Heaps and Priority Queues

Computer Science E-119
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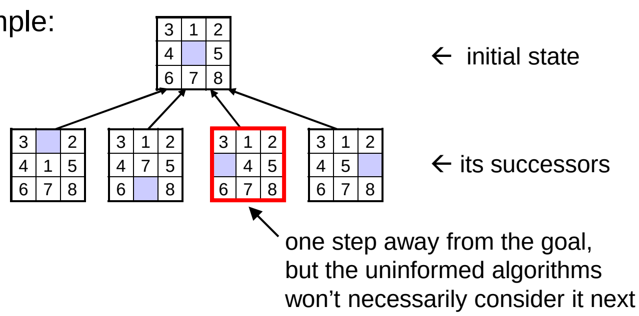
State-Space Search Revisited

- Earlier, we considered three algorithms for state-space search:
  - breadth-first search (BFS)
  - depth-first search (DFS)
  - iterative-deepening search (IDS)
- These are all *uninformed* search algorithms.
  - always consider the states in a certain order
  - do not consider how close a given state is to the goal
- 8 Puzzle example:

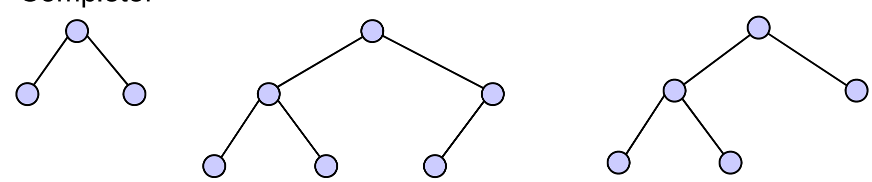


Informed State-Space Search

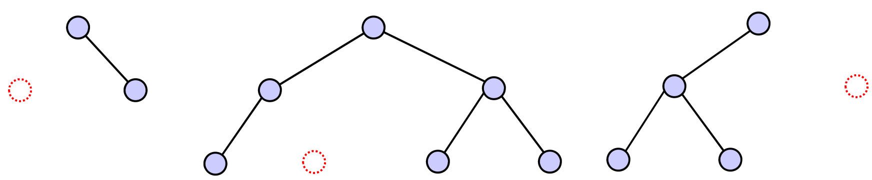
- *Informed* search algorithms attempt to consider more promising states first.
- These algorithms associate a *priority* with each successor state that is generated.
  - base priority on an estimate of nearness to a goal state
  - when choosing the next state to consider, select the one with the highest priority
- Use a *priority queue* to store the yet-to-be-considered search nodes. Key operations:
  - insert: add an item to the priority queue, ordering it according to its priority
  - remove: remove the highest priority item
- How can we efficiently implement a priority queue?
  - use a type of binary tree known as a *heap*

Complete Binary Trees

- A binary tree of height h is *complete* if:
  - levels 0 through $h - 1$ are fully occupied
  - there are no "gaps" to the left of a node in level h
- Complete:

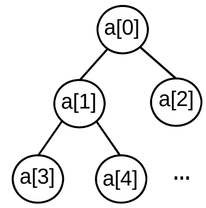


- Not complete (= missing node):

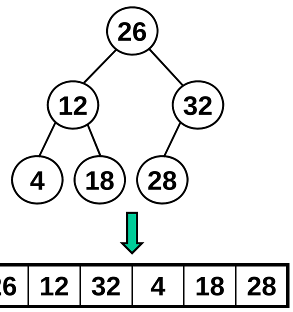


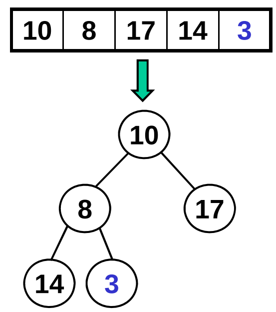
Representing a Complete Binary Tree

- A complete binary tree has a simple array representation.
- The nodes of the tree are stored in the array in the order in which they would be visited by a level-order traversal (i.e., top to bottom, left to right).



