Let's Walk and Explore

A Senior Project submitted to
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of
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Dedication

Dedicated to all the special people in my life, especially my mother, wife, daughters, and sons.

Acknowledgments

First and foremost, all thanks and praise are for God. After which I would like to show my appreciation to the following people for their encouragement, assistance, and instruction. Professor Branden Stone for his advising and guidance. Professor Malik Ndiaye for providing me with the article that led to the writing of this thesis and his assisting me in the nascent stages of this thesis. Professor Delia Mellis was helpful to the completion this project: she made sure that I had the articles and the books I needed when I needed them. Ms. Dorothy Crane Was an extra pair of eyes: she helped with the sentence structure in my introduction. And, John Zoccoli for ameliorating some of the hardships in the writing of this thesis. Finally, I am greatly appreciative of Bard College and the Bard Prison Initiative (B.P.I.) for providing me with this wonderful opportunity. Max, Dan, and Professor Lagemann I owe all three of you a lot.

1
Introduction

In this thesis, we will explore the Combinatorial Trace Method; that is, we will attempt to take any graph G, and attempt to compute the graph's eigenvalues and also to find a formula, which in most cases will be some nice combinatorial formula, for the number of closed walks of length n in G.

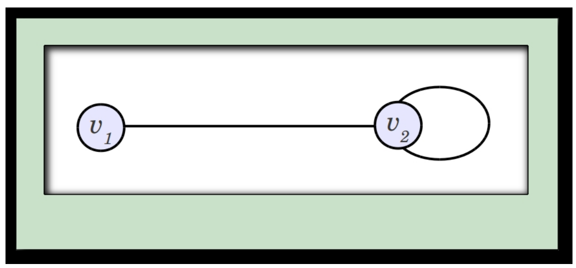
1.0.1 History and Motivation

The motivation for this project comes from a paper of Mike Krebs and Natilie Martinez[5]. At the end of their paper, they state that a good undergraduate research project would be to "take any graph G, attempt both to compute the graph's eigenvalues and also to find a formula for the number of closed walks of length n." Therefore, in this thesis, we will follow the authors' suggestion and take the following graphs: paths, cycles, completes, and some unclassified graphs and then attempt to not only compute the graph's eigenvalues but also find a formula for the number of closed walks of length n in those graphs. Before we attempt the preceding, here is an example from Krebs and Matinez's paper that should give the reader an idea of what we will be attempting.

Example 1.0.1. Let the Fibonacci numbers F_n be defined recursively by $F_{-1} = 1$, $F_0 = 0$, and $F_{n+2} = F_{n+1} + F_n$. The closely-related Lucas numbers obey the same recursion with the initial values $L_0 = 2$, $L_1 = 1$. Both the Fibonacci and Lucas numbers are well studied and occur naturally in many counting problems. The classical Binet formula for Lucas numbers states that

$$L_n = \varphi^n + \overline{\varphi}^n$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\overline{\varphi} = \frac{1-\sqrt{5}}{2}$. The claim is that L_n counts the number of closed walks in G of length n. Consider the following graph.



Notice that the adjacency matrix of this graph is

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

To show the claim, let v_2^n be the number of closed walks of length n based at vertex v_2. Also note that we have $v_2^0 = 1$ (the trivial walk) and $v_2^1 = 1$ (traversing the loop). For $n \geq 2$, a walk of length n based at v_2 ends either going around the loop or traveling back and forth across the edge. Hence, $v_2^n = v_2^{n-1} + v_2^{n-2}$. Therefore, $v_2^n = F_{n+1}$. Now, let v_1^n be the number of closed walks of length n based at the vertex v_1. We then have $v_1^0 = 1 = F_{-1}$ and $v_1^1 = 0 = F_0$. For $n \geq 2$, a closed walk of length n based at v_1 must consist of the edge from v_1 to v_2, followed by a closed walk of length $n-2$ based at v_2, returning by the edge to v_1. Hence $v_1^n = v_2^{n-2} = F_{n-1}$.

Now we can use the well-known (and easily verified) identity $L_n = F_{n+1} + F_{n-1}$ to find the total number of closed walks in G of length n. In particular we have $v_1^n + v_2^n = L_n$.

At this point, we should turn our attention to Fibonacci numbers, and take note that the Binet formula in this case is

$$F_n = \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}. \tag{1.0.1}$$

Observing this equation, we see that this equation is not amenable to the combinatorial trace method because neither side is a power sum. However, squaring both sides and using $\varphi * \overline{\varphi} = 1$, they establish that (1.0.1) is equivalent to

$$5F_n^2 + 2(-1)^n = (\varphi^2)^n - (\overline{\varphi}^2)^n.$$

At this juncture, the right-hand side is now a power sum, so we are now in combinatorial trace method territory.

