Problems for Vectors in Space, n-Vectors

1. When he was young, Captain Conundrum mowed lawns on weekends to help pay his college tuition bills. He charged his customers according to the size of their lawns at a rate of 5¢ per square foot and meticulously kept a record of the areas of their lawns in an ordered list:

$$A = (200, 300, 50, 50, 100, 100, 200, 500, 1000, 100)\,.$$

He also listed the number of times he mowed each lawn in a given year, for the year 1988 that ordered list was

$$f = (20, 1, 2, 4, 1, 5, 2, 1, 10, 6)\,.$$

(a) Pretend that A and f are vectors and compute $A \cdot f$.

(b) What quantity does the dot product $A \cdot f$ measure?

(c) How much did Captain Conundrum earn from mowing lawns in 1988? Write an expression for this amount in terms of the vectors A and f.

(d) Suppose Captain Conundrum charged different customers different rates. How could you modify the expression in part 1c to compute the Captain's earnings?

2. (2) Find the angle between the diagonal of the unit square in $\mathbb{R}^2$ and one of the coordinate axes.
   (3) Find the angle between the diagonal of the unit cube in $\mathbb{R}^3$ and one of the coordinate axes.
   (n) Find the angle between the diagonal of the unit (hyper)-cube in $\mathbb{R}^n$ and one of the coordinate axes.
   (∞) What is the limit as $n \to \infty$ of the angle between the diagonal of the unit (hyper)-cube in $\mathbb{R}^n$ and one of the coordinate axes?

3. Consider the matrix $M = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and the vector $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

   (a) Sketch X and MX in $\mathbb{R}^2$ for several values of X and θ.

   (b) Compute $\frac{||MX||}{||X||}$ for arbitrary values of X and θ.

   (c) Explain your result for (b) and describe the action of M geometrically.

4. Suppose in $\mathbb{R}^2$ I measure the x direction in inches and the y direction in miles. Approximately what is the real-world angle between the vectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$? What is the angle between these two vectors according to the dot-product? Give a definition for an inner product so that the angles produced by the inner product are the actual angles between vectors.

5. (Lorentzian Strangeness). For this problem, consider $\mathbb{R}^n$ with the Lorentzian inner product and metric defined above.

   (a) Find a non-zero vector in two-dimensional Lorentzian space-time with zero length.

   (b) Find and sketch the collection of all vectors in two-dimensional Lorentzian space-time with zero length.

   (c) Find and sketch the collection of all vectors in three-dimensional Lorentzian space-time with zero length.