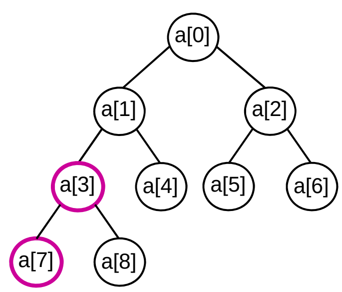
- Examples:





Navigating a Complete Binary Tree in Array Form

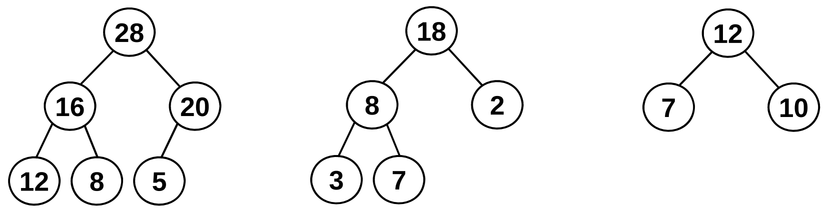
- The root node is in `a[0]`
- Given the node in `a[i]`:
  - its left child is in `a[2*i + 1]`
  - its right child is in `a[2*i + 2]`
  - its parent is in `a[(i - 1)/2]` (using integer division)



- Examples:
  - the left child of the node in `a[1]` is in `a[2*1 + 1]` = `a[3]`
  - the right child of the node in `a[3]` is in `a[2*3 + 2]` = `a[8]`
  - the parent of the node in `a[4]` is in `a[(4-1)/2]` = `a[1]`
  - the parent of the node in `a[7]` is in `a[(7-1)/2]` = `a[3]`

Heaps

- Heap: a complete binary tree in which each interior node is greater than or equal to its children
- Examples:



- The largest value is always at the root of the tree.
- The smallest value can be in *any* leaf node – there's no guarantee about which one it will be.
- Strictly speaking, the heaps that we will use are *max-at-top* heaps. You can also define a *min-at-top* heap, in which every interior node is less than or equal to its children.

A Class for Items in a Heap

```
public class HeapItem {
    private Object data;
    private double priority;
    ...
    public int compareTo(HeapItem other) {
        // error-checking goes here…
        double diff = priority – other.priority;
        if (diff > 1e-6)
            return 1;
        else if (diff < -1e-6)
            return -1;
        else
            return 0;
    }
}
```

- `HeapItem` objects group together a data item and its priority.

A Class for Items in a Heap (cont.)

```
public int compareTo(HeapItem other) {
    // error-checking goes here…
    double diff = priority – other.priority;
    if (diff > 1e-6)
        return 1;
    else if (diff < -1e-6)
        return -1;
    else
        return 0;
}
```

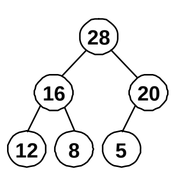
- The `compareTo` method returns:
  - -1 if the calling object has a lower priority than the other object
  - 1 if the calling object has a higher priority than the other object
  - 0 if they have the same priority

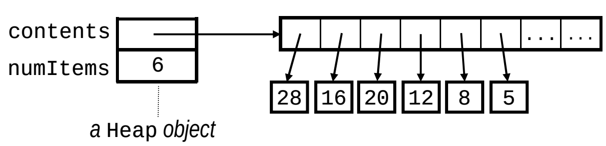
<u>numeric comparison</u>	<u>comparison using `compareTo`</u>
item1 < item2	item1.compareTo(item2) < 0
item1 > item2	item1.compareTo(item2) > 0
item1 == item2	item1.compareTo(item2) == 0

Heap Implementation (~cscie119/examples/heaps/Heap.java)

```
public class Heap {
    private HeapItem[] contents;
    private int numItems;

    public Heap(int maxSize) {
        contents = new HeapItem[maxSize];
        numItems = 0;
    }
    …
}
```

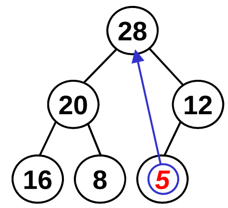




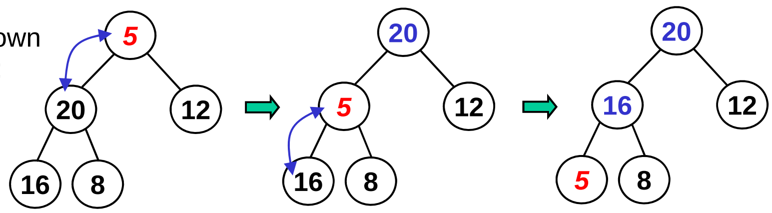
Note: we're just showing the priorities of the items, and we're showing them as integers.

