Half Space, Albedo Problem, Isotropic Scattering

Exponential Random Flight

This is code to accompany the book:

A Hitchhiker's Guide to Multiple Scattering

www.eugenedeon.com

Path Setup

Put a file at ~/.hitchhikerpath with the path to your hitchhiker repo so that these worksheets can find the MC data from the C++ simulations for verification

```
SetDirectory[Import["~/.hitchhikerpath"]]
```

Notation

α - single-scattering albedo
Σt - extinction coefficient
z - depth in medium (positive inside the scattering half space)
u = cos θ - direction cosine

Caseology

Caseology quantities

$$\text{CaseN0}[c\_, v0\_] := \frac{1}{2}\, c\, v0^3 \left(\frac{c}{v0^2 - 1} - \frac{1}{v0^2}\right)$$

```
Casev0[c_?NumericQ] :=
 FindRoot[c v ArcTanh[1/v] - 1 == 0, {v, 1.00000000001, 10^10}, Method → "Brent"][[1]][[
  2]]
```

$$\text{CaseN}[c\_, v\_] := v \left(\text{Case}\lambda[v, c]^2 + \left(\frac{\pi\, c\, v}{2}\right)^2\right)$$

$$\text{Case}\lambda[v\_, c\_] := 1 - c\, v\, \text{ArcTanh}[v]$$

$$\text{Case}\psi 0[u\_, v0\_, c\_, z\_] := \frac{c}{2}\, \frac{v0}{v0 - \text{Sign}[z]\, u}$$

H-function

H-function (Stibbs-Weir)

In[1376]:=
```
halfspaceAlbedoProblemIsotropic`H[α_, u_] :=
 Exp[(-u)/Pi NIntegrate[Log[1 - α t Cot[t]]/(Cos[t]^2 + u^2 Sin[t]^2), {t, 0, Pi/2}]]
```

In[1377]:=
```
halfspaceAlbedoProblemIsotropic`Hmoment[α_, j_] :=
 NIntegrate[H[α, u] u^j, {u, 0, 1}]
```

Approximate H-function

In[1379]:=
```
Clear[n, y];
halfspaceAlbedoProblemIsotropic`H2[α_, u_] :=
 (1 - (1 - y) u (n + (1 - n/2 - n u) Log[(1 + u)/u]))^-1 /. n → (1 - y)/(1 + y) /. y → (1 - α)^(1/2)
```

In[1381]:=
```
halfspaceAlbedoProblemIsotropic`HmomentApprox[α_, j_] :=
 1/(j + 1) 2/(1 + y) (1 + j/(2 (j + 2)) (1 - y)/(1 + y)) /. y → Sqrt[1 - α]
```

Albedo

Exact Solution

In[1494]:=
```
halfspaceAlbedoProblemIsotropic`albedoexact[α_, ui_] :=
 1 - Sqrt[1 - α] halfspaceAlbedoProblemIsotropic`H[α, ui]
```

In[1499]:=
```
halfspaceAlbedoProblemIsotropic`albedoexact[α_] :=
 NIntegrate[2 u halfspaceAlbedoProblemIsotropic`albedoexact[α, u], {u, 0, 1}]
```

Single-scattered Albedo (Exact Solution)

In[1408]:=
```
halfspaceAlbedoProblemIsotropic`albedoSingleScatterExact[α_, ui_] :=
 1/2 α (1 + ui Log[ui/(1 + ui)])
```

In[1508]:=
```
halfspaceAlbedoProblemIsotropic`albedoSingleScatterExact[α_] := 2/3 (α - α Log[2])
```

Double-scattered Albedo (Exact Solution)

In[1409]:=
```
halfspaceAlbedoProblemIsotropic`albedoDoubleScatterExact[α_, ui_] :=
 -1/8 α^2 (-1 + Log[e^(2 ui^2 (Log[1-ui] Log[ui]-PolyLog[2, ui/(-1+ui)]))] +
    2 ui (ui Log[1 + 1/ui]^2 + Log[4 ui/(1 + ui)] - ui Log[1 - ui] Log[1 + ui]) +
    2 ui^2 (PolyLog[2, -1/ui] + PolyLog[2, (1 + ui)/(-1 + ui)]))
```

In[1410]:=
```
halfspaceAlbedoProblemIsotropic`albedoDoubleScatterExact[α_] :=
 1/24 c^2 (4 + π^2 - 16 Log[2])
```

Approximate General Albedos

Classical Diffusion

In[1519]:=
```
halfspaceAlbedoProblemIsotropic`albedoClassicalDiffusion[c_, ui_] :=
 c / ((1 + 2/3 (3 (1 - c))^(1/2)) (1 + (3 (1 - c))^(1/2) ui))
```

Rigorous Diffusion

In[1520]:=
```
halfspaceAlbedoProblemIsotropic`albedoRigorousDiffusion[c_, ui_] :=
 c / ((1 + 2 (1-c)/#) (1 + # ui)) &[1/Casev0[c]]
```

