of the second rectangle is $f(x_2)$, and the height of the third rectangle is $f(x_3)$.



Therefore, in this example, the 3rd right Riemann sum equals

$$R_3 = f(x_1)(x_1 - x_0) + f(x_2)(x_2 - x_1) + f(x_3)(x_3 - x_2) = (f(x_1) + f(x_2) + f(x_3))\,\Delta x.$$

In general, if $[a, b]$ is subdivided into n intervals with partition points $\{x_0, x_1, \ldots, x_n\}$, then the nth right Riemann sum equals

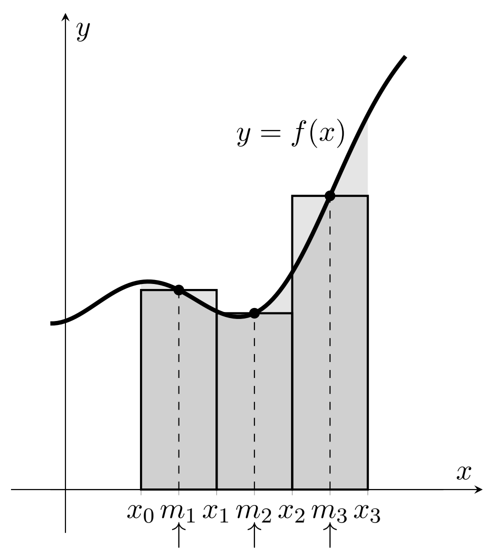
$$R_n = (f(x_1) + f(x_2) + \ldots + f(x_n))\,\Delta x = \sum_{k=1}^{n} f(x_k)\Delta x.$$

MIDPOINT RIEMANN SUM

The nth midpoint Riemann sum M_n is the sum of the areas of the rectangles whose heights are the functional values of the midpoints of the endpoints of each subinterval. For the sake of notation, we denote by m_k the midpoint of two consecutive partition points x_{k-1} and x_k; that is,

$$m_k = \frac{x_{k-1} + x_k}{2}.$$

We now consider the following illustration. We subdivide the interval into three subintervals corresponding to three rectangles. Since we are considering midpoints of the endpoints, the height of the first rectangle is $f(m_1)$, the height of the second rectangle is $f(m_2)$, and the height of the third rectangle is $f(m_3)$.



Therefore, in this example, the 3rd midpoint Riemann sum equals

$$M_3 = f(m_1)(x_1 - x_0) + f(m_2)(x_2 - x_1) + f(m_3)(x_3 - x_2) = (f(m_1) + f(m_2) + f(m_3))\,\Delta x.$$

In general, if $[a, b]$ is subdivided into n intervals with partition points $\{x_0, x_1, \ldots, x_n\}$, then the nth midpoint Riemann sum equals

$$L_n = (f(m_1) + f(m_2) + \ldots + f(m_n))\,\Delta x = \sum_{k=1}^{n} f(m_k)\Delta x,$$

where $m_k = \dfrac{x_{k-1} + x_k}{2}$.

Teaching Tip

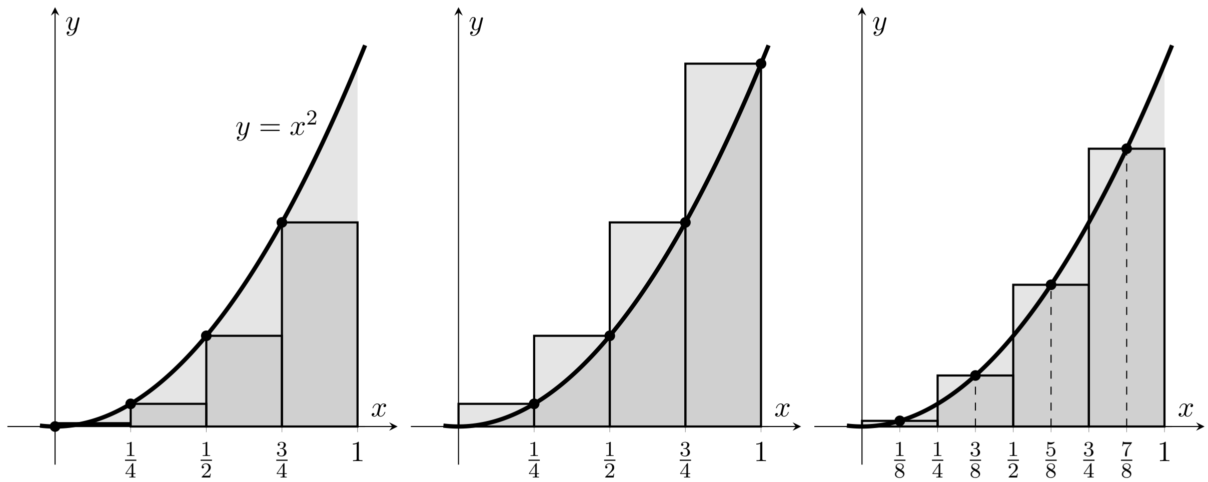
Please allow the students to use a calculator to compute the Riemann sums.

EXAMPLE 2: Find the 4th left, right, and midpoint Riemann sums of the following functions with respect to a regular partitioning of the given intervals.

(a) $f(x) = x^2$ on $[0, 1]$ (b) $f(x) = \sin x$ on $[0, \pi]$.

Solution.

(a) First, note that $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. Hence, $\mathcal{P}_4 = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$. We then compute the midpoints of the partition points: $\left\{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right\}$.



The 4th left Riemann sum equals

$$\begin{aligned} L_4 &= \left(f(0) + f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)\right) \cdot \Delta x \\ &= \left(0 + \frac{1}{16} + \frac{1}{4} + \frac{9}{16}\right) \cdot \frac{1}{4} = 0.21875. \end{aligned}$$

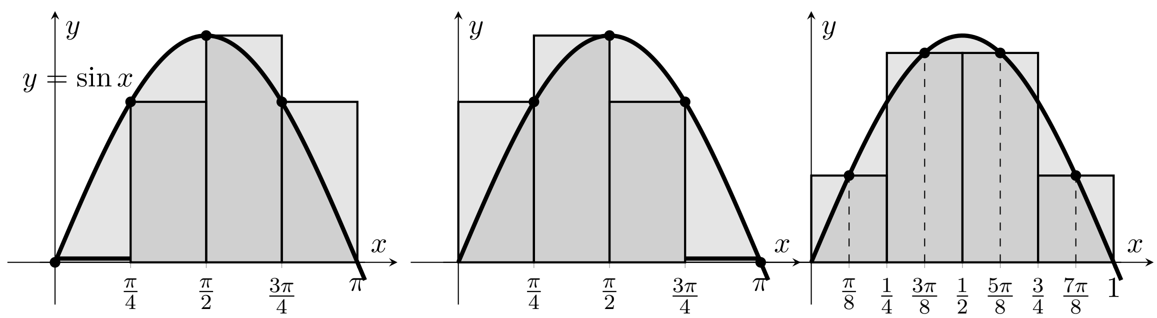
The 4th right Riemann sum equals

$$\begin{aligned} R_4 &= \left(f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right) + f(1)\right) \cdot \Delta x \\ &= \left(\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1\right) \cdot \frac{1}{4} = 0.46875. \end{aligned}$$

Lastly, the 4th midpoint Riemann sum equals

$$\begin{aligned} M_4 &= \left(f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right) \cdot \Delta x \\ &= \left(\frac{1}{64} + \frac{9}{64} + \frac{25}{64} + \frac{49}{64}\right) \cdot \frac{1}{4} = 0.328125. \end{aligned}$$

(b) First, note that $\Delta x = \dfrac{\pi - 0}{4} = \dfrac{\pi}{4}$. Hence, $\mathcal{P}_4 = \left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, 1\right\}$. We then compute the midpoints of the partition points: $\left\{\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}\right\}$.



The 4th left Riemann sum equals

$$\begin{aligned} L_4 &= \left(f(0) + f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right) + f\left(\frac{3\pi}{4}\right)\right) \cdot \Delta x \\ &= \left(\sin 0 + \sin\frac{\pi}{4} + \sin\frac{\pi}{2} + \sin\frac{3\pi}{4}\right) \cdot \frac{\pi}{4} = 1.896... \end{aligned}$$

The 4th right Riemann sum equals

$$\begin{aligned} R_4 &= \left(f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right) + f\left(\frac{3\pi}{4}\right) + f(\pi)\right) \cdot \Delta x \\ &= \left(\sin\frac{\pi}{4} + \sin\frac{\pi}{2} + \sin\frac{3\pi}{4} + \sin\pi\right) \cdot \frac{\pi}{4} = 1.896... \end{aligned}$$

Finally, the 4th midpoint Riemann sum equals

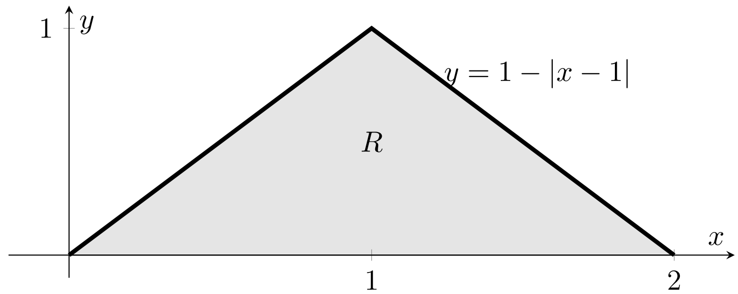
$$\begin{aligned} M_4 &= \left(f\left(\frac{\pi}{8}\right) + f\left(\frac{3\pi}{8}\right) + f\left(\frac{5\pi}{8}\right) + f\left(\frac{7\pi}{8}\right)\right) \cdot \Delta x \\ &= \left(\sin\frac{\pi}{8} + \sin\frac{3\pi}{8} + \sin\frac{5\pi}{8} + \sin\frac{7\pi}{8}\right) \cdot \frac{\pi}{4} = 2.052... \end{aligned}$$

Teaching Tip

The discussions below are supplementary and may be skipped without affecting the flow of your lecture. They can, however, strengthen the learning of the students, and are encouraged to be taught if there is still sufficient time.

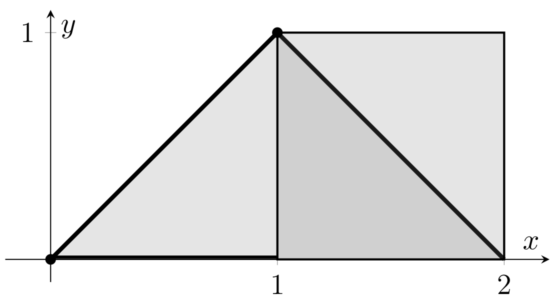
The following example shows that arbitrarily increasing the number of partition points does not necessarily give a better approximation of the true area of the region.

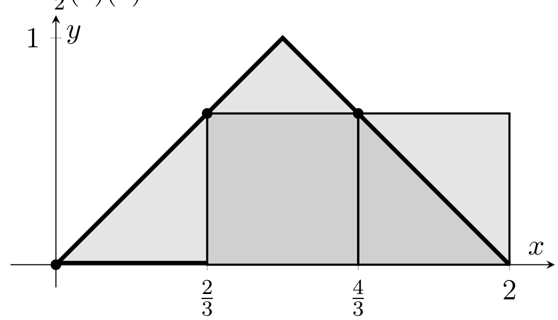
EXAMPLE 3: Let $f(x) = 1 - |x - 1|$ and consider the closed region R bounded by $y = f(x)$ and the x-axis on the interval $[0, 2]$.



Show that relative to regular partitioning, the second left Riemann sum L_2 is a better approximation of the area of R than the third left Riemann sum L_3 of f on $[0, 2]$.

Solution. First, observe that the exact area of R is $\frac{1}{2}(1)(2) = 1$.





Now, the step size and partition points, repectively, are $\Delta x = 1$ and $\mathcal{P}_2 = \{0, 1, 2\}$ for L_2 and $\Delta x = \frac{2}{3}$ and $\mathcal{P}_3 = \{0, \frac{2}{3}, \frac{4}{3}, 2\}$ for L_3. Using the formula for the left Riemann sum, we have the following computations:

$$L_2 = (f(0) + f(1)) \cdot \Delta x = (0 + 1) \cdot 1 = 1$$

while

$$L_3 = \left(f(0) + f\left(\frac{2}{3}\right) + f\left(\frac{4}{3}\right)\right) \cdot \Delta x = \left(0 + \frac{2}{3} + \frac{2}{3}\right) \cdot \frac{2}{3} = \frac{4}{3} \cdot \frac{2}{3} = \frac{8}{9}.$$

Clearly, L_2 is closer (in fact, equal) to the exact value of 1, than L_3.

We ask the students in the exercises to show that M_2 and R_2 are better approximations of the area of R than M_3 and R_3, respectively.

Point of Discussion

Ask the class how they were convinced that increasing the number of partition points (and thereby increasing the number of rectangles) would make the Riemann approximation to the area of R better and better, if the previous example clearly shows the opposite?

REFINEMENT

To deal with the question above, we define the concept of a *refinement* of a partition:

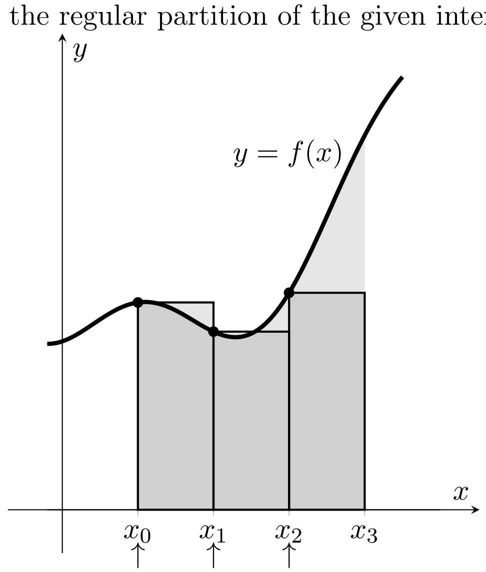
Definition 6. *A partition $\mathcal{Q}$ of an interval I is a refinement of another partition $\mathcal{P}$ of I if $\mathcal{P} \subseteq \mathcal{Q}$, meaning, $\mathcal{Q}$ contains all partition points of $\mathcal{P}$ and more.*

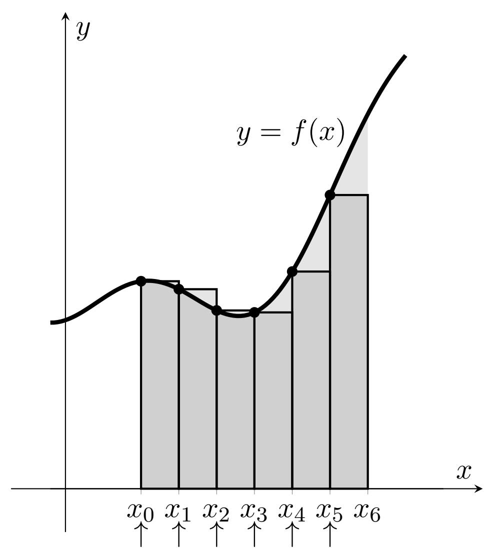
EXAMPLE 4: For the interval $I = [0, 2]$, $\mathcal{P}_3 = \{0, \frac{2}{3}, \frac{4}{3}, 2\}$ is not a refinement of $\mathcal{P}_2 = \{0, 1, 2\}$. However, $\mathcal{P}_4 = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$ is a refinement of $\mathcal{P}_2$ while $\mathcal{P}_6 = \{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2\}$ is a refinement of $\mathcal{P}_3$.

EXAMPLE 5: $\mathcal{P}_{2n}$ is always a refinement of $\mathcal{P}_n$. In fact $\mathcal{P}_{2n}$ contains all partition points of $\mathcal{P}_n$ and all the midpoints therein.

Theorem 15. *Suppose $\mathcal{Q}$ is a refinement of $\mathcal{P}$. Then, any (left, midpoint, right) Riemann sum approximation of a region R relative to $\mathcal{Q}$ is equal to or is better than the same kind of Riemann sum approximation relative to $\mathcal{P}$.*

It is not hard to convince ourselves about the validity of the above theorem. Consider the diagrams below. The one on the left illustrates L_3 while the right one illustrates L_6 relative to the regular partition of the given interval.





Remark 1: The theorem also conforms to the procedure in the classical method of exhaustion wherein they use the areas of inscribed regular n-gons to approximate the area of a circle. The sequence of n-gons the Greeks considered was such that the next n-gon would have twice the number of sides as the previous one.

Remark 2: A consequence of the theorem is that for any positive integer n, the sequence

$$L_n, L_{2n}, L_{4n}, L_{8n}, L_{16n}, \ldots$$

is a monotone sequence converging to the exact area of the region. This means that if A is the exact area of the region and $L_n \leq A$, then

$$L_n \leq L_{2n} \leq L_{4n} \leq L_{8n} \leq L_{16n} \leq \ldots \leq A.$$

For instance, in Example 3, $A = 1$ and $L_3 = \frac{8}{9}$. The above inequalities imply that $L_3 \leq L_6 \leq A$. Indeed,

$$\begin{aligned} L_6 &= \left(f(0) + f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right) + f(1) + f\left(\frac{4}{3}\right) + f\left(\frac{5}{3}\right)\right) \cdot \Delta x \\ &= \left(0 + \frac{1}{3} + \frac{2}{3} + 1 + \frac{2}{3} + \frac{1}{3}\right) \cdot \frac{1}{3} = 1. \end{aligned}$$

IRREGULAR PARTITION

Sometimes, the partition of an interval is irregular, that is, the lengths of the subintervals are not equal. This kind of partitioning is usually used when you want to obtain a refinement of a partition (and thereby get a better approximation) without computing for a lot more points.

For example, suppose that you think that a Riemann sum relative to the partition $\mathcal{P} = \{0, \frac{2}{3}, \frac{4}{3}, 2\}$ is already close to the exact value, then you can simply insert one more point, say 1, to get the partition $\mathcal{P}' = \{0, \frac{2}{3}, 1, \frac{4}{3}, 2\}$. Since $\mathcal{P}'$ is a refinement of $\mathcal{P}$, then a Riemann sum relative to it should have a better value than that of $\mathcal{P}$ and you just have to compute for one more functional value, $f(1)$, rather than 3 more values.

To get Riemann sums relative to irregular partitions, the idea is the same, you just have to be careful about the variable step sizes.

Consider an irregular partition $\mathcal{P} = \{x_0, x_1, x_2, x_3, \ldots, x_n\}$ of an interval. In general it may not be the case that $x_1 - x_0 = x_2 - x_1$. So, we define the step sizes Δx_k.

For each $k \in \{1, 2, \ldots, n\}$, define the kth step size Δx_k to be the length of the kth subinterval, i.e.

