Z - Transform

Definition:

$$X(Z) = \sum_{n=0}^{\infty} X(nT)\, Z^{-n}$$

where T is the sampling time, $n = 0, 1, 2, ---$

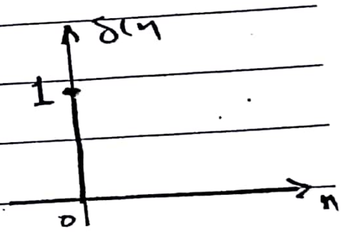
* for sequence

$$X(Z) = \sum_{n=0}^{\infty} X(n)\, Z^{-n}$$

where $T = 1$ sec.

* Z-Transform for some elementary functions

1] impulse function ($\delta(t)$ or $\delta(n)$)

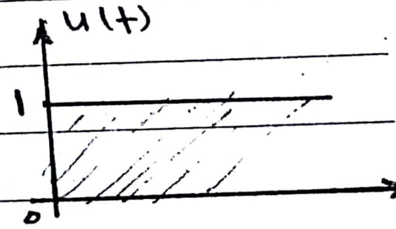


$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$\delta(z) = \sum_{n=0}^{\infty} \delta(n)\, Z^{-n} = 1$$

$$\boxed{\therefore\ \delta(n) \xrightarrow{Z.T} 1}$$

2] unit step function



$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$u(z) = \sum_{n=0}^{\infty} u(nT)\, Z^{-n} = \sum_{n=0}^{\infty} 1 . Z^{-n} = 1 + Z^{-1} + Z^{-2} + ---$$

$$= \underbrace{1 + 1/z + (1/z)^2 + ---}_{\text{geometric series}}$$

$$U(z) = \frac{1}{1 - \text{base}} = \frac{1}{1 - 1/z} = \frac{z}{z-1}$$

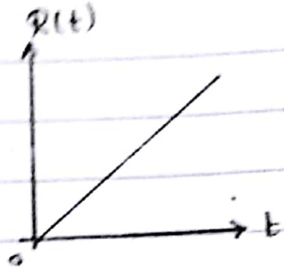
$\text{base} = 1/z$

$$\boxed{u(t) \xrightarrow{\ Z.T\ } \frac{z}{z-1}}$$

* **Properties of Z-Transform**

Property	X[n] or x(t)	X(z)
- Multiply by "Const"	$a\,x(t)$	$a\,X(z)$
- linearity	$x(t) + h(t)$	$X(z) + H(z)$
- Multiply by t	$t\,x(t)$	$-zT \frac{d}{dz} X(z)$ (T: Sampling)
- Multiply by n	$n\,x(n)$	$-z \frac{d}{dz} X(z)$
- Multiply by a^n	$a^n x(n)$ (a: Const)	$X(z/a)$
- Multiply by exp	$e^{-at} x(t)$	$X(z e^{aT})$
	$e^{-an} x(n)$	$X(z e^{a})$
- Shifting	$x(t - NT)$	$z^{-N} X(z)$ shift
	$x(n - N)$	$z^{-N} X(z)$
	$N = 0, 1, 2, 3, \ldots$	
	$x(t + NT)$	$z^{N} X(z) - z^{N} x(0) - z^{N-1} x(T) \ldots$
	$x(n + N)$	$z^{N} X(z) - z^{N} x(0) - z^{N-1} x(1) \ldots$

3] Ramp Function



$$R(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$R(t) = t\,u(t)$$

but

$$u(t) \xrightarrow{Z.T} \frac{Z}{Z-1} \quad \text{AND} \quad t\,u(t) \longrightarrow -ZT \frac{d}{dZ} \frac{Z}{Z-1}$$

$$\cancel{-}ZT \cdot \frac{\cancel{Z}-1-\cancel{Z}}{(Z-1)^2}$$

$$\therefore t\,u(t) \xrightarrow{Z.T} \frac{ZT}{(Z-1)^2}$$

$$\boxed{\therefore R(t) \xrightarrow{Z.T} \frac{ZT}{(Z-1)^2}}$$

4] Polynomial function

$$x(n) = \begin{cases} a^n & n = 0, 1, 2, \ldots \\ 0 & n < 0 \end{cases}$$

$$x(n) = a^n u(n)$$

but

$$u(n) = \frac{Z}{Z-1} \quad \text{AND} \quad a^n u(n) \longrightarrow \frac{Z/a}{Z/a - 1} = \frac{Z}{Z-a}$$

$$\boxed{\therefore x(n) \xrightarrow{Z.T} \frac{Z}{Z-a}}$$

5] exponential function

$$x(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$x(t) = e^{-at} u(t)$$

but

$$u(t) = \frac{z}{z-1} \quad \text{AND} \quad e^{-at} u(t) \longrightarrow \frac{z e^{aT}}{z e^{aT} - 1} \cdot \frac{e^{-aT}}{e^{-aT}}$$

$$\boxed{x(t) \xrightarrow{Z.T} \frac{z}{z - e^{-aT}}}$$

6] Sinusoidal function

$$x(t) = \begin{cases} \sin wt & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Note:

$$e^{jwT} = \cos wT + j \sin wT$$

$$e^{-jwT} = \cos wT - j \sin wT$$

then

$$\cos wT = \tfrac{1}{2} \left(e^{jwT} + e^{-jwT} \right)$$

$$\sin wT = \tfrac{1}{2j} \left(e^{jwT} - e^{-jwT} \right)$$

$$x(t) = \tfrac{1}{2} \left(e^{jwT} - e^{-jwt} \right)$$

$$x(z) = \tfrac{1}{2} \left(\frac{z}{z - e^{jwT}} - \frac{z}{z - e^{-jwT}} \right)$$

$$= \frac{z}{2} \left(\frac{z - e^{-jwT} - z + e^{jwT}}{z^2 - z \underbrace{\left(e^{jwT} + e^{-jwT} \right)} + 1} \right)$$

$$\boxed{x(t) \xrightarrow{Z.T} \frac{z \sin wT}{z^2 - 2z \cos wT + 1}}$$

Assignment 1

* Obtain Z-Transform of

① $x_1(t) = t\,e^{-2t}$, Assum sampling time $T = 0.2\,sec$

Sol: $t \xrightarrow{Z.T} \frac{TZ}{(Z-1)^2}$

$e^{-2t}\,t \xrightarrow{Z.T} \frac{T(Ze^{2T})}{(Ze^{2T}-1)^2}$ but $T = 0.2\,sec$

$x_1(t) = \frac{0.298\,Z}{(1.49Z-1)^2}$

② $x_2(n) = e^{-3n}\sin\omega n$

Sol: $\sin\omega n \xrightarrow{Z.T} \frac{Z\sin\omega}{Z^2 - 2Z\cos\omega + 1}$

$e^{-3n}\sin\omega n \xrightarrow{Z.T} \frac{(Ze^3)\sin\omega}{(Ze^3)^2 - 2(Ze^3)\cos\omega + 1}$

③ $X_3(n) = 5^n e^{-2n}$

Sol. $e^{-2n} \xrightarrow{Z.T} \frac{Z}{Z - e^{-2}}$

$5^n e^{-2n} \xrightarrow{Z.T} \frac{Z/5}{Z/5 - e^{-2}}$

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④ $X_4(t) = 5\,u(t-2T)$

Sol: $u(t) \xrightarrow{Z.T} \frac{Z}{Z-1}$

$5\,u(t-2T) \xrightarrow{Z.T} 5\,Z^{-2} \cdot \frac{Z}{Z-1} \longrightarrow \frac{5}{Z(Z-1)}$

⑤ $X_5(n) = 7^n \cdot n$

Sol: $n \xrightarrow{Z.T} \frac{z}{(z-1)^2}$

$7^n \cdot n \xrightarrow{Z.T} \frac{(Z/7)}{((Z/7)-1)^2}$

⑥ $X_6(n) = 3\,\delta(n) - 2\,\delta(n-1)$

Sol: $\delta(n) \xrightarrow{Z.T} 1$

$3\delta(n) - 2\delta(n-1) \xrightarrow{Z.T} 3 - 2Z^{-1}$

* Inverse Z-Transform

$X(Z)$	$x(n)$
$\frac{Z}{Z-1}$	$u(n)$
$\frac{Z}{Z-a}$	a^n
$\frac{Z}{Z-e^{-aT}}$	e^{-at}
$\frac{Z}{Z-e^{-a}}$	e^{-an}
$\frac{Z \sin \omega T}{Z^2-2Z\cos \omega T+1}$	$\sin \omega T$
$\frac{Z(Z-\cos \omega T)}{Z^2-2Z\cos \omega T+1}$	$\cos \omega T$

I] Partial fraction Method :

Ex: find the inverse Z-Transform of $X(Z) = \frac{Z}{(Z-1)(Z+2)}$

sol:

$$X(Z) = \frac{Z}{(Z-1)(Z+2)} \Rightarrow \frac{X(Z)}{Z} = \frac{1}{(Z-1)(Z+2)}$$

$$\frac{X(Z)}{Z} = \frac{A}{Z-1} + \frac{B}{Z+2}$$

$$A = \lim_{Z\to 1} \frac{1}{Z+2} = 1/3$$

$$B = \lim_{Z\to -2} \frac{1}{Z-1} = -1/3$$

$$\frac{X(Z)}{Z} = \frac{1/3}{Z-1} + \frac{-1/3}{Z+2}$$

$$X(Z) = 1/3 \frac{Z}{Z-1} - 1/3 \frac{Z}{Z-(-2)}$$

$$x(n) = \frac{1}{3}u(n) - \frac{1}{3}(-2)^n$$

Ex$_2$: find the inverse Z-Transform of $X(Z) = \frac{10Z+5}{(Z-1)(Z+2)}$

Sol: $X(Z) = \frac{10Z+5}{(Z-1)(Z+2)} = \frac{A}{Z-1} + \frac{B}{Z+2}$

$$A = \lim_{Z\to 1} \frac{10Z+5}{Z+2} = 5$$

$$B = \lim_{Z\to -2} \frac{10Z+5}{Z-1} = 5$$

$$X(Z) = \frac{5}{Z-1} + \frac{5}{Z+2} \cdot Z Z^{-1}$$

$$X(Z) = 5Z^{-1}\frac{Z}{Z-1} + 5Z^{-1}\frac{Z}{Z-(-2)}$$

$$x(n) = 5u(n-1) + 5(-2)^{n-1}$$

Ex$_3$: find the inverse Z-Transform of $X(Z) = \frac{1}{(Z-1)^2(Z+2)}$

Sol: $X(Z) = \frac{1}{(Z-1)^2(Z+2)} = \frac{A}{(Z-1)^2} + \frac{B}{Z-1} + \frac{C}{Z+2}$

$$A = \lim_{Z\to 1} \frac{1}{Z+2} = \frac{1}{3}$$

$$B = \lim_{Z\to 1} \frac{d}{dZ}\frac{1}{Z+2} = \lim_{Z\to 1} \frac{-1}{(Z+2)^2} = -\frac{1}{9}$$

$$C = \lim_{Z\to -2} \frac{1}{(Z-1)^2} = \frac{1}{9}$$

$$X(Z) = \frac{1/3}{(Z-1)^2} + \frac{-1/9}{(Z-1)} + \frac{1/9}{(Z+2)} \cdot Z \cdot Z^{-1}$$

$$X(Z) = \frac{1}{3}Z^{-1}\frac{Z}{(Z-1)^2} - \frac{1}{9}Z^{-1}\frac{Z}{Z-1} + \frac{1}{9}Z^{-1}\frac{Z}{Z-(-2)}$$

$$x(n) = \frac{1}{3}(n-1) - \frac{1}{9}u(n-1) + \frac{1}{9}(-2)^{n-1}$$

$\underline{Ex_4}$: Find the inverse Z-Transform of $X(Z) = \dfrac{Z^2+Z+2}{(Z-1)(Z^2-Z+1)}$