Grosjean Modified Diffusion 1

In[1523]:=
```
halfspaceAlbedoProblemIsotropic`albedoGrosjeanDiffusion1[c_, ui_] :=
 c/2 (1 - ui Log[(1 + ui)/ui]) (1 - c (13 - 5 c)/(13 - 5 c + 16/3 (2 - c) K)) +
  c^2 (13 - 5 c)/((1 + K ui) (13 - 5 c + 16/3 (2 - c) K)) -
  5/4 K c^2 (2 - c) (1 - 2 ui + 2 ui^2 Log[(1+ui)/ui])/(13 - 5 c + 16/3 (2 - c) K) /. K → (3 (1 - c)/(2 - c))^(1/2)
```

Grosjean Modified Diffusion 2

In[1553]:=
```
halfspaceAlbedoProblemIsotropic`albedoGrosjeanDiffusion2[c_, ui_] :=
 c/2 (1 - ui Log[(1 + ui)/ui]) (1 - c/(1 + 2 (2-c) K/3)) +
  c^2/((1 + K ui) (1 + 2 (2-c) K/3)) /. K → (3 (1 - c)/(2 - c))^(1/2)
```

Approximate Normal Incidence Albedos

[Pomraning 1965]

In[1567]:=
```
halfspaceAlbedoProblemIsotropic`albedoNormalPomraning[α_] :=
 2/((1 + #) Log[1 - #^2]) (Log[1 + #] - #) &[1 / Casev0[α]]
```

[Prahl 2002]

In[1528]:=
```
halfspaceAlbedoProblemIsotropic`albedoNormalPrahl[α_] :=
 (1 - #) (1 - 0.128 #)/(1 + 1.83 #) &[Sqrt[1 - α]]
```

Approximate WhiteSky Albedos

Integral of H2

In[1501]:=
```
halfspaceAlbedoProblemIsotropic`albedoH2[α_] :=
 NIntegrate[2 u (1 - Sqrt[1 - α] halfspaceAlbedoProblemIsotropic`H2[α, u]), {u, 0, 1}]
```

[Pomraning 1965]

In[1566]:=
```
halfspaceAlbedoProblemIsotropic`albedoWhiteskyPomraning[α_] :=
 -4
 ---------------- (Log[1 + #] - #)^2 &[1 / Casev0[α]]
 #^2 Log[1 - #^2]
```

[van de Hulst 1980]

In[1527]:=
```
halfspaceAlbedoProblemIsotropic`albedoWhiteskyVanDeHulst[α_] :=
 (1 - #) (1 - 0.139 #)
 ---------------------- &[Sqrt[1 - α]]
 1 + 1.17 #
```

BRDF

Exact Solution

In[1629]:=
```
halfspaceAlbedoProblemIsotropic`BRDF[α_, ui_, uo_] :=
 1
 ---------- α halfspaceAlbedoProblemIsotropic`H2[α, ui]
 4 Pi (ui + uo)
  halfspaceAlbedoProblemIsotropic`H2[α, uo]
```

In[1641]:=
```
halfspaceAlbedoProblemIsotropic`BRDFsinglescatter[α_, ui_, uo_] := α 1/(4 Pi) 1/(ui + uo)
```

In[1639]:=
```
halfspaceAlbedoProblemIsotropic`BRDFdoublescatter[α_, ui_, uo_] :=
 α^2 (ui ArcCoth[1 + 2 ui] + uo ArcCoth[1 + 2 uo])
 ---- ---------------------------------------------
 4 Pi ui + uo
```

Load MC Data

Delta Incidence

In[1272]:=
```
halfspaceAlbedoProblemIsotropic`deltafs = FileNames[
    "code/3D_medium/halfspace/albedoProblem/data/albedoproblem_delta*.txt"];
```

In[1292]:=
```
halfspaceAlbedoProblemIsotropic`indexdelta[x_] := Module[{data, α, Σt, ui},
    data = Import[x, "Table"];
    Σt = data[[2, 5]];
    α = data[[2, 3]];
    ui = data[[1, -1]];
    {α, Σt, ui, data}];
halfspaceAlbedoProblemIsotropic`simulationsdelta =
  halfspaceAlbedoProblemIsotropic`indexdelta /@
   halfspaceAlbedoProblemIsotropic`deltafs;
halfspaceAlbedoProblemIsotropic`alphas =
 Union[#[[1]] & /@ halfspaceAlbedoProblemIsotropic`simulationsdelta]
```

Out[1294]= {0.01, 0.1, 0.3, 0.5, 0.7, 0.8, 0.9, 0.95, 0.99, 0.999}

In[1298]:=
```
halfspaceAlbedoProblemIsotropic`muts =
 Union[#[[2]] & /@ halfspaceAlbedoProblemIsotropic`simulationsdelta]
```

Out[1298]= {1}

In[1299]:=
```
halfspaceAlbedoProblemIsotropic`uis =
 Union[#[[3]] & /@ halfspaceAlbedoProblemIsotropic`simulationsdelta]
```

Out[1299]= {0.1, 0.25, 0.5, 1}

WhiteSky Illumination

In[1475]:=
```
halfspaceAlbedoProblemIsotropic`whiteskyfs = FileNames[
    "code/3D_medium/halfspace/albedoProblem/data/albedoproblem_whitesky*.txt"];
```

