CS532100 Numerical Optimization Homework 2

109062710 Bijon Setyawan Raya

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1. Consider the linear least square problem:

$$\min_{\vec{x}\in\mathbb{R}^2} ||A\vec{x}-\vec{b}||^2$$

where

$$A = \begin{bmatrix} 4 & 8 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}, \vec{b} = \begin{pmatrix} 21/4 \\ 0 \\ 0 \end{pmatrix}$$

(a) (10%) Write its normal equation.

$$A^T A\vec{x} = A^T\vec{b} \tag{1}$$

$$\begin{bmatrix} 4 & 2 & 1 \\ 8 & 4 & 2 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ 8 & 4 & 2 \end{bmatrix} \begin{bmatrix} 21/4 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{3}$$

$$x_1 + 2x_2 = 1 \tag{4}$$

Then, we can express x_1 and x_2 as

$$\vec{x} = \begin{bmatrix} 1-2t \\ t \end{bmatrix} \tag{5}$$

(b) (10%) Express $\vec{b} = \vec{b}_1 + \vec{b}_2$ such that $\vec{b}_1$ is in the subspace spanned by A's column vectors, and $\vec{b}_2$ is orthogonal to A's column vectors. From matrix A, we know that its basis is its first column which is $\begin{bmatrix} 4 & 2 & 1 \end{bmatrix}^T$. From its basis column, we can express $\vec{b}$ in $\vec{b}_1 + \vec{b}_2$. Let,

$$\vec{b}_1 = \begin{bmatrix} 4s \\ 2s \\ s \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \tag{6}$$

Then we can express $\vec{b}$ such as

$$\vec{b} = \vec{b}_1 + \vec{b}_2 \tag{7}$$

$$\begin{bmatrix} 21/4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4s+t_1 \\ 2s+t_2 \\ s+t_3 \end{bmatrix} \tag{8}$$

Since $\vec{b}_1$ is a column space of A, and $\vec{b}_2$ is orthogonal to $\vec{b}_1$, then

$$\begin{bmatrix} 4s \\ 2s \\ s \end{bmatrix} \begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix} = 0 \tag{9}$$

$$4s \cdot t_1 + 2s \cdot t_2 + s \cdot t_3 = 0 \tag{10}$$

We can add $4s \cdot t_1 + 2s \cdot t_2 + s \cdot t_3 = 0$ at the last index of our matrix equality. Then

$$\begin{bmatrix} 4s + t_1 \\ 2s + t_2 \\ s + t_3 \\ 4s \cdot t_1 + 2s \cdot t_2 + s \cdot t_3 \end{bmatrix} = \begin{bmatrix} 21/4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the equality above, we then have $s = 1$, $t_1 = 5/4$, $t_2 = -2$, and $t_3 = -1$. Thus,

$$\vec{b}_1 = \begin{bmatrix} 4s \\ 2s \\ s \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 5/4 \\ -2 \\ -1 \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} 21/4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 5/4 \\ -2 \\ -1 \end{bmatrix} \tag{12}$$

(c) (10%) Show that $\vec{z} \in \mathbb{R}^2$ is a least square solution for $A\vec{x} = \vec{b}$ if and only if $\vec{z}$ is part of a solution to the larger linear system:

$$\begin{bmatrix} 0 & A^T \\ A & I \end{bmatrix} \begin{bmatrix} \vec{z} \\ \vec{y} \end{bmatrix} = \begin{bmatrix} 0 \\ \vec{b} \end{bmatrix}$$

We are going to show that $\vec{z} \in \mathbb{R}^2$ is a least square solution for $A\vec{x} = \vec{b}$ $\iff$ $\vec{z}$ is part of a solution to the larger linear system

i. First, we are going to show that if $\vec{z} \in \mathbb{R}^2$ is a least square solution for $A\vec{x} = \vec{b}$ then $\vec{z}$ is part of a solution to the larger linear system.

