Quadratic Equations

1 Questions

1. Represent the following situations in the form of a quadratic equation:

   (i) The area of a rectangle is 46. The length of the rectangle is 3 more than half its width. [Solution on page 2]

   (ii) A train travels a distance of 45 km at a constant speed from point A to point B. On the way back (that is, B to A), the train reduces its constant speed to 3 km/h less than what it initially had been. Hence, it takes the train 3 hours more to cover the same distance. [Solution on page 3]

Determine the vertex of the following quadratic functions:

(i) $f(x) = (x+5)^2 - 2$ [Solution on page 3]

(ii) $g(x) = x^2 + 6x + 3$ [Solution on page 3]

(iii) $h(x) = x^2 - 14x$ [Solution on page 3]

Provide a proof for the quadratic formula. [Solution on page 4]
Find the solutions for the following quadratic equations:

(i) $x^2 + 5x + 6 = 0$ [Solution on page 4]

(ii) $x^2 - 125x = 0$ [Solution on page 5]

(iii) $4x^2 - 2(a^2 + b^2)x + a^2b^2 = 0$ [Solution on page 5]

(iv) $x^2 + \left(a + \frac{1}{a}\right)x = -1$ [Solution on page 5]

Find two consecutive natural numbers whose product is 6. [Solution on page 5 and 6]
The sum of the reciprocals of two natural numbers is $\frac{1}{6}$. The result of subtracting 3 from the first number and adding 2 to the second number is the same. Find the two numbers. [Solution on page 6 and 7]
Starting at one of its x-intercepts, a particle travels along the parabola $f(x) = x^2 + 7x - 18$ to reach the other x-intercept. Find the shortest distance between the two intercepts. [Solution on page 7]
Find the discriminant of the following quadratic functions and determine the nature of their roots:

(i) $f(x) = 2x^2 + 7x + 15$ [Solution on page 7]

(ii) $g(x) = 7x^2 + 33$ [Solution on page 7]

(iii) $h(x) = x^2 + 7x - 3$ [Solution on page 7]

Is it possible to construct a rectangle whose length is 3 times its width and whose area is 500? If so, then find its dimensions. [Solution on page 8]

Assume that p, q, r and s are real numbers such that $pr = 2(q+s)$. Consider the quadratic equations $x^2 + px + q = 0$ and $x^2 + rx + s = 0$. Show that at least one of these equations has real roots. [Solution on page 8]

2 Solutions

1.

(i) Let the width of the rectangle be x and the length of the rectangle be y. From the information given about the dimensions, $y = \frac{x}{2} + 3$ (Equation 1). We know that the area of a rectangle is its length times its width. So, we have

$$\begin{aligned} xy &= 46 \\ x\left(\frac{x}{2} + 3\right) &= 46 \quad \text{(From Equation 1)} \\ \frac{x^2}{2} + 3x &= 46 \\ \frac{x^2 + 6x}{2} &= 46 \\ x^2 + 6x &= 46 \cdot 2 \\ x^2 + 6x - 92 &= 0 \end{aligned}$$

$$\boxed{\boldsymbol{x^2 + 6x - 92 = 0}}$$

(ii) Let the original speed and time it takes to cover the distance by x and y respectively. Since we know that distance is speed multiplied by the time taken, we have $xy = 45$

(Equation 1). By the same logic, we also have $(x-3)(y+3) = 45$. Note that:

$$\begin{aligned}
(x-3)(y+3) &= 45 \\
xy + 3x - 3y - 9 &= 45 \\
45 + 3x - 3\left(\frac{45}{x}\right) - 9 &= 45 \quad \text{(From Equation 1)} \\
3x - \frac{135}{x} - 9 &= 0 \\
3x^2 - 135 - 9x &= 0 \\
3x^2 - 9x - 135 &= 0
\end{aligned}$$

$$\boxed{\mathbf{3x^2 - 9x - 135 = 0}}$$

2. For this question, we will be using the fact that the vertex of a quadratic function in the form $f(x) = (x-h)^2 + k$ is (h, k).

(i) In this function, we have $h = -5$ and $k = -2$. So the vertex is $\boxed{\mathbf{(-5, -2)}}$.

(ii) Note that:

$$\begin{aligned}
x^2 + 6x + 3 &= (x^2 + 6x + 9) + 3 - 9 \\
&= (x+3)^2 - 6
\end{aligned}$$

So, we have $h = -3$ and $k = -6$. Therefore, the vertex is $\boxed{\mathbf{(-3, -6)}}$.