We can go one step further in calculating that $L_{2n} = 5F_n^2 + 2(-1)^n$. Therefore our combinatorial equation for the above graph is

$$L_{2n} = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

2

Notation and Basic Definitions

Before we begin work with the combinatorial trace method, it would be prudent to first establish some basic definitions and theorems. This will give the reader some background information and help move along our discussion.

We will begin this paper with the most fundamental definition to this project: a graph.

Definition 2.0.2. A graph G consists of two nonempty sets $V(G)$ and $E(G)$ where the nonempty set $V(G)$ is a set of vertices and the nonempty set $E(G)$, disjoint from $V(G)$, is a set of edges. An edge, e_i, of G is said to join two vertices v_i and v_{i+1} of G and is abbreviated v_iv_{i+1}.

Example 2.0.3. Let G=$(V(G), E(G))$ where

$$V(G)=\{v_1, v_2, v_3\},$$

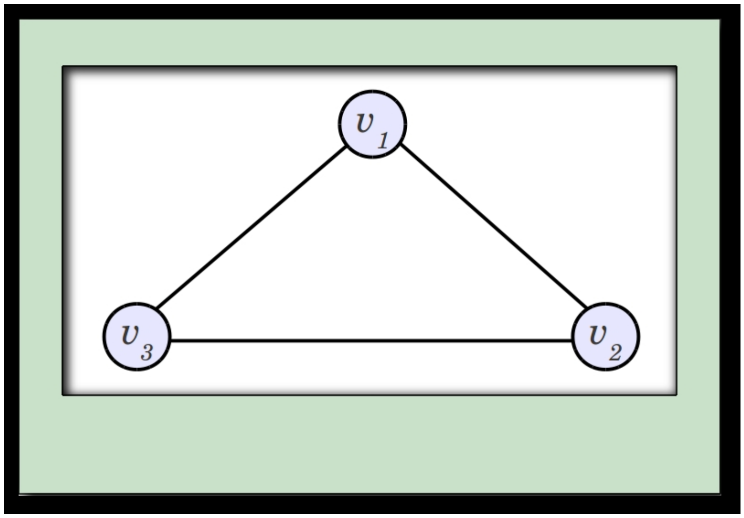
and

$$E(G)=\{e_1, e_2, e_3\},$$

and may be ordered as follows:

$$\{v_1e_1v_2, v_2e_2v_3, v_3e_3v_1\}.$$

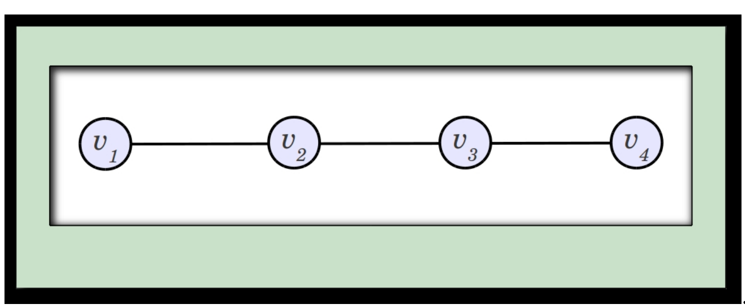
We obtain the graph seen in the following figure.



Next, crucial to our discussion is a walk in a graph G, which is defined as follows:

Definition 2.0.4. [1] A walk in G is a finite non-null sequence $W = v_1, e_1, v_2, e_2, ..., v_{n-1}, e_n, v_n$, whose terms are alternately vertices and edges, such that, for $1 \leq i \leq n$, to the ends of e_i are v_1 and v_n. We say that W is a walk from v_1 to v_n, or a $(v_1, v_n) - walk$. The integer n is the length of W: the number of edges one must traverse between v_1 and v_n. Also, a walk is closed if it has positive length and its beginning and end are the same.

Example 2.0.5. Let us consider a graph of the P_n family, which is a simple path on n vertices; namely, we look at P_4 where our graph looks as follows.



.

We see that a walk can be defined on this graph as

$$W = \{v_1, e_1, v_2, e_2, v_3\},$$

and a closed walk can be defined as

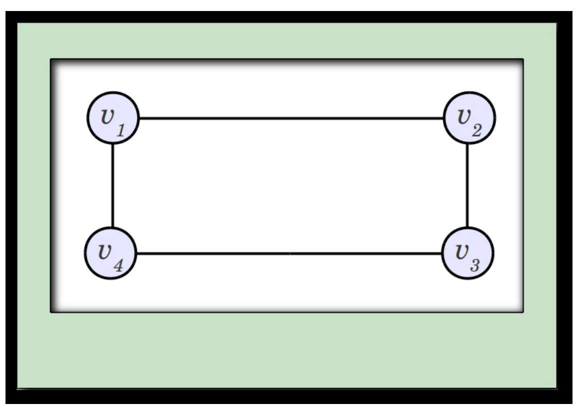
$$C = \{v_1, e_1, v_2, e_2, v_3, e_2, v_2, e_1, v_1\}.$$

Now that we have established what a graph and a walk is, let us define another term that is crucial to our study: the adjacency matrix.