$$A\vec{x} = \vec{b} \tag{13}$$

$$A^T A\vec{x} = A^T\vec{b} \tag{14}$$

$$\begin{bmatrix} 4 & 2 & 1 \\ 8 & 4 & 2 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ 8 & 4 & 2 \end{bmatrix} \begin{bmatrix} 21/4 \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{16}$$

$$x_1 + 2x_2 = 1 \tag{17}$$

Then, we can express x_1 and x_2 as

$$\vec{x} = \begin{bmatrix} 1 - 2t \\ t \end{bmatrix} \tag{18}$$

Plugging $\vec{x}$ into $\vec{z}$ in the larger linear system, we then have

$$\begin{bmatrix} 0 & A^T \\ A & I \end{bmatrix} \begin{bmatrix} \vec{z} \\ \vec{y} \end{bmatrix} = \begin{bmatrix} 0 \\ \vec{b} \end{bmatrix} \tag{19}$$

$$\begin{bmatrix} 0 & 0 & 4 & 2 & 1 \\ 0 & 0 & 8 & 4 & 2 \\ 4 & 8 & 1 & 0 & 0 \\ 2 & 4 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1-2t \\ t \\ \vec{y}_1 \\ \vec{y}_2 \\ \vec{y}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 21/4 \\ 0 \\ 0 \end{bmatrix} \tag{20}$$

Solving $\vec{y}$, we have

$$\vec{y} = \begin{bmatrix} 5/4 \\ -2 \\ -1 \end{bmatrix} \tag{21}$$

From $\vec{y}$, we can tell that $\vec{z} \in \mathbb{R}^2$ is a least square solution for $A\vec{x} = \vec{b}$.
Thus, the premise, if $\vec{z} \in \mathbb{R}^2$ is a least square solution for $A\vec{x} = \vec{b}$ then $\vec{z}$ is part of a solution to the larger linear system, holds.

ii. Second, we are going to show that if $\vec{z}$ is a part of the solution to the larger linear system then $\vec{z} \in \mathbb{R}^2$ is a least square solution for $A\vec{x} = \vec{b}$.
The larger linear system can be expressed as

$$\begin{bmatrix} 0 & A^T \\ A & I \end{bmatrix} \begin{bmatrix} \vec{z} \\ \vec{y} \end{bmatrix} = \begin{bmatrix} 0 \\ \vec{b} \end{bmatrix} \tag{22}$$

$$\begin{bmatrix} 0 & 0 & 4 & 2 & 1 \\ 0 & 0 & 8 & 4 & 2 \\ 4 & 8 & 1 & 0 & 0 \\ 2 & 4 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{z}_1 \\ \vec{z}_2 \\ \vec{y}_1 \\ \vec{y}_2 \\ \vec{y}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 21/4 \\ 0 \\ 0 \end{bmatrix} \tag{23}$$

Solving y_1, y_2, and y_3, we express them as

$$\vec{y} = \begin{bmatrix} 5/4 \\ -2 \\ -1 \end{bmatrix} \tag{24}$$

After acquiring $\vec{y}$, we can acquire $\vec{z}$ with the third, the fourth, and the fifth row in the matrix. Thus,

$$4\vec{z}_1 + 8\vec{z}_2 + \vec{y}_1 = \frac{21}{4} \tag{25}$$
$$2\vec{z}_1 + 4\vec{z}_2 + \vec{y} + 2 = 0 \tag{26}$$
$$\vec{z}_1 + 2\vec{z}_2 + \vec{y}_3 = 0 \tag{27}$$

Solving the equalities above, we have

$$\vec{z}_1 + 2\vec{z}_2 = 1 \tag{28}$$

We can express the equality above in matrix as,

$$\vec{z} = \begin{bmatrix} 1 - 2t \\ t \end{bmatrix} \tag{29}$$

Clearly, $\vec{z}$ is a part of the solution to the larger linear system. Thus, the premise, if $\vec{z}$ is a part of the solution to the larger linear system then $\vec{z} \in \mathbb{R}^2$ is a least square solution for $A\vec{x} = \vec{b}$, holds.

By showing the premises above, we can conclude that the statement, $\vec{z} \in \mathbb{R}^2$ is a least square solution for $A\vec{x} = \vec{b}$ if and only if $\vec{z}$ is part of a solution to the larger linear system, holds.

