

# CS532100 Numerical Optimization Homework 3

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Due Jan 12

1. (25%) Consider the problem

$$\begin{array}{ll} \min_{x_1, x_2} & 0.1 \times (x_1 - 3)^2 + x_2^2 \\ \text{s.t.} & x_1^2 + x_2^2 - 1 \leq 0 \end{array} \quad (1)$$

- (a) Write down the KKT conditions for (1).

Let's define the Lagrangian Form first. The Lagrangian form is

$$L(x_1, x_2, \lambda) = 0.1(x_1 - 3)^2 + x_2^2 - \lambda(1 - x_1^2 - x_2^2)$$

The KKT conditions are

- i.  $\nabla_{x_1, x_2} L(x_1^*, x_2^*, \lambda^*) = 0$
  - ii.  $(x_1^*)^2, (x_2^*)^2$
  - iii.  $\lambda^* \geq 0$
  - iv.  $\lambda^*(1 - (x_1^*)^2 - (x_2^*)^2) = 0$
- (b) Solve the KKT conditions and find the optimal solutions, including the Lagrangian parameters. [Answer here!](#)
- (c) Compute the reduced Hessian and check the second order conditions for the solution. [Answer here!](#)

2. (20%) Consider the problem

$$\begin{array}{ll} \min_{x_1, x_2, x_3} & (x_1 - 1)^2 + (x_2 - 1)^2 + (x_3 - 1)^2 \\ \text{s.t.} & 0 \leq x_1, x_2, x_3 \leq 2. \end{array} \quad (2)$$

Find the optimization solution  $x^*$  for (2) with gradient projection method, with initial guess at  $\vec{x}_0 = (x_1, x_2, x_3) = (0, 0, 2)^T$ .

(Find out all segments and the minimizers of all segments, and determine whether the solution you got from this method is optimal solution. Justify your answer.) [Here!](#)

3. (15%) Consider the problem

$$\begin{array}{ll} \min_{x_1, x_2} & z = 8x_1 + 5x_2 \\ \text{s.t.} & 2x_1 + x_2 \leq 100 \\ & 3x_1 + 4x_2 \leq 240 \\ & x_1, x_2 \geq 0 \end{array} \quad (3)$$

Formulate this problem to the equation of the interior point method, and derive the gradient of the Lagrangian and the Jacobian of the function F. (The "gradient" means F and the Jacobian is the derivative of F.)

First, the primal and the dual problems should be defined.

The primal problem is

$$\begin{array}{ll}\min_{x_1, x_2} & 8x_1 + 5x_2 \\ \text{s.t.} & 2x_1 + x_2 + x_3 = 100 \\ & 3x_1 + 4x_2 + x_4 = 240 \\ & x_1, x_2, x_3, x_4 \geq 0\end{array}$$

The dual problem is

$$\begin{array}{ll}\max_{y_1, y_2} & 100x_1 + 240x_2 \\ \text{s.t.} & = 8 \\ & = 5 \\ & x_1, x_2, x_3, x_4 \geq 0\end{array}$$

4. (20%) Consider the following constrained optimization problem

$$\begin{array}{ll}\min_{x_1, x_2} & x_1 + x_2 \\ \text{s.t.} & x_1^2 + x_2^2 = 2\end{array} \quad (4)$$

- (a) Write the augmented Lagrangian penalty function  $L$  and Hessian of  $L$  of this problem.

The augmented Lagrangian penalty function is

$$x_1 + x_2 - \rho(x_1^2 + x_2^2 - 2) + \frac{\mu}{2}(x_1^2 + x_2^2 - 2)^2$$

The gradient of  $L$  is

$$\nabla L(x_1, x_2, \rho) = \begin{bmatrix} 1 - 2\rho x_1 + 2\mu x_1(x_1^2 + x_2^2 - 2) \\ 1 - 2\rho x_2 + 2\mu x_2(x_1^2 + x_2^2 - 2) \end{bmatrix}$$

The Hessian of  $L$  is

$$\nabla^2 L(x_1, x_2, \rho) = \begin{bmatrix} -2\rho + 4\mu x_1 & \\ 3 & 5 \end{bmatrix}$$

- (b) To make the augmented Lagrangian function  $L$  exact, what is the penalty parameter  $\mu$  should be ?
5. (5%) Find the condition number  $\kappa(A)$  of matrix A. Describe how ill- conditioned and good-conditioned matrices behave in matrix computation.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 3 & 0 \end{bmatrix}$$

In order to calculate the condition number of matrix  $A$ , we need to know that  $\text{cond}(A) = \|A\| \cdot \|A^{-1}\|$ .

However, we should know that  $\|A\|$  the n-norm of matrix  $A$ . Since the norm-2 is the most common one, we are going to use  $\|A\|_2$  and  $\|A^{-1}\|_2$

The inverse of matrix  $A$  is

$$A^{-1} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 0 & 1/3 \\ 0 & 1/2 & 0 \end{bmatrix}$$

Calculating both  $\|A\|_2$  and  $\|A\|_2^{-1}$ , we get  $\|A\|_2 = 4$  and  $\|A^{-1}\|_2 = 0.5$ . Thus,  $\text{cond}_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2 = 2$ .

Other results:

- (a) Norm-1  $\rightarrow \text{cond}_1(A) = 2$
- (b) Norm- $\infty \rightarrow \text{cond}_\infty(A) = 2$
- (c) Frobenius norm  $\rightarrow \text{cond}_{fro}(A) = 3.505$
- (d) Max norm  $\rightarrow \text{cond}_{\max}(A) = 2$

Ill-conditioned number is a condition where a small change in the input value will result in a large change in the output value.

As oppose to ill-conditioned number, well-conditioned number is a condition where a little or large change to the input in a function will result in a little change to the output.

Therefore, a condition number of 0 would mean that the matrix removes any error, which isn't possible. However

In fact, the smallest possible condition number is 1, where an error is neither magnified nor diminished.

6. (15%) The problem 15.4 in textbook shows an example of Maratos effect.

$$\begin{aligned} \min_{x_1, x_2} f(x_1, x_2) &= 2(x_1^2 + x_2^2 - 1) - x_1 \\ \text{s.t. } x_1^2 + x_2^2 - 1 &= 0 \end{aligned}$$

The optimal solution is  $\vec{x}^* = (1, 0)$ . Suppose  $\vec{x}_k = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$ ,  $\vec{p}_k = \begin{pmatrix} \sin^2 \theta \\ -\sin \theta \cos \theta \end{pmatrix}$ . The Maratos effect says although  $\vec{p}_k$  is a good step, but the filter method will reject it. In the textbook and slides, it says the remedy to this problem is using the second order correction. Read the textbook or the slides to understand the reason why the second order correction can help avoiding this problem. And explain it in your own words. [Here!](#)