

Review of probability and statistics

Machine Learning I (2023-2024)
UMONS

Exercise 1

An economics consulting firm has created a model to predict recessions. The model predicts a recession with probability 80% when a recession is indeed coming and with probability 10% when no recession is coming. The unconditional probability of falling into a recession is 20%. If the model predicts a recession, what is the probability that a recession will indeed come?

Solution:

Let $R \in \{0, 1\}$ be a Bernoulli random variable indicating whether we fell into a recession ($R = 1$ means we fell into a recession, $R = 0$ means we did not). Let $M \in \{0, 1\}$ be another Bernoulli random variable indicating the outcome of the prediction model ($M = 1$ means that the model predicted that a recession was coming, $M = 0$ means that it did not).

We know that $\mathbb{P}(R = 1) = 0.2$, $\mathbb{P}(M = 1|R = 1) = 0.8$ and $\mathbb{P}(M = 1|R = 0) = 0.1$. We are interested in finding the probability that a recession will come, conditional on the fact that the model predicted it, i.e. we are looking for $\mathbb{P}(R = 1|M = 1)$. We have:

$$\begin{aligned}\mathbb{P}(R = 1|M = 1) &= \frac{\mathbb{P}(R = 1, M = 1)}{\mathbb{P}(M = 1)} && \text{(Conditional probability)} \\ &= \frac{\mathbb{P}(M = 1|R = 1)\mathbb{P}(R = 1)}{\mathbb{P}(M = 1)} && \text{(Conditional probability)} \\ &= \frac{\mathbb{P}(M = 1|R = 1)\mathbb{P}(R = 1)}{\mathbb{P}(M = 1|R = 1)\mathbb{P}(R = 1) + \mathbb{P}(M = 1|R = 0)\mathbb{P}(R = 0)} && \text{(Law of total probability)} \\ &= \frac{\mathbb{P}(M = 1|R = 1)\mathbb{P}(R = 1)}{\mathbb{P}(M = 1|R = 1)\mathbb{P}(R = 1) + (1 - \mathbb{P}(M = 0|R = 0))(1 - \mathbb{P}(R = 1))} \\ &= \frac{0.8 \times 0.2}{0.8 \times 0.2 + 0.1 \times 0.8} \\ &= \frac{2}{3}\end{aligned}$$

Exercise 2

Answer the questions for the following joint distributions between random variables X and Y .

2.1

Given the following joint PMF:

	$X = 0$	$X = 1$
$Y = 0$	0.14	0.26
$Y = 1$	0.21	0.39

- Compute the marginal PMF of X and the marginal PMF of Y .
- Compute the conditional PMF of Y given $X = 0$.
- Given $s_1(X, Y) = X^2 + 3Y + 1$, compute the joint expectation $\mathbb{E}_{XY}[s_1(X, Y)]$ and the conditional expectation $\mathbb{E}_{Y|X}[s_1(X, Y)|X = 0]$.
- Given $s_2(X, Y) = XY^3 - 4X + 2Y$, compute the joint expectation $\mathbb{E}_{XY}[s_2(X, Y)]$ and the conditional expectation $\mathbb{E}_{XY}[s_2(X, Y)|Y = 1]$.
- Are X and Y independent?

Solution:

Marginal PMFs

$$p_X(0) = p_{XY}(0, 0) + p_{XY}(0, 1) = 0.14 + 0.21 = 0.35$$

$$p_X(1) = p_{XY}(1, 0) + p_{XY}(1, 1) = 0.26 + 0.39 = 0.65$$

$$p_Y(0) = p_{XY}(0, 0) + p_{XY}(1, 0) = 0.14 + 0.26 = 0.4$$

$$p_Y(1) = p_{XY}(0, 1) + p_{XY}(1, 1) = 0.21 + 0.39 = 0.6$$

Conditional PMFs

$$p_{Y|X}(1|0) = \frac{p_{XY}(0,1)}{p_X(0)} = \frac{0.21}{0.35} = 0.6$$

$$p_{Y|X}(0|0) = \frac{p_{XY}(0,0)}{p_X(0)} = \frac{0.14}{0.35} = 0.4$$

Expectations

1) s_1

$$\begin{aligned}\mathbb{E}_{XY}[s_1(X, Y)] &= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} s_1(X, Y) p_{XY}(x, y) \\ &= s_1(0, 0) p_{XY}(0, 0) + s_1(0, 1) p_{XY}(0, 1) + s_1(1, 0) p_{XY}(1, 0) + s_1(1, 1) p_{XY}(1, 1) \\ &= 0.14 + 4 \times 0.21 + 2 \times 0.26 + 5 \times 0.39 = 3.45\end{aligned}$$

$$\begin{aligned}\mathbb{E}_{Y|X}[s_1(X, Y)|X = 0] &= \sum_{y \in \mathcal{Y}} s_1(0, Y) p_{Y|X}(y|x = 0) \\ &= s_1(0, 0) p_{Y|X}(0|0) + s_1(0, 1) p_{Y|X}(1|0) \\ &= 0.4 + 4 \times 0.6 = 2.8\end{aligned}$$