2. In Note05 (Page 16), memoryless BFGS iteration matrix H_{k+1} can be derived from considering the Hestenes–Stiefel form of the nonlinear conjugate gradient method. Recalling that $\vec{s}_k = \alpha_k \vec{p}_k$, we have that the search direction for this method is given by

$$\vec{p}_{k+1} = -\nabla f_{k+1} + \frac{\nabla f_{k+1}^T \vec{y}_k}{\vec{y}^T \vec{p}_k} \vec{p}_k \tag{30}$$

$$= -\nabla f_{k+1} + \frac{\nabla f_{k+1}^T \vec{y}_k}{\vec{y}^T \vec{s}_k} \vec{s}_k \tag{31}$$

$$= -(I - \frac{\vec{s}_k \vec{y}_k^T}{\vec{y}^T \vec{s}_k}) \nabla f_{k+1} \tag{32}$$

$$= -\hat{H}_{k+1} \nabla f_{k+1} \tag{33}$$

However, the matrix $\hat{H}_{k+1}$ is neither symmetric nor positive definite.

(a) (10%) Please show that the matrix $\hat{H}_{k+1}$ is singular.
(You can only prove it for the case $\nabla f_k, \vec{p}_k, \vec{y}_k, \vec{s}_k \in \mathbb{R}^2$ for all $k \in \mathbb{N}$.)
Your answer here!

(b) (0%) Please read the reference book (Page 180) to understand the derivation of the inverse hessian formula in Note05 (Page 16).
(you don't need to write anything in this subproblem.)

$$H_{k+1} = (I - \frac{\vec{s}_k \vec{y}_k^T}{\vec{y}_k^T \vec{s}_k})(I - \frac{\vec{y}_k \vec{s}_k^T}{\vec{y}_k^T \vec{s}_k}) + \frac{\vec{s}_k \vec{s}_k^T}{\vec{y}_k^T \vec{s}_k}$$

3. (10%) The total least square problem is to solve the following problem

$$\min_{\vec{x}, \|\vec{x}\|=1} \vec{x}^T A^T A \vec{x}$$

where A is an $m \times n$ matrix. Here we assume $m > n$. Let $A = U\Sigma V^T$ be the SVD of A, where U is the matrix of left singular vectors, V is the matrix of right singular vectors, and Σ is a diagonal matrix with diagonal elements $\sigma_1, \sigma_2, \ldots, \sigma_n$. Moreover, U and V are orthogonal matrices, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Show the solution of the above problem is the σ^2.

Let $v_1, v_2, \ldots, v_n$ be an orthonormal basis of $\mathbb{R}^n$, and set $x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ where $c_1, c_2, \ldots, c_n \in \mathbb{R}$.

$A^TA = V\Sigma^TU^TU\Sigma V^T = V\Sigma^2V^T$ with eigenvalues $\sigma_1^2 \geq \sigma_2^2 \geq \ldots \geq \sigma_n^2 \geq 0$.

$$\vec{x}^TA^TA\vec{x} = \vec{x}(c_1\sigma_1^2v_1 + c_2\sigma_2^2v_2 + \cdots + c_n\sigma_n^2v_n) \quad (34)$$
$$= c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \cdots + c_n^2\sigma_n^2 \quad (35)$$
$$\geq \sigma_n^2(c_1^2 + c_2^2 + \cdots + c_n^2) = \sigma_n^2 \quad \text{for } ||\vec{x}|| = 1 \quad (36)$$

Thus,

$$\min_{\vec{x}, \|\vec{x}\|=1} \vec{x}^TA^TA\vec{x} = \sigma_n^2$$

4. Consider the following linear programming problem:

$$\begin{aligned} \max_{x_1,x_2} \quad & z = x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 4 \\ & 4x_1 + 2x_2 \leq 12 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned} \quad (37)$$

(a) (10%) Please refer Note08 (Page 2) to draw the figure of the constraints by any means, and use that to solve the problem.

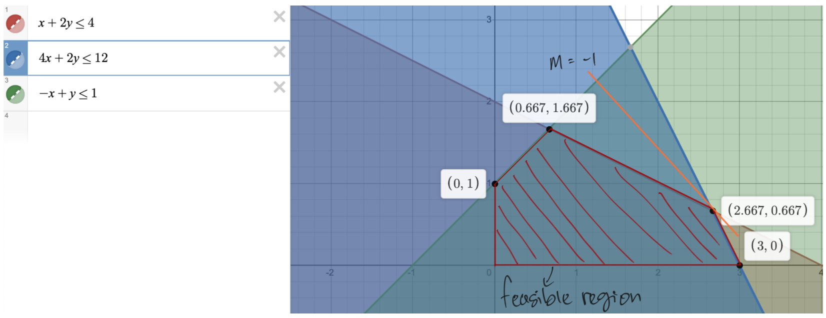

Figure 1: Feasible region of $\vec{z}$.