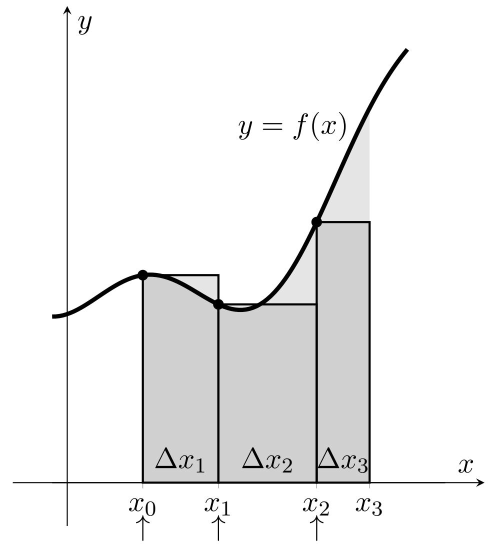
$$\Delta x_k = x_k - x_{k-1}.$$

With this notation, the left Riemann sum with respect to the partition $\mathcal{P}$ is

$$\begin{aligned} L_{\mathcal{P}} &= f(x_0)\Delta x_1 + f(x_1)\Delta x_2 + \ldots + f(x_{n-2})\Delta x_{n-1} + f(x_{n-1})\Delta x_n \\ &= \sum_{k=1}^{n} f(x_{k-1})\Delta x_k. \end{aligned}$$

Now, suppose we are given $y = f(x)$ and a partition $\mathcal{P} = \{x_0, x_1, x_2, x_3\}$. Then the left Riemann sum with respect to the partition $\mathcal{P}$ is

$$L_{\mathcal{P}} = f(x_0)\Delta x_1 + f(x_1)\Delta x_2 + f(x_2)\Delta x_3.$$



Very similarly, the right Riemann sum is given by

$$\begin{aligned} R_{\mathcal{P}} &= f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \ldots + f(x_{n-1})\Delta x_{n-1} + f(x_n)\Delta x_n \\ &= \sum_{k=1}^{n} f(x_k)\Delta x_k, \end{aligned}$$

and the midpoint Riemann sum is given by

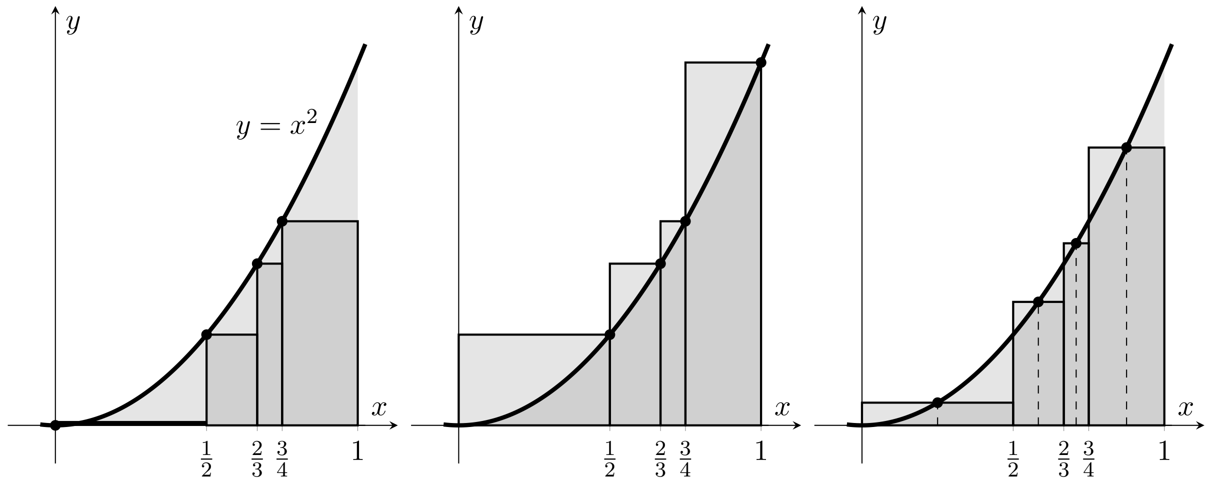
$$\begin{aligned} M_{\mathcal{P}} &= f(m_1)\Delta x_1 + f(m_2)\Delta x_2 + \ldots + f(m_{n-1})\Delta x_{n-1} + f(m_n)\Delta x_n \\ &= \sum_{k=1}^{n} f(m_k)\Delta x_k, \end{aligned}$$

with the same convention that $m_k = \dfrac{x_{k-1} + x_k}{2}$, the midpoint of the kth interval.

EXAMPLE 6: Relative to the partition $\mathcal{P} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1\}$, find the left, right, and midpoint Riemann sums of $f(x) = x^2$ on the interval $[0, 1]$.

Solution. Observe that $\mathcal{P}$ partitions $[0, 1]$ into 4 irregular subintervals: $[0, \frac{1}{2}]$, $[\frac{1}{2}, \frac{2}{3}]$, $[\frac{2}{3}, \frac{3}{4}]$, $[\frac{3}{4}, 1]$. The step sizes are $\Delta x_1 = \frac{1}{2}$, $\Delta x_2 = \frac{1}{6}$, $\Delta x_3 = \frac{1}{12}$, $\Delta x_4 = \frac{1}{4}$. This implies that the Riemann sums are

$$\begin{aligned} L_{\mathcal{P}} &= f(0)\Delta x_1 + f\left(\frac{1}{2}\right)\Delta x_2 + f\left(\frac{2}{3}\right)\Delta x_3 + f\left(\frac{3}{4}\right)\Delta x_4 \\ &= 0 \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{6} + \frac{4}{9} \cdot \frac{1}{12} + \frac{9}{16} \cdot \frac{1}{4} = 0.2193. \\ R_{\mathcal{P}} &= f\left(\frac{1}{2}\right)\Delta x_1 + f\left(\frac{2}{3}\right)\Delta x_2 + f\left(\frac{3}{4}\right)\Delta x_3 + f(1)\Delta x_4 \\ &= \frac{1}{4} \cdot \frac{1}{2} + \frac{4}{9} \cdot \frac{1}{6} + \frac{9}{16} \cdot \frac{1}{12} + 1 \cdot \frac{1}{4} = 0.4959. \\ M_{\mathcal{P}} &= f\left(\frac{1}{4}\right)\Delta x_1 + f\left(\frac{7}{12}\right)\Delta x_2 + f\left(\frac{17}{24}\right)\Delta x_3 + f\left(\frac{7}{8}\right)\Delta x_4 \\ &= \frac{1}{16} \cdot \frac{1}{2} + \frac{49}{144} \cdot \frac{1}{6} + \frac{289}{576} \cdot \frac{1}{12} + \frac{49}{64} \cdot \frac{1}{4} = 0.3212. \end{aligned}$$



(C) EXERCISES

Students may use calculators to compute for Riemann sums.

1. Let $f(x) = x^3$ be defined on $[0, 1]$. Find the left Riemann sum relative to the regular partitions $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$. From these three approximations, could you guess what is the area of the region bounded by $y = x^3$ and the x-axis on $[0, 1]$?

2. Let $f(x) = 3x - x^2$ be defined on $[0, 2]$. Find the right Riemann sum relative to the regular partitions $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$. From these three approximations, could you guess what is the area of the region bounded by $y = 3x - x^2$ and the x-axis on $[0, 2]$?

3. Let $f(x) = \sqrt{x}$ be defined on $[0, 1]$. Find the midpoint Riemann sum relative to the regular partitions $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$. From these three approximations, could you guess what is the area of the region bounded by $y = \sqrt{x}$ and the x-axis on $[0, 1]$?

4. Let $f(x) = \dfrac{1}{x+1}$ be defined on $[0, 1]$. Find the left, right, and midpoint Riemann sums relative to the regular partitions $\mathcal{P}_2$ and $\mathcal{P}_3$.

5. Let $f(x) = \tan x$ be defined on $\left[0, \dfrac{\pi}{4}\right]$. Find the left, right, and midpoint Riemann sums relative to the regular partitions $\mathcal{P}_2$ and $\mathcal{P}_3$. (Do not forget to put your calculators to radian measure mode.)

6. In Example 3, show that relative to regular partitioning, the second right and midpoint Riemann sums R_2 and M_2 are better approximations of the area of R than the third right and midpoint Riemann sums R_3 and M_3 of $f(x) = 1 - |x - 1|$ on $[0, 2]$.

$\star$7. Define $f(x) = 3x + \sqrt{x}$ on the interval $[0, 9]$. Find the left and right Riemann sums relative to the regular partition $\mathcal{P}_3$. Next, find the left and right Riemann sums of f relative to the partition $\mathcal{Q}_1 = \mathcal{P}_3 \cup \{1\}$ and $\mathcal{Q}_2 = \mathcal{Q}_1 \cup \{4\}$. Explain the advantage of using the partitions Q_1 and Q_2 instead of the regular partitions $\mathcal{P}_4$ and $\mathcal{P}_5$.

$\star$8. If the graph of $y = f(x)$ is increasing, which kind of Riemann sum underestimates the exact area of the region? Which overestimates it? What about if the graph is decreasing?

$\star$9. Relative to the same partition, which among the left, right, and midpoint Riemann sums provides the best approximation of the exact area of a region? (Without much loss of generality, you can assume that the curve is monotone because if it is not monotone on a subinterval, then a sufficient refinement of the partition on that subinterval would result in monotone portions of the graph on each subinterval.)

Solution to starred exercises

7. We first collect the functional values that we will be needing.

x	$f(x) = 3x + \sqrt{x}$
0	0
1	4
3	10.73
4	14
6	20.45
9	30

Now, $\mathcal{P}_3 = \{0, 3, 6, 9\}$. So,

- $L_3 = 3(f(0) + f(3) + f(5)) = 93.54$
- $R_3 = 3(f(3) + f(6) + f(9)) = 183.54$

Now, $\mathcal{Q}_1 = \{0, 1, 3, 6, 9\}$. So,

- $L_{\mathcal{Q}_1} = 1 \cdot f(0) + 2 \cdot f(1) + 3 \cdot f(3) + 3 \cdot f(6) = 101.54$
- $R_{\mathcal{Q}_1} = 1 \cdot f(1) + 2 \cdot f(3) + 3 \cdot f(6) + 3 \cdot f(9) = 176.81$

From the given, $\mathcal{Q}_2 = \{0, 1, 3, 4, 6, 9\}$. So,

- $1 \cdot f(0) + 2 \cdot f(1) + 1 \cdot f(3) + 2 \cdot f(4) + 3 \cdot f(6) = 108.08$
- $1 \cdot f(1) + 2 \cdot f(3) + 1 \cdot f(4) + 2 \cdot f(6) + 3 \cdot f(9) = 170.36$

It is attractive to consider $\mathcal{Q}_1$ and $\mathcal{Q}_2$ because we are sure that we get better approximations (as they are refinements of $\mathcal{P}_3$). Moreover, we already have values for $f(3)$ and $f(6)$ from $\mathcal{P}_3$. We just have to reuse them for the Riemann sums for $\mathcal{Q}_1$ and $\mathcal{Q}_2$. In contrast, if we had used $\mathcal{P}_4 = \{0, 9/4, 9/2, 27/4, 9\}$, we need to get the values of $f(9/4)$, $f(9/2)$ and $f(27/4)$ and we are still not assured that the corresponding Riemann sums are better than those corresponding to $\mathcal{P}_3$.

8. If the graph is increasing the rectangles of the left Riemann sum are inscribed, hence it underestimates the true area of the region. On the other hand, the rectangles of the right Riemann sum are circumscribed by the region. Hence, they overestimate the area of the region. The opposite situation happens for decreasing graphs.

9. For a monotone curve (i.e., either increasing or decreasing), the midpoint Riemann sum provides the best approximation to the area of the region. If you bisect these rectangles in the middle, you will see that one is inscribed and the other is circumscribed. Therefore, the underestimation of one is offset by the overestimation of the other.

TOPIC 15.2: The Formal Definition of the Definite Integral

DEVELOPMENT OF THE LESSON

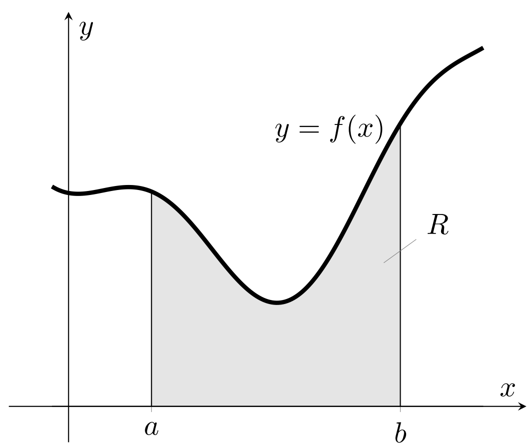
(A) INTRODUCTION

We recall from the last lesson that Riemann sums of $y = f(x)$ on $[a, b]$ provide for approximations of the exact area of the region bounded by $y = f(x)$ and the x-axis. We have also hinted from previous discussions that this approximation gets better and better as we double the partition points.

In this lesson, we formally define the definite integral as the limit of these Riemann sums when the number of partition points goes to infinity. We then associate this value with the exact area of the region described above, if the limit exists.

(B) LESSON PROPER

Again, we work with a continuous positive function $y = f(x)$ defined on a closed and bounded interval $[a, b]$. The objective of this lesson is to find the area of the region R bounded by $y = f(x)$ from above, the x-axis from below, the line $x = a$ from the left and $x = b$ from the right.



To avoid complications, we just consider the case where the partition on the interval is regular. We recall that $\mathcal{P}_n = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ (where $x_0 = a$ and $x_k = x_{k-1} + \Delta x$ with $\Delta x = \frac{b-a}{n}$) partitions $[a, b]$ into n congruent subintervals.

For each subinterval $k = 1, 2, \ldots, n$, let x_k^* be any point in the kth subinterval $[x_{k-1}, x_k]$. Then, the Riemann sum, defined by this choice of points, relative to the partition $\mathcal{P}$ is

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_{n-1}^*)\Delta x + f(x_n^*)\Delta x = \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

The class should realize that

- If this is a left Riemann sum, then $x_k^* = x_{k-1}$;
- If this is a right Riemann sum, then $x_k^* = x_k$; and finally,
- If this is a midpoint Riemann sum, then $x_k^* = \frac{1}{2}(x_{k-1} + x_k)$.

In any case, we know that the above Riemann sum is only an approximation of the exact area of R. To make this estimate exact, we let n approach infinity. This limit of the Riemann sum is what we call the ***definite integral*** of f over $[a, b]$:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \int_a^b f(x)\,dx,$$

if this limit exists. The value of this integral does not depend on the kind (left, right, or midpoint) of Riemann sum being used.

Teaching Tip

To make this lesson easy to remember, you may relate the anecdote on why the integral sign $\int$ looks like an elongated s: This is because the integral sign is nothing but a limit l of a sum s and $\int$ is the symbol that you get by joining the letters l and s.

As the definite integral is the limit of a Riemann sum, many authors also refer to it as a Riemann integral.

Remember that Δx actually depends on n: $\Delta x = \dfrac{b-a}{n}$. So, this term cannot be taken out from the limit operator. After taking the limit, this Δx becomes our differential dx.

The integral sign $\int$ and the differential dx act as delimiters, which indicate that everything between them is the *integrand* - the upper boundary of the region whose area is what this integral is equal to. The numbers a and b are called the *lower and upper limits of integration*, respectively. Recall that the integral sign $\int$ was earlier used to denote the process of antidifferentiation. There is a reason why the same symbol ($\int$) is being used – we shall see later that antiderivatives are intimately related to finding areas below curves.

> **Geometric Interpretation of the Definite Integral**
> Let f be a positive continuous function on $[a, b]$. Then the definite integral
>
> $$\int_a^b f(x)\,dx$$
>
> is the area of the region bounded by $y = f(x)$, the x-axis, $x = a$, and $x = b$.
>
> Consequently, the definite integral **does not** depend on the variable x. Changing this variable only changes the name of the x-axis but not the area of the region. Therefore,
>
> $$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du.$$

COMPUTING DEFINITE INTEGRALS BY APPEALING TO GEOMETRIC FORMULAS

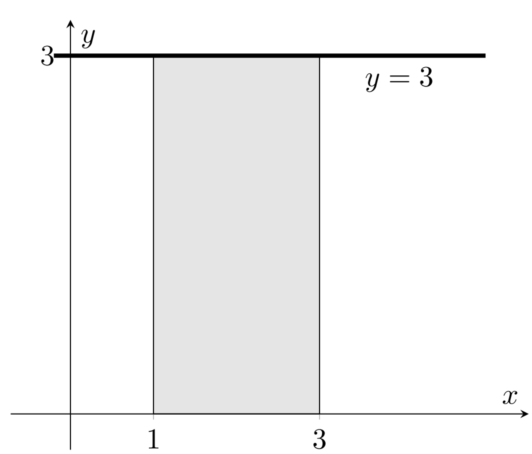
Using the geometric interpretation of the definite integral, we can always think of a definite integral as an area of a region. If we're lucky that the region has an area that is easy to compute using elementary geometry, then we are able to solve the definite integral without resorting to its definition.

EXAMPLE 1: Find the exact values of the following definite integrals:

1. $\displaystyle\int_1^2 3\,dx$
2. $\displaystyle\int_0^2 (1 - |x-1|)\,dx$
3. $\displaystyle\int_1^3 (3x+1)\,dx$
4. $\displaystyle\int_{-1}^1 \sqrt{1-x^2}\,dx$

Solution. Using the above definition of the definite integral, we just draw the region and find its area using elementary geometry.

1. The graph of $y = 3$ is a horizontal line.



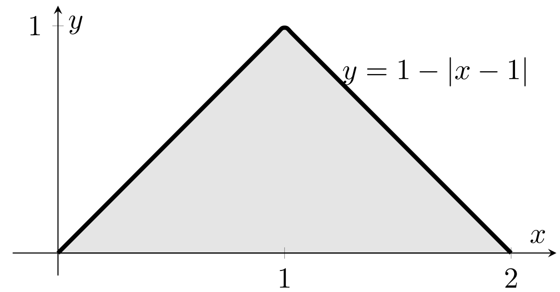
Since the region is a rectangle, its area equals $L \times W = 2 \times 3 = 6$. Therefore,

$$\int_1^3 3\,dx = 6.$$

2. The graph of $y = 1 - |x-1|$ is as given.

   The shaded region is a triangle with base $b = 2$ and height $h = 1$. Its area equals $\frac{1}{2}bh = 1$. Therefore,

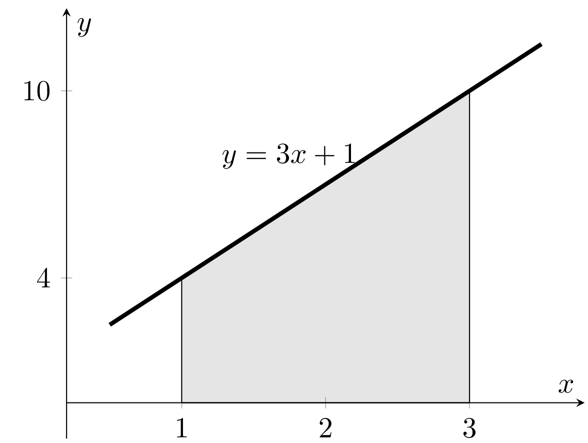
   $$\int_0^2 (1 - |x-1|)\,dx = 1.$$



3. The graph of $y = 3x + 1$ is a line slanting to the right.

   The shaded region is a trapezoid with bases $b_1 = 4$ and $b_2 = 10$ and height $h = 2$. Its area equals $\frac{1}{2}(b_1 + b_2)h = \frac{1}{2}(4+10)2 = 14$. Therefore,

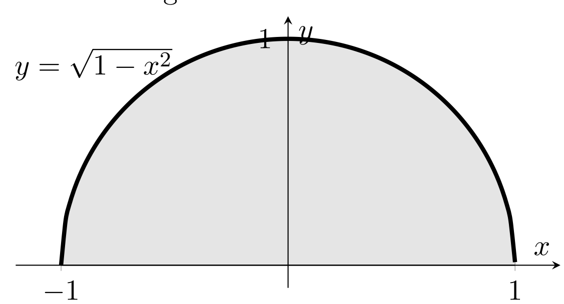
   $$\int_1^3 (3x+1)\,dx = 14.$$



4. The graph of $y = \sqrt{1-x^2}$ is a semicircle centered at the origin with radius 1.

   The area of the shaded region is $\dfrac{1}{2}\pi(1)^2$.
   Therefore,

   $$\int_{-1}^1 \sqrt{1-x^2}\,dx = \frac{\pi}{2}.$$



Teaching Tip

The discussion below about limits at infinity of rational functions is *optional*. It is not a learning competency, but it is an underlying concept in the formal definition of a definite integral.

