Brief Introduction to Bayesian Inference

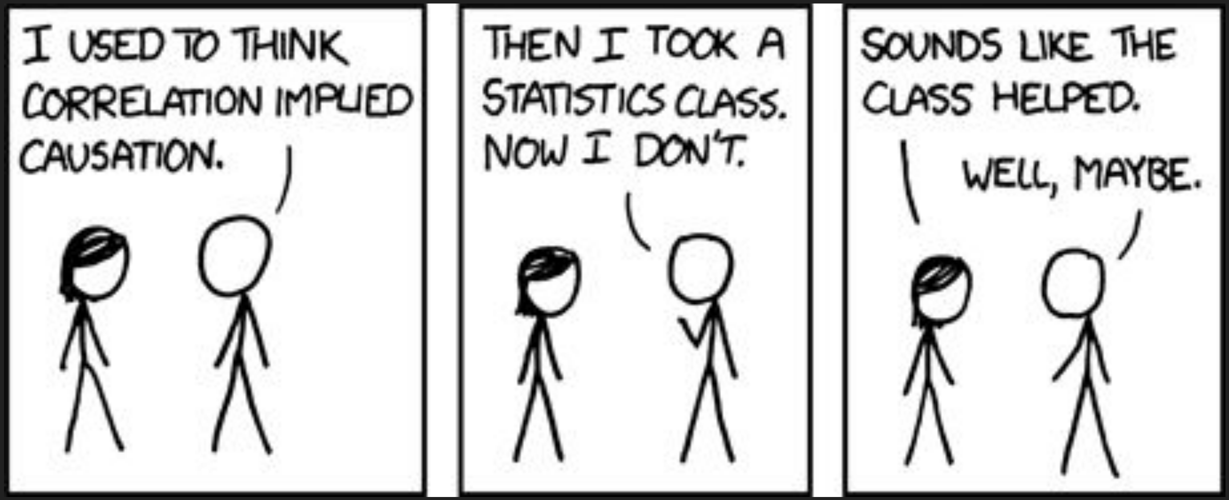
Byron J. Smith

BMI 206

2020-11-19

Slides: https://bit.ly/36GJRrE

Code: https://bit.ly/36M3YVl



With help from XKCD

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- Measure (continuous) parameters
  - "Wearing a mask decreases your risk of becoming infected with SARS-CoV-2 by 55.2%."

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- Measure (continuous) parameters
  - "Wearing a mask decreases your risk of becoming infected with SARS-CoV-2 by 55.2%."
- ...But what about questions somewhere in between these two extremes?
  - "Does wearing a mask affect your risk of becoming infected?"

“Traditional”: Null hypothesis statistical testing

"Does wearing a mask affect your risk of becoming infected?"

- Run an (e.g.) t-test
    - Implies a null hypothesis: $H_0 : \mu_{\text{mask}} = \mu_{\text{not}}$
- Calculate p-value
- If $p < 0.05$: we “reject the null hypothesis”
        - Conclude that wearing a mask *does* affect your infection risk
- Else $p > 0.05$: we cannot reject

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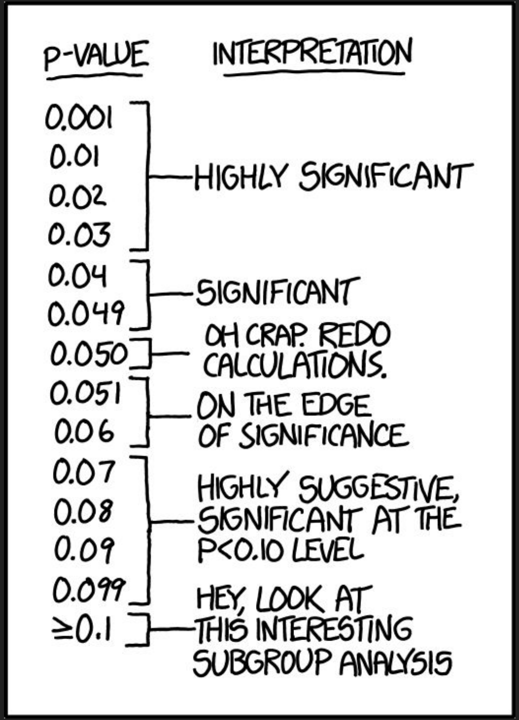