From the figure above, we can see that $x = 2.667$ and $y = 0.667$. Thus, the optimal value is $z = x + y = 3.3333$.

(b) (10%) Derive its dual problem and solve the dual problem by any means. Compare the solutions of the primal and the dual problems.

i. The primal problem is as follow:
Let z be the optimal value, x_i be the variables that we are planning to maximize, and ω_i be the slack variables.
To solve the primal problem, we then convert it into a table.
The initial state

$$\begin{array}{l} \zeta = x_1 + x_2 \\ \hline \omega_1 = 4 - x_1 - 2x_2 \\ \omega_2 = 12 - 4x_1 - 2x_2 \\ \omega_3 = 1 + x_1 - x_2 \end{array}$$

The first iteration

$$\begin{array}{c} \zeta = 3 + \frac{1}{2}x_2 - \frac{1}{4}\omega_2 \\ \hline \omega_1 = 1 - \frac{3}{2}x_2 + \frac{1}{4}\omega_2 \\ x_1 = 3 - \frac{1}{2}x_2 - \frac{1}{4}\omega_2 \\ \omega_3 = 4 - \frac{3}{2}x_2 - \frac{1}{4}\omega_2 \end{array}$$

The second iteration

$$\begin{array}{c} \zeta = \frac{10}{3} - \frac{1}{6}\omega_2 - \frac{1}{3}\omega_1 \\ \hline x_2 = \frac{2}{3} + \frac{1}{6}\omega_2 - \frac{2}{3}\omega_1 \\ x_1 = \frac{8}{3} - \frac{1}{3}\omega_2 + \frac{1}{3}\omega_1 \\ \omega_3 = 3 - \frac{1}{2}\omega_2 + \omega_1 \end{array}$$

The values of the slack variables are

$$\omega_1 = 0, \omega_2 = 0, \omega_3 = 3$$

The values of the decision variables are

$$x_1 = \frac{8}{3} x_2 = \frac{2}{3}$$

The objective value

$$z = x_1 + x_2 = \frac{10}{3}$$

ii. The dual problem is

$$\begin{aligned} \min_{y_1,y_2,y_3} \quad & z = 4y_1 + 12y_2 + y_3 \\ \text{s.t.} \quad & y_1 + 4y_2 - y_3 \geq 1 \\ & 2y_1 + 2y_2 + y_3 \geq 1 \\ & y_1, y_2, y_3 \geq 0 \end{aligned} \tag{38}$$

Let s_i be the surplus variables and a_i be the artificial variables. Including these variables in the dual problem, we then have

$$\begin{aligned} \min \quad & z = 4y_1 + 12y_2 + y_3 + 0s_1 + 0s_2 + Ma_1 + Ma_2 \\ \text{s.t.} \quad & y_1 + 4y_2 - y_3 - s_1 + a_1 = 1 \\ & 2y_1 + 2y_2 + y_3 - s_2 + a_2 = 1 \\ & y_1, y_2, y_3, s_1, s_2, a_1, a_2 \geq 0 \end{aligned} \tag{39}$$

Since we want to solve the dual problem above, we then convert the equations above into a table and 2 iterations are needed to find the optimal solution.

		y_1	y_2	y_3	s_1	s_2	a_1	a_2		$\frac{b_i}{a_{i1}}$
Basis	C_B	-4	-12	-1	0	0	$-M$	$-M$	b_i	
a_1	$-M$	1	4	-1	-1	0	1	0	1	$\frac{1}{4}$ → leave
a_2	$-M$	2	2	1	0	-1	0	1	1	$\frac{1}{2}$
Z_j		$-3M$	$-6M$	0	M	M	$-M$	$-M$	$-2M$	
$C_j - Z_j$		$-4+3M$	$-12+6M$ enter	-1	$-M$	$-M$	0	0		

Figure 2: The initial state.

		y_1	y_2	y_3	s_1	s_2	a_2	
Basis	C_B	-4	-12	-1	0	0	$-M$	
y_2	-12	$\frac{1}{4}$	1	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	$\frac{1}{4}$
a_2	$-M$	2	2	1	0	-1	1	1

$\times(-2)$

Figure 3: The first iteration (1).