LIMITS AT INFINITY OF RATIONAL FUNCTIONS

Consider the rational function $f(x) = \dfrac{3x+1}{x+3}$. We describe the behavior of this function for large values of x using a table of values:

x	$f(x) = \frac{3x+1}{x+3}$
100	2.92233
$1,000$	2.99202
$10,000$	2.99920
$100,000$	2.99992

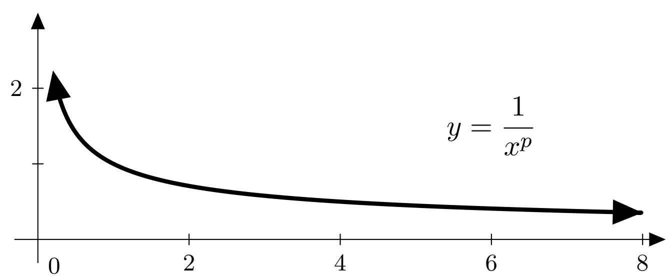
Clearly, as x goes very large, the value of y approaches the value of 3. We say that the limit of $f(x) = \dfrac{3x+1}{x+3}$ as x approaches infinity is 3, and we write

$$\lim_{x\to\infty} \frac{3x+1}{x+3} = 3.$$

To compute limits at infinity of rational functions, it is very helpful to know the following theorem:

Theorem 16. *If* $p > 0$*, then* $\displaystyle\lim_{x\to\infty} \frac{1}{x^p} = 0.$

We illustrate the theorem using the graph of $y = \dfrac{1}{x^p}$.



Observe that as x takes large values, the graph approaches the x-axis, or the $y = 0$ line. This means that the values of y can take arbitrarily small values by making x very large. This is what we mean by $\displaystyle\lim_{x\to\infty} \frac{1}{x^p} = 0$.

The technique in solving the limit at infinity of rational functions is to *divide by the largest power of* x in the rational function and apply the above theorem.

EXAMPLE 2: Compute the limits of the following rational functions.

(a) $\displaystyle\lim_{x\to\infty} \frac{2x+4}{5x+1}$

(b) $\displaystyle\lim_{x\to\infty} \frac{4-x+x^2}{3x^2-2x+7}$

(c) $\displaystyle\lim_{x\to\infty} \frac{20x+1}{3x^3-5x+1}$

(d) $\displaystyle\lim_{x\to\infty} \frac{3x^2+4}{8x-1}$

Solution.

(a) The highest power of x here is x^1. So,

$$\lim_{x\to\infty} \frac{2x+4}{5x+1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \frac{2+\frac{4}{x}}{5+\frac{1}{x}}.$$

Using limit theorems and Theorem 16,

$$\lim_{x\to\infty} \frac{2x+4}{5x+1} = \frac{\lim\limits_{x\to\infty} 2 + 4\lim\limits_{x\to\infty} \frac{1}{x}}{\lim\limits_{x\to\infty} 5 + \lim\limits_{x\to\infty} \frac{1}{x}} = \frac{2+0}{5+0} = \frac{2}{5}.$$

(b) The highest power of x here is x^2. So, by Theorem 16,

$$\lim_{x\to\infty} \frac{4-x+x^2}{3x^2-2x+7} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{4}{x^2}+\frac{1}{x}+1}{3-\frac{2}{x}+\frac{7}{x^2}} = \frac{0-0+1}{3-0+0} = \frac{1}{3}.$$

(c) The highest power of x here is x^3. Again, using Theorem 16,

$$\lim_{x\to\infty} \frac{20x+1}{3x^3-5x+1} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{20}{x^2}+\frac{1}{x^3}}{3-\frac{5}{x^2}+\frac{1}{x^3}} = \frac{0+0}{3-0+0} = 0.$$

(d) The highest power of x here is x^2. So,

$$\lim_{x\to\infty} \frac{3x^2+4}{8x-1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{3+\frac{4}{x^2}}{\frac{8}{x}-\frac{1}{x^2}}.$$

Notice that the numerator approaches 3 but the denominator approaches 0. Therefore, the limit *does not exist*.

Limits at Infinity of Rational Functions

Suppose the degrees of the polynomials $p(x)$ and $q(x)$ are m and n, respectively. There are only three cases that could happen in solving for $\lim\limits_{x\to\infty} \frac{p(x)}{q(x)}$.

- If $m = n$, then $\lim\limits_{x\to\infty} \frac{p(x)}{q(x)}$ exists. In fact, the limit is nonzero and equals the ratio of the leading coefficient of $p(x)$ to the leading coefficient of $q(x)$.
- If $m < n$, then $\lim\limits_{x\to\infty} \frac{p(x)}{q(x)}$ exists and is equal to 0.
- If $m > n$, then $\lim\limits_{x\to\infty} \frac{p(x)}{q(x)}$ DNE.

COMPUTING AREAS USING THE FORMAL DEFINITION OF THE DEFINITE INTEGRAL

The problem with the definition of the definite integral is that it contains a summation, which needs to be evaluated completely before we can apply the limit. For simple summations like $\sum_{k=1}^{n} k^p$, where p is a positive integer, formulas exist which can be verified by the principle of mathematical induction. However, in general, formulas for complex summations are scarce.

The next simple examples illustrate how a definite integral is computed by definition.

EXAMPLE 3: Show that $\displaystyle\int_1^3 3x+1\,dx = 14$ using the definition of the definite integral as a limit of a sum.

Solution. Let us first get the right (the choice of "right" here is arbitrary) Riemann sum of $f(x) = 3x+1$ relative to the regular partition $\mathcal{P}_n$ of $[1,3]$ into n subintervals. Note that $\Delta x = \frac{3-1}{n} = \frac{2}{n}$. Since we are looking for the *right* Riemann sum, the partition points are $x_k = x_0 + k\Delta x = 1 + k\frac{2}{n} = 1 + \frac{2k}{n}$. Thus, by the formula of the right Riemann sum, we have

$$R_n = \sum_{k=1}^{n} f(x_k)\Delta x_k = \sum_{k=1}^{n} (3x_k+1)\Delta x_k.$$

Using the definitions of Δx and x_k above, we obtain

$$R_n = \sum_{k=1}^{n} \left(3\left(1+\frac{2k}{n}\right)+1\right)\frac{2}{n} = \sum_{k=1}^{n}\left(4+\frac{6k}{n}\right)\frac{2}{n} = \sum_{k=1}^{n}\left(\frac{8}{n}+\frac{12k}{n^2}\right).$$

We apply properties of the summation: distributing the summation symbol over the sum, factoring out those which are independent of the index k, and finally, applying the formulas,

$$\sum_{k=1}^{n} 1 = n \quad \text{and} \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

This implies

$$R_n = \frac{8}{n}\sum_{k=1}^{n} 1 + \frac{12}{n^2}\sum_{k=1}^{n} k = \frac{8}{n}\cdot n + \frac{12}{n^2}\cdot\frac{n(n+1)}{2} = 8 + 6\left(1+\frac{1}{n}\right).$$

Finally, by definition

$$\int_1^3 3x+1\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(8+6\left(1+\frac{1}{n}\right)\right).$$

Using Theorem 16, $\displaystyle\lim_{n\to\infty}\frac{1}{n} = 0$. Therefore, it follows that $\displaystyle\int_1^3 3x+1\,dx = 14$, as desired.

For the next example, we need the following formula for the sum of the first n perfect squares:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \tag{3.4}$$

This can be proven by the principle of mathematical induction or by simplifying the telescoping sum $\displaystyle\sum_{k=1}^{n}((k+1)^3 - k^3)$.

EXAMPLE 4: Use the definition of the definite integral as a limit of a Riemann sum to show that $\displaystyle\int_0^1 x^2\,dx = \frac{1}{3}$.

Solution. For convenience, let us again use the right Riemann sum relative to the partition $\mathcal{P}_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$ of $[0,1]$. Clearly, $\Delta x = \frac{1}{n}$ and $x_k = x_0 + k\Delta x = 0 + k \cdot \frac{1}{n} = \frac{k}{n}$. By the formula of the right Riemann sum, we have

$$R_n = \sum_{k=1}^{n} f(x_k)\Delta x_k = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n}.$$

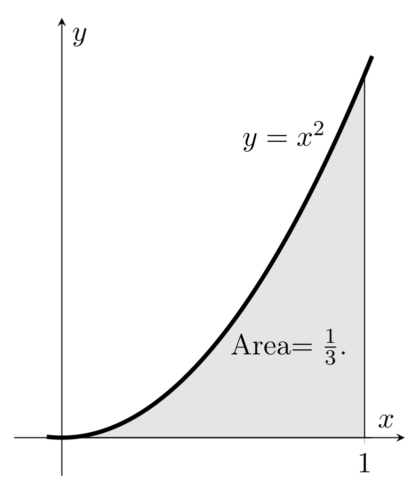
Simplifying, and using formula (3.4), we obtain

$$R_n = \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{1}{n^3} \cdot \frac{2n^3 + 3n^2 + n}{6} = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}.$$

Hence, the definite integral is just the limit of the above expression as n tends to infinity. We use Theorem 16 to evaluate the limits of the last two terms in the expression. Therefore, we have

$$\int_0^1 x^2\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}.$$

This illustrates the power of the definite integral in computing areas of non-polygonal regions. Using the previous remark, the region bounded above by $y = x^2$, below by the x-axis, and at the sides by the vertical lines $x = 0$ and $x = 1$, has an area equal to $\dfrac{1}{3}$.



Teaching Tip

As previously discussed, the computation of definite integrals is, in general, very cumbersome. We just have to wait a little more because a very powerful theorem (Fundamental Theorem of Calculus) in the next section will relate this with the antiderivative/indefinite integral. This connection will make the computation of integrals so much easier than finding the limit of the sum. This will also explain the connection of the antiderivative with areas, as presented in the motivation of the lecture Riemann sums, and the reason why antiderivatives and the definite integral share the same notation ($\int$).

PROPERTIES OF THE DEFINITE INTEGRAL

As a limit of a sum, the definite integral shares the common properties of the limit and of the summation.

Theorem 17 (Linearity of the Definite Integral)**.** *Let f and g be positive continuous functions on $[a,b]$ and let $c \in \mathbb{R}$. Then*

(a) $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

(b) $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Teaching Tip

Although the limit is distributive over products and quotients, the summation is **not**. Therefore, the integral is also **NOT** distributive over products nor over quotients. This means

$$\int_a^b f(x)g(x)\,dx \neq \int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$$

and

$$\int_a^b \frac{f(x)}{g(x)}\,dx \neq \frac{\int_a^b f(x)\,dx}{\int_a^b g(x)\,dx}.$$

EXAMPLE 5: Suppose we are given that $\displaystyle\int_a^b f(x)\,dx = 2$ and $\displaystyle\int_a^b g(x)\,dx = 7$. Find the exact value of the following:

(a) $\displaystyle\int_a^b 3f(x)\,dx$

(b) $\displaystyle\int_a^b f(x) - g(x)\,dx$

(c) $\displaystyle\int_a^b 2f(x) + g(x)\,dx$

(d) $\displaystyle\int_a^b 3f(x) - 2g(x)\,dx$

Solution. Using the properties of the integral,

(a) $\displaystyle\int_a^b 3f(x)\,dx = 3\int_a^b f(x)\,dx = 3(2) = 6.$

(b) $\displaystyle\int_a^b f(x) - g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = 2 - 7 = -5.$

(c) $\displaystyle\int_a^b 2f(x) + g(x)\,dx = 2\int_a^b f(x) + \int_a^b g(x)\,dx = 2(2) + 7 = 11.$

(d) $\displaystyle\int_a^b 3f(x) - 2g(x)\,dx = 3\int_a^b f(x)\,dx - 2\int_a^b g(x)\,dx = 3(2) - 2(7) = -8.$

Another important property of the definite integral is called *additivity*. Before we proceed, we first define the following:

Definition 7. *Let f be a continuous positive function on $[a, b]$. Then*

(a) $\displaystyle\int_a^a f(x)\,dx = 0,$ *and*

(b) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$

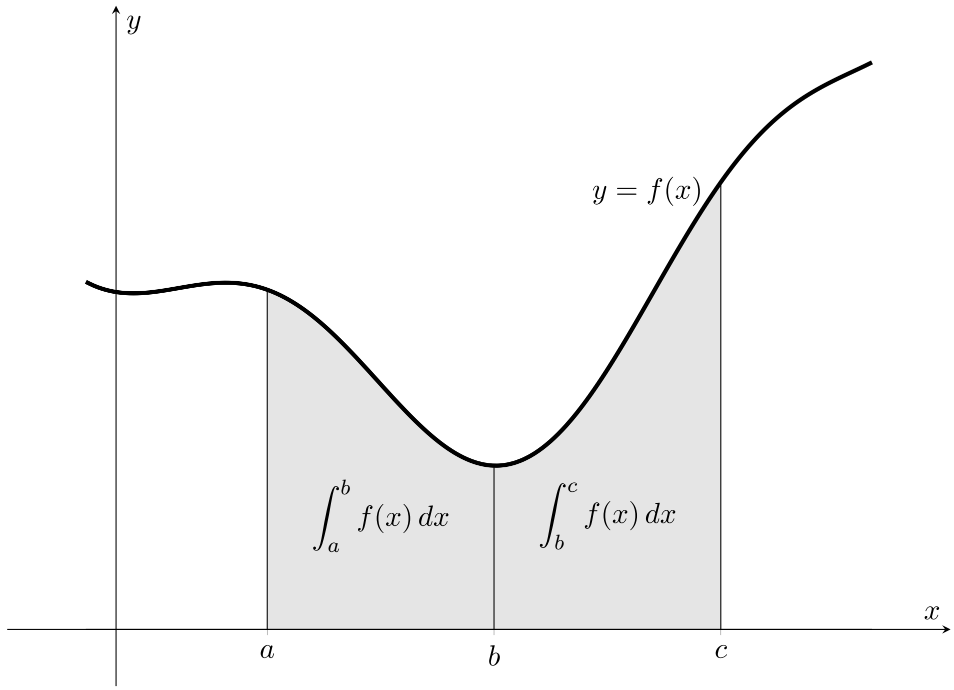
The first one is very intuitive if you visualize the definite integral as an area of a region. Since the left and right boundaries are the same ($x = a$), then there is no region and the area therefore is 0 . The second one gives meaning to a definite integral whenever the lower limit of integration is bigger than the upper limit of integration. We will see later that this is needed so the property of additivity will be consistent with our intuitive notion.

Theorem 18 (Additivity of the Definite Integral)**.** *Let f be a positive continuous function on a closed and bounded interval I containing distinct numbers a, b and c. Then*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

no matter how a, b and c are ordered on the interval I.

In the case when $a < b < c$, the theorem can be visualized as follows: If we interpret the definite integral as the area under the curve $y = f(x)$, then the theorem certainly makes sense. However, the rigorous proof of the theorem requires advanced techniques in analysis and will not be presented here.



If we accept the claim as true for the case when $a < b < c$, we can confirm the validity of the theorem in 5 more cases: $a < c < b,\ b < a < c,\ b < c < a,\ c < a < b,\ c < b < a$. We shall only show this for the case $b < c < a$ since the other cases are proven similarly.

Since $b < c < a$, then it is clear from the geometric interpretation of the definite integral that

$$\int_b^c f(x)\,dx + \int_c^a f(x)\,dx = \int_b^a f(x)\,dx.$$

Using the second formula in the above definition,

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \quad \text{and} \quad \int_c^a f(x)\,dx = -\int_a^c f(x)\,dx.$$

Substituting these into the above equation yields

$$\int_b^c f(x)\,dx - \int_a^c f(x)\,dx = -\int_a^b f(x)\,dx,$$

which when rearranged gives the desired result.

EXAMPLE 6: Suppose that $\int_0^2 f(x)\,dx = I$ and $\int_0^1 f(x)\,dx = J$. Find $\int_1^2 f(x)\,dx$.

Solution. By additivity,

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx.$$

Substituting the given values yields

$$I = \int_1^2 f(x)\,dx + J.$$

This implies that $\int_1^2 f(x)\,dx = I - J$.

EXAMPLE 7: Suppose we are given that $\int_a^b f(x)\,dx = 3$, $\int_d^c f(x) = 10$ and $\int_d^b f(x)\,dx \;=\; 4$. Find $\int_c^a f(x)\,dx$.

Solution. By additivity,

$$\begin{aligned}\int_c^a f(x)\,dx &= \int_c^d f(x)\,dx + \int_d^b f(x)\,dx + \int_b^a f(x)\,dx \\ &= -\int_d^c f(x)\,dx + \int_d^b f(x)\,dx - \int_a^b f(x)\,dx \\ &= -10 + 4 - 3 = -9.\end{aligned}$$

THE DEFINITE INTEGRAL AS A NET SIGNED AREA

We always assumed that the function that we are considering is always positive. What happens if the function has a negative part? How do we interpret this geometrically?

General Geometric Interpretation of a Definite Integral

Suppose that f is a continuous function on $[a, b]$. (Notice that we dropped the assumption that f must be positive.) Then the definite integral

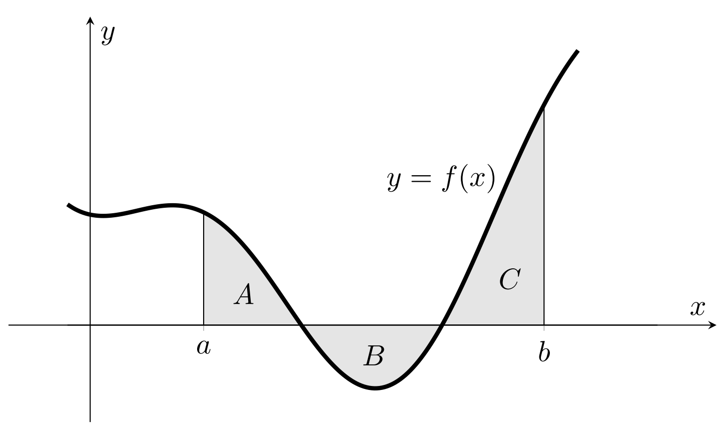
$$\int_a^b f(x)\,dx$$

is the net signed area of the region with boundaries $y = f(x)$, $x = a$, $x = b$, and the x-axis.

The *net signed* area equals the sum of all the areas above the x-axis minus the sum of all the areas below the x-axis. In effect, we are associating positive areas for the regions above the x-axis and negative areas for the regions below the x-axis.

