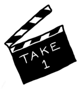
24 Problems: Kernel, Range, Nullity, Rank

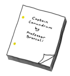
1. Let $L: V \to W$ be a linear transformation. Show that $\ker L = \{0_V\}$ if and only if L is one-to-one:

   (a) First, suppose that $\ker L = \{0_V\}$. Show that L is one-to-one. Think about methods of proof–does a proof by contradiction, a proof by induction, or a direct proof seem most appropriate?

   (b) Now, suppose that L is one-to-one. Show that $\ker L = \{0_V\}$. That is, show that 0_V is in $\ker L$, and then show that there are no other vectors in $\ker L$.



`Hint for 1`



2. Let $\{v_1, \ldots, v_n\}$ be a basis for V. Explain why

$$L(V) = \operatorname{span}\{L(v_1), \ldots, L(v_n)\}.$$

3. Suppose $L\colon \mathbb{R}^4 \to \mathbb{R}^3$ whose matrix M in the standard basis is row equivalent to the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Explain why the first three columns of the original matrix M form a basis for $L(\mathbb{R}^4)$.
Find and describe and algorithm (*i.e.* a general procedure) for finding a basis for $L(\mathbb{R}^n)$ when $L\colon \mathbb{R}^n \to \mathbb{R}^m$.
Finally, use your algorithm to find a basis for $L(\mathbb{R}^4)$ when $L\colon \mathbb{R}^4 \to \mathbb{R}^3$ is the linear transformation whose matrix M in the standard basis is

$$\begin{pmatrix} 2 & 1 & 1 & 4 \\ 0 & 1 & 0 & 5 \\ 4 & 1 & 1 & 6 \end{pmatrix}.$$

4. Claim: If $\{v_1, \ldots, v_n\}$ is a basis for $\ker L$, where $L\colon V \to W$, then it is always possible to extend this set to a basis for V.

   Choose a simple yet non-trivial linear transformation with a non-trivial kernel and verify the above claim for the transformation you choose.

5. Let $P_n(x)$ be the space of polynomials in x of degree less than or equal to n, and consider the derivative operator $\frac{\partial}{\partial x}$. Find the dimension of the kernel and image of $\frac{\partial}{\partial x}$.

   Now, consider $P_2(x, y)$, the space of polynomials of degree two or less in x and y. (Recall that xy is degree two, y is degree one and x^2y is degree three, for example.) Let $L = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. (For example, $L(xy) = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(xy) = y + x$.) Find a basis for the kernel of L. Verify the dimension formula in this case.