		y_1	y_2	y_3	s_1	s_2	a_2		$\frac{b_i}{a_{i1}}$
Basis	C_B	-4	-12	-1	0	0	$-M$	b_i	
y_2	-12	$\frac{1}{4}$	1	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	$\frac{1}{4}$	1
a_2	$-M$	$2-\frac{1}{2}=\frac{3}{2}$	0	$1+\frac{1}{2}=\frac{3}{2}$	$\frac{1}{2}$	-1	1	$\frac{1}{2}$	$\frac{1}{2} \div \frac{3}{2} = \frac{1}{3}$ → leave
Z_j		$-3-\frac{3}{2}M$	-12	$3-\frac{3}{2}M$	$3-\frac{1}{2}M$	M	$-M$		
$C_j - Z_j$		$-1+\frac{3}{2}M$	0 enter	$-4+\frac{3}{2}M$	$-3+\frac{1}{2}M$	$-M$	0		

Figure 4: The first iteration (2).

		y_1	y_2	y_3	s_1	s_2	
Basis	C_B	-4	-12	-1	0	0	
y_2	-12	$\frac{1}{4}$	1	$-\frac{1}{4}$	$-\frac{1}{4}$	0	$\frac{1}{4}$
y_1	-4	1	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{2}\times\frac{2}{3}=\frac{1}{3}$

Figure 5: The last iteration (1).

		y_1	y_2	y_3	s_1	s_2	
Basis	C_B	-4	-12	-1	0	0	
y_2	-12	0	1	$-\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6} \rightarrow y_2=\frac{1}{6}$
y_1	-4	1	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{3} \rightarrow y_1=\frac{1}{3}$
Z_j		-4	-12	2	$\frac{8}{3}$	$\frac{2}{3}$	$\boxed{-\frac{10}{3}}$
C_j-Z_j		0	0	-3	$-\frac{8}{3}$	$-\frac{2}{3}$	

$\because$ All $C_j - Z_j$ are less or equal to 0

$\therefore$ Reach optimal solution

Figure 6: The last iteration (2).

Thus, the objective value for the dual problem is $-(-\frac{10}{3}) = \frac{10}{3}$. From the solutions that we get from the primal problem and the dual problem, we can infer that the values of surplus variables in the dual problem equal to the values of decision variables in the primal problem.

As for the optimal values of the primal slack variables are the first three entries in the $C_j - Z_j$ row. Thus,

$$\omega_1 = -(\text{first entry of } C_j - Z_j) = 0 \tag{40}$$
$$\omega_2 = -(\text{second entry of } C_j - Z_j) = 0 \tag{41}$$
$$\omega_3 = -(\text{third entry of } C_j - Z_j) = 3 \tag{42}$$

Most importantly, we can observe that the optimal value of the objective function is the same for both primal and dual.

(c) (10%) Verify the complementarity slackness condition. Suppose that $x = (x_1, x_2, \ldots, x_n)$ is a primal feasible solution and $y = (y_1, y_2, \ldots, y_n)$ is a dual feasible solution. Let $(\omega_1, \omega_2, \ldots, \omega_m)$ be the corresponding primal slack variables, and $(z_1, z_2, \ldots, z_n)$ be the corresponding dual slack variables. Then x and y are optimal for their respective problems if and only if:

i. $z_j \cdot x_j = 0$, for $j = 1, 2, \ldots, n$

ii. $\omega_i \cdot y_i = 0$, for $i = 1, 2, \ldots, m$

For the primal problem,

$$z_1 \cdot x_1 = 0 \cdot \frac{8}{3} = 0 \tag{43}$$

$$z_2 \cdot x_2 = 0 \cdot \frac{2}{3} = 0 \tag{44}$$

For the dual problem,

$$\omega_1 \cdot y_1 = 0 \cdot \frac{1}{3} = 0 \tag{45}$$

$$\omega_2 \cdot y_2 = 0 \cdot \frac{1}{6} = 0 \tag{46}$$

(d) (10%) Transform the problem to the standard form.

$$\begin{aligned} \min_{x_1, x_2} \quad & z = -x_1 - x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_3 = 4 \\ & 4x_1 + 2x_2 + x_4 = 12 \\ & -x_1 + x_2 + x_5 = 1 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned} \tag{47}$$

(e) (10%) Solve it by the simplex method, as provided in Figure 1, using $\vec{x}_0 = (0, 0)$. Indicate $B_k, N_k, \vec{s}_k, \vec{d}_k, p_k, q_k, \gamma_k$ in each step.