For example, consider the following graph of $y = f(x)$ on $[a, b]$. If the areas of the shaded regions are A, B and C, as shown, then

$$\int_a^b f(x)\,dx = A + C - B.$$



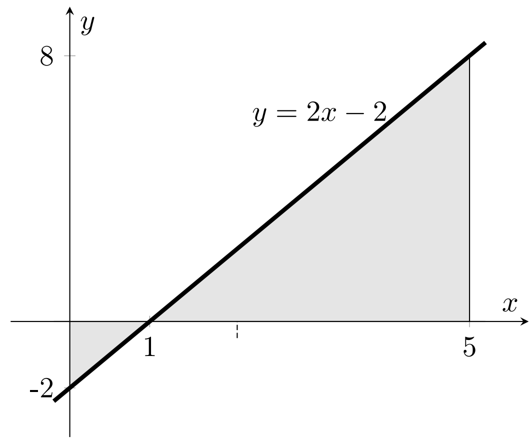
EXAMPLE 8: Interpret the following integrals as signed areas to obtain its value.

(a) $\displaystyle\int_0^5 2x - 2\,dx$ (b) $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x\,dx$

Solution. (a) The graph of $y = 2x - 2$ is shown on the right.

From 0 to 1, the area of the triangle (below the x-axis) is $\frac{1}{2}(1)(2) = 1$. From 1 to 5, the area of the triangle (above the x-axis) is $\dfrac{1}{2}(4)(8) = 16$. Therefore, the net signed area equals

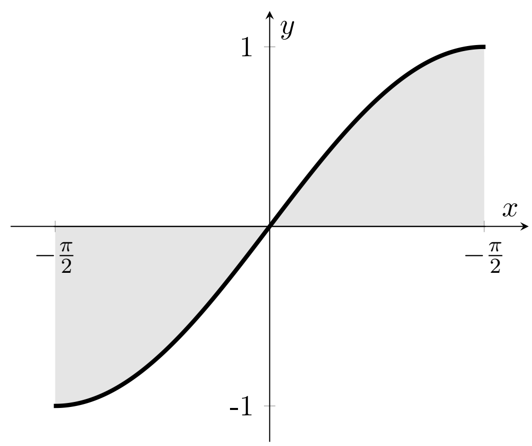
$$\int_0^5 2x - 2\,dx = 16 - 1 = 15.$$



(b) The graph of $y = \sin x$ is shown on the right.

Observe that because $y = \sin x$ is symmetric with respect to the origin, the region (below the x-axis) from $-\pi/2$ to 0 is congruent to the region (above the x-axis) from 0 to $\pi/2$. Therefore, the net signed area equals

$$\int_{-\pi/2}^{\pi/2} \sin x\,dx = 0.$$



(C) EXERCISES

1. Explain why, for an even function $f(x)$, we have $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.
2. Explain why, for an odd function $f(x)$, we have $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.
3. Evaluate the following integrals by considering the areas they represent.

a. $\displaystyle\int_1^1 x^3 \sin x\,dx$

c. $\displaystyle\int_{-1}^{1} x^3\,dx$

b. $\displaystyle\int_{-5}^{-5} \sqrt{1-x}\,dx$

d. $\displaystyle\int_{-4}^{4} x^7 + x^5 - 3x\,dx$

e. $\displaystyle\int_{-\pi/4}^{\pi/4} \tan x\,dx$

f. $\displaystyle\int_{-2}^{2} \frac{x}{2+x^2-\cos x}\,dx$

g. $\displaystyle\int_{-2}^{1} 5\,dx$

h. $\displaystyle\int_{a}^{b} c\,dx$

i. $\displaystyle\int_{0}^{3} 2x\,dx$

j. $\displaystyle\int_{2}^{5} 7-3x\,dx$

k. $\displaystyle\int_{-1}^{3} 3x+2\,dx$

l. $\displaystyle\int_{-4}^{-1} 1-x\,dx$

m. $\displaystyle\int_{-1}^{4} |x|\,dx$

n. $\displaystyle\int_{0}^{4} 1-|x|\,dx$

o. $\displaystyle\int_{-1}^{5} 2-|x-3|\,dx$

p. $\displaystyle\int_{3}^{1} |x-2|\,dx$

q. $\displaystyle\int_{0}^{2} \sqrt{4-x^2}\,dx$

r. $\displaystyle\int_{0}^{2} \sqrt{1-(x-1)^2}\,dx$

s. $\displaystyle\int_{0}^{2} 3-\sqrt{4-x^2}\,dx$

t. $\displaystyle\int_{1}^{2} 2+\sqrt{2x-x^2}\,dx$

4. Suppose that $\displaystyle\int_1^4 f(x)\,dx = 4$ and $\displaystyle\int_4^7 f(x)\,dx$. What is the value of $\displaystyle\int_1^7 f(x)\,dx$? Assume further that $\displaystyle\int_4^0 f(x)\,dx = 3$. What is the value of $\displaystyle\int_0^1 f(x)\,dx$?

5. Define
$$P(x) = \int_0^x f(x)\,dx.$$
Assuming that $P(1) = 3$, $P(2) = 5$, and $P(3) = 6$, determine the values of

(i) $P(0)$ (ii) $\displaystyle\int_1^2 f(x)\,dx$ (iii) $\displaystyle\int_1^3 f(x)\,dx$?

6. Given that $\displaystyle\int_0^1 x^2\,dx = \frac{1}{3}$, evaluate $\displaystyle\int_{-1}^1 1+2x+3x^2+4x^3\,dx$.

7. Use the definition of the definite integral as a limit of a Riemann sum to find the area of the region bounded by $y = x^2$, $y = 0$, $x = 1$ and $x = 2$.

⋆8. Prove by mathematical induction that $\displaystyle\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}$ for all $n \in \mathbb{N}$. Use the definition of the definite integral as a limit of a Riemann sum to show that $\displaystyle\int_0^1 x^3\,dx = \frac{1}{4}$.

<u>Answer to starred exercise</u>

8. We prove that
$$\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4} \tag{3.5}$$
for all $n \in \mathbb{N}$. The base case is trivial. Suppose that m is a positive integer such that

$\sum_{k=1}^{m} k^3 = \frac{m^2(m+1)^2}{4}$. Then

$$\sum_{k=1}^{m+1} k^3 = \sum_{k=1}^{m} k^3 + (m+1)^3 = \frac{m^2(m+1)^2}{4} + (m+1)^3$$
$$= (m+1)^2\left(\frac{m^2}{4} + (m+1)\right) = \frac{(m+1)^2(m+2)^2}{4}.$$

This proves the inductive step. Hence the formula (3.5) is true for all positive integers n.

Let us use the right Riemann sum relative to the partition $\mathcal{P}_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$ of $[0,1]$. Clearly, $\Delta x = \frac{1}{n}$, and $x_k = x_0 + k\Delta x = 0 + k \cdot \frac{1}{n} = \frac{k}{n}$. Now, by the formula of the right Riemann sum, we have

$$R_n = \sum_{k=1}^{n} f(x_k)\Delta x_k = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^3 \frac{1}{n}.$$

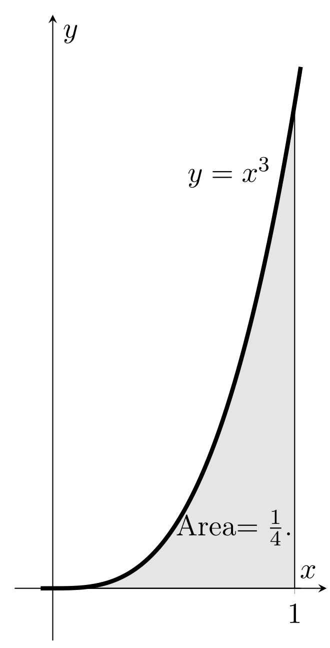
Simplifying, and using the above formula (3.5), we obtain

$$R_n = \frac{1}{n^4}\sum_{k=1}^{n} k^3 = \frac{1}{n^4} \cdot \frac{n^2(n+1)^2}{4} = \frac{1}{n^4} \cdot \frac{n^4 + 2n^3 + n^2}{4} = \frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}.$$

Finally, the definite integral is just the limit of the above expression as n tends to infinity.

$$\int_0^1 x^3\,dx = \lim_{n\to\infty} R_n$$
$$= \lim_{n\to\infty}\left(\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}\right) = \frac{1}{4}.$$

The region bounded above by $y = x^3$, below by the x-axis, and at the sides by the vertical lines $x = 0$ and $x = 1$, has area equal to $\frac{1}{4}$.



LESSON 16: The Fundamental Theorem of Calculus

TIME FRAME: 2 hours

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Illustrate the Fundamental Theorem of Calculus; and
2. Compute the definite integral of a function using the Fundamental Theorem of Calculus.

LESSON OUTLINE:

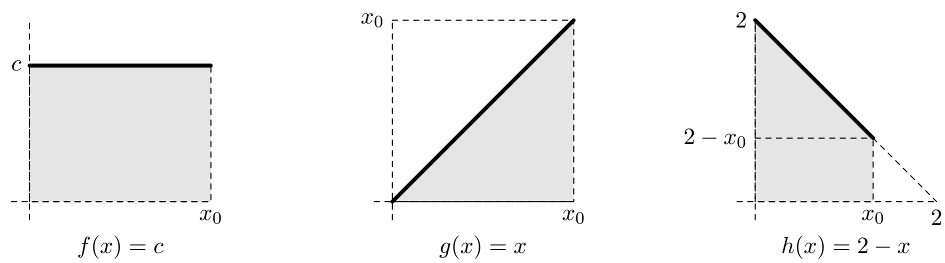
1. Statement of the theorem
2. Examples
3. Revisiting the Table of Integrals
4. Integrands with absolute values

TOPIC 16.1: Illustration of the Fundamental Theorem of Calculus

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

If you review the past lessons, you will see that the only similarity between definite and indefinite integrals is their use of the notation "$\int$". The indefinite integral is the inverse process of differentiation while the definite integral is the process of finding the area of a plane region by taking the limit of a sum. The only vague connection we have established is the relationship between the antiderivative of a function and the area of the region below its curve:



Then,

Function	Antiderivative	Area of shaded region
$f(x) = c$	cx	cx_0
$g(x) = x$	$\frac{1}{2}x^2$	$\frac{1}{2}x_0 \cdot x_0$
$h(x) = 2 - x$	$2x - \frac{1}{2}x^2$	$\frac{1}{2}(2 - x_0 + 2)(x_0)$

The connection, in fact, lies in the powerful result which forms the basis of the theory of Riemann integration – the Fundamental Theorem of Calculus. It stresses the inverse relationship between differentiation and integration. Very loosely, the theorem says that the integral of the derivative of a function returns the same function.

(B) LESSON PROPER

Fundamental Theorem of Calculus (FTOC)
Let f be a continuous function on $[a, b]$ and let F be an antiderivative of f, that is, $F'(x) = f(x)$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

EXAMPLE 1: Note that $F(x)=\dfrac{x^3}{3}$ is an antiderivative of $f(x)=x^2$ (since $F'(x)=f(x)$.) Hence, by FTOC,

$$\int_0^1 x^2\,dx=\frac{x^3}{3}\bigg|_0^1=F(1)-F(0)=\frac{1}{3}-0=\frac{1}{3},$$

as we have seen before.

Vertical Bar Notation

We adopt the following notation:

$$F(x)\bigg|_a^b=F(b)-F(a).$$

For example,

$$(1+x-x^2)\bigg|_1^2=(1+2-2^2)-(1+1-1^2)=(-1)-(1)=-2,$$

and

$$\sin x\bigg|_{\pi/4}^{\pi/2}=\sin(\pi/2)-\sin(\pi/4)=1-\frac{\sqrt{2}}{2}=\frac{2-\sqrt{2}}{2}.$$

Using the above notation, the FTOC now states: If F is an antiderivative of f, then

$$\int_a^b f(x)\,dx=F(x)\bigg|_a^b.$$

The constant of integration that was necessary for indefinite integration will now just cancel out:

$$\big(F(x)+C\big)\bigg|_a^b=\big(F(b)+C\big)-\big(F(a)+C\big)=F(b)-F(a)=F(x)\bigg|_a^b.$$

The next examples will validate that FTOC works by redoing the examples in the previous section.

EXAMPLE 2: Without referring to the graphs of the integrands, find the exact values of the following definite integrals:

1. $\displaystyle\int_1^2 3\,dx$
2. $\displaystyle\int_0^2 (1-|x-1|)\,dx$
3. $\displaystyle\int_1^3 (3x+1)\,dx$

$\star$4. $\displaystyle\int_{-1}^1 \sqrt{1-x^2}\,dx$

Solution. We integrate using the Fundamental Theorem of Calculus.

1. $\displaystyle\int_1^2 3\,dx = 3x\Big|_1^2 = 3(2-1) = 3.$
2. The solution for this problem takes a few more steps because the absolute value function is essentially a piecewise function. Recall that if E is any expression, then $|E| = E$ if $E \geq 0$, while $|E| = -E$ if $E < 0$. With this in mind,

$$\begin{aligned} 1 - |x-1| &= \begin{cases} 1-(x-1) & \text{if } x-1 \geq 0 \\ 1-[-(x-1)] & \text{if } x-1 < 0 \end{cases} \\ &= \begin{cases} 2-x & \text{if } x \geq 1 \\ x & \text{if } x < 1. \end{cases} \end{aligned}$$

   Therefore, by additivity,

$$\begin{aligned} \int_0^2 (1-|x-1|)\,dx &= \int_1^2 2 - x\,dx + \int_0^1 x\,dx \\ &= \left(2x - \frac{x^2}{2}\right)\Bigg|_1^2 + \left(\frac{x^2}{2}\right)\Bigg|_0^1 \\ &= \left[\left(4 - \frac{4}{2}\right) - \left(2 - \frac{1}{2}\right)\right] + \left[\frac{1}{2} - 0\right] = 1. \end{aligned}$$

3. $\displaystyle\int_1^3 (3x+1)\,dx = \left(\frac{3x^2}{2} + x\right)\Bigg|_1^3 = \left(\frac{27}{2} + 3\right) - \left(\frac{3}{2} + 1\right) = 14.$
4. (For very advanced learners) There is a technique of integration needed to integrate $\sqrt{1-x^2}$. This is called ***trigonometric substitution***, and the student will learn this in college. For now, we convince ourselves that

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\sin^{-1} x + x\sqrt{1-x^2}\right) + C$$

   by differentiating the right-hand side and observing that it yields the integrand of the left-hand side. Hence, by FTOC,

$$\int_{-1}^1 \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\sin^{-1} x + x\sqrt{1-x^2}\right)\Bigg|_{-1}^1 = \frac{1}{2}\left(\frac{\pi}{2} + 0\right) - \frac{1}{2}\left(-\frac{\pi}{2} - 0\right) = \frac{\pi}{2}.$$

Because, FTOC evaluates the definite integral using antiderivatives and not by "net signed area," we don't have to look at the graph and look at those regions below the x-axis, i.e., FTOC works even if the graph has a "negative part."

EXAMPLE 3: Evaluate the following integrals using FTOC.

(a) $\displaystyle\int_0^5 (2x-2)\,dx$ (b) $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x\,dx$

Solution.

(a) $\displaystyle\int_0^5 (2x-2)\,dx = \left(2\frac{x^2}{2} - 2x\right)\Bigg|_0^5 = (25-10)-(0-0) = 15.$

(b) $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x\,dx = (-\cos x)\Big|_{-\pi/2}^{\pi/2} = (0-0) = 0.$

These answers are the same as when we appealed to the geometrical solution of the integral in the previous section.

TOPIC 16.2: Computation of Definite Integrals using the Fundamental Theorem of Calculus

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

In the previous section, we illustrated how the Fundamental Theorem of Calculus works. If f is a continuous function on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b)\Big|_a^b = F(b) - F(a),$$

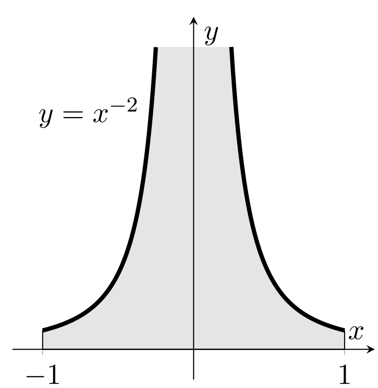
where $f(x) = F'(x)$.

For this lesson, we start with a remark about the applicability of the FTOC and proceed with answering some exercises, either individually or by group.

Remark 1: If the function is not continuous on its interval of integration, the FTOC will not guarantee a correct answer. For example, we know that an antiderivative of $f(x) = x^{-2}$ is $F(x) = -x^{-1}$. So, if we apply the FTOC to f on $[-1, 1]$, we get

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{-1}^{1} = (-1 - (+1)) = -2.$$

This is absurd as the region described by $\displaystyle\int_{-1}^{1} x^{-2}\,dx$ is given below:



Clearly, the area should be positive. The study of definite integrals of functions which are discontinuous on an interval (which may not be closed nor bounded) is called *improper integration* and will be studied in college.

(B) **LESSON PROPER**

Table of Integrals

Observe that for FTOC to work, the student must be able to produce an antiderivative for the integrand. This is why the student must be comfortable with the first few lessons in this chapter. A very common and indispensable formula is the Power Rule:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, n \neq -1.$$

For other cases, we recall the table of integrals for reference.

1. $\displaystyle\int dx = x + C$
2. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad \text{if } n \neq -1$
3. $\displaystyle\int af(x)\;dx = a\int f(x)\,dx$
4. $\displaystyle\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
5. $\displaystyle\int e^x\,dx = e^x + C.$
6. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C.$
7. $\displaystyle\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C.$
8. $\displaystyle\int \sin x\,dx = -\cos x + C$
9. $\displaystyle\int \cos x\,dx = \sin x + C$
10. $\displaystyle\int \sec^2 x\,dx = \tan x + C$
11. $\displaystyle\int \csc^2 x\,dx = -\cot x + C$
12. $\displaystyle\int \sec x \tan x\,dx = \sec x + C$
13. $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$
14. $\displaystyle\int \tan x\,dx = -\ln|\cos x| + C$
15. $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$
16. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$
17. $\displaystyle\int \csc x\,dx = \ln|\csc x - \cot x| + C$
18. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$

EXAMPLE 1: Using FTOC, evaluate the following definite integrals:

1. $\displaystyle\int_1^4 \sqrt{x}\,dx$
2. $\displaystyle\int_1^2 \frac{x^3 - 2x^2 + 4x - 2}{x}\,dx$
3. $\displaystyle\int_0^{\pi/4} \cos x + \tan x\,dx$
4. $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{1 - x^2}}$

Solution.

1. $\displaystyle\int_1^4 \sqrt{x}\,dx = \int_1^4 x^{1/2}\,dx = \left.\frac{x^{3/2}}{3/2}\right|_1^4 = \frac{2}{3}(8 - 1) = \frac{14}{3}.$

2. We first divide the numerator by the denominator to express the fraction as a sum.

$$\begin{aligned}\int_1^2 \frac{x^3 - 2x^2 + 4x - 2}{x}\,dx &= \int_1^2 \left(x^2 - 2x + 4 - \frac{2}{x}\right)dx \\ &= \left(\frac{x^3}{3} - x^2 + 4x - 2\ln|x|\right)\Bigg|_1^2 \\ &= \left(\frac{8}{3} - 4 + 8 - 2\ln 2\right) - \left(\frac{1}{3} - 1 + 4 - 0\right) \\ &= \frac{10}{3} - 2\ln 2.\end{aligned}$$

3. $\displaystyle\int_0^{\pi/4} \cos x + \tan x\,dx = (\sin x + \ln|\sec x|)\Bigg|_0^{\pi/4} = \frac{\sqrt{2}}{2} + \ln(\sqrt{2})$

4. $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\Bigg|_0^{1/2} = \sin^{-1}(1/2) - \sin^{-1}(0) = \frac{\pi}{6} - 0 = \frac{\pi}{6}.$

INTEGRANDS WITH ABSOLUTE VALUES

Solving definite integrals with absolute values in the integrands has been discussed in passing in previous examples. We will now give a more in-depth discussion.