2) s_2

$$\begin{aligned}\mathbb{E}_{XY}[s_2(X, Y)] &= s_2(0, 0) p_{XY}(0, 0) + s_2(0, 1) p_{XY}(0, 1) + s_2(1, 0) p_{XY}(1, 0) + s_2(1, 1) p_{XY}(1, 1) \\ &= 0 + 2 \times 0.21 - 4 \times 0.26 - 0.39 = -1.01\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_{X|Y}[s_2(X, Y)|Y = 1] &= \sum_{x \in \mathcal{X}} s_2(x, 1)p_{X|Y}(x|y = 1) \\
&= s_2(0, 1)p_{X|Y}(0|1) + s_2(1, 1)p_{X|Y}(1|1) \\
&= s_2(0, 1)\frac{p_{XY}(0,1)}{p_Y(1)} + s_2(1, 1)\frac{p_{XY}(1,1)}{p_Y(1)} \\
&= 2 \times \frac{0.21}{0.6} - \frac{0.39}{0.6} = 0.05
\end{aligned}$$

Independence

Two discrete random variables X and Y are independent iff

$$p_{XY}(x, y) = p_X(x)p_Y(y) \quad \forall x \in \mathcal{X}, y \in \mathcal{Y}.$$

We must check that the equality holds for all realizations of the random variables X and Y :

$$\begin{aligned}
p_{XY}(0, 0) &= 0.14 = p_X(0)p_Y(0) \\
p_{XY}(0, 1) &= 0.21 = p_X(0)p_Y(1) \\
p_{XY}(1, 0) &= 0.26 = p_X(1)p_Y(0) \\
p_{XY}(1, 1) &= 0.39 = p_X(1)p_Y(1)
\end{aligned}$$

The random variables X and Y are independent.

2.2

Given the following joint PMF:

	$X = 0$	$X = 1$	$X = 2$
$Y = 1$	0.1	0.2	0.3
$Y = 2$	0.05	0.15	0.2

- a) Compute the marginal PMF of X and the marginal PMF of Y .
- b) Compute the conditional PMF of Y given $X = 1$.
- c) Are X and Y independent?

Solution:

Marginal PMFs

$$p_X(0) = 0.15$$

$$p_X(1) = 0.35$$

$$p_X(2) = 0.5$$

$$p_Y(1) = 0.6$$

$$p_Y(2) = 0.4$$

Conditional PMFs

$$p_{Y|X}(1|1) = 0.57$$

$$p_{Y|X}(2|1) = 0.43$$

Independence

$$p_{XY}(0, 1) = 0.1 \neq 0.09 = p_X(0)p_Y(1) \rightarrow \text{Not independent.}$$

Exercise 3

Alex and Bob each flip a different fair coin twice. Denote “1” as head, and “0” as tail. Let X be the maximum of the two numbers Alex gets, and let Y be the minimum of the two numbers Bob gets.

- a) Find the marginal PMF $p_X(x)$ and $p_Y(y)$.
- b) Find the joint PMF $p_{X,Y}(x,y)$.
- c) Find the conditional PMF $p_{X|Y}(x|y)$. Does $p_{X|Y}(x|y) = p_X(x)$? Why ?

Solution:

For both Alex and Bob, the sample space for flipping a fair coin twice is $\Omega = \{00, 01, 10, 11\}$. if X and Y are the random variables respectively denoting the maximum of the two numbers Alex gets and the minimum of the two numbers Bob gets, then $\mathcal{X} = \{0, 1\}$ and $\mathcal{Y} = \{0, 1\}$.