“Science by p-value”

<u>Shortcomings:</u>

- Misleading when:
  - Intuition/reality does not match test assumptions (E.g. small sample size)
  - When our null hypothesis is trivially wrong
- Ignores prior information
- Ignores effect size

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Odds Are,

It’s Wrong

Science fails to face the shortcomings of statistics

By Tom Siegfried

For better or for worse, science has long been married to mathematics. Generally it has been for the better. Especially since the days of Galileo and Newton, math has nurtured science. Rigorous mathematical methods have secured science’s fidelity to fact and conferred a timeless reliability to its findings.

“There is increasing concern,” declared epidemiologist John Ioannidis in a highly cited 2005 paper in *PLoS Medicine*, “that in modern research, false findings may be the majority or even the vast majority of published research claims.”

Ioannidis claimed to prove that more than half of published findings are false, but his analysis came under fire for statis-

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STATISTICS

Measurement error and the replication crisis

The assumption that measurement error always reduces effect sizes is false

By **Eric Loken**[1] and **Andrew Gelman**[2]

reliable measurement. In epidemiology, it

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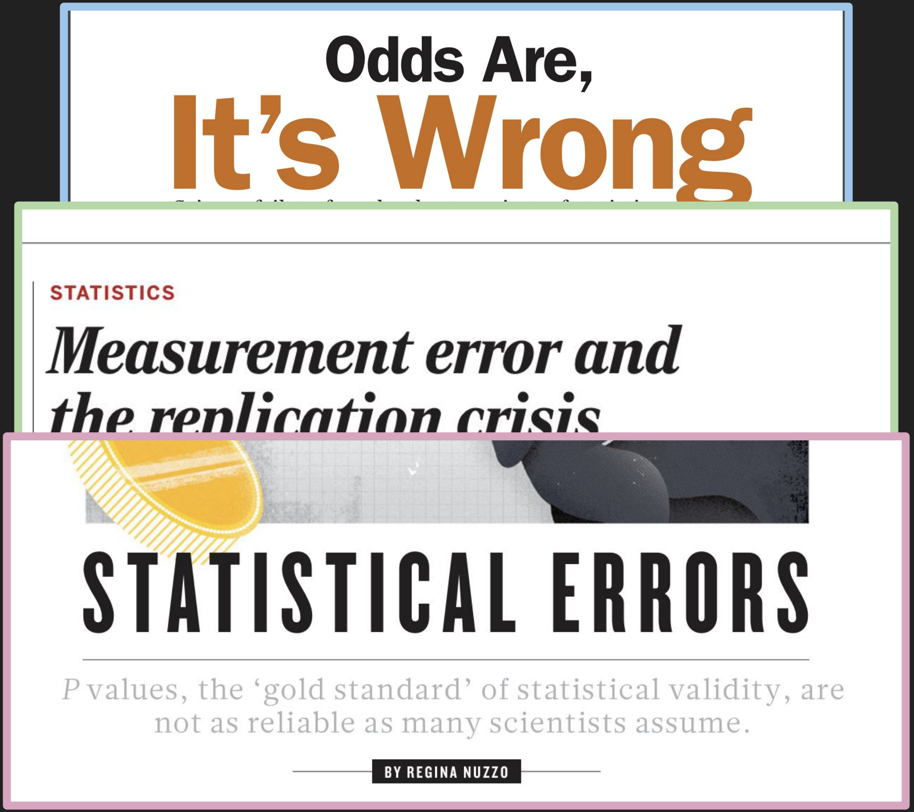

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What we would prefer:

"How does wearing a mask affect the risk of becoming infected?"

