Probabilistic Modelling of Sequences: Learning

DT2119 Speech and Speaker Recognition

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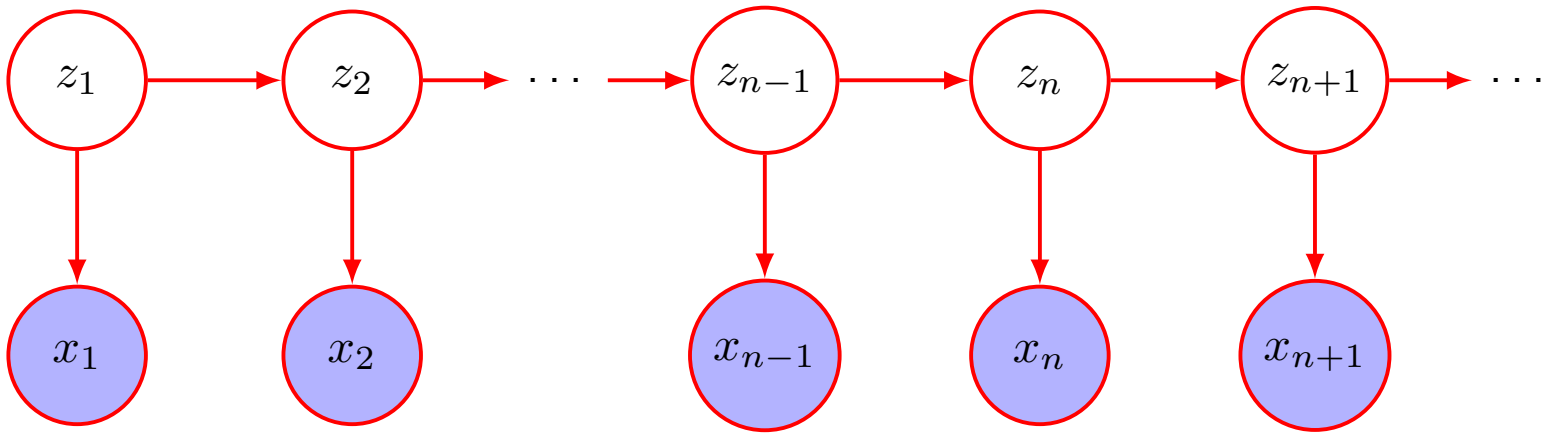
HMM Inference: Learning

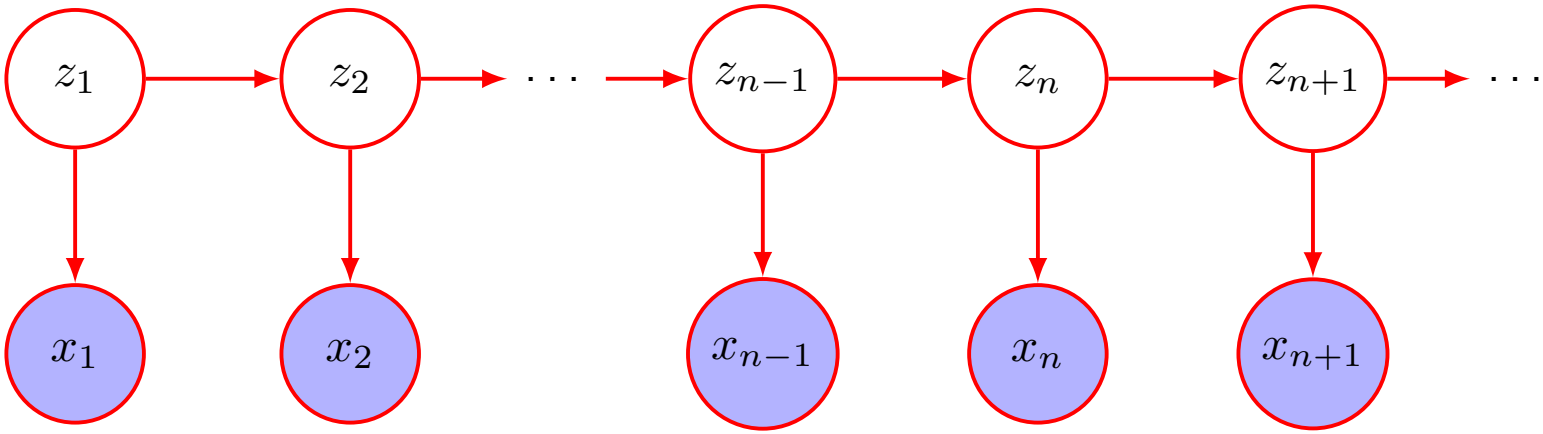


- Given observations X update model parameters

$$\theta = \{\pi, A, \phi\}$$

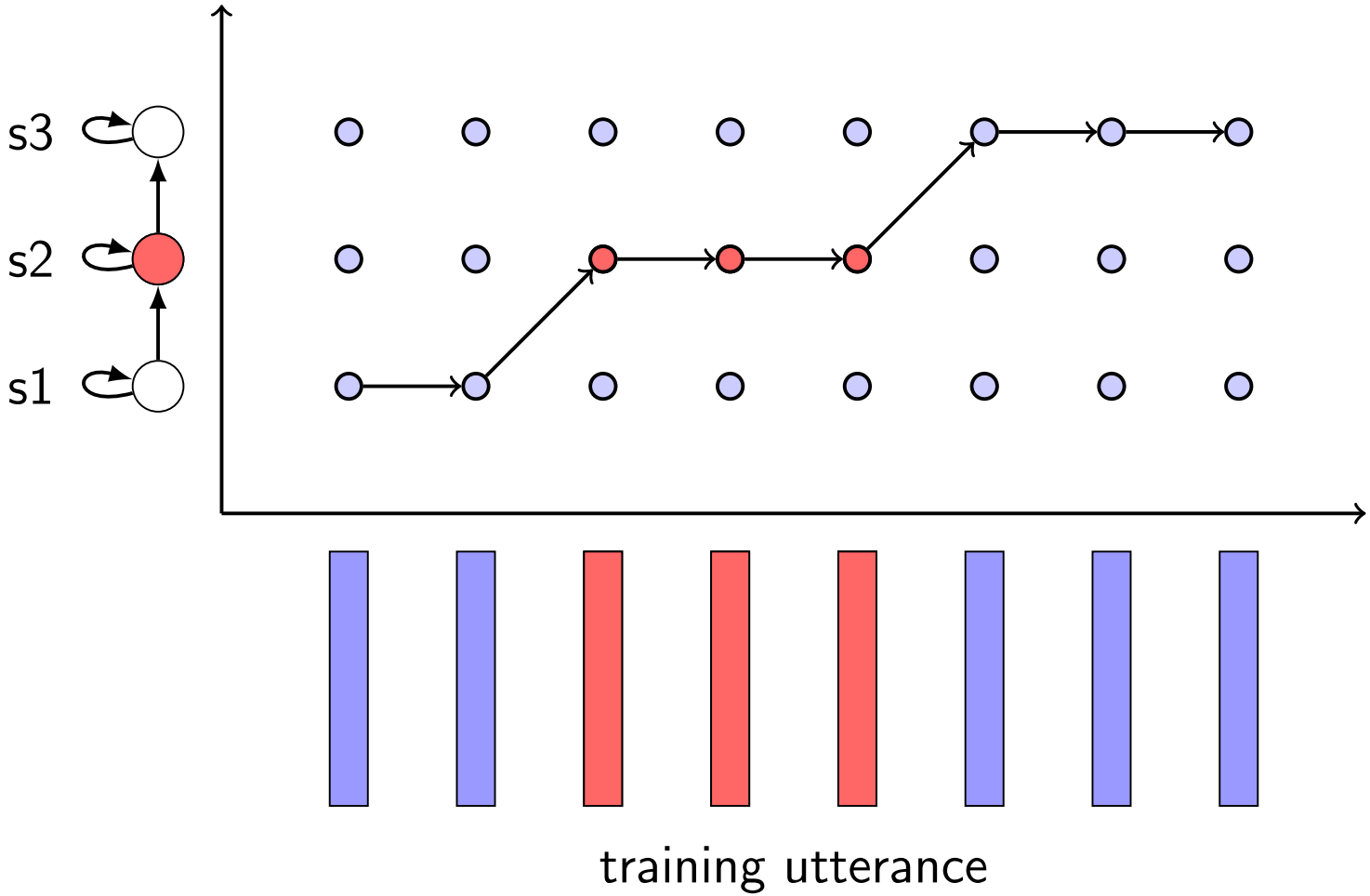
- to maximise either:
  - model fit to data (e.g. likelihood, posterior)
  - classification performance (discriminative training)