Removing the Largest Item from a Heap

- Remove and return the item in the root node.
- In addition, we need to move the largest remaining item to the root, while maintaining a complete tree with each node >= children
- Algorithm:
  1. make a copy of the largest item
  2. move the last item in the heap to the root (see diagram at right)
  3. "sift down" the new root item until it is >= its children (or it's a leaf)
  4. return the largest item



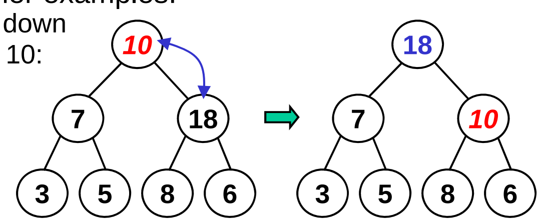
sift down the 5:



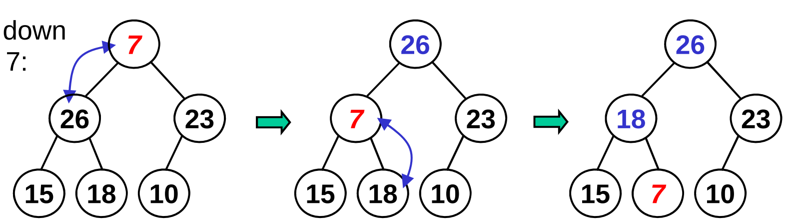
Sifting Down an Item

- To sift down item *x* (i.e., the item whose key is *x*):
  1. compare *x* with the larger of the item's children, *y*
  2. if $x < y$, swap *x* and *y* and repeat
- Other examples:

sift down the 10:



sift down the 7:



`siftDown()` Method

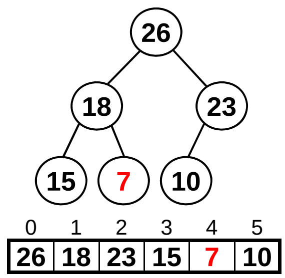
```
private void siftDown(int i) {
    HeapItem toSift = contents[i];

    int parent = i;
    int child = 2 * parent + 1;
    while (child < numItems) {
        // If the right child is bigger, compare with it.
        if (child < numItems - 1  &&
          contents[child].compareTo(contents[child + 1]) < 0)
            child = child + 1;

        if (toSift.compareTo(contents[child]) >= 0)
            break;  // we're done

        // Move child up and move down one level in the tree.
        contents[parent] = contents[child];
        parent = child;
        child = 2 * parent + 1;
    }
    contents[parent] = toSift;
}
```

- We don't actually swap items. We wait until the end to put the sifted item in place.



toSift: 7

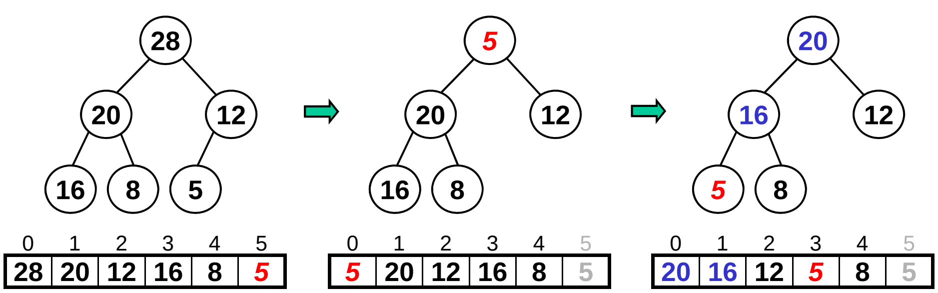
parent	child
0	1
1	3
1	4
4	9

`remove()` Method

```
public HeapItem remove() {
    HeapItem toRemove = contents[0];

    contents[0] = contents[numItems - 1];
    numItems--;
    siftDown(0);

    return toRemove;
}
```



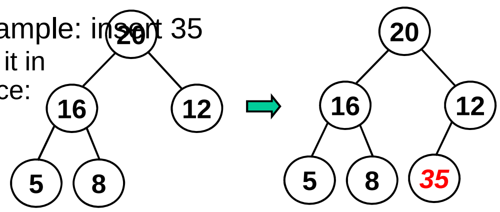
numItems: **6**	numItems: **5**	numItems: **5**
toRemove: **28**	toRemove: **28**	toRemove: **28**