↓

There is a 95% probability that wearing a mask decreases your risk of becoming infected by 50% or more

Bayes's Rule

$$P(\theta|X) = \frac{P(X|\theta)\, P(\theta)}{P(X)}$$

Bayes's Rule

Parameters

$$P(\theta|X) = \frac{P(X|\theta)\,P(\theta)}{P(X)}$$

Bayes's Rule

Data

$$P(\theta|X) = \frac{P(X|\theta)\,P(\theta)}{P(X)}$$

Bayes's Rule

"Posterior"

$$\boxed{P(\theta|X)} = \frac{P(X|\theta)\,P(\theta)}{P(X)}$$

Bayes's Rule

Have you seen this before?

$$P(\theta|X) = \frac{\boxed{P(X|\theta)}\,P(\theta)}{P(X)}$$

Bayes's Rule

"Likelihood"

$$P(\theta|X) = \frac{\boxed{\mathcal{L}(\theta|X)}P(\theta)}{P(X)}$$

Bayes's Rule

"Prior"

$$P(\theta|X) = \frac{P(X|\theta)\boxed{P(\theta)}}{P(X)}$$

Bayes's Rule

"Normalizing Constant"

$$P(\theta|X) = \frac{P(X|\theta)\, P(\theta)}{\boxed{P(X)}}$$

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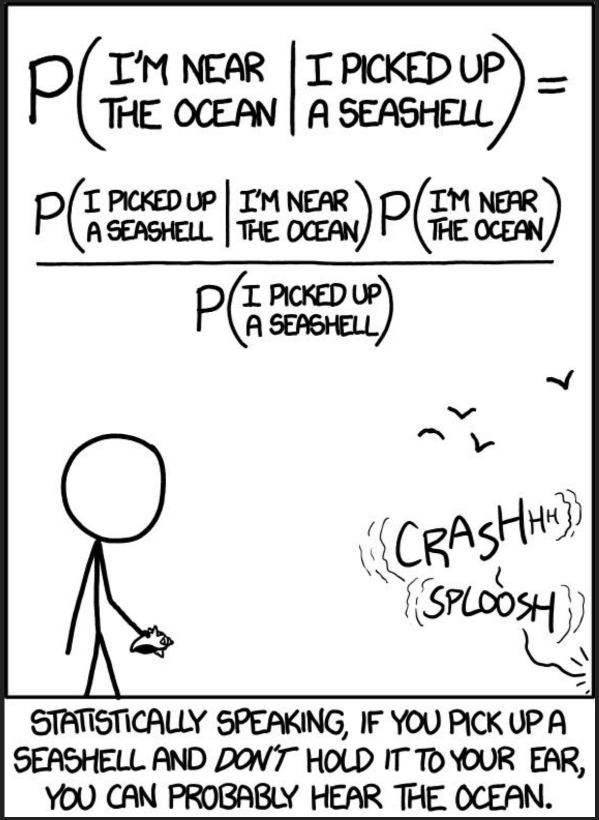

Does wearing a mask affect COVID-19 risk?

Rephrase: What is the relative risk to wearers vs. non-wearers?