$\underline{Sol}$: $X(Z) = \dfrac{Z^2+Z+2}{(Z-1)(Z^2-Z+1)} = \dfrac{A}{(Z-1)} + \dfrac{BZ+C}{Z^2-Z+1}$

$$A = \lim_{Z\to 1} \frac{Z^2+Z+2}{Z^2-Z+1} = 4$$

$\underline{but}$ $\dfrac{Z^2+Z+2}{(Z-1)(Z^2-Z+1)} = \dfrac{A}{Z-1} + \dfrac{BZ+C}{Z^2-Z+1}$

$$\therefore Z^2+Z+2 = A(Z^2-Z+1) + (BZ+C)(Z-1)$$
$$= AZ^2 - AZ + A + BZ^2 - BZ + CZ - C$$
$$= (A+B)Z^2 + (C-A-B)Z + (A-C)$$

* Compare coefficient of both sides :

Coeff Z^2 : $A+B = 1 \Rightarrow A = 4$ then $B = -3$

Coeff Z^0 : $A - C = 2 \Rightarrow A = 4$ then $C = 2$

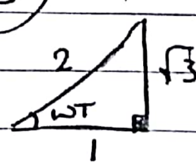
$$\therefore X(Z) = \frac{4}{Z-1} + \frac{-3Z+2}{Z^2-Z+1} \quad Z Z^{-1}$$

$$= 4Z^{-1}\frac{Z}{Z-1} + Z^{-1}\frac{Z(-3Z+2)}{Z^2-Z+1}$$

$$\cos \omega t \underset{Z^{-1}}{\overset{Z.T}{\rightleftarrows}} \frac{Z(Z-\cos \omega T)}{Z^2 - 2Z\cos \omega T + 1}$$

$2\cos \omega T = 1 \quad \therefore \cos \omega T = 1/2$

$\sin \omega T = \sqrt{3}/2$



$$X(Z) = 4Z^{-1}\frac{Z}{Z-1} - 3Z^{-1}\frac{Z(Z-2/3)}{Z^2-Z+1}$$

note
$2/3 = 4/6 = 1/6 + 3/6$
$= 1/6 + 1/2$

$$= 4Z^{-1}\frac{Z}{Z-1} - 3Z^{-1}\frac{Z(Z-1/2-1/6)}{Z^2-Z+1}$$

$$X(Z) = 4Z^{-1}\frac{Z}{Z-1} - 3Z^{-1}\frac{Z(Z-1/2)}{Z^2-Z+1} - \frac{3Z^{-1}Z(-1/6)}{Z^2-Z+1}$$

$$= 4Z^{-1}\frac{Z}{Z-1} - 3Z^{-1}\frac{Z(Z-1/2)}{Z^2-Z+1} + 1/2\, Z^{-1}\frac{Z}{Z^2-Z+1} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

$$\sin \omega t \xrightarrow[Z^{-T}]{Z \cdot T} \frac{Z \sin \omega T}{Z^2 - 2Z\cos\omega T + 1}$$

$$= 4Z^{-1}\frac{Z}{Z-1} - 3Z^{-1}\frac{Z(Z-1/2)}{Z^2-Z+1} + \frac{\sqrt{3}}{2\sqrt{3}}Z^{-1}\frac{Z}{Z^2-Z+1}$$

$$X(t) = 4u(t-T) - 3\cos\omega(t-T) + \frac{1}{\sqrt{3}}\sin\omega(t-T)$$

OR: $X(n) = 4u(n-1) - 3\cos\omega(n-1) + \frac{1}{\sqrt{3}}\sin\omega(n-1)$

2] Direct long Division Method

$$X(Z) = \sum_{n=0}^{\infty} X(n)\, Z^{-n}$$

$$X(Z) = X(0) + X(1)Z^{-1} + X(2)Z^{-2} + \ldots \quad \text{---} (\mathrm{I})$$

EX: find $X(n)$ for $n=0,1,2,3,4$ when $X(Z)$ is given by:

$$X(Z) = \frac{Z^2+2Z}{Z^3+Z^2+Z+1} \times \frac{Z^{-3}}{Z^{-3}}$$

First, rewrite $X(Z)$ as a ratio of polynomial in Z^{-1} as follow:

$$X(Z) = \frac{Z^{-1}+2Z^{-2}}{1+Z^{-1}+Z^{-2}+Z^{-3}}$$

morning glory

$X(z)$

$$z^{-1} + z^{-2} - 2z^{-3} + z^{-5}$$

$$1 + z^{-1} + z^{-2} + z^{-3} \;\Big)\; \cancel{z^{-1}} + 2z^{-2}$$

$$-$$

$$\ominus \cancel{z^{-1}} \overset{\ominus}{+} z^{-2} \overset{\ominus}{+} z^{-3} \overset{\ominus}{+} z^{-4}$$

$$\cancel{z^{-2}} - z^{-3} - z^{-4}$$

$$\ominus \cancel{z^{-2}} \overset{\ominus}{+} z^{-3} \overset{\ominus}{+} z^{-4} \overset{\ominus}{+} z^{-5}$$

$$-2\cancel{z^{-3}} - 2\cancel{z^{-4}} - z^{-5}$$

$$\overset{\oplus}{-} \cancel{2z^{-3}} \overset{\oplus}{-} \cancel{2z^{-4}} \overset{\oplus}{-} 2z^{-5} \overset{\oplus}{-} 2z^{-6}$$

$$z^{-5} + 2z^{-6}$$

$$\therefore X(z) = z^{-1} + z^{-2} - 2z^{-3} + z^{-5} + \cdots \quad \text{---} \quad (\mathrm{II})$$

Compare (I) & (II)

$x(0) = 0$

$x(1) = 1$

$x(2) = 1$

$x(3) = -2$

$x(4) = 0$

* Z-Transform method for Solving Difference Equations:

Consider the linear difference equation

$$y(n) + a_1 y(n-1) + \dots + a_n y(n-N) = b_0 u(n) + b_1 u(n-1) + \dots + b_n u(n-N)$$

where $u(n)$ and $y(n)$ are the systems input and output at nth iteration.

if $Z[y(n)] = Y(z)$

then

$X(n+1), X(n+2) \dots$ and $X(n-1), X(n-2) \dots$

can be expressed in terms of $X(z)$ and initial conditions

	function	Z-Transform
	⋮	
shifting	$X(n+2)$	$z^2 X(z) - z^2 X(0) - z X(1)$
	$X(n+1)$	$z X(z) - z X(0)$
	$X(n)$	$X(z)$
	$X(n-1)$	$z^{-1} X(z)$
	$X(n-2)$	$z^{-2} X(z)$
	⋮	

Ex: Solve the following difference equ. using Z-Transform

$$y(n) + y(n-1) = u(n)$$

Sol.

$$y(n) + y(n-1) = u(n)$$

Take Z-Transform

$$Y(z) + z^{-1} Y(z) = \frac{z}{z-1}$$

$$Y(z)\left[1 + z^{-1}\right] = \frac{z}{z-1}$$

$$Y(z)\left[\frac{z+1}{z}\right] = \frac{z}{z-1}$$

$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z+1)}$$

$$\frac{Y(z)}{z} = \frac{A}{z-1} + \frac{B}{z+1}$$

$$A = \lim_{z \to 1} \frac{z}{z+1} = 1/2$$

$$B = \lim_{z \to -1} \frac{z}{z-1} = +1/2$$

$$\frac{Y(z)}{z} = \frac{1/2}{z-1} + \frac{1/2}{z-(-1)}$$

$$Y(z) = 1/2 \frac{z}{z-1} + 1/2 \frac{z}{z-(-1)}$$

$$\therefore y(n) = 1/2\, u(n) + 1/2 \dots$$

Ex: Solve the following difference equ $y(n+1)+2y(n)=0$ where $y(0)=0.5$.

Sol:

$$y(n+1)+2y(n)=0$$

* Take Z-Transform

$$Z\,Y(Z)-Z\,Y(0)+2\,Y(Z)=0$$

$$Y(Z)\,[Z+2]-0.5\,Z=0$$

$$Y(Z)=\frac{+0.5\,Z}{Z-(-2)}$$

* Take Z-inverse

$$y(n)=\tfrac{1}{2}(-2)^n$$

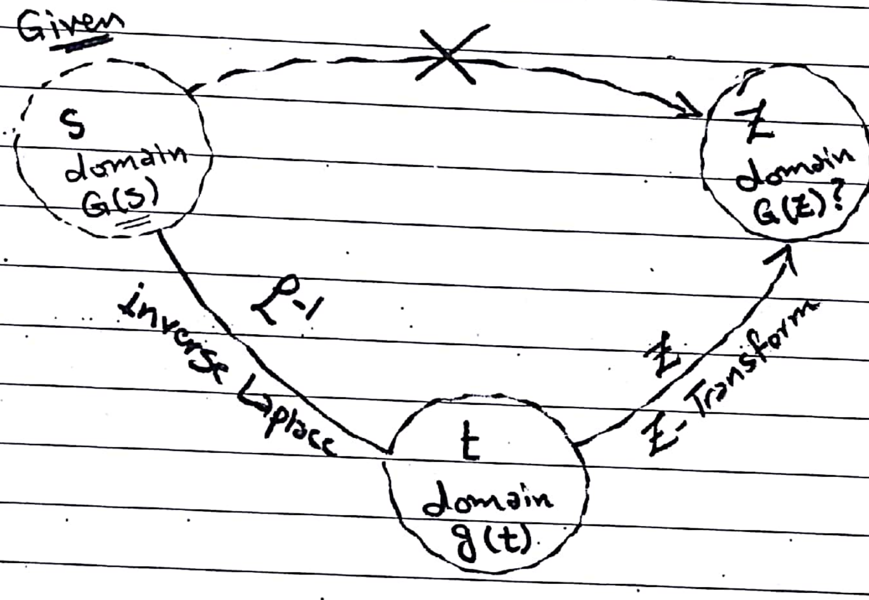
Digital Equivalent Transfer function of Analog system

Given:

Analog T.F $G(s)$.

Required:

Digital Equivalent T.F $G(z)$.



I impulse Invariant Method: required $G(z) = ??$

impulse i/p $\quad \delta(t) \longrightarrow \boxed{G(s)} \longrightarrow \quad y(t) = ??$ impulse response

$$y(t) = \delta(t) * g(t) = g(t)$$ impulse response

$$g(t) = \mathcal{L}^{-1}\, G(s) \quad \text{---} \; (1)$$

$$\therefore \; G(z) = \mathcal{Z}\, g(t) \quad \text{---} \; (2)$$

$$G(z) = \mathcal{Z}\, \mathcal{L}^{-1}\, G(s)$$

z-Transform   Laplace inverse

* Table of Z-Transform

$X(t)$ or $x(n)$	$X(s)$	$X(z)$
$\delta(t)$	1	1
$u(t)$	$\frac{1}{s}$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
$e^{\pm at}$	$\frac{1}{s \mp a}$	$\frac{z}{z - e^{\pm aT}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$\frac{z(z - \cos \omega T)}{z^2 - 2z \cos \omega T + 1}$

Ex: find the impulse response and the digital Transfer function for the following system:

$$G(s) = \frac{1}{s+1}$$

Sol: impulse response:

$$\delta(t) \longrightarrow \boxed{\frac{1}{s+1}} \longrightarrow g(t) = ??$$

$$g(t) = \mathcal{L}^{-1} G(s) = \mathcal{L}^{-1} \frac{1}{s+1} \qquad \therefore g(t) = e^{-t}$$

digital Transfer function:

$$G(z) = \mathcal{Z}\, g(t) = \mathcal{Z}\, e^{-t} \qquad \therefore G(z) = \frac{z}{z - e^{-T}}$$