Definition 2.0.6. [2] The adjacency matrix A of a graph G with $V(G) = \{v_1, v_2, ..., v_n\}$ is the $n \times n$ matrix (a_{ij}), where $a_{ij}=1$ if $v_i v_j \in E(G)$ and $a_{ij} = 0$, otherwise.

We should take note that the adjacency matrix A–if G has no loops– of a graph G is a symmetric matrix consisting of zeros and ones with zeros across the diagonal of the matrix.

Example 2.0.7. The following is the C_4 graph, a cycle on 4 vertices.



Let A be its adjacency matrix, then the adjacency matrix of C_4 is

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Notice that A is a symmetric matrix with a zero diagonal.

The next proposition is given special attention because it will come in handy in our proof of the combinatorial trace method.

Proposition 2.0.8. *If the square matrix A is symmetric, then so is A^n.*

Proof. We know that A is symmetric if and only if $A{=}A^t$. To show that A^n is symmetric, we need to show $A^n{=}(A^n)^t$. Notice that

$$(A^n)^t = \overbrace{(AA...A)^t}^{n\ times} = \overbrace{A^tA^t...A^t}^{n\ times}.$$

By the property of transpose, we also have that $(AB)^t{=}B^tA^t$. Since $A{=}A^t$,

$$A_n^tA_{n-1}^t...A_1^t = A_nA_{n-1}...A_1 = A^n,$$

which shows that A^n is symmetric. □

Example 2.0.9. Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

then

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \text{ and } A^3 = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix}.$$

At this juncture, we now know what a graph, walk, and an adjacency matrix are. The following proposition will aid us and ameliorate the rigorous work of counting walks in a graph G.

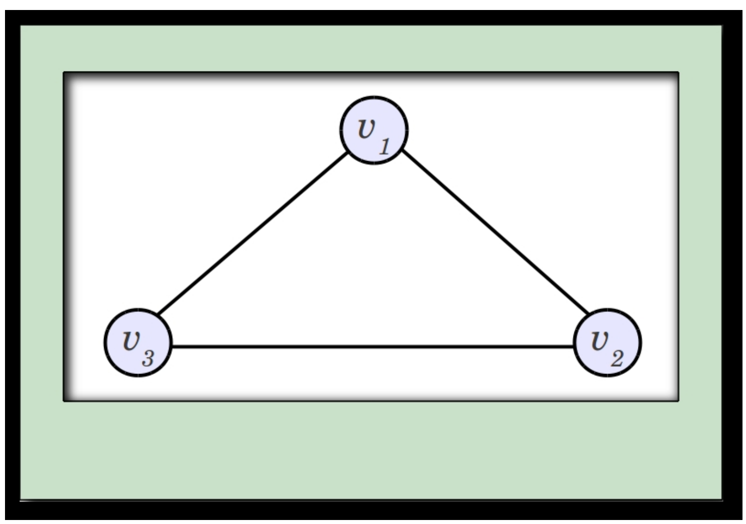
Proposition 2.0.10. [2] *If A is the adjacency matrix of a graph G with $V(G){=}\{v_1, v_2, ...v_n\}$, then the [i,j] entry of A^n, $n{\geq}1$, is the number of different walks from v_i to v_j of length n in G.*

Proof. The proof is by induction on n. The result is obvious for $n = 1$ since there exists a walk from v_i to v_j of length 1 if and only if there is an edge that connects v_i to v_j, which can be denoted $v_iv_j \in E(G)$. Let $A^{n-1} = [a_{ij}^{(n-1)}]$ and assume $a_{ij}^{(n-1)}$ is the number of different v_i–v_j walks of length $n-1$ in G. Furthermore, let $A^n = [a_{ij}^{(n)}]$. Since $A^n = A^{n-1}A$, we have the following:

$$a_{ij}^{(n)} = \sum_{k=1}^{n} a_{ik}^{(n-1)} a_{kj}. \tag{2.0.1}$$

Every walk from v_i to v_j of length n in G consists of a walk from v_i to v_k of length $n-1$, where v_k is adjacent to v_j, followed by the edge v_kv_j and the vertex v_j. Thus by the inductive hypothesis and Equation (2.0.1), we have the desired result. □

Example 2.0.11. Let G be a cycle graph on three vertices, denoted by C_3, which looks as follows.



We can count any walk of length n from any vertex, by simply multiplying C_3's adjacency matrix n times and looking at the a_{ij} entry corresponding to the vertex we are searching. Thus, the adjacency matrix of C_3, denoted by A, is as follows:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } A^3 = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix}.$$

Thus, if we look at the graph G, we see, from our entry $a_{11}^{(3)}$, that there are two closed walks of length 3 in our graph G from v_1 to v_1; namely, we let our walks be ordered as $W_1{=}v_1v_2v_3v_1$ and $W_2{=}v_1v_3v_2v_1$. And, as the preceding theorem states this holds for any $a_{ij}^{(n)}$ for the number of walks of length n in G.