HMM Inference: Learning



- problem: incomplete data, state sequence Z
- there is no closed-form solution
- only iterative procedures: given θ^{old} how to estimate θ^{new}

Viterbi training (simple approach)



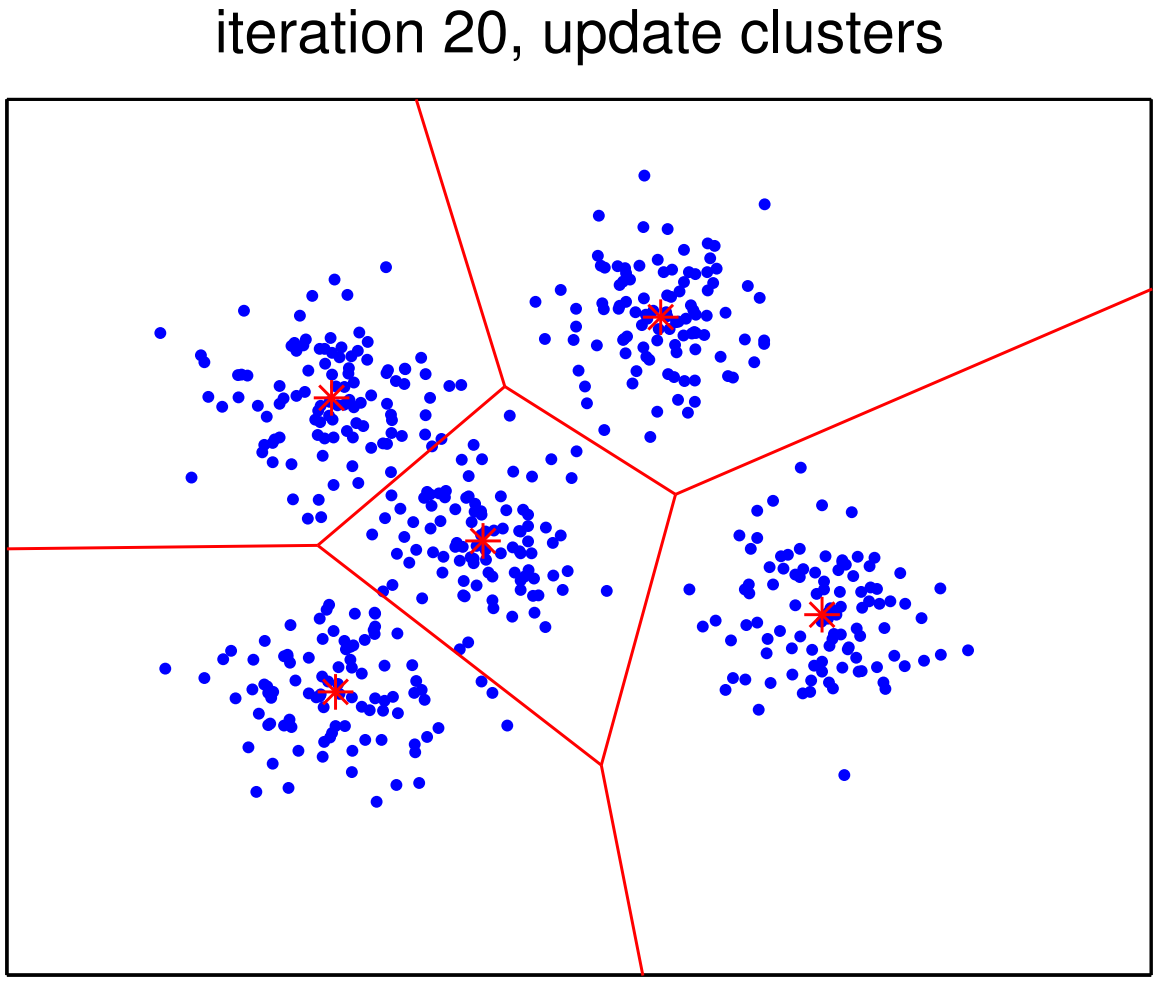
problem: sensitive to misalignments

but still used for ANN/DNN training (Lab3)

Parallel with K-means

Data: k (number of desired clusters), n data points $\mathbf{x}_i$
Result: k clusters
initialization: assign initial value to k centroids $\mathbf{c}_i$;
repeat
    assign each point $\mathbf{x}_i$ to closest centroid $\mathbf{c}_j$;
    compute new centroids as mean of each group of points;
until *centroids do not change*;
return k clusters;

K-means: example



HMM Inference: Learning

Latent variables $\rightarrow$ Expectation Maximisation

- locally maximise the likelihood of the complete data X, Z
- close form and efficient solution with froward-backward or Baum-Welch algorithm[1]
- general idea: sum over all possible paths weighted by posterior probability of the path
- also: every observation vector contributes to all parameter updates

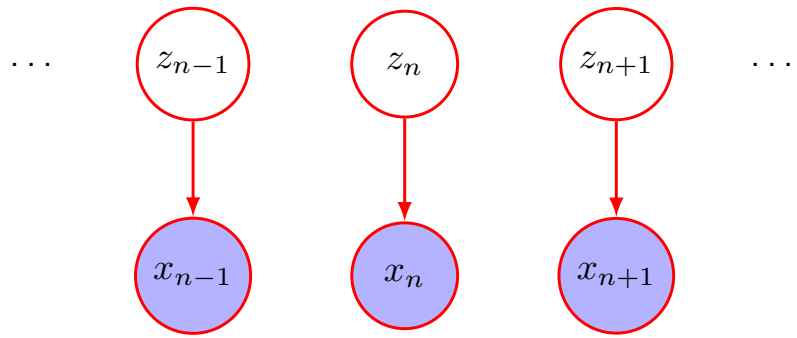
[1] L. E. Baum, T. Petrie, G. Soules, and N. Weiss. "A maximization technique occurring in the statistical analysis of probabilistic functions of Markov chains". In: *Ann. Math. Statist.* 41.1 (1970), pp. 164–171.