θ Odds ratio of COVID-19 among wearers and non-wearers

X

$P(X|\theta)$

$P(\theta)$

$P(X)$

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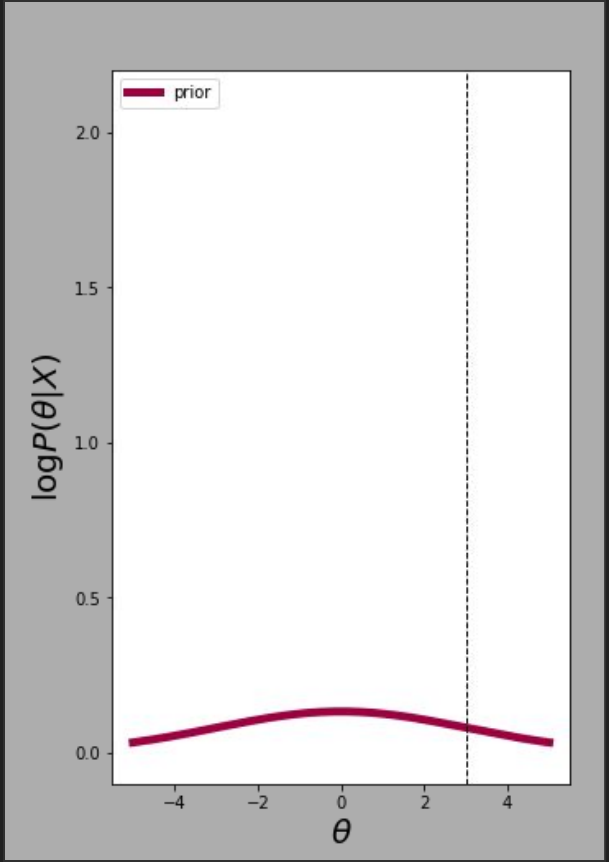
$P(X)$	Marginal probability of this observation

$$P(\theta|X) = \frac{P(X|\theta)\,P(\theta)}{P(X)}$$

Does wearing a mask affect COVID-19 risk?

- As we collect more data, the posterior distribution converges on the observed odds ratio



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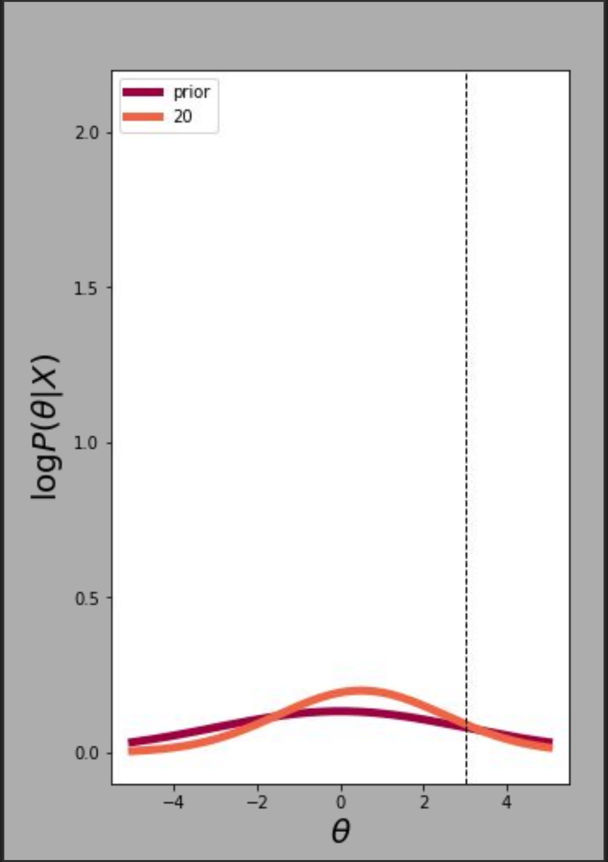