As has been said, for any continuous expression E, its absolute value can always be written in piecewise form:

$$|E| = \begin{cases} E & \text{if } E \geq 0 \\ -E & \text{if } E < 0. \end{cases}$$

Therefore, the first step in solving this kind of integral is to eliminate the absolute value bars. This is done by dividing the interval of integration $[a, b]$ into two subintervals according to the piecewise version of the function. Hence, one integrand is either purely positive (or zero) and the other is purely negative.

EXAMPLE 2: Evaluate the following definite integrals:

1. $\displaystyle\int_{-1}^2 |x-3|\,dx$
2. $\displaystyle\int_3^7 |x-3|\,dx$
3. $\displaystyle\int_1^4 |x-3|\,dx$
4. $\displaystyle\int_0^2 4x + |2x-1|\,dx$
5. $\displaystyle\int_0^{2\pi/3} |\cos x|\,dx$
6. $\displaystyle\int_{-2}^3 |x^2-1|\,dx$

Solution. For items 1-3, observe that by definition,

$$|x-3| = \begin{cases} x-3, & \text{if } x-3 \geq 0 \\ -(x-3), & \text{if } x-3 < 0, \end{cases} = \begin{cases} x-3, & \text{if } x \geq 3, \\ -x+3, & \text{if } x < 3. \end{cases}$$

1. Since $x-3$ is always negative on the interval of integration $[-1,2]$, we replace $|x-3|$ with $-(x-3)$. Hence,

$$\int_{-1}^{2} |x-3|\,dx = \int_{-1}^{3} -x+3\,dx = \left(-\frac{x^2}{2}+3x\right)\Bigg|_{-1}^{2} = \left(-\frac{4}{2}+6\right)-\left(-\frac{1}{2}-3\right) = \frac{15}{2}.$$

2. Since $x-3$ is always nonnegative on $[3,7]$, we replace $|x-3|$ with $x-3$. So,

$$\int_{3}^{7} |x-3|\,dx = \int_{3}^{7} x-3\,dx = \left(\frac{x^2}{2}-3x\right)\Bigg|_{3}^{7} = \left(\frac{49}{2}-21\right)-\left(\frac{9}{2}-9\right) = 8.$$

3. Since $x-3$ is neither purely positive nor purely negative on $[1,4]$, we need to divide this interval into $[1,3]$ and $[3,4]$. On the first, we replace $|x-3|$ with $-x+3$, while on the second, we replace $|x-3|$ with $x-3$.

$$\begin{aligned}\int_{1}^{4} |x-3|\,dx &= \int_{1}^{3} -x+3\,dx + \int_{3}^{4} x-3\,dx \\ &= \left(-\frac{x^2}{2}+3x\right)\Bigg|_{1}^{3} + \left(\frac{x^2}{2}-3x\right)\Bigg|_{3}^{4} \\ &= \left[\left(-\frac{9}{2}+9\right)-\left(-\frac{1}{2}+3\right)\right] + \left[\left(\frac{16}{2}-12\right)-\left(\frac{9}{2}-9\right)\right] \\ &= \left[\frac{9}{2}-\frac{5}{2}\right] + \left[-\frac{8}{2}+\frac{9}{2}\right] = \frac{5}{2}.\end{aligned}$$

4. We split the interval of integration into $[0,\frac{1}{2}]$ and $[\frac{1}{2},2]$ since

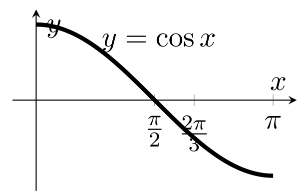
$$|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1 \geq 0 \\ -(2x-1) & \text{if } 2x-1 < 0. \end{cases}$$

Hence, $4x+|2x-1| = \begin{cases} 4x+(2x-1) & \text{if } x \geq \frac{1}{2} \\ 4x-(2x-1) & \text{if } x < \frac{1}{2}. \end{cases}$

Therefore,

$$\begin{aligned}\int_{0}^{2} 4x+|2x-1|\,dx &= \int_{0}^{\frac{1}{2}} 2x+1\,dx + \int_{\frac{1}{2}}^{2} 6x-1\,dx \\ &= (x^2+x)\Big|_{0}^{1/2} + (3x^2-x)\Big|_{1/2}^{2} \\ &= \left[\left(\frac{1}{4}+\frac{1}{2}\right)-(0+0)\right] + \left[(12-2)-\left(\frac{3}{4}-\frac{1}{2}\right)\right] = \frac{21}{2}.\end{aligned}$$

5. The cosine function is nonnegative on $[0, \pi/2]$ and negative on $[\pi/2, 2\pi/3]$. See graph below.

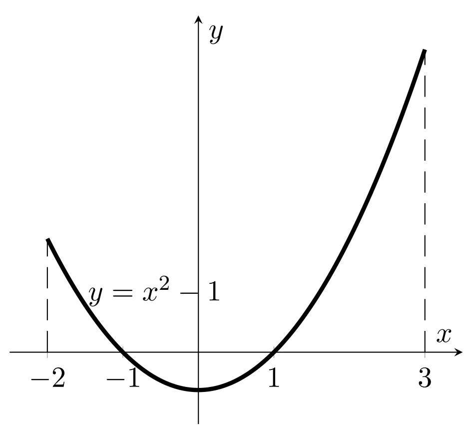


Hence, $|\cos x| = \begin{cases} \cos x, & \text{if } \cos x \geq 0, \\ -\cos x, & \text{if } \cos x < 0 \end{cases} = \begin{cases} \cos x, & \text{if } x \in [0, \pi/2], \\ -\cos x, & \text{if } x \in [\pi/2, 2\pi/3]. \end{cases}$

Therefore,

$$\begin{aligned}
\int_0^{2\pi/3} |\cos x|\, dx &= \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{2\pi/3} -\cos x\, dx \\
&= \sin x \Big|_0^{\pi/2} + (-\sin x)\Big|_{\pi/2}^{2\pi/3} \\
&= (1-0) + [(-\sqrt{3}/2) - (-1)] = 2 - \frac{\sqrt{3}}{2}.
\end{aligned}$$

6. The quadratic function $y = x^2 - 1$ is below the x-axis only when $x \in (-1, 1)$. This conclusion can be deduced by solving the inequality $x^2 - 1 < 0$ or by referring to the graph below.



This implies that we have to divide $[-2, 3]$ into the three subintervals $[-2, -1]$, $[-1, 1]$, and $[1, 3]$. On $[-1, 1]$, we replace $|x^2 - 1|$ with $-x^2 + 1$ while on the other subintervals, we simply replace $|x^2-1|$ with x^2-1. Hence,

$$\begin{aligned}
\int_{-2}^{3}|x^2-1|\,dx &= \int_{-2}^{-1} x^2-1\,dx + \int_{-1}^{1} -x^2+1\,dx + \int_{1}^{3} x^2-1\,dx \\
&= \left(\frac{x^3}{3}-x\right)\Big|_{-2}^{-1} + \left(-\frac{x^3}{3}+x\right)\Big|_{-1}^{1} + \left(\frac{x^3}{3}-x\right)\Big|_{1}^{3} \\
&= \left[\left(-\frac{1}{3}+1\right)-\left(-\frac{8}{3}+2\right)\right] + \left[\left(-\frac{1}{3}+1\right)-\left(\frac{1}{3}-1\right)\right] \\
&\quad + \left[\left(\frac{27}{3}-3\right)-\left(\frac{1}{3}-1\right)\right] \\
&= \frac{28}{3}.
\end{aligned}$$

(C) EXERCISES

Evaluate the following definite integrals:

1. $\displaystyle\int_0^{\frac{\pi}{2}} \sin x\,dx$
2. $\displaystyle\int_0^4 (1+3x-x^2)\,dx$
3. $\displaystyle\int_1^{64} \frac{1+\sqrt[3]{x}}{\sqrt[3]{x}}\,dx$
4. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1+\cos^2 x}{\cos^2 x}\,dx$

⋆5. $\displaystyle\int_{\frac{\pi}{3}}^{\pi} \sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\,dx$

⋆6. $\displaystyle\int_{\frac{2\pi}{3}}^{\pi} \frac{\tan x}{\sec x-\tan x}\,dx$

7. $\displaystyle\int_{-1}^{0} |3x+1\,|dx$
8. $\displaystyle\int_0^3 (|3x-6|-1)\,dx$

⋆9. $\displaystyle\int_0^4 |x^2-4x+3|\,dx$

⋆10. $\displaystyle\int_0^{\pi} \left|\sin x-\frac{1}{2}\right|\,dx$

11. $\displaystyle\int_1^2 \frac{1}{x^2}\left(1+\frac{1}{x}\right)^2\,dx$
12. $\displaystyle\int_{-\pi}^{\pi} f(x)\,dx$ if $f(x)=\begin{cases} x & \text{if } x\le 0 \\ \sin x & \text{if } x>0. \end{cases}$

Solutions to starred exercises

5. Observe that by the identity

$$\sin 2\theta = 2\sin\theta\cos\theta,$$

we can rewrite the integrand into $\sin(x/2)\cos(x/2)=\frac{1}{2}\sin x$. Hence,

$$\int_{\frac{\pi}{3}}^{\pi} \sin(x/2)\cos(x/2)\,dx = \int_{\frac{\pi}{3}}^{\pi} \frac{1}{2}\sin x\,dx.$$

Since $-\frac{1}{2}\cos x$ is an antiderivative for the integrand, FTOC says that

$$\int_{\frac{\pi}{3}}^{\pi} \frac{1}{2}\sin x\,dx = -\frac{1}{2}\cos x\Big|_{\frac{\pi}{3}}^{\pi} = -\frac{1}{2}\left(-1-\frac{1}{2}\right) = \frac{3}{4}.$$

6. We use trigonometric identities to rewrite the integrand.

$$\begin{aligned}\frac{\tan x}{\sec x - \tan x}\cdot\frac{\sec x+\tan x}{\sec x+\tan x} &= \frac{\sec x\tan x+\tan^2 x}{\sec^2 x-\tan^2 x}\\ &= \sec x\tan x+\tan^2 x = \sec x\tan x+\sec^2 x-1.\end{aligned}$$

Note that $\sec x+\tan x-x$ is an antiderivative of the integrand. Hence, by FTOC,

$$\begin{aligned}\int_{\frac{2\pi}{3}}^{\pi}\frac{\tan x}{\sec x-\tan x}\,dx &= (\sec x+\tan x-x)\Big|_{\frac{2\pi}{3}}^{\pi}\\ &= (-1-0-\pi)-(-2-\sqrt{3}-(2\pi)/3) = 1+\sqrt{3}-\pi/3.\end{aligned}$$

9. Since $x^2-4x+3=(x-1)(x-3)$, a table of signs will show that this expression is negative when $x\in(1,3)$ and nonnegative otherwise. Therefore, we have to split the integral over three intervals: $[0,1]$, $[1,3]$ and $[3,4]$. Using the definition of the absolute value,

$$\begin{aligned}\int_0^4 |x^2-4x+3|\,dx &= \int_0^1 |x^2-4x+3|\,dx+\int_1^3 |x^2-4x+3|\,dx+\int_3^4 |x^2-4x+3|\,dx\\ &= \int_0^1 (x^2-4x+3)\,dx+\int_1^3 -(x^2-4x+3)\,dx+\int_3^4 (x^2-4x+3)\,dx\\ &= \left(\frac{x^3}{3}-2x^2+3x\right)\Bigg|_0^4-\left(\frac{x^3}{3}-2x^2+3x\right)\Bigg|_1^3+\left(\frac{x^3}{3}-2x^2+3x\right)\Bigg|_3^4\\ &= \left(\frac{4}{3}-0\right)-\left(0-\frac{4}{3}\right)+\left(\frac{4}{3}-0\right)=4.\end{aligned}$$

10. On $[0,\pi]$, $\sin x-\dfrac{1}{2}\ge 0 \Leftrightarrow x\in\left[\dfrac{\pi}{6},\dfrac{5\pi}{6}\right]$.Therefore,

$$\begin{aligned}\int_0^{\pi}\left|\sin x-\frac{1}{2}\right|dx &= \int_0^{\frac{\pi}{6}}\left|\sin x-\frac{1}{2}\right|dx+\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\left|\sin x-\frac{1}{2}\right|dx+\int_{\frac{5\pi}{6}}^{\pi}\left|\sin x-\frac{1}{2}\right|dx\\ &= \int_0^{\frac{\pi}{6}}-\left(\sin x-\frac{1}{2}\right)dx+\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\left(\sin x-\frac{1}{2}\right)dx+\int_{\frac{5\pi}{6}}^{\pi}-\left(\sin x-\frac{1}{2}\right)dx\\ &= \left(\cos x+\frac{x}{2}\right)\Bigg|_0^{\frac{\pi}{6}}+\left(-\cos x-\frac{x}{2}\right)\Bigg|_{\frac{\pi}{6}}^{\frac{5\pi}{6}}+\left(\cos x+\frac{x}{2}\right)\Bigg|_{\frac{5\pi}{6}}^{\pi}\\ &= \left(\left(\frac{\sqrt{3}}{2}+\frac{\pi}{12}\right)-1\right)+\left(\left(\frac{\sqrt{3}}{2}-\frac{5\pi}{12}\right)-\left(-\frac{\sqrt{3}}{2}-\frac{\pi}{12}\right)\right)\\ &\qquad +\left(\left(-1+\frac{\pi}{2}\right)-\left(-\frac{\sqrt{3}}{2}+\frac{5\pi}{12}\right)\right)\\ &= 2\sqrt{3}-2-\frac{\pi}{6}.\end{aligned}$$

LESSON 17: Integration Technique: The Substitution Rule for Definite Integrals

TIME FRAME: 3 hours

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Illustrate the substitution rule; and
2. Compute the definite integral of a function using the substitution rule

LESSON OUTLINE:

1. Review of the substitution rule in solving indefinite integrals
2. Two ways of solving a definite integral through substitution
3. The Substitution Rule [1]

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[1] for Definite Integrals (SRDI)

TOPIC 17.1: Illustration of the Substitution Rule for Definite Integrals

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

As mentioned before, a variety of integrals, especially those whose integrands are not immediately integrable, become easily solvable after we apply the substitution rule. In this section, we illustrate how the substitution rule is still applicable in definite integration. That is, if in a definite integral, the integrand is not integrable at first sight, one must check whether a substitution is possible. Just like in the indefinite case, the goal is for the substitution to eventually yield an integrable version of the integrand.

We take note this early, however, that not all initially nonintegrable integrands can be made integrable by using substitution. Still, the substitution rule is a very handy and powerful tool for a great variety of integrals, both definite and indefinite.

(B) LESSON PROPER

How do we evaluate definite integrals $\int_a^b f(x)\,dx$ where the substitution technique may be applied?

Let us first recall how a substitution is done. Consider the indefinite integral

$$\int (x-2)^{54}\,dx.$$

One option in solving this is by expanding $f(x) = (x-2)^{54}$, meaning, multiplying $(x-2)$ by itself 54 times. Fortunately, in Section 3, we were taught the substitution rule, which allows us to solve the given integral without having to perform tedious multiplication at all, that is, with $y = x - 2$, we get

$$\begin{aligned} \int (x-2)^{54}\,dx &= \int y^{54}\,dy \\ &= \frac{1}{55}y^{55} + C \\ &= \frac{1}{55}(x-2)^{55} + C. \end{aligned}$$

Now, let us consider the definite integral

$$\int_1^3 (x-2)^{54}\,dx.$$

How do we evaluate this through substitution? There are two ways of approaching the solution of a definite integral through substitution.

Method 1.

We first consider the definite integral as an indefinite integral and apply the substitution technique. The answer (antiderivative of the function) is expressed in terms of the original variable and the FTOC is applied using the limits of integration $x = a$ and $x = b$.

To illustrate, to integrate $\int_1^3 (x-2)^{54}\, dx$, we first apply the substitution technique to the indefinite integral using the substitution $y = x - 2$ and express the antiderivative in terms of x :

$$\int (x-2)^{54}\, dx = \frac{1}{55}(x-2)^{55} + C.$$

We apply the FTOC using the original limits of integration $x = 1$ and $x = 3$, so we have

$$\int_1^3 (x-2)^{54}\, dx = \frac{1}{55}(x-2)^{55} + C\Big|_1^3 = \frac{1}{55}(1)^{55} - \frac{1}{55}(-1)^{55} = \frac{2}{55}.$$

Note that for definite integrals, we can omit the constant of integration C in the antiderivative since this will cancel when we evaluate at the limits of integration.

Method 2.

In the second method, the substitution is applied directly to the definite integral and the limits or bounds of integration are also changed according to the substitution applied. How is this done? If the substitution $y = g(x)$ is applied, then the limits of integration $x = a$ and $x = b$ are changed to $g(a)$ and $g(b)$, respectively. The FTOC is then applied to the definite integral where the integrand is a function of y and using the new limits of integration $y = g(a)$ and $y = g(b)$.

To illustrate this method, let us consider the same definite integral $\int_1^3 (x-2)^{54}\, dx$. Applying the substitution technique, we let $y = (x-2)$ so $dy = dx$. For the limits of integration in the given definite integral, these are changed in accordance to the substitution $y = x - 2$:

$$\text{If } \ x = 1, \text{ then } y = 1 - 2 = -1 \ \text{ and if } \quad x = 3 \text{ then } y = 3 - 2 = 1.$$

We then apply the FTOC to the definite integrable involving the new variable y yielding:

$$\begin{aligned}\int_1^3 (x-2)^{54}\,dx &= \int_{-1}^{1} y^{54}\,dy \\ &= \frac{y^{55}}{55} + C\Big|_{-1}^{1} \\ &= \frac{(1)^{55}}{55} - \frac{(-1)^{55}}{55} = \frac{2}{55}.\end{aligned}$$

This alternative solution pays special attention to the bounds of integration in performing a substitution. The two methods, of course, give the same result.

Why must special attention be given to the bounds of integration when performing a substitution? The bounds a and b in the definite integral $\int_a^b f(x)\,dx$ refer to values of x. In the final step of definite integration, the resulting expression is evaluated at $x = a$ and at $x = b$. If substitution is applied, there is a change in variable, hence, the limits of integration must correspond correctly to the variable.

Let us summarize:

In applying the substitution technique of integration to the definite integral

$$\int_a^b f(x)\,dx,$$

the integrand "$f(x)\,dx$" is replaced by an expression in terms of y where y is a function of x, say $y = g(x)$ which implies $dy = g'(x)dx$. The antiderivative, say $F(y)$ is thus expressed as a function of y.

In the first method, the variable y is then expressed in terms of x , giving the antiderivative $F(g(x))$ and this is evaluated with the original bounds $x = a$ and $x = b$.