- a) From the sample space, we find:

$$\begin{aligned}p_X(0) &= \frac{1}{4} \\p_X(1) &= \frac{3}{4} \\p_Y(0) &= \frac{3}{4} \\p_Y(1) &= \frac{1}{4}\end{aligned}$$

- b) By definition of the joint pmf and as the random variables X and Y are independent, we have:

$$p_{XY}(0,0) = p_X(0)p_Y(0) = \frac{1}{4} \times \frac{3}{4} = \frac{3}{16}$$

$$p_{XY}(0,1) = p_X(0)p_Y(1) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

$$p_{XY}(1,0) = p_X(1)p_Y(0) = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$$

$$p_{XY}(1,1) = p_X(1)p_Y(1) = \frac{3}{4} \times \frac{1}{4} = \frac{3}{16}$$

We can check that $\sum_{x \in \mathcal{X}, y \in \mathcal{Y}} p_{XY}(x,y) = \frac{3}{16} + \frac{1}{16} + \frac{9}{16} + \frac{3}{16} = 1$.

- c) As the variables X and Y are independent, we have that $p_{X|Y}(x|y) = p_X(x)$ and $p_{Y|X}(y|x) = p_Y(y)$. Therefore,

$$p_{X|Y}(0|0) = p_{X|Y}(0|1) = p_X(0) = \frac{1}{4}.$$

Exercise 4

We have a population of people, 47% of whom were men and the remaining 53% were women. Suppose that the average height of the men was 70 inches, and the women was 71 inches. What is the average height of the entire population? [Hint: Use the law of total expectation]

Solution:

Let M be a Bernoulli random variable with support $\mathcal{M} \in \{0, 1\}$ indicating whether an individual is either male or female ($M = 1$ means that the individual is male, $M = 0$ means that the individual is female). Let H be a continuous random variable with support $\mathcal{H} \in \mathbb{R}^+$ indicating the height of an individual of the population.

We know that $p_M(1) = 0.47$ and that $p_M(0) = 0.53$. Moreover, $\mathbb{E}[H|M = 1] = 70$ and $\mathbb{E}[H|M = 0] = 71$. We are interested in finding the average height of the entire population, i.e $\mathbb{E}[H]$.

From the law of total expectation, we have:

$$\begin{aligned}\mathbb{E}[H] &= \mathbb{E}[H|M = 1]p_M(1) + \mathbb{E}[H|M = 0]p_M(0) \\ &= 70 \times 0.47 + 71 \times 0.53 \\ &= 70.53 \text{ inches}\end{aligned}$$

Exercise 5

Let X_1, X_2, \dots, X_n be a collection of n random variables, and a_1, a_2, \dots, a_n , a set of constants, we have

$$\text{Var} \left(\sum_{i=1}^n a_i X_i \right) = \sum_{i=1}^n \sum_{j=1}^n a_i a_j \text{Cov}(X_i, X_j).$$

Prove the above fact. You can use the fact that, for a set of numbers e_1, e_2, \dots, e_n ,

$$\left(\sum_{i=1}^n e_i \right)^2 = \sum_{i=1}^n \sum_{j=1}^n e_i e_j.$$

Solution:

By expanding the expression of the variance, and by successively applying the properties of the expectation, we get :

$$\begin{aligned} \text{Var} \left(\sum_{i=1}^n a_i X_i \right) &= \mathbb{E} \left[\left(\sum_{i=1}^n a_i X_i - \mathbb{E} \left[\left(\sum_{i=1}^n a_i X_i \right) \right] \right)^2 \right] \\ &= \mathbb{E} \left[\left(\sum_{i=1}^n a_i X_i \right)^2 - 2 \mathbb{E} \left[\sum_{i=1}^n a_i X_i \right] \left(\sum_{i=1}^n a_i X_i \right) + \left(\mathbb{E} \left[\sum_{i=1}^n a_i X_i \right] \right)^2 \right] \\ &= \mathbb{E} \left[\sum_{i=1}^n \sum_{j=1}^n a_i a_j X_i X_j - 2 \left(\sum_{i=1}^n a_i \mathbb{E}[X_i] \right) \left(\sum_{i=1}^n a_i X_i \right) + \left(\mathbb{E} \left[\sum_{i=1}^n a_i X_i \right] \right)^2 \right] \\ &= \mathbb{E} \left[\sum_{i=1}^n \sum_{j=1}^n a_i a_j X_i X_j \right] - 2 \left(\sum_{i=1}^n a_i \mathbb{E}[X_i] \right) \left(\sum_{i=1}^n a_i \mathbb{E}[X_i] \right) + \left(\mathbb{E} \left[\sum_{i=1}^n a_i X_i \right] \right)^2 \\ &= \mathbb{E} \left[\sum_{i=1}^n \sum_{j=1}^n a_i a_j X_i X_j \right] - 2 \left(\sum_{i=1}^n a_i \mathbb{E}[X_i] \right)^2 + \left(\sum_{i=1}^n a_i \mathbb{E}[X_i] \right)^2 \\ &= \mathbb{E} \left[\sum_{i=1}^n \sum_{j=1}^n a_i a_j X_i X_j \right] - \left(\sum_{i=1}^n a_i \mathbb{E}[X_i] \right)^2 \\ &= \mathbb{E} \left[\sum_{i=1}^n \sum_{j=1}^n a_i a_j X_i X_j \right] - \left(\sum_{i=1}^n \sum_{j=1}^n a_i a_j \mathbb{E}[X_i] \mathbb{E}[X_j] \right) \\ &= \sum_{i=1}^n \sum_{j=1}^n a_i a_j \left(\mathbb{E}[X_i X_j] - \mathbb{E}[X_i] \mathbb{E}[X_j] \right) \\ &= \sum_{i=1}^n \sum_{j=1}^n a_i a_j \text{Cov}(X_i, X_j) \end{aligned}$$

