Feature Extraction and Comparing Sequences

DT2119 Speech and Speaker Recognition

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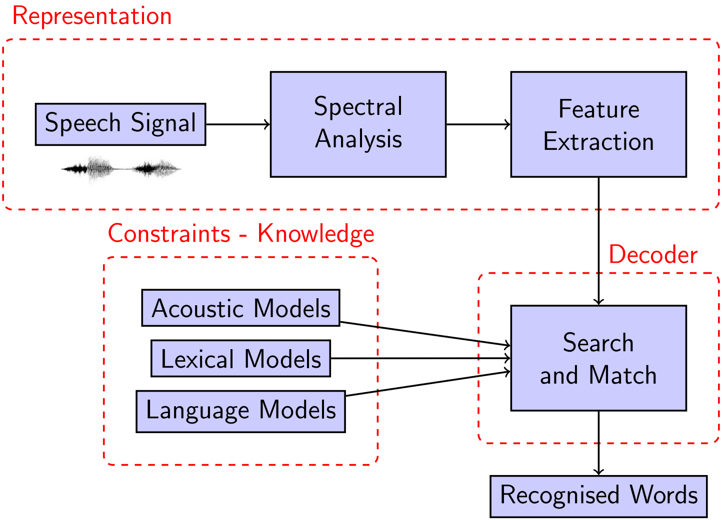
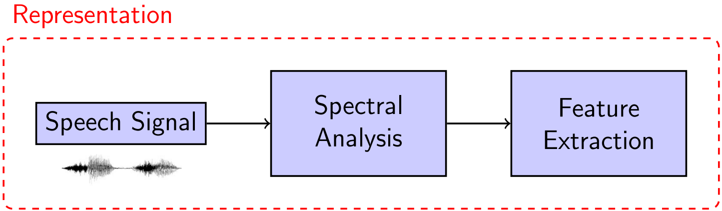
Components of ASR System



Speech Signal Representations



Goals:

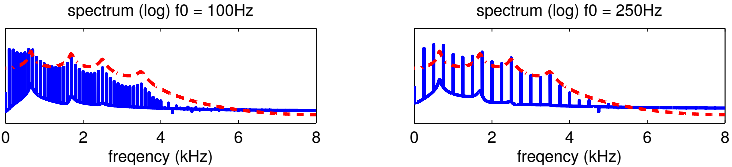
- disregard irrelevant information
- optimise relevant information for modelling

Means:

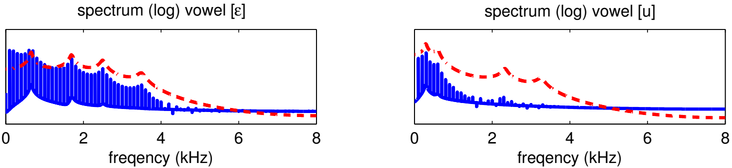
- try to model essential aspects of speech production
- imitate auditory processes
- consider properties of statistical modelling

F_0 and Formants

- Varying F_0 (vocal fold oscillation rate)



- Varying Formants (vocal tract shape)



Other Transforms

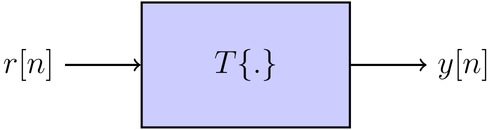
Laplace Transform

$$X(s) = \int_0^\infty x(t)e^{-st}dt, s \in \mathbb{C}$$

z-Transform

$$X(z) = \sum_{-\infty}^{+\infty} x[n]z^{-n}, z \in \mathbb{C}$$

Linear Prediction Coefficients (LPC)



approximate $y[n]$ as a linear combination of p previous samples:

$$\hat{y}[n] = \sum_{k=1}^{p} a_k y[n-k]$$

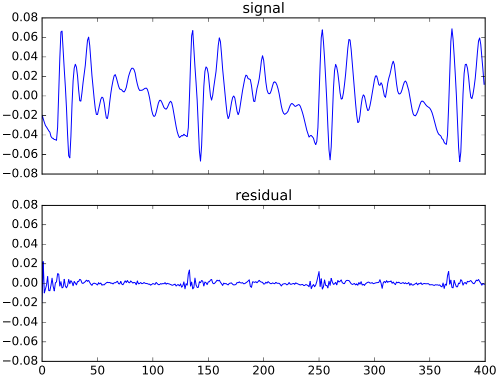
The error is called residual: $r[n] = \hat{y}[n] - y[n]$
The output of the signal is:

$$y[n] = \sum_{k=1}^{p} a_k y[n-k] + r[n]$$

LPC and Speech coding

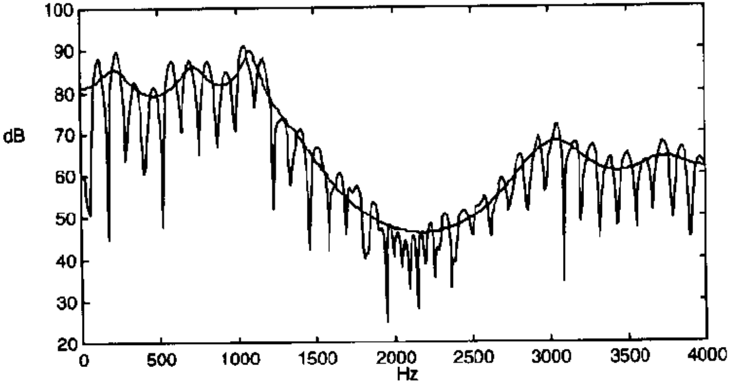
$$y[n] = \sum_{k=1}^{p} a_k y[n-k] + r[n]$$

- We only need to send $a_1, \ldots, a_p$ and $r[n]$.
- $r[n]$ can be coded with fewer bits



LPC Example

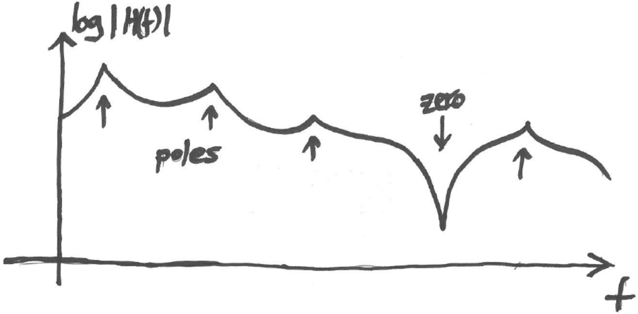
$$y[n] = \sum_{k=1}^{p} a_k y[n-k] + r[n]$$



Infinite Impulse Response (IIR) Systems

Auto regressive (AR): y depends on (delayed) samples of the input, as well as the output at previous times (feedback)

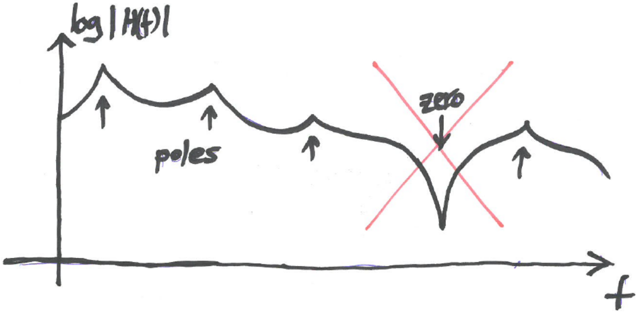
$$\begin{aligned} y[n] &= \frac{1}{a_0}\sum_{i=0}^{P} b_i x[n-i] && \leftarrow \text{zeros} \\ &\quad -\frac{1}{a_0}\sum_{j=1}^{Q} a_j y[n-j] && \leftarrow \text{poles} \end{aligned}$$