(iii) Note that:

$$\begin{aligned}
x^2 - 14x &= (x^2 - 14x + 49) - 49 \\
&= (x-7)^2 - 49
\end{aligned}$$

So, we have $h = 7$ and $k = -49$. Therefore, the vertex is $\boxed{\mathbf{(7, -49)}}$.

3. Consider the standard form of any quadratic function: $f(x) = ax^2 + bx + c$ where a, b and c are constants. Set this equal to 0 as we are trying to solve for x when $f(x) = 0$. We have:

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
a\left(x^2 + \frac{b}{a}x\right) + c &= 0 \\
a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c &= 0 \\
a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - a\left(\frac{b^2}{4a^2}\right) + c &= 0 \\
a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c &= 0 \\
a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a} - c \\
a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\
x &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

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4.

(i) We have:

$$\begin{aligned}
x^2 + 5x + 6 &= 0 \\
x^2 + 2x + 3x + 6 &= 0 \\
x(x+2) + 3(x+2) &= 0 \\
(x+2)(x+3) &= 0
\end{aligned}$$

So, either $x + 2 = 0$ or $x + 3 = 0$. Therefore, the solutions are $\boxed{\boldsymbol{x = -3, x = -2}}$.

(ii) We have:

$$\begin{aligned} x^2 - 125x &= 0 \\ x(x-125) &= 0 \end{aligned}$$

So, either $x = 0$ or $x - 125 = 0$. Therefore, the solutions are $\boxed{x = 0, x = 125}$.

(iii) We have:

$$\begin{aligned} 4x^2 - 2(a^2+b^2)x + a^2b^2 &= 0 \\ 4x^2 - 2a^2x - 2b^2x + a^2b^2 &= 0 \\ 2x(2x-a^2) - b^2(2x-a^2) &= 0 \\ (2x-b^2)(2x-a^2) &= 0 \end{aligned}$$

So, either $2x - b^2 = 0$ or $2x - a^2 = 0$. Therefore, the solutions are $\boxed{x = \frac{a^2}{2}, x = \frac{b^2}{2}}$.

(iv) We have:

$$\begin{aligned} x^2 + \left(a + \frac{1}{a}\right)x &= -1 \\ x^2 + ax + \frac{x}{a} &= -1 \\ x^2 + ax + \frac{x}{a} + 1 &= 0 \\ x(x+a) + \frac{1}{a}(x+a) &= 0 \\ (x+a)(x+\frac{1}{a}) &= 0 \end{aligned}$$

So, either $x + a = 0$ or $x + \frac{1}{a} = 0$. Therefore, the solutions are $\boxed{x = -a, x = \frac{-1}{a}}$.

5. Let the two consecutive numbers be x and $x+1$. First, we need to find x when $x(x+1) = 6$. So, we have:

$$\begin{aligned} x(x+1) &= 6 \\ x^2 + x &= 6 \\ x^2 + x - 6 &= 0 \\ x^2 + 3x - 2x - 6 &= 0 \\ x(x+3) - 2(x+3) &= 0 \\ (x-2)(x+3) &= 0 \end{aligned}$$

So, either $x - 2 = 0$ or $x + 3 = 0$. Thus, $x = 2, x = -3$. Since we are only considering natural numbers, we disregard $x = -3$. Therefore, the two consecutive natural numbers whose product is 6 are $\boxed{\mathbf{2, 3}}$.

6. Solution 1:

Let the two numbers be a and b. We have to two equations, $\frac{1}{a} + \frac{1}{b} = \frac{1}{6}$ (Equation 1) and $a - 3 = b + 2$ (Equation 2). Note that:

$$\begin{aligned} a - 3 &= b + 2 \quad \text{(Equation 2)} \\ a &= b + 5 \quad \text{(Equation 3)} \end{aligned}$$

$$\begin{aligned} \frac{1}{a} + \frac{1}{b} &= \frac{1}{6} \quad \text{(Equation 1)} \\ \frac{1}{b+5} + \frac{1}{b} &= \frac{1}{6} \\ \frac{b + (b+5)}{b(b+5)} &= \frac{1}{6} \\ \frac{2b+5}{b^2+5b} &= \frac{1}{6} \\ 6(2b+5) &= b^2 + 5b \\ 12b + 30 &= b^2 + 5b \\ b^2 - 7b - 30 &= 0 \\ b^2 - 10b + 3b - 30 &= 0 \\ b(b-10) + 3(b-10) &= 0 \\ (b+3)(b-10) &= 0 \end{aligned}$$

So, either $b+3 = 0$ or $b-10 = 0$. Thus, $b = -3, b = 10$. Since we are only considering natural numbers, we disregard $b = -3$. From Equation 3, we have $a = 10 + 5 = 15$. Therefore, the two numbers are $\boxed{\mathbf{15, 10}}$.