In addition, the preceding theorem has another immediate consequence; namely, the trace–denoted by $tr(A^n)$– of a square matrix A^n is the sum of the diagonal entries of A^n, and the trace gives us the total number of closed walks of length n in A^n.

Nevertheless, the following proposition is one we will need to prove the combinatorial trace method. Since we are discussing matrices and operations on matrices, it makes sense to list and prove it here.

Proposition 2.0.12. *If A and B are square matrices of the same size, then $tr(AB) = tr(BA)$.*

Proof. Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix}$$

then the i^{th} term on the diagonal of AB equals

$$\sum_{j=1}^{n} a_{ij}b_{ji}$$

and hence

$$tr(AB) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji}.$$

Further, the i^{th} term on the diagonal of BA equals

$$\sum_{j=1}^{n} b_{ij}a_{ji}$$

and

$$tr(BA) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}a_{ji}.$$

Thus,

$$tr(BA) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}a_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ij}a_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ji}b_{ij} = tr(BA).$$

as desired. □

Example 2.0.13. Let

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 6 \\ 1 & 3 \end{bmatrix}.$$

Then

$$tr(AB) = (3+3)(5+3) = (6)(8) = 48 \tag{2.0.2}$$

and

$$tr(BA) = (5+3)(3+3) = (8)(6) = 48. \tag{2.0.3}$$

So we see that $tr(AB) = tr(BA)$.

Let us now discuss eigenvectors, eigenvalues, and eigenspaces.

Definition 2.0.14. An *eigenvector* of an $n \times n$ matrix A is a nonzero vector x such that $Ax = \lambda x$ for some scalar λ. A scalar λ is called an *eigenvalue* of A if there is a nontrivial solution x of $Ax = \lambda x$.

Example 2.0.15. Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and $u = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, then

$$Au = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4u. \tag{2.0.4}$$

Thus, x is an eigenvector corresponding to the eigenvalue -4.

We can also show that λ is an eigenvalue of an $n \times n$ matrix A as the following example highlights.

Example 2.0.16. Let $\lambda = 7$, then λ is an eigenvalue of A if and only if the equation

$$Ax = 7x \tag{2.0.5}$$

has a nontrivial solution. But (2.0.5) is equivalent to $Ax - 7x = 0$, or

$$(A - 7I)x = 0.$$

To solve this homogeneous equation, form the matrix

$$Ax - 7I = \left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}\right) = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

The columns of $A - 7I$ are obviously linearly dependent, so (2.0.5) has nontrivial solutions. Thus, 7 is an eigenvalue of A.

Note: The set of all solutions of $(A - \lambda I)x = 0$ is just the null-space of the matrix $A - \lambda I$; this set is a subspace of $\mathbb{R}^n$ and is called the eigenspace of A corresponding to λ.

Example 2.0.17. Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & 1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. We can also find a basis for the corresponding eigenspace.

First, we know that

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & 1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}.$$

And, now we can row reduce the augmented matrix for $(A - 2I)x = 0$

$$\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6. \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So $x_1 = \frac{1}{2}x_2 - 3x_3$. Now, because $(A - 2I)x = 0$ has free variables, it is clear that 2 is an eigenvalue of A. The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \text{ } x_2 \textit{ and } x_3 \text{ are free.}$$

The eigenspace is a two-dimensional subspace in $\mathbb{R}^3$. A basis is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Theorem 2.0.18. [3] *If $v_1, \ldots, v_n$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ of an $n \times n$ matrix A, then the set $\{v_1, \ldots, v_n\}$ is linearly independent.*

Proof. Suppose $v_1, \ldots, v_n$ is linearly dependent. Since v_1 is nonzero, there is a theorem [3] that says that one of the vectors in the set is a linear combination of the preceding vectors. Let p be the least index such that v_{p+1} is a linear combination of the preceding (linearly independent) vectors. Then there exists scalars $c_1, \ldots, c_p$ not all equal to zero such that

$$c_1 v_1 + \ldots + c_p v_p = v_{p+1} \tag{2.0.6}$$

Multiplying both sides of (2.0.6) by A and using the fact that $Av_k = \lambda_k v_k$ for each k, we obtain

$$\begin{aligned} c_1 A v_1 + \ldots + c_p A v_p &= A v_{p+1} \\ c_1 \lambda_1 v_1 + \ldots + c_p \lambda_p v_p &= \lambda_{p+1} v_{p+1} \end{aligned} \tag{2.0.7}$$

Multiplying both sides of (2.0.6) by λ_{p+1} and subtracting the results from (2.0.7), we have

$$c_1(\lambda_1 - \lambda_{p+1}) v_1 + \ldots + c_p(\lambda_p - \lambda_{p+1}) v_p = 0 \tag{2.0.8}$$

Since $v_1, v_2, \ldots, v_p$ is linearly independent, the weights in (2.0.8) are all zero. But none of the factors $\lambda_i - \lambda_{p+1}$ are zero, because the eigenvalues are distinct. Hence, $c_i = 0$ for $i = 1, \ldots, n$, which is a contradiction. And, (2.0.6) says that $v_{p+1} = 0$, which is impossible. Hence, $v_1, ..., v_n$ cannot be linearly dependent and therefore must be linearly independent. $\square$

Next, we will give an example of the preceding theorem and list more theorems that our example will highlight.

