Half Rod, Albedo Problem, Isotropic Scattering

This is code to accompany the book:

A Hitchhiker's Guide to Multiple Scattering

www.eugenedeon.com

Path Setup

Put a file at ~/.hitchhikerpath with the path to your hitchhiker repo so that these worksheets can find the MC data from the C++ simulations for verification

```
SetDirectory[Import["~/.hitchhikerpath"]]
```

Exponential Random Flight

Notation

α - single-scattering albedo
Σt - extinction coefficient
x - position coordinate in rod (source at x = 0)

Analytic solutions

Half rod reflectance/albedo (R)

```
Clear[α, g];
```

```
halfrodalbedoisoscatter`R[α_] := 2 (-α/2 - Sqrt[1 - α] + 1)/α
```

```
Series[halfrodalbedoisoscatter`R[α], {α, 0, 5}]
```

$$\frac{\alpha}{4}+\frac{\alpha^2}{8}+\frac{5\,\alpha^3}{64}+\frac{7\,\alpha^4}{128}+\frac{21\,\alpha^5}{512}+O[\alpha]^6$$

```
halfrodalbedoisoscatter`R[α_, n_] := α^n 2 (-1)^n Binomial[1 / 2, n + 1]
```

Internal distribution, 'radiance'

```
halfrodalbedoisoscatter`LR[x_, α_, Σt_] := e^(-Sqrt[1-α] Σt x)
```

```
halfrodalbedoisoscatter`LL[x_, α_, Σt_] := α Exp[-Σt √(1-α) x] / (-α + 2 √(1-α) + 2)
```

Fluence

```
halfrodalbedoisoscatter`ϕ[x_, α_, Σt_] :=
 halfrodalbedoisoscatter`LR[x, α, Σt] + halfrodalbedoisoscatter`LL[x, α, Σt]
```

n-th collided fluence

```
halfrodalbedoisoscatter`ϕ[x_, α_, Σt_, n_] :=
 α^n (SeriesCoefficient[halfrodalbedoisoscatter`ϕ[x, A, Σt], {A, 0, n}] /. A → α)
```

Moments

```
halfrodalbedoisoscatter`ϕm[α_, Σt_, k_] :=
 2 (1-α)^(-1-k/2) (-1 + √(1-α) + α) Σt^(-1-k) Gamma[1+k] / α
```

Only accurate for n even

```
halfrodalbedoisoscatter`ϕm[α_, Σt_, k_, n_] :=
 α^n (2 (-1)^n Σt^(-1-k) Gamma[1+k] (-(2+n) Gamma[(1-k)/2] -
      ((k+n) Binomial[-k/2, 1+n] + 2 (2+n) Binomial[-k/2, 2+n])
       Gamma[-1/2 - k/2 - n] Gamma[2+n])) / (Gamma[-1/2 - k/2 - n] Gamma[3+n])
```

load MC data

```
halfrodalbedoisoscatter`ppoints[xs_, dx_, maxx_, Σt_] :=
 Table[{dx (i-1) + 0.5 dx, (1/Σt) xs[[i]]}, {i, 1, Length[xs]}][[1 ;; -2]]
```

```
halfrodalbedoisoscatter`fs = FileNames[
    "code/rod/halfrod/albedoProblem/data/halfrod_albedoproblem_isotropicscatter
      _exp*"];
```

```
halfrodalbedoisoscatter`index[x_] := Module[{data, α, Σt},
    data = Import[x, "Table"];
    Σt = data[[1, 11]];
    α = data[[2, 3]];
    {α, Σt, data}];
halfrodalbedoisoscatter`simulations =
  halfrodalbedoisoscatter`index /@ halfrodalbedoisoscatter`fs;
```

```
halfrodalbedoisoscatter`alphas =
 Union[#[[1]] & /@ halfrodalbedoisoscatter`simulations]
```

```
{0.1, 0.3, 0.5, 0.7, 0.9, 0.95, 0.98, 0.99, 0.999}
```

```
halfrodalbedoisoscatter`muts =
 Union[#[[2]] & /@ halfrodalbedoisoscatter`simulations]
```

```
{1, 3}
```

```
halfrodalbedoisoscatter`numcollorders =
 halfrodalbedoisoscatter`simulations[[1]][[3]][[2, 11]]
```

```
20
```

