Review of probability and statistics

Machine Learning I (2023-2024)
UMONS

Exercise 1

An economics consulting firm has created a model to predict recessions. The model predicts a recession with probability 80% when a recession is indeed coming and with probability 10% when no recession is coming. The (unconditional) probability of falling into a recession is 20%. If the model predicts a recession, what is the probability that a recession will indeed come?

Exercise 2

Answer the questions for the following joint distributions between random variables X and Y.

2.1

Given the following joint PMF:

	$X=0$	$X=1$
$Y=0$	0.14	0.26
$Y=1$	0.21	0.39

a) Compute the marginal PMF of X and the marginal PMF of Y.

b) Compute the conditional PMF of Y given $X=0$.

c) Given $s_1(X,Y) = X^2 + 3Y + 1$, compute the joint expectation $\mathbb{E}_{XY}[s_1(X,Y)]$ and the conditional expectation $\mathbb{E}_{Y|X}[s_1(X,Y)|X=0]$.

d) Given $s_2(X,Y) = XY^3 - 4X + 2Y$, compute the joint expectation $\mathbb{E}_{XY}[s_2(X,Y)]$ and the conditional expectation $\mathbb{E}_{XY}[s_2(X,Y)|Y=1]$.

e) Are X and Y independent?

2.2

Given the following joint PMF:

	$X = 0$	$X = 1$	$X = 2$
$Y = 1$	0.1	0.2	0.3
$Y = 2$	0.05	0.15	0.2

a) Compute the marginal PMF of X and the marginal PMF of Y.

b) Compute the conditional PMF of Y given $X = 1$.

c) Are X and Y independent?

Exercise 3

Alex and Bob each flip a different fair coin twice. Denote "1" as head, and "0" as tail. Let X be the maximum of the two numbers Alex gets, and let Y be the minimum of the two numbers Bob gets.

a) Find the marginal PMF $p_X(x)$ and $p_Y(y)$.

b) Find the joint PMF $p_{X,Y}(x,y)$.

c) Find the conditional PMF $p_{X|Y}(x|y)$. Does $p_{X|Y}(x|y) = p_X(x)$? Why?

Exercise 4

We have a population of people, 47% of whom were men and the remaining 53% were women. Suppose that the average height of the men was 70 inches, and the women was 71 inches. What is the average height of the entire population? [Hint: Use the law of total expectation]

Exercise 5

Let $X_1, X_2, \ldots, X_n \in \mathbb{R}$ be a collection of n random variables, and $a_1, a_2, \ldots, a_n$, a set of constants, we have

$$\operatorname{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \operatorname{Cov}(X_i, X_j).$$

Prove the above fact. You can use the fact that, for a set of numbers $e_1, e_2, \ldots, e_n$,

$$\left(\sum_{i=1}^{n} e_i\right)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} e_i e_j.$$

Exercise 6

We observe a sample of real values $y_1, y_2, \ldots, y_n$ where $y_i \geq 0$ for $i = 1, 2, \ldots, n$. Let us assume they are all i.i.d. observations of a random variable Y with an exponential distribution:

$$p(y; \alpha) = \alpha e^{-\alpha y}$$

where $\alpha > 0$ is called the rate.

a) Write down the formula of the likelihood function as a function of the observed data and the unknown parameter α.

b) Write down the formula of the log-likelihood

c) Compute the maximum likelihood estimate (MLE) of α.

Exercise 7

We observe a sample of i.i.d. pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ where $(x_i, y_i) \in \mathcal{X} \times \mathbb{R}$ for $i = 1, 2, \ldots, n$. We assume that the conditional PDF $p(y; x)$ is normally distributed with a variance fixed at σ^2. Given an input x, the mean $\mu_\theta(x)$ is determined by a model μ_θ with parameters $\theta \in \Theta$. More specifically, the conditional PDF is given by:

$$\begin{aligned} p(y; x, \theta) &= \mathcal{N}(y; \mu_\theta(x), \sigma^2) \\ &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu_\theta(x)}{\sigma}\right)^2}. \end{aligned}$$

Note that the specific sets $\mathcal{X}$ and Θ are not relevant to our problem. For example, we could have $\mathcal{X} = \mathbb{R}$ and $\Theta = \mathbb{R}^2$ for a uni-dimensional linear regression task with one coefficient and one bias.

a) Write down the formula of the likelihood function as a function of the observed data and parameters θ.

b) Write down the formula of the log-likelihood.

c) Can you prove that maximizing the likelihood is equivalent to minimizing the mean squared error $\frac{1}{n}\sum_{i=1}^{n}(\mu_\theta(x_i) - y_i)^2$ (with respect to θ)?

Complementary exercise

Find the marginal PDF $f_X(x)$ if the joint PDF $f_{XY}(x,y)$ is defined as:

$$f_{XY}(x,y) = \frac{e^{-|y-x|-x^2/2}}{2\sqrt{2\pi}}$$

Complementary exercise

Let p_X be a normal distribution $\mathcal{N}(\mu, \sigma^2)$ where $\mu \in \mathbb{R}$, and $\sigma > 0$. Consider the two scenarios where $n = 10$ or $n = 1000$. For each scenario,

1. repeat the following procedure 1000 times:
   (a) Generate n i.i.d. realizations $X_1, X_2, \ldots, X_n$ where $X_i \sim p_X$.
   (b) Compute $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.
2. compute the mean and variance of the 1000 values computed in 1(b)
3. plot a histogram of these 1000 values, and add vertical lines at the true mean and the computed mean.

Experiment with different values of μ and σ, and confirm that you obtain $E[\bar{X}_n] = \mu$ and $\text{Var}(\bar{X}_n) = \frac{\sigma^2}{n}$.