In[1476]:=
```
halfspaceAlbedoProblemIsotropic`indexwhitesky[x_] := Module[{data, α, Σt},
    data = Import[x, "Table"];
    Σt = data[[2, 5]];
    α = data[[2, 3]];
    {α, Σt, data}];
halfspaceAlbedoProblemIsotropic`simulationswhitesky =
  halfspaceAlbedoProblemIsotropic`indexwhitesky /@
   halfspaceAlbedoProblemIsotropic`whiteskyfs;
```

Util

In[1632]:=
```
halfspaceAlbedoProblemIsotropic`plotpoints[data_, du_] :=
  Table[{du i - 0.5 du, data[[i]]}, {i, 1, Length[data]}];
```

Albedo Benchmarks

Normal Incidence

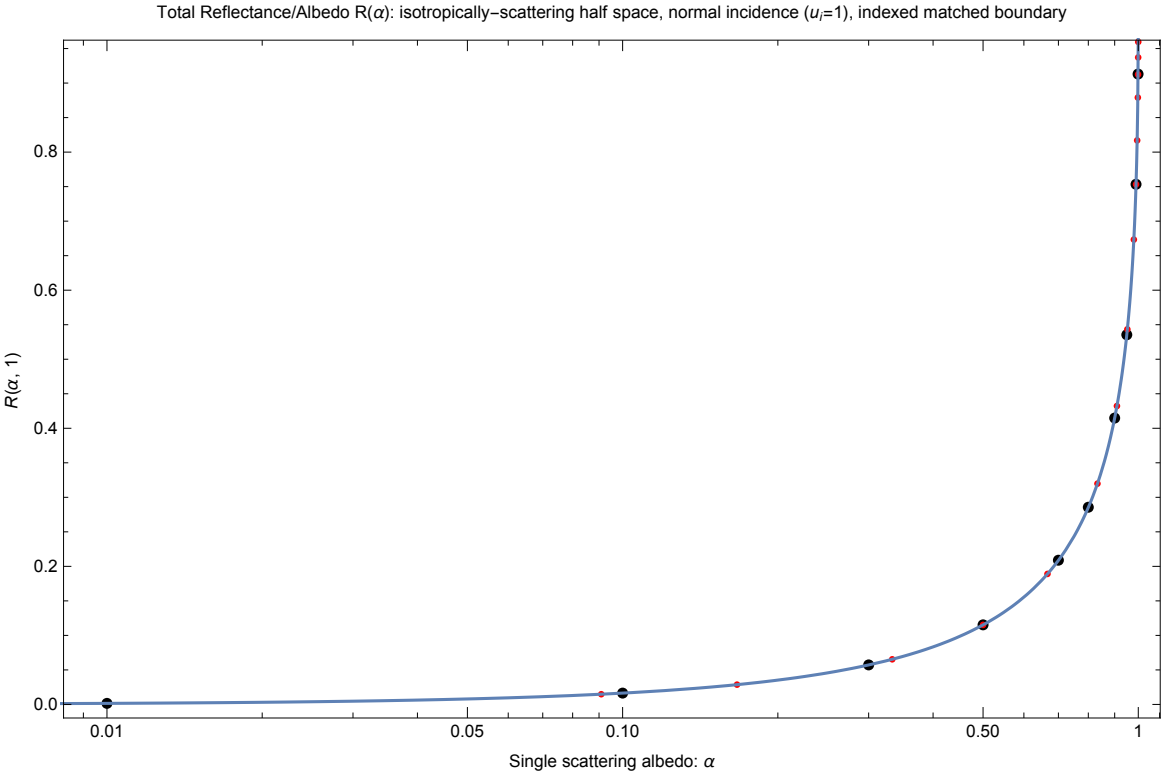
In[1313]:=
```
MCMLBenchmarkData =
  {{0.9990009990009991`, 0.912484`}, {0.09090909090909091`, 0.0147879`},
   {0.16666666666666666`, 0.0286725`}, {0.3333333333333333`, 0.0653595`},
   {0.5`, 0.11524`}, {0.6666666666666666`, 0.189042`},
   {0.9090909090909091`, 0.432273`}, {0.9950248756218907`, 0.816965`},
   {0.8333333333333334`, 0.319804`}, {0.998003992015968`, 0.879038`},
   {0.9995002498750625`, 0.937142`}, {0.9998000399920016`, 0.959626`},
   {0.9999000099990001`, 0.971259`}, {0.9523809523809523`, 0.543346`},
   {0.9803921568627451`, 0.673321`}, {0.9900990099009901`, 0.75327`}};
```

In[1312]:=
```
halfspaceAlbedoProblemIsotropic`normalRs = {#[[1]], #[[4]][[3, 3]]} & /@
  Select[halfspaceAlbedoProblemIsotropic`simulationsdelta, #[[3]] == 1 &]
```

Out[1312]=
```
{{0.01, 0.0015482}, {0.1, 0.0164138}, {0.3, 0.0572445},
 {0.5, 0.115279}, {0.7, 0.208883}, {0.8, 0.28562}, {0.95, 0.535438},
 {0.999, 0.913071}, {0.99, 0.753413}, {0.9, 0.414801}}
```