In the second method, we proceed with the substitution as above and the new bounds are computed through the same equation used to perform the substitution. Thus, if $y = g(x)$, then the new bounds are

$$y = g(a) \text{ and } y = g(b)$$

and the definite integral is now expressed as

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(y)\,dy.$$

This is known as the Substitution Rule for Definite Integrals.

EXAMPLE 1: Compute $\displaystyle\int_0^2 (2x-1)^3\,dx$.

Solution. Method 1. Let $y = 2x - 1$. It follows that $dy = 2\,dx$. Hence, $dx = \dfrac{1}{2}\ dy$. Evaluating the definite integral, we have

$$\begin{aligned}
\int (2x-1)^3\ dx &= \int y^3 \cdot \frac{1}{2}\ dy \\
&= \int \frac{1}{2} y^3\ dy \\
&= \frac{1}{2} \int y^3\ dy \\
&= \frac{1}{2} \cdot \frac{y^4}{4} + C \\
&= \frac{1}{8} y^4 + C \\
&= \frac{1}{8}(2x-1)^4 + C.
\end{aligned}$$

So, by FTOC,

$$\int (2x-1)^3\ dx = \frac{1}{8}(2x-1)^4 \Big|_0^2 = \frac{1}{8}(3)^4 - \frac{1}{8}(-1)^4 = 10.$$

Method 2. Let $y = 2x - 1$, and so $dy = 2\,dx$. Hence, $dx = \dfrac{1}{2}\ dy$. The bounds are then transformed as follows:

If $x = 0$, then

$$y = 2(0) - 1 = -1.$$

If $x = 2$, then

$$y = 2(2) - 1 = 3.$$

The substitution yields the transformed definite integral

$$\int_{-1}^{3} y^3 \cdot \frac{1}{2}\ dy.$$

Evaluating the above definite integral,

$$\begin{aligned}
\int_{-1}^{3} y^3 \cdot \frac{1}{2} dy &= \int_{-1}^{3} \frac{1}{2} y^3\ dy \\
&= \frac{1}{2} \int_{-1}^{3} y^3\ dy \\
&= \frac{1}{2} \cdot \frac{1}{4} y^4 \Big|_{-1}^{3} \\
&= \frac{1}{8}[(3)^4 - (-1)^4] \\
&= \frac{1}{8}(80) \\
&= 10.
\end{aligned}$$

EXAMPLE 2: Compute $\int_{-2}^{-1} \sqrt{2-7x}\, dx$.

Solution. Let $y = 2-7x$. It follows that $dy = -7\, dx$ or $dx = -\frac{1}{7}\, dy$. For the transformed bounds: If $x = -2$, then

$$y = 2 - 7(-2) = 16.$$

If $x = -1$, then

$$y = 2 - 7(-1) = 9.$$

The substitution yields the transformed definite integral

$$\begin{aligned} \int_{16}^{9} \sqrt{y} \cdot -\frac{1}{7}\, dy &= \int_{16}^{9} y^{1/2} \cdot -\frac{1}{7}\, dy \\ &= -\frac{1}{7} \int_{16}^{9} y^{1/2}\, dy \\ &= \frac{1}{7} \int_{9}^{16} y^{1/2}\, dy. \end{aligned}$$

Hence, we have

$$\begin{aligned} \int_{-2}^{-1} \sqrt{2-7x}\, dx &= \frac{1}{7} \int_{9}^{16} y^{1/2}\, dy \\ &= \frac{1}{7} \cdot \frac{2}{3} y^{3/2} \Big|_{9}^{16} \\ &= \frac{2}{21} [16^{3/2} - 9^{3/2}] \\ &= \frac{2}{21} (64 - 27) \\ &= \frac{74}{21}. \end{aligned}$$

EXAMPLE 3: Evaluate $\int_{0}^{1} 14\, \sqrt[3]{1+7x}\, dx$.

Solution. Let $y = 1 + 7x$. Then $dy = 7\, dx$, and $14dx = 2\, dy$.
If $x = 0$, then $y = 1$. If $x = 1$, then $y = 8$.
Hence,

$$\int_{0}^{1} 14\, \sqrt[3]{1+7x}\, dx = \int_{1}^{8} 2y^{1/3}\, dy = \frac{1}{4/3}\, y^{4/3} \Big|_{1}^{8} = \frac{3}{4}\, (8^{4/3} - 1^{4/3}) = \frac{3}{4}\, (16 - 1) = \frac{45}{4}.$$

EXAMPLE 4: Evaluate $\displaystyle\int_0^2 \frac{9x^2}{(x^3+1)^{3/2}}\,dx$.

Solution. Let $y = x^3 + 1$. Then $dy = 3x^2\,dx$, and $9x^2dx = 3\,dy$.
If $x = 0$, then $y = 1$. If $x = 2$, then $y = 9$.
Hence,

$$\begin{aligned}
\int_0^2 \frac{9x^2}{(x^3+1)^{3/2}}\,dx &= \int_1^9 \frac{3}{y^{3/2}}\,dy \\
&= \int_1^9 3y^{-3/2}\,dy \\
&= \frac{3}{-1/2}\,y^{-1/2}\,\Big|_1^9 \\
&= -6(9^{-1/2}-1) \\
&= -6\left(\frac{1}{3}-1\right) \\
&= 4.
\end{aligned}$$

EXAMPLE 5: Evaluate the integral $\displaystyle\int_4^9 \frac{\sqrt{x}}{(30-x^{\frac{3}{2}})^2}\,dx$.

Solution. Notice that if we let $y = 30 - x^{\frac{3}{2}}$, then we have $dy = -\frac{3}{2}x^{\frac{1}{2}}\,dx$ so that $-\frac{2}{3}\,dy = \sqrt{x}\,dx$, which is the numerator of the integrand. Converting the limits of integration, we have $x = 4$ implying $y = 22$ and $x = 9$ implying $y = 3$. Thus,

$$\begin{aligned}
\int_4^9 \frac{\sqrt{x}}{(30-x^{\frac{3}{2}})^2}\,dx &= \int_{22}^3 \left(\frac{1}{y^2}\right)\left(-\frac{2}{3}\,dy\right) \\
&= -\frac{2}{3}\int_{22}^3 y^{-2}\,dy \\
&= -\frac{2}{3}\left(\frac{y^{-1}}{-1}\right)\Big|_{22}^3 \\
&= \left[\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\right]-\left[\left(\frac{2}{3}\right)\left(\frac{1}{22}\right)\right] \\
&= \left(\frac{2}{3}\right)\left[\frac{1}{3}-\frac{1}{22}\right] \\
&= \frac{19}{99}.
\end{aligned}$$

EXAMPLE 6: Evaluate the integral $\displaystyle\int_0^{\frac{\pi}{4}} \sin^3 2x\cos 2x\,dx$.

Solution. Let $y = \sin 2x$. Differentiating both sides we have, $dy = 2\cos 2x\ dx$ and $\cos 2x\ dx = \dfrac{dy}{2}$. Now, when $x = 0$ it implies that $y = \sin 0 = 0$ and when $x = \dfrac{\pi}{4}$ implies $y = \sin\dfrac{\pi}{2} = 1$. Thus,

$$\int_0^{\frac{\pi}{4}} \sin^3 2x \cos 2x\ dx = \int_0^{\frac{\pi}{4}} (\sin 2x)^3 \cos 2x\ dx = \int_0^1 y^3 \cdot \frac{dy}{2} = \frac{1}{2}\left(\frac{y^4}{4}\right)\Bigg|_0^1 = \frac{1}{8}.$$

EXAMPLE 7: Evaluate $\displaystyle\int_{-1}^{0} x^2\ e^{x^3+1}\ dx$.

Solution. Recall that $D_t(e^t) = e^t$, and therefore $\displaystyle\int e^t dt = e^t + c$. In other words, the derivative and the antiderivative of the exponential are both the exponential itself.

Now, let $y = x^3 + 1$. Then $dy = 3x^2\,dx$, so that $x^2 dx = \dfrac{1}{3}\,dy$.
If $x = -1$, then $y = 0$. If $x = 0$, then $y = 1$. Hence,

$$\int_{-1}^{0} x^2\ e^{x^3+1}\ dx \ = \ \int_0^1 \frac{1}{3}e^y\ dy = \ \frac{1}{3}e^y\ \Bigg|_0^1 = \ \frac{1}{3}(e^1 - e^0) = \ \frac{1}{3}(e-1) = \ \frac{e-1}{3}.$$

If we are dealing with an integral whose lower limit of integration is greater than the upper limit, we can use the property that says

$$\int_a^b f(x)\ dx \ = \ -\int_b^a f(x)\ dx.$$

EXAMPLE 8: Evaluate the definite integral $\displaystyle\int_2^0 2x(1+x^2)^3\ dx$.

Solution. First, note that

$$\int_2^0 2x(1+x^2)^3\ dx = -\int_0^2 2x(1+x^2)^3\ dx$$

Let $u = 1 + x^2$ so that $du = 2x\,dx$ which is the other factor. Changing the $x-$limits, we have when $x = 0$, then $u = 1 + 0^2 = 1$ and when $x = 2$, then $u = 1 + 2^2 = 5$. Thus,

$$-\int_0^2 2x(1+x^2)^3\ dx = -\int_1^5 u^3\ du = -\left(\frac{u^4}{4}\right)\Bigg|_1^5 = -\frac{5^4}{4} - \frac{1^4}{4} = -\frac{625-1}{4} = -\frac{624}{4} = -156.$$

(C) **EXERCISES**

Evaluate the following definite integrals.

1. $\displaystyle\int \frac{2x}{1+x^2}\,dx$ **Ans:** $\ln|1+x^2|+C$

2. $\displaystyle\int -2xe^{x^2-4}\,dx$ **Ans:** $-e^{x^2-4}+C$

3. $\displaystyle\int \frac{x^2}{\sqrt[4]{x^3+2}}\,dx$ **Ans:** $\frac{4}{9}(x^3+2)^{\frac{3}{4}}+C$

4. $\displaystyle\int \frac{\cos x}{1+\sin x}\,dx$ **Ans:** $\ln|1+\sin x|+C$

5. $\displaystyle\int \frac{dx}{\sqrt{x}(1+\sqrt{x})^2}$ **Ans:** $-\frac{2}{1+\sqrt{x}}+C$

6. $\displaystyle\int z^4(1+z^5)^3\,dz$ **Ans:** $\frac{1}{20}(1+z^5)^4+C$

7. $\displaystyle\int (\sin x+\cos x)^2\,dx$ **Ans:** $x+\sin^2 x+C$

8. $\displaystyle\int \frac{x}{e^{x^2}}\,dx$ **Ans:** $-\dfrac{1}{e^{x^2}}+C$

9. $\displaystyle\int \frac{\cos x}{\sqrt{1+\sin x}}\,dx$ **Ans:** $2\sqrt{1+\sin x}+C$

10. $\displaystyle\int \frac{t^3}{\sqrt{t^4+16}}\,dt$ **Ans:** $\frac{1}{2}\sqrt{t^4+16}+C$

11. $\displaystyle\int \cos^4 2w\sin^3 2w\,dw$ **Ans:** $-\frac{1}{10}\cos^5(2w)+\frac{1}{14}\cos^7(2w)+C$

12. $\displaystyle\int \cos^3\frac{x}{3}\,dx$ **Ans:** $3\sin(\frac{x}{3})-\sin^3(\frac{x}{3})+C$

13. $\displaystyle\int \frac{3x}{(x^2+1)^7}\,dx$ **Ans:** $-\frac{1}{(x^2+1)^6}$

14. $\displaystyle\int \frac{e^x}{(e^x+5)^3}\,dx$ **Ans:** $-\frac{1}{2(e^x+5)^2}+C$

15. $\displaystyle\int \frac{1}{x\ln x}\,dx$ **Ans:** $\ln|\ln x|+C$

16. $\displaystyle\int \frac{20y}{(y^2+1)^{20}}\,dy$ **Ans:** $-\frac{10}{19(y^2+1)^{19}}+C$

17. $\displaystyle\int xe^{-3x^2}\,dx$ **Ans:** $-\frac{1}{6}e^{-3x^2}+C$

18. $\displaystyle\int \frac{12x^3}{3x^4+1}\,dx$ **Ans:** $\ln|3x^4+1|+C$

19. $\displaystyle\int_0^{\frac{\pi}{2}} e^{\sin x}\cos x\,dx$ **Ans:** $e-1$

20. $\displaystyle\int_1^6 \frac{dx}{\sqrt{x+3}}$ **Ans:** 2

21. $\displaystyle\int_{-1}^2 \frac{(x+1)\,dx}{\sqrt{x^2+2x+4}}$ **Ans:** $\sqrt{3}$

22. $\displaystyle\int_0^5 x^3\sqrt{x^4+1}\,dx$ ***Ans:*** $\frac{626^{\frac{2}{3}}-1}{6}$

23. $\displaystyle\int_{-1}^1 x^2e^{x^3}\,dx$ ***Ans:*** $\frac{e^2-1}{3e}$

24. $\displaystyle\int_{-1}^3 \frac{3x\,dx}{x^2+2}$ ***Ans:*** $\frac{3}{2}\left(\ln\frac{11}{3}\right)$

25. $\displaystyle\int_0^2 (e^x+1)^2e^x\,dx$ ***Ans:*** $\frac{(e+1)^3-8}{3}$

26. $\displaystyle\int_0^1 (e^x+1)^2\,dx$ ***Ans:*** $\frac{e^2+4e-3}{2}$

27. $\displaystyle\int_0^{\frac{1}{2}} \frac{t}{\sqrt{1-t^2}}\,dt$ ***Ans:*** $\frac{2-\sqrt{3}}{2}$

28. $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2\theta\tan^2\theta\,d\theta$ ***Ans:*** $\frac{\pi}{8}-\frac{1}{4}$

29. $\displaystyle\int_0^1 e^{x+e^x}\,dx$ ***Ans:*** e^e-e

30. $\displaystyle\int_1^{e^2} \frac{(\ln x)^8+1}{x}\,dx$ ***Ans:*** $\frac{530}{9}$

31. $\displaystyle\int_0^3 \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$ ***Ans:*** $\ln\left|\frac{e^6+1}{2e^3}\right|$

32. $\displaystyle\int_0^2 (-3x+4)e^{-3x^2+8x}\,dx$ ***Ans:*** $\frac{e^4-1}{2}$

33. $\displaystyle\int_3^8 \frac{x-1}{\sqrt{x+1}}\,dx$ ***Ans:*** $\frac{26}{3}$

34. $\displaystyle\int_{-1}^1 (x+2)(x^2+4x-3)^3\,dx$ ***Ans:*** -160

35. $\displaystyle\int_0^{15} \frac{x}{\sqrt{x+1}}\,dx$ ***Ans:*** 36

Solutions to Selected Exercises

1. Letting $u=1+x^2$ implies that $du=2dx$. Thus,

$$\begin{aligned}\int\frac{2x}{1+x^2}\,dx &= \int\frac{1}{u}\,du\\ &= \ln|u|+C\\ &= \ln|1+x^2|+C\end{aligned}$$

3. Letting $u = x^3 + 2$ implies that $du = 3x^3 dx$. Thus, $x^2 dx = \frac{du}{3}$. Hence,

$$\begin{aligned}
\int \frac{x^2}{\sqrt[4]{x^3+2}}\, dx &= \int \frac{1}{\sqrt[4]{u}} \left(\frac{du}{3}\right) \\
&= \frac{1}{3}\int u^{-\frac{1}{4}}\, du \\
&= \frac{1}{3}\left(\frac{u^{\frac{3}{4}}}{\frac{3}{4}}\right) + C \\
&= \frac{4}{9}(x^3+2)^{\frac{3}{4}} + C
\end{aligned}$$

4. Put $u = 1 + \sin x$. It will yield $du = \cos x\; dx$, thus

$$\begin{aligned}
\int \frac{\cos x}{1+\sin x}\, dx &= \int \frac{1}{u}\, du \\
&= \ln|u| + C \\
&= \ln|1+\sin x| + C
\end{aligned}$$

5. Letting $u = 1 + \sqrt{x}$ implies that $du = \frac{1}{2\sqrt{x}}\, dx$. Hence, $\frac{dx}{\sqrt{x}} = 2du$, we have

$$\begin{aligned}
\int \frac{dx}{\sqrt{x}(1+\sqrt{x})^2} &= \int \frac{1}{u^2}(2du) \\
&= 2\int u^{-2}\, du \\
&= 2\left(\frac{u^{-2+1}}{-2+1}\right) + C \\
&= -\frac{2}{1+\sqrt{x}} + C
\end{aligned}$$

7. Simplifying $(\sin x + \cos x)^2$ yields $(\sin^2 x + 2\sin x\cos x + \cos^2 x)$. Thus,

$$\int (\sin x + \cos x)^2\, dx = \int (\sin^2 x + 2\sin x \cos x + \cos^2 x)\, dx.$$

Now,

- $\int \sin^2 x\, dx = \int \frac{1-\cos(2x)}{2} dx = \int \frac{1}{2} dx - \int \frac{\cos(2x)}{2} dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$
- $\int (2\sin x \cos x)\, dx = 2\int u\, du = u^2 + C = \sin^2 x + C,$ where $u = \sin x$
- $\int \cos^2 x\, dx = \int (1 - \sin^2 x)\, dx$

$$\begin{aligned}
&= \int dx - \int \frac{1-\cos(2x)}{2} dx \\
&= x - \left[\frac{1}{2}x - \frac{1}{4}\sin(2x)\right] + C \\
&= x - \frac{1}{2}x + \frac{1}{4}\sin(2x) + C
\end{aligned}$$

Thus,

$$\begin{aligned}
\int (\sin x + \cos x)^2 \, dx &= \int (\sin^2 x + 2\sin x \cos x + \cos^2 x) \, dx \\
&= \int \sin^2 x \, dx + \int 2\sin x \cos x \, dx + \int \cos^2 x \, dx \\
&= \left[\frac{1}{2}x - \frac{1}{4}\sin(2x)\right] + \sin^2 x + \left[x - \frac{1}{2}x + \frac{1}{4}\sin(2x)\right] + C \\
&= x + \sin^2 x + C
\end{aligned}$$

11. Let $u = \cos 2w$. Thus $du = -2\sin 2w \; dw$ and $\dfrac{du}{-2} = \sin 2w \, dw$. Now, using the trigonometric identity $\sin^2 2w = 1 - \cos^2 2w$,

$$\begin{aligned}
\int \cos^4 2w \sin^3 2w \; dw &= \int \cos^4 2w(1 - \cos^2 2w)(\sin 2w) \; dw \\
&= \int (\cos^4 2w - \cos^6 2w) \sin 2w \; dw \\
&= \int (u^4 - u^6) \left(\frac{du}{-2}\right) \\
&= -\frac{1}{2}\left(\frac{u^5}{5} - \frac{u^7}{7}\right) + C \\
&= -\frac{1}{10}\cos^5 2w + \frac{1}{14}\cos^7 2w + C
\end{aligned}$$

16. Letting $u = y^2 + 1$ implies that $du = 2y \, dy$ which yields $10 \, du = 20y \, dy$. Thus,

$$\begin{aligned}
\int \frac{20y}{(y^2+1)^{20}} \; dy &= \int \frac{10 \; du}{u^{20}} \\
&= 10 \int u^{-20} \; du \\
&= 10\left(\frac{u^{-19}}{-19}\right) + C \\
&= -\frac{10}{19(y^2+1)^{19}} + C
\end{aligned}$$

LESSON 18: Application of Definite Integrals in the Computation of Plane Areas

TIME FRAME: 6 sessions

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Compute the area of a plane region using the definite integral; and
2. Solve problems involving areas of plane regions.