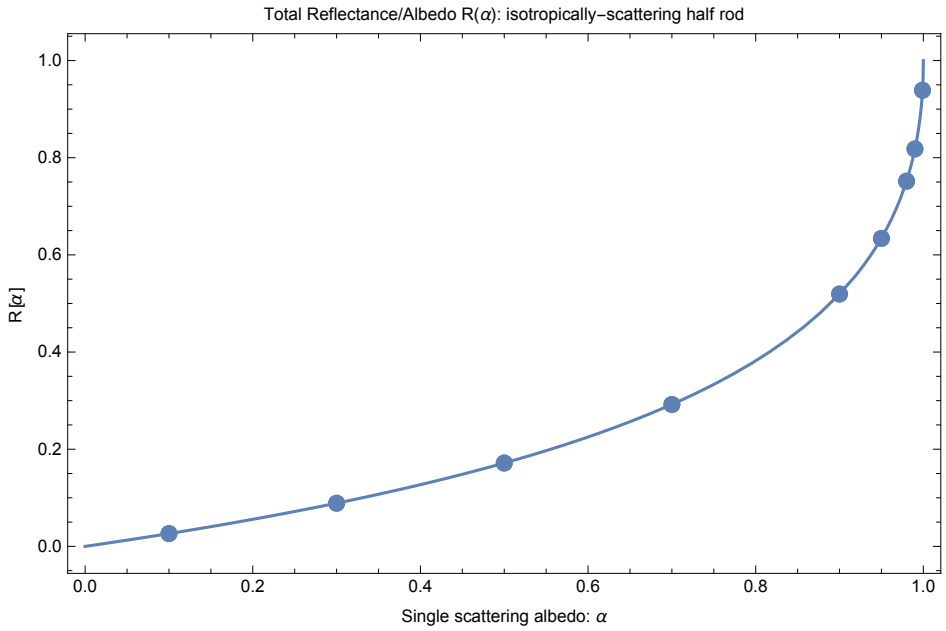
Halfrod Albedo

```
MCalbedo[f_] := Module[{data, α},
  data = Import[f, "Table"];
  α = data[[2, 3]];
  {α, data[[3, 3]]}
 ]
```

```
MCalbedos = Table[MCalbedo[f], {f, halfrodalbedoisoscatter`fs}]
```

```
{{0.1, 0.0262874}, {0.1, 0.0262874}, {0.3, 0.0888128},
 {0.3, 0.0888128}, {0.5, 0.17156}, {0.5, 0.17156}, {0.7, 0.291991},
 {0.7, 0.291991}, {0.95, 0.633904}, {0.95, 0.633904}, {0.98, 0.751703},
 {0.98, 0.751703}, {0.999, 0.938793}, {0.999, 0.938793},
 {0.99, 0.818082}, {0.99, 0.818082}, {0.9, 0.519448}, {0.9, 0.519448}}
```

```
Clear[α]; vizrodalbedoiso = Show[
  Plot[halfrodalbedoisoscatter`R[c], {c, 0, 1}],
  ListPlot[MCalbedos]
  , Frame → True, ImageSize → 500,
  FrameLabel → {{"R[α]",}, {"Single scattering albedo: α",
     "Total Reflectance/Albedo R(α): isotropically-scattering half rod"}}
 ]
```



n-th collided albedo

```
Manipulate[
 If[Length[halfrodalbedoisoscatter`simulations] > 0,
  Module[{data, Rs, ns, analytic, j, numcollorders},
   data = SelectFirst[
      halfrodalbedoisoscatter`simulations, #[[1]] == α && #[[2]] == Σt &][[3]];
   numcollorders = data[[2, 11]];
   Rs = N[{data[[5]]}];
   ns = Table[n, {n, 0, numcollorders - 1}];
   analytic = Table[halfrodalbedoisoscatter`R[α, n], {n, ns}];
   j = Join[{ns}, {analytic}, Rs];
   TableForm[
    Join[{{"n", "analytic", "MC"}}, Transpose[j]]
   ]
  ]
  ,
  Text[
   "Uh oh!  Couldn't find MC data.  Try to evaluate this entire notebook and
     ensure the data path is setup correctly."]
 ]
 , {α, halfrodalbedoisoscatter`alphas}, {Σt, halfrodalbedoisoscatter`muts}]
```