Example 2.0.19. We can find the eigenvalues of a matrix A as well as the eigenvectors of A, and we can show that the eigenvectors of A are linearly independent. First, let $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

Now, we must find all λ such that the matrix equation

$$(A - \lambda I)x = 0$$

has a nontrivial solution. By the Invertible Matrix Theorem, this problem is equivalent to finding all λ such that the matrix $A - \lambda I$ is not invertible, where

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}.$$

This matrix fails to be invertible when its determinant is zero. So, the eigenvalues of A are the solutions of the equation

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix} = 0.$$

Recall that

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

So,

$$\det(A - \lambda I) = (2-\lambda)(-6-\lambda) - (3)(3) = -12 + 6\lambda - 2\lambda + \lambda^2 - 9 = \lambda^2 + 4\lambda - 21 = (\lambda - 3)(\lambda + 7).$$

If $\det(A - \lambda I) = 0$, then $\lambda = 3$ or $\lambda = -7$. So the eigenvalues of A are 3 and -7.

Now that we have our eigenvalues, we can find our eigenvectors.

For the basis for $\lambda = 3$, we get that

$$\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \sim \begin{bmatrix} -3 & 9 \\ 0 & 0 \end{bmatrix}.$$

So, $x_1 = 3x_2$, and we get the following equality

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \end{bmatrix},$$

which gives us the eigenvector

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Now, for the basis for $\lambda = -7$

$$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 9 & 3 \\ 0 & 0 \end{bmatrix}.$$

So, $x_1 = -\frac{1}{3}x_2$, and we obtain the following equality

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}x_2 \\ x_2 \end{bmatrix},$$

which gives us the following eigenvector

$$\begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

The eigenspace is a two-dimensional subspace in $\mathbb{R}^2$. A basis is

$$\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}.$$

We should take note of the fact that if we are dealing with a graph G with n vertices, then its adjacency matrix is an $n \times n$, symmetric matrix. Therefore, the following theorem highlights the number of eigenvalues that we should have for such a matrix.

Proposition 2.0.20. *An $n \times n$ symmetric matrix A of real numbers has n real eigenvalues.*

Proof. Let A be an $n\times n$ matrix with the property that $A^T{=}A$, let x be any vector in $\mathbb{C}^n$, and let $q = \overline{x}^T Ax$. Since A is an $n\times n$ matrix, the equation $\det(A-\lambda I)=0$ has at most n roots because $\det(A-\lambda I)=0$ is a polynomial of degree n.

First we can show that q is a real number by verifying that $\overline{q}=q$:

$$\overline{q} = \overline{\overline{x}^T Ax} = x^T\overline{Ax} = x^T A\overline{x} = (x^T A\overline{x})^T = \overline{x}^T A^T x = \overline{x}^T Ax = q.$$

So, q is real. Now, let us show that λ is real. Recall from Definition 2.0.14 that $Ax=\lambda x$, and that $\overline{x}^T Ax$ is real. Therefore, we get

$$\overline{x}^T Ax = \overline{x}^T\lambda x = \lambda\overline{x}^T x.$$

We know that q is a real number and that $\overline{x}^T x$ is also a real number. Therefore, it follows that λ is real. Thus, an $n\times n$ symmetric matrix A has n real eigenvalues. □

Now, based on the fact that if a graph is diagonalizable that it can be written in the form $A=PDP^{-1}$, and, subsequently, $A^n{=}PD^nP^{-1}$, which will be crucial to our proof of the combinatorial trace method, we will take a glance at the famous Diagonalization Theorem as it is noted in [3]:

The Diagonalization Theorem. An $n\times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

Proof. First, observe that if P is any $n\times n$ matrix with columns $v_1,\ldots,v_n$ and if D is any Diagonal $n\times n$ matrix with diagonal entries $\lambda_1,\ldots,\lambda_n$, then

$$AP = A[v_1v_2\ldots v_n] = [Av_1Av_2\ldots Av_n] \tag{2.0.9}$$

while

$$PD = P \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} = [\lambda_1 v_1, \lambda_2 v_2, \dots, \lambda_n v_n]. \tag{2.0.10}$$

Now, suppose that A is diagonalizable and $A = PDP^{-1}$. Right-multiplying this relation by P, we have $AP = PD$. In this case, equation (2.0.9) and (2.0.10) imply that

$$[Av_1 Av_2 \dots Av_n] = [\lambda_1 v_1 \lambda_2 v_2 \dots \lambda_n v_n]. \tag{2.0.11}$$

