CS532100 Numerical Optimization Homework 2

109062710 Bijon Setyawan Raya

Due Dec 10

1. Consider the linear least square problem:

$$\min_{\vec{x} \in \mathbb{R}^2} ||A\vec{x} - \vec{b}||^2$$

where

$$A = \begin{bmatrix} 4 & 8 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}, \vec{b} = \begin{pmatrix} 21/4 \\ 0 \\ 0 \end{pmatrix}$$

(a) (10%) Write its normal equation.

$$A^T A\vec{x} = A^T\vec{b} \tag{1}$$

$$\begin{bmatrix} 4 & 2 & 1 \\ 8 & 4 & 2 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ 8 & 4 & 2 \end{bmatrix} \begin{bmatrix} 21/4 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{3}$$

$$x_1 + 2x_2 = 1 \tag{4}$$

Then, we can express x_1 and x_2 as

$$\vec{x} = \begin{bmatrix} 1 - 2t \\ t \end{bmatrix} \tag{5}$$

(b) (10%) Express $\vec{b} = \vec{b}_1 + \vec{b}_2$ such that $\vec{b}_1$ is in the subspace spanned by A's column vectors, and $\vec{b}_2$ is orthogonal to A's column vectors. From matrix A, we know that its basis is its first column which is $\begin{bmatrix} 4 & 2 & 1 \end{bmatrix}^T$. From its basis column, we can express $\vec{b}$ in $\vec{b}_1 + \vec{b}_2$. Let,

$$\vec{b}_1 = \begin{bmatrix} 4s \\ 2s \\ s \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \tag{6}$$

Then we can express $\vec{b}$ such as

$$\vec{b} = \vec{b}_1 + \vec{b}_2 \tag{7}$$

$$\begin{bmatrix} 21/4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4s + t_1 \\ 2s + t_2 \\ s + t_3 \end{bmatrix} \tag{8}$$

Since $\vec{b}_1$ is a column space of A, and $\vec{b}_2$ is orthogonal to $\vec{b}_1$, then

$$\begin{bmatrix} 4s \\ 2s \\ s \end{bmatrix} \begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix} = 0 \tag{9}$$

$$4s \cdot t_1 + 2s \cdot t_2 + s \cdot t_3 = 0 \tag{10}$$

We can add $4s \cdot t_1 + 2s \cdot t_2 + s \cdot t_3 = 0$ at the last index of our matrix equality. Then

$$\begin{bmatrix} 4s + t_1 \\ 2s + t_2 \\ s + t_3 \\ 4s \cdot t_1 + 2s \cdot t_2 + s \cdot t_3 \end{bmatrix} = \begin{bmatrix} 21/4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the equality above, we then have $s = 1$, $t_1 = 5/4$, $t_2 = -2$, and $t_3 = -1$. Thus,

$$\vec{b}_1 = \begin{bmatrix} 4s \\ 2s \\ s \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 5/4 \\ -2 \\ -1 \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} 21/4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 5/4 \\ -2 \\ -1 \end{bmatrix} \tag{12}$$

(c) (10%) Show that $\vec{z} \in \mathbb{R}^2$ is a least square solution for $A\vec{x} = \vec{b}$ if and only if $\vec{z}$ is part of a solution to the larger linear system:

$$\begin{bmatrix} 0 & A^T \\ A & I \end{bmatrix} \begin{bmatrix} \vec{z} \\ \vec{y} \end{bmatrix} = \begin{bmatrix} 0 \\ \vec{b} \end{bmatrix}$$

We are going to show that $\vec{z} \in \mathbb{R}^2$ is a least square solution for $A\vec{x} = \vec{b}$ $\iff$ $\vec{z}$ is part of a solution to the larger linear system

i. First, we are going to show that if $\vec{z} \in \mathbb{R}^2$ is a least square solution for $A\vec{x} = \vec{b}$ then $\vec{z}$ is part of a solution to the larger linear system.