α 0.98

Σt 1 3

n	analytic	MC
0	1.	0.
1	0.245	0.244497
2	0.12005	0.120324
3	0.0735306	0.074076
4	0.050442	0.050198
5	0.0370749	0.037007
6	0.0285477	0.028276
7	0.0227311	0.022734
8	0.0185637	0.018692
9	0.0154636	0.015586
10	0.0130878	0.013021
11	0.0112228	0.011227
12	0.0097293	0.00972
13	0.00851313	0.008508
14	0.00750858	0.007559
15	0.00666856	0.006809
16	0.00595856	0.005858
17	0.00535277	0.005461
18	0.00483158	0.004784
19	0.00437983	0.004334

Compare Deterministic and MC

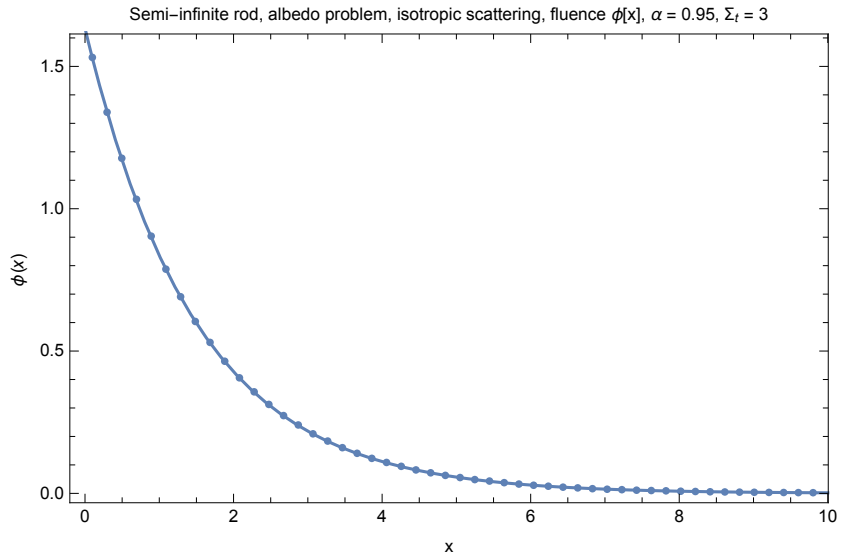
Internal distribution

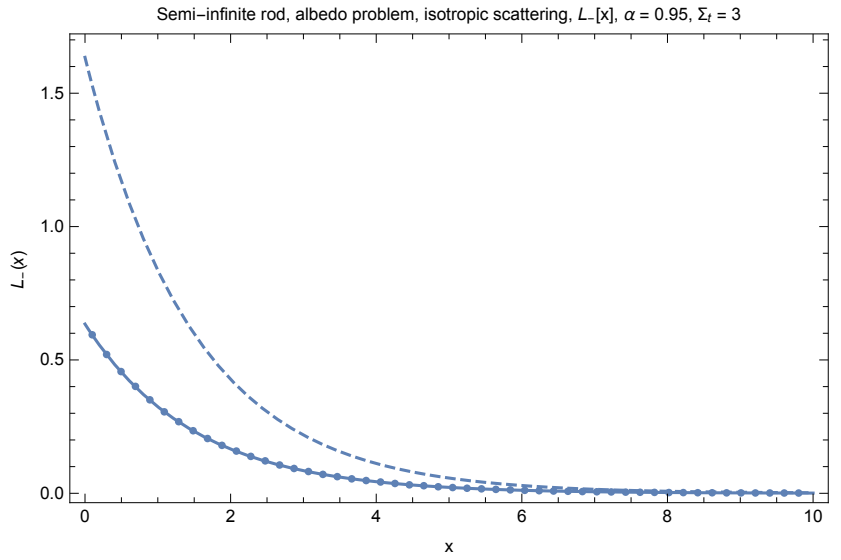
```
Clear[alpha, Σt];
Manipulate[
 If[Length[halfrodalbedoisoscatter`simulations] > 0,
```