Inserting an Item in a Heap

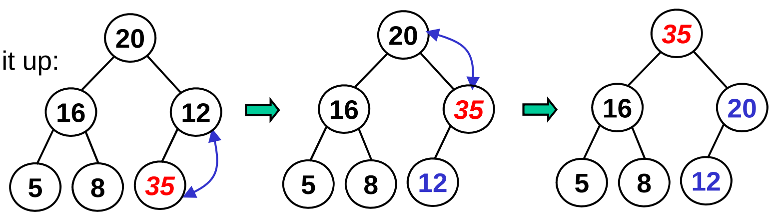
- Algorithm:
  1. put the item in the next available slot (grow array if needed)
  2. "sift up" the new item
     until it is <= its parent (or it becomes the root item)

- Example: insert 35

put it in place:

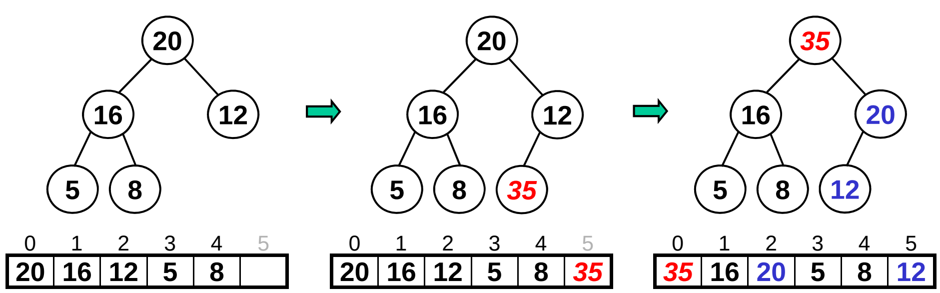


sift it up:



`insert()` Method

```
public void insert(HeapItem item) {
    if (numItems == contents.length) {
        // code to grow the array goes here…
    }
    contents[numItems] = item;
    siftUp(numItems);
    numItems++;
}
```



numItems: **5**
item: **35**

numItems: **5**
item: **35**

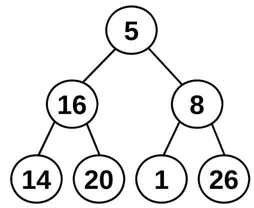
numItems: **6**

Converting an Arbitrary Array to a Heap

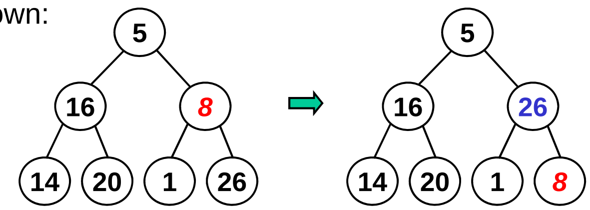
- Algorithm to convert an array with n items to a heap:
  1. start with the parent of the last element:
     `contents[i]`, where `i = ((n – 1) – 1)/2 = (n – 2)/2`
  2. sift down `contents[i]` and all elements to its left

- Example:

0	1	2	3	4	5	6
5	16	8	14	20	1	26

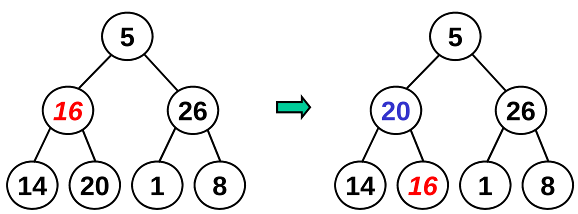


- Last element's parent = `contents[(7 – 2)/2]` = `contents[2]`.
  Sift it down:

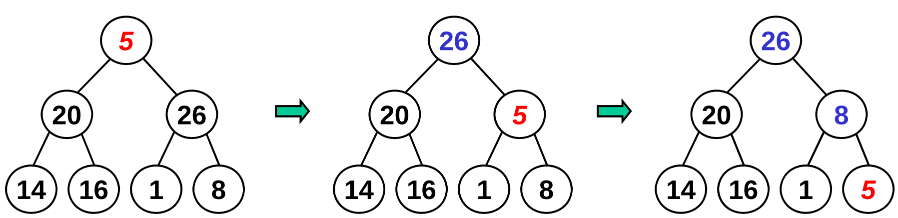


Converting an Array to a Heap (cont.)

