21 Problems: Orthonormal Bases

1. Let $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

   (a) Write D in terms of the vectors e_1 and e_2, and their transposes.

   (b) Suppose $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible. Show that D is similar to

   $$M = \frac{1}{ad - bc} \begin{pmatrix} \lambda_1 ad - \lambda_2 bc & -(\lambda_1 - \lambda_2)ab \\ (\lambda_1 - \lambda_2)cd & -\lambda_1 bc + \lambda_2 ad \end{pmatrix}.$$

   (c) Suppose the vectors $\begin{pmatrix} a & b \end{pmatrix}$ and $\begin{pmatrix} c & d \end{pmatrix}$ are orthogonal. What can you say about M in this case? (Hint: think about what M^T is equal to.)

2. Suppose $S = \{v_1, \ldots, v_n\}$ is an *orthogonal* (not orthonormal) basis for $\mathbb{R}^n$. Then we can write any vector v as $v = \sum_i c^i v_i$ for some constants c^i. Find a formula for the constants c^i in terms of v and the vectors in S.

Hint

3. Let u, v be independent vectors in $\mathbb{R}^3$, and $P = \text{span}\{u, v\}$ be the plane spanned by u and v.

   (a) Is the vector $v^{\perp} = v - \frac{u \cdot v}{u \cdot u} u$ in the plane P?

   (b) What is the angle between $v^{\perp}$ and u?

   (c) Given your solution to the above, how can you find a third vector perpendicular to both u and $v^{\perp}$?

   (d) Construct an orthonormal basis for $\mathbb{R}^3$ from u and v.

   (e) Test your abstract formulae starting with

$$u = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \text{ and } v = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}.$$