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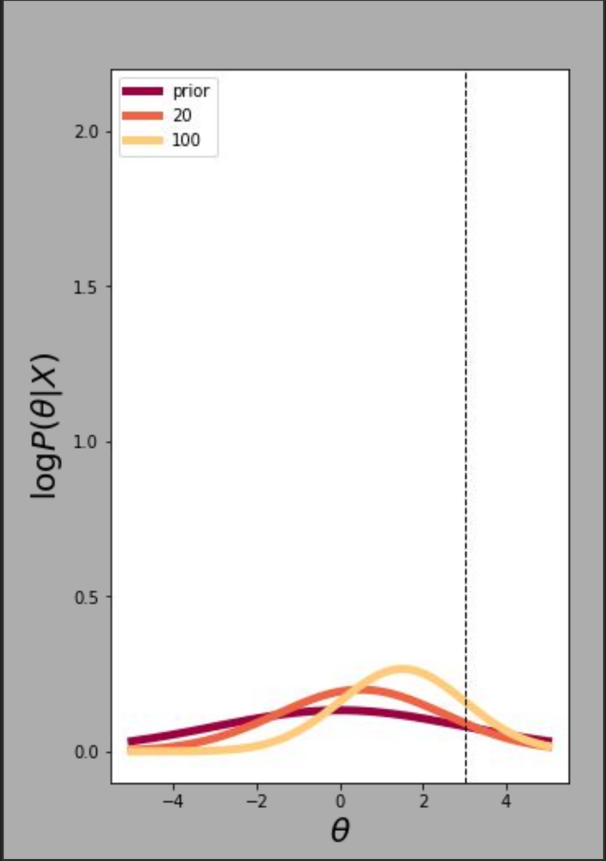

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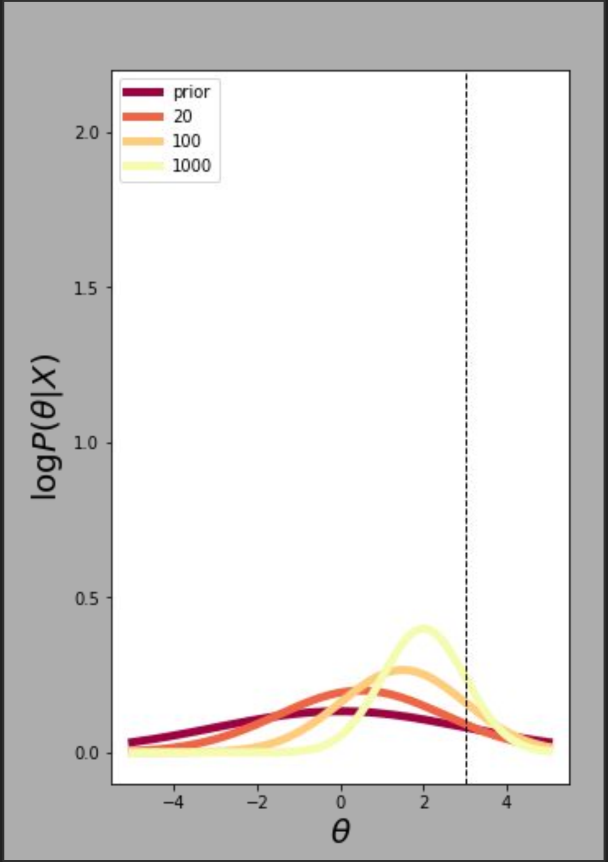

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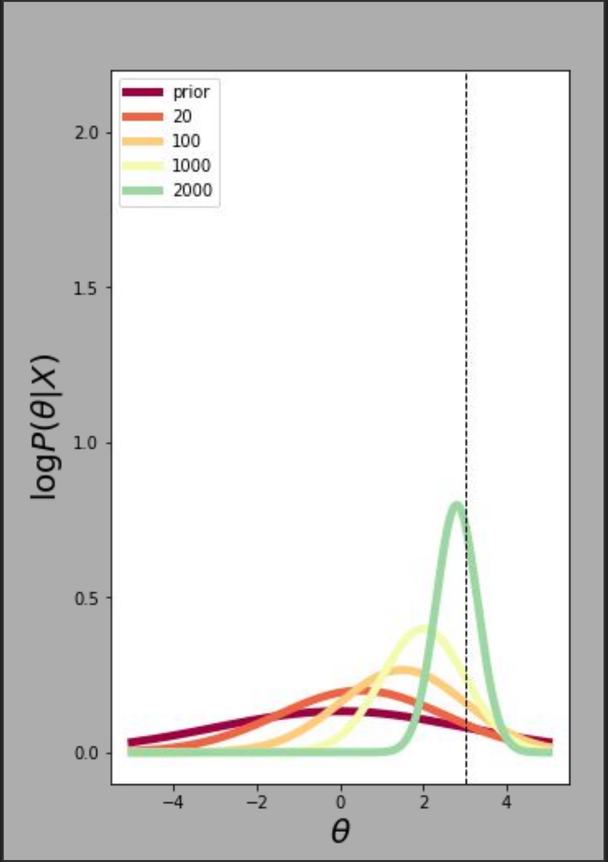