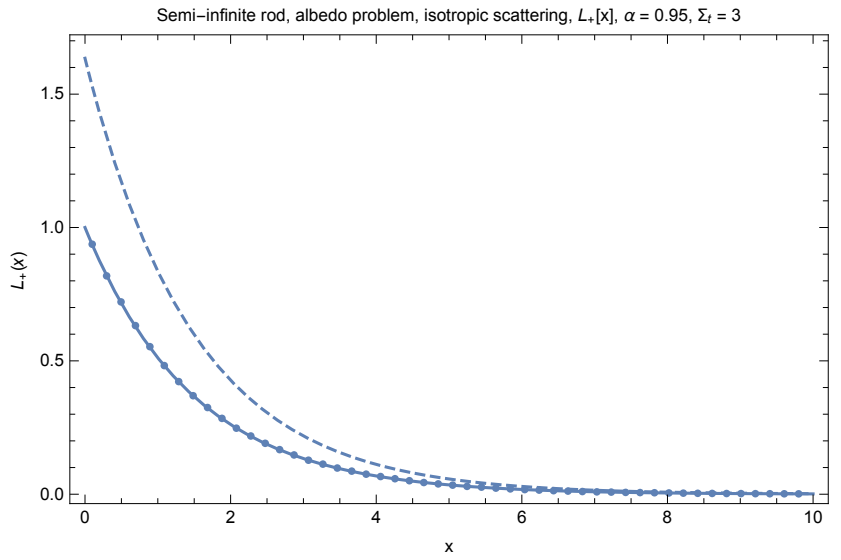
```
Module[{data, maxx, dx, numcollorders, nummoments, pointsCL, plotpointsCL,
  pointsCR, plotpointsCR, plotpointsϕ, plotϕ, plotLL, plotLR},
 data = SelectFirst[halfrodalbedoisoscatter`simulations,
    #[[1]] == α && #[[2]] == Σt &][[3]];
 maxx = data[[2, 5]];
 dx = data[[2, 7]];
 numcollorders = data[[2, 11]];
 nummoments = data[[2, 13]];

 pointsCL = data[[7]];
 (* divide by Σt to convert collision density into L *)
 plotpointsCL = halfrodalbedoisoscatter`ppoints[pointsCL, dx, maxx, Σt];
 pointsCR = data[[9]];
 plotpointsCR = halfrodalbedoisoscatter`ppoints[pointsCR, dx, maxx, Σt];
 (* divide by Σt to convert collision density into fluence *)
 plotpointsϕ =
  halfrodalbedoisoscatter`ppoints[pointsCL + pointsCR, dx, maxx, Σt];

 plotϕ = Show[
   ListPlot[plotpointsϕ, PlotRange → All, PlotStyle → PointSize[.01]],
   Plot[halfrodalbedoisoscatter`ϕ[x, α, Σt], {x, 0, maxx}, PlotRange → All]
   , Frame → True,
   FrameLabel -> {{ϕ[x],},
     {"x", "Semi-infinite rod, albedo problem, isotropic scattering, fluence
         ϕ[x], α = " <> ToString[α] <> ", Σt = " <> ToString[Σt]}}
  ];

 plotLL = Show[
   ListPlot[plotpointsCL, PlotRange → All, PlotStyle → PointSize[.01]],
   Plot[halfrodalbedoisoscatter`LL[x, α, Σt], {x, 0, maxx}, PlotRange → All],
   Plot[halfrodalbedoisoscatter`ϕ[x, α, Σt],
    {x, 0, maxx}, PlotRange → All, PlotStyle → Dashed]
   , Frame → True,
   FrameLabel ->
    {{L-[x],}, {"x", "Semi-infinite rod, albedo problem, isotropic
         scattering, L-[x], α = " <>
       ToString[α] <> ", Σt = " <> ToString[Σt]}}, PlotRange → All
  ];

 plotLR = Show[
   ListPlot[plotpointsCR, PlotRange → All, PlotStyle → PointSize[.01]],
   Plot[halfrodalbedoisoscatter`LR[x, α, Σt], {x, 0, maxx}, PlotRange → All],
   Plot[halfrodalbedoisoscatter`ϕ[x, α, Σt],
    {x, 0, maxx}, PlotRange → All, PlotStyle → Dashed]
   , Frame → True,
   FrameLabel ->
    {{L+[x],}, {"x", "Semi-infinite rod, albedo problem, isotropic
         scattering, L+[x], α = " <>
       ToString[α] <> ", Σt = " <> ToString[Σt]}}, PlotRange → All
  ];
 Show[GraphicsGrid[{{plotϕ}, {plotLL}, {plotLR}}], ImageSize → 500]
]
,
Text[
 "Uh oh!  Couldn't find MC data.  Try to evaluate this entire notebook and
   ensure the data path is setup correctly."]
]
, {α, halfrodalbedoisoscatter`alphas}, {Σt, halfrodalbedoisoscatter`muts}]
```