LESSON OUTLINE:

1. Areas of Plane Regions
2. Application of Definite Integrals

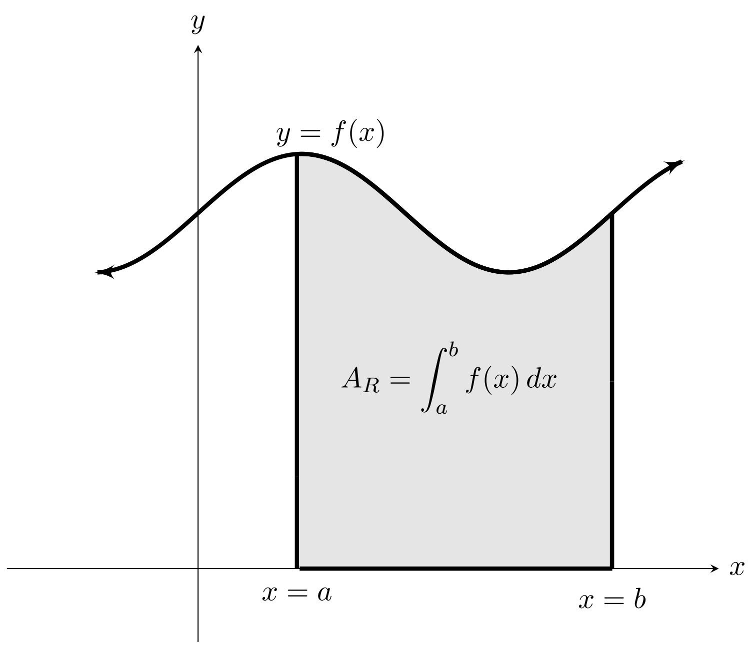
TOPIC 18.1: Areas of Plane Regions Using Definite Integrals

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

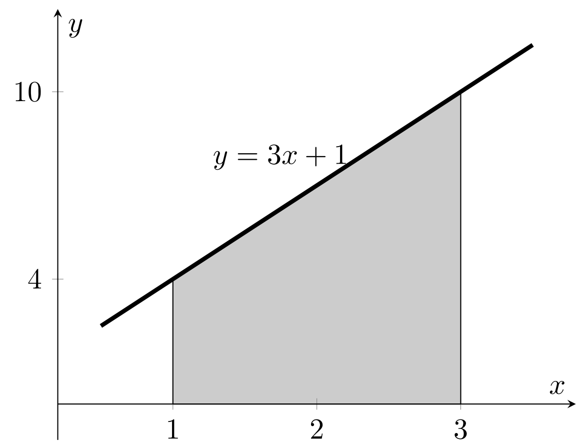
We have learned how to evaluate definite integrals. One of the many applications of the evaluation of definite integrals is in determining the areas of plane regions bounded by curves.

Consider a continuous function f. If the graph of $y = f(x)$ over the interval $[a, b]$ lies entirely above the x-axis, then $\int_a^b f(x)\,dx$ gives the area of the region bounded by the curves $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$. This is illustrated in the figure below:



EXAMPLE 1: Find the area of the plane region bounded by $y = 3x + 1$, $x = 1$, $x = 3$, and the x-axis.

Solution. The graph of the plane region is shown in the figure below.



This plane region is clearly in the first quadrant of the Cartesian plane (see figure above) and hence immediately from the previous discussion, we obtain

$$\begin{aligned} A_R &= \int_a^b f(x)\,dx \\ &= \int_1^3 (3x+1)\,dx. \end{aligned}$$

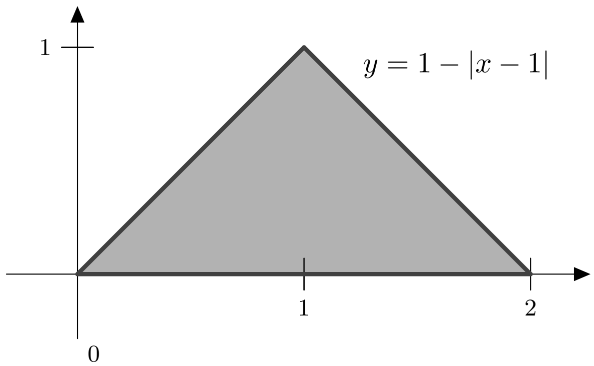
Evaluating the integral and applying the Fundamental Theorem of Calculus, we get

$$\begin{aligned} A_R &= \int_1^3 (3x+1)\,dx \\ &= \left(\frac{3x^2}{2} + x\right)\Big|_1^3 \\ &= \left(\frac{27}{3} + 3\right) - \left(\frac{3}{2} + 1\right) \\ &= 14 \text{ square units.} \end{aligned}$$

Recall that in the previous discussion, we evaluated $\displaystyle\int_1^3 (3x+1)\,dx$ and got the value 14. As we previously mentioned, this is the reason why we use the same symbol since antiderivatives are intimately related to finding the areas below curves.

EXAMPLE 2: Find the area of the plane region bounded above by $y = 1 - |x - 1|$ and below by the x-axis.

Solution. The graph of the plane region is shown below.



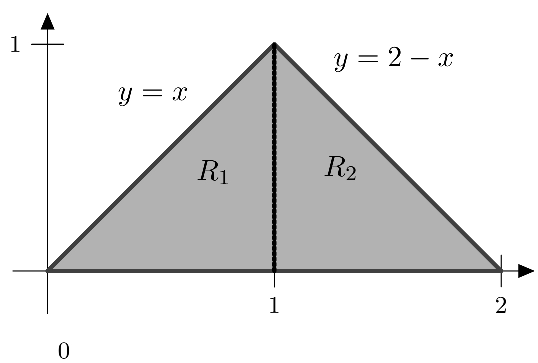
Observe that the line from the point $(0, 0)$ to $(1, 1)$ is given by

$$y = 1 - [-(x - 1)] = x$$

and the line from the point $(1, 1)$ to $(0, 2)$ is given by

$$y = 1 - (x - 1) = 2 - x.$$

Clearly, we have two subregions here, Region 1 (R_1) which is bounded above by $y = x$, and Region 2 (R_2) which is bounded above by $y = 2 - x$.

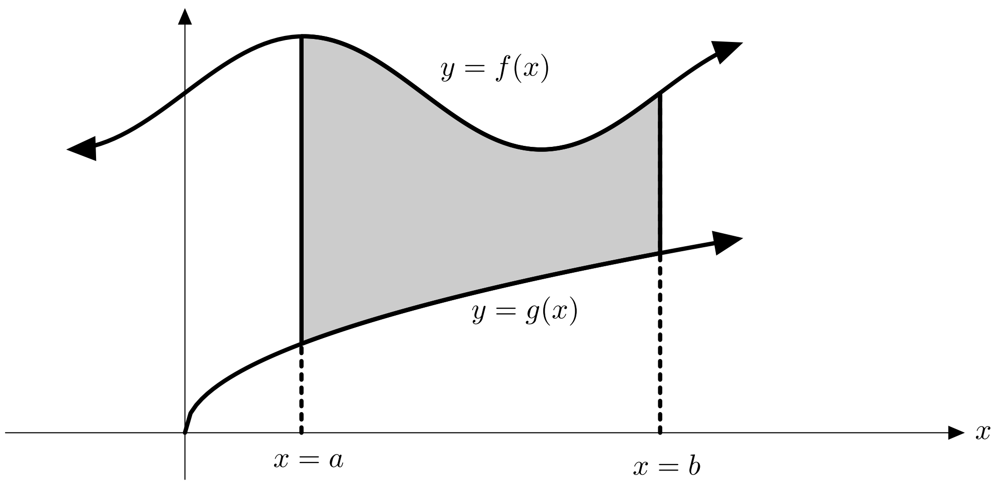


Hence, the area of the entire plane region is given by

$$\begin{aligned} \mathrm{A} &= \mathrm{A}_{R_1} + \mathrm{A}_{R_2} \\ &= \int_0^1 x\,dx + \int_1^2 (2 - x)\,dx \\ &= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2 \\ &= \frac{1^2}{2} - \frac{0^2}{2} + \left[2(2) - \frac{2^2}{2}\right] - \left[2(1) - \frac{1^2}{2}\right] \\ &= 1 \text{ square unit.} \end{aligned}$$

(B) **LESSON PROPER**

We now generalize the problem from finding the area of the region bounded by above by a curve and below by the x-axis to finding the area of a plane region bounded by several curves (such as the one shown below).



The height or distance between two curves at x is

$$h = (y\text{-coordinate of the upper curve}) - (y\text{-coordinate of the lower curve}).$$

Now, if $y = f(x)$ is the upper curve and $y = g(x)$ is the lower curve, then

$$h = f(x) - g(x).$$

Area between two curves

If f and g are continuous functions on the interval $[a, b]$ and $f(x) \geq g(x)$ for all $x \in [a, b]$, then the area of the region R bounded above by $y = f(x)$, below by $y = g(x)$, and the vertical lines $x = a$ and $x = b$ is

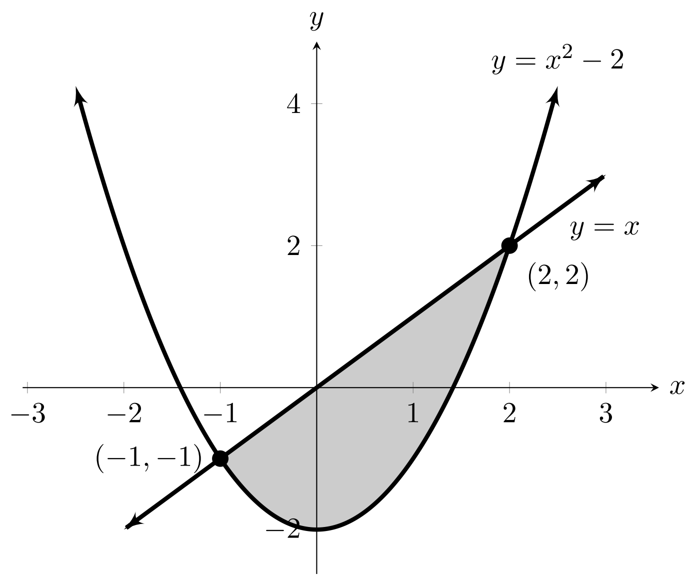
$$A_R = \int_a^b [f(x) - g(x)]\, dx.$$

EXAMPLE 3: Find the area of the plane region bounded by the curves $y = x^2 - 2$ and $y = x$.

Solution. We start by finding the points of intersection of the two curves. Substituting $y = x$ into $y = x^2 - 2$, we obtain

$$\begin{aligned} x &= x^2 - 2 \\ \Longrightarrow \quad 0 &= x^2 - x - 2 \\ \Longrightarrow \quad 0 &= (x-2)(x+1). \end{aligned}$$

Thus, we have $x = 2$ or $x = -1$. When $x = 2$, $y = 2$ while when $x = -1$, $y = -1$. Hence, we have the points of intersection $(2, 2)$ and $(-1, -1)$. The graphs of the two curves, along with their points of intersection, are shown below.



The function $f(x) - g(x)$ will be $x - (x^2 - 2)$. Our interval is $I = [-1, 2]$ and so $a = -1$ and $b = 2$. Therefore, the area of the plane region is

$$\begin{aligned}
A_R &= \int_a^b [f(x) - g(x)]\,dx \\
&= \int_{-1}^{2} [x - (x^2 - 2)]\,dx \\
&= \left[\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]_{-1}^{2} \\
&= \left[\frac{2^2}{2} - \frac{2^3}{3} + 2(2)\right] - \left[\frac{-1^2}{2} - \frac{(-1)^3}{3} + 2(-1)\right] \\
&= \left[2 - \frac{8}{3} + 4\right] - \left[\frac{1}{2} - \frac{-1}{3} - 2\right] \\
&= \frac{9}{2} \text{ square units.}
\end{aligned}$$

EXAMPLE 4: Find the area of the plane region bounded by the curves $y = x^2$, $x = -1$, $x = 2$, and $y = -1$.

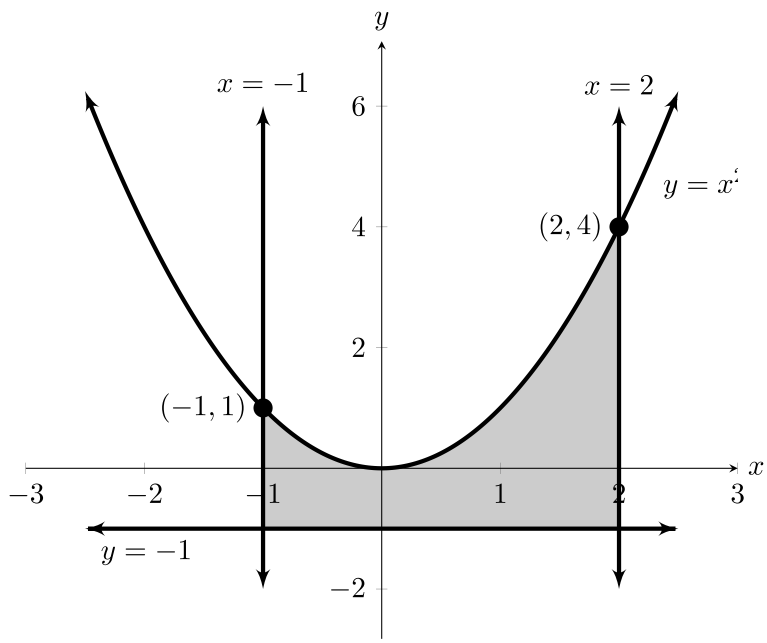
Solution. First, we find the points of intersection of the curves. With respect to the curves $y = x^2$ and $x = -1$, we have

$$y = (-1)^2 = 1.$$

Hence, these curves intersect at the point $(-1, 1)$. For the curves $y = x^2$ and $x = 2$, we have

$$y = 2^2 = 4.$$

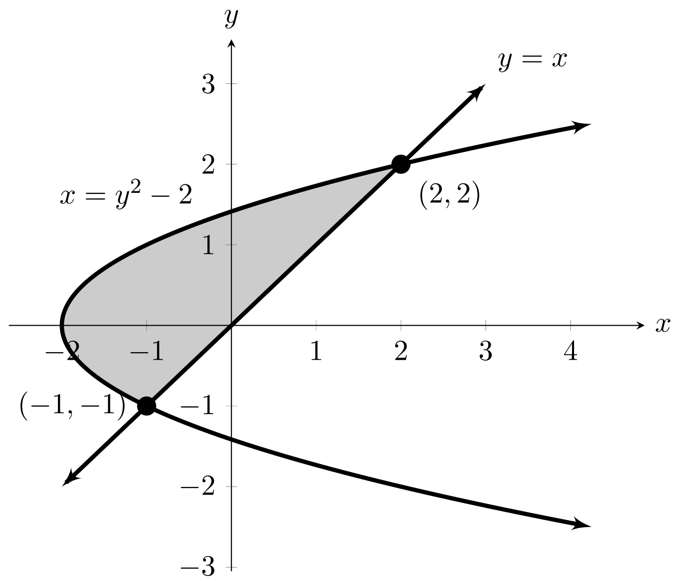
Thus, they intersect at the point $(2, 4)$. Now, for the curves $x = -1$ and $y = -1$, they intersect at $(-1, -1)$. While for $x = 2$ and $y = -1$, they intersect at $(2, -1)$. The graphs of these curves are shown below and the required region is shaded.



The function $f(x) - g(x)$ will be $x^2 - (-1) = x^2 + 1$. Our interval is $I = [-1, 2]$ and so $a = -1$ and $b = 2$. Therefore, the area of the plane region is

$$\begin{aligned}
A_R &= \int_a^b [f(x) - g(x)]\,dx \\
&= \int_{-1}^{2} (x^2 + 1)\,dx \\
&= \left(\frac{x^3}{3} + x\right)\Bigg|_{-1}^{2} \\
&= \left[\frac{2^3}{3} + 2\right] - \left[\frac{-1^3}{3} - 1\right] \\
&= \left[\frac{8}{3} + 2\right] - \left[\frac{-1}{3} - 1\right] \\
&= 6 \text{ square units.}
\end{aligned}$$

In the formula for the area of a plane region, the upper curve $y = f(x)$ is always above the lower curve $y = g(x)$ on $[a, b]$. Hence, the height of any vertical line on the region will always have the same length that is given by the function $f(x) - g(x)$. What if this is not true anymore? Consider the figure below:



To the left of $x = -1$, the upper curve is the part of the parabola located above the x-axis while the lower curve is the part of the parabola below the x-axis. On the other hand, to the right of $x = 1$, the upper curve is the parabola while the lower curve is the line $y = x$. Hence, in this case, we need to split the region into subregions in such a way that in each subregion the difference of the upper and lower curves is the same throughout the subregion.

Teacher Tip

The teacher must illustrate this well in the class so that the students would be able to visualize the concept properly.

EXAMPLE 5: Set up the definite integral for the area of the region bounded by the parabola $x = y^2 - 2$ and the the line $y = x$. (Refer to the above figure.)

Solution. First, we find the points of intersection of the parabola and the line. Substituting $x = y$ in $x = y^2 - 2$, we have

$$\begin{aligned} & y &=& \; y^2 - 2 \\ \Longrightarrow \;& 0 &=& \; y^2 - y - 2 \\ \Longrightarrow \;& 0 &=& \; (y-2)(y+1). \end{aligned}$$

Hence, we have $y = 2$ or $y = -1$. If $y = 2$, $x = 2$, and $x = -1$ when $y = -1$. Thus, we have two points of intersection $(2, 2)$ and $(-1, -1)$.

Note that the equation of the parabola $x = y^2 - 2$ gives us two expressions for y: $y = \sqrt{x+2}$ (points on the parabola located above the x-axis) and $y = -\sqrt{x+2}$ (points on the parabola located below the x-axis).