Exercise 6

We observe a sample of real values y_1, y_2, \dots, y_n where $y_i \geq 0$ for $i = 1, 2, \dots, n$. Let us assume they are all i.i.d. observations of a random variable Y with an exponential distribution:

$$p(y; \alpha) = \alpha e^{-\alpha y}$$

where $\alpha > 0$ is called the rate.

- Write down the formula of the likelihood function as a function of the observed data and the unknown parameter α .
- Write down the formula of the log-likelihood
- Compute the maximum likelihood estimate (MLE) of α .

Solution:

a)

$$\begin{aligned}\mathcal{L}(\alpha) &= \mathcal{L}(\alpha; y_1, \dots, y_n) \\ &= p(y_1, y_2, \dots, y_n; \alpha) \\ &= p(y_1; \alpha) p(y_2; \alpha) \dots p(y_n; \alpha) \quad (y_i, i = 1, \dots, n \text{ are i.i.d. random variables.}) \\ &= \prod_{i=1}^n p(y_i; \alpha) \\ &= \prod_{i=1}^n \alpha e^{-\alpha y_i}.\end{aligned}$$

b)

$$\begin{aligned}\log \mathcal{L}(\alpha) &= \log \left(\prod_{i=1}^n \alpha e^{-\alpha y_i} \right) \\ &= \sum_{i=1}^n (\log \alpha - \alpha y_i) \\ &= n(\log \alpha - \alpha \bar{y}),\end{aligned}$$

where $\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$.

c)

$$\begin{aligned}\hat{\alpha} &= \operatorname{argmax}_{\alpha \in \mathbb{R}} \mathcal{L}(\alpha) \\ &= \operatorname{argmax}_{\alpha \in \mathbb{R}} \log \mathcal{L}(\alpha) \\ &= \operatorname{argmax}_{\alpha \in \mathbb{R}} (n(\log \alpha - \alpha \bar{y}))\end{aligned}$$

Taking the derivative with respect to α and equaling to zero :

$$\left(\log \mathcal{L} \right)'(\alpha) = n \left(\frac{1}{\alpha} - \bar{y} \right)$$

$$\begin{aligned}\left(\log \mathcal{L} \right)'(\alpha) &= 0 \\ \iff n \left(\frac{1}{\alpha} - \bar{y} \right) &= 0 \\ \iff \alpha &= \frac{1}{\bar{y}}\end{aligned}$$

Thus we have that the MLE $\hat{\alpha} = \frac{1}{\bar{y}}$. To check that $\hat{\alpha}$ is indeed a maximum, we can verify that the second derivative of the log-likelihood computed in $\hat{\alpha}$ is always negative :

$$\left(\log \mathcal{L}\right)''(\hat{\alpha}) = -\frac{n}{\hat{\alpha}^2} < 0 \quad \forall \hat{\alpha} \in \mathbb{R}$$

Exercise 7

We observe a sample of i.i.d. pairs $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ where $(x_i, y_i) \in \mathcal{X} \times \mathbb{R}$ for $i = 1, 2, \dots, n$. We assume that the conditional PDF $p(y; x)$ is normally distributed with a variance fixed at σ^2 . Given an input x , the mean $\mu_\theta(x)$ is determined by a model μ_θ with parameters $\theta \in \Theta$:

$$\begin{aligned} p(y; x, \theta) &= \mathcal{N}(\mu_\theta(x), \sigma^2) \\ &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{y - \mu_\theta(x)}{\sigma} \right)^2}. \end{aligned}$$

- Write down the formula of the likelihood function as a function of the observed data and parameters θ .
- Write down the formula of the log-likelihood.
- Can you prove that maximizing the likelihood is equivalent to minimizing the mean squared error $\frac{1}{n} \sum_{i=1}^n (\mu_\theta(x_i) - y_i)^2$ (with respect to θ)?