α 0.95

Σt 1 3


Semi−infinite rod, albedo problem, isotropic scattering, fluence ϕ[x], α = 0.95, Σt = 3
ϕ(x)
x
0.0
0.5
1.0
1.5
0
2
4
6
8
10


Semi−infinite rod, albedo problem, isotropic scattering, L−[x], α = 0.95, Σt = 3
L−(x)
x
0.0
0.5
1.0
1.5
0
2
4
6
8
10


Semi−infinite rod, albedo problem, isotropic scattering, L+[x], α = 0.95, Σt = 3
L+(x)
x
0.0
0.5
1.0
1.5
0
2
4
6
8
10

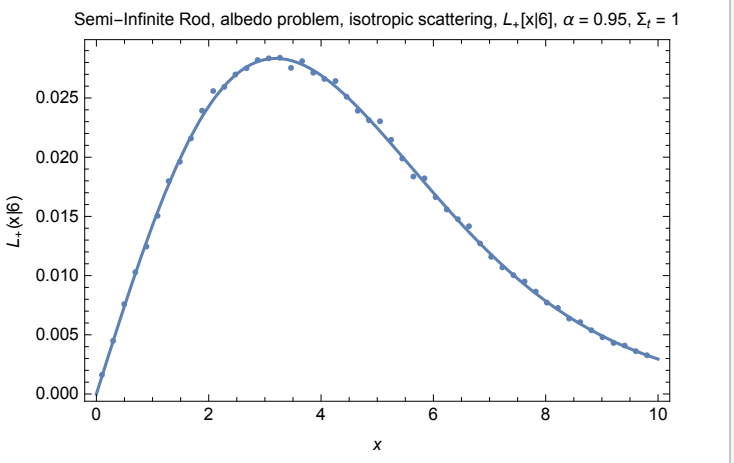
N-th order Radiance/Angular flux

```
Manipulate[
 If[Length[halfrodalbedoisoscatter`simulations] > 0,
  Module[{data, nthL, nthR, maxx, dx, numcollorders, LnR},
   data = SelectFirst[
      halfrodalbedoisoscatter`simulations, #[[1]] == α && #[[2]] == Σt &][[3]];
   maxx = data[[2, 5]];
   dx = data[[2, 7]];
   numcollorders = data[[2, 11]];
   nthL = data[[13 + numcollorders + 1 ;; 13 + 2 numcollorders]];
   nthR = data[[13 + 2 numcollorders + 2 ;; -1]];

   Clear[c, x];
   LnR = SeriesCoefficient[halfrodalbedoisoscatter`LR[x, c, Σt], {c, 0, n}] α^n;
   Show[
    ListPlot[halfrodalbedoisoscatter`ppoints[nthR[[n + 1]], dx, maxx, Σt],
     PlotRange → All, PlotStyle → PointSize[.01]],
    Plot[LnR, {x, 0, maxx}, PlotRange → All]
    , Frame → True,
    FrameLabel -> {{L+["x|" <> ToString[n]],},
      {x, "Semi-Infinite Rod, albedo problem, isotropic scattering, L+[x|" <>
        ToString[n] <> "], α = " <> ToString[α] <>
        ", Σt = " <> ToString[Σt]}}, PlotRange → All
   ]
  ],
  Text[
   "Uh oh!  Couldn't find MC data.  Try to evaluate this entire notebook and
     ensure the data path is setup correctly."]
 ]
 , {α, halfrodalbedoisoscatter`alphas}, {Σt, halfrodalbedoisoscatter`muts},
 {n, Range[If[NumberQ[halfrodalbedoisoscatter`numcollorders],
    halfrodalbedoisoscatter`numcollorders, 1]]}]
```