$$A\vec{x} = \vec{b} \tag{13}$$

$$A^T A\vec{x} = A^T\vec{b} \tag{14}$$

$$\begin{bmatrix} 4 & 2 & 1 \\ 8 & 4 & 2 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ 8 & 4 & 2 \end{bmatrix} \begin{bmatrix} 21/4 \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{16}$$

$$x_1 + 2x_2 = 1 \tag{17}$$

Then, we can express x_1 and x_2 as

$$\vec{x} = \begin{bmatrix} 1 - 2t \\ t \end{bmatrix} \tag{18}$$

Plugging $\vec{x}$ into $\vec{z}$ in the larger linear system, we then have

$$\begin{bmatrix} 0 & A^T \\ A & I \end{bmatrix} \begin{bmatrix} \vec{z} \\ \vec{y} \end{bmatrix} = \begin{bmatrix} 0 \\ \vec{b} \end{bmatrix} \tag{19}$$

$$\begin{bmatrix} 0 & 0 & 4 & 2 & 1 \\ 0 & 0 & 8 & 4 & 2 \\ 4 & 8 & 1 & 0 & 0 \\ 2 & 4 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1-2t \\ t \\ \vec{y}_1 \\ \vec{y}_2 \\ \vec{y}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 21/4 \\ 0 \\ 0 \end{bmatrix} \tag{20}$$

Solving $\vec{y}$, we have

$$\vec{y} = \begin{bmatrix} 5/4 \\ -2 \\ -1 \end{bmatrix} \tag{21}$$

From $\vec{y}$, we can tell that $\vec{z} \in \mathbb{R}^2$ is a least square solution for $A\vec{x} = \vec{b}$.
Thus, the premise, if $\vec{z} \in \mathbb{R}^2$ is a least square solution for $A\vec{x} = \vec{b}$ then $\vec{z}$ is part of a solution to the larger linear system, holds.

ii. Second, we are going to show that if $\vec{z}$ is a part of the solution to the larger linear system then $\vec{z} \in \mathbb{R}^2$ is a least square solution for $A\vec{x} = \vec{b}$.
The larger linear system can be expressed as

$$\begin{bmatrix} 0 & A^T \\ A & I \end{bmatrix} \begin{bmatrix} \vec{z} \\ \vec{y} \end{bmatrix} = \begin{bmatrix} 0 \\ \vec{b} \end{bmatrix} \tag{22}$$

$$\begin{bmatrix} 0 & 0 & 4 & 2 & 1 \\ 0 & 0 & 8 & 4 & 2 \\ 4 & 8 & 1 & 0 & 0 \\ 2 & 4 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{z}_1 \\ \vec{z}_2 \\ \vec{y}_1 \\ \vec{y}_2 \\ \vec{y}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 21/4 \\ 0 \\ 0 \end{bmatrix} \tag{23}$$

Solving y_1, y_2, and y_3, we express them as

$$\vec{y} = \begin{bmatrix} 5/4 \\ -2 \\ -1 \end{bmatrix} \tag{24}$$

After acquiring $\vec{y}$, we can acquire $\vec{z}$ with the third, the fourth, and the fifth row in the matrix. Thus,

$$4\vec{z}_1 + 8\vec{z}_2 + \vec{y}_1 = \frac{21}{4} \tag{25}$$

$$2\vec{z}_1 + 4\vec{z}_2 + \vec{y} + 2 = 0 \tag{26}$$

$$\vec{z}_1 + 2\vec{z}_2 + \vec{y}_3 = 0 \tag{27}$$

Solving the equalities above, we have

$$\vec{z}_1 + 2\vec{z}_2 = 1 \tag{28}$$

We can express the equality above in matrix as,

$$\vec{z} = \begin{bmatrix} 1 - 2t \\ t \end{bmatrix} \tag{29}$$

Clearly, $\vec{z}$ is a part of the solution to the larger linear system. Thus, the premise, if $\vec{z}$ is a part of the solution to the larger linear system then $\vec{z} \in \mathbb{R}^2$ is a least square solution for $A\vec{x} = \vec{b}$, holds.

By showing the premises above, we can conclude that the statement, $\vec{z} \in \mathbb{R}^2$ is a least square solution for $A\vec{x} = \vec{b}$ if and only if $\vec{z}$ is part of a solution to the larger linear system, holds.