Solution 2:

Let the result of subtracting 3 from the first number and adding 2 to the second number be

x. The first number is $x+3$ and the second number is $x-2$. We have:

$$\begin{aligned}
\frac{1}{x+3}+\frac{1}{x-2} &= \frac{1}{6} \\
\frac{(x-2)+(x+3)}{(x+3)(x-2)} &= \frac{1}{6} \\
\frac{2x+1}{x^2+x-6} &= \frac{1}{6} \\
6(2x+1) &= x^2+x-6 \\
12x+6 &= x^2+x-6 \\
x^2-11x-12 &= 0 \\
x^2-12x+x-12 &= 0 \\
x(x-12)+(x-12) &= 0 \\
(x-12)(x+1) &= 0
\end{aligned}$$

So, either $x-12=0$ or $x+1=0$. Thus, $x=-1, x=12$. The second number is either $-1-2=-3$ or $12-2=10$. Since $x=-1$ leads to the second number being a non-natural number, we disregard $x=-1$. So, the first number is $12+3=15$ and the second number is $12-2=10$. Therefore, the two numbers are $\boxed{\mathbf{15, 10}}$.

7. First, let's find the x-intercepts of the function $f(x)$. Set $f(x)$ equal to 0. We have:

$$\begin{aligned}
x^2+7x-18 &= 0 \\
x^2+9x-2x-18 &= 0 \\
x(x+9)-2(x+9) &= 0 \\
(x+9)(x-2) &= 0
\end{aligned}$$

So, either $x-2=0$ or $x+9=0$. Thus, $x=-9, x=2$. So, the x-intercepts are $(-9,0)$ and $(2,0)$. The shortest path between any two points on the xy plane is a straight line. Therefore, the shortest distance is $\boxed{\mathbf{2-(-9)=11}}$.

8. The discriminant of any quadratic function in the standard form, $f(x)=ax^2+bx+c$ is b^2-4ac.

(i) In this function, we have $a=2, b=6, c=15$. Therefore, the discriminant of $f(x)$ is $6)^2-4(2)(15)=36-120\boxed{\mathbf{=-84}}$.

(ii) In this function, we have $a=7, b=0, c=33$. Therefore, the discriminant of $g(x)$ is $(0)^2-4(7)(33)\boxed{\mathbf{=-924}}$.

(iii) In this function, we have $a=1, b=7, c=-3$. Therefore, the discriminant of $h(x)$ is $(7)^2-4(1)(-3)=49+12\boxed{\mathbf{=61}}$.

9. Let the length and the width of the rectangle be x and y. We have, $x = 3y$ (Equation 1) and $xy = 500$. Note that:

$$\begin{aligned} xy &= 500 \\ (3y)(y) &= 500 \quad \text{(From Equation 1)} \\ 3y^2 &= 500 \\ 3y^2 - 500 &= 0 \end{aligned}$$

Consider the function $f(y) = 3y^2 - 500$. If this function has real roots, then the rectangle is possible. The discriminant of this function is $0^2 - 4(3)(-500) = 6000$ which is greater than 0. Therefore, $f(y)$ has real roots and the rectangle is possible.

10. Let $f(x) = x^2 + px + q$ and $g(x) = x^2 + rx + s$. The discriminant of $f(x)$, $\underset{1}{\mathrm{D}}$, is $p^2 - 4q$ and the discriminant of $g(x)$, $\underset{2}{\mathrm{D}}$, is $r^2 - 4s$. Note that:

$$\begin{aligned} \underset{1}{\mathrm{D}} + \underset{2}{\mathrm{D}} &= p^2 - 4q + r^2 - 4s \\ &= p^2 + r^2 - 4(q + s) \\ &= p^2 + r^2 - 2pr \\ &= (p - r)^2 \geq 0 \end{aligned}$$

Since $\underset{1}{\mathrm{D}} + \underset{2}{\mathrm{D}} \geq 0$, at least one of them has to be greater than or equal to zero. Therefore, at least one of the equations has real roots.

Author: Pradyumna Bangalore Sheshadri
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