We first set up the integral of subregion R_1 (part of the region located on the left of $x = -1$). The upper curve here as we have observed earlier is $y = \sqrt{x+2}$ and the lower curve is $y = -\sqrt{x+2}$. Thus, the difference of the upper and lower curve is given by

$$\textit{upper curve} - \textit{lower curve} = \sqrt{x+2} - [-\sqrt{x+2}] = 2\sqrt{x+2}.$$

The interval I is $[-2, -1]$. Hence, the area of R_1 is

$$A_{R_1} = \int_{-2}^{-1} 2\sqrt{x+2}\, dx.$$

For subregion R_2 (part of the region located to the right of $(-1, -1)$, the upper curve is $y = \sqrt{x+2}$ and the lower curve is $y = x$. So, we have

$$\textit{upper curve} - \textit{lower curve} = \sqrt{x+2} - x.$$

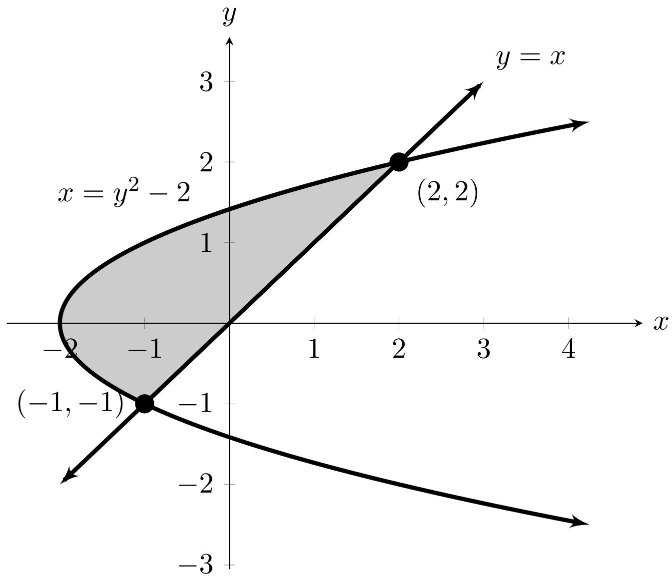
The interval I is $[-1, 2]$. Hence, the area of R_2 is

$$A_{R_2} = \int_{-1}^{-2} [\sqrt{x+2} - 2]\, dx.$$

Finally, to get the area of the region R, we simply add the areas of subregions R_1 and R_2, i.e.,

$$\begin{aligned} A_R &= A_{R_1} + A_{R_2} \\ &= \int_{-2}^{-1} 2\sqrt{x+2}\, dx + \int_{-1}^{-2} [\sqrt{x+2} - 2]\, dx. \end{aligned}$$

Consider again the figure below.



Observe that if we use horizontal rectangles, the length of a rectangle at an arbitrary point y would be the same throughout the region. Indeed, using horizontal rectangles is an alternative method of solving the problem.

If u and v are continuous functions on the interval $[c, d]$ and $v(y) \geq u(y)$ for all $y \in [c, d]$, then the area of the region R bounded on the left by $x = u(y)$, on the right by $x = v(y)$, and the horizontal lines $y = c$ and $y = d$ is

$$A_R = \int_c^d [v(y) - u(y)]\, dy.$$

Observe that the function $v(y) - u(y)$ is always given by

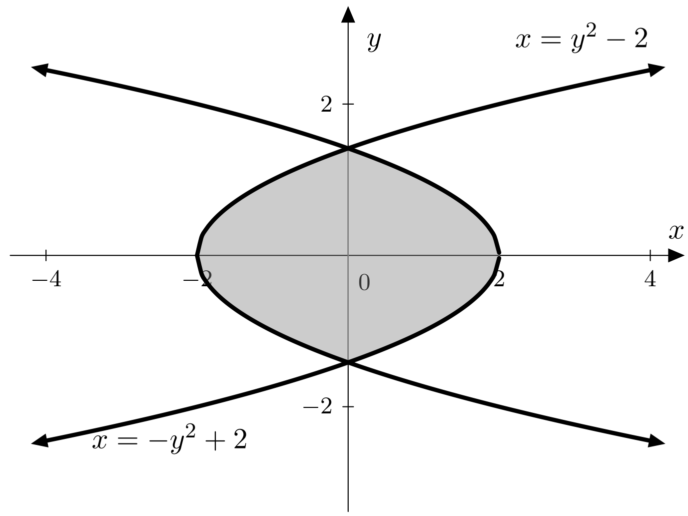
right curve – left curve.

EXAMPLE 6: Set up the integral of the area of the region bounded by the curves $x = -y^2 + 2$ and $x = y^2 - 2$.

Solution. We first find the points of intersection of the two curves. We have

$$\begin{aligned} -y^2 + 2 &= y^2 - 2 \\ \implies \quad 0 &= 2y^2 - 4 \\ \implies \quad 0 &= 2(y^2 - 2) \\ \implies \quad 0 &= 2(y - \sqrt{2})(y + \sqrt{2}). \end{aligned}$$

We get $y = -\sqrt{2}$ and $y = \sqrt{2}$. Using the curve $x = y^2 - 2$, we have the points of intersection: $(0, -\sqrt{2})$ and $(0, \sqrt{2})$. The graphs of the two curves are shown below.



Let us now determine the height of a vertical rectangle. Observe that to the left the y-axis the upper curve is $x = y^2 - 2$ while the upper curve to the right of the y-axis is $x = -y^2 + 2$. Hence, we have to split the region into two subregions to get the area of the shaded region.

However, notice that if we use a horizontal rectangle, we have the same curve on the right and the same curve on the left. The function $v(y) - u(y)$ is given by

$$\textit{right curve} - \textit{left curve} = (y^2 - 2) - (-y^2 + 2).$$

Our interval $I = [-\sqrt{2}, \sqrt{2}]$. Therefore, the area of the plane region is given by the integral

$$A_R = \int_c^d [v(y) - u(y)]\, dy = \int_{-\sqrt{2}}^{\sqrt{2}} [(y^2 - 2) - (-y^2 + 2)]\, dy = \int_{-\sqrt{2}}^{\sqrt{2}} [2y^2 - 4]\, dy.$$

(C) **EXERCISES**

1. Find the area of the region bounded by the graphs of

   a. $x = y^2 - 4$ and $x - 2y = 1$.

   b. $y = 9 - x^2$ and $y = x + 3$.

   c. $2y^2 + 9x = 36$ and $14y = 9x$.

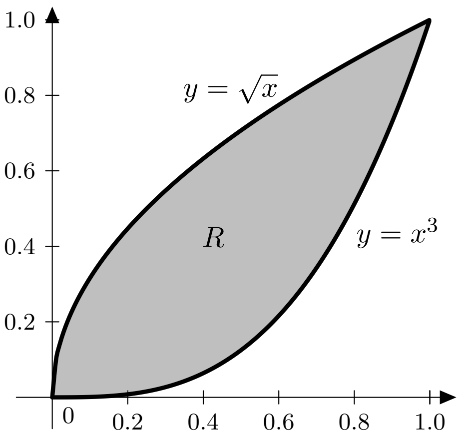
   d. $y = \sin x$ and $y = \cos x$, $x = 0$, and $x = \frac{\pi}{4}$.

   e. $y = x + \sin x$ and $y = x$, $x = 0$, and $x = \pi$.

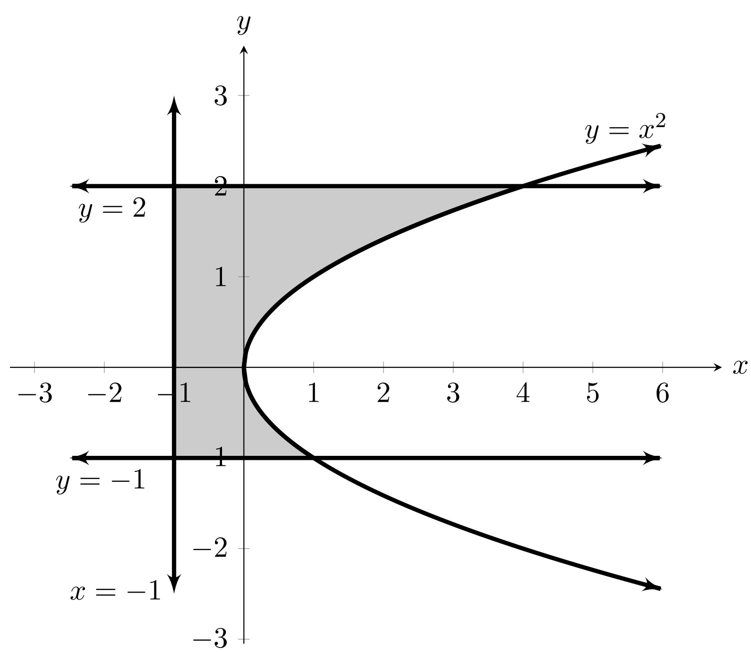
2. Given the region R bounded by the curves $y = \sqrt{x}$ and $y = x^3$, Set up the definite integral for the area of R using

   a. vertical rectangles; and

   b. horizontal rectangles.



3. Find the area of the region bounded by $x = y^2$, $x = -1$, $y = 2$ and $y = -1$. (See figure below.)



4. Derive a formula for the area of a circle of radius r,

   a. using horizontal rectangles; and

   b. using vertical rectangles.

5. Derive a formula for the area of a sector of a circle with central angle α radians.

6. Find the area of the region bounded by the graphs of

   a. $xy = 8$ and $y = 6 - x$.

   b. $y(1 + x^2) = 1$ and $15y = x^2 - 1$

TOPIC 18.2: Application of Definite Integrals: Word Problems

Teaching Tip

This is an optional and enrichment topic. These are situational problems for students' appreciation of the application of definite integrals.

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

We have learned how to find areas of regions bounded by curves. We will use this concept in situational problems.

Parcels of land are shaped in the form of regular polygons – usually triangles and quadrilaterals. However, there are possibilities that one can acquire a piece of land with an irregular shape. This can happen in places where the property being acquired is near a river. River-currents normally erode the soil, changing the shape of the riverbank. Sometimes, land is divided irregularly resulting in irregular shapes of the land parcels.

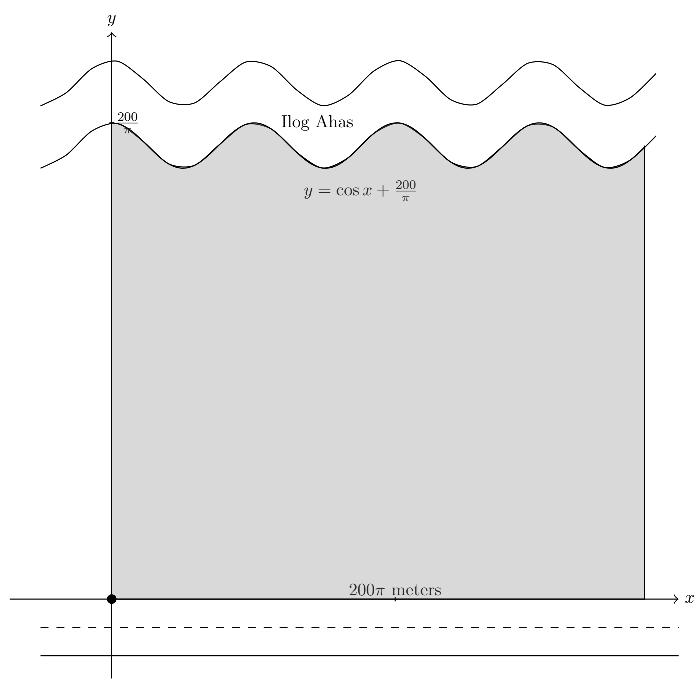
(B) LESSON PROPER

WORD PROBLEMS

EXAMPLE 1: Juan wants to acquire a lot 200π meters wide and with length bounded from the road side to the banks of "Ilog Ahas", which follows the equation $y = \cos x + \dfrac{200}{\pi}$. (Refer to the figure.)

a. Find the area of the lot.

b. If the price per square meter is ₱500, how much is the cost of land?



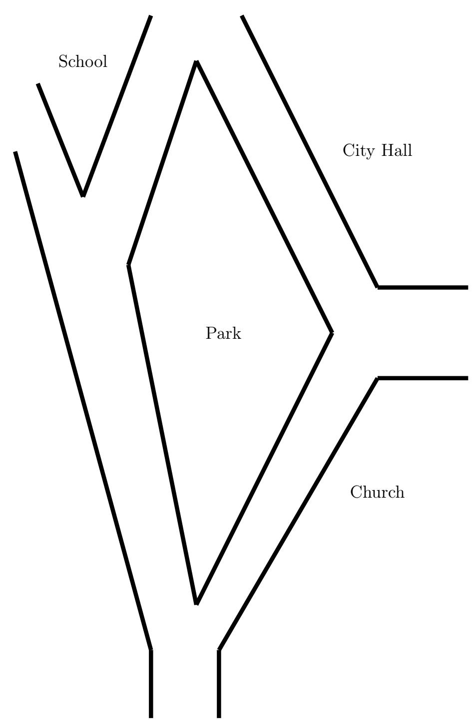
Solution.

a. Suppose we place the x-axis along the side of the road and the y-axis on one side of the lot, as shown. Note that the y-axis is placed such that it runs along the farthest side of "Ilog Ahas". We can now apply definite integrals to find the area of the region. (Refer to the figure.)

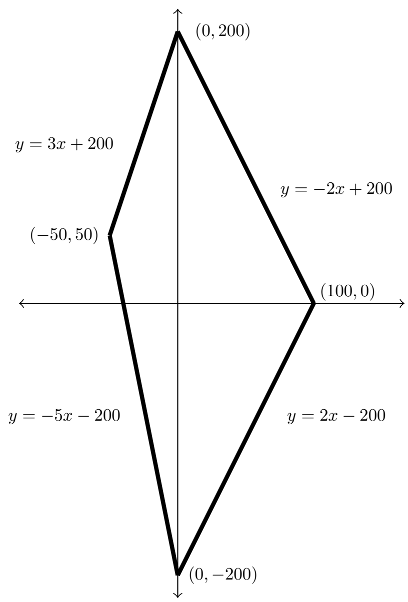
$$
\begin{aligned}
A &= \int_0^{200\pi} \left(\cos x + \frac{200}{\pi} \right) dx \\
&= \left(\sin x + \frac{200}{\pi} \right) \Big|_0^{200\pi} \\
&= \sin(200\pi) + \frac{200}{\pi}(200\pi) - \sin 0 - \frac{200}{\pi}(0) \\
&= 40,000 \text{ square meters.}
\end{aligned}
$$

b. The price of the lot is $500(40,000) = ₱20,000,000$.

EXAMPLE 2: Consider the figure on the right which shows the shape of a park in a certain city. The Mayor of the city asked the city engineer to cover the entire park with frog grass that costs ₱150 per square foot. Determine how much budget the Mayor should allocate to cover the entire park with frog grass.



Solution. To determine the area, the city engineer first places the $x-$axis and $y-$axis accordingly, as shown. The points of the park's vertices are then determined. He discovered that the lines are $y = -2x + 200$, $y = 2x - 200$, $y = 3x + 200$ and $y = -5x - 200$. (Refer to the figure.)



Using vertical rectangles, the city engineer has to split the region into two subregions. Subregion R_1 is the one to the left of the y-axis whose upper curve is $y = 3x + 200$ and $y = -5x - 200$ as the lower curve. Subregion R_2 is the one to the right of the y-axis whose upper curve is $y = -2x + 200$ and $y = 2x - 200$ as the lower curve. The length of the vertical rectangles on R_1 is

$$3x + 200 - (-5x - 200) = 8x + 400$$

while the height of the vertical rectangles on R_2 is

$$-2x + 200 - (2x - 200) = -4x + 400.$$

Hence, the area of the region will be

$$\begin{aligned} A_R = A_{R_1} + A_{R_2} &= \int_{-50}^{0} (8x + 400)\,dx + \int_{0}^{100} (-4x + 400)\,dx \\ &= (4x^2 + 400x)\Big|_{-50}^{0} + (-2x^2 + 400x)\Big|_{0}^{100} \\ &= 30,000 \text{ square feet.} \end{aligned}$$

Therefore, the cost of covering the entire park with frog grass that costs ₱ 150 per square foot is ₱ $150(30,000) =$ ₱ $4,500,000$.

(C) EXERCISES

1. Jeremy wants to acquire a lot which is bounded by a river on one side and a road on the other side, both of which are mathematically inclined. The river follows the equation $y = \cos x$ and the road follows the equation $y = \sin x$. The shortest distance from the road to the river is $\frac{100}{\pi}$ meters while the distance from one end to the other on the side of the road is 100π meters. Find the area of the lot.
2. A circular park is to be covered with bermuda grass which costs ₱50 per square foot. Use the concept of definite integrals to find the total cost of covering the park. (You can check your answer by using $A = \pi r^2$.)

CHAPTER 3 EXAM

I. Evaluate the following integrals.

1. $\displaystyle\int \left(33x^2 - 26x + 11\right)\, dx$
2. $\displaystyle\int \frac{2^x - 3}{2^{2x}}\, dx$
3. $\displaystyle\int \frac{9}{81y^2 + 1}\, dy$
4. $\displaystyle\int \frac{2}{1 - 7x}\, dx$
5. $\displaystyle\int_{-1}^{4} (1 - 3x)\, dx$
6. $\displaystyle\int_{-1}^{-2} \left(6x^2 - 6x - 18\right)\, dt$

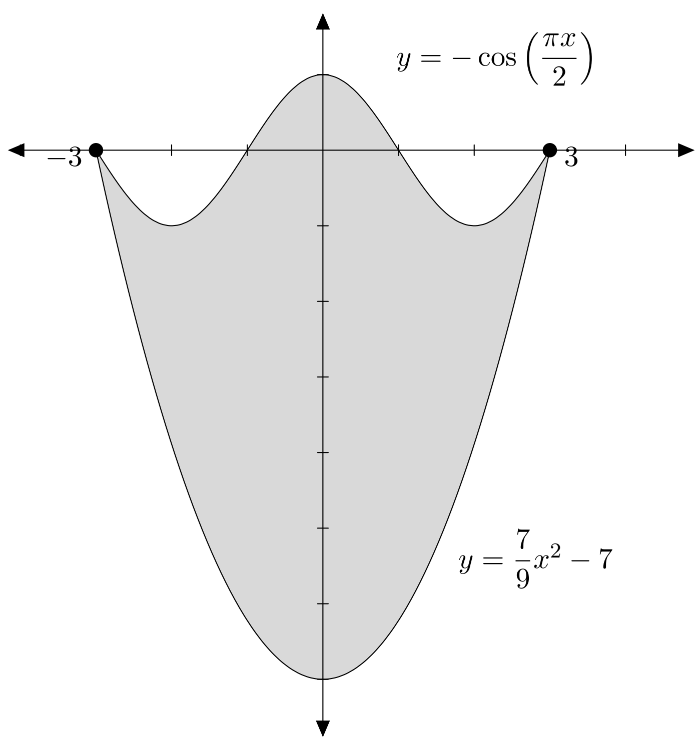
II. Find a particular solution to the separable differential equation

$$(x+3)\sin y\, dx - \frac{x^2}{\tan y}\, dy = 0$$

if we know that $y = \pi$ when $x = 1$.

III. Find the 3rd Midpoint Riemann Sum of $f(x) = 4x$ with respect to the regular partition of the interval $[0, 3]$.

IV. Set up the integral of the area of the region in the figure. $y = -\cos\left(\frac{\pi x}{2}\right)$ and $y = \frac{7x^2}{9} - 7$



Bibliography

[1] H. Anton, I. Bivens, S. Davis, *Calculus: Early Transcendentals*, John Wiley and Sons, 7th Edition, 2002.

[2] R. Barnett, M. Ziegler, K. Byleen *Calculus for Business, Economics, Life Sciences and Social Sciences*, Pearson Education (Asia) Pre Ltd, 9th Edition, 2003.

[3] L. Leithold, *College Algebra and Trigonometry*, Addison Wesley Longman Inc., 1989, reprinted-*Pearson Education Asia Pte. Ltd*, 2002.

[4] L. Leithold, *The Calculus 7*, Harpercollins College Div., 7th edition, December 1995.

[5] Math 53 Module Committee, *Math 53 Elementary Analysis I Course Module*, Institute of Mathematics, UP Diliman, 2012.

[6] J. Stewart, *The Calculus: Early Transcendentals*, Brooks/Cole, 6th Edition, 2008.

[7] S. Tan, *Applied Calculus for the Managerial, Life and Social Sciences*, Brooks/Cole, Cengage Learning, 9th Edition, 2014.

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