Ex: Find the impulse response and the digital Transfer function for the following system:

$$G(s) = \frac{1}{s(s+2)}$$

Sol: impulse response:

$$g(t) = \mathcal{L}^{-1} G(s) = \mathcal{L}^{-1} \frac{1}{s(s+2)} = \mathcal{L}^{-1}\left\{ \frac{A}{s} + \frac{B}{s+2} \right\}$$

$$A = \lim_{s \to 0} \frac{1}{s+2} = \frac{1}{2}$$

$$B = \lim_{s \to -2} \frac{1}{s} = -\frac{1}{2}$$

$$\therefore g(t) = \mathcal{L}^{-1}\left\{ \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \frac{1}{s+2} \right\}$$

$$\therefore g(t) = \frac{1}{2} u(t) - \frac{1}{2} e^{-2t}$$

digital Transfer function

$$G(z) = \mathcal{Z}\, g(t) = \mathcal{Z}\left\{ \frac{1}{2} u(t) - \frac{1}{2} e^{-2t} \right\}$$

$$G(z) = \frac{1}{2} \frac{z}{z-1} - \frac{1}{2} \frac{z}{z - e^{-2T}}$$

Note that

For Analog system

Analog i/p $\longrightarrow$ $\boxed{\text{Analog System } G(s)}$ $\longrightarrow$ Analog o/p

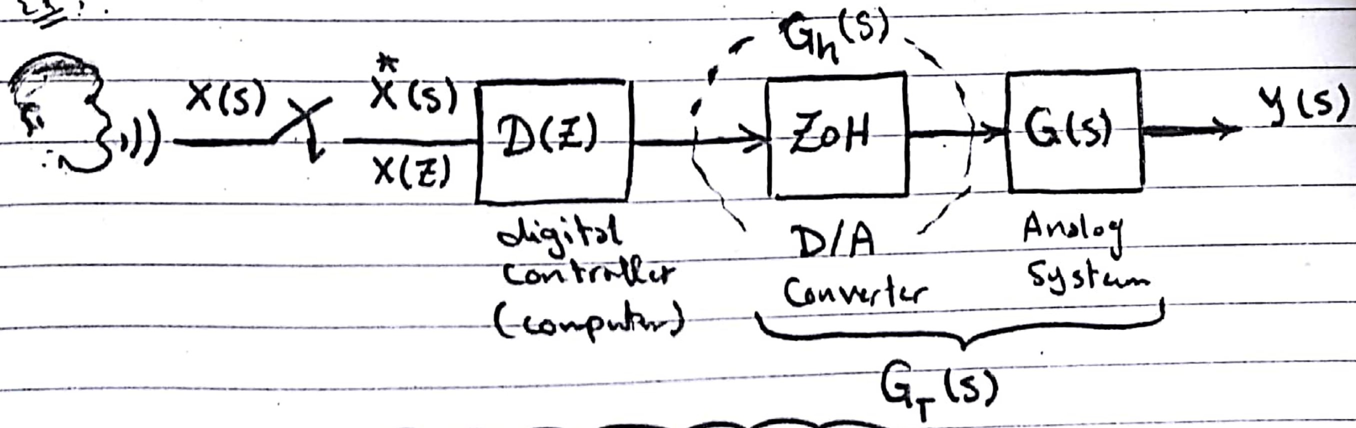
For Discrete system

Discrete i/p $\longrightarrow$ $\boxed{\text{Discrete system } G(Z)}$ $\longrightarrow$ Discrete o/p

Note that

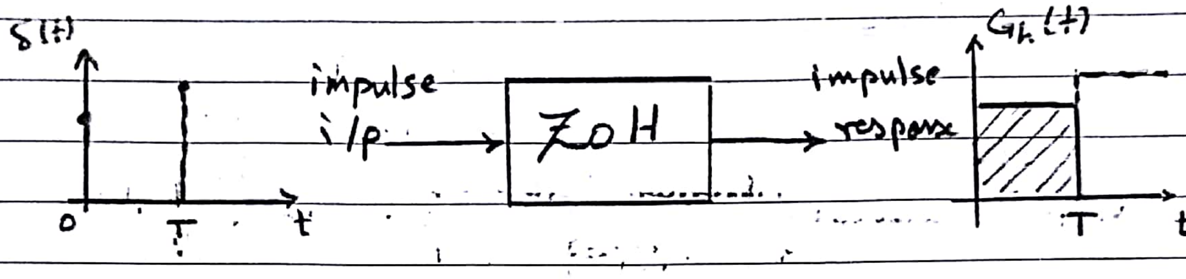
Every Analog T.F should be preconnected by Zero order Hold (ZoH) between the Computer (digital Controller) and the analog T.F.

Ex:

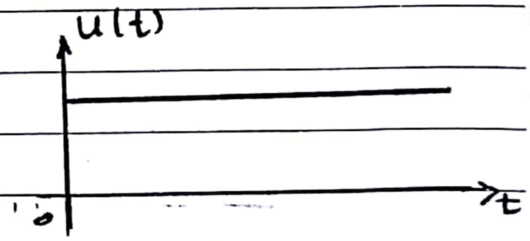


$$\underbrace{G_T(s)}_{} = \underbrace{G_h(s)}_{\text{Required}} \cdot \underbrace{G(s)}_{\text{Given}}$$

Zero order Hold (ZOH)

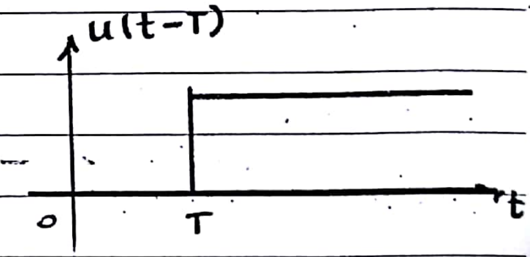


$$G_h(t) = u(t) - u(t-T)$$

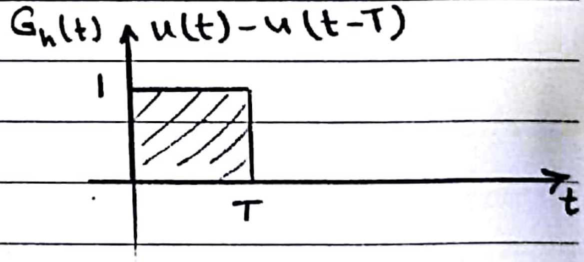


* Take Laplace Transform

$$G_h(s) = \frac{1}{s} - \frac{e^{-sT}}{s}$$



$$\therefore G_h(s) = \frac{1 - e^{-sT}}{s}$$



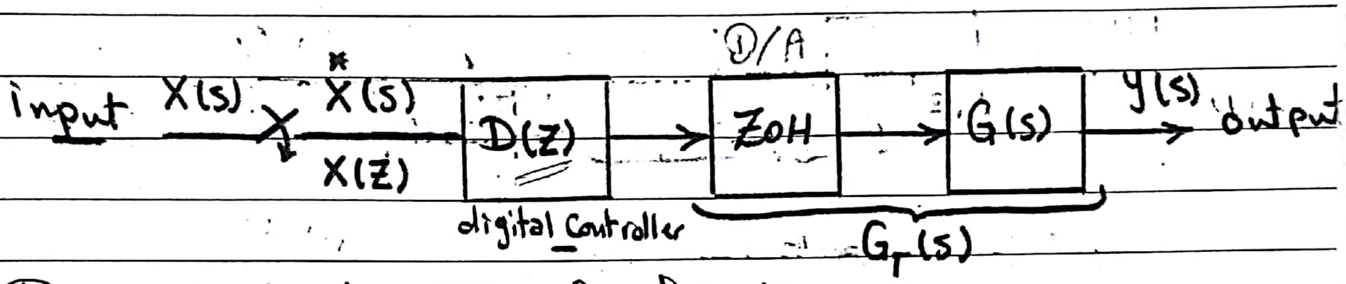
Note

$$X(t) \xrightarrow{L.T} X(s)$$

$$X(t-T) \xrightarrow{L.T} e^{-sT} X(s)$$

* Pulse Transfer function

① open loop system



Required: digital Transfer function

$$T.F = \frac{y(z)}{X(z)} = D(z)\, G_T(z)$$

but:

$$G_T(z) = \mathcal{Z}\, G_T(s) = \mathcal{Z}\, G_h(s) \cdot G(s)$$

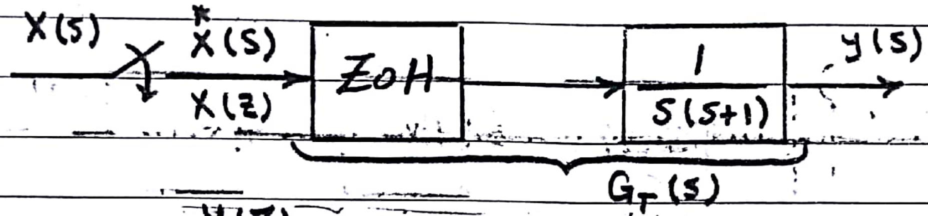
$$= \mathcal{Z}\, \frac{1-e^{-sT}}{s} \cdot G(s)$$

but $z = e^{sT}$

$$G_T(z) = \mathcal{Z}\, \frac{1-z^{-1}}{s}\, G(s)$$

$$\therefore G_T(z) = (1-z^{-1})\, \mathcal{Z}\, \frac{G(s)}{s}$$

Ex1: find the digital Transfer function for the following open Loop system. Assum Sampling time = 1 sec.



Sol

$$T.F = \frac{Y(Z)}{X(Z)} = G_T(Z)$$

$$G_T(Z) = (1 - Z^{-1})\, \mathcal{Z}\, \frac{G(S)}{S}$$

$$= \frac{Z-1}{Z}\, \mathcal{Z}\, \frac{1}{S^2(S+1)} = \frac{Z-1}{Z}\, \mathcal{Z}\left\{\frac{A}{S^2} + \frac{B}{S} + \frac{C}{S+1}\right\}$$

$$A = \lim_{S \to 0} \frac{1}{S+1} = 1$$

$$B = \lim_{S \to 0} \frac{d}{dS}\left(\frac{1}{S+1}\right) = \lim_{S \to 0} \frac{-1}{(S+1)^2} = -1$$

Note: $e^{-1} = 0.36$

$$C = \lim_{S \to -1} \frac{1}{S^2} = 1$$

$$\therefore G_T(Z) = \frac{Z-1}{Z}\, \mathcal{Z}\left(\underbrace{\frac{1}{S^2}}_{t} - \underbrace{\frac{1}{S}}_{u} + \underbrace{\frac{1}{S+1}}_{e^{-t}}\right)$$

$$= \frac{Z-1}{\cancel{Z}}\left(\frac{T\cancel{Z}}{(Z-1)^2} - \frac{\cancel{Z}}{Z-1} + \frac{\cancel{Z}}{Z - e^{-T}}\right)$$

$$G_T(Z) = \frac{1}{Z-1} - 1 + \frac{Z-1}{Z-0.36}$$

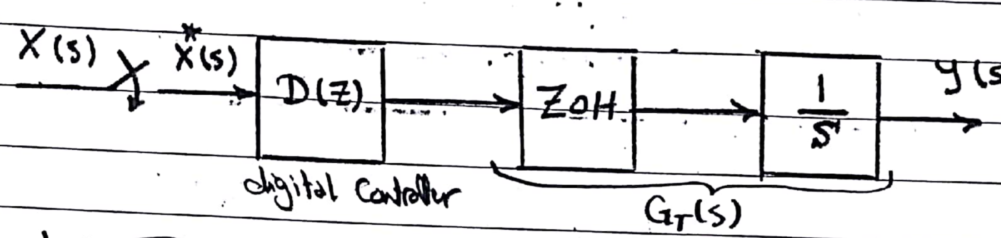
$$= \frac{Z - 0.36 + (Z-1)^2 - (Z-1)(Z-0.36)}{(Z-1)(Z-0.36)}$$

$$= \frac{Z - 0.36 + Z^2 - 2Z + 1 - Z^2 + 1.36Z - 0.36}{Z^2 - 1.36Z + 0.36}$$

$$T.F = G_T(Z) = \frac{0.36Z + 0.28}{Z^2 - 1.36Z + 0.36}$$

Ex: Find the digital Transfer function for the following open Loop system, where $D(z) = \frac{z-1}{z}$, then find the step response, Assume sampling time $T_s = 1\,sec$