Equating columns, we find that

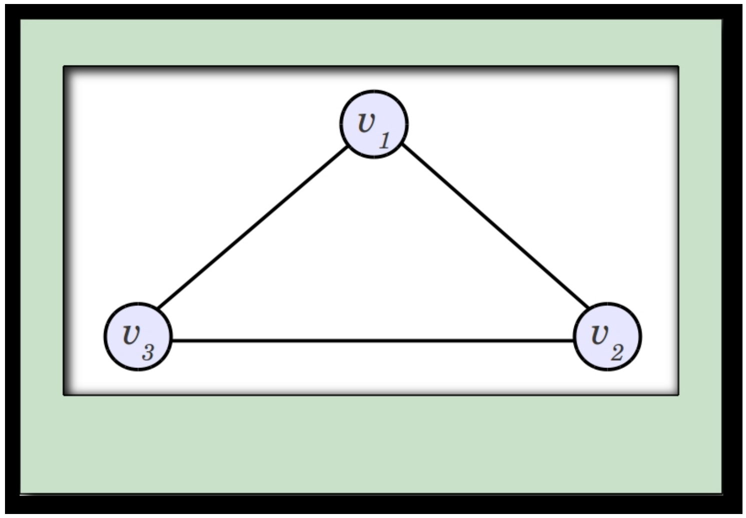
$$Av_1 = \lambda_1 v_1,\ Av_2 = \lambda_2 v_2,\ \dots Av_n = \lambda_n v_n. \tag{2.0.12}$$

Since P is invertible, its columns $v_1, \dots, v_n$ must be linearly independent. Also, since these columns are nonzero, the equation in (2.0.11) shows that $\lambda_1, \dots, \lambda_n$ are eigenvalues and $v_1, \dots, v_n$ are corresponding eigenvectors. This argument proves the "only if" parts of the first and second statements, along with the third statement, of the theorem.

Conversely, given any n eigenvectors $v_1, \dots, v_n$ use them to construct the columns of P and use corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ to construct D. By equations, (2.0.9), (2.0.10), and (2.0.11), $AP = PD$. This is true without any condition on the eigenvectors. If, in fact, the eigenvectors are linearly independent, then P is invertible and $AP = PD$ implies that $A = PDP^{-1}$. □

In other words, a matrix A is diagonalizable if and only if there are enough eigenvectors to form a basis of $\mathbb{R}^n$. We call such a basis an *eigenvector basis* of $\mathbb{R}^n$.

Example 2.0.21. Take the graph C_3, a cycle on 3 vertices.



Its adjacency matrix is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

We can find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

There are four steps to implement the description in the preceding theorem.

Step 1: <u>Find the eigenvalues of A.</u>

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \\ &= -\lambda \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & -\lambda \end{bmatrix} + \begin{bmatrix} 1 & -\lambda \\ 1 & 1 \end{bmatrix} \\ &= -\lambda(\lambda^2 - 1) - (-\lambda - 1) + (1 + \lambda) \\ &= (\lambda + 1)[-\lambda_2 + \lambda + 2] \\ &= (\lambda + 1)^2(\lambda - 2). \end{aligned}$$

Therefore, the eigenvalues are $\lambda = 2, \ \lambda = -1$, and $\lambda = -1$.

Step 2: Find three linearly independent eigenvectors of A.

Three are needed because, as stated earlier, A is a 3×3 matrix. This is the critical step. If it fails, then the preceding theorem says that A cannot be diagonalizable. The basis for each eigenspace is as follows:

For the basis for $\lambda = 2$, we get

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2. \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3. \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0. \end{bmatrix}, \sim \begin{bmatrix} -6 & 0 & 6 \\ 0 & -3 & 3 \\ 0 & 0 & 0. \end{bmatrix}.$$

So $x_1 = x_3$ and $x_2 = x_3$. As a result, we get the following equality.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and we obtain the following eigenvector

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Now, for the basis for $\lambda = -1$ we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1. \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0. \end{bmatrix}.$$

So $x_1 = -x_2 - x_3$, and obtain the equality

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

which leads to us obtaining the following two eigenvectors.

$$v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The reader can check that $\{v_1, v_2, v_3\}$ is a linearly independent set.

Step 3: Construct P from the vectors in step 2.

The order of the vectors is unimportant. Using the order chosen in step 2, we get

$$P[v_1v_2v_3] = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1. \end{bmatrix}$$

Step 4: Construct D from the Corresponding eigenvalues.

In this step, it is essential that the order of the eigenvalues matches the order chosen for the columns of P:

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1. \end{bmatrix}$$

To check that PD works, and to avoid computing P^{-1}, we can verify that $AP = PD$. This is equivalent to $A = PDP^{-1}$ when P is invertible–in this case, P is invertible because the detP$\neq 0$; in fact, the detP$= 3$. So,

$$AP = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & -1 & 0 \\ 2 & 0 & -1. \end{bmatrix}.$$

and

$$PD = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & -1 & 0 \\ 2 & 0 & -1. \end{bmatrix}.$$

Therefore, $AP = PD$, so PD really works.