Expectation Maximization for Mixture Models

$$P(x|\theta) = \sum_{k=1}^{K} \pi_k P(x|\theta_k),$$

$$\theta = \{\pi_1, \ldots, \pi_k, \theta_1, \ldots, \theta_K\},$$

$$\sum_{k=1}^{K} \pi_k = 1$$



- augment the data with the latent variables: $z_i \in \{1, \ldots, K\}$ assignment of each data point x_i to a component of the mixture
- interpret the mixture as marginal of the joint

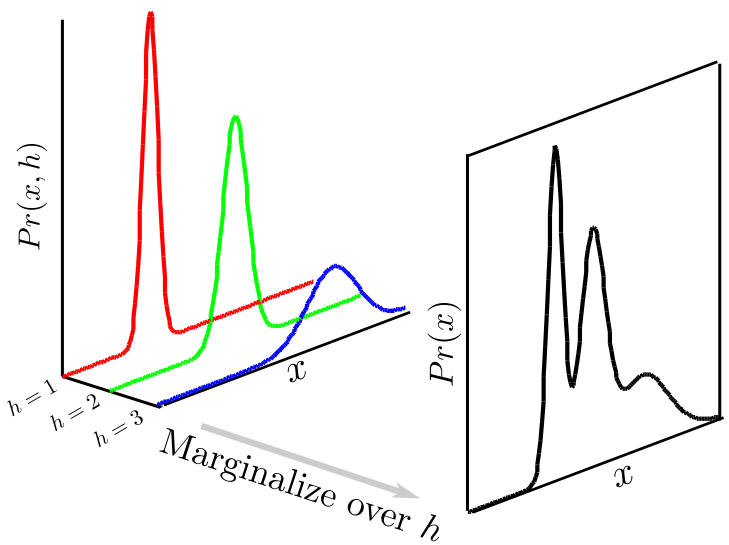
$$P(x|\theta) = \sum_{z} P(x, z|\theta)$$

Example: Mixture of Gaussians

Using the discrete hidden/latent variable z and $P(x, z)$:

$$p(x|\theta) = \sum_{k=1}^{K} p(x, z = k|\theta) = \sum_{k=1}^{K} p(x \mid z = k)P(z = k)$$

$$= \sum_{k=1}^{K} \pi_k \mathcal{N}(x; \mu_k, \sigma_k^2)$$



$\leftarrow$ **mixture density**

Figures taken from **Computer Vision: models, learning and inference** by Simon Prince.

Expectation Maximization: Idea

Ideally we would like to maximize:

$$\log p(X|\theta) = \log \left\{ \sum_Z p(X, Z|\theta) \right\}$$

with $X = \{x_1, \ldots, x_N\}, \quad Z = \{z_1, \ldots, z_N\}$

. . . but log of sum hard to optimize

Instead optimize likelihood of complete data:

$$\log p(X, Z|\theta)$$

Z not known, but we can compute posterior given current model $p(Z|X, \theta^{\text{old}})$

Optimize the expected value of the likelihood:

$$\mathcal{Q}(\theta, \theta^{\text{old}}) = \sum_Z p(Z|X, \theta^{\text{old}}) \log p(X, Z|\theta)$$

Expectation Maximization in Practice

1. Initialization: choose initial value of θ^{old}
2. Expectation step: evaluate posterior $p(Z|X, \theta^{\text{old}})$
3. Maximization step: evaluate θ^{new} with:

$$\begin{aligned} \theta^{\text{new}} &= \arg\max_{\theta} \mathcal{Q}(\theta, \theta^{\text{old}}) \\ &= \arg\max_{\theta} \sum_{Z} p(Z|X, \theta^{\text{old}}) \log p(X, Z|\theta) \end{aligned}$$

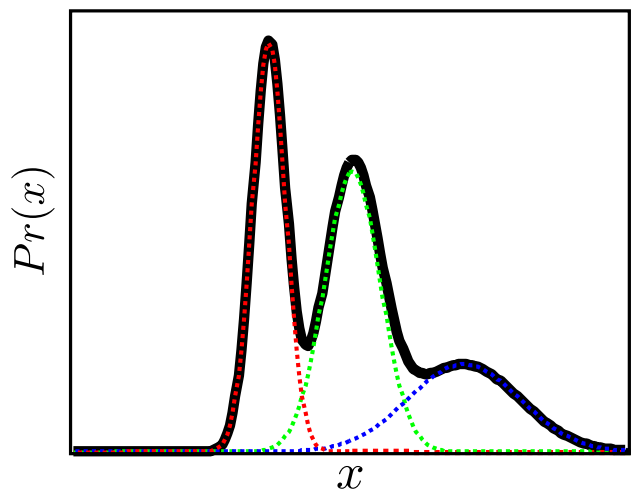
4. Check for convergence, otherwise $\theta^{\text{old}} \leftarrow \theta^{\text{new}}$ and go to step 2.

Mixture of Gaussians

This distribution is a weighted sum of K Gaussian distributions

$$p(x|\theta) = \sum_{k=1}^{K} \pi_k \mathcal{N}(x; \mu_k, \sigma_k^2)$$

$$\theta = \{\pi_1, \ldots, \pi_K, \mu_1, \ldots, \mu_K, \sigma_1^2, \ldots, \sigma_K^2\}$$



This model can describe **complex multi-modal** probability distributions by combining simpler distributions.

EM for two Gaussians

Assume: We know the pdf of x has this form:

$$P(x) = \pi_1 \mathcal{N}(x; \mu_1, \sigma_1^2) + \pi_2 \mathcal{N}(x; \mu_2, \sigma_2^2)$$

where $\pi_1 + \pi_2 = 1$ and $\pi_k > 0$ for components $k = 1, 2$.

Unknown: Values of the parameters

$$\theta = \{\pi_1, \mu_1, \sigma_1, \mu_2, \sigma_2\}.$$

Have: Observed N samples $x_1, \ldots, x_N$ drawn from $P(x)$.

Want to: Estimate θ from $x_1, \ldots, x_N$.

EM for two Gaussians

For each sample x_n introduce a *hidden variable* z_n

$$z_n = \begin{cases} 1 & \text{if sample } x_n \text{ was drawn from } \mathcal{N}(x; \mu_1, \sigma_1^2) \\ 2 & \text{if sample } x_n \text{ was drawn from } \mathcal{N}(x; \mu_2, \sigma_2^2) \end{cases}$$

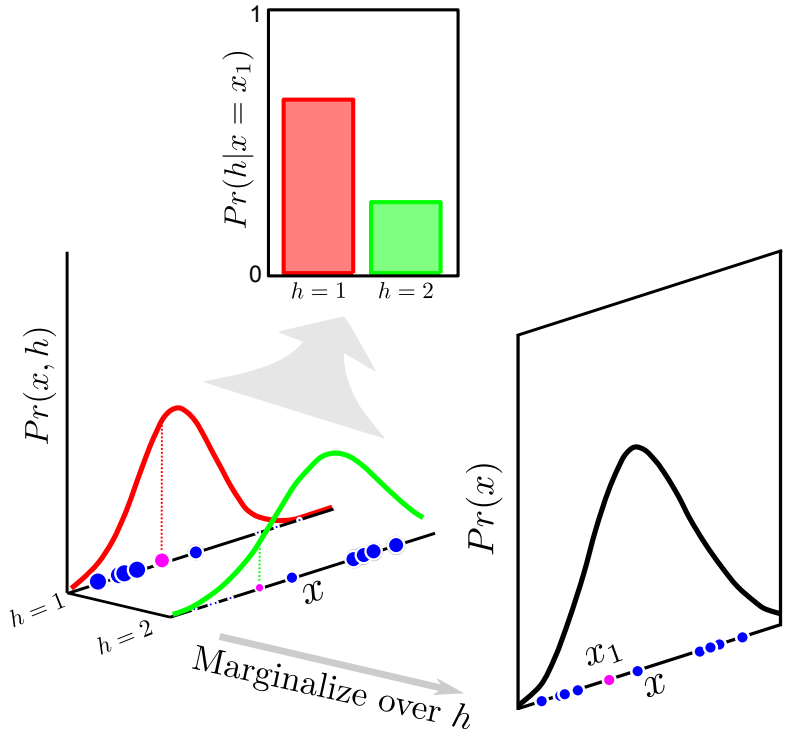
and come up with initial values

$$\theta^{(0)} = \{\pi_1^{(0)}, \mu_1^{(0)}, \sigma_1^{(0)}, \mu_2^{(0)}, \sigma_2^{(0)}\}$$

for each of the parameters.

