The H-function

This is code to accompany the book:

A Hitchhiker's Guide to Multiple Scattering

www.eugenedeon.com

Analytic Solution

H-function (Stibbs-Weir)

```
Clear[H];
```

$$\texttt{H[}\alpha\texttt{\_, u\_] := Exp}\left[\frac{-u}{Pi}\ \texttt{NIntegrate}\left[\frac{\texttt{Log[1} - \alpha\ \texttt{t Cot[t]]}}{\texttt{Cos[t]}^2 + u^2\ \texttt{Sin[t]}^2}, \{t, 0, \frac{Pi}{2}\}\right]\right]$$

Special cases for no absorption ($\alpha = 1$):

[Wiener and Hopf 1931]

$$\texttt{HWienerHopf[u\_] :=}\ (1+u)\ \texttt{Exp}\left[\frac{u}{Pi}\ \texttt{NIntegrate}\left[\frac{\texttt{Log}\left[\texttt{Sin[t]}^2 / (1 - \texttt{t Cot[t]})\right]}{\texttt{Cos[t]}^2 + u^2\ \texttt{Sin[t]}^2}, \{t, 0, \frac{Pi}{2}\}\right]\right]$$

[Placzek 1947]

$$\texttt{HPlaczek[u\_] :=}\ \frac{1}{(1+u)^{1/2}}\ \texttt{Exp}\left[\frac{1}{Pi}\ \texttt{NIntegrate}\left[\frac{\texttt{t ArcTan[u Tan[t]]}}{\texttt{1 - t Cot[t]}}, \{t, 0, \frac{Pi}{2}\}\right]\right]$$

In[157]:=
```
Quiet[Table[{halfspaceAlbedoProblemIsotropic`H[1, u],
    HWienerHopf[u], HPlaczek[u]}, {u, 0, 1, 0.1}]] // TableForm
```

Out[157]//TableForm=

1.	1.	1.
1.24735	1.24735	1.24735
1.45035	1.45035	1.45035
1.64252	1.64252	1.64252
1.82928	1.82928	1.82928
2.01278	2.01278	2.01278
2.19413	2.19413	2.19413
2.37397	2.37397	2.37397
2.5527	2.5527	2.5527
2.73059	2.73059	2.73059
2.90781	2.90781	2.90781

Approximate H-function

Hapke approximation 1

$$\texttt{H1[w\_, u\_] :=}\ \frac{1 + 2u}{1 + 2u\sqrt{1 - w}}$$

Hapke approximation 2

```
Clear[H2];
H2[w_, u_] := (1 - (1 - y) u (n + (1 - n/2 - n u) Log[(1 + u)/u]))^-1 /. n → (1 - y)/(1 + y) /. y → (1 - w)^(1/2)

V[s_, c_] := 1 - c/(2 s) Log[(1 + s)/(1 - s)]

vexact[c_?NumericQ] :=
 FindRoot[V[s, c] == 0, {s, 10^-10, 1 - 10^-10}, Method → "Brent"][[1]][[2]]
```

Grosjean 1958b approximation

```
HGrosjean[w_, u_] := 1 + (w u)/2 Log[(1 + u)/u] + (2 w^2 u)/((1 + K u) ((13-5 w)/8 + 2/3 (2 - w) K)) -
   (w^2 u)/((13-5 w)/8 + 2/3 (2 - w) K) (17/32 + 15/16 u - u (1 + 15/16 u) Log[(1 + u)/u]) /. K → ((3 (1 - w))/(2 - w))^(1/2)
```

NSE 1976 vol 27, 607 - 608 (two more approximations are given)

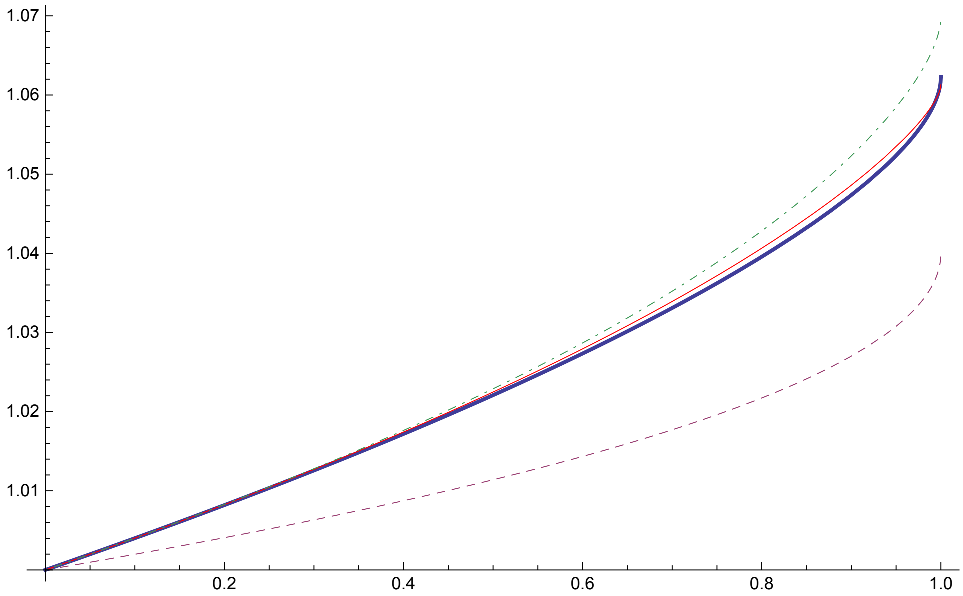
```
H32[c_, u_] := (1 - c/2 ((1 - A u + B u^2) u Log[(1 + u)/u] + A u + (u/2 - u^2) B))^-1 //.
  {A → α - 2/3 B, B → (K Log[1 + K] + (K + Log[1 - K]) α)/(K/b + (2/3 - 1/K) Log[1 - K] - 1), α → 4/c (1 - Sqrt[1 - c]) - 2,
   b → 1/(Sqrt[1 - c]/(2 K) - c/(4 K^2) Log[1 - K^2]), K → vexact[c]}

H32[0.5, 1]

1.24997
```