2. In Note05 (Page 16), memoryless BFGS iteration matrix H_{k+1} can be derived from considering the Hestenes–Stiefel form of the nonlinear conjugate gradient method. Recalling that $\vec{s}_k = \alpha_k \vec{p}_k$, we have that the search direction for this method is given by

$$\begin{aligned} \vec{p}_{k+1} &= -\nabla f_{k+1} + \frac{\nabla f_{k+1}^T \vec{y}_k}{\vec{y}^T \vec{p}_k} \vec{p}_k && (30) \\ &= -\nabla f_{k+1} + \frac{\nabla f_{k+1}^T \vec{y}_k}{\vec{y}^T \vec{s}_k} \vec{s}_k && (31) \\ &= -(I - \frac{\vec{s}_k \vec{y}_k^T}{\vec{y}^T \vec{s}_k}) \nabla f_{k+1} && (32) \\ &= -\hat{H}_{k+1} \nabla f_{k+1} && (33) \end{aligned}$$

However, the matrix $\hat{H}_{k+1}$ is neither symmetric nor positive definite.

(a) (10%) Please show that the matrix $\hat{H}_{k+1}$ is singular.
(You can only prove it for the case $\nabla f_k, \vec{p}_k, \vec{y}_k, \vec{s}_k \in \mathbb{R}^2$ for all $k \in \mathbb{N}$.)
Your answer here!

(b) (0%) Please read the reference book (Page 180) to understand the derivation of the inverse hessian formula in Note05 (Page 16).
(you don't need to write anything in this subproblem.)

$$H_{k+1} = (I - \frac{\vec{s}_k \vec{y}_k^T}{\vec{y}_k^T \vec{s}_k})(I - \frac{\vec{y}_k \vec{s}_k^T}{\vec{y}_k^T \vec{s}_k}) + \frac{\vec{s}_k \vec{s}_k^T}{\vec{y}_k^T \vec{s}_k}$$

3. (10%) The total least square problem is to solve the following problem

$$\min_{\vec{x}, \|\vec{x}\|=1} \vec{x}^T A^T A \vec{x}$$

where A is an $m \times n$ matrix. Here we assume $m > n$. Let $A = U\Sigma V^T$ be the SVD of A, where U is the matrix of left singular vectors, V is the matrix of right singular vectors, and Σ is a diagonal matrix with diagonal elements $\sigma_1, \sigma_2, \ldots, \sigma_n$. Moreover, U and V are orthogonal matrices, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Show the solution of the above problem is the σ^2.

Your answer here!

4. Consider the following linear programming problem:

$$\begin{aligned} \max_{x_1,x_2} \quad & z = x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \le 4 \\ & 4x_1 + 2x_2 \le 12 \\ & -x_1 + x_2 \le 1 \\ & x_1, x_2 \ge 0 \end{aligned} \tag{34}$$

(a) (10%) Please refer Note08 (Page 2) to draw the figure of the constraints by any means, and use that to solve the problem.

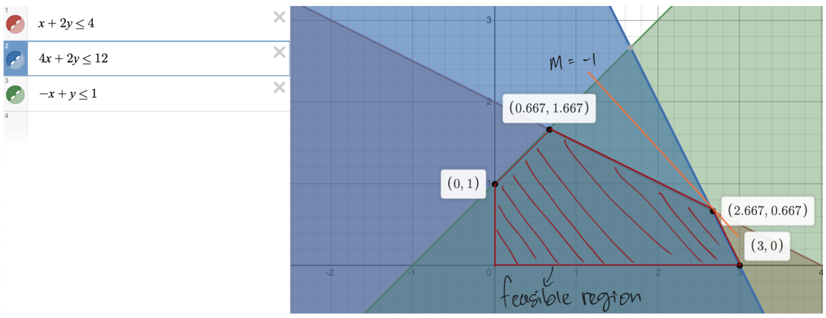

Figure 1: Feasible region of $\vec{z}$.