EM for two Gaussians: E-step

The responsibility of k-th Gaussian for each sample x (indicated by the size of the projected data point)



Look at each sample x along hidden variable h in the E-step

Figure from **Computer Vision: models, learning and inference** by Simon Prince.

EM for two Gaussians: E-step (cont.)

E-step: Compute the *"posterior probability"* that x_n was generated by component k given the current estimate of the parameters $\theta^{(t)}$. (responsibilities)

for $n = 1, \ldots N$

for $k = 1, 2$

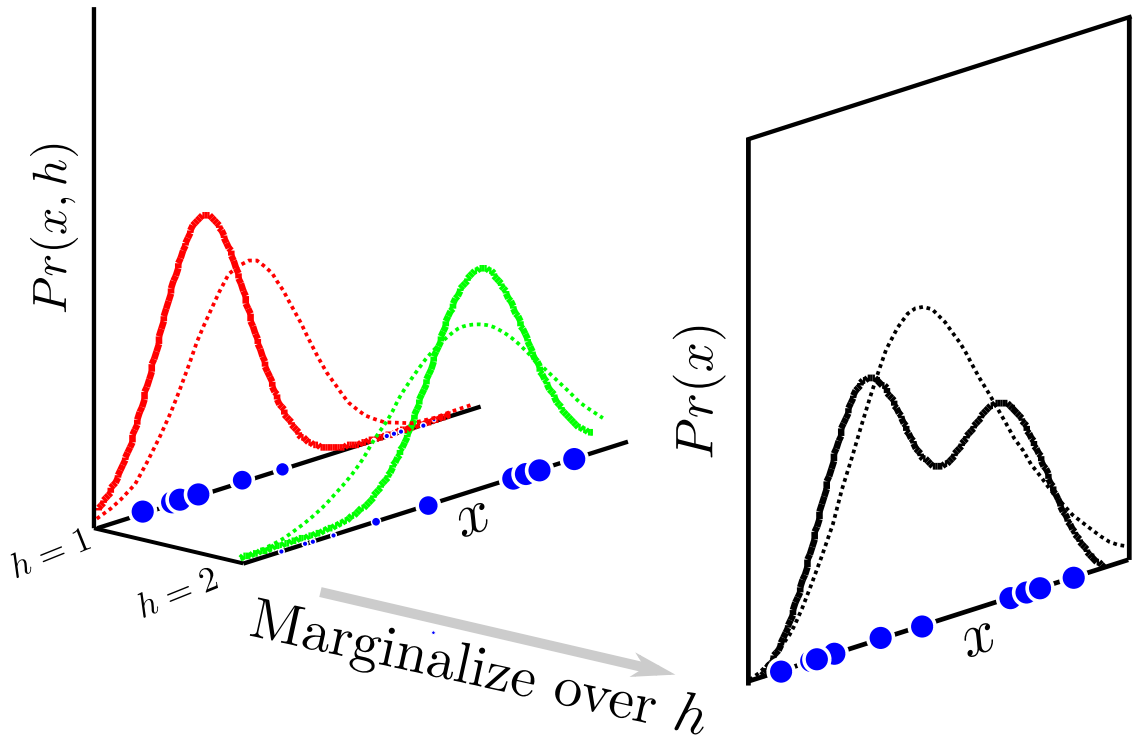
$$\gamma_n(k) = P(z_n = k \,|\, x_n, \theta^{(t)})$$

$$= \frac{\pi_k^{(t)} \mathcal{N}(x_n; \mu_k^{(t)}, \sigma_k^{(t)})}{\pi_1^{(t)} \mathcal{N}(x_n; \mu_1^{(t)}, \sigma_1^{(t)}) + \pi_2^{(t)} \mathcal{N}(x_n; \mu_2^{(t)}, \sigma_2^{(t)})}$$

Note: $\gamma_n(1) + \gamma_n(2) = 1$ and $\pi_1 + \pi_2 = 1$

EM for two Gaussians: M-step

Fitting the Gaussian model for each of k-th constituent.
Sample x_i contributes according to the responsibility $\gamma_i(k)$.



(dashed and solid lines for fit before and after update)

Look along samples x for each z in the M-step

Figure from **Computer Vision: models, learning and inference** by Simon Prince.

EM for two Gaussians: M-step (cont.)

M-step: Compute the *Maximum Likelihood* of the parameters of the mixture model given out data's membership distribution, the γ_n's:

for $k = 1, 2$

$$\mu_k^{(t+1)} = \frac{\sum_{n=1}^{N} \gamma_n(k) x_n}{\sum_{n=1}^{N} \gamma_n(k)},$$

$$\left(\sigma_k^{(t+1)}\right)^2 = \frac{\sum_{n=1}^{N} \gamma_n(k)(x_n - \mu_k^{(t+1)})^2}{\sum_{n=1}^{N} \gamma_n(k)},$$

$$\pi_k^{(t+1)} = \frac{\sum_{n=1}^{N} \gamma_n(k)}{N}.$$

Example of Expectation Maximization

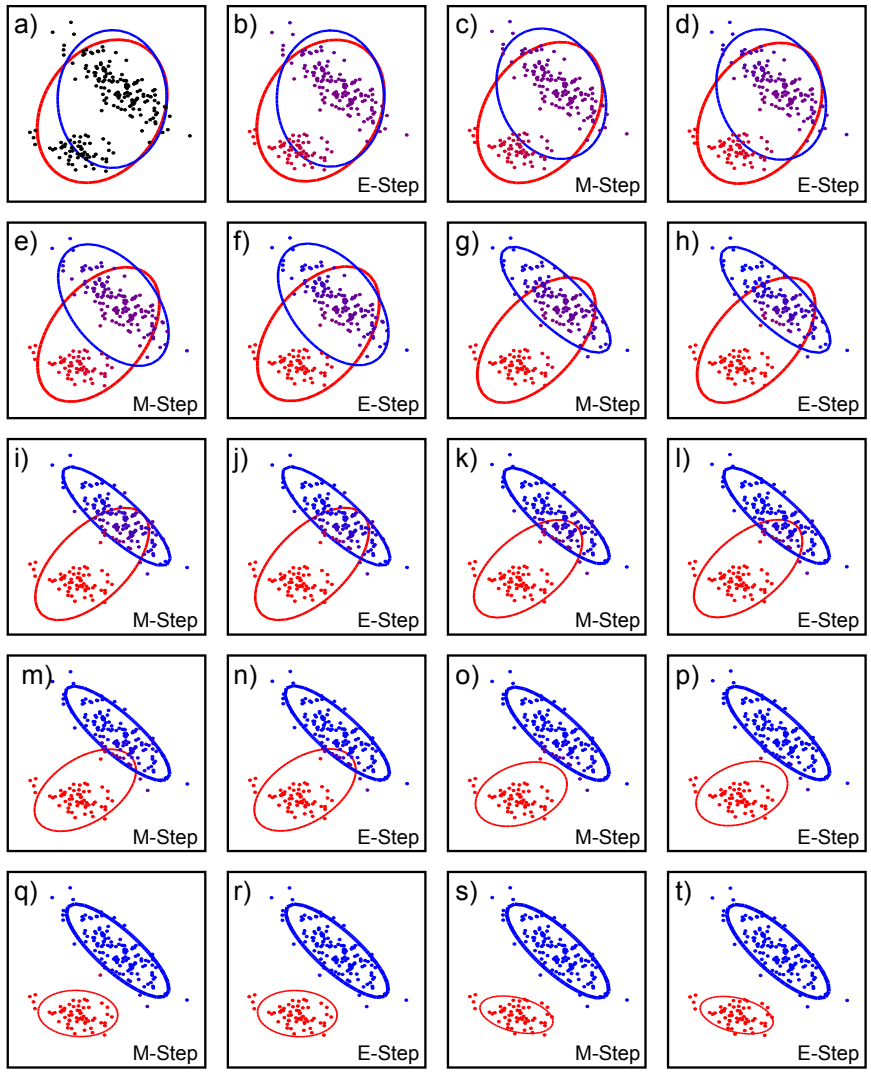
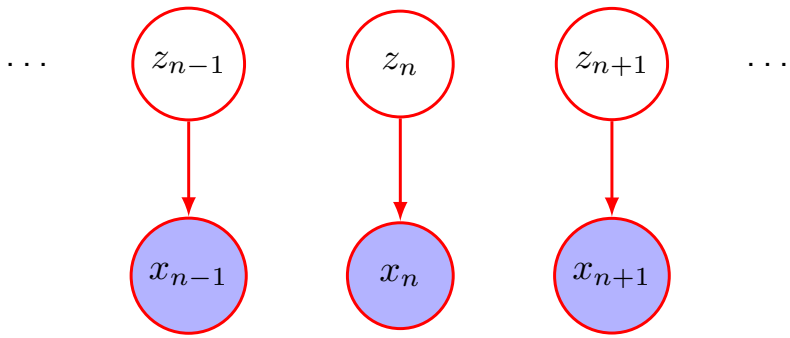


