Review of probability and statistics

Machine Learning I (2022-2023)
UMONS

Exercise 1

An economics consulting firm has created a model to predict recessions. The model predicts a recession with probability 80% when a recession is indeed coming and with probability 10% when no recession is coming. The unconditional probability of falling into a recession is 20%. If the model predicts a recession, what is the probability that a recession will indeed come?

Exercise 2

For the following joint distributions between random variables Y and X, find both marginal distributions and the conditional distribution requested. Are the two random variables independent?

2.1

Find the marginal distributions and the distribution of Y conditional on $X = 0$.

	$X = 0$	$X = 1$
$Y = 0$	0.14	0.26
$Y = 1$	0.21	0.39

Also, compute the joint expectation $\mathbb{E}_{XY}[s_i(X,Y)]$ for the following functions:

1) $s_1(X,Y) = X^2 + 3Y + 1$.
2) $s_2(X,Y) = XY^3 - 4X + 2Y$.

Additionally, compute the expectation of s_1 conditional on $X = 0$, i.e. $\mathbb{E}_{Y|X}[s_1(X,Y)|X = 0]$, and the expectation of s_2 conditional on $Y = 1$, i.e. $\mathbb{E}_{X|Y}[s_2(X,Y)|Y = 1]$.

2.2

Find the marginal distributions and the distribution of X conditional on $Y = 1$.

	$X = 0$	$X = 1$
$Y = 1$	0.45	0.25
$Y = 3$	0.05	0.25

2.3

Find the marginal distributions and the distribution of Y conditional on $X = 1$.

	$X = 0$	$X = 1$	$X = 2$
$Y = 1$	0.1	0.2	0.3
$Y = 2$	0.05	0.15	0.2

2.4

Find the marginal distributions and the distribution of Y conditional on $X = 2$.

	$X = 0$	$X = 1$	$X = 2$
$Y = 1$	0.05	0.04	0.01
$Y = 2$	0.1	0.08	0.02
$Y = 3$	0.35	0.28	0.07

Exercise 3

Alex and Bob each flips a fair coin twice. Denote "1" as head, and "0" as tail. Let X be the maximum of the two numbers Alex gets, and let Y be the minimum of the two numbers Bob gets.

a) Find the marginal pmf $p_X(x)$ and $p_Y(y)$.

b) Find the joint pmf $p_{X,Y}(x,y)$.

c) Find the conditional pmf $p_{X|Y}(x|y)$. Does $p_{X|Y}(x|y) = p_X(x)$? Why ?

Exercise 4

We have a population of people, 47% of whom were men and the remaining 53% were women. Suppose that the average height of the men was 70 inches, and the women was 71 inches. What is the average height of the entire population? [Hint: Use the law of total expectation]

Exercise 5

Let $X_1, X_2, \ldots, X_n$ be a collection of n random variables, and $a_1, a_2, \ldots, a_n$, a set of constants, we have

$$\operatorname{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \operatorname{Cov}(X_i, X_j).$$

Prove the above fact. You can use the fact that, for a set of numbers $e_1, e_2, \ldots, e_n$,

$$\left(\sum_{i=1}^{n} e_i\right)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} e_i e_j.$$

Exercise 6

Let p_X be a normal distribution $\mathcal{N}(\mu, \sigma^2)$ where $\mu \in \mathbb{R}$, and $\sigma > 0$. Consider the two scenarios where $n = 10$ or $n = 1000$. For each scenario,

1. repeat the following procedure 1000 times:
   (a) Generate n i.i.d. realizations $X_1, X_2, \ldots, X_n$ where $X_i \sim p_X$.
   (b) Compute $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.
2. compute the mean and variance of the 1000 values computed in 1(b)
3. plot a histogram of these 1000 values, and add vertical lines at the true mean and the computed mean.

Experiment with different values of μ and σ, and confirm that you obtain $E[\bar{X}_n] = \mu$ and $\mathrm{Var}(\bar{X}_n) = \frac{\sigma^2}{n}$.

Exercise 7

You observe a sample of real values $y_1, y_2, \ldots, y_n$ where $y_i > 1$ for $i = 1, 2, \ldots, n$. Let us assume they are all i.i.d. observations of a random variable Y with the following probability density function:

$$p(y;\alpha) = \begin{cases} \alpha e^{-\alpha y}, & \text{if } y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

1. Write down the formula for the log-likelihood as a function of the observed data and the unknown parameter α.

2. Compute the maximum likelihood estimate (MLE) of α.

Complementary exercise

Find the marginal pdf $f_X(x)$ if the joint pdf $f_{XY}(x,y)$ is defined as :

$$f_{XY}(x,y) = \frac{e^{-|y-x|-x^2/2}}{2\sqrt{2\pi}}$$