(b) (10%) Derive its dual problem and solve the dual problem by any means. Compare the solutions of the primal and the dual problems.

i. The primal problem is as follow:
Let z be the optimal value, x_i be the variables that we are planning to maximize, and ω_i be the slack variables.
To solve the primal problem, we then convert it into a table.
The initial state

$$\begin{array}{l} \zeta = x_1 + x_2 \\ \hline \omega_1 = 4 - x_1 - 2x_2 \\ \omega_2 = 12 - 4x_1 - 2x_2 \\ \omega_3 = 1 + x_1 - x_2 \end{array}$$

The first iteration

$$\begin{array}{l} \zeta = 3 + \frac{1}{2}x_2 - \frac{1}{4}\omega_2 \\ \hline \omega_1 = 1 - \frac{3}{2}x_2 + \frac{1}{4}\omega_2 \\ x_1 = 3 - \frac{1}{2}x_2 - \frac{1}{4}\omega_2 \\ \omega_3 = 4 - \frac{3}{2}x_2 - \frac{1}{4}\omega_2 \end{array}$$

The second iteration

$$\begin{array}{l} \zeta = \frac{10}{3} - \frac{1}{6}\omega_2 - \frac{1}{3}\omega_1 \\ \hline x_2 = \frac{2}{3} + \frac{1}{6}\omega_2 - \frac{2}{3}\omega_1 \\ x_1 = \frac{8}{3} - \frac{1}{3}\omega_2 + \frac{1}{3}\omega_1 \\ \omega_3 = 3 - \frac{1}{2}\omega_2 + \omega_1 \end{array}$$

From the table above, we can see that the values of $x_1, x_2, \text{and} z$ to be

$$x_1 = \frac{8}{3}, x_2 = \frac{2}{3}, z = x_1 + x_2 = \frac{10}{3}$$

The values of the slack variables are

$$\omega_1 = 0, \omega_2 = 0, \omega_3 = 3 \tag{35}$$

ii. The dual problem is

$$\begin{aligned} \min_{y_1,y_2,y_3} \quad & z = 4y_1 + 12y_2 + y_3 \\ \text{s.t.} \quad & y_1 + 4y_2 - y_3 \geq 1 \\ & 2y_1 + 2y_2 + y_3 \geq 1 \\ & y_1, y_2, y_3 \geq 0 \end{aligned} \tag{36}$$

Let s_i be the surplus variables and a_i be the artificial variables. Including these variables in the dual problem, we then have

$$\begin{aligned} \min \quad & z = 4y_1 + 12y_2 + y_3 + 0s_1 + 0s_2 + Ma_1 + Ma_2 \\ \text{s.t.} \quad & y_1 + 4y_2 - y_3 - s_1 + a_1 = 1 \\ & 2y_1 + 2y_2 + y_3 - s_2 + a_2 = 1 \\ & y_1, y_2, y_3, s_1, s_2, a_1, a_2 \geq 0 \end{aligned} \tag{37}$$

Since we want to solve the dual problem above, we then convert the equations above into a table and 5 iterations are needed to find the optimal solution.

		y_1	y_2	y_3	s_1	s_2	a_1	a_2		$\frac{b_i}{a_{i1}}$
Basis	C_B	-4	-12	-1	0	0	-M	-M	b_i	
a_1	-M	1	4	-1	-1	0	1	0	1	$\frac{1}{4}$ → leave
a_2	-M	2	2	1	0	-1	0	1	1	$\frac{1}{2}$
Z_j		-3M	-6M	0	M	M	-M	-M	-2M	
$C_j - Z_j$		-4+3M	-12+6M enter	-1	-M	-M	0	0		

Figure 2: The initial state.

		y_1	y_2	y_3	s_1	s_2	a_2	
Basis	C_B	-4	-12	-1	0	0	-M	
y_2	-12	$\frac{1}{4}$	1	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	$\frac{1}{4}$
a_2	-M	2	2	1	0	-1	1	1

Figure 3: The first iteration.

		y_1	y_2	y_3	s_1	s_2	a_2		
Basis	C_B	-4	-12	-1	0	0	$-M$	b_i	$\frac{b_i}{a_{i1}}$
y_2	-12	$\frac{1}{4}$	1	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	$\frac{1}{4}$	1
a_2	$-M$	$2-\frac{1}{2}=\frac{3}{2}$	0	$1+\frac{1}{2}=\frac{3}{2}$	$\frac{1}{2}$	-1	1	$\frac{1}{2}$	$\frac{1}{2}\div\frac{3}{2}=\frac{1}{3}$ → leave
z_j		$-3-\frac{3}{2}M$	-12	$3-\frac{3}{2}M$	$3-\frac{1}{2}M$	M	$-M$		
c_j-z_j		$-1+\frac{3}{2}M$ enter	0	$-4+\frac{3}{2}M$	$-3+\frac{1}{2}M$	$-M$	0		