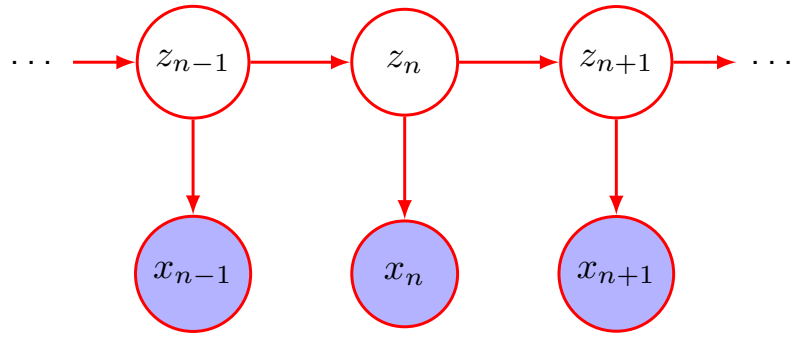
Figure from **Computer Vision: models, learning and inference** by Simon Prince.

Mixture Models vs Hidden Markov Models

Mixture Model



Hidden Markov Model



EM Algorithm

Baum-Welch Algorithm (EM)

1. Posterior of latent variable z_n depends on full sequence X

$$\gamma_n(k) = P(z_n = k \,|\, X, \theta^{(t)}) \neq P(z_n = k \,|\, x_n, \theta^{(t)})$$

2. We also need the posterior of two subsequent latent variables

$$\xi_n(i,j) = P(z_{n-1} = s_i, z_n = s_j | X, \theta)$$

Baum-Welch: Goal

Estimating parameters:

$$\theta = \{A, \pi, \phi\}$$

- transition prob. $A : a_{ij} = P(z_{n+1} = s_j | z_n = s_i)$
- prior prob. $\pi_i = P(z_1 = s_i)$
- emission prob. $\phi_i(x_n) = p(x_n | z_n = s_i)$
  For example, for
  - Discrete HMMs, $\phi_i(x_n) = \text{Cat}(x_n | \lambda_{i1}, \ldots \lambda_{id})$
  - Gaussian HMMs, $\phi_i(x_n) = \mathcal{N}(x_n | \mu_i, \Sigma_i)$
  - GMM-HMMs, $\phi_i(x_n) = \sum_{k=1}^{K} w_k \mathcal{N}(x_n | \mu_{ik}, \Sigma_{ik})$

Baum-Welch E-Step

Estimate posterior $P(Z|X, \theta^{\text{old}})$ or, at least the sufficient statistics given:

- current model parameters
- full sequence of observations X

1. Posterior of latent variable z_n

$$\gamma_n(i) = P(z_n = s_i|X, \theta)$$

2. Posterior of two subsequent latent variables

$$\xi_n(i, j) = P(z_{n-1} = s_i, z_n = s_j|X, \theta)$$

Note: Huang, Acero and Hon call this $\gamma_n(i, j)$

Note: posteriors conditioned to full sequence X not only x_n

Baum-Welch M-Step

Weighted Maximum Likelihood estimates:

Emission probabilities:

Same as mixture model given $\gamma_n(i) = P(z_n = s_j|X, \theta)$

Transition probabilities

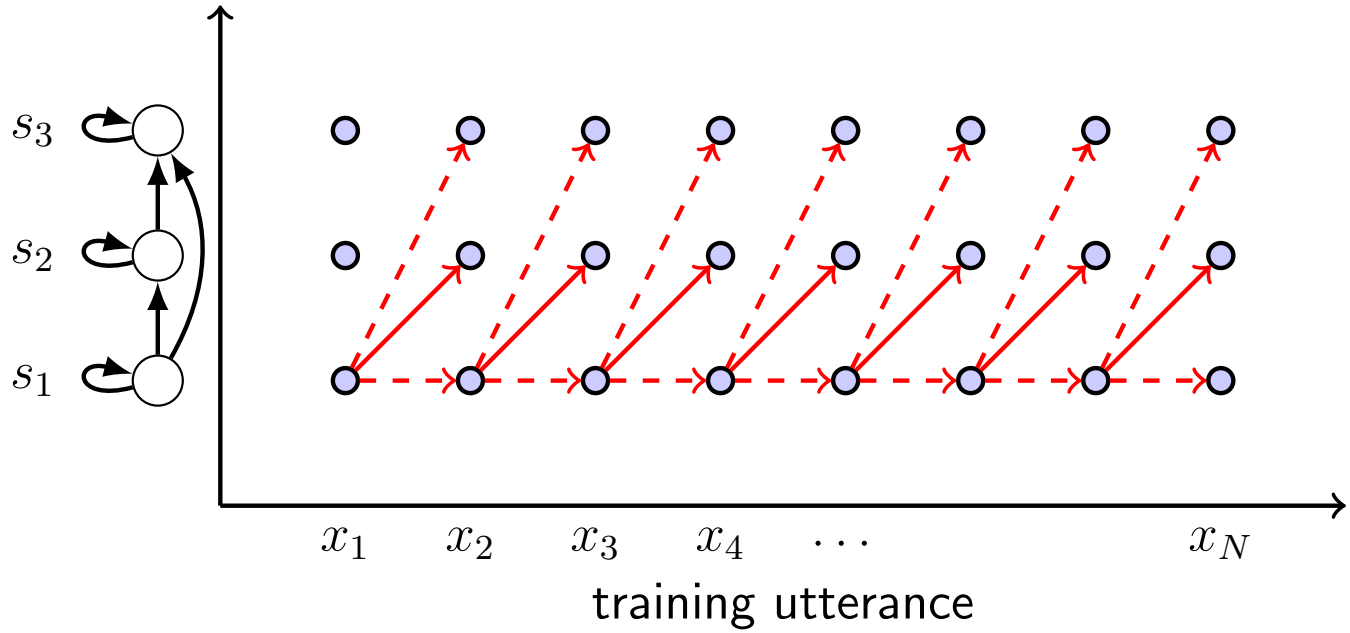
$$a_{ij}^{\text{new}} = \frac{E\left[s_i \to s_j|X, \theta^{\text{old}}\right]}{E\left[s_i \to s_{\text{any}}|X, \theta^{\text{old}}\right]} = \frac{\sum_{n=2}^{N} \xi_n(i,j)}{\sum_{n=2}^{N}\sum_{k=1}^{M} \xi_n(i,k)}$$

or, equivalently,

$$= \frac{E\left[s_i \to s_j|X, \theta^{\text{old}}\right]}{E\left[s_i|X, \theta^{\text{old}}\right]} = \frac{\sum_{n=2}^{N} \xi_n(i,j)}{\sum_{n=1}^{N-1} \gamma_n(i)}$$

Expectations are over the posteriors $P(Z|X, \theta^{\text{old}})$.