Sol:



digital Transfer function:

$$T.f = \frac{y(z)}{x(z)} = D(z) \cdot G_T(z)$$

but:

$$G_T(z) = (1 - z^{-1}) \mathcal{Z} \frac{G(s)}{s}$$

$$\therefore G_T(z) = \frac{z-1}{z} \mathcal{Z} \frac{1}{s^2} \sim t$$

$$G_T(z) = \frac{\cancel{z-1}}{\cancel{z}} \cdot \frac{Tz}{(z-1)^{\cancel{2}}} \qquad T_s = 1\,sec$$

then

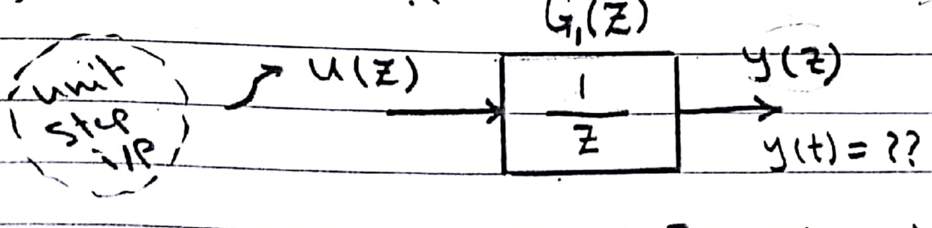
$$G_T(z) = \frac{1}{z-1}$$

$$\therefore T.f = \frac{z-1}{z} \cdot \frac{1}{z-1}$$

$\therefore$ digital Transfer function

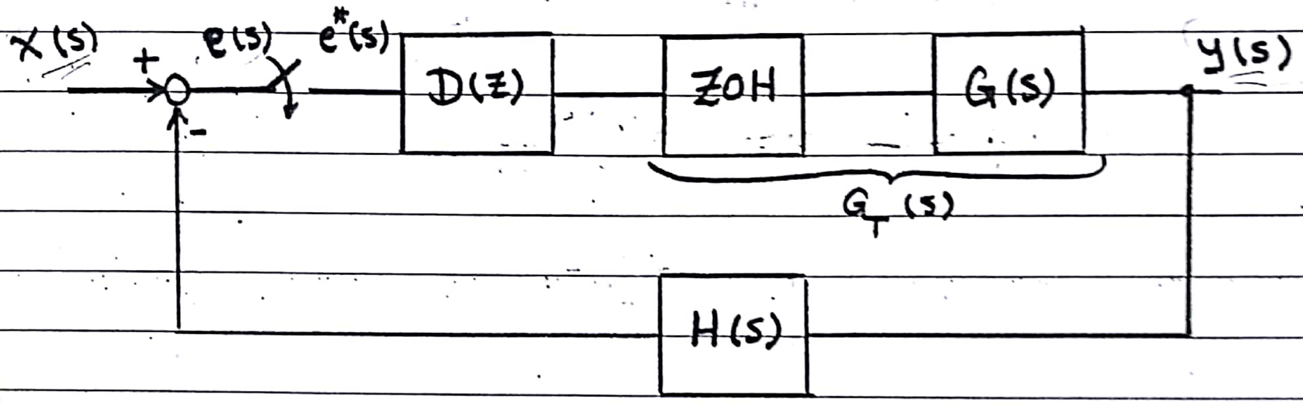
$$T.f = \frac{1}{z}$$

*) Step response = ??



$$y(z) = u(z) \cdot G_1(z) = \frac{z}{z-1} \cdot \frac{1}{z} \Rightarrow y(z) = \frac{1}{z-1}$$

② Closed Loop System:



*) digital Transfer function:

$$T.F = \frac{Y(Z)}{X(Z)} = \frac{D(Z)\, G_T(Z)}{1 + D(Z)\, G_T H(Z)}$$

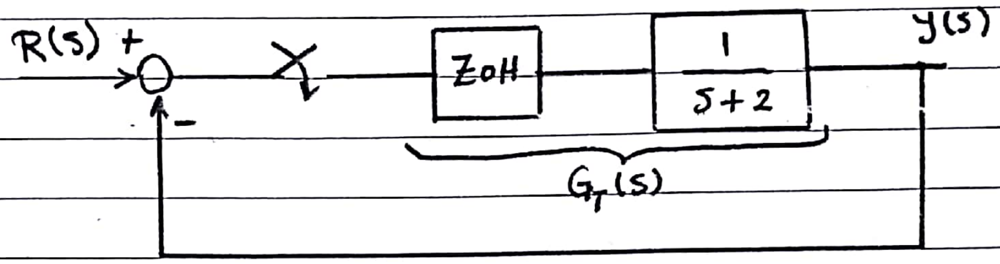
where

$$\boxed{G_T(Z) = (1 - z^{-1})\, \mathcal{Z}\, \frac{G(s)}{s}}$$

AND

$$\boxed{G_T H(Z) = (1 - z^{-1})\, \mathcal{Z}\, \frac{G(s) \cdot H(s)}{s}}$$

Ex: find the digital Transfer function for the following closed Loop system, Assume: sampling time $T_s = 0.5$ sec, then find step response.



Sol:

digital Transfer Function

$$T.f = \frac{y(z)}{R(z)} = \frac{G_T(z)}{1+G_T(z)}$$

but

$$G_T(z) = (1-z^{-1})\, \mathcal{Z}\, \frac{G(s)}{s}$$

$$= \frac{z-1}{z}\, \mathcal{Z}\, \frac{1}{s(s+2)}$$

$$= \frac{z-1}{z}\, \mathcal{Z}\left(\frac{A}{s} + \frac{B}{s+2}\right)$$

$$A = \lim_{s\to 0} \frac{1}{s+2} = \frac{1}{2}$$

$$B = \lim_{s\to -2} \frac{1}{s} = -\frac{1}{2}$$

u $\qquad$ e^{-2t}

$$\therefore\ G_T(z) = \frac{z-1}{z}\, \mathcal{Z}\left(\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s+2}\right)$$

$$= \frac{z-1}{\cancel{z}}\left(\frac{1}{2}\,\frac{\cancel{z}}{z-1} - \frac{1}{2}\,\frac{\cancel{z}}{z-e^{-2T}}\right) \quad \text{but } T_s = \frac{1}{2}$$

$$= \frac{1}{2}\left(1 - \frac{z-1}{z-0.36}\right)$$

$$= \frac{1}{2}\left(\frac{\cancel{z}-0.36-\cancel{z}+1}{z-0.36}\right)$$

then

$$G_T(z) = \frac{0.32}{z - 0.36}$$

but

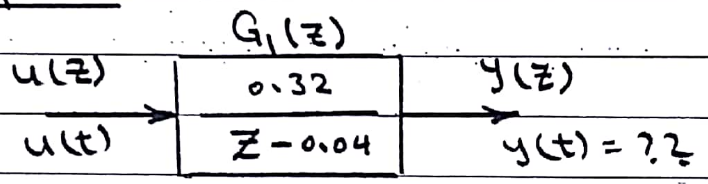
$$T.F = \frac{G_T(z)}{1 + G_T(z)} = \frac{\frac{0.32}{z-0.36}}{1 + \frac{0.32}{z-0.36}} = \frac{0.32}{z - 0.36 + 0.32}$$

then

the digital Transfer function

$$\boxed{T.F = \frac{0.32}{z - 0.04}}$$

* unit step response:



$$y(z) = u(z) \cdot G(z)$$

$$y(z) = \frac{z}{z-1} \cdot \frac{0.32}{z-0.04} = \frac{0.32z}{(z-1)(z-0.04)}$$

$$\frac{y(z)}{z} = \frac{0.32}{(z-1)(z-0.04)} = \frac{A}{z-1} + \frac{B}{z-0.04}$$

$$A = \lim_{z \to 1} \frac{0.32}{z-0.04} = 1/3$$

$$B = \lim_{z \to 0.04} \frac{0.32}{z-1} = -1/3$$

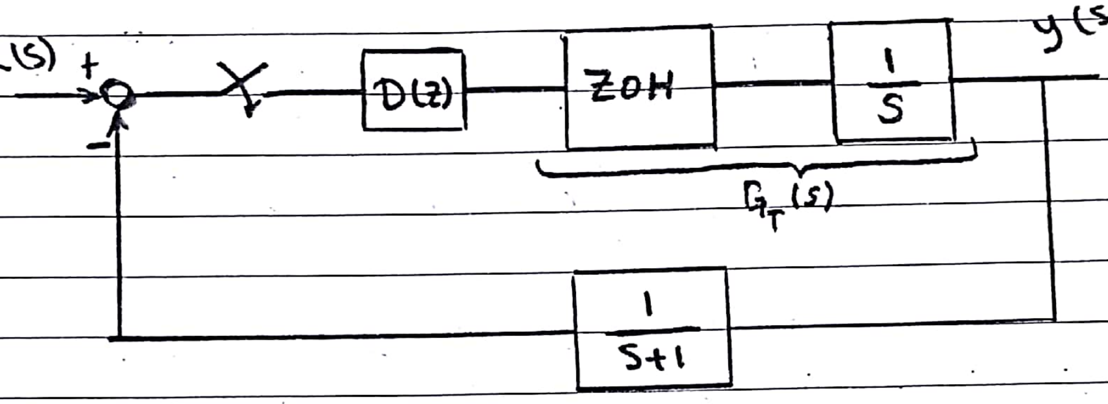
$$y(z) = 1/3 \, \frac{z}{z-1} - 1/3 \, \frac{z}{z-0.04}$$

∴ step response:

$$\boxed{y(n) = 1/3 \, u(n) - 1/3 \, (0.04)^n}$$

morning glory

Ex: find the digital Transfer function for the following closed loop system, where $D(z) = \frac{z-1}{z}$, Assume $T_s = 1$ sec