Let $\mathcal{B} = \begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$ be the index set of the basic variables and $\mathcal{N} = \begin{bmatrix} 1 & 2 \end{bmatrix}$ the index set of the non-basic variables.

i. Iteration 1

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, N = \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ -1 & 1 \end{bmatrix}$$

$$\vec{x}_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{48}$$

$$\vec{X}_B = B^{-1} b = b = \begin{bmatrix} 4 \\ 12 \\ 1 \end{bmatrix} \tag{49}$$

$$\vec{x} = \begin{bmatrix} \vec{x}_B \\ \vec{x}_N \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{50}$$

Solve $\vec{v}$ in $B^T\vec{v} = \vec{c}_B$. Thus,

$$\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Pricing vector

$$\vec{p} = \vec{c}_N - N^T\vec{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Solve $B\vec{s} = A(:,1)$ to check if it is bounded

$$\vec{s} = B^{-1}\vec{A}_1 = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} > 0 \tag{51}$$

$$\vec{d} = \begin{bmatrix} d_3 \\ d_4 \\ d_5 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ -1 \\ 1 \\ 0 \end{bmatrix} \tag{52}$$

Ratio test

$$\alpha = \min(|\frac{4}{-1}|, |\frac{12}{-4}|) = 3$$

Thus,

$$\vec{x}_B^{new} = \vec{x}_B^{old} - \alpha\vec{s} = \begin{bmatrix} 4 \\ 12 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \tag{53}$$

$$\vec{x}^{new} = \begin{bmatrix} x_3 \\ x_1 \\ x_5 \\ x_4 \\ x_2 \end{bmatrix} \tag{54}$$

$$\vec{B}^{new} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & -1 & 1 \end{bmatrix} \tag{55}$$

$$N^{new} = \begin{bmatrix} 0 & 2 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \tag{56}$$

$$A^{new} = \begin{bmatrix} 0 & 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 4 & 0 \\ 0 & 1 & 0 & -1 & 1 \end{bmatrix} \tag{57}$$

The index set of the new basic variables are $\mathcal{B} = \{3, 1, 5\}$ and the non-basic variables are $\mathcal{N} = \{4, 2\}$

ii. Iteration 2

$$\vec{x}_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{58}$$

$$\vec{X}_B = B^{-1}b = b \tag{59}$$

$$= \begin{bmatrix} 1 & -1/4 & 0 \\ 0 & 1/4 & 0 \\ 0 & 1/4 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \tag{60}$$

Solve $\vec{v}$ in $B^T\vec{v} = \vec{c}_B$. Thus,

$$B^T\vec{v} = \vec{c}_B \tag{61}$$

$$= \begin{bmatrix} 1 & -1/4 & 0 \\ 0 & 1/4 & 0 \\ 0 & 1/4 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \tag{62}$$

$$\vec{v} = \begin{bmatrix} 0 \\ -1/4 \\ 0 \end{bmatrix} \tag{63}$$

Pricing vector

$$\vec{p} = \vec{c}_N - N^T\vec{v} \tag{64}$$

$$= \begin{bmatrix} 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1/4 \\ 0 \end{bmatrix} \tag{65}$$

$$= \begin{bmatrix} 1/4 \\ -1/2 \end{bmatrix} \tag{66}$$

Solve $B\vec{s} = A(:, 2)$ to check if it is bounded

$$\vec{s} = B^{-1}\vec{A}_2 \tag{67}$$

$$\begin{bmatrix} 1 & -1/4 & 0 \\ 0 & 1/4 & 0 \\ 0 & 1/4 & 1 \end{bmatrix} \begin{matrix} 2 \\ 2 \\ 1 \end{matrix} \begin{matrix} 3/2 \\ = 1/2 > 0 \\ 3/2 \end{matrix} \tag{68}$$

$$\vec{d} = \begin{bmatrix} d_3 \\ d_1 \\ d_5 \\ d_4 \\ d_2 \end{bmatrix} = \begin{bmatrix} -3/2 \\ -1/2 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} \tag{69}$$