α 0.95

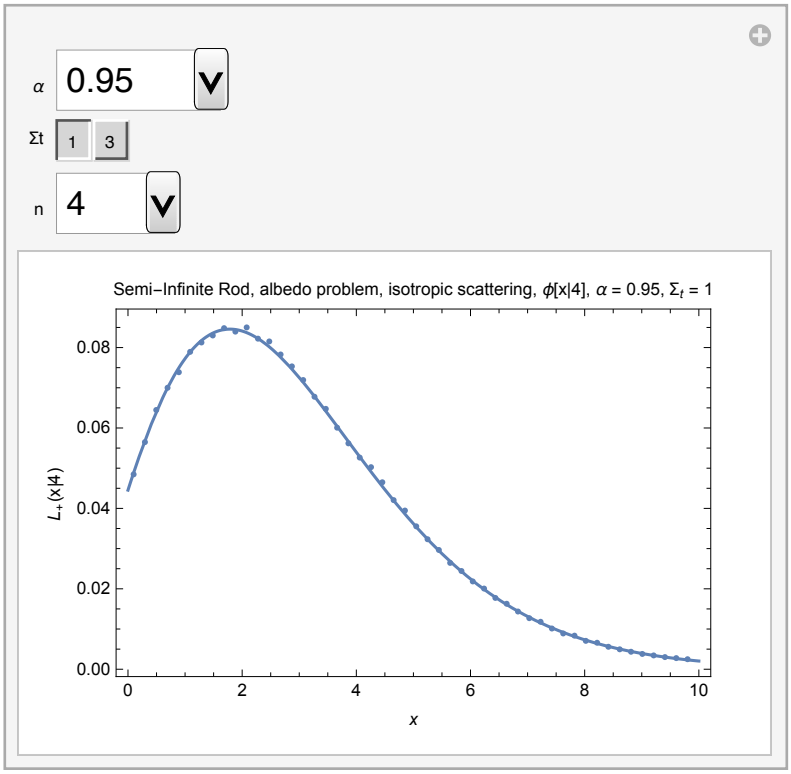
Σt 1 3

n 6


Semi−Infinite Rod, albedo problem, isotropic scattering, $L_+[x|6]$, $\alpha = 0.95$, $\Sigma_t = 1$
$L_+(x|6)$
0.025
0.020
0.015
0.010
0.005
0.000
0
2
4
6
8
10
x

N-th order Fluence / scalar flux

```
Manipulate[
 If[Length[halfrodalbedoisoscatter`simulations] > 0,
  Module[{data, maxx, dx, numcollorders, nthL, nthR, ϕn},
   data = SelectFirst[
      halfrodalbedoisoscatter`simulations, #[[1]] == α && #[[2]] == Σt &][[3]];
   maxx = data[[2, 5]];
   dx = data[[2, 7]];
   numcollorders = data[[2, 11]];
   nthL = data[[13 + numcollorders + 1 ;; 13 + 2 numcollorders]];
   nthR = data[[13 + 2 numcollorders + 2 ;; -1]];

   Clear[c];
   ϕn = SeriesCoefficient[halfrodalbedoisoscatter`ϕ[x, c, Σt], {c, 0, n}] α^n;
   Show[
    ListPlot[halfrodalbedoisoscatter`ppoints[nthR[[n + 1]] + nthL[[n + 1]],
      dx, maxx, Σt], PlotRange → All, PlotStyle → PointSize[.01]],
    Plot[ϕn, {x, 0, maxx}, PlotRange → All]
    , Frame → True,
    FrameLabel -> {{L+["x|" <> ToString[n]],},
      {x, "Semi-Infinite Rod, albedo problem, isotropic scattering, ϕ[x|" <>
        ToString[n] <> "], α = " <> ToString[α] <>
        ", Σt = " <> ToString[Σt]}}, PlotRange → All
   ]
  ],
  Text[
   "Uh oh!  Couldn't find MC data.  Try to evaluate this entire notebook and
     ensure the data path is setup correctly."]
 ]
 , {α, halfrodalbedoisoscatter`alphas}, {Σt, halfrodalbedoisoscatter`muts},
 {n, Range[If[NumberQ[halfrodalbedoisoscatter`numcollorders],
    halfrodalbedoisoscatter`numcollorders, 1]]}]
```