Compare various approximations

```
u = 0.02;
Plot[
 {
  H[c, u],
  H1[c, u],
  H2[c, u],
  HGrosjean[c, u]
 }
 , {c, 0, 1}, PlotStyle → {Thick, Dashed, Red, DotDashed}
]
```



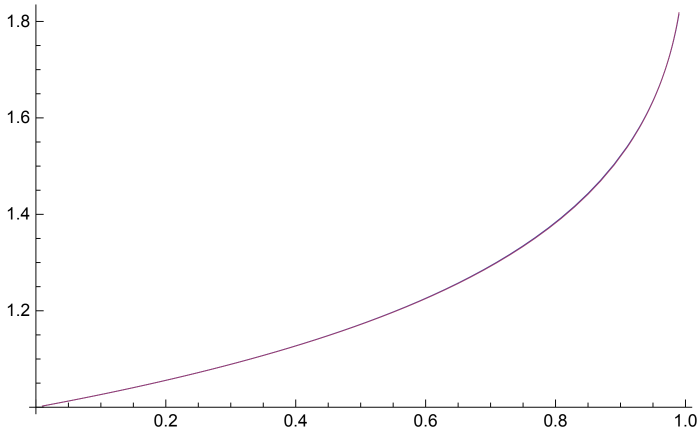
H-function moments

$$\text{Hmoment1[c\_] := } \left(1 - \sqrt{1 - c}\right) \frac{2}{c}$$

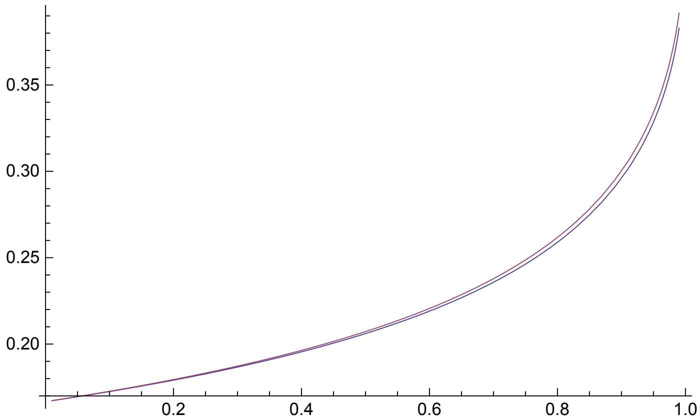
$$\text{HmomentApprox[c\_, j\_] := } \frac{1}{j + 1} \frac{2}{1 + y} \left(1 + \frac{j}{2\,(j + 2)} \frac{1 - y}{1 + y}\right) \text{ /. } y \to \sqrt{1 - c}$$

```
a
```

```
Plot[{
  NIntegrate[ H2[c, u], {u, 0, 1}],
  Hmoment1[c]
 }, {c, 0.01, .99}]
```




```
j = 5;
Plot[{
  NIntegrate[ u^j H2[c, u], {u, 0, 1}],
  HmomentApprox[c, j]
 }, {c, 0.01, .99}]
```



Benchmark H-function data

```
Hfuncdata = Table[H[c, u],
   {c, {0.001, 0.01, 0.05, 0.1, 0.2, 0.3, 0.5, 0.7, 0.8, 0.9, 0.95, 0.98, 0.99,
     0.995, 0.999, 0.9999, 0.99999, 0.999999}}, {u, {0.01, 0.1, 0.2, 0.5, 1}}];

Transpose[Join[
   {Table[c, {c, {0.001, 0.01, 0.05, 0.1, 0.2, 0.3, 0.5, 0.7, 0.8, 0.9, 0.95, 0.98,
        0.99, 0.995, 0.999, 0.9999, 0.99999, 0.999999}}]}, Transpose[Hfuncdata]]];
```

```
Grid[%]
```

0.001	1.00002	1.00012	1.00018	1.00027	1.00035
0.01	1.00023	1.0012	1.0018	1.00276	1.00349
0.05	1.00116	1.00609	1.00914	1.01409	1.01785
0.1	1.00235	1.01238	1.01864	1.02892	1.03682
0.2	1.00478	1.02562	1.03892	1.06118	1.07865
0.3	1.0073	1.03987	1.06115	1.09756	1.12684
0.5	1.01272	1.07237	1.11346	1.18774	1.25126
0.7	1.01887	1.11303	1.18252	1.31795	1.44475
0.8	1.02242	1.13881	1.22864	1.41326	1.59822
0.9	1.0266	1.17214	1.29143	1.55603	1.8501
0.95	1.02923	1.19523	1.33734	1.67179	2.07712
0.98	1.03131	1.21513	1.37876	1.78629	2.32579
0.99	1.03226	1.22488	1.39977	1.8486	2.47279
0.995	1.03289	1.23162	1.41463	1.89463	2.58735
0.999	1.03367	1.24042	1.43442	1.95869	2.75607
0.9999	1.03408	1.24518	1.44532	1.99545	2.85822
0.99999	1.03421	1.24667	1.44876	2.00728	2.89196
0.999999	1.03424	1.24713	1.44985	2.01104	2.90278