Ratio test

$$\alpha = \min(|-\frac{2}{3}|, |6|, |\frac{8}{3}|) = \frac{2}{3}$$

Thus,

$$\vec{x}^{new} = \begin{bmatrix} x_2 \\ x_1 \\ x_5 \\ x_3 \\ x_2 \end{bmatrix} \tag{70}$$

$$\vec{B}^{new} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 4 & 0 \\ 1 & -1 & 1 \end{bmatrix} \tag{71}$$

$$N^{new} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{72}$$

$$A^{new} = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 \\ 1 & 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix} \tag{73}$$

The index set of the new basic variables are $\mathcal{B} = \{2, 1, 5\}$ and the non-basic variables are $\mathcal{N} = \{4, 3\}$

iii. Iteration 3

$$\vec{x}_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{74}$$

$$\vec{X}_B = B^{-1}b = b \tag{75}$$

$$= \begin{bmatrix} 2/3 & -1/6 & 0 \\ -1/3 & 1/3 & 0 \\ -1 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 8/3 \\ 3 \end{bmatrix} \tag{76}$$

Solve $\vec{v}$ in $B^T\vec{v} = \vec{c}_B$. Thus,

$$B^T\vec{v} = \vec{c}_B \tag{77}$$

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 4 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \tag{78}$$

$$\vec{v} = \begin{bmatrix} -1/3 \\ -1/6 \\ 0 \end{bmatrix} \tag{79}$$

Pricing vector

$$\vec{p} = \vec{c}_N - N^T\vec{v} \tag{80}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1/3 \\ -1/6 \\ 0 \end{bmatrix} \tag{81}$$

$$= \begin{bmatrix} 1/6 \\ 1/3 \end{bmatrix} \tag{82}$$

Since $\vec{p}$ is positive, the stopping condition is reached...

Thus,

$$\vec{x} = \begin{bmatrix} \vec{x}_1 \\ \vec{x}_2 \end{bmatrix} = \begin{bmatrix} 8/3 \\ 2/3 \end{bmatrix} \tag{83}$$

$$z = \frac{8}{3} + \frac{2}{3} = \frac{10}{3} \tag{84}$$

The maximum value is $10/3 = 3.333$.

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(1) Given a basic feasible point $\vec{x}_0$ and the corresponding index set $\mathcal{B}_0$ and $\mathcal{N}_0$.

(2) For $k = 0, 1, \ldots$

(3) Let $B_k = A(:, \mathcal{B}_k), N_k = A(:, \mathcal{N}_k)$, $\vec{x}_B = \vec{x}_k(\mathcal{B}_k), \vec{x}_N = \vec{x}_k(\mathcal{N}_k)$, and $\vec{c}_B = \vec{c}_k(\mathcal{B}_k), \vec{c}_N = \vec{c}_k(\mathcal{N}_k)$.

(4) Compute $\vec{s}_k = \vec{c}_N - N_k^T {B_k^{-1}}^T \vec{c}_B$ (pricing)

(5) If $\vec{s}_k \geq 0$, return the solution $\vec{x}_k$. (found optimal solution)

(6) Select $q_k \in \mathcal{N}_k$ such that $\vec{s}_k(i_q) < 0$, where i_q is the index of q_k in $\mathcal{N}_k$

(7) Compute $\vec{d}_k = B_k^{-1} A_k(:, q_k)$. (search direction)

(8) If $\vec{d}_k \leq 0$, return `unbounded`. (unbounded case)

(9) Compute $[\gamma_k, i_p] = \min_{i, \vec{d}_k(i) > 0} \frac{\vec{x}_B(i)}{\vec{d}_k(i)}$ (ratio test)
(The first return value is the minimum ratio;)
(the second return value is the index of the minimum ratio.)

(10) $x_{k+1} \begin{pmatrix} \mathcal{B} \\ \mathcal{N} \end{pmatrix} = \begin{pmatrix} \vec{x}_B \\ \vec{x}_N \end{pmatrix} + \gamma_k \begin{pmatrix} -\vec{d}_k \\ \vec{e}_{i_q} \end{pmatrix}$
($\vec{e}_{i_q} = (0, \ldots, 1, \ldots, 0)^T$ is a unit vector with i_qth element 1.)

(11) Let the i_pth element in $\mathcal{B}$ be p_k. (pivoting)
$\mathcal{B}_{k+1} = (\mathcal{B}_k - \{p_k\}) \cup \{q_k\}$, $\mathcal{N}_{k+1} = (\mathcal{N}_k - \{q_k\}) \cup \{p_k\}$

Figure 7: The simplex method for solving (minimization) linear programming