Math 42 HW 7

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1

I found interesting the power and capabilities of spreadsheet tools such as excel. Specifically, the optimization of profit for market demand seems to be a complex problem that would need to be done by an individual highly trained in mathematics and programming, but excel makes it a much more simple task.

2

I found interesting the metric used to measure the data density of a display and how dense some can get. Just as I find it interesting the limit to minimizing the ink on a graph, I also am interested how many data points is too much on a model. I do really like the idea of putting related graphs into one large model to display a lot of information in a smaller space, and will explore more how to do this using seaborn.

3

a

$$x' = sin(x)$$

$$0 = sin(x)$$

$$x = n\pi$$

Thus the fixed points are $x = n\pi$ where n is an integer.

$$(sin(x))' = cos(x)$$

$$cos(0) = 1$$

$$cos(\pi) = -1$$

$$cos(2\pi) = 1$$

$$cos(3\pi) = -1$$

$$\vdots$$

The fixed points are unstable when n is 0 or an even integer and the fixed points are stable when n is an odd integer.

$$x' = rx(1 - \frac{x}{\pi})$$

$$0 = x(1 - \frac{x}{\pi})$$

$$x = 0, \pi$$

Thus the fixed points are $x = 0, \pi$.

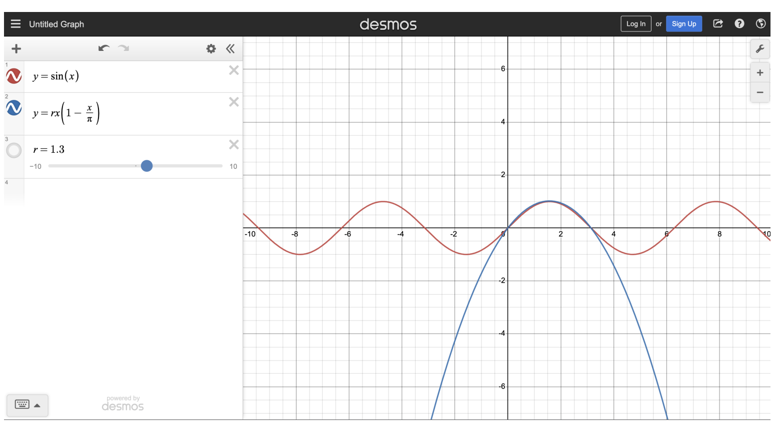
$$(rx(1 - \frac{x}{\pi}))' = r(1 - \frac{2x}{\pi})$$

$$r(1 - \frac{2(0)}{\pi}) = r$$

$$r(1 - \frac{2(\pi)}{\pi}) = r(1 - \pi)$$

The fixed point $x = 0$ is unstable while the fixed point $x = \pi$ is stable. The two differentials share the fixed points $x = 0$ and $x = \pi$ and thus their graphs become very close to each other at those points.

b



The two graphs share the same shape and are nearly identical when r = 1.3 between the points $x = 0$ and $x = \pi$, the two fixed points that they share.

4

a

$$N' = -aN\ln(bN)$$
$$0 = -aN\ln(bN)$$
$$N = \frac{1}{b}$$

The fixed point $N = \frac{1}{b}$

$$(-aN\ln(bN))' = -a(\ln(bN) + 1)$$
$$-a(\ln(b(\frac{1}{b})) + 1) = -a$$

The fixed point at $x = \frac{1}{b}$ is stable.

b

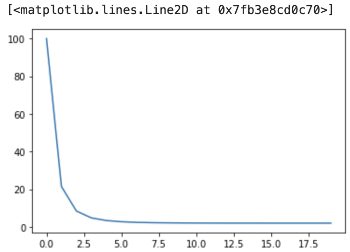
c

$a = 0.5, b = 0.5, N(0) = 100$

```
[87]: def tumor_growth(initial, a, b, t):
          c = np.log(np.log(initial * b))
          return np.exp(np.exp(-a*t + c)) / b
```

```
[111]: tumor_model_1 = []
       for i in range(20):
           tumor_model_1.append(tumor_growth(100, 0.5, 0.5, i))
       plt.plot(list(range(0, 20)), tumor_model_1)
```

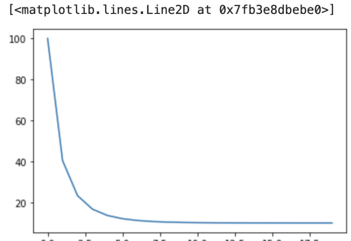
```
[111]: [<matplotlib.lines.Line2D at 0x7fb3e8cd0c70>]
```