Linear Prediction

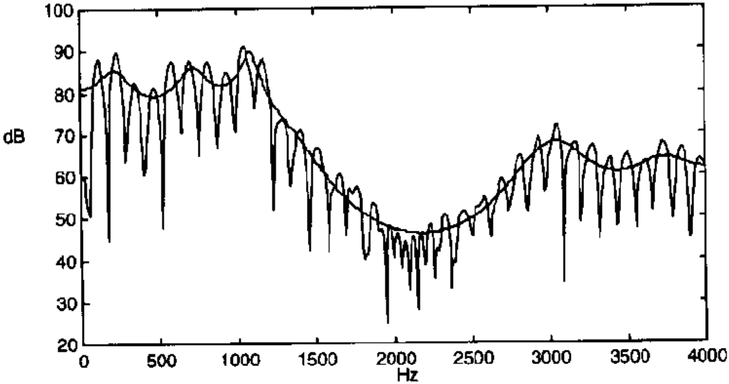
Auto regressive (AR), but only depends on current input

$$
\begin{aligned}
y[n] &= \frac{b_0}{a_0} x[n] && \leftarrow \text{no zeros} \\
&\quad - \frac{1}{a_0} \sum_{j=1}^{Q} a_j y[n-j] && \leftarrow \text{poles}
\end{aligned}
$$



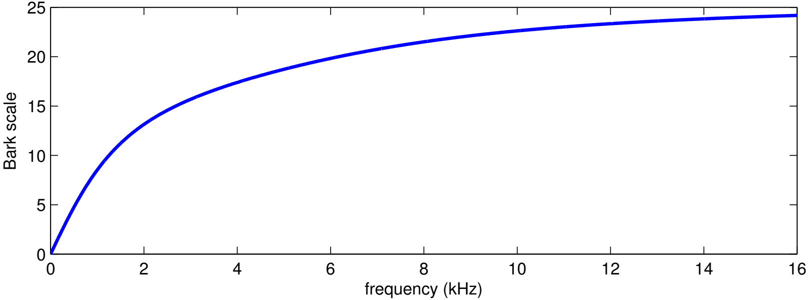
LPC Limitations

- better match at spectral peaks than at valleys (all-pole model)
- not accurate if transfer function contain zeros (nasals, fricatives. . .)



Perceptual Linear Prediction

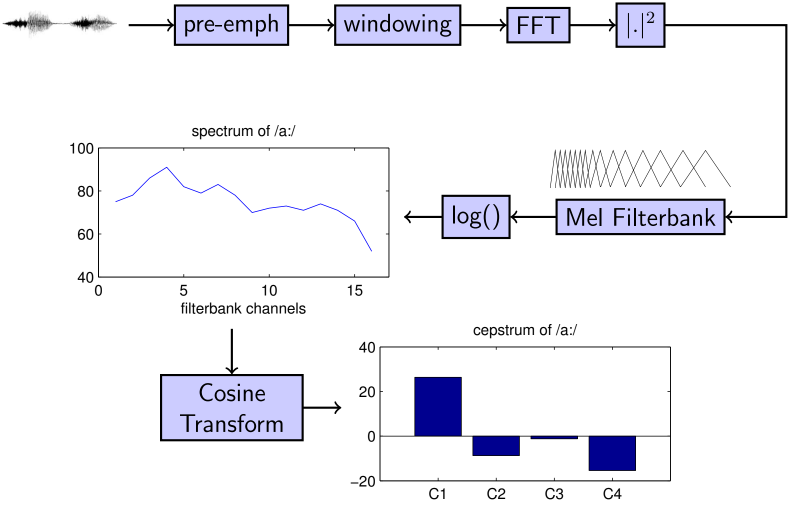
- Transform to the Bark frequency scale before computing the LPC coefficients
- Cubic root of energy instead of logarithm



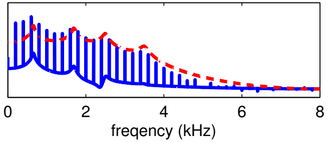
Mel Frequency Cepstrum Coefficients

- *de facto* standard in ASR
- imitate aspects of auditory processing
- does not assume all-pole model of the spectrum
- uncorrelated: easier to model statistically

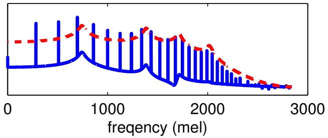
MFCCs Calculation



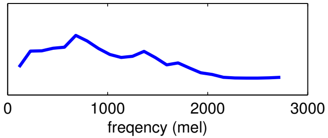
Mel Frequency Cepstral Coefficients



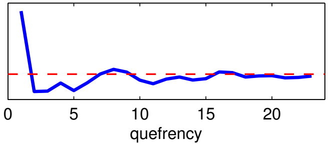
Linear to Mel frequency



Filterbank ($\sim$ 20-25 filters) + $\log()$

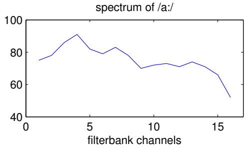


Discrete Cosine Transform

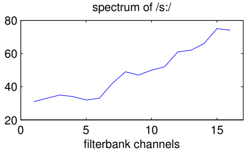


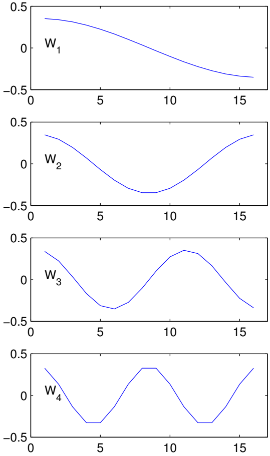
MFCC: Cosine Transform

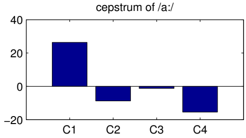
$$C_j = \sqrt{\frac{2}{N}} \sum_{i=1}^{N} A_i \cos\left(\frac{j\pi(i-0.5)}{N}\right)$$



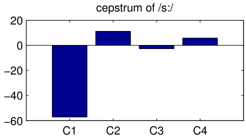
A_i







C_j



MFCC Rationale

- signals combined in a convolutive way: $a[n] * b[n] * c[n]$
- in the spectral domain: $A(z)B(z)C(z)$
- taking the $\log$: $\log(A(z)) + \log(B(z)) + \log(C(z))$
- to analise the different contribution perform Fourier transform (DCT if not interested in phase information).
- Terminology:
  - frequency vs quefrency
  - spectrum vs cepstrum
  - filter vs lifter
  - ...

MFCC Advantages[1]

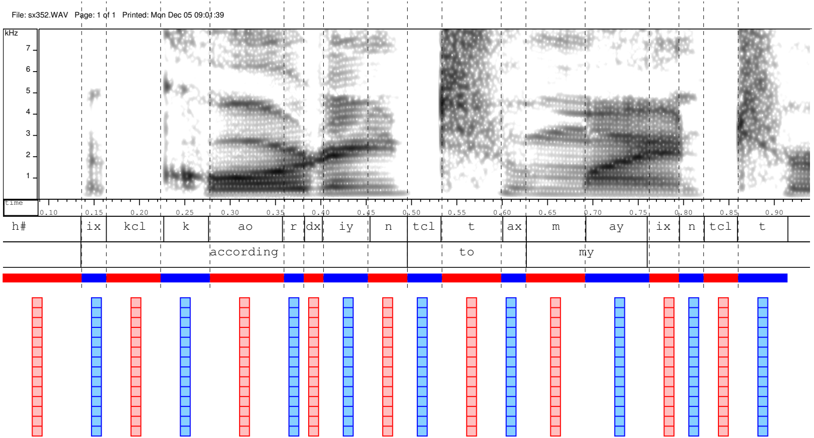
- fairly uncorrelated coefficients (simpler statistical models)
- high phonetic discrimination (empirically shown)
- do not assume all-pole model
- low number of coeff. enough to capture coarse structure of spectrum
- Cepstral Mean Subtraction corresponds to channel removal

[1] B. Bogert, M. Healy, and J. Tukey. "The Quefrency Alanysis of Time Series for Echoes: Cepstrum, Pseudo-autocovariance, Cross-Cepstrum and Saphe Cracking". In: *Proc. Symp. Time Series Analysis*. Ed. by M. Rosemblatt. John Wiley & Sons, 1963, pp. 209–243.