Example: Transition Probability



$$a_{12}^{\text{new}} = \frac{E\left[s_1 \to s_2 | X, \theta^{\text{old}}\right]}{E\left[s_1 \to s_{\text{any}} | X, \theta^{\text{old}}\right]} = \frac{\sum_{n=2}^{N} \xi_n(1,2)}{\sum_{n=2}^{N} \sum_{k=1}^{3} \xi_n(1,k)}$$

$$= \frac{E\left[s_1 \to s_2 | X, \theta^{\text{old}}\right]}{E\left[s_1 | X, \theta^{\text{old}}\right]} = \frac{\sum_{n=2}^{N} \xi_n(1,2)}{\sum_{n=1}^{N-1} \gamma_n(1)}$$

Example: Transition Probability

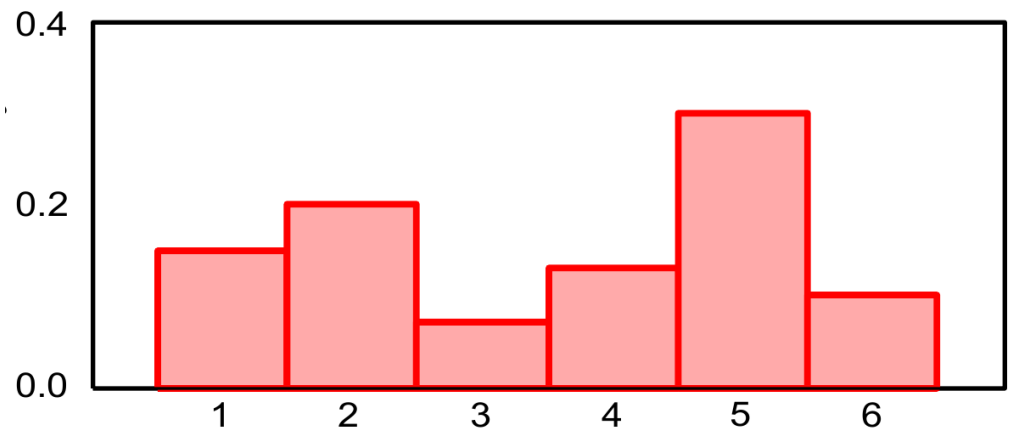
- $\sum_{n=2}^{N} \xi_n(i,j)$ is the expected number of transitions between state s_i and s_j (given X and θ^{old})
- $\sum_{n=1}^{N-1} \gamma_n(i)$ is the expected number of times we are in state s_i (given X and θ^{old})
- we never take a hard decision on when the transition happened

Emission probabilities

- Discrete HMMs (DHMMs)
  - vector quantisation
- Continuous HMMs
  - Single Gaussian $\phi_j(x_n) = N(x_n|\mu_j, \Sigma_j)$
  - Gaussian Mixture
- Semi-continuous HMMs (SCHMMs)

Discrete HMMs

- quantise feature vectors
- observation: sequence of discrete symbols
- $\phi_j(x_n)$ simple discrete probability distribution
- problem: quantisation error



Discrete HMMs: learn $\phi_j(x_n)$

Remember that

$$\gamma_n(j) = P(z_n = s_j | X, \theta)$$

are the posteriors of the latent variable
Update rule:

$$\phi_j(x_n = k) = \frac{E[x_n = k, z_n = s_j]}{E[z_n = s_j]} = \frac{\sum_{n:(x_n=k)} \gamma_n(j)}{\sum_{n=1}^{N} \gamma_n(j)}$$

HMMs with Gaussian Emission Probability

$$\phi_j(x_n) = N(x_n|\mu_j, \Sigma_j)$$

Update rules:

$$\mu_j = \frac{\sum_{n=1}^{N} \gamma_n(j) x_n}{\sum_{n=1}^{N} \gamma_n(j)}$$

$$\Sigma_j = \frac{\sum_{n=1}^{N} \gamma_n(j) \left(x_n - \mu_j\right) \left(x_n - \mu_j\right)^T}{\sum_{n=1}^{N} \gamma_n(j)}$$

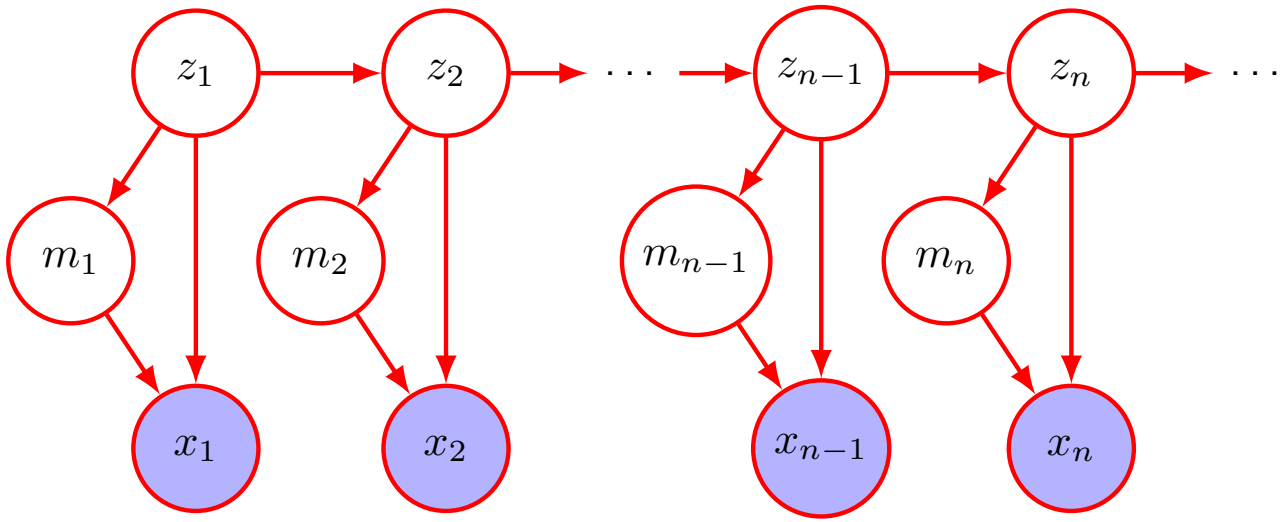
HMMs with Mixture Emission Probability

Often the Emission probability is modelled as a Mixture of Gaussians

$$\phi_j(x_n) = \sum_{k=1}^{K} w_{jk} N(x_n | \mu_{jk}, \Sigma_{jk})$$

$$\sum_{k=1}^{M} w_{jk} = 1$$

HMMs with Mixture Emission Probability



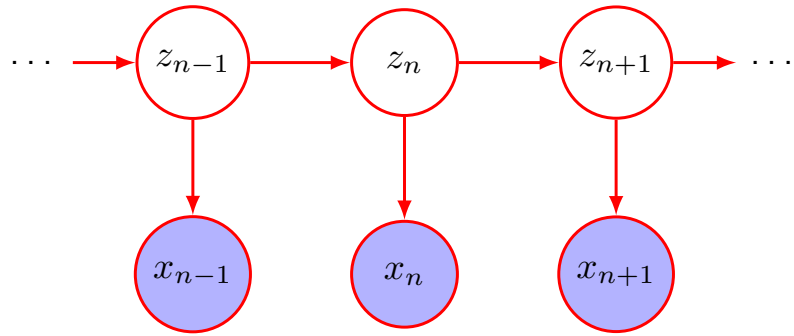
Emission:

$$
\begin{aligned}
p(x_n|z_n, m_n) &= \mathcal{N}(x_n; \mu_{z_n,m_n}, \Sigma_{z_n,m_n}) \\
p(m_n|z_n) &= W(m_n, z_n)
\end{aligned}
$$