$a = 0.5, b = 0.1, N(0) = 100$

```
[112]: tumor_model_2 = []
       for i in range(20):
           tumor_model_2.append(tumor_growth(100, 0.5, 0.1, i))
       plt.plot(list(range(0, 20)), tumor_model_2)
```

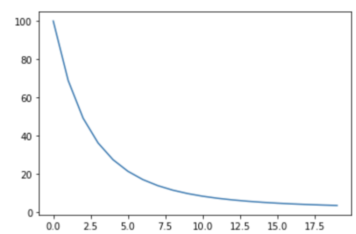
```
[112]: [<matplotlib.lines.Line2D at 0x7fb3e8dbebe0>]
```



$a = 0.1, b = 0.5, N(0) = 100$

```
[113]: tumor_model_3 = []
       for i in range(20):
           tumor_model_3.append(tumor_growth(100, 0.1, 0.5, i))
       plt.plot(list(range(0, 20)), tumor_model_3)
```

```
[113]: [<matplotlib.lines.Line2D at 0x7fb3e8e54670>]
```



d

This model seems to disagree with most models of tumor growth because it seems to be exponentially decaying rather than exponentially growing.

5

a

$$0 = s(1-x)x^a - (1-s)x(1-x)^a$$
$$0 = x(1-x)[sx^{a-1} - (1-s)(1-x)^{a-1}]$$
$$x^* = 0, 1$$
$$0 = sx^{a-1} - (1-s)(1-x)^{a-1}$$
$$sx^{a-1} = (1-s)(1-x)^{a-1}$$
$$\frac{s}{1-s} = \left(\frac{1-x}{x}\right)^{a-1}$$
$$\left(\frac{s}{1-s}\right)^{\frac{1}{a-1}} = \frac{1-x}{x}$$
$$\left(\frac{s}{1-s}\right)^{\frac{1}{a-1}} = \frac{1}{x} - 1$$
$$\left(\frac{s}{1-s}\right)^{\frac{1}{a-1}} + 1 = \frac{1}{x}$$
$$x^* = \frac{1}{\left(\frac{s}{1-s}\right)^{\frac{1}{a-1}} + 1}$$

b

$$\frac{d}{dx}(s(1-x)x^a - (1-s)x(1-x)^a)$$
$$f'(x) = s(-x^a + ax^{a-1}(1-x)) - (1-s)((1-x)^a - ax(1-x)^{a-1})$$
$$a = 2, f'(0) = s - 1 < 0$$
$$a = 2, f'(1) = -s < 0$$

Thus the fixed points at $x^* = 0, 1$ are stable.

c

$$f'(x^*) = s(-x^a + ax^{a-1}(1-x)) - (1-s)((1-x)^a - ax(1-x)^{a-1})$$
$$f'(x^*) = (1-s)(1-x^*)^{a-1}[a - x^*(1+a) - 1 + x^*(1+a)]$$
$$f'(x^*) = (1-s)(1-x^*)^{a-1}(a-1)$$
$$0 < f'(x^*) < 1$$

Thus, the third fixed point is unstable.

d

One assumption is that everyone in the population either speaks X or Y. Another assumption is that an individual can only speak one language. To relax these assumptions, one may be able to say that an individual can be in both x and y, causing the proportion total to be greater than 1 but including everyone who speaks both languages.

6

a

$$\begin{cases} x' = 0.15x - 0.01x^2 - 0.03xy \\ y' = 0.2y - 0.04xy - 0.02y^2 \end{cases} \tag{1}$$

$$0 = 0.15x - 0.01x^2 - 0.03xy$$
$$0 = x[0.15 - 0.01x - 0.02y]$$
$$x = 0, y = \frac{15-x}{3}$$

$$0 = 0.2y - 0.04xy - 0.02y^2$$
$$0 = y[0.2 - 0.04x - 0.02y]$$
$$y = 0, y = 10 - 2x$$

$$x = 0, y = 0, (\mathbf{0}, \mathbf{0})$$

$$x = 0, y = 10 - 2x, (\mathbf{0}, \mathbf{10})$$

$$y = 0, y = \frac{15 - x}{3}, (\mathbf{15}, \mathbf{0})$$

$$\frac{15 - x}{3} = 10 - 2x, (\mathbf{3}, \mathbf{4})$$

$$J = \begin{bmatrix} 0.15 - 0.02x - 0.02y & -0.03x \\ -0.04y & 0.2 - 0.04x - 0.04y \end{bmatrix}$$