- Next, sift down `contents[1]`:



- Finally, sift down `contents[0]`:

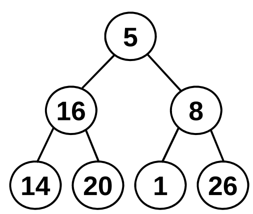


Creating a Heap from an Array

```
public class Heap {
    private HeapItem[] contents;
    private int numItems;
    ...
    public Heap(HeapItem[] arr) {
        // Note that we don't copy the array!
        contents = arr;
        numItems = arr.length;
        makeHeap();
    }

    private void makeHeap() {
        int last = contents.length - 1;
        int parentOfLast = (last - 1)/2;
        for (int i = parentOfLast; i >= 0; i--)
            siftDown(i);
    }
    ...
}
```

Time Complexity of a Heap



- A heap containing n items has a height <= $\log_2 n$.
- Thus, removal and insertion are both $O(\log n)$.
  - remove: go down at most $\log_2 n$ levels when sifting down from the root, and do a constant number of operations per level
  - insert: go up at most $\log_2 n$ levels when sifting up to the root, and do a constant number of operations per level
- This means we can use a heap for a $O(\log n)$-time priority queue.
- Time complexity of creating a heap from an array?

Using a Heap to Sort an Array

- Recall selection sort: it repeatedly finds the smallest remaining element and swaps it into place:

0	1	2	3	4	5	6
5	**16**	**8**	**14**	**20**	**1**	**26**

0	1	2	3	4	5	6
1	**16**	**8**	**14**	**20**	**5**	**26**

0	1	2	3	4	5	6
1	***5***	**8**	**14**	**20**	**16**	**26**

…

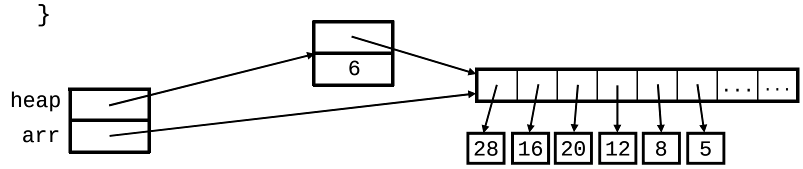
- It isn't efficient ($O(n^2)$), because it performs a linear scan to find the smallest remaining element ($O(n)$ steps per scan).
- Heapsort is a sorting algorithm that repeatedly finds the *largest* remaining element and puts it in place.
- It *is* efficient ($O(n \log n)$), because it turns the array into a heap, which means that it can find and remove the largest remaining element in $O(\log n)$ steps.

Heapsort (~cscie119/examples/heaps/HeapSort.java)

```
public class HeapSort {
    public static void heapSort(HeapItem[] arr) {
        // Turn the array into a max-at-top heap.
        // The heap object will hold a reference to the
        // original array, with the elements rearranged.
        Heap heap = new Heap(arr);

        int endUnsorted = arr.length - 1;
        while (endUnsorted > 0) {
            // Get the largest remaining element and put it
            // where it belongs -- at the end of the portion
            // of the array that is still unsorted.
            HeapItem largestRemaining = heap.remove();
            arr[endUnsorted] = largestRemaining;

            endUnsorted--;
        }
    }
}
```

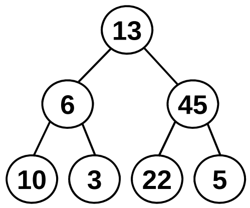


Heapsort Example

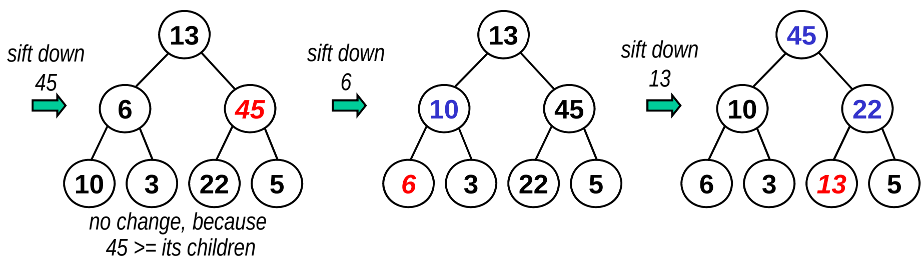
- Sort the following array:

0	1	2	3	4	5	6
13	6	45	10	3	22	5

- Here's the corresponding complete tree:

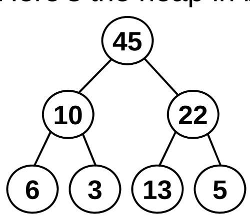


- Begin by converting it to a heap:



Heapsort Example (cont.)