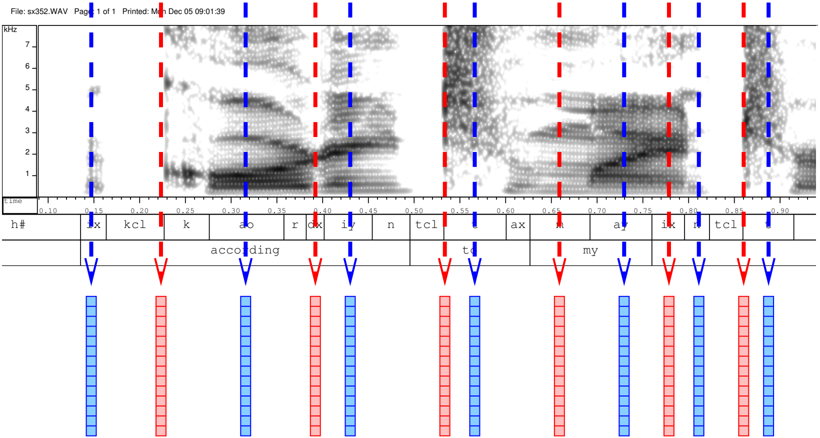
MFCCs: typical values

- 12 Coefficients C1–C12
- Energy (could be C0)
- Delta coefficients (derivatives in time)
- Delta-delta (second order derivatives)
- total: 39 coefficients per frame (analysis window)

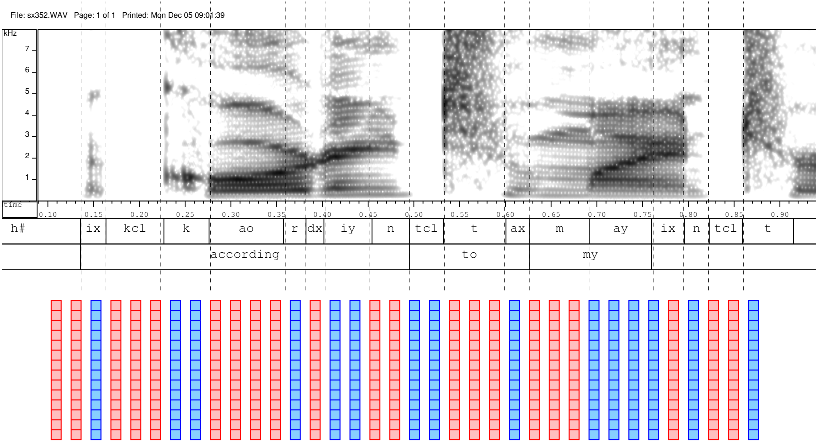
Segment-Based Processing



Landmark-Based Processing



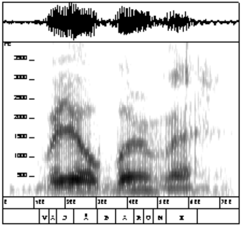
Frame-Based Processing



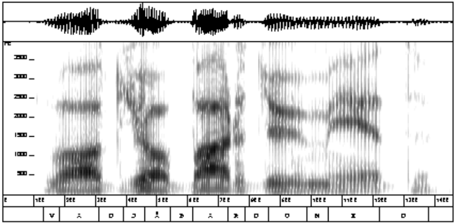
Frame-wise distance metrics

distance	$d(x, y)$
city block:	$\sum_i \lvert x_i - y_i \rvert$
Euclidean:	$\sqrt{\sum_i (x_i - y_i)^2}$
Mahalanobis:	$\sum_i (x_i - \mu_y)^2 / \sigma_y$
probability function:	$f(X \in x \vert \mu_y, \Sigma_y)$
artificial neural networks:	$f\left(\sum_i w_i x_i - \theta\right)$

Comparing Utterances



Va jobbaru me



Vad jobbar du med

"What is your occupation"
("What work you with")

Combining frame-wise scores into utterance scores

Template Matching

- oldest technique
- simple comparison of template patterns
- compensate for varying speech rate (Dynamic Programming)

Hidden Markov Models (HMMs)

- most used technique
- models of segmental structure of speech
- recognition by Viterbi search (Dynamic Programming)

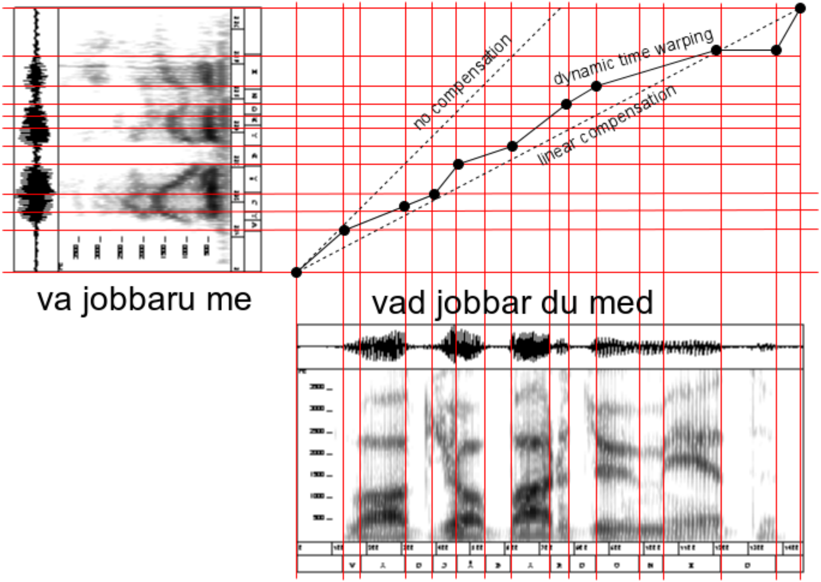
Recurrent Neural Networks

- often used in combination with HMMs

Template Matching: Why?

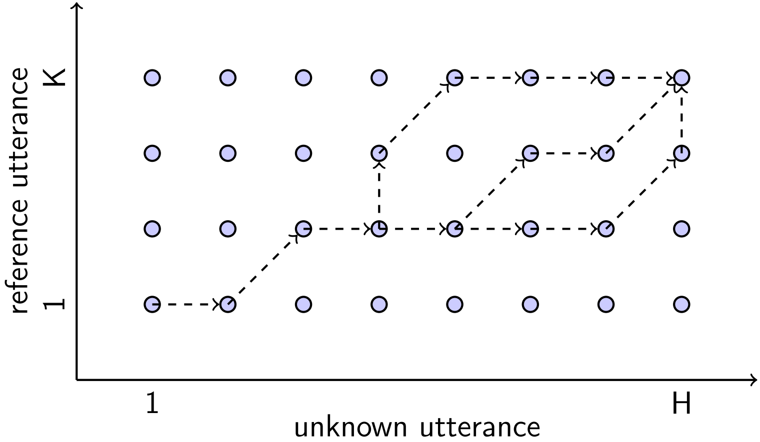
1. Historical: first technique used in ASR
2. Pedagogical: explain the problem and Dynamic Programming
3. Also called Dynamic Time Warping (DTW)

Template Matching



Dynamic Programming

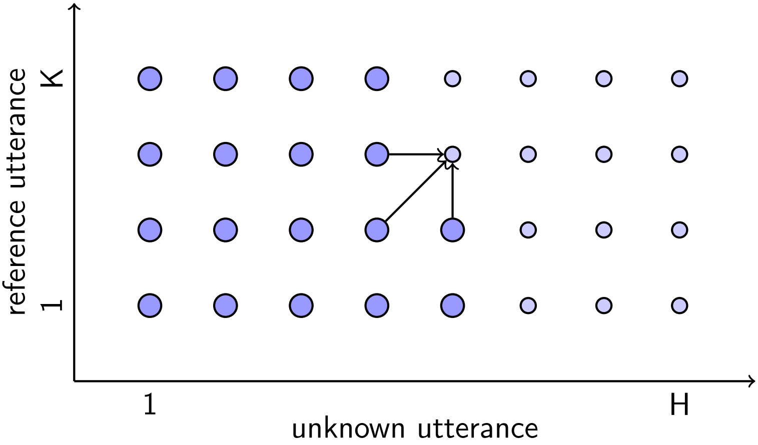
- compare any possible alignment
- problem: exponential with H and K!