Sol:

digital Transfer function

$$T.F = \frac{Y(z)}{R(z)} = \frac{D(z) \cdot G_T(z)}{1 + D(z)\, G_T H(z)}$$

$$G_T(z) = (1 - z^{-1})\, \mathcal{Z}\, \frac{G(s)}{s}$$

$$= \frac{z-1}{z}\, \mathcal{Z}\, \frac{1}{s^2} \to t$$

$$= \frac{\cancel{z-1}}{\cancel{z}} \cdot \frac{T\cancel{z}}{(z-1)^{\cancel{2}}}$$

$$\therefore G_T(z) = \frac{1}{z-1} \quad \text{then} \quad D(z) \cdot G_T(z) = \frac{1}{z}$$

$$G_T H(z) = (1 - z^{-1})\, \mathcal{Z}\, \frac{G(s) \cdot H(s)}{s}$$

$$= \frac{z-1}{z}\, \mathcal{Z}\, \frac{1}{s^2(s+1)}$$

$$= \frac{z-1}{z}\, \mathcal{Z}\left(\frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}\right)$$

$$A = \lim_{s \to 0} \frac{1}{s+1} = 1$$

$$B = \lim_{s \to 0} \frac{d}{ds}\left(\frac{1}{s+1}\right) = \lim_{s \to 0} \frac{-1}{(s+1)^2} = -1$$

$$C = \lim_{s \to -1} \frac{1}{s^2} = 1$$

$$\therefore G_1H(z) = \frac{z-1}{z} Z\left(\underbrace{\frac{1}{s^2}}_{t} - \underbrace{\frac{1}{s}}_{u} + \underbrace{\frac{1}{s+1}}_{e^{-t}}\right)$$

$$= \frac{\cancel{z-1}}{\cancel{z}}\left(\frac{T\cancel{z}}{(z-1)^2} - \frac{\cancel{z}}{z-1} + \frac{\cancel{z}}{z-e^{-T}}\right)$$

but $T = 1\,sec$

$$G_TH(z) = \left(\frac{1}{z-1} - 1 + \frac{z-1}{z-0.36}\right)$$

then

$$D(z).G_TH(z) = \frac{z-1}{z}\left(\frac{1}{z-1} - 1 + \frac{z-1}{z-0.36}\right)$$

$$= \left(\frac{1}{z} - \frac{z-1}{z} + \frac{(z-1)^2}{z(z-0.36)}\right)$$

$$= \frac{z-0.36-(z-1)(z-0.36)+(z-1)^2}{z(z-0.36)}$$

$$= \frac{z-0.36-\cancel{z^2}+1.36z-0.36+\cancel{z^2}-2z+1}{z(z-0.36)}$$

$$\boxed{D(z).G_TH(z) = \frac{0.36z+0.28}{z(z-0.36)}}$$

but

$$T.F = \frac{D(z).G_T(z)}{1+D(z)G_TH(z)}$$

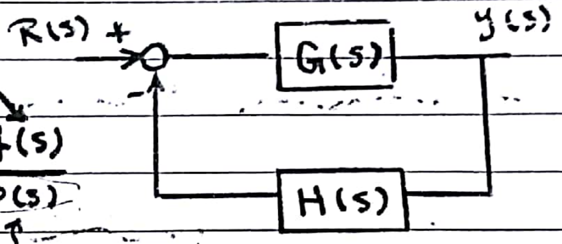
$$T.F = \frac{1/z}{1+\frac{0.36z+0.28}{z(z-0.36)}} = \frac{z-0.36}{z(z-0.36)+0.36z+0.28}$$

$$T.F = \frac{z - 0.36}{z^2 - \cancel{0.36 z} + \cancel{0.36 z} + 0.28}$$

$$T.F = \frac{z - 0.36}{z^2 + 0.28}$$

* stability analysis for Discrete time control

* for Analog system

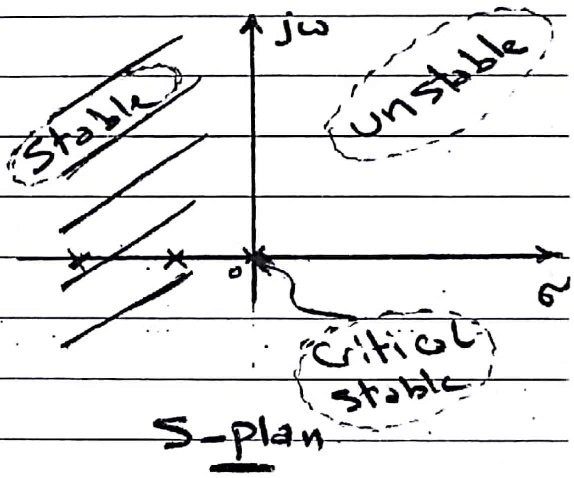


$$T.F = \frac{y(s)}{R(s)} = \frac{G(s)}{1+GH(s)} = \frac{q(s)}{P(s)}$$

zeros → $q(s)$, poles → $P(s)$

then, the stability of the system is given by the c/cs equ:

$$P(s) = 1 + GH(s) = 0$$



but

$$S = \sigma + jw \quad \text{---} (I)$$

for stable $\sigma < 0$

* Mapping from S-plan to Z-plan:

$$Z = e^{ST} \quad \text{---} (II)$$

* from (I) in (II)

$$Z = e^{(\sigma + jw)T}$$

$$= e^{\sigma T} \cdot e^{jwT} = |Z| \angle wT \quad \text{---} (III)$$

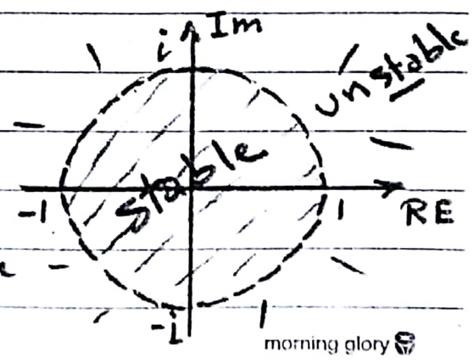
$$= |Z| \angle Z$$

*) the boundary between stable region & unstable region given by:

$$\sigma = 0 \quad \text{---} (IV)$$

* from (IV) in (III);

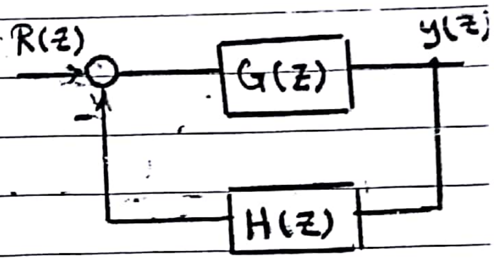
$$Z = |1| \angle wT$$



* for stable: $\sigma < 0$ then system stable inside the unit circle

* Stability analysis of closed Loop System in z-plan:

* for the following closed loop system the transfer function can be given by:



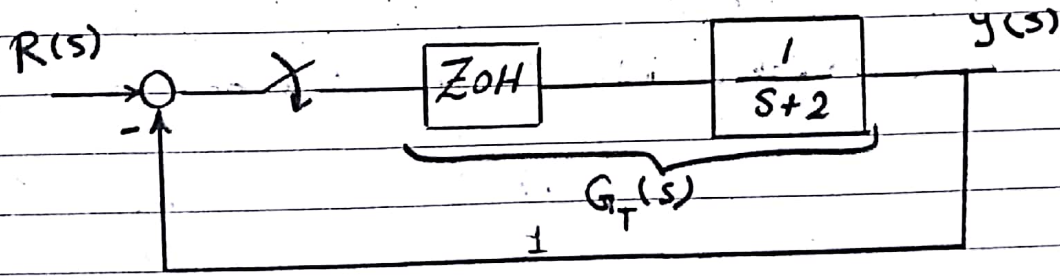
$$T.F = \frac{G(z)}{1+GH(z)} = \frac{q(z)}{P(z)}$$

* The stability of the system defined by the c/cs eqn:

$$\boxed{P(z) = 1 + GH(z) = 0}$$

- if all poles lie within the unit circle in z-plan then → system is stable.
- if any pole lies outside the unit circle in z-plan then → system is unstable.
- if a simple pole lies at $z=1$ or complex conjugate on unit circle then → system is criticle stable.

Ex1: Check the stability for the following closed loop system assume sampling time $T = 0.5$ sec.



Sol:

The digital Transfer function:

$$T.F = \frac{G_T(z)}{1 + G_T(z)}$$

$$G_T(Z) = (1 - Z^{-1})\ \mathcal{Z}\ \frac{G(s)}{s}$$

$$= \frac{Z-1}{Z}\ \mathcal{Z}\ \frac{1}{s(s+2)} = \frac{Z-1}{Z}\ \mathcal{Z}\left(\frac{A}{s} + \frac{B}{s+2}\right)$$

$$A = \lim_{s \to 0} \frac{1}{s+2} = \frac{1}{2}$$

$$B = \lim_{s \to -2} \frac{1}{s} = -\frac{1}{2}$$

$$\therefore G_T(Z) = \frac{Z-1}{Z}\ \mathcal{Z}\left(\frac{1}{2}\,\underbrace{\frac{1}{s}}_{u} - \frac{1}{2}\,\underbrace{\frac{1}{s+2}}_{e^{-2t}}\right)$$

$$G_T(Z) = \frac{Z-1}{Z}\left[\frac{1}{2}\,\frac{Z}{Z-1} - \frac{1}{2}\,\frac{Z}{Z-e^{-2T}}\right]$$

<u>but</u> $T = 0.5$ sec

$$\therefore G_T(Z) = \frac{1}{2}\left[1 - \frac{Z-1}{Z-0.36}\right]$$

$$\therefore G_T(Z) = \frac{1}{2}\left[\frac{Z-0.36-Z+1}{Z-0.36}\right] \qquad \therefore G_T(Z) = \frac{0.32}{Z-0.36}$$

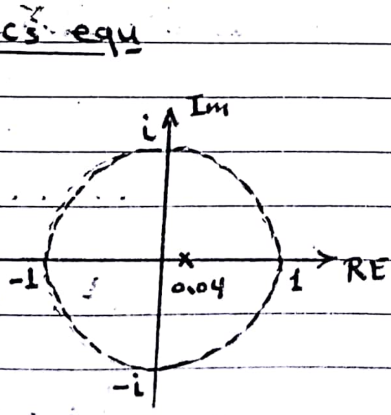
$$T.F = \frac{G_T(Z)}{1+G_T(Z)} = \frac{\frac{0.32}{Z-0.36}}{1+\frac{0.32}{Z-0.36}} = \frac{0.32}{Z-0.36+0.32}$$

$$\therefore T.F = \frac{0.32}{Z-0.04} = \frac{q(Z)}{P(Z)}$$

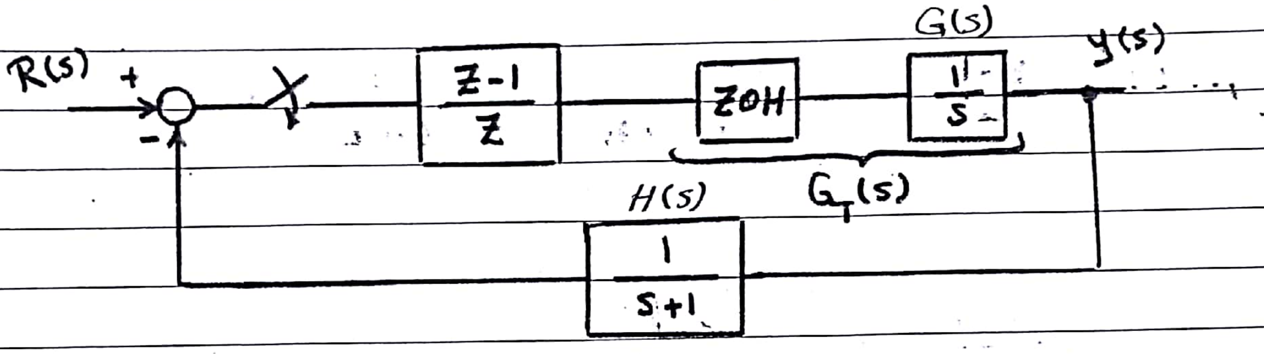
- <u>The stability of the system defined by the Cl.c's equ</u>

$$P(Z) = Z - 0.04 = 0 \qquad \therefore Z = 0.04$$

Since all poles lie inside the unit circle then system is stable.



Ex₂: Check the stability for the following closed loop system, assume sampling time $T = 1$ sec.