Semi-Continuous HMMs

- All Gaussian distributions in a pool of pdfs
- each $\phi_j(x_n)$ is a discrete probability distribution over the pool of Gaussians
- similar to quantisation, but probabilistic
- used for sharing parameters

Calculate sufficient statistics



$$\begin{aligned} \gamma_n(i) &= P(z_n = s_i | X, \theta) \\ \xi_n(i,j) &= P(z_{n-1} = s_i, z_n = s_j | X, \theta) \end{aligned}$$

We can do this with the help of the forward and backward variables:

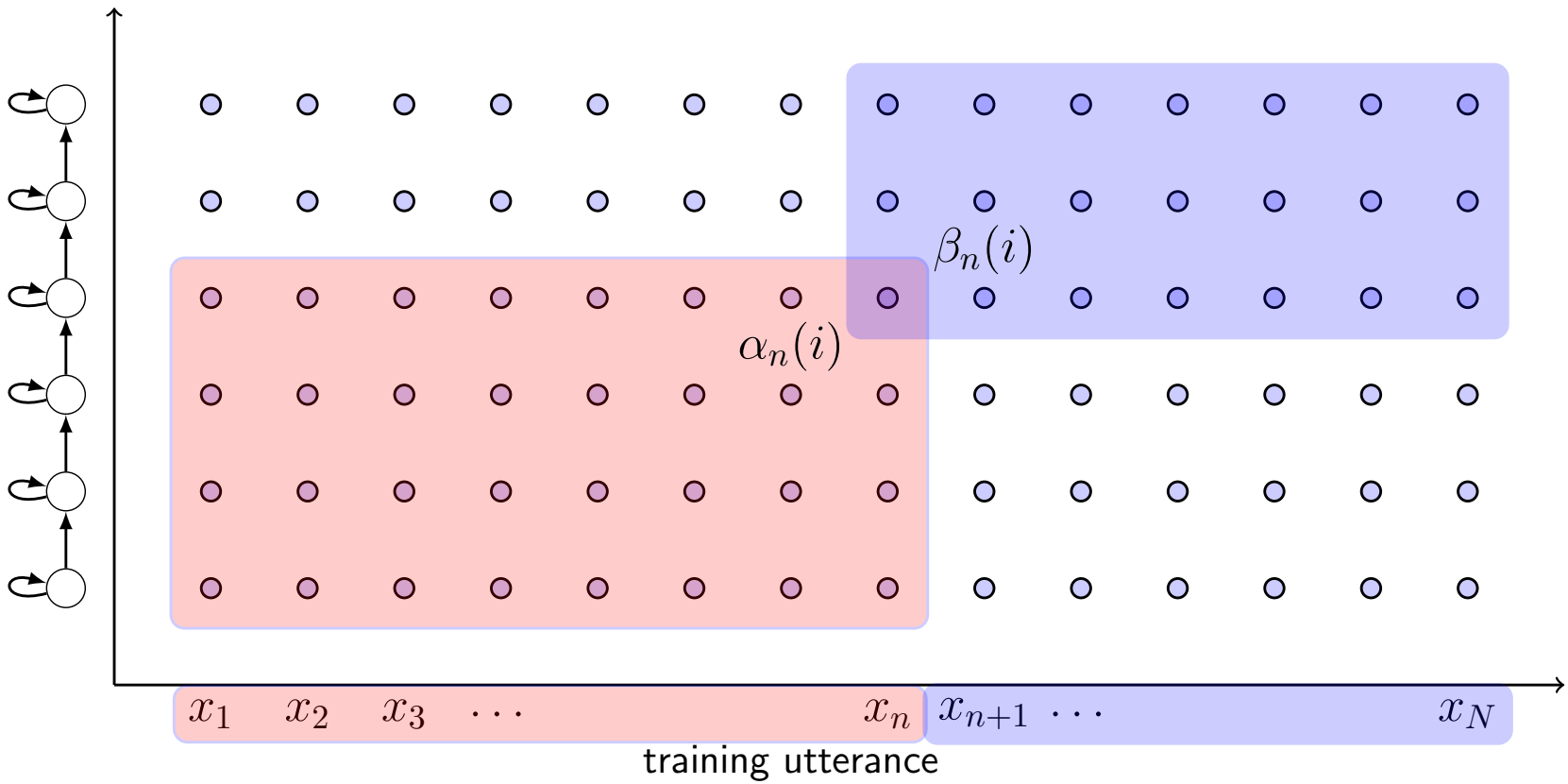
$$\begin{aligned} \alpha_n(i) &= P(x_1, \ldots, x_n, z_n = s_i | \theta) \\ \beta_n(i) &= P(x_{n+1}, \ldots, x_N | z_n = s_i, \theta) \end{aligned}$$

Calculate γ (forward-backward)

$$
\begin{aligned}
\gamma_n(i) &= P(z_n = s_i|X,\theta) \\
&= \frac{p(X, z_n = s_j|\theta)}{p(X|\theta)} \\
&= \frac{p(x_1,\ldots,x_n,x_{n+1},\ldots,x_N,z_n = s_j|\theta)}{p(X|\theta)} \\
&= \frac{p(x_1,\ldots,x_n,x_{n+1},\ldots,x_N|z_n = s_j,\theta)P(z_n = s_j|\theta)}{p(X|\theta)} \\
&= \frac{p(x_1,\ldots,x_n|z_n=s_j,\theta)p(x_{n+1},\ldots,x_N|z_n=s_j,\theta)P(z_n=s_j|\theta)}{p(X|\theta)} \\
&= \frac{p(x_1,\ldots,x_n,z_n = s_j|\theta)p(x_{n+1},\ldots,x_N|z_n = s_j,\theta)}{p(X|\theta)} \\
&= \frac{\alpha_n(i)\beta_n(i)}{\sum_i \alpha_N(i)}
\end{aligned}
$$