Dynamic Programming

Dynamic Time Warping (DTW) algorithm

for $h = 1$ to H **do**
  for $k = 1$ to K **do**
    $AccD[h,k] = LocD[h,k] + \min(AccD[h-1,k], AccD[h-1,k-1], AccD[h,k-1])$



DP Example: Spelling

- observations are letters
- local distance: 0 (same letter), 1 (different letter)
- Unknown utterance: ALLDRIG
- Reference1: ALDRIG
- Reference2: ALLTID
- Problem: find closest match

Distance char-by-char:

- ALLDRIG–ALDRIG $= 5$
- ALLDRIG–ALLTID $= 4$

DP Example: Solution

LocD[h,k]=

G	1	1	1	1	1	1	0
I	1	1	1	1	1	0	1
R	1	1	1	1	0	1	1
D	1	1	1	0	1	1	1
L	1	0	0	1	1	1	1
A	0	1	1	1	1	1	1
	A	L	L	D	R	I	G

AccD[h,k]=

G	5	4	4	3	2	1	0
I	4	3	3	2	1	0	1
R	3	2	2	1	0	1	2
D	2	1	1	0	1	2	3
L	1	0	0	1	2	3	4
A	0	1	2	3	4	5	6
	A	L	L	D	R	I	G

Distance ALLDRIG–ALDRIG: *AccD*[H,K] $= 0$

Distance ALLDRIG–ALLTID? (5min)

DP Example: Solution

LocD[h,k]=

D	1	1	1	0	1	1	1
I	1	1	1	1	1	0	1
T	1	1	1	1	1	1	1
L	1	0	0	1	1	1	1
L	1	0	0	1	1	1	1
A	0	1	1	1	1	1	1
	A	L	L	D	R	I	G

AccD[h,k]=

D	5	3	3	2	3	3	3
I	4	2	2	2	2	2	3
T	3	1	1	1	2	3	4
L	2	0	0	1	2	3	4
L	1	0	0	1	2	3	4
A	0	1	2	3	4	5	6
	A	L	L	D	R	I	G

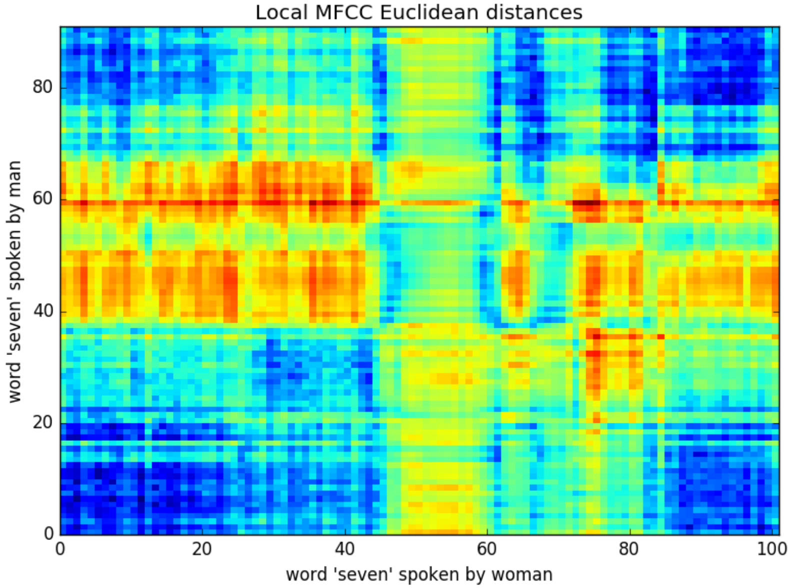
Distance ALLDRIG–ALDRIG: *AccD*[H,K] $= 0$
Distance ALLDRIG–ALLTID: *AccD*[H,K] $= 3$

Best path: Backtracking

Sometimes we want to know the path

1. at each point [h,k] remember the minimum distance predecessor (back pointer)
2. at the end point [H,K] follow the back pointers until the start

Real Example from Lab1



Accumulated MFCC Euclidean distances

Properties of Template Matching

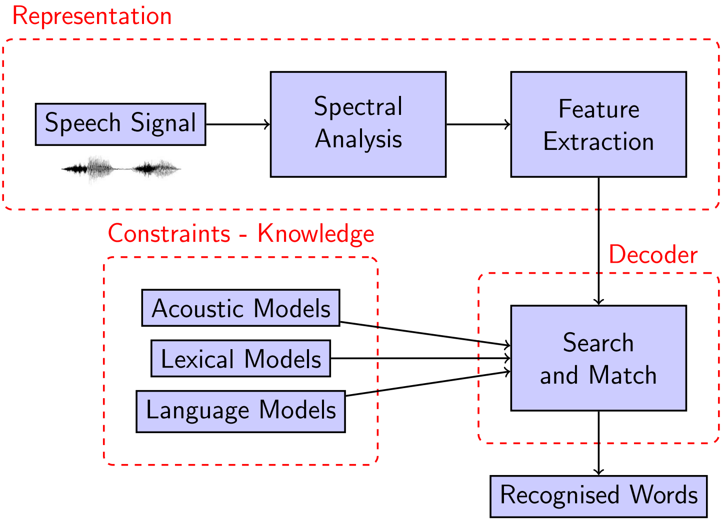
Pros:

+ No need for phonetic transcriptions
+ within-word co-articulation for free
+ high time resolution

Cons:

- cross-word co-articulation not modelled
- needs word segmentation (isolated words)
- requires recordings of every word
- not easy to model variation
- does not scale up with vocabulary size

Components of ASR System



Assuming you know:

- Probability theory
- Basic probability distributions
- Basic machine learning

Bayes' Rule

if

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

then

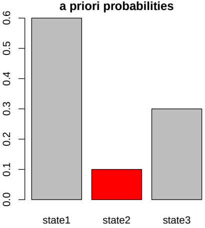
$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

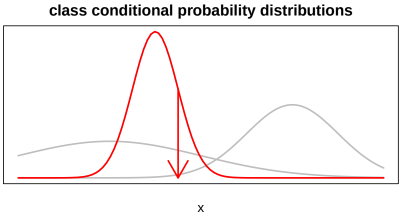
and

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

The Probabilistic Model of Classification

- "Nature" assumes one of c states ω_j with *a priori* probability $P(\omega_j)$
- When in state ω_j, "nature" emits observations $\hat{\mathbf{x}}$ with distribution $p(\mathbf{x}|\omega_j)$

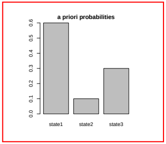


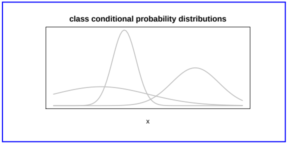


Problem

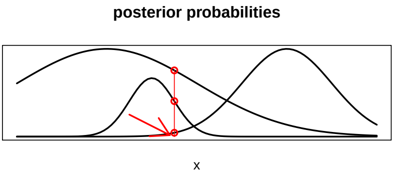
- If I observe $\hat{\mathbf{x}}$ and I know $P(\omega_j)$ and $p(\mathbf{x}|\omega_j)$ for each j
- what can I say about the state of "nature" ω_j?