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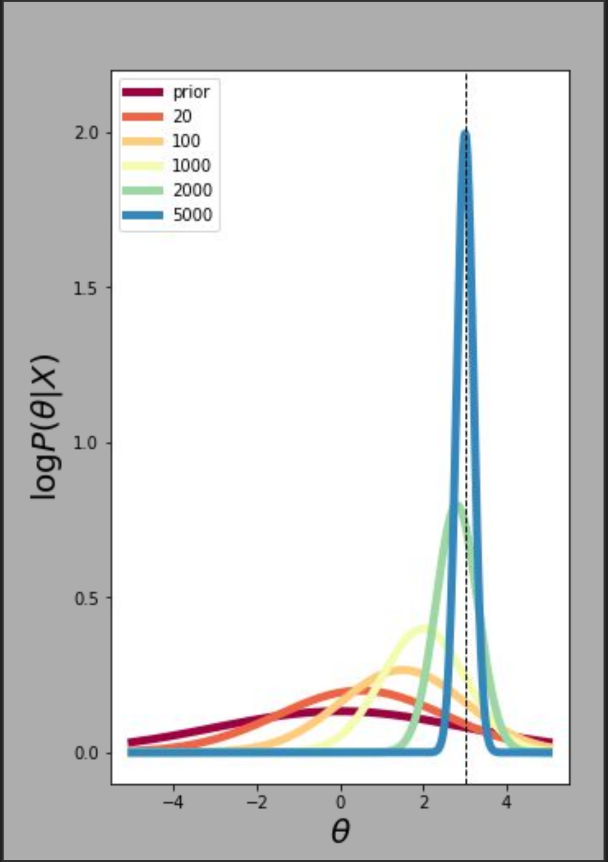

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How to calculate the posterior probability?

- Hard

$$P(\theta|X) = \frac{P(X|\theta)\,P(\theta)}{\int_{\Theta} P(X|\theta')\,P(\theta')\,d\theta'}$$

How to calculate the posterior probability?

- Hard
- Easier: *Sampling* from the posterior

$$P(\theta|X) = \frac{P(X|\theta)\,P(\theta)}{\int_\Theta P(X|\theta')\,P(\theta')\,d\theta'} \propto P(X|\theta)\,P(\theta)$$

How to calculate the posterior probability?

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- Markov-Chain Monte Carlo (Metropolis-Hastings algorithm)

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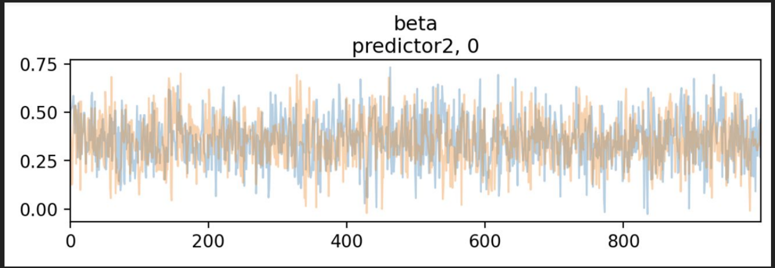
Sampling θ from the posterior