Solution:

a)

$$\begin{aligned} \mathcal{L}(\theta) &= p(y_1, \dots, y_n; x_1, \dots, x_n, \theta) \\ &= \prod_{i=1}^n p(y_i; x_i, \theta) \\ &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{y_i - \mu_\theta(x_i)}{\sigma} \right)^2} \end{aligned}$$

b)

$$\begin{aligned} \log \mathcal{L}(\theta) &= \sum_{i=1}^n \log p(y_i; x_i, \theta) \\ &= \sum_{i=1}^n \log(-\sigma\sqrt{2\pi}) - \frac{1}{2} \left(\frac{y_i - \mu_\theta(x_i)}{\sigma} \right)^2 \end{aligned}$$

c)

$$\begin{aligned} \arg \max_{\theta} \mathcal{L}(\theta) &= \arg \max_{\theta} \log \mathcal{L}(\theta) \\ &= \arg \max_{\theta} \sum_{i=1}^n \log(-\sigma\sqrt{2\pi}) - \frac{1}{2} \left(\frac{y_i - \mu_\theta(x_i)}{\sigma} \right)^2 \\ &= \arg \max_{\theta} \sum_{i=1}^n -\frac{1}{2} \left(\frac{y_i - \mu_\theta(x_i)}{\sigma} \right)^2 \quad (\text{The first term is constant with respect to } \theta) \\ &= \arg \max_{\theta} -\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu_\theta(x_i))^2 \quad (\text{By linearity of the sum}) \\ &= \arg \max_{\theta} -\frac{1}{n} \sum_{i=1}^n (y_i - \mu_\theta(x_i))^2 \quad (\text{Multiplying by } \frac{2\sigma^2}{n}, \text{ which is constant with respect to } \theta \text{ and positive}) \\ &= \arg \min_{\theta} \frac{1}{n} \sum_{i=1}^n (y_i - \mu_\theta(x_i))^2 \quad (\text{Maximizing } x \text{ is equivalent to minimizing } -x) \end{aligned}$$

Complementary exercise

Find the marginal PDF $f_X(x)$ if the joint PDF $f_{XY}(x,y)$ is defined as:

$$f_{XY}(x,y) = \frac{e^{-|y-x|-x^2/2}}{2\sqrt{2\pi}}$$

Solution:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y) dy \\ &= \int_{-\infty}^{\infty} \frac{e^{-|y-x|-x^2/2}}{2\sqrt{2\pi}} dy \\ &= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|y-x|} e^{-x^2/2} dy \\ &= \frac{e^{-x^2/2}}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|y-x|} dy \end{aligned}$$

We have that : $|y-x| = \begin{cases} y-x, & \text{if } y \geq x \\ x-y, & \text{if } y \leq x \end{cases}$, and thus :

$$\begin{aligned} f_X(x) &= \frac{e^{-x^2/2}}{2\sqrt{2\pi}} \int_{-\infty}^x e^{y-x} dy + \int_x^{\infty} e^{x-y} dy \\ &= \frac{e^{-x^2/2}}{2\sqrt{2\pi}} \left(e^{-x} \left[e^y \right]_{-\infty}^x + e^x \left[-e^{-y} \right]_x^{\infty} \right) \\ &= \frac{e^{-x^2/2}}{2\sqrt{2\pi}} \left(e^0 - e^{-\infty} - e^{-\infty} + e^0 \right) \\ &= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \end{aligned}$$

Complementary exercise

Let p_X be a normal distribution $\mathcal{N}(\mu, \sigma^2)$ where $\mu \in \mathbb{R}$, and $\sigma > 0$. Consider the two scenarios where $n = 10$ or $n = 1000$. For each scenario,

1. repeat the following procedure 1000 times:
 - (a) Generate n i.i.d. realizations X_1, X_2, \dots, X_n where $X_i \sim p_X$.
 - (b) Compute $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$.
2. compute the mean and variance of the 1000 values computed in 1(b)
3. plot a histogram of these 1000 values, and add vertical lines at the true mean and the computed mean.

Experiment with different values of μ and σ , and confirm that you obtain $E[\bar{X}_n] = \mu$ and $\text{Var}(\bar{X}_n) = \frac{\sigma^2}{n}$.