Bayes decision theory

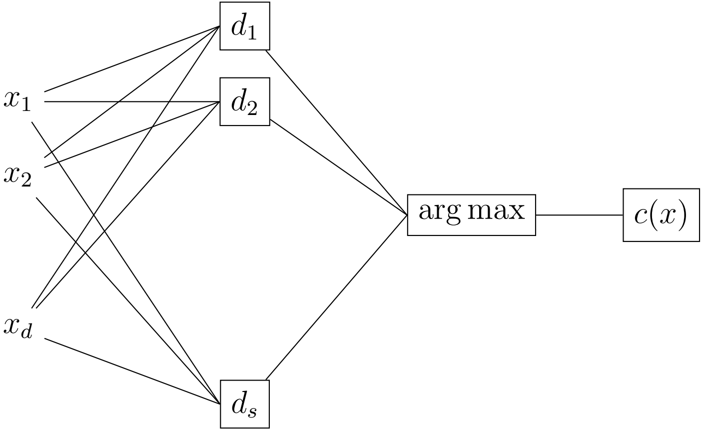




$$P(\omega_j|\mathbf{x}) = \frac{\color{blue}{p(\mathbf{x}|\omega_j)}\ \color{red}{P(\omega_j)}}{p(\mathbf{x})}$$



Classifiers: Discriminant Functions



$$d_i(\mathbf{x}) = p(\mathbf{x}|\omega_i)\ P(\omega_i)$$

Classifiers: Decision Boundaries

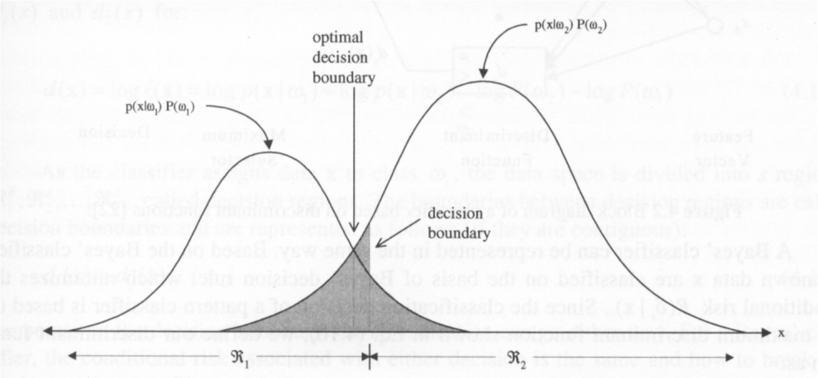


Figure from Huang, Acero, Hon.

Decision Boundaries in Two Dimensions

https://github.com/giampierosalvi/GaussianDecisionBoundaries

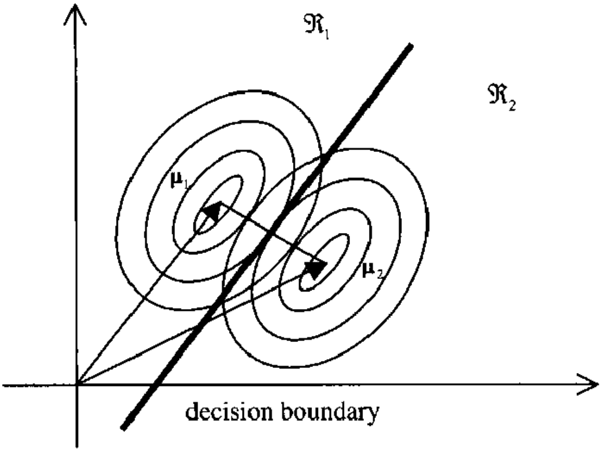


Figure from Huang, Acero, Hon.

Bayes' Rule and Speech Recognition

$A =$ words, $B =$ sounds:

- During training we know the words and can compute $P(\text{sounds}|\text{words})$ using frequentist approach (repeated observations)
- during recognition we want $\widehat{\text{words}} = \arg\max P(\text{words}|\text{sounds})$
- using Bayes' rule:

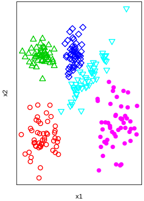
$$P(\text{words}|\text{sounds}) = \frac{P(\text{sounds}|\text{words})P(\text{words})}{P(\text{sounds})}$$

where
$P(\text{words})$: *a priori* probability of the words (Language Model)
$P(\text{sounds})$: *a priori* probability of the sounds (constant, can be ignored)

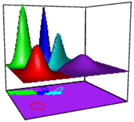
Estimation Theory

- so far we assumed we know $P(\omega_j)$ and $p(\mathbf{x}|\omega_j)$
- how can we obtain them from collections of data?
- this is the subject of Estimation Theory

Parameter estimation



$\longrightarrow$

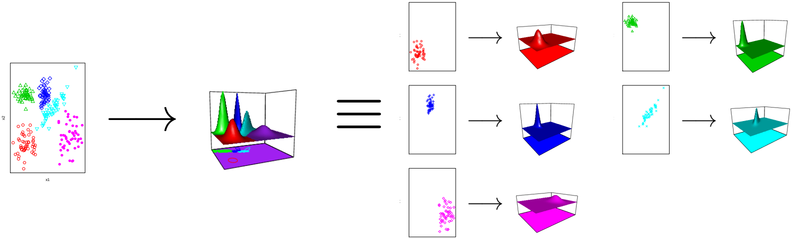


Assumptions:

- samples from class ω_i do not influence estimate for class $\omega_j,\ i \neq j$
- samples from the same class are independent and identically distributed (i.i.d.)

Parameter estimation (cont.)

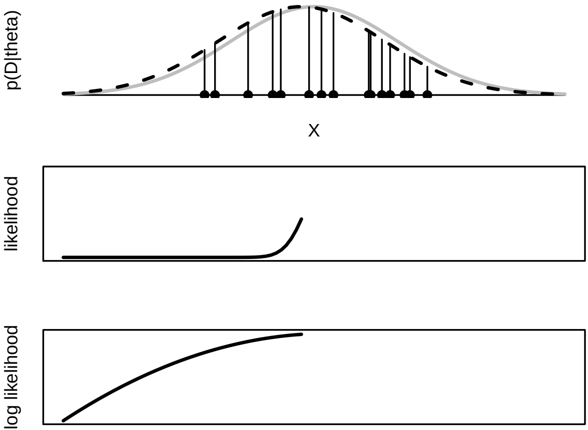
- class independence assumption:



- Maximum likelihood estimation
- Maximum a posteriori estimation
- Bayesian estimation

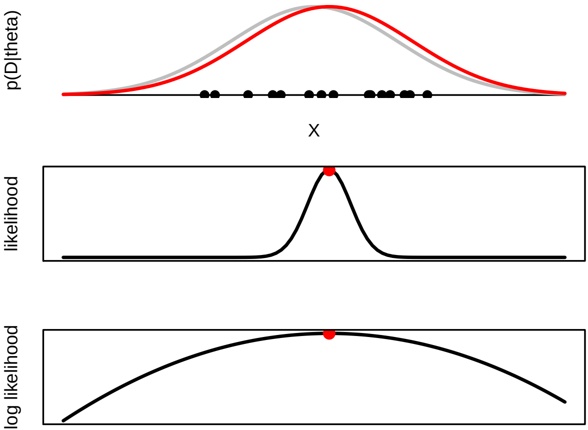
Maximum likelihood estimation

- Find parameter vector $\hat{\theta}$ that maximises $p(\mathcal{D}|\theta)$ with $\mathcal{D} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$
- i.i.d. $\rightarrow p(\mathcal{D}|\theta) = \prod_{k=1}^{n} p(\mathbf{x}_k|\theta)$



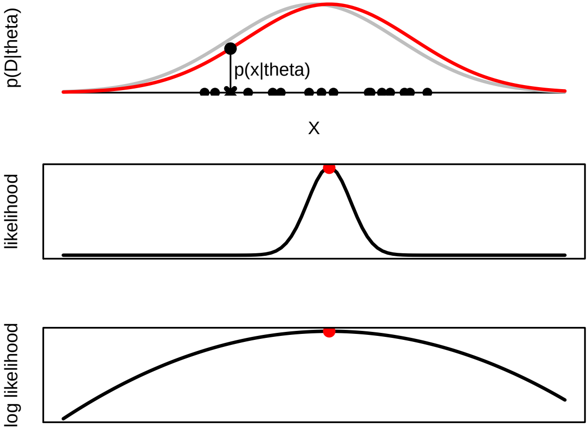
Maximum likelihood estimation

- Find parameter vector $\hat{\theta}$ that maximises $p(\mathcal{D}|\theta)$ with $\mathcal{D} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$
- i.i.d. $\rightarrow p(\mathcal{D}|\theta) = \prod_{k=1}^{n} p(\mathbf{x}_k|\theta)$