Figure 4: The second iteration.

		y_1	y_2	y_3	s_1	s_2	
Basis	C_B	-4	-12	-1	0	0	
y_2	-12	$\frac{1}{4}$	1	$-\frac{1}{4}$	$-\frac{1}{4}$	0	$\frac{1}{4}$
y_1	-4	1	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{2}\times\frac{2}{3}=\frac{1}{3}$

Figure 5: The third iteration.

		y_1	y_2	y_3	s_1	s_2	
Basis	C_B	-4	-12	-1	0	0	
y_2	-12	0	1	$-\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$ → $y_2=\frac{1}{6}$
y_1	-4	1	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$ → $y_1=\frac{1}{3}$
z_j		-4	-12	2	$\frac{8}{3}$	$\frac{2}{3}$	
c_j-z_j		0	0	-3	$-\frac{8}{3}$	$-\frac{2}{3}$	

∵ All $c_j - z_j$ are less or equal to 0

∴ Reach optimal solution

Figure 6: The last iteration.

(c) (10%) Verify the complementarity slackness condition. Answer here!

(d) (10%) Transform the problem to the standard form.

$$\begin{aligned} \max_{x_1,x_2} \quad & z = x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_3 = 4 \\ & 4x_1 + 2x_2 + x_4 = 12 \\ & -x_1 + x_2 + x_5 = 1 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned} \tag{38}$$

(e) (10%) Solve it by the simplex method, as provided in Figure 1, using $\vec{x}_0 = (0, 0)$. Indicate $B_k, N_k, \vec{s}_k, \vec{d}_k, p_k, q_k, \gamma_k$ in each step. Answer here!

---

(1) Given a basic feasible point $\vec{x}_0$ and the corresponding index set $\mathcal{B}_0$ and $\mathcal{N}_0$.

(2) For $k = 0, 1, \ldots$

(3) Let $B_k = A(:, \mathcal{B}_k), N_k = A(:, \mathcal{N}_k)$, $\vec{x}_B = \vec{x}_k(\mathcal{B}_k), \vec{x}_N = \vec{x}_k(\mathcal{N}_k)$, and $\vec{c}_B = \vec{c}_k(\mathcal{B}_k), \vec{c}_N = \vec{c}_k(\mathcal{N}_k)$.

(4) Compute $\vec{s}_k = \vec{c}_N - N_k^T (B_k^{-1})^T \vec{c}_B$ (pricing)

(5) If $\vec{s}_k \geq 0$, return the solution $\vec{x}_k$. (found optimal solution)

(6) Select $q_k \in \mathcal{N}_k$ such that $\vec{s}_k(i_q) < 0$, where i_q is the index of q_k in $\mathcal{N}_k$

(7) Compute $\vec{d}_k = B_k^{-1} A_k(:, q_k)$. (search direction)

(8) If $\vec{d}_k \leq 0$, return `unbounded`. (unbounded case)

(9) Compute $[\gamma_k, i_p] = \min\limits_{i, \vec{d}_k(i) > 0} \dfrac{\vec{x}_B(i)}{\vec{d}_k(i)}$ (ratio test)
(The first return value is the minimum ratio;
the second return value is the index of the minimum ratio.)

(10) $x_{k+1} \begin{pmatrix} \mathcal{B} \\ \mathcal{N} \end{pmatrix} = \begin{pmatrix} \vec{x}_B \\ \vec{x}_N \end{pmatrix} + \gamma_k \begin{pmatrix} -\vec{d}_k \\ \vec{e}_{i_q} \end{pmatrix}$
($\vec{e}_{i_q} = (0, \ldots, 1, \ldots, 0)^T$ is a unit vector with i_qth element 1.)

(11) Let the i_pth element in $\mathcal{B}$ be p_k. (pivoting)
$\mathcal{B}_{k+1} = (\mathcal{B}_k - \{p_k\}) \cup \{q_k\}$, $\mathcal{N}_{k+1} = (\mathcal{N}_k - \{q_k\}) \cup \{p_k\}$

Figure 7: The simplex method for solving (minimization) linear programming