In[1388]:=
```
Quiet[Show[
  Show[
   ListLogLinearPlot[halfspaceAlbedoProblemIsotropic`normalRs,
    PlotStyle → {PointSize[0.01], Black}],
   ListLogLinearPlot[MCMLBenchmarkData, PlotStyle → {PointSize[0.006], Red}],
   LogLinearPlot[Quiet[halfspaceAlbedoProblemIsotropic`albedoexact[c, 1]],
    {c, 0.001, .9999}, PlotRange → All]
  ], Frame → True, FrameLabel → {{R[α, 1],}, {"Single scattering albedo: α",
      "Total Reflectance/Albedo R(α): isotropically-scattering half
        space, normal incidence (u_i=1), indexed matched boundary"}}
 ]]
```

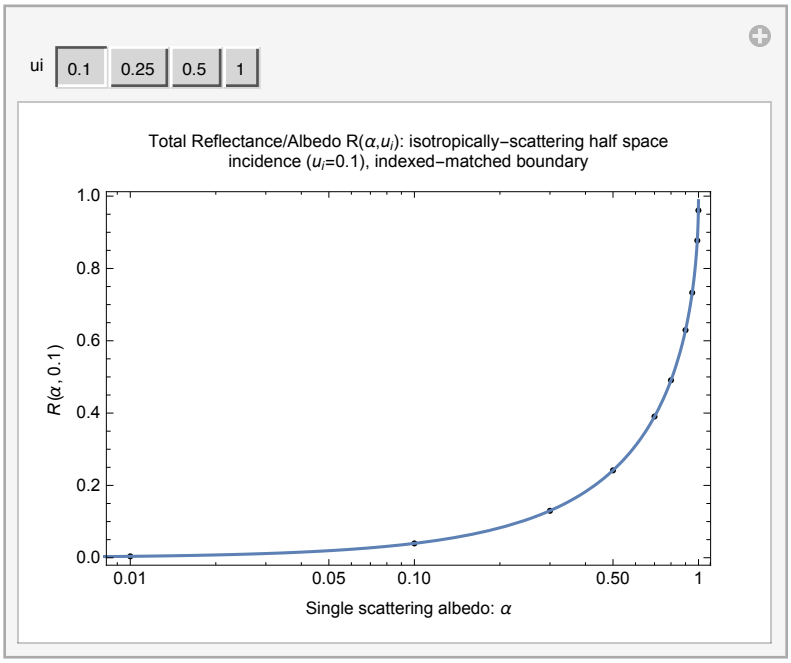
Out[1388]=



Albedo - Delta Illumination - General incidence

In[1430]:=
```
Manipulate[
 Module[{Rs},
  Rs = {#[[1]], #[[4]][[3, 3]]} & /@
    Select[halfspaceAlbedoProblemIsotropic`simulationsdelta, #[[3]] == ui &];
  Quiet[Show[
    Show[
     ListLogLinearPlot[Rs, PlotStyle → {PointSize[0.01], Black}],
     LogLinearPlot[Quiet[halfspaceAlbedoProblemIsotropic`albedoexact[c, ui]],
      {c, 0.001, .9999}, PlotRange → All]
    ], Frame → True, FrameLabel → {{R[α, ui],},
      {"Single scattering albedo: α", "Total Reflectance/Albedo R(α,uᵢ):
          isotropically-scattering half space\nincidence (uᵢ=" <>
        ToString[ui] <> "), indexed-matched boundary"}}
   ]]
 ]
 , {ui, halfspaceAlbedoProblemIsotropic`uis}]
```

