# The H-function

This is code to accompany the book:

A Hitchhiker's Guide to Multiple Scattering



www.eugenedeon.com

## Analytic Solution

### H-function (Stibbs-Weir)

In[1331]:=
```
Clear[H];
H[α_, u_] := Exp[-u/Pi NIntegrate[Log[1 - α t Cot[t]]/(Cos[t]^2 + u^2 Sin[t]^2), {t, 0, Pi/2}]]
```

$$H[\alpha\_, u\_] := \mathrm{Exp}\left[\frac{-u}{Pi}\, \mathrm{NIntegrate}\left[\frac{\mathrm{Log}[1-\alpha\, t\, \mathrm{Cot}[t]]}{\mathrm{Cos}[t]^2 + u^2\, \mathrm{Sin}[t]^2}, \{t, 0, \frac{Pi}{2}\}\right]\right]$$

### Approximate H-function

Hapke approximation 1

$$H1[w\_, u\_] := \frac{1 + 2\, u}{1 + 2\, u \sqrt{1 - w}}$$

Hapke approximation 2

```
Clear[H2];
```

$$H2[w\_, u\_] := \left(1 - (1 - y)\, u \left(n + \left(1 - \frac{n}{2} - n\, u\right) \mathrm{Log}\left[\frac{1 + u}{u}\right]\right)\right)^{-1} /.\ n \to \frac{1 - y}{1 + y} /.\ y \to (1 - w)^{1/2}$$

$$V[s\_, c\_] := 1 - \frac{c}{2\, s}\, \mathrm{Log}\left[\frac{1 + s}{1 - s}\right]$$

```
vexact[c_?NumericQ] :=
 FindRoot[V[s, c] == 0, {s, 10^-10, 1 - 10^-10}, Method → "Brent"][[1]][[2]]
```

### Grosjean 1958b approximation

$$HGrosjean[w\_, u\_] := 1 + \frac{w\, u}{2}\, \mathrm{Log}\left[\frac{1 + u}{u}\right] + \frac{2\, w^2\, u}{(1 + K\, u) \left(\frac{13-5\, w}{8} + \frac{2}{3}\, (2 - w)\, K\right)} - \frac{w^2\, u}{\left(\frac{13-5\, w}{8} + \frac{2}{3}\, (2 - w)\, K\right)} \left(\frac{17}{32} + \frac{15}{16}\, u - u \left(1 + \frac{15}{16}\, u\right) \mathrm{Log}\left[\frac{1 + u}{u}\right]\right) /.\ K \to \left(\frac{3\, (1 - w)}{2 - w}\right)^{1/2}$$

## NSE 1976 vol 27, 607 - 608 (two more approximations are given)

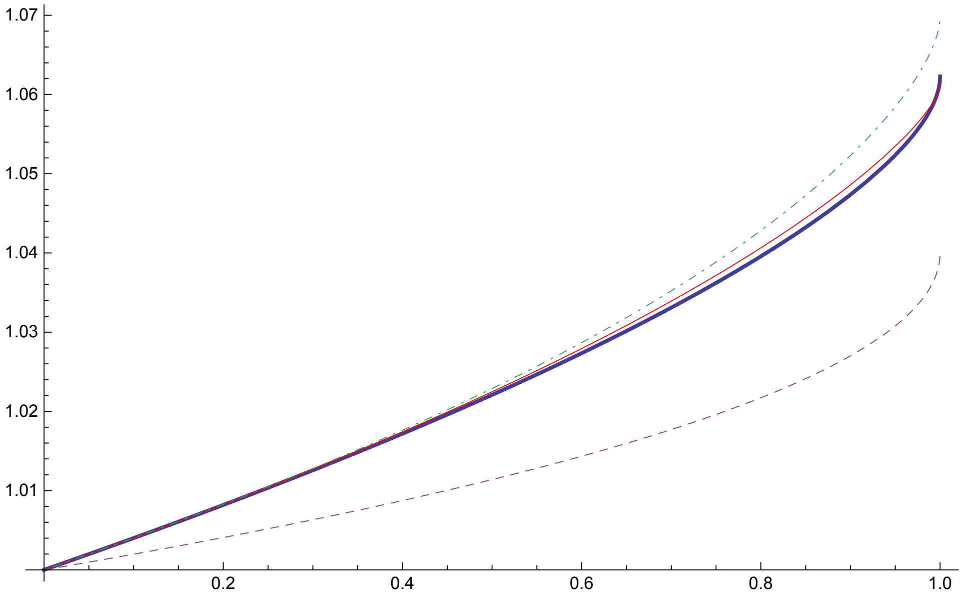
$$\text{H32}[c\_, u\_] := \left(1 - \frac{c}{2}\left(\left(1 - A\,u + B\,u^2\right) u\,\text{Log}\left[\frac{1+u}{u}\right] + A\,u + \left(\frac{u}{2} - u^2\right) B\right)\right)^{-1} \;/.\; \left\{A \to \alpha - \frac{2}{3} B,\; B \to \frac{K\,\text{Log}[1+K] + (K + \text{Log}[1-K])\,\alpha}{\frac{K}{b} + \left(\frac{2}{3} - \frac{1}{K}\right)\text{Log}[1-K] - 1},\; \alpha \to \frac{4}{c}\left(1 - \sqrt{1-c}\right) - 2,\; b \to 1 \Big/ \left(\frac{\sqrt{1-c}}{2\,K} - \frac{c}{4\,K^2}\,\text{Log}\left[1 - K^2\right]\right),\; K \to \text{vexact}[c]\right\}$$

```
H32[0.5, 1]
```

```
1.24997
```