Maximum likelihood estimation

- Find parameter vector $\hat{\theta}$ that maximises $p(\mathcal{D}|\theta)$ with $\mathcal{D} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$
- i.i.d. $\rightarrow p(\mathcal{D}|\theta) = \prod_{k=1}^{n} p(\mathbf{x}_k|\theta)$



ML estimation of Gaussian mean

$$N(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \text{ with } \theta = \{\mu, \sigma^2\}$$

Log-likelihood of data (i.i.d. samples):

$$\log P(\mathcal{D}|\theta) = \sum_{i=1}^{N} \log N(x_i|\mu,\sigma^2) = -N\log\left(\sqrt{2\pi\sigma}\right) - \sum_{i=1}^{N} \frac{(x_i-\mu)^2}{2\sigma^2}$$

$$0 = \frac{d\log P(\mathcal{D}|\theta)}{d\mu} = \sum_{i=1}^{N} \frac{(x_i-\mu)}{\sigma^2} = \frac{\sum_{i=1}^{N} x_i - N\mu}{\sigma^2} \iff$$

$$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

ML estimation of Gaussian parameters

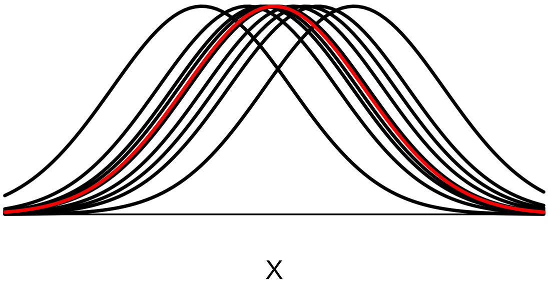
$$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \hat{\mu})^2$$

- same result by minimizing the sum of square errors!
- but we make assumptions explicit

Problem: few data points

10 repetitions with 5 points each



Maximum a Posteriori Estimation

$$\hat{\mu}, \hat{\sigma}^2 = \arg\max_{\mu,\sigma^2} \left[\prod_{i=1}^{N} P(x_i|\mu, \sigma^2) P(\mu, \sigma^2) \right]$$

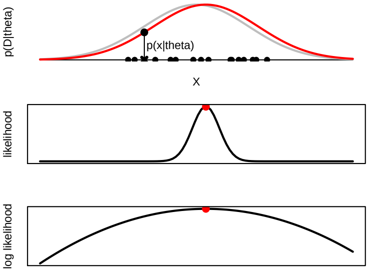
where the prior $P(\mu, \sigma^2)$ needs a nice mathematical form for closed solution

$$\begin{aligned} \hat{\mu}_{\text{MAP}} &= \frac{N}{N+\gamma}\,\hat{\mu}_{\text{ML}} + \frac{\gamma}{N+\gamma}\,\delta \\ \hat{\sigma}^2_{\text{MAP}} &= \frac{N}{N+3+2\alpha}\hat{\sigma}^2_{\text{ML}} + \frac{2\beta + \gamma(\delta + \hat{\mu}_{\text{MAP}})^2}{N+3+2\alpha} \end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ are parameters of the prior distribution

ML, MAP and Point Estimates

- Both ML and MAP produce point estimates of θ
- Assumption: there is a true value for θ
- advantage: once $\hat{\theta}$ is found, everything is known



Overfitting

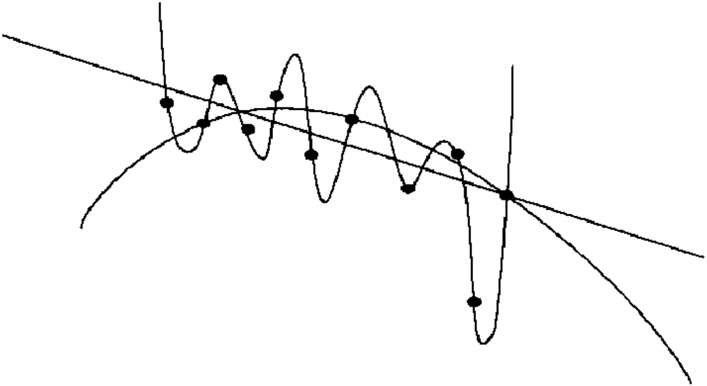


Figure from Huang, Acero, Hon.

Overfitting: Phoneme Discrimination

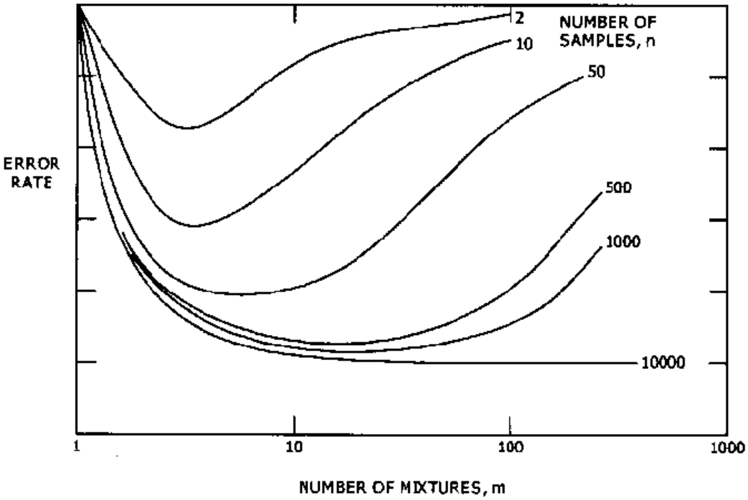


Figure from Huang, Acero, Hon.

Bayesian estimation

- Consider θ as a random variable
- characterize θ with the posterior distribution $P(\theta|\mathcal{D})$ given the data

$$\begin{array}{llcl} \text{ML:} & \mathcal{D} & \rightarrow & \hat{\theta}_{\text{ML}} \\ \text{MAP:} & \mathcal{D}, P(\theta) & \rightarrow & \hat{\theta}_{\text{MAP}} \\ \text{Bayes:} & \mathcal{D}, P(\theta) & \rightarrow & P(\theta|\mathcal{D}) \end{array}$$

- for new data points, instead of $P(\mathbf{x}_{\text{new}}|\hat{\theta}_{\text{ML}})$ or $P(\mathbf{x}_{\text{new}}|\hat{\theta}_{\text{MAP}})$, compute:

$$P(\mathbf{x}_{\text{new}}|\mathcal{D}) = \int_{\theta\in\Theta} P(\mathbf{x}_{\text{new}}|\theta)P(\theta|\mathcal{D})d\theta$$

Bayesian estimation (cont.)

Pros:

- better use of the data
- makes a priori assumptions explicit
- easily implemented recursively
  - use posterior $p(\theta|\mathcal{D})$ as new prior
- reduce overfitting

Cons:

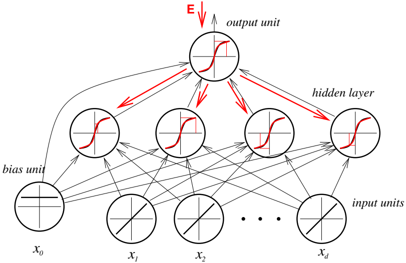
- definition of noninformative priors can be tricky
- often requires numerical integration

Other Training Strategies: Discriminative Training

- Maximum Mutual Information Estimation
- Minimum Error Rate Estimation
- Neural Networks

Multi layer neural networks

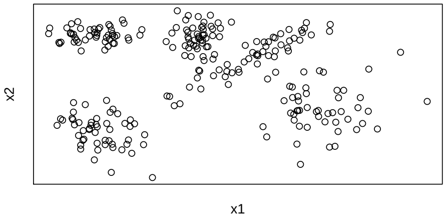
Multi layer
neural networks



- Backpropagation algorithm

Unsupervised Learning

- so far we assumed we knew the class ω_i for each data point
- what if we don't?
- class independence assumption loses meaning