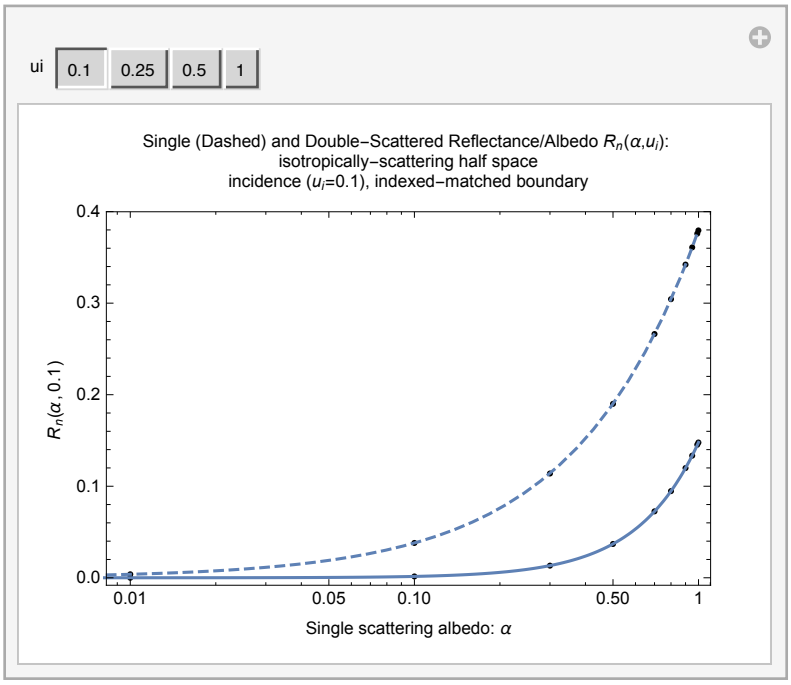
Out[1430]=



Single & Double-Scattered Albedo - Delta Illumination - General incidence

In[1498]:=
```
Manipulate[
 Module[{RsSingle, RsDouble},
  RsSingle = {#[[1]], #[[4]][[5, 2]]} & /@
    Select[halfspaceAlbedoProblemIsotropic`simulationsdelta, #[[3]] == ui &];
  RsDouble = {#[[1]], #[[4]][[5, 3]]} & /@
    Select[halfspaceAlbedoProblemIsotropic`simulationsdelta, #[[3]] == ui &];
  Quiet[Show[
    Show[
     ListLogLinearPlot[RsSingle, PlotStyle → {PointSize[0.01], Black}],
     ListLogLinearPlot[RsDouble, PlotStyle → {PointSize[0.01], Black}],
     LogLinearPlot[
      Quiet[halfspaceAlbedoProblemIsotropic`albedoSingleScatterExact[c, ui]],
      {c, 0.001, .9999}, PlotRange → All, PlotStyle → Dashed],
     LogLinearPlot[Quiet[
       halfspaceAlbedoProblemIsotropic`albedoDoubleScatterExact[c, ui]],
      {c, 0.001, .9999}, PlotRange → All]
    ], Frame → True, FrameLabel → {{Rn[α, ui],}, {"Single scattering albedo: α",
       "Single (Dashed) and Double-Scattered Reflectance/Albedo Rn(α,ui):
          \nisotropically-scattering half space\nincidence (ui=" <>
        ToString[ui] <> "), indexed-matched boundary"}}
   ]]
 ]
 , {ui, halfspaceAlbedoProblemIsotropic`uis}]
```

Out[1498]=

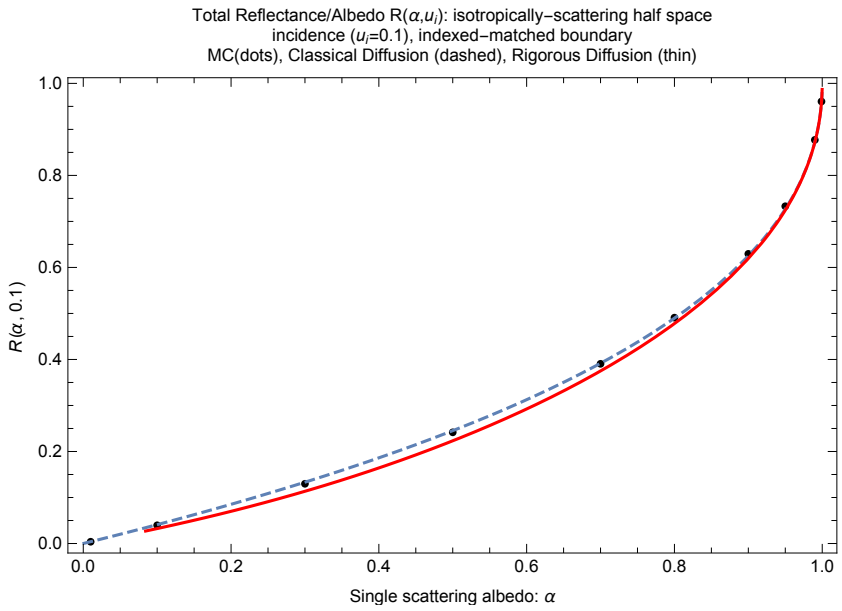


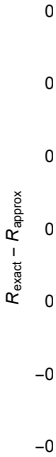
Approximate General Albedos

In[1557]:=
```
Manipulate[
 Module[{Rs},
  Rs = {#[[1]], #[[4]][[3, 3]]} & /@
    Select[halfspaceAlbedoProblemIsotropic`simulationsdelta, #[[3]] == ui &];
  GraphicsRow[{Quiet[Show[
      Show[
       ListPlot[Rs, PlotStyle → {PointSize[0.01], Black}],
       Plot[halfspaceAlbedoProblemIsotropic`albedoClassicalDiffusion[c, ui],
        {c, 0.001, .9999},
        PlotRange → All, PlotStyle → Dashed],
       Plot[halfspaceAlbedoProblemIsotropic`albedoRigorousDiffusion[c, ui],
        {c, 0.001, .9999},
        PlotRange → All, PlotStyle → Red]
      ], Frame → True,
      FrameLabel → {{R[α, ui],}, {"Single scattering albedo: α",
          "Total Reflectance/Albedo R(α,uᵢ): isotropically-scattering
             half space\nincidence (uᵢ=" <> ToString[ui] <>
           "), indexed-matched boundary\nMC(dots), Classical Diffusion
             (dashed), Rigorous Diffusion (thin)"}},
      ImageSize → 500
     ]],
    Quiet[Show[
      Show[
       Plot[halfspaceAlbedoProblemIsotropic`albedoexact[c, ui] -
         halfspaceAlbedoProblemIsotropic`albedoClassicalDiffusion[
          c, ui], {c, 0.001, .9999},
        PlotRange → All, PlotStyle → Dashed],
       Plot[halfspaceAlbedoProblemIsotropic`albedoexact[c, ui] -
         halfspaceAlbedoProblemIsotropic`albedoRigorousDiffusion[
          c, ui], {c, 0.001, .9999},
        PlotRange → All, PlotStyle → Red],
       Plot[halfspaceAlbedoProblemIsotropic`albedoexact[c, ui] -
         halfspaceAlbedoProblemIsotropic`albedoGrosjeanDiffusion1[
          c, ui], {c, 0.001, .9999},
        PlotRange → All, PlotStyle → DotDashed]
      ], Frame → True, FrameLabel → {{"R_exact - R_approx",},
        {"Single scattering albedo: α", "Total Reflectance/Albedo R(α,uᵢ):
            isotropically-scattering half space\nincidence (uᵢ=" <>
          ToString[ui] <> "), indexed-matched boundary\nClassical
            Diffusion (dashed), Rigorous Diffusion
            (thin), Grosjean1 (DotDashed)"}}
     ]]
   }]
 ]
 , {ui, halfspaceAlbedoProblemIsotropic`uis}]
```