Compare moments of ϕ

Divide these results, which are collision density moments, by Σt to produce radiance/fluence moments:

```
Manipulate[
 If[Length[halfrodalbedoisoscatter`simulations] > 0,
  Module[{data, nummoments, ϕmoments, ks, analytic, j},
   data = SelectFirst[
      halfrodalbedoisoscatter`simulations, #[[1]] == α && #[[2]] == Σt &][[3]];
   nummoments = data[[2, 13]];
   ϕmoments = N[{data[[11]]}/Σt];
   ks = {Table[k, {k, 0, nummoments - 1}]};
   analytic = Table[halfrodalbedoisoscatter`ϕm[α, Σt, k], {k, ks}];
   j = Join[ks, analytic, ϕmoments];
   TableForm[
    Join[{{"k", "analytic", "MC"}}, Transpose[j]]
   ]
  ],
  Text[
   "Uh oh!  Couldn't find MC data.  Try to evaluate this entire notebook and
     ensure the data path is setup correctly."]
 ]
 , {α, halfrodalbedoisoscatter`alphas}, {Σt, halfrodalbedoisoscatter`muts}]
```

α 0.9

Σt 1 3

k	analytic	MC
0	1.60169	1.60147
1	1.68833	1.68786
2	3.5593	3.55933
3	11.2555	11.2588
4	47.4574	47.4567
5	250.122	249.805
6	1581.91	1576.1
7	11 672.4	11 583.1
8	98 430.2	97 005.
9	933 790.	908 873.

n-th collided moments of ϕ

```
Manipulate[
 If[Length[halfrodalbedoisoscatter`simulations] > 0,
  Module[{data, ϕmoments, ks, analytic, j, nummoments},
   data = SelectFirst[
      halfrodalbedoisoscatter`simulations, #[[1]] == α && #[[2]] == Σt &][[3]];
   nummoments = data[[2, 13]];
   ϕmoments = N[{data[[13 + n]]}/Σt];
   ks = {Table[k, {k, 0, nummoments - 1}]};
   analytic =
    Table[Quiet[N[halfrodalbedoisoscatter`ϕm[α, Σt, k, n]]], {k, ks}];
   j = Join[ks, analytic, ϕmoments];
   TableForm[
    Join[{{"k", "analytic", "MC"}}, Transpose[j]]
   ]
  ],
  Text[
   "Uh oh!  Couldn't find MC data.  Try to evaluate this entire notebook and
     ensure the data path is setup correctly."]
 ]
 , {α, halfrodalbedoisoscatter`alphas}, {Σt, halfrodalbedoisoscatter`muts},
 {n, Range[If[NumberQ[halfrodalbedoisoscatter`numcollorders],
    halfrodalbedoisoscatter`numcollorders, 1]]}]
```

α 0.95

Σt 1 3

n 3

k	analytic	MC
0	0.156292	0.156609
1	Indeterminate	0.138916
2	0.185566	0.186225
3	Indeterminate	0.323477
4	0.681401	0.682787
5	Indeterminate	1.68771
6	4.77643	4.7642
7	Indeterminate	15.0691
8	53.6496	52.5697
9	Indeterminate	199.489