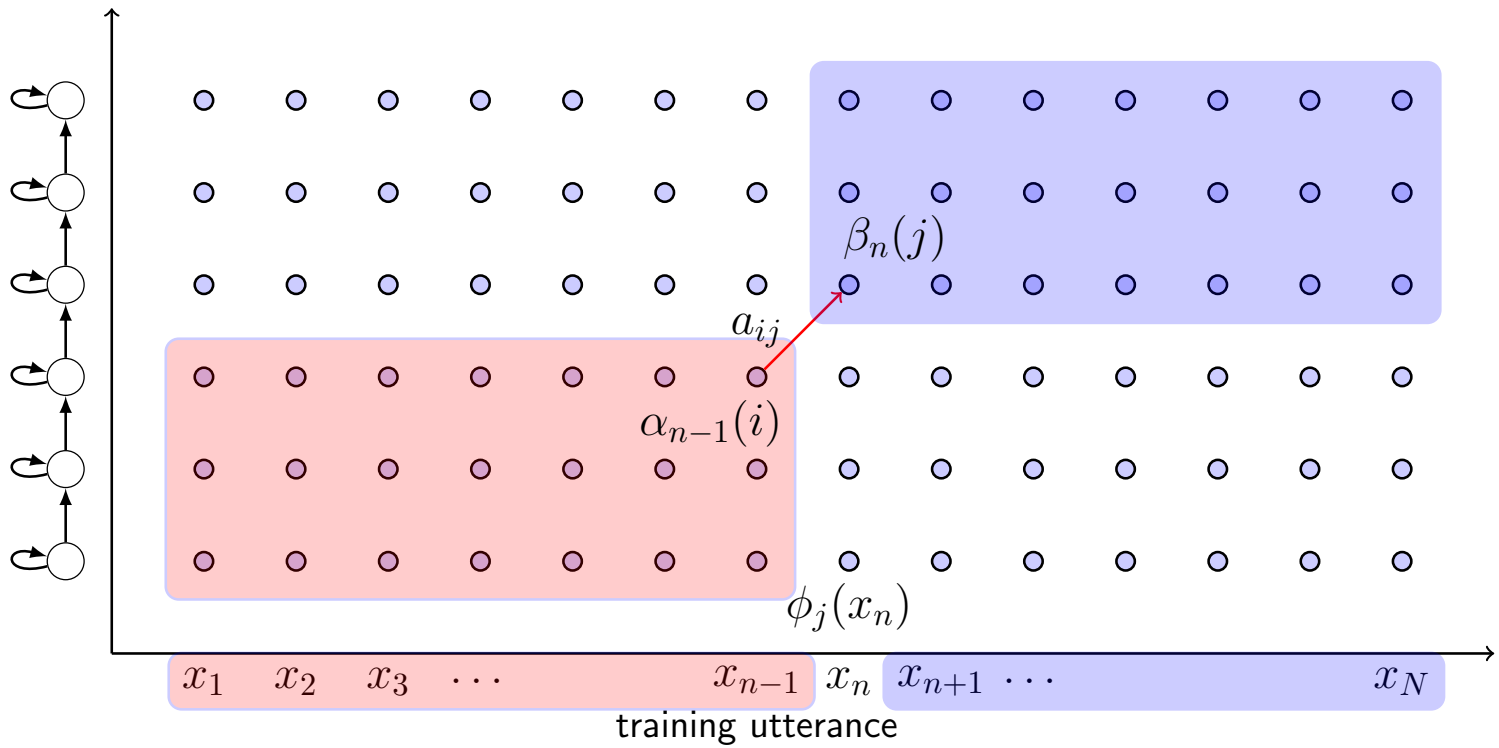
Calculate γ: Illustration

$$\gamma_n(i) = P(z_n = s_i|X, \theta) = \frac{\alpha_n(i)\beta_n(i)}{\sum_i \alpha_N(i)}$$



Calculate ξ (forward-backward)

$$
\begin{aligned}
\xi_n(i,j) &= P(z_n = s_j, z_{n-1} = s_i | X, \theta) \\
&= \frac{P(z_n = s_j, z_{n-1} = s_i, X | \theta)}{P(X|\theta)} \\
\ldots &= \frac{\alpha_{n-1}(i)\ a_{ij}\ \phi_j(x_n)\ \beta_n(j)}{\sum_{k=1}^{M} \alpha_N(k)}
\end{aligned}
$$



Baum-Welch: Properties

instance of Expectation Maximisation:

- iterative procedure
- guaranteed to convert to local maximum of the likelihood $P(X|\theta^{\text{new}})$
- sensitive to initialisation
- update formulae for emission probability model $\phi_j(x_n)$ same as for mixture models (with new version of posteriors)

Numerical Problems

Product of many probabilities.
Solution: work in log domain

$$
\begin{aligned}
\alpha'_1(j) &= \pi'_j + \phi'_j(x_1) \\
\alpha'_n(j) &= \log \sum_{i=1}^{M} e^{\left(\alpha'_{n-1}(i) + a'_{ij}\right)} + \phi'_j(x_n) \\
\beta'_N(i) &= 0 \\
\beta'_n(i) &= \log \sum_{j=1}^{M} e^{\left(a'_{ij} + \phi'_j(x_{n+1}) + \beta'_{n+1}(j)\right)}
\end{aligned}
$$

In Lab2 `logsumexp()`.

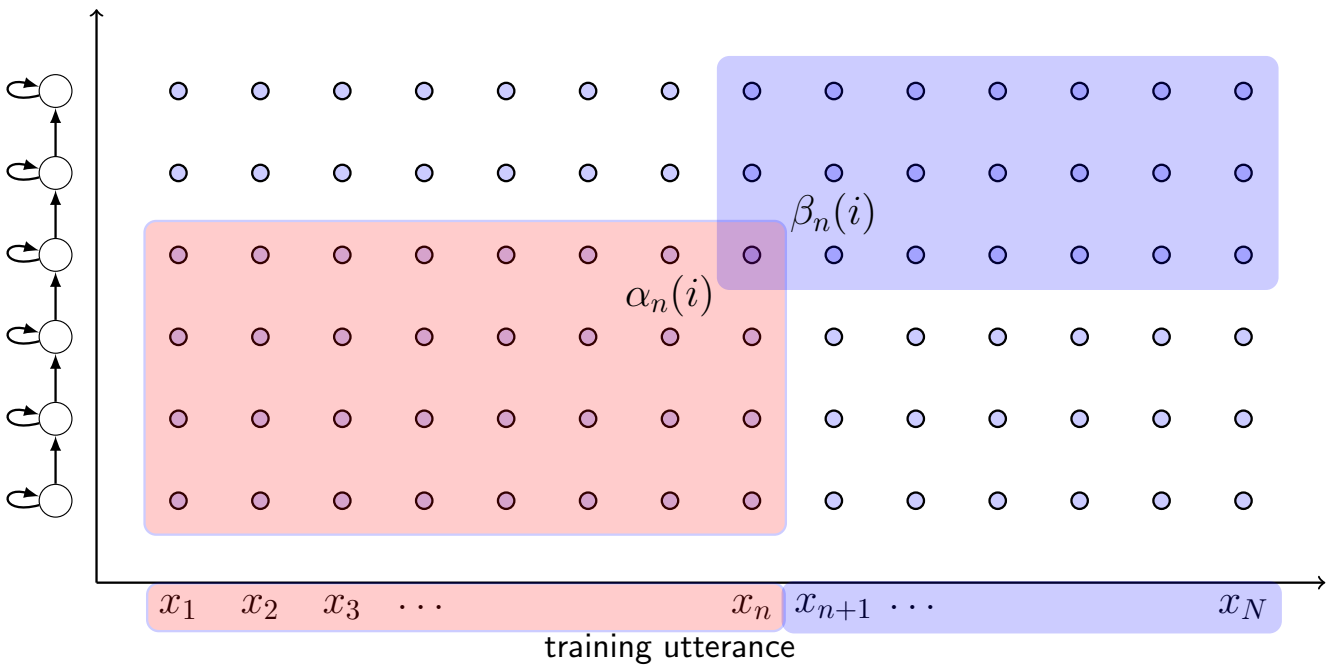
Train on several utterances

Set of utterances $X^1, X^2, \ldots, X^U$

- cannot concatenate them: need to calculate α, β, γ and ξ each time

Each utterance corresponds to several models

- reuse model states (sentence $\rightarrow$ words $\rightarrow$ phonemes)



Concatenating HMMs

Utterance to words:

sil one zero one three sil

Words to phones

sil w ah n sp z iy r ow sp w ah n sp th r iy sp sil

Phones to states

sil0 sil1 sil2 w0 w1 w2 ah0 ah1 ah2 n0 n1 n2 sp0 z0 z1 z2 iy0
iy1 iy2 r0 r1 r2 ow0 ow1 ow2 sp0 w0 w1 w2 ah0 ah1 ah2 n0
n1 n2 sp0 th0 th1 th2 r0 r1 r2 iy0 iy1 iy2 sp0 sil0 sil1 sil2