Out[1557]=

ui 0.1 0.25 0.5 1

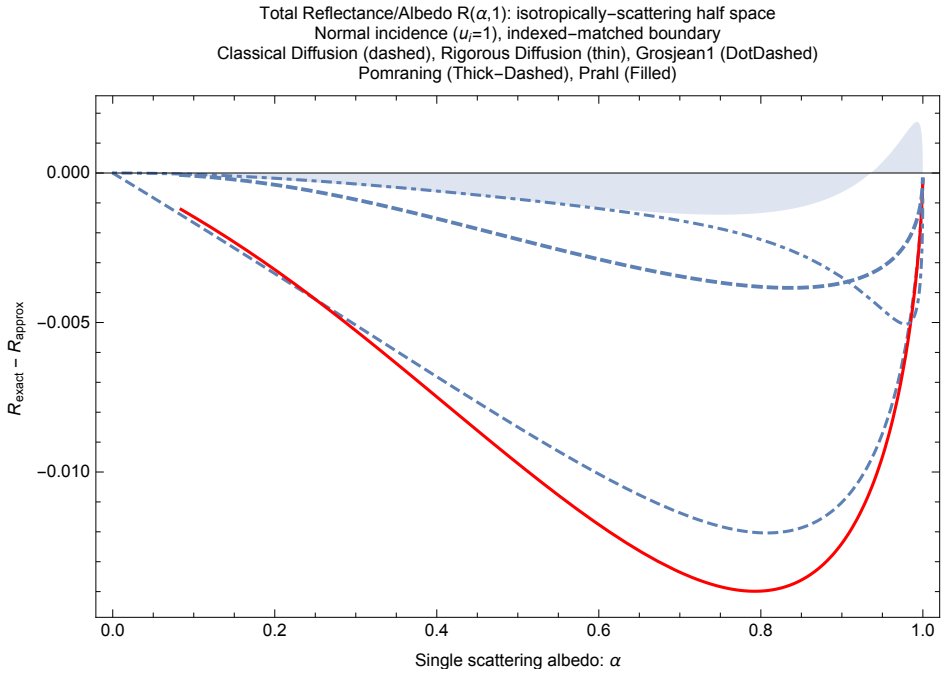




Normal incidence - Approximate Albedos

In[1571]:=
```
Quiet[Show[
  Show[
   Plot[halfspaceAlbedoProblemIsotropic`albedoexact[c, 1] -
     halfspaceAlbedoProblemIsotropic`albedoClassicalDiffusion[c, 1], {c,
     0.001, .9999},
    PlotRange → All, PlotStyle → Dashed],
   Plot[halfspaceAlbedoProblemIsotropic`albedoexact[c, 1] -
     halfspaceAlbedoProblemIsotropic`albedoRigorousDiffusion[c, 1], {c,
     0.001, .9999},
    PlotRange → All, PlotStyle → Red],
   Plot[halfspaceAlbedoProblemIsotropic`albedoexact[c, 1] -
     halfspaceAlbedoProblemIsotropic`albedoGrosjeanDiffusion1[c, 1], {c,
     0.001, .9999},
    PlotRange → All, PlotStyle → DotDashed],
   Plot[halfspaceAlbedoProblemIsotropic`albedoexact[c, 1] -
     halfspaceAlbedoProblemIsotropic`albedoNormalPomraning[c], {c,
     0.001, .9999},
    PlotRange → All, PlotStyle → {Thick, Dashed}],
   Plot[halfspaceAlbedoProblemIsotropic`albedoexact[c, 1] -
     halfspaceAlbedoProblemIsotropic`albedoNormalPrahl[c], {c, 0.001, .9999},
    PlotRange → All, PlotStyle → Opacity[0], Filling → Axis]
  ], Frame → True, ImageSize → 500,
  FrameLabel → {{"Rexact - Rapprox",},
    {"Single scattering albedo: α", "Total Reflectance/Albedo R(α,1):
       isotropically-scattering half space\nNormal incidence
       (ui=1), indexed-matched boundary\nClassical Diffusion
       (dashed), Rigorous Diffusion (thin), Grosjean1
       (DotDashed)\nPomraning (Thick-Dashed), Prahl (Filled)"}}
]]
```