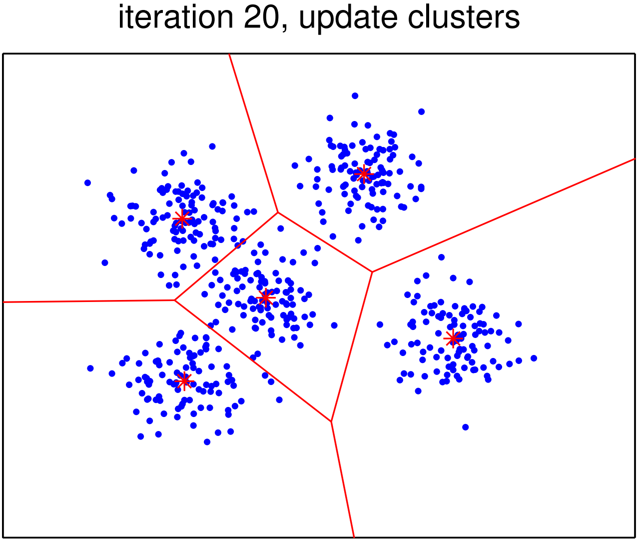
Vector Quantisation, K-Means

- describes each class with a centroid
- a point belongs to a class if the corresponding centroid is closest (Euclidean distance)
- iterative procedure
- guaranteed to converge
- not guaranteed to find the optimal solution
- used in vector quantization

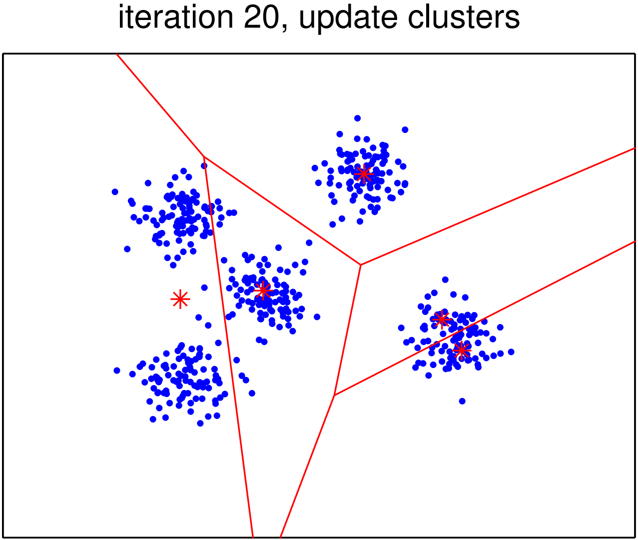
K-means: algorithm

Data: k (number of desired clusters), n data points $\mathbf{x}_i$
Result: k clusters
initialization: assign initial value to k centroids $\mathbf{c}_i$;
repeat
    assign each point $\mathbf{x}_i$ to closest centroid $\mathbf{c}_j$;
    compute new centroids as mean of each group of points;
until *centroids do not change*;
return k clusters;

K-means: example



K-means: sensitivity to initial conditions



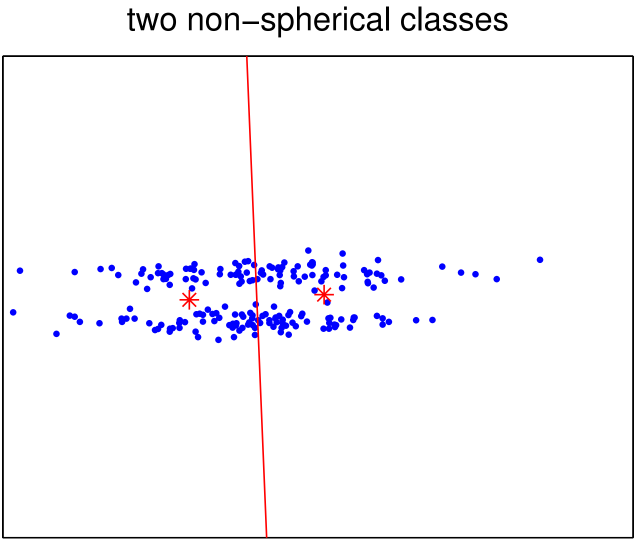
Solution: LBG Algorithm

- Linde–Buzo–Gray
- start with one centroid
- adjust to mean
- split centroid (with ϵ)
- K-means
- split again…

K-means: limits of Euclidean distance

- the Euclidean distance is isotropic (same in all directions in $\mathbb{R}^p$)
- this favours spherical clusters
- the size of the clusters is controlled by their distance

K-means: non-spherical classes



Probabilistic Clustering

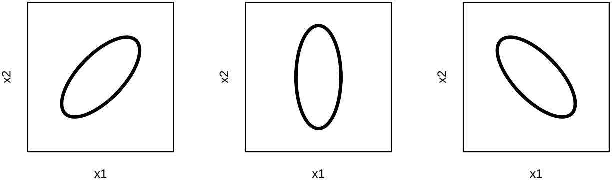
- model data as a mixture of probability distributions (Gaussian)
- each distribution corresponds to a cluster
- clustering corresponds to parameter estimation

Gaussian distributions

$$f_k(\mathbf{x}_i|\mu_k, \Sigma_k) = \frac{exp\left\{-\frac{1}{2}(\mathbf{x}_i - \mu_k)^T \Sigma_k^{-1}(\mathbf{x}_i - \mu_k)\right\}}{(2\pi)^{\frac{p}{2}} \left|\Sigma_k\right|^{\frac{1}{2}}}$$

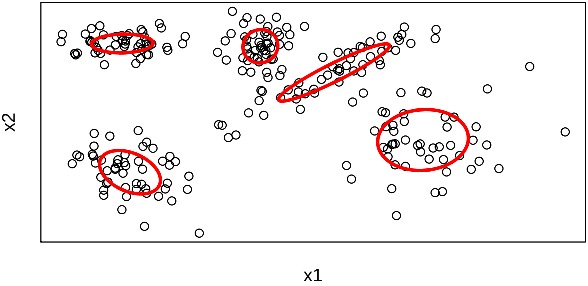
Eigenvalue decomposition of the covariance matrix:

$$\Sigma_k = \lambda_k D_k A_k D_k^T$$



Mixture of Gaussian distributions

Σ_k	Distribution	Volume	Shape	Orientation
λI	Spherical	Equal	Equal	N/A
$\lambda_k I$	Spherical	Variable	Equal	N/A
$\lambda D A D^T$	Ellipsoidal	Equal	Equal	Equal
$\lambda D_k A D_k^T$	Ellipsoidal	Equal	Equal	Variable
$\lambda_k D_k A D_k^T$	Ellipsoidal	Variable	Equal	Variable
$\lambda_k D_k A_k D_k^T$	Ellipsoidal	Variable	Variable	Variable



Fitting the model

- given the data $D = \{\mathbf{x}_i\}$
- given a certain model $\mathcal{M}$ and its parameters θ
- maximize the model fit to the data as expressed by the likelihood

$$\mathcal{L} = p(D|\theta)$$

Unsupervised Case

- release class independence assumption:
- learn the mixture at once
- problem of missing data
- solution: Expectation Maximization

Expectation Maximization

Fitting model parameters with missing (latent) variables

$$P(\mathbf{x}|\theta) = \sum_{k=1}^{K} \pi_k P(x|\theta_k),$$

$$\text{with } \theta = \{\pi_1, \ldots, \pi_k, \theta_1, \ldots, \theta_K\}$$

- very general idea (applies to many different probabilistic models)
- augment the data with the missing variables: h_{ik} probability of assignment of each data point x_i to each component of the mixture k
- optimize the Likelihood of the complete data:

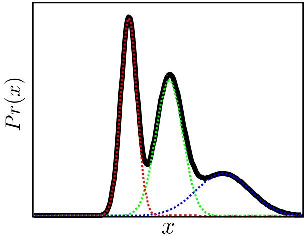
$$P(\mathbf{x}, \mathbf{h}|\theta)$$

Mixture of Gaussians

This distribution is a weight sum of K Gaussian distributions

$$P(x) = \sum_{k=1}^{K} \pi_k \mathcal{N}(x; \mu_k, \sigma_k^2)$$

where $\pi_1 + \cdots + \pi_K = 1$
and $\pi_k > 0$ $(k = 1, \ldots, K)$.



This model can describe **complex multi-modal** probability distributions by combining simpler distributions.