For (0,0):

$$\begin{bmatrix} 0.15 & 0 \\ 0 & 0.2 \end{bmatrix}$$

with eigenvalues 0.15 and 0.2. Thus, the fixed point at (0,0) is an **unstable source**.
For (0, 10):

$$\begin{bmatrix} -0.15 & 0 \\ -0.4 & -0.2 \end{bmatrix}$$

with eigenvalues -0.2 , -0.15. Thus, the fixed point at (0, 10) is a **stable sink**.
For (15, 0):

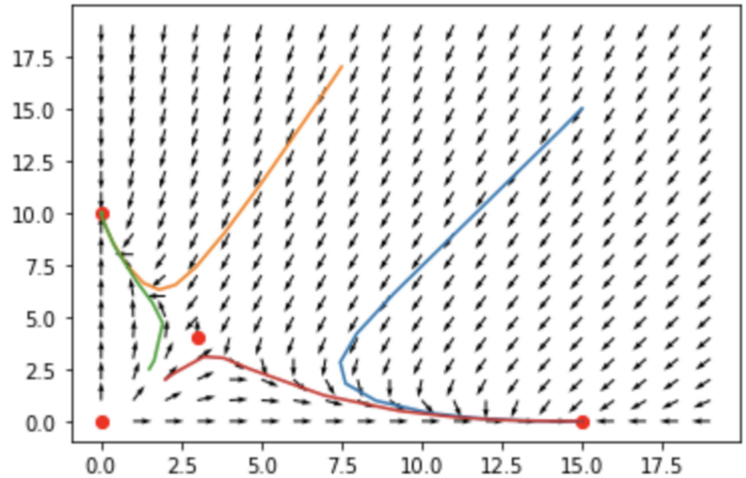
$$\begin{bmatrix} -0.15 & 0 - 0.45\ 0 & -0.4 \end{bmatrix}$$

with eigenvalues -0.15, -0.4 . Thus, the fixed point at (15, 0) is a **stable sink**.
For (3,4):

$$\begin{bmatrix} -0.03 & -0.09 \\ -0.16 & -0.08 \end{bmatrix}$$

with eigenvalues 0.06757651, -0.17757651. Thus, the fixed point at (3, 4) is a **saddle point**.

b



c

The two populations cannot exist simultaneously forever. Depending on what the initial state is, one population will reach 0 and the other population will reach a carrying capacity. If population X gets out competed, then Y will end up with a population of 10, and if Y gets out competed, then X will end up with a population of 15.

7

$$\begin{cases} x' = 0.1x - 0.01x^2 - 0.005xy \\ y' = 0.2y - 0.005xy - 0.04y^2 \end{cases} \tag{2}$$

$$0 = 0.1x - 0.01x^2 - 0.005xy$$
$$0 = x[0.1 - 0.01x - 0.005y]$$
$$x = 0, y = 20 - 2x$$

$$0 = 0.2y - 0.005xy - 0.04y^2$$
$$0 = y[0.2 - 0.005x - 0.04y]$$
$$y = 0, x = 40 - 8y$$

$$x = 0, y = 0, (\mathbf{0}, \mathbf{0})$$

$$x = 0, y = 5, (\mathbf{0}, \mathbf{5})$$

$$y = 0, x = 10, (\mathbf{10}, \mathbf{0})$$

$$(\mathbf{8}, \mathbf{4})$$

$$J = \begin{bmatrix} 0.1 - 0.02x - 0.005y & -0.005x \\ -0.005y & 0.2 - 0.005x - 0.08y \end{bmatrix}$$

For (0,0):

$$\begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}$$

with eigenvalues 0.1 and 0.2. Thus, the fixed point at (0,0) is an **unstable source**.

For (0,5):

$$\begin{bmatrix} 0.075 & 0 \\ -0.025 & -0.2 \end{bmatrix}$$

with eigenvalues -0.2 and 0.075. Thus, the fixed point at (0,5) is an **saddle point**.

For (10,0):

$$\begin{bmatrix} -0.1 & -0.05 \\ -0.025 & 0.15 \end{bmatrix}$$

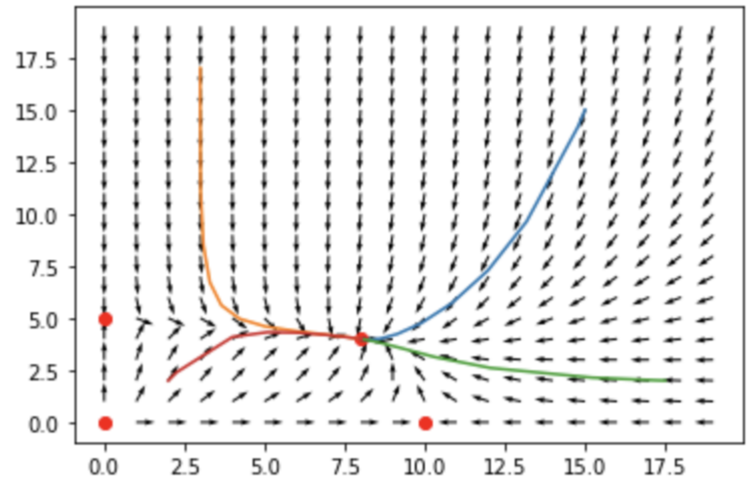
with eigenvalues -0.1 and 0.15. Thus, the fixed point at (10,0) is an **saddle point**.

