Discrete Control System

Chapter 1 : Z - Transform

$$\boxed{X(z) = \sum_{n=0}^{\infty} X(nT)\, Z^{-n}}$$

, where $n = 0, 1, 2, \dots$

T is sampling time

~~for~~ Sequence $\Rightarrow$ $T_s = 1\ sec$

$$X(z) = \sum_{n=0}^{\infty} x(n)\, Z^{-n}$$

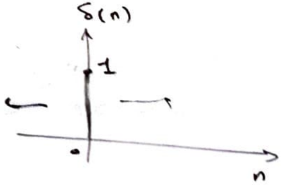
Real

Given

* Z-Transform for Some elementary functions

① Impulse function:

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



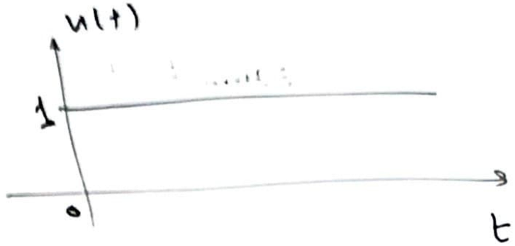
$$\delta(z) = \sum_{n=0}^{\infty} \delta(n)\, Z^{-n}$$

$$= \sum_{n=0}^{\infty} 1 \cdot Z^{-n} = 1$$

$$\boxed{\delta(n) \xrightarrow{Z \cdot T} 1}$$

① unit step:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$u(z) = \sum_{n=0}^{\infty} 1 \cdot z^{-n}$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots$$

$$= \underbrace{1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots}_{\text{Geometric series}}$$

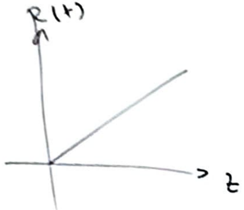
$$\text{Sum} = \frac{1}{1 - \text{base}}$$

$$= \frac{1}{1 - 1/z} = \frac{z}{z-1}$$

$$u(k) \xrightarrow{Z.T} \frac{z}{z-1}$$

3) Ramp function:

$$R(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$u(t) \xrightarrow{Z.T} \frac{z}{z-1}$$

$$R(t) = t\,u(t)$$

$$t\,u(t) \longrightarrow -zT \frac{d}{dz} \frac{z}{z-1}$$

$$-z.T \cdot \frac{(z-1)-z}{(z-1)^2}$$

$$-zT \frac{\cancel{z}-1-\cancel{z}}{(z-1)^2}$$

$$R(t) \xrightarrow{z.T} \frac{zT}{(z-1)^2}$$

4) Polynomial function:

$$x(n) = \begin{cases} a^n & , n = 0, 1, 2 \dots \\ 0 & , n < 0 \end{cases}$$

$$u(n) \xrightarrow{Z-T} \frac{z}{z-1}$$

$$x(n) = a^n u(n)$$

$$x(n) \xrightarrow{Z.T} \frac{z/a}{z/a - 1}$$

$$X(n) \xleftarrow{Z.T} \frac{z}{z-a}$$

5 exponential function:

$$x(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$u(t) \xrightarrow{z.T} \frac{z}{z-1}$$

$$e^{-at} = e^{-at}\, u(t)$$

$$X(z) \xrightarrow{z.T} \frac{z\, e^{aT}}{z e^{aT} - 1}$$

$$X(z) \xrightarrow{z-T} \frac{z}{z - e^{-aT}}$$

⑥ Sinusoidal function:

$$x(t) = \begin{cases} \sin wt & t > 0 \\ 0 & t < 0 \end{cases}$$

$$e^{jwt} = \cos wt + j \sin wt$$

$$e^{-jwt} = \cos wt - j \sin wt$$

So,

$$\cos wt = \frac{1}{2}\left(e^{jwt} + e^{-jwt}\right)$$

$$\sin wt = \frac{1}{2j}\left(e^{jwt} - e^{-jwt}\right)$$

$$x(t) = \frac{1}{2j}\left(e^{jwt} - e^{-jwt}\right)$$

$$X(z) = \frac{z}{2j}\left[\frac{1}{(z - e^{jwT})} - \frac{1}{(z - e^{-jwT})}\right]$$

$$= \frac{z}{2j}\left[\frac{\cancel{z} - e^{-jwT} - \cancel{z} + e^{jwT}}{z^2 - z e^{-jwT} - z e^{jwT} + 1}\right]$$

$$= \frac{z}{2j} \frac{e^{jwT} - e^{-jwT}}{z^2 - z\left(e^{-jwT} + e^{jwT}\right) + 1}$$

$$x(t) \xrightarrow{Z.T} \frac{z \sin wT}{z^2 - 2z \cos wT + 1}$$

$$x(t) = \begin{cases} \cos wt & t > 0 \\ 0 & t < 0 \end{cases}$$

Inverse Z-Transform:

① Partial fraction method:

Ex: find the inverse Z-Transform of $X(z) = \dfrac{z}{(z+1)(z+2)}$

Sol:

$$X(z) = \frac{z}{(z+1)(z+2)}$$

$$\frac{X(z)}{z} = \frac{1}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$$

$$A = \lim_{z \to -1} \frac{1}{z+2} = 1$$

$$B = \lim_{z \to -2} \frac{1}{z+1} = -1$$

Inverse Z-Transform:

① Partial fraction method:

Eg: find the inverse Z-Transform of $X(z) = \dfrac{z}{(z+1)(z+2)}$

Sol:

$$X(z) = \frac{z}{(z+1)(z+2)}$$

$$\frac{X(z)}{z} = \frac{1}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$$

$$A = \lim_{z \to -1} \frac{1}{z+2} = 1$$

$$B = \lim_{z \to -2} \frac{1}{z+1} = -1$$

$$X(z) = \frac{z}{z+1} - \frac{z}{z+2}$$

$$X(z) = \frac{z}{z-(-1)} - \frac{z}{z-(-2)}$$

$$x(n) = (-1)^n - (-2)^n$$

Inverse Z-Transform:

① Partial fraction method:

Ex: find the inverse Z-Transform of $X(z) = \dfrac{10z+5}{(z-1)(z+2)}$

Sol:

$$X(z) = \frac{10z+5}{(z-1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2}$$

$$A = \lim_{z\to 1} \frac{10z+5}{z+2} = 5$$

$$B = \lim_{z\to -2} \frac{10z+5}{z-1} = 5$$

$$X(z) = \frac{5}{z-1} + \frac{5}{z+2} \cdot z\, z^{-1}$$

$$= 5z^{-1}\frac{z}{z-1} + 5z^{-1}\frac{z}{z-(-2)}$$

$$\boxed{x(n) = 5u(n-1) + 5(-2)^{n-1}}$$

Inverse Z-Transform:

① partial fraction method:

Ex: find the inverse Z-Transform of $X(z) = \dfrac{1}{(z-1)^2(z+2)}$

Sol:

$$X(z) = \frac{1}{(z-1)^2(z+2)} = \frac{A}{(z-1)^2} + \frac{B}{z-1} + \frac{C}{z+2}$$

$$A = \lim_{z\to 1} \frac{1}{z+2} = 1/3$$

$$B = \lim_{z\to 1} \frac{d}{dz} \frac{1}{(z+2)}$$

$$= \lim_{z\to 1} \frac{-1}{(z+2)^2} = -1/9$$

$$C = \lim_{z\to -2} \frac{1}{(z-1)^2} = 1/9$$

$$X(z) = 1/3 \frac{1}{(z-1)^2} - 1/9 \frac{1}{z-1} + 1/9 \frac{1}{z+2} \quad z.z^{-1}$$

$$= 1/3 z^{-1} \frac{z}{(z-1)^2} - 1/9 z^{-1} \frac{z}{z-1} + 1/9 z^{-1} \frac{z}{z-(-2)}$$

$$x(n) = -1/3 (n-1) - 1/9 u(n-1) + 1/9 (-2)^{n-1}$$

① Direct long Division

$$x(z) = \sum_{n=0}^{\infty} x(n) z^{-n} \quad \text{—— (I)}$$

Ex. find $x(n)$ for $n = 0, 1, 2, 3, 4$ when $X(z)$ is given by

$$X(z) = \frac{z^2 + 2z}{z^3 + z^2 + z + 1} * \frac{z^{-3}}{z^{-3}}$$

$$= \frac{z^{-1} + 2z^{-2}}{1 + z^{-1} + z^{-2} + z^{-3}}$$

$$X(z) = \underline{X(0)} + \underline{X(1)} z^{-1} + \underline{X(2)} z^{-2} + \underline{X(3)} z^{-3} + \underline{X(4)} z^{-4} \quad \text{—— (II)}$$

$$z^{-1} + z^{-2} - 2z^{-3} + z^{-5} \quad \text{—— (III)}$$

$$\begin{array}{r|l}
1 + z^{-1} + z^{-2} + z^{-3} & \cancel{z^{-1}} + 2z^{-2} \\
& \ominus \cancel{z^{-1}} \pm z^{-2} \pm z^{-3} \pm z^{-4} \\ \hline
& \cancel{z^{-2}} - z^{-3} - z^{-4} \\
& \ominus \cancel{z^{-2}} \pm z^{-3} \pm z^{-4} \pm z^{-5} \\ \hline
& -2\cancel{z^{-3}} - 2\cancel{z^{-4}} - z^{-5} \\
& \oplus -2\cancel{z^{-3}} \oplus -2\cancel{z^{-4}} \oplus -2z^{-5} \oplus -2z^{-6} \\ \hline
& z^{-5} + 2z^{-6}
\end{array}$$

Compare II & III

$X(0) = 0$

$X(1) = 1$

$X(2) = 1$

$X(3) = -2$

$X(4) = 0$