Sol: the digital Transfer function:

$$T.F = \frac{Y(z)}{R(z)} = \frac{D(z).G_T(z)}{1+D(z)\,G_TH(z)}$$

where

$$\therefore G_T(z) = (1-z^{-1})\,\mathcal{Z}\,\frac{G(s)}{s}$$

And

$$G_TH(z) = (1-z^{-1})\,\mathcal{Z}\,\frac{G(s).H(s)}{s}$$

$$\therefore G_T(z) = \frac{z-1}{z}\,\mathcal{Z}\,\frac{1}{s^2} \rightarrow t$$

$$= \frac{\cancel{z-1}}{\cancel{z}} \cdot \frac{T\cancel{z}}{(z-1)^{\cancel{2}}}$$

but $T = 1$sec

$$\therefore G_T(z) = \frac{1}{z-1} \Rightarrow D(z).G_T(z) = \frac{z-1}{z} \cdot \frac{1}{z-1}$$

$$\therefore D(z).G_T(z) = 1/z$$

$$G_TH(z) = \frac{z-1}{z}\,\mathcal{Z}\,\frac{1}{s^2(s+1)} = \frac{z-1}{z}\,\mathcal{Z}\left(\frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}\right)$$

$$A = \lim_{s \to 0} \frac{1}{s+1} = 1$$

$$B = \lim_{s \to 0} \frac{d}{ds}\frac{1}{s+1} = \lim_{s \to 0} \frac{-1}{(s+1)^2} = -1$$

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$$C = \lim_{s \to -1} \frac{1}{s^2} = 1$$

$$G_T H(z) = \frac{z-1}{z} \, \mathcal{Z}\left(\overset{t}{\frac{1}{s^2}} - \overset{u}{\frac{1}{s}} + \overset{e^{-t}}{\frac{1}{s+1}} \right)$$

$$G_T H(z) = \frac{z-1}{\cancel{z}} \left(\frac{T\cancel{z}}{(z-1)^2} - \frac{\cancel{z}}{(z-1)} + \frac{\cancel{z}}{z-e^{-T}} \right)$$

but $T = 1\,\text{sec}$

$$G_T H(z) = \left(\frac{1}{z-1} - 1 + \frac{z-1}{z-0.36} \right)$$

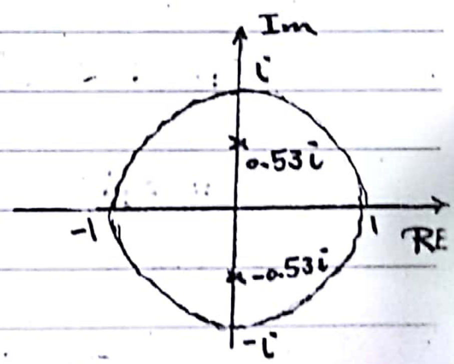
$$\Rightarrow D(z) \cdot G_T H(z) = \frac{z-1}{z} \left(\frac{1}{z-1} - 1 + \frac{z-1}{z-0.36} \right)$$

$$= \frac{1}{z} - \frac{z-1}{z} + \frac{(z-1)^2}{z(z-0.36)}$$

$$= \frac{z-0.36 - (z-1)(z-0.36) + (z-1)^2}{z(z-0.36)}$$

$$= \frac{z - 0.36 - \cancel{z^2} + 1.36z - 0.36 + \cancel{z^2} - 2z + 1}{z(z-0.36)}$$

$$D(z) \cdot G_T H(z) = \frac{0.36z + 0.28}{z(z-0.36)}$$



$$\therefore T.F = \frac{D(z)\, G_T(z)}{1 + D(z)\, G_T H(z)}$$

$$\therefore T.F = \frac{1/z}{1 + \frac{0.36z + 0.28}{z(z-0.36)}} = \frac{z - 0.36}{z(z-0.36) + 0.36z + 0.28}$$

$$= \frac{z - 0.36}{z^2 - \cancel{0.36z} + \cancel{0.36z} + 0.28} = \frac{z - 0.36}{z^2 + 0.28} = \frac{q(z)}{p(z)}$$

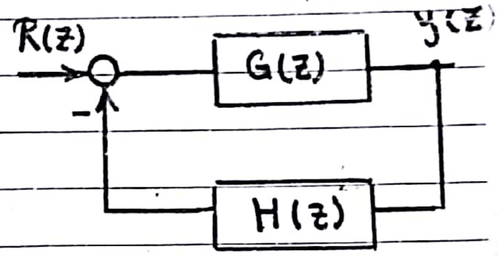
* the stability of the system defined by the c/cs eqn:

$$P(z) = z^2 + 0.28 = 0 \Rightarrow z = \pm 0.53\, i$$

Since all poles lie inside the unit circle then system is stable.

*) Jury Stability Test

- For the following closed loop system the transfer function can be given by:



$$T.F = \frac{G(z)}{1 + GH(z)} = \frac{q(z)}{P(z)}$$

* The stability of the system defined by the C/cs equ:

$$\boxed{P(z) = 1 + GH(z) = 0}$$

* Assume that the C/cs equ $P(z)$ is polynomial in z as follow:

$$\boxed{P(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_{n-1} z + a_n}$$

*) Conditions for stability:

1. $|a_n| < a_0$

2. $P(z)\Big|_{z=1} > 0$

3. $P(z)\Big|_{z=-1}$ → for n even $P(-1) > 0$
   → for n odd $P(-1) < 0$

4. Construct Jury Table

$|b_{n-1}| > |b_0|$

$|c_{n-2}| > |c_0|$

$\vdots$

$|q_2| > |q_0|$ no. of coefficient equal '3' stop

z^0	z^1	z^2	z^3	$\dots$	z^{n-1}	z^n
a_n	a_{n-1}	a_{n-2}	a_{n-3}	$\dots$	a_1	a_0
a_0	a_1	a_2	a_3	$\dots$	a_{n-1}	a_n
b_{n-1}	b_{n-2}	b_{n-3}	$\dots$	b_1	b_0	
b_0	b_1	b_2	$\dots$	b_{n-2}	b_{n-1}	
c_{n-2}	c_{n-3}	$\dots$	c_1	c_0		
c_0	c_1	$\dots$	c_{n-3}	c_{n-2}		
	$\vdots$					
q_2	q_1	q_0				

no. of coefficient equal '3' stop

* iff all conditions satisfied then system is stable.

$$b_{n-1} = \begin{vmatrix} a_n & a_0 \\ a_0 & a_n \end{vmatrix}, \quad b_{n-2} = \begin{vmatrix} a_n & a_1 \\ a_0 & a_{n-1} \end{vmatrix}, \quad \dots, \quad b_0 = \begin{vmatrix} a_n & a_{n-1} \\ a_0 & a_1 \end{vmatrix}$$

$$c_{n-2} = \begin{vmatrix} b_{n-1} & b_0 \\ b_0 & b_{n-1} \end{vmatrix}, \quad c_{n-3} = \begin{vmatrix} b_{n-1} & b_1 \\ b_0 & b_{n-2} \end{vmatrix}, \quad \dots, \quad c_0 = \begin{vmatrix} b_{n-1} & b_{n-2} \\ b_0 & b_1 \end{vmatrix}$$

Ex: Construct the Jury Table and check the stability of the c/cs eqn

$$P(z) = z^4 - 1.2z^3 + 0.07z^2 + 0.3z - 0.08$$

Sol:

+ Conditions of stability

1- $|a_n| < a_0$, $|-0.08| < 1$ Satisfy

2- $P(z)\big|_{z=1} = (1)^4 + 1.2(1)^3 + 0.07(1)^2 + 0.3(1) - 0.08 = 0.09 > 0$ Satisfy

3- $P(z)\big|_{z=-1} = (-1)^4 - 1.2(-1)^3 + 0.07(-1)^2 + 0.3(-1) - 0.08 = 1.89 > 0$

$n = 4$ (even) && $P(z)\big|_{z=-1} > 0$ Satisfy

4- Construct Jury Table:

z^0	z^1	z^2	z^3	z^4
-0.08	0.3	0.07	-1.2	1
1	-1.2	0.07	0.3	-0.08
-0.994	1.176	-0.756	-0.204	
-0.204	-0.756	1.176	-0.994	
0.946	-1.184	0.315		

Coefficients equal '3' stop

$b_3 = \begin{vmatrix} -0.08 & 1 \\ 1 & -0.08 \end{vmatrix}$ $\therefore b_3 = -0.994$, $b_2 = \begin{vmatrix} -0.08 & -1.2 \\ 1 & 0.3 \end{vmatrix}$ $\therefore b_2 = 1.176$

$b_1 = \begin{vmatrix} -0.08 & 0.07 \\ 1 & 0.07 \end{vmatrix}$ $b_1 = -0.756$, $b_0 = \begin{vmatrix} -0.08 & 0.3 \\ 1 & -1.2 \end{vmatrix}$ $\therefore b_0 = -0.204$

Check

$|b_3| > |b_0| \Rightarrow 0.994 > 0.204$ Satisfy

$C_2 = \begin{vmatrix} -0.994 & -0.204 \\ -0.204 & -0.994 \end{vmatrix}$ $C_2 = 0.946$

$C_1 = \begin{vmatrix} -0.994 & -0.756 \\ -0.204 & 1.176 \end{vmatrix}$ $C_1 = -1.184$

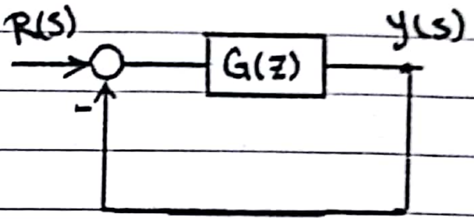
$C_0 = \begin{vmatrix} -0.994 & 1.176 \\ -0.204 & -0.756 \end{vmatrix}$ $C_0 = 0.315$

$|C_2| > |C_0| \Rightarrow 0.946 > 0.315$ satisfy

Since all Conditions satisfied then system is stable

Ex: for the following unity feed back closed loop system find Rang of K for stability Give that the open loop Transfer function of the system is

$$G(z) = \frac{K(0.3679z + 0.2642)}{(z-0.3679)(z-1)}$$



Sol:

the digital Transfer function:

$$T.F = \frac{G(z)}{1+G(z)} = \frac{q(z)}{P(z)}$$

∴ the stability of the system is given by the c/cs equation:

$$P(z) = 1 + G(z) = 0$$

$$\therefore P(z) = 1 + \frac{K(0.3679z + 0.2642)}{(z-0.3679)(z-1)} = 0$$

$$P(z) = (z-0.367)(z-1) + K(0.3679z + 0.2642) = 0$$

$$P(z) = z^2 - 1.367z + 0.367 + 0.3679Kz + 0.2642K = 0$$

$$P(z) = z^2 + (0.3679K - 1.367)z + 0.2642K + 0.367 = 0$$

Conditions of stability

1- $|a_n| < a_0 \Rightarrow |0.2642K + 0.367| < 1$

$$-1 < 0.264K + 0.367 < 1$$

$$-1.367 < 0.264K < 0.63$$

$$-5.1775 < K < 2.3925 \quad (1)$$

2- $P(z)\big|_{z=1} > 0$

$$P(z)\big|_{z=1} = (1)^2 + (0.3679K - 1.367)(1) + 0.2642K + 0.367 > 0$$
$$= \cancel{1} + 0.3679K - \cancel{1.367} + 0.2642K + \cancel{0.367} > 0$$

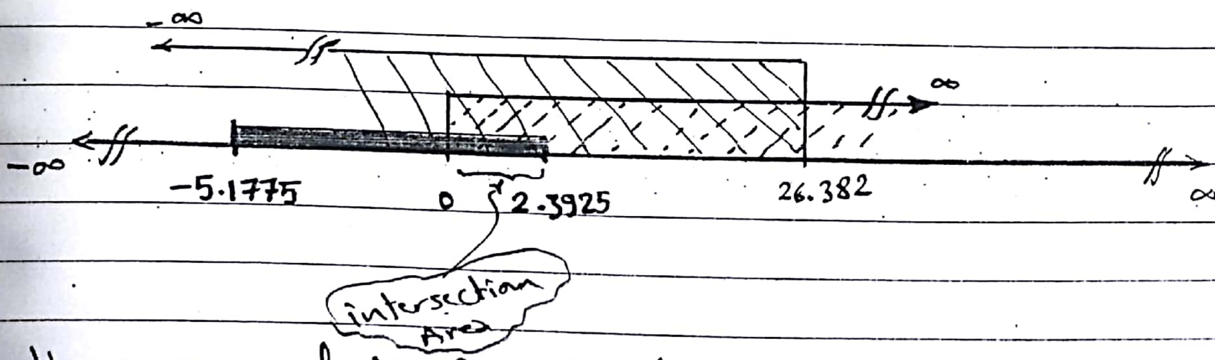
$$K > 0 \quad --- \ (2)$$

3- $P(z)\big|_{z=-1} > 0$ for n even

$$\therefore P(z)\big|_{z=-1} = (-1)^2 + (0.3679K - 1.367)(-1) + 0.2642K + 0.367 > 0$$

$$K < 26.382$$

* we select the Range that satisfy the three conditions (the intersection between them).