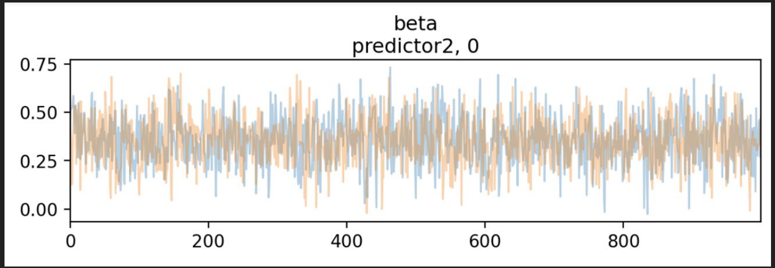
- Markov-Chain Monte Carlo (Metropolis-Hastings algorithm)

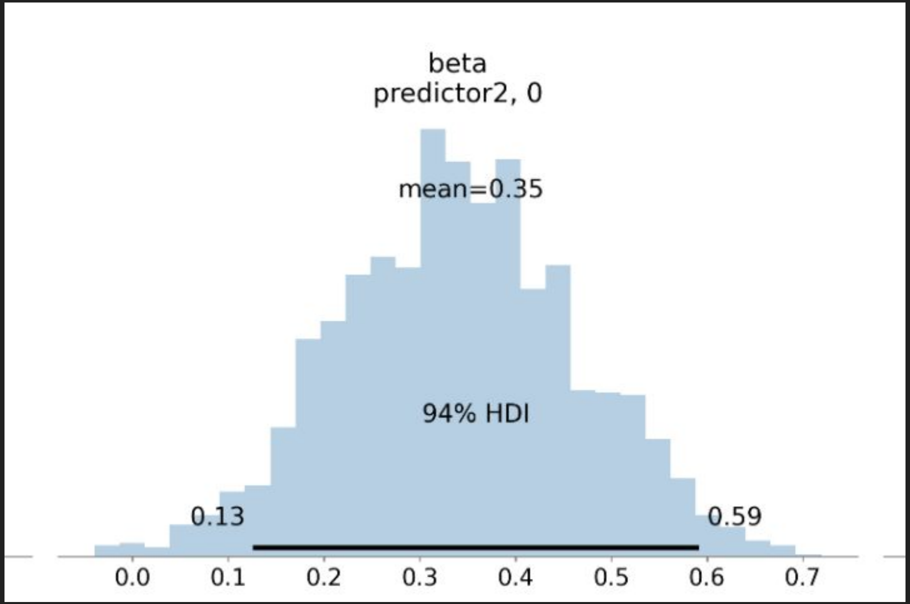
$$\theta_1, \theta_2, \theta_3, \ldots, \theta_n$$



Sampling θ from the posterior

- Markov-Chain Monte Carlo (Metropolis-Hastings algorithm)
- Use samples from the posterior to calculate
  - Expectations
  - Credible intervals





Why doesn’t everyone do it this way?

- Computation
  - Scales unfavorably with number of parameters, size of data
  - May require many samples due to “poor mixing”
- The Prior
  - “Introduces bias”
  - Subject to criticism
- Alternatives
  - Maximum likelihood

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- Computation
  - Huge progress in last 20 years.
  - HMC, Variational Inference
- The Prior
  - Useful for incorporating expert knowledge, constraints, intuition.
  - Makes assumptions explicit
- Alternatives
  - Challenging to assess uncertainty from point estimates

Logistic Regression

$$
\begin{aligned}
y_i &\sim \text{Bernoulli}(p_i) \\
\text{logit}(p_i) &= \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_K x_{iK} \\
\beta_k &\sim \text{Normal}(0, 1)
\end{aligned}
$$

Try it out!

https://bit.ly/36M3YVl

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\beta_k &\sim \text{Normal}(0, 10)
\end{aligned}
$$

Logistic Regression

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$$\text{logit}(p_i) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_K x_{iK}$$
$$\beta_k \sim \text{Laplace}(1)$$

Democratized statistical modeling

- Emphasizes parameter estimates and uncertainty over p-values
- "Inference Button": flexible, well built software to sample from and interpret models
  - STAN, PyMC3, Pyro, Turing.jl
- Only scratched the surface in this session

- Every Statistics XKCD (44 and counting): https://bit.ly/32YH4Jc