For (8,4):

$$\begin{bmatrix} -0.08 & -0.04 \\ -0.02 & -0.16 \end{bmatrix}$$

with eigenvalues -0.07101021 and -0.16898979. Thus, the fixed point at (8,4) is a **stable sink**.

b



c

The two species in this model will coexist no matter the starting state. Eventually, there will be a distribution of 8 X and 4 Y as the fixed point at (8, 4) is the only sink. The fundamental difference between this model and the last is that these two species compete much less, so they are able to coexist together without killing off the other species.

8

8.1

a

Logistic model:

$$\frac{dx}{dt} = rx(1 - \frac{x}{K})$$

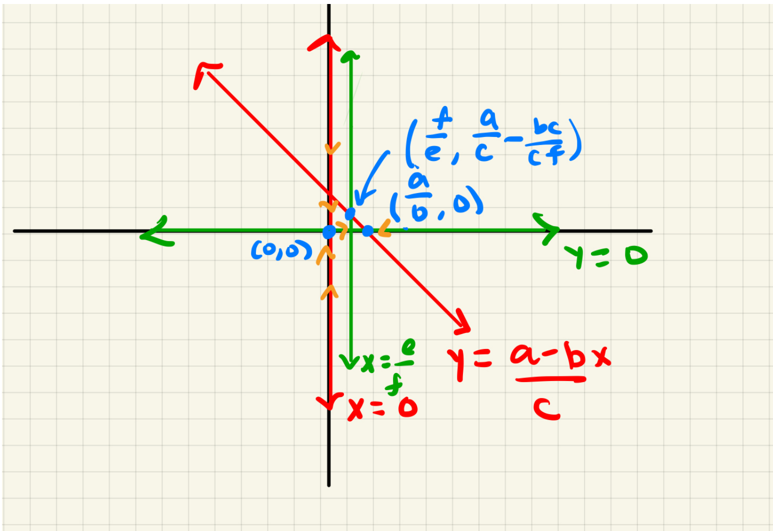
$$\frac{dx}{dt} = rx - \frac{rx^2}{K}$$

Using our prey model:

$$\frac{dx}{dt} = ax - bx^2 - cxy$$

$$a = r, b = \frac{r}{K}$$

b



$$0 = ax - bx^2 - cxy$$

$$0 = x[a - bx - cy]$$

$$x = 0, y = \frac{a - bx}{c}$$

$$0 = exy - fy$$

$$0 = y[ex - f]$$

$$y = 0, x = \frac{f}{e}$$

$$x = 0, y = 0, (\mathbf{0}, \mathbf{0})$$

$$y = 0, x = \frac{a}{b}, (\frac{\mathbf{a}}{\mathbf{b}}, \mathbf{0})$$

$$x = \frac{a - cy}{b}, x = \frac{f}{e}, (\frac{\mathbf{f}}{\mathbf{e}}, \frac{\mathbf{a}}{\mathbf{c}} - \frac{\mathbf{bf}}{\mathbf{ce}})$$

c

$$J = \begin{bmatrix} a - 2bx - cy & -cx \\ ey & ex - f \end{bmatrix}$$

For $(0, 0)$:

$$\begin{bmatrix} a & 0 \\ 0 & -f \end{bmatrix}$$

The eigenvalues are a and -d, meaning that the fixed point at (0, 0) is a **saddle point**.
For $(\frac{a}{b}, 0)$:

$$\begin{bmatrix} -a & \frac{-ca}{b} \\ 0 & \frac{ac}{b} - f \end{bmatrix}$$

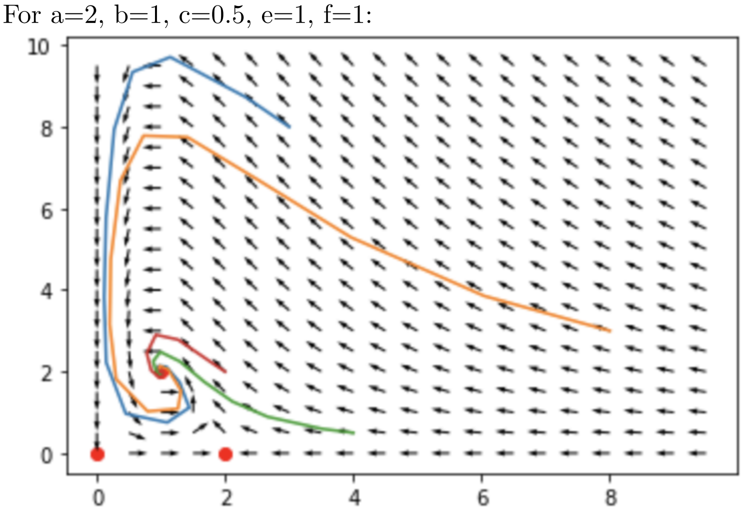
The eigenvalues are both negative, meaning that the fixed point at $(\frac{a}{b}, 0)$ is **unstable**. For $(\frac{f}{e}, \frac{a}{c} - \frac{bf}{ce})$:

$$\begin{bmatrix} \frac{-2bf}{e} & \frac{-cf}{e} \\ \frac{ea}{c} - \frac{hb}{c} & 0 \end{bmatrix}$$

Thus the fixed point at $(\frac{f}{e}, \frac{a}{c} - \frac{bf}{ce})$ is a **spiral sink**.

d

For a=2, b=1, c=0.5, e=1, f=1:



8.2

a

Logistic model:

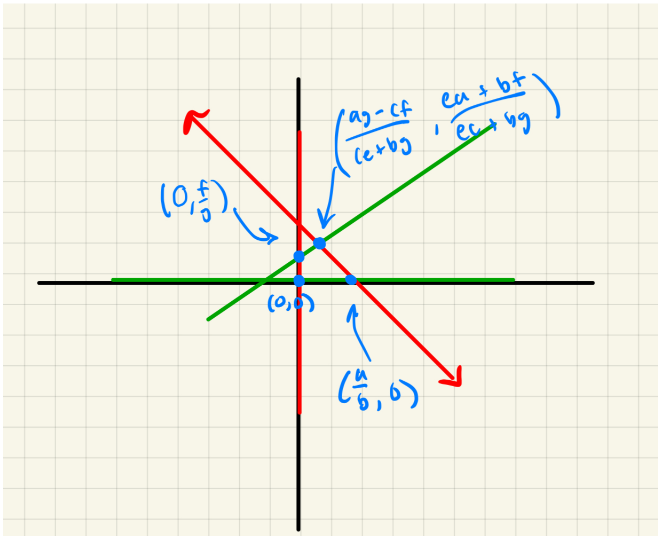
$$\frac{dy}{dt} = ry(1 - \frac{y}{K})$$
$$\frac{dy}{dt} = ry - \frac{ry^2}{K}$$

Using our predator model:

$$\frac{dy}{dt} = exy + fy - gy^2$$

$$f = r, g = \frac{r}{K}$$

b



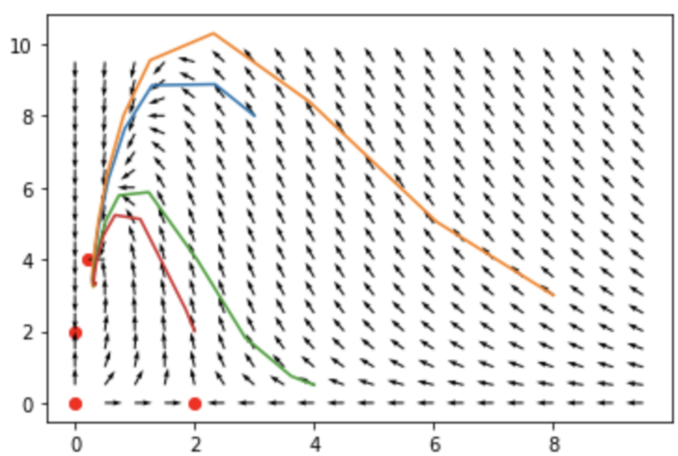
Fixed points: $(0,0), (0, \frac{f}{g}), (\frac{a}{b}, 0), (\frac{ag-cf}{ce+bg}, \frac{ea+bf}{ec+bg})$

c

$$J = \begin{bmatrix} a - 2bx - cy & -cx \\ ey & ex + f - 2gy \end{bmatrix}$$

The fixed point at $(\frac{ag-cf}{ce+bg}, \frac{ea+bf}{ec+bg})$ is a **spiral sink**.

d



a=2, b=1, c=0.5, e=2, f=1, g=0.5

9

For my teams final project, we will be using the cycling data.