then: range of K for stability is given by

$$0 < K < 2.3925$$

Chapter 1

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State Space Representation

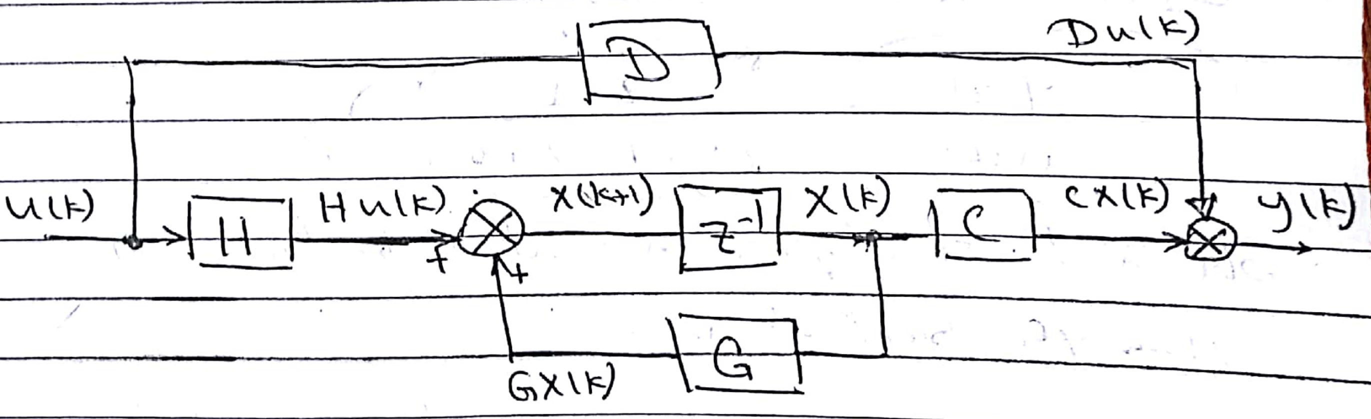
Given :

$$X(k+1) = GX(k) + Hu(k)$$

$$y(k) = CX(k) + Du(k)$$

where

$X(k)$ is the state vector
$Y(k)$ is the out vector
$u(k)$ is the input vector
$G(k)$ is the state matrix
$H(k)$ is the input matrix
$C(k)$ is the output matrix
$D(k)$ is the direct Transmission matrix



Block diagram of State Space

* Unity matrix I $\quad I = \underset{1\times1}{[1]}$, $I = \underset{2\times2}{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}$, $I = \underset{3\times3}{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}$

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II * Canonical Controlable form

* Consider the following

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}} \quad \text{--- (1)}$$

then

$$\underline{X}(k+1) = \overbrace{\left[\begin{array}{c|c} 0 & I \\ \hline -a_n \quad -a_{n-1} & \cdots \; -a_1 \end{array}\right]}^{G} \underline{X}(k) + \overbrace{\left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right]}^{H} u(k)$$

$\underbrace{-a_n \quad -a_{n-1} \quad \cdots \quad -a_1}_{\text{Coefficients}}$

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معكوسة

$$y(k) = \big[\underbrace{b_n - a_n b_0 \; ; \; b_{n-1} - a_{n-1} b_0 \; ; \cdots \; b_1 - a_1 b_0}\big] \, \underline{X}(k) + b_0 u(k)$$

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EX: Consider the following system

$$\frac{Y(z)}{U(z)} = \frac{z + 1}{z^{2} + 1.3z + 0.4}$$

obtain the state space in Contrallable form Then Draw the Block diagram.

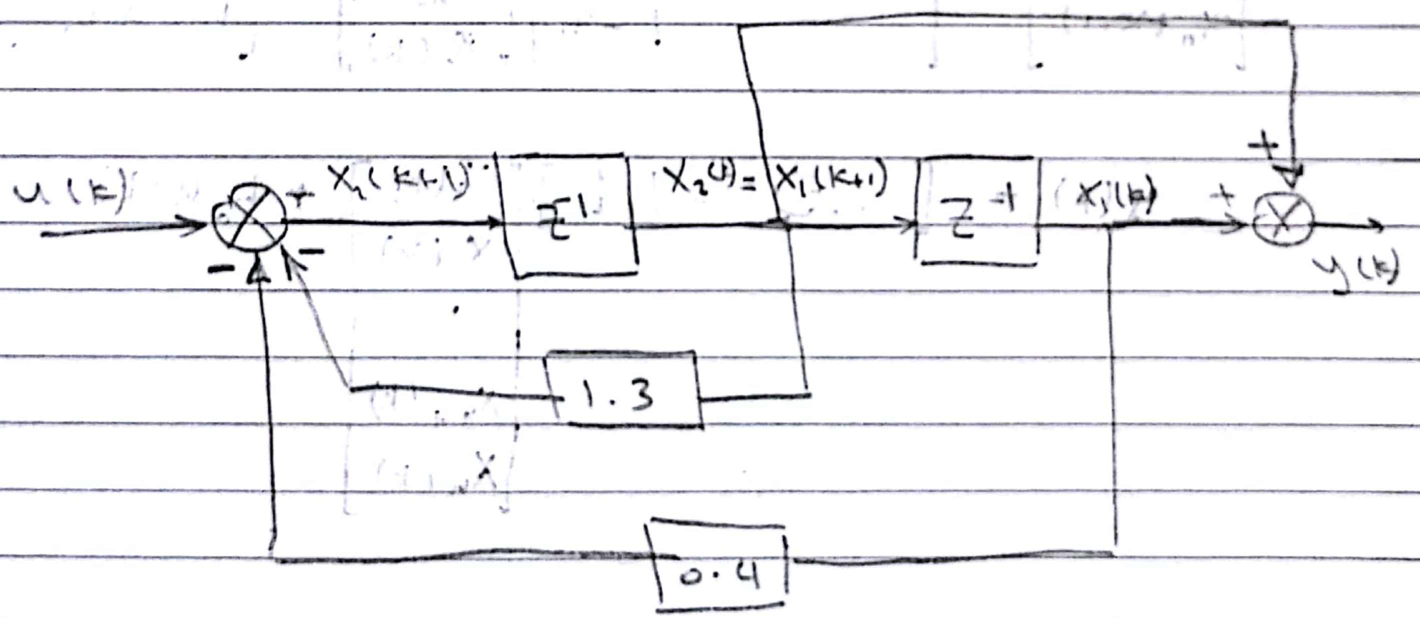
Sol,

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \overset{G}{\begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix}} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \overset{H}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} u(k)$$

I

$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Block diagram of Canonical Controllable form



Note that

no. of delay = order of domenator = matrix order

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2] Observable Canonical Form

* Consider the following:

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

then:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \left[\begin{array}{c:c} 0 & -a_n \\ \hdashline & -a_{n-1} \\ I & \vdots \\ & -a_2 \\ & -a_1 \end{array}\right] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix} u(k)$$

$$y(k) = \left[\begin{array}{c:c} 0 & 1 \end{array}\right] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

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Example: Consider the following system

$$\frac{Y(z)}{U(z)} = \frac{z+1}{z^2 + 1.3z + 0.4}$$

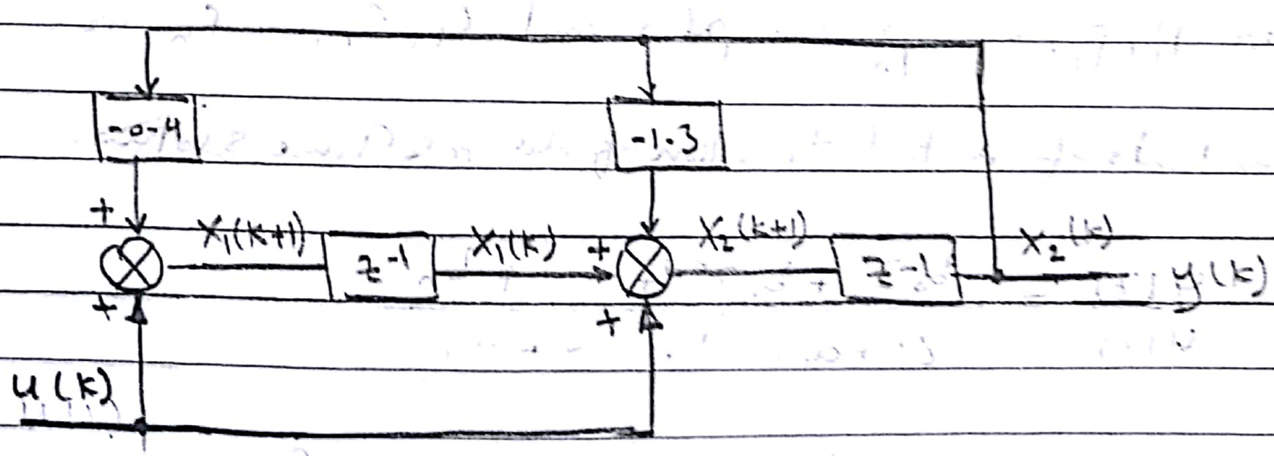
obtain State space in observable form then Draw Block diagram

Sol: order of polynomial = no. of state variables = no. of delays

$$\begin{bmatrix} X_1(k+1) \\ X_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & -0.4 \\ 1 & -1.3 \end{bmatrix} \begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix}$$

Block diagram:



3] Diagonal Canonical form

* Consider the following,

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

then,

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 \\ 0 & 0 & P_{n-1} & 0 \\ 0 & 0 & 0 & P_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

Where $P_1, P_2, \cdots, P_n$ are poles and $C_1, C_2, \cdots C_n$ are constants of partial fraction of discrete Time system.

$$\frac{Y(z)}{U(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{z^n + a_1 z^{n-1} + \cdots + a_n}$$

$$= b_0 + \frac{C_1}{z - P_1} + \frac{C_2}{z - P_2} + \cdots + \frac{C_n}{z - P_n}$$

Example:

Consider the following system

$$\frac{Y(z)}{U(z)} = \frac{z+1}{z^2+1.3z+0.4}$$

obtain state space in diagonal form then draw the Block diagram

Sol:

$$\frac{Y(z)}{U(z)} = \frac{z+1}{z^2+1.3z+0.4} = \frac{A}{z+0.8} + \frac{B}{z+0.5}$$

$$= \frac{5/3}{z+0.5} + \frac{-2/3}{z+0.8}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.8 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 5/3 & -2/3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Block diagram H.W

Pulse Transfer function

Given

$$X(k+1) = G X(k) + H u(k) \quad \text{---} (1)$$

i/p → $u(k)$

$$y(k) = C X(k) + D u(k) \quad \text{---} (2)$$

o/p → $y(k)$

Required

$$T.F = \frac{Y(z)}{U(z)} = ??$$

Taking Z-Transform for both equations (1), (2)

$$Z X(z) - Z X(0) = G X(z) + H U(z) \quad \text{---} (3)$$

$$Y(z) = C X(z) + D U(z) \quad \text{---} (4)$$

for zero initial condition then $X(0) = 0$

$$Z X(z) = G X(z) + H U(z)$$

$$Z X(z) - G X(z) = H U(z)$$

$$X(z)[ZI - G] = H U(z)$$

$$X(z) = (ZI - G)^{-1} H U(z) \quad \text{---} (5)$$

from equ (5) into (4)

$$Y(z) = C (ZI - G)^{-1} H U(z) + D U(z)$$

$$\therefore T.F = \frac{Y(z)}{U(z)} = C[ZI - G]^{-1} H + D$$

#

Example:

Given:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \overset{G}{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \overset{H}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} u(k)$$

$$y(k) = \overset{C}{\begin{bmatrix} 0 & 1 \end{bmatrix}} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

find the pulse Transfer function $\dfrac{Y(z)}{U(z)} = ??$

Solution

$$T.F = C\,[ZI - G]^{-1} H + \cancel{D}$$

$$[ZI - G] = Z \overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} Z & \cancel{Z} - 1 \\ 2 & Z+3 \end{bmatrix}$$

$$[ZI - G]^{-1} = \frac{1}{|ZI - G|} \operatorname{adj} [ZI - G]$$

$$= \frac{1}{Z^2 + 3Z + 2} \begin{bmatrix} Z+3 & 1 \\ -2 & Z \end{bmatrix}$$

$$= \frac{1}{Z^2 + 3Z + 2} \overset{C}{\begin{bmatrix} 0 & 1 \end{bmatrix}} \overbrace{\begin{bmatrix} Z+3 & 1 \\ -2 & Z \end{bmatrix}} \overset{H}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

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$$\therefore T.F = \frac{1}{z^2+3z+2}\begin{bmatrix} -2 & -z \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore T.F = \frac{-z}{z^2+3z+2}$$ #

Discretization of Continuous-Time State Space Equation

Given

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\}$$ "Continuous time" state space equations

Required T = sampling time

$$\left.\begin{aligned} x[(k+1)T] &= G\,x(kT) + H\,u(kT) \\ y(kT) &= C\,x(kT) + D\,u(kT) \end{aligned}\right\}$$ Discrete time state space equations

Discrete time space matrices

$$G = e^{AT} = \mathcal{L}^{-1}[sI - A]^{-1}$$

$$H = \left(\int_0^T e^{A\lambda}\, d\lambda\right) B$$

$$C = C$$

$$D = D$$

Calculations using analog (continuous) time space model

Example: obtain the discrete time state and output equations and the pulse Transfer functions

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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$$= \begin{bmatrix} \frac{1}{2}\lambda + \frac{1}{4}e^{-2\lambda} \\ -\frac{1}{2}e^{-2\lambda} \end{bmatrix}_0^T$$

$$= \begin{bmatrix} \frac{1}{2}T + \frac{1}{4}e^{-2T} \\ -\frac{1}{2}e^{-2T} \end{bmatrix} - \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \end{bmatrix}$$

$$H = \left[\int_0^T e^{A\lambda} d\lambda\right] B = \begin{bmatrix} \frac{1}{2}T + \frac{1}{4}e^{-2T} - \frac{1}{4} \\ -\frac{1}{2}e^{-2T} + \frac{1}{2} \end{bmatrix}$$

Then

$$\underline{X}[(k+1)T] = \begin{bmatrix} 1 & \frac{1}{2} - \frac{1}{2}e^{-2T} \\ 0 & e^{-2T} \end{bmatrix} \underline{X}(kT) + \begin{bmatrix} \frac{1}{2}T + \frac{1}{4}e^{-2T} - \frac{1}{4} \\ -\frac{1}{2}e^{-2T} + \frac{1}{2} \end{bmatrix} u(kT)$$

$$y(kT) = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{X}(kT)$$

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Controllability

Given

$$x(k+1) = \overset{G}{A}\, x(k) + \overset{H}{B}\, u(k)$$

The system is said to be Completely Controllable If and only If The controllability matrix (M) has full Rank.

$$M = [B : AB : A^2B] \quad \text{for } A = 3\times3$$

or $M = [B : AB]$ for $A = 2\times2$

IFF $|M| \neq 0$ then system is Completely Controllable.

Full Rank

Example:

Check the controllability for the following system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ -0.8 \end{bmatrix}}_{B} u(k)$$

Solution:

The controllable matrix $M = [B : AB]$

$$AB = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -0.8 \end{bmatrix} = \underbrace{\begin{bmatrix} -0.8 \\ 0.64 \end{bmatrix}}_{AB}$$

$$|M| = \begin{vmatrix} 1 & -0.8 \\ -0.8 & 0.64 \end{vmatrix} = 0.64 * 1 - (-0.8 * -0.8) = 0$$

Since $|M| = 0$ then system is un controllable

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Special Case

If the matrix A is a diagonal matrix,

then If matrix B has no zero element then system is completely controllable

else If matrix B has a zero element then system is uncontrollable.

Example:

Check the controllability for the following systems

(a)

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{B} u(k)$$

Since matrix A in diagonal form and matrix B has no zero elements

then system is completely controllable.

(b)

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix} u(k)$$

zero element

Since matrix A in diagonal form and matrix B has a zero element

then system is not controllable.

Observability

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Given

$$X(k+1) = \overset{G}{A}\,X(k) + \overset{H}{B}\,u(k)$$

$$y(k) = C\,X(k)$$

The system is said to be completely observable IFF the observability matrix (N) has full Rank

$$N = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}, \quad \text{for } A = 3\times 3$$

or

$$N = \begin{bmatrix} C \\ CA \end{bmatrix}, \quad \text{for } A = 2\times 2$$

Example:

Check the observability of the following system

$$\begin{bmatrix} X_1(k+1) \\ X_2(k+1) \\ X_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} X_1(k) \\ X_2(k) \\ X_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} X_1(k) \\ X_2(k) \\ X_3(k) \end{bmatrix}$$

Note

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

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Solution

the observable matrix $N = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$

$$CA = \underset{1\times3}{\begin{bmatrix} 4 & 5 & 1 \end{bmatrix}} \underset{3\times3}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}} = \underset{1\times3}{\overbrace{\begin{bmatrix} -6 & -7 & -1 \end{bmatrix}}^{CA}}$$

$$CA^2 = CA \cdot A = \underset{1\times3}{\begin{bmatrix} -6 & -7 & -1 \end{bmatrix}} \underset{3\times3}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}} = \underset{1\times3}{\overbrace{\begin{bmatrix} 6 & 5 & -1 \end{bmatrix}}^{CA^2}}$$

$$N = \begin{bmatrix} 4 & 5 & 1 \\ -6 & -7 & -1 \\ 6 & 5 & -1 \end{bmatrix}$$

$$|N| = 4(7+5) - 5(6+6) + 1(-30+42)$$

$$= 48 - 60 + 12 = 0$$

Since $|N| = 0$ then system is not observable

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Special Case

If the matrix A is a diagonal matrix.

then If matrix C has no zero elements

then system is completely observable

else If matrix C has a zero element

then system is unobservable

Example:

Check the observability for the following system

(a)
$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \overset{A}{\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \overset{B}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} u(k)$$

$$y(k) = \overset{C}{\begin{bmatrix} 1 & 5 \end{bmatrix}} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Since matrix A in diagonal form and matrix C has no zero elements then system is completely observable.

Design Via Pole Placement

Given:

$$x(k+1) = Ax(k) + Bu(k)$$

Required:

Design a state feedback matrix (K) [$u = -Kx(k)$] such that the desired eign values (closed loop poles are $\mu_1, \mu_2, ---, \mu_n$).

Method 1

A) using Transformation Method : "system in canonical controllable form"

* step 1: check the controllability

$$M = [\underbrace{B \quad AB}_{\text{for } (2\times 2)} \quad A^2B] \text{ for } (3\times 3)$$

- iff the system is completely controllable go to Next step.

* Step 2: Calculate the characteristic equation for the original system.

$$|ZI - A| = 0$$

$$Z^n + a_1 Z^{n-1} + --- + a_{n-1} Z + a_n = 0$$

Get $a_1, a_2, ---, a_n$

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* Step 3: write desired characteristic equation

$$(Z-\mu_1)(Z-\mu_2)\cdots(Z-\mu_n)=0$$

$$Z^n+\alpha_1 Z^{n-1}+\alpha_2 Z^{n-2}+\cdots+\alpha_{n-1} Z+\alpha_n=0$$

Get $\alpha_1, \alpha_2, \ldots, \alpha_n$

* Step 4: Calculate T

$$W=\begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & 1 \\ a_{n-2} & a_{n-3} & \cdots 1 & 0 \\ \vdots & & & \vdots \\ a_1 & & & 0 \\ 1 & & & 0 \end{bmatrix}$$

$$T=MW$$

Important Note:

If the system is in Canonical Controllable form:

$$T=I \quad , \quad T^{-1}=I$$

* Step 5:

$$K=[\alpha_n-a_n \vdots \alpha_{n-1}-a_{n-1} \vdots \cdots \alpha_1-a_1]T^{-1}$$

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Example:

Given $X(k+1) = G X(k) + H u(k)$

where $G = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}$, $H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Design a state feedback matrix K ($u = -KX$) such that the desired closed loop poles are:

$$Z_1 = 0.5 + j0.5 \quad , \quad Z_2 = 0.5 - j0.5$$

Solution:

(1) Using Transformation method:

Step 1: check controllability

$$M = [B \quad AB]$$

$$AB = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\therefore M = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \qquad \therefore |M| = -1 \neq 0$$

$\therefore$ System is completely controllable

Step 2: Characteristic equation of original system

$$|ZI - A| = 0$$

$$\begin{vmatrix} Z & -1 \\ 0.16 & Z+1 \end{vmatrix} = 0$$

$$Z^2 + Z + 0.16 = 0$$

$a_1 Z$

$a_1 = 1$, $a_2 = 0.16$

Step3: characteristic equation of desired poles

$$(z-\mu_1)(z-\mu_2)=0$$

$$(z-(0.5+j0.5))(z-(0.5-j0.5))=0$$

$$(z-0.5-j0.5)(z-0.5+j0.5)=0$$

$$z^2-0.5z+\cancel{j0.5z}-0.5z+0.25-\cancel{0.25j}-\cancel{j0.5z}+\cancel{0.25j}+0.25=0$$

$$z^2-z+0.5=0$$

$$\alpha_1=-1, \quad \alpha_2=0.5$$

Step4: Calculate T,

but System in canonical controllable form

then $T = I$ & $T^{-1} = I$

Step5:

$$K=[\alpha_2-a_2 \quad \alpha_1-a_1]$$

$$K=[0.34 \quad -2].$$

B) Direct Method :

- If the order n of the system is low then substitute $K = [\underbrace{k_1 : k_2 : \dots : k_n}_{for\ (1 \times n)}]$ into charachteristic equation:

$$|zI - \underline{G + HK}| = 0$$

then: compare the coefficients of powers in z of this characteristic equation with the desired characteristic equation

$$z^n + \alpha_1 z^{n-1} + \alpha_2 z^{n-2} + \dots + \alpha_n = 0$$

Example:

Given:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}}^{G} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{H} u(k)$$

Design a state feedback matrix K ($u = -Kx$) such that the desired closed loop poles are

$$z_{1,2} = 0.5 \pm j0.5$$

Solution:

$$K = [k_1 \quad k_2]$$

Step 1: check controllability (as before)

$$|M| = -1$$

Step 2:

The characteristic equation:

$$|zI - G + HK| = 0$$

$$\left| \begin{vmatrix} z & -1 \\ 0.16 & z+1 \end{vmatrix} + \underset{2\times 1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \underset{1\times 2}{\begin{bmatrix} K_1 & K_2 \end{bmatrix}} \right| = 0$$

$$\left| \begin{vmatrix} z & -1 \\ 0.16 & z+1 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ K_1 & K_2 \end{vmatrix} \right| = 0$$

$$\begin{vmatrix} z & -1 \\ K_1+0.16 & K_2+z+1 \end{vmatrix} = 0$$

$$z^2 + (K_2+1)z + (K_1+0.16) = 0 \quad \text{—— I}$$

Step 3:

Characteristic equation of desired poles

$$(z-\mu_1)(z-\mu_2) = 0$$

$$z^2 - z + 0.5 = 0 \quad \text{—— II}$$

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Compare (I) & (II)

$k_2 + 1 = -1$ $\boxed{k_2 = -2}$

$k_1 + 0.16 = 0.5$ $\boxed{k_1 = 0.34}$

$K = [0.34 \quad -2]$