Mixture of Gaussians

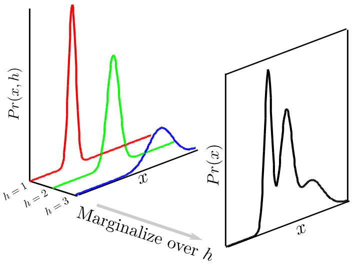
$$P(x) = \sum_{k=1}^{K} \pi_k \mathcal{N}(x; \mu_k, \sigma_k^2)$$

- Learning the parameters of this model from training data $x_1, \ldots, x_n$ is not trivial - using the usual straightforward maximum likelihood approach.
- Instead learn parameters using the **Expectation-Maximization** (EM) algorithm.

Mixture of Gaussians as a marginalization

We can interpret the Mixture of Gaussians model with the introduction of a discrete hidden/latent variable h and $P(x, h)$:

$$P(x) = \sum_{k=1}^{K} P(x, h = k) = \sum_{k=1}^{K} P(x \mid h = k) P(h = k)$$
$$= \sum_{k=1}^{K} \pi_k \, \mathcal{N}(x; \mu_k, \sigma_k^2)$$



$\leftarrow$ **mixture density**

Figures taken from **Computer Vision: models, learning and inference** by Simon Prince.

EM for two Gaussians

Assume: We know the pdf of x has this form:

$$P(x) = \pi_1 \mathcal{N}(x; \mu_1, \sigma_1^2) + \pi_2 \mathcal{N}(x; \mu_2, \sigma_2^2)$$

where $\pi_1 + \pi_2 = 1$ and $\pi_k > 0$ for components $k = 1, 2$.

Unknown: Values of the parameters (Many!)

$$\Theta = (\pi_1, \mu_1, \sigma_1, \mu_2, \sigma_2).$$

Have: Observed n samples $x_1, \ldots, x_n$ drawn from $P(x)$.

Want to: Estimate Θ from $x_1, \ldots, x_n$.

How would it be possible to get them all???

EM for two Gaussians

For each sample x_i introduce a *hidden variable* h_i

$$h_i = \begin{cases} 1 & \text{if sample } x_i \text{ was drawn from } \mathcal{N}(x; \mu_1, \sigma_1^2) \\ 2 & \text{if sample } x_i \text{ was drawn from } \mathcal{N}(x; \mu_2, \sigma_2^2) \end{cases}$$

and come up with initial values

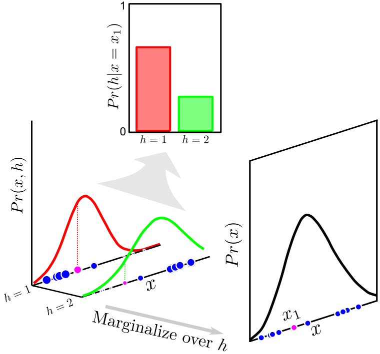
$$\Theta^{(0)} = (\pi_1^{(0)}, \mu_1^{(0)}, \sigma_1^{(0)}, \mu_2^{(0)}, \sigma_2^{(0)})$$

for each of the parameters.

EM is an *iterative algorithm* which updates $\Theta^{(t)}$ using the following two steps...

EM for two Gaussians: E-step

The responsibility of k-th Gaussian for each sample x (indicated by the size of the projected data point)



Look at each sample x along hidden variable h in the E-step

Figure from **Computer Vision: models, learning and inference** by Simon Prince.

EM for two Gaussians: E-step (cont.)

E-step: Compute the *"posterior probability"* that x_i was generated by component k given the current estimate of the parameters $\Theta^{(t)}$. (responsibilities)

for $i = 1, \ldots n$

for $k = 1, 2$

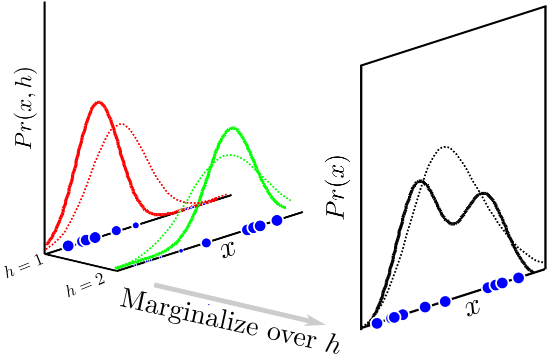
$$\gamma_{ik}^{(t)} = P(h_i = k \,|\, x_i, \Theta^{(t)})$$

$$= \frac{\pi_k^{(t)}\,\mathcal{N}(x_i; \mu_k^{(t)}, \sigma_k^{(t)})}{\pi_1^{(t)}\,\mathcal{N}(x_i; \mu_1^{(t)}, \sigma_1^{(t)}) + \pi_2^{(t)}\,\mathcal{N}(x_i; \mu_2^{(t)}, \sigma_2^{(t)})}$$

Note: $\gamma_{i1}^{(t)} + \gamma_{i2}^{(t)} = 1$ and $\pi_1 + \pi_2 = 1$

EM for two Gaussians: M-step

Fitting the Gaussian model for each of k-th constinuetnt.
Sample x_i contributes according to the responsibility γ_{ik}.



(dashed and solid lines for fit before and after update)

Look along samples x for each h in the M-step

Figure from **Computer Vision: models, learning and inference** by Simon Prince.

EM for two Gaussians: M-step (cont.)

M-step: Compute the *Maximum Likelihood* of the parameters of the mixture model given out data's membership distribution, the $\gamma_i^{(t)}$'s:

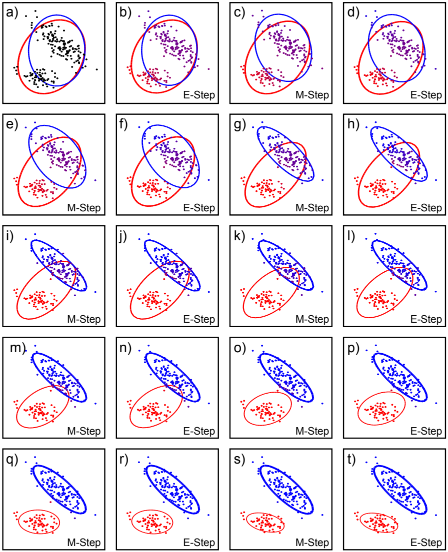
for $k = 1, 2$

$$\mu_k^{(t+1)} = \frac{\sum_{i=1}^n \gamma_{ik}^{(t)} x_i}{\sum_{i=1}^n \gamma_{ik}^{(t)}},$$

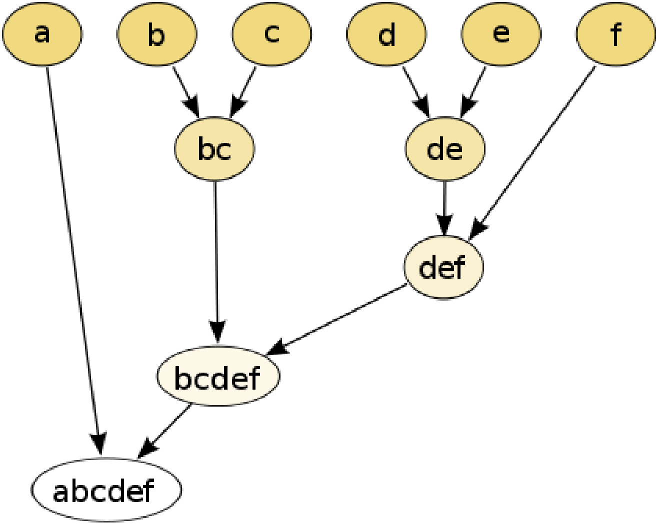
$$\sigma_k^{(t+1)} = \sqrt{\frac{\sum_{i=1}^n \gamma_{ik}^{(t)} (x_i - \mu_k^{(t+1)})^2}{\sum_{i=1}^n \gamma_{ik}^{(t)}}},$$

$$\pi_k^{(t+1)} = \frac{\sum_{i=1}^n \gamma_{ik}^{(t)}}{n}.$$

EM in practice



Hierarchical Clustering



(Figure from Wikipedia)