Out[1571]=

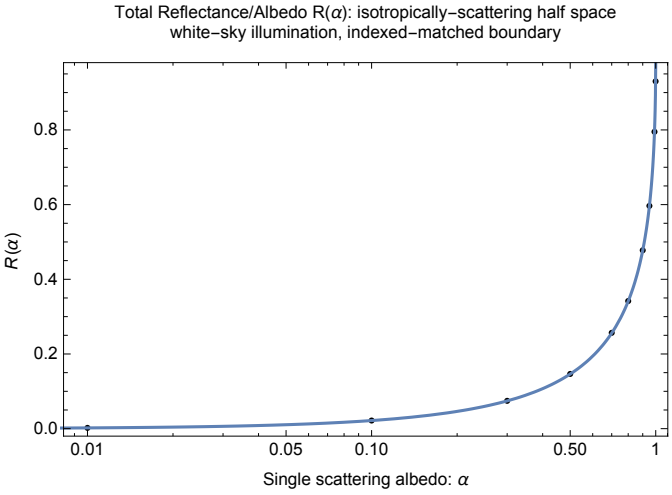


Albedo - WhiteSky Illumination - Integral using H2 approximation

In[1502]:=

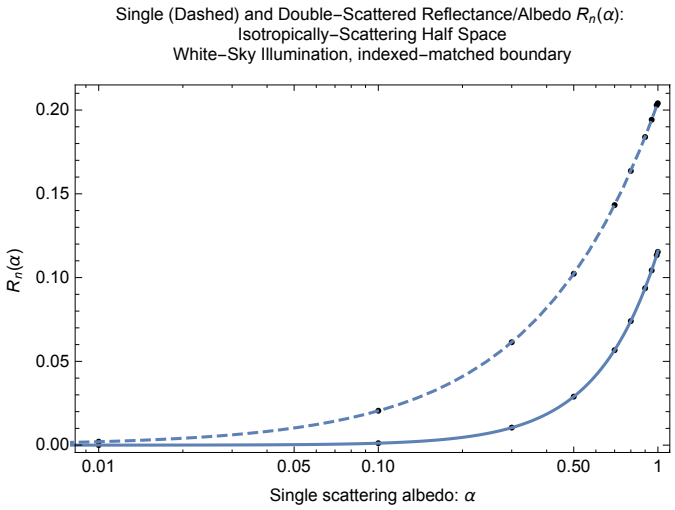
```
Module[{Rs},
 Rs = {#[[1]], #[[3]][[3, 3]]} & /@
   halfspaceAlbedoProblemIsotropic`simulationswhitesky;
 Quiet[Show[
   Show[
    ListLogLinearPlot[Rs, PlotStyle → {PointSize[0.01], Black}],
    LogLinearPlot[Quiet[halfspaceAlbedoProblemIsotropic`albedoH2[c]],
     {c, 0.001, .9999}, PlotRange → All]
   ], Frame → True, FrameLabel → {{R[α],}, {"Single scattering albedo: α",
      "Total Reflectance/Albedo R(α): isotropically-scattering half
        space\nwhite-sky illumination, indexed-matched boundary"}}
  ]]
]
```

Out[1502]=



In[1510]:=
```
Module[{RsSingle, RsDouble},
 RsSingle = {#[[1]], #[[3]][[5, 2]]} & /@
   halfspaceAlbedoProblemIsotropic`simulationswhitesky;
 RsDouble = {#[[1]], #[[3]][[5, 3]]} & /@
   halfspaceAlbedoProblemIsotropic`simulationswhitesky;
 Quiet[Show[
   Show[
    ListLogLinearPlot[RsSingle, PlotStyle → {PointSize[0.01], Black}],
    ListLogLinearPlot[RsDouble, PlotStyle → {PointSize[0.01], Black}],
    LogLinearPlot[
     Quiet[halfspaceAlbedoProblemIsotropic`albedoSingleScatterExact[c]],
     {c, 0.001, .9999}, PlotRange → All, PlotStyle → Dashed],
    LogLinearPlot[Quiet[
      halfspaceAlbedoProblemIsotropic`albedoDoubleScatterExact[c]],
     {c, 0.001, .9999}, PlotRange → All]
   ], Frame → True, FrameLabel → {{Rn[α],}, {"Single scattering albedo: α",
      "Single (Dashed) and Double-Scattered Reflectance/Albedo Rn(α):
        \nIsotropically-Scattering Half Space\nWhite-Sky
        Illumination, indexed-matched boundary"}}
  ]
 ]
]
```