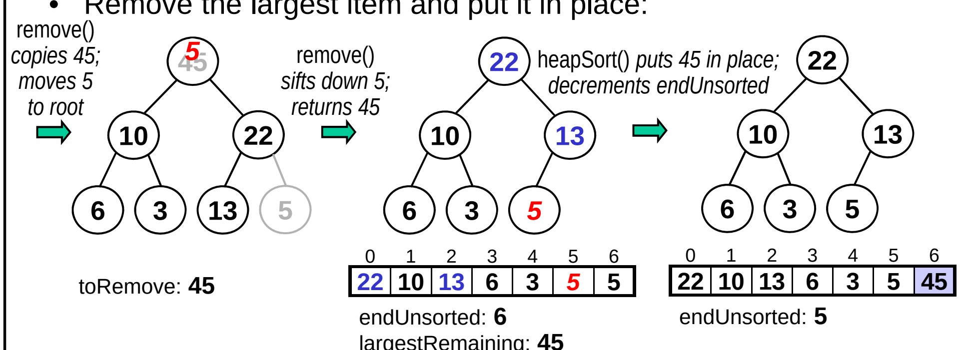
- Here's the heap in both tree and array forms:



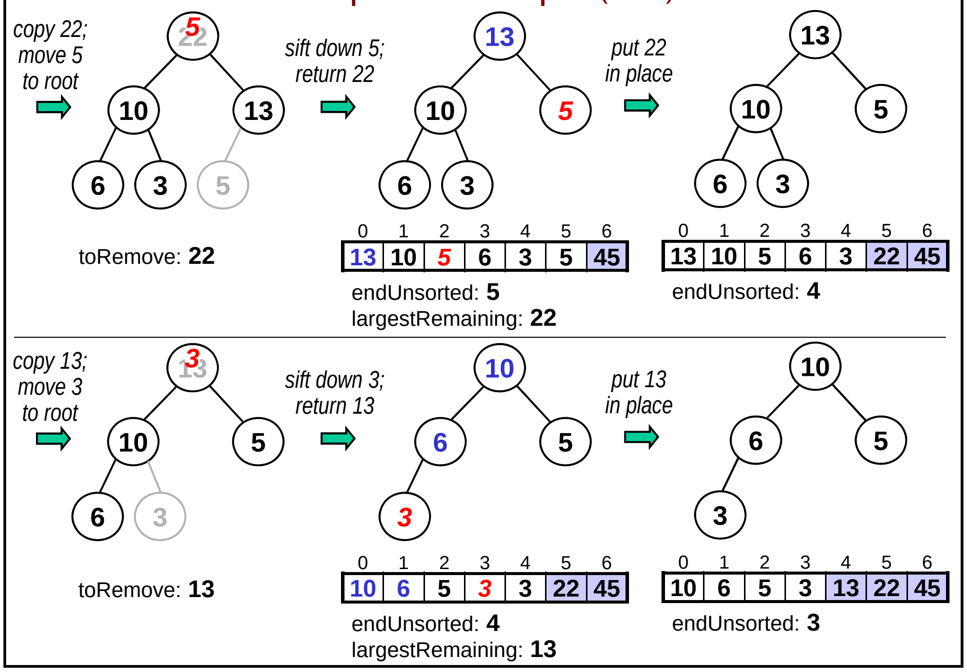
0	1	2	3	4	5	6
45	10	22	6	3	13	5

endUnsorted: **6**

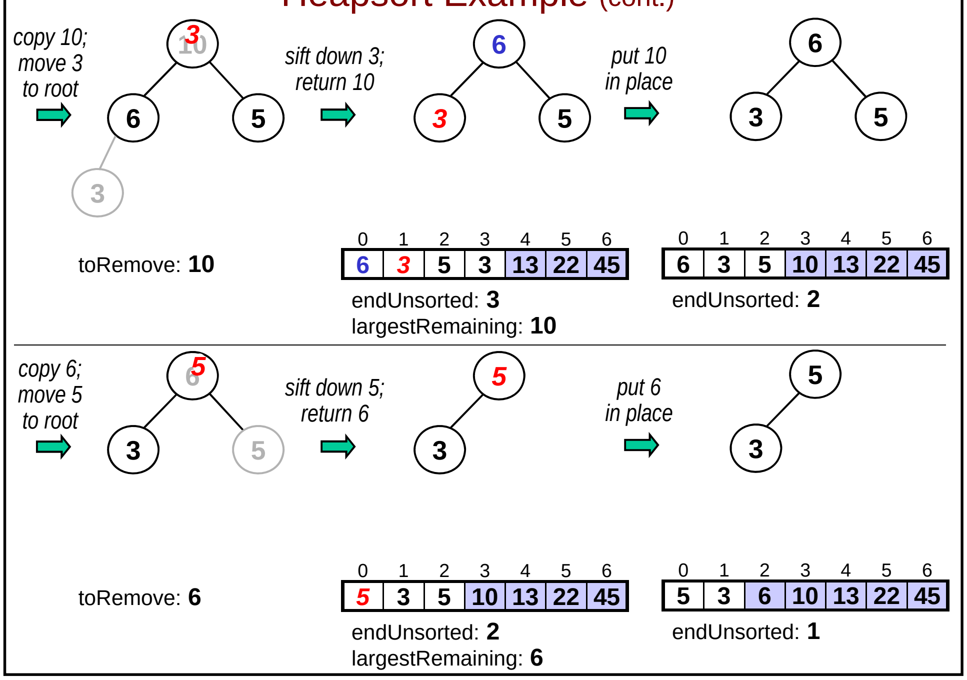
- Remove the largest item and put it in place:



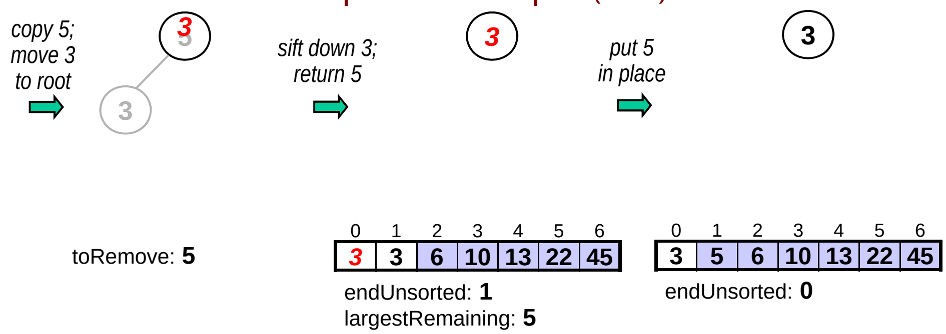
Heapsort Example (cont.)



Heapsort Example (cont.)



Heapsort Example (cont.)



How Does Heapsort Compare?

algorithm	best case	avg case	worst case	extra memory
selection sort	$O(n^2)$	$O(n^2)$	$O(n^2)$	$O(1)$
insertion sort	$O(n)$	$O(n^2)$	$O(n^2)$	$O(1)$
Shell sort	$O(n \log n)$	$O(n^{1.5})$	$O(n^{1.5})$	$O(1)$
bubble sort	$O(n^2)$	$O(n^2)$	$O(n^2)$	$O(1)$
quicksort	$O(n \log n)$	$O(n \log n)$	$O(n^2)$	$O(1)$
mergesort	$O(n \log n)$	$O(n \log n)$	$O(n \log n)$	$O(n)$
heapsort	$O(n \log n)$	$O(n \log n)$	$O(n \log n)$	$O(1)$

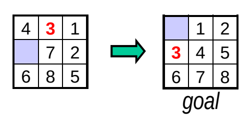
- Heapsort matches mergesort for the best worst-case time complexity, but it has better space complexity.
- Insertion sort is still best for arrays that are almost sorted.
    - heapsort will scramble an almost sorted array before sorting it
- Quicksort is still typically fastest in the average case.

State-Space Search: Estimating the Remaining Cost

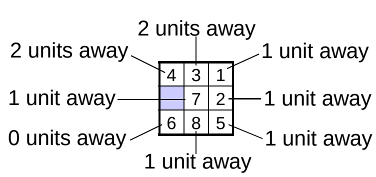
- As mentioned earlier, informed search algorithms associate a priority with each successor state that is generated.
- The priority is based in some way on the remaining cost – i.e., the cost of getting from the state to the closest goal state.
  - for the 8 puzzle, remaining cost = # of steps to closest goal
- For most problems, we can't determine the exact remaining cost.
  - if we could, we wouldn't need to search!
- Instead, we estimate the remaining cost using a *heuristic function* h(x) that takes a state x and computes a cost estimate for it.
  - heuristic = rule of thumb
- To find optimal solutions, we need an *admissable* heuristic – one that never overestimates the remaining cost.