Now that we have the tools that we need, let us prove the motivating theorem of this project. We call it the Combinatorial Trace Method.

Theorem 2.0.22. *Combinatorial Trace Theorem. Given a finite graph G, the number of closed walks of length n in G equals the sum of the n^{th} powers of the eigenvalues of the adjacency matrix of G.*

Proof. Let G be a simple, finite graph with vertex set $\{v_0,...,v_n\}$. Let A be the adjacency matrix of G where the $(ij)^{th}$entry is 1 if v_i is adjacent to v_j, and 0 otherwise. Let's call $\lambda_0, ..., \lambda_n$ the eigenvalues of G that are eigenvalues of A.

By Proposition 2.0.10, the $(ij)^{th}$entry of A^n is the number of walks in G of length n from v_i to v_j. So, the $(ii)^{th}$entry of A^n equals the number of walks in G of length n from v_i to v_i that are closed walks.

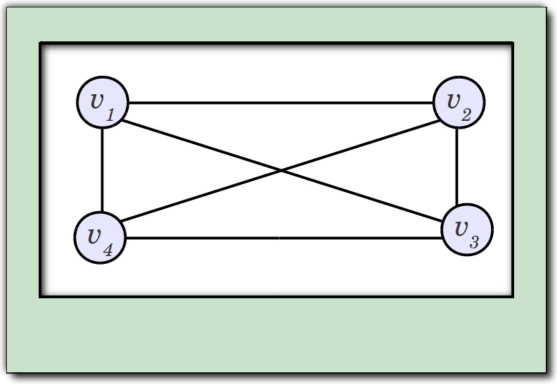
Now, the number of closed walks of length n in G equals tr(A^n). We can compute tr(A^n) in two ways. On the one hand, we have tr(A^n)=$\sum_{i=1}^{n} A_i^n$. And, on the other hand, first note that the adjacency matrix of G is symmetric; therefore, the adjacency matrix of G is diagonalizable: A=PDP^{-1}. As a result, A^n=PD^nP^{-1}, and

$$\begin{aligned}
tr(A^n) &= tr(PD^nP^{-1}) \\
&= tr((PD^n)(P^{-1})) \\
&= tr((P)(P^{-1}D^n)) \\
&= tr((PP^{-1})D^n) \\
&= tr(ID^n) \\
&= tr(D^n).
\end{aligned}$$

So, tr(A^n)=$\sum_{i=1}^{n} D(i,n)$, where $D(i,n)$ denote the number of closed walks of length n in G that begin and end at x_i. Therefore, $\sum_{i=1}^{n} \lambda_i^n$= $\sum_{i=1}^{n} D(i,n)$. □

Now that we have our theorem, we will give an example. The following example will show that we can use the eigenvalues from the adjacency matrix of a graph G to develop power sums to help us count the number of closed walks of length n in a graph. Also, we will see in this example that we can take the trace of the n^{th} power of the adjacency matrix of G to find the number of closed walks of length n in G.

Example 2.0.23. Take the following complete graph on four vertices, which we denote as K_4.



The adjacency matrix for this graph is:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

And,

$$A^2 = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}.$$

First let us establish that the number of closed walks of length 2 in K_4 equals the sum of the 2^{nd} power of the eigenvalues. The eigenvalues for this adjacency matrix are: 3, -1, -1, and -1. Therefore, by taking the sum of the squares of these eigenvalues, we get

$$(3)^2 + (-1)^2 + (-1)^2 + (-1)^2 = 12$$

Thus, there are twelve closed walks of length 2 in K_4. Let $v_a \longrightarrow v_a$ denote a closed walk at a vertex a, then there are three closed walks for $v_1 \longrightarrow v_1$, three for $v_2 \longrightarrow v_2$, three for $v_3 \longrightarrow v_3$, and three for $v_4 \longrightarrow v_4$. They are:

$$W_1 = v_1v_2v_1 \ W_2 = v_1v_3v_1 \ W_3 = v_1v_4v_1$$

$$W_4 = v_2v_1v_2 \ W_5 = v_2v_3v_2 \ W_6 = v_2v_4v_2$$

$$W_7 = v_3v_1v_3 \ W_8 = v_3v_2v_3 \ W_9 = v_3v_4v_3$$

$$W_{10} = v_4v_1v_4 \ W_{11} = v_4v_2v_4 \ W_{12} = v_4v_3v_4$$

In addition, according to Proposition 2.0.10, we can also count the number of closed walks in G by looking at the trace of A^n. The [i,j] entry of A^n, $n \geq 1$, is the number of different walks from v_i to v_j of length n in G. Therefore, we need only look at the [i,i] entry of A^n. By looking at the trace of A^2 we get

$$a_{11}^2 = 3,\ a_{22}^2 = 3,\ a_{33}^2 = 3, \text{ and } a_{44}^2 = 3$$

and

$$3 + 3 + 3 + 3 = 12,$$

which equals what we derived from the sum of the squares of the eigenvalues.