## Compare various approximations

```
u = 0.02;
Plot[
 {
  H[c, u],
  H1[c, u],
  H2[c, u],
  HGrosjean[c, u]
 }
 , {c, 0, 1}, PlotStyle → {Thick, Dashed, Red, DotDashed}
]
```



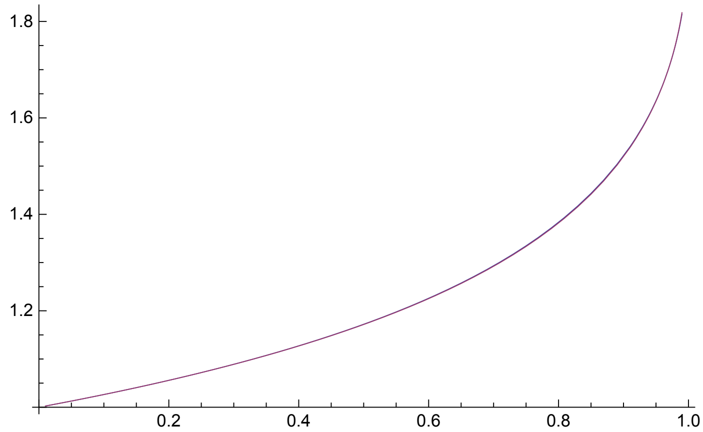

## H-function moments

$$\text{Hmoment1}[c\_] := \left(1 - \sqrt{1-c}\right) \frac{2}{c}$$

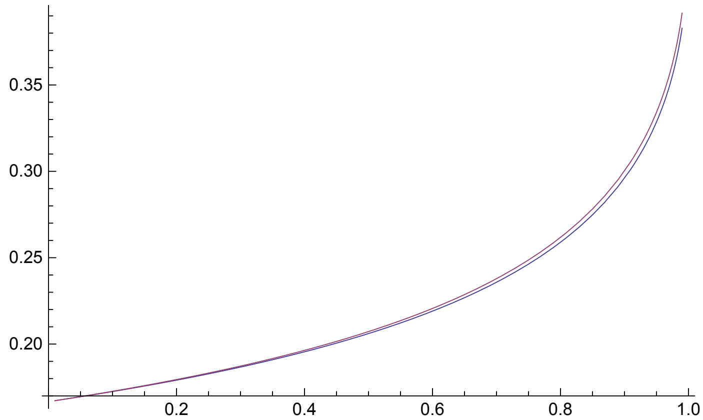
$$\text{HmomentApprox}[c\_, j\_] := \frac{1}{j+1}\,\frac{2}{1+y}\left(1 + \frac{j}{2\,(j+2)}\,\frac{1-y}{1+y}\right) \;/.\; y \to \sqrt{1-c}$$

```
a
```

```
Plot[{
  NIntegrate[H2[c, u], {u, 0, 1}],
  Hmoment1[c]
 }, {c, 0.01, .99}]
```




```
j = 5;
Plot[{
  NIntegrate[u^j H2[c, u], {u, 0, 1}],
  HmomentApprox[c, j]
 }, {c, 0.01, .99}]
```




# Benchmark H-function data

```
Hfuncdata = Table[H[c, u],
   {c, {0.001, 0.01, 0.05, 0.1, 0.2, 0.3, 0.5, 0.7, 0.8, 0.9, 0.95, 0.98, 0.99,
     0.995, 0.999, 0.9999, 0.99999, 0.999999}}, {u, {0.01, 0.1, 0.2, 0.5, 1}}];

Transpose[Join[
   {Table[c, {c, {0.001, 0.01, 0.05, 0.1, 0.2, 0.3, 0.5, 0.7, 0.8, 0.9, 0.95, 0.98,
        0.99, 0.995, 0.999, 0.9999, 0.99999, 0.999999}}]}, Transpose[Hfuncdata]]];
```

```
Grid[%]
```

| | | | | | |
|---|---|---|---|---|---|
| 0.001 | 1.00002 | 1.00012 | 1.00018 | 1.00027 | 1.00035 |
| 0.01 | 1.00023 | 1.0012 | 1.0018 | 1.00276 | 1.00349 |
| 0.05 | 1.00116 | 1.00609 | 1.00914 | 1.01409 | 1.01785 |
| 0.1 | 1.00235 | 1.01238 | 1.01864 | 1.02892 | 1.03682 |
| 0.2 | 1.00478 | 1.02562 | 1.03892 | 1.06118 | 1.07865 |
| 0.3 | 1.0073 | 1.03987 | 1.06115 | 1.09756 | 1.12684 |
| 0.5 | 1.01272 | 1.07237 | 1.11346 | 1.18774 | 1.25126 |
| 0.7 | 1.01887 | 1.11303 | 1.18252 | 1.31795 | 1.44475 |
| 0.8 | 1.02242 | 1.13881 | 1.22864 | 1.41326 | 1.59822 |
| 0.9 | 1.0266 | 1.17214 | 1.29143 | 1.55603 | 1.8501 |
| 0.95 | 1.02923 | 1.19523 | 1.33734 | 1.67179 | 2.07712 |
| 0.98 | 1.03131 | 1.21513 | 1.37876 | 1.78629 | 2.32579 |
| 0.99 | 1.03226 | 1.22488 | 1.39977 | 1.8486 | 2.47279 |
| 0.995 | 1.03289 | 1.23162 | 1.41463 | 1.89463 | 2.58735 |
| 0.999 | 1.03367 | 1.24042 | 1.43442 | 1.95869 | 2.75607 |
| 0.9999 | 1.03408 | 1.24518 | 1.44532 | 1.99545 | 2.85822 |
| 0.99999 | 1.03421 | 1.24667 | 1.44876 | 2.00728 | 2.89196 |
| 0.999999 | 1.03424 | 1.24713 | 1.44985 | 2.01104 | 2.90278 |