Heuristic Function for the Eight Puzzle

- Manhattan distance = horizontal distance + vertical distance
  - example: For the board at right, the Manhattan distance of the 3 tile from its position in the goal state = 1 column + 1 row = 2



- Use h(x) = sum of the Manhattan distances of the tiles in x from their positions in the goal state
  - for our example:

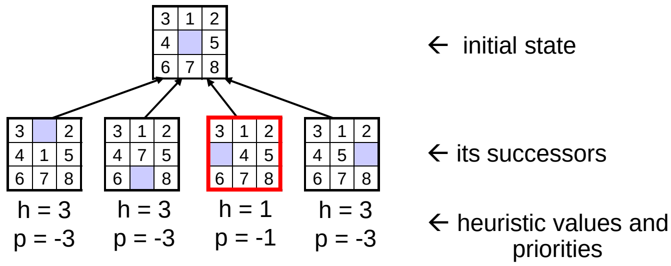


h(x) = 1 + 1 + 2 + 2
+ 1 + 0 + 1 + 1 = 9

- This heuristic is admissible because each of the operators (move blank up, move blank down, etc.) moves a single tile a distance of 1, so it will take at least h(x) steps to reach the goal.

Greedy Search

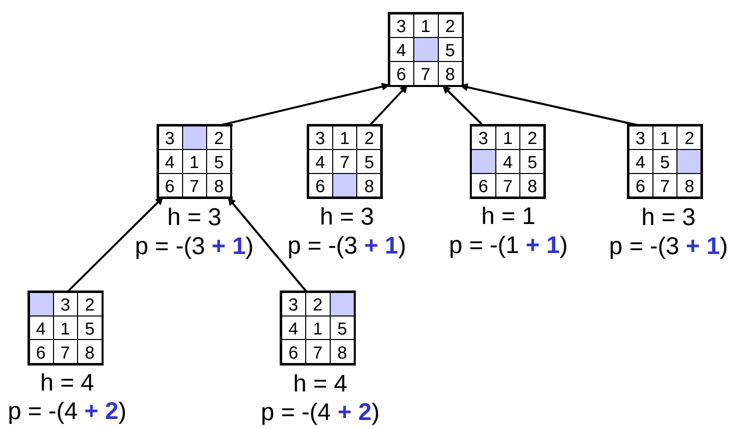
- Priority of state x, p(x) = -1 * h(x)
  - mult. by -1 so states closer to the goal have higher priorities



- Greedy search would consider the highlighted successor before the other successors, because it has the highest priority.
- Greedy search is:
  - incomplete: it may not find a solution
    - it could end up going down an infinite path
  - not optimal: the solution it finds may not have the lowest cost
    - it fails to consider the cost of getting *to* the current state

A* Search

- Priority of state x, p(x) = -1 * (h(x) + g(x))
  where g(x) = the cost of getting from the initial state to x



- Incorporating g(x) allows A* to find an optimal solution – one with the minimal *total* cost.

Characteristics of A*

- It is complete and optimal.
  - provided that h(x) is admissable, and that g(x) increases or stays the same as the depth increases
- Time and space complexity are still typically exponential in the solution depth, d – i.e., the complexity is $O(b^d)$ for some value b.
- However, A* typically visits far fewer states than other optimal state-space search algorithms.

solution depth	iterative deepening	A* w/ Manhattan dist. heuristic
4	112	12
8	6384	25
12	364404	73
16	did not complete	211
20	did not complete	676

Source: Russell & Norvig, *Artificial Intelligence: A Modern Approach*, Chap. 4.

The numbers shown are the average number of search nodes visited in 100 randomly generated problems for each solution depth.

The searches do *not* appear to have excluded previously seen states.

- Memory usage can be a problem, but it's possible to address it.

Implementing Informed Search (~cscie119/examples/search)

- Add new subclasses of the abstract `Searcher` class.
- For example:

```
public class GreedySearcher extends Searcher {
    private Heap nodePQueue;

    public void addNode(SearchNode node) {
        nodePQueue.insert(
          new HeapItem(node, -1 * node.getCostToGoal()));
    }
    …
```