3

The Combinatorial Formation

As stated in the introduction, the combinatorial trace method is a technique that allows us to establish equalities between expressions and power sums. Therefore, let us now explore finding expressions and power sums to count the number of closed walks of length n on a graph G.

Moving forward, we should take what we gathered from Example 2.0.23; that is, we have two ways–using the trace or using the powers of the eigenvalues of the adjacency matrix–to count the number of closed walks of length n in a graph G. Therefore, using these two methods, we will attempt to establish a formula, which we hope will be some nice combinatorial formula. We will attempt this with the graphs P_v, some unclassified graphs, C_v graphs, and K_v graphs. We should take note that the number of closed walks of length n of a graph is

$$\sum_{i=0}^{n} \lambda_i^n$$

where λ_i^n is the n^{th} powers of the eigenvalues of any adjacency matrix of the graph. Thus, from Theorem 2.0.22, the number of closed walks of length n in G equals the sum of the n^{th} power of the eigenvalues of the adjacency matrix of G.

We begin with our first family of graphs: P_v, which are simple paths on v vertices.

3.1 GRAPH: P_v (Paths on v vertices)

At the outset, the reader should take note that all paths, denoted by P_v are bipartite, which is defined in the following definition.

Definition 3.1.1. A graph is bipartite when the set of vertices can be partitioned into two sets X and Y such that every edge is between a vertex in X and a vertex in Y.

In other words, a graph is bipartite when it is 2-colorable; that is, when its vertices can be colored with a minimum of two colors without adjacent vertices having the same color.

Now, when a graph G is bipartite, it has no closed walks of odd length. So, no path has a closed walk of odd length. As a result, we will only have to deal with walks of length $2n$. Now, let us state this as a proposition.

Proposition 3.1.2. *A bipartite graph implies no closed walks of odd length.*

Proof. Because G is bipartite the vertices of G can be partitioned into two sets X and Y such that every edge is between a vertex in X and a vertex in Y. Consequently, every time you leave a vertex in a particular set and return to that vertex, you must add two edges in your calculation of the length of the closed walk. Hence, every closed walk has an even number of edges and there can be no closed walks of odd length. $\square$

Because the graph P_1 is trivial, we will begin with simply stating a few facts about this graph and move onto the graph P_2.

As already stated, this is the most simple path. The reader should note that there is only one vertex and no edges. Because there are no edges in this graph, there is no way to traverse this graph: leave out from a vertex by one edge and return by that same edge or another. Thus, the only walk in this graph is the trivial walk because there are no edges to traverse.

Now, we move onto P_2 graphs.

3.1.1 Examination of P_2

The first observation that the reader should make is that, in relation to P_1, P_2 has an added edge and an added vertex. As a result, we can traverse P_2. The second observation that the reader should make is that P_2 is bipartite. What does this mean? What significance does this have in counting the number of closed walks of length n? First, let us take a glance at the following picture of the graph P_2, then we will explain the significance of a bipartite graph.

Proposition 3.1.3. *The number of closed walks of even length in the graph P_2 is 2.*

Proof. First note that the adjacency matrix of P_2 is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Now we see that the eigenvalues for P_2 are

$$\begin{aligned} \det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) &= \det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) \\ &= \lambda^2 - 1. \end{aligned}$$

Setting this equal to zero yields

$$\lambda^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm\sqrt{1}.$$

Therefore, $\lambda_0 = \sqrt{1}$ and $\lambda_1 = -\sqrt{1}$. By Theorem 2.0.22 (The Combinatorial Trace Method), we get that the number of closed walks of length $2n$ is:

$$(\sqrt{1})^{2n} + (-\sqrt{1})^{2n} = (1)^n + (1)^n = 2. \tag{3.1.1}$$

□

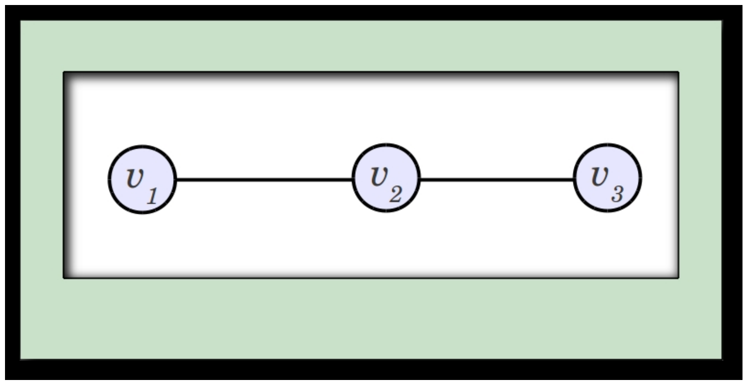
Combinatorial Proof of Proposition 3.1.3. If we look at P_2, we see that we can partition v_1 