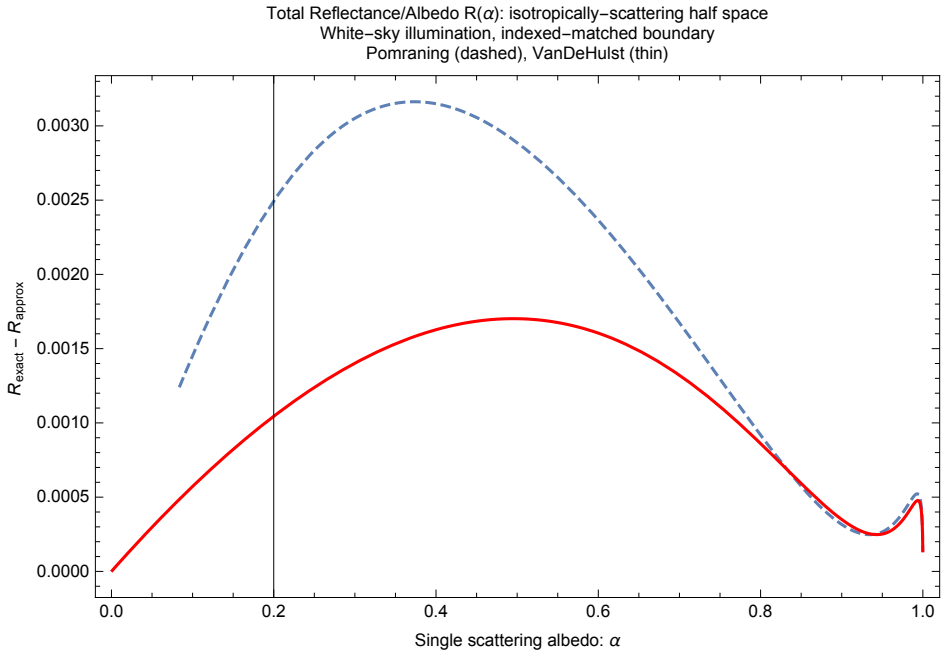
Out[1510]=



Approximate White-sky Albedos

In[1582]:=
```
Quiet[Show[
  Show[
   Plot[halfspaceAlbedoProblemIsotropic`albedoH2[c] -
     halfspaceAlbedoProblemIsotropic`albedoWhiteskyPomraning[c], {c,
     0.001, .9999},
    PlotRange → All, PlotStyle → Dashed],
   Plot[halfspaceAlbedoProblemIsotropic`albedoH2[c] -
     halfspaceAlbedoProblemIsotropic`albedoWhiteskyVanDeHulst[c], {c,
     0.001, .9999},
    PlotRange → All, PlotStyle → Red]
  ], Frame → True, ImageSize → 500,
  FrameLabel → {{"Rexact - Rapprox",}, {"Single scattering albedo: α",
     "Total Reflectance/Albedo R(α): isotropically-scattering half
       space\nWhite-sky illumination, indexed-matched
       boundary\nPomraning (dashed), VanDeHulst (thin)"}}
 ]]
```

Out[1582]=



BRDF Benchmarks

In[1668]:=
```
Manipulate[
 Module[{emerging, imdata, sim, du, emergingS, emergingD, stride},
  sim = SelectFirst[halfspaceAlbedoProblemIsotropic`simulationsdelta,
    #[[3]] == ui && #[[1]] == α && #[[2]] == 1 &];
  imdata = sim[[4]];
  du = imdata[[2, 7]];
  stride = 3;
  emerging = halfspaceAlbedoProblemIsotropic`plotpoints[
     imdata[[7]], du][[1 ;; -1 ;; stride]];
  emergingS = halfspaceAlbedoProblemIsotropic`plotpoints[
     imdata[[9]], du][[1 ;; -1 ;; stride]];
  emergingD = halfspaceAlbedoProblemIsotropic`plotpoints[
     imdata[[11]], du][[1 ;; -1 ;; stride]];
  Show[
   Show[
    Plot[2 Pi u halfspaceAlbedoProblemIsotropic`BRDF[α, ui, u],
     {u, 0, 1}, PlotRange → All],
    Plot[2 Pi u halfspaceAlbedoProblemIsotropic`BRDFsinglescatter[α, ui, u],
     {u, 0, 1}, PlotStyle → Dashed],
    Plot[2 Pi u halfspaceAlbedoProblemIsotropic`BRDFdoublescatter[α, ui, u],
     {u, 0, 1}, PlotStyle → Thick],
    ListPlot[emerging, PlotStyle → {PointSize[0.004], Black}],
    ListPlot[emergingS, PlotStyle → {PointSize[0.004], Black}],
    ListPlot[emergingD, PlotStyle → {PointSize[0.004], Black}]
   ], Frame → True, ImageSize → 500,
   FrameLabel → {{2 π u_o f_r[u_i, u_o],}, {u_o, "Emerging Distribution f_r(u_i,u_o):
          isotropically-scattering half space\nincidence (u_i=" <>
       ToString[ui] <> "), indexed-matched boundary\nTotal (thin),
         Single-Scatter (dashed), Double-scatter (thick)"}}
  ]
 ]
 ,
 {{α, 0.8}, halfspaceAlbedoProblemIsotropic`alphas},
 {{ui, 0.25}, halfspaceAlbedoProblemIsotropic`uis}
]
```

Out[1668]=

α 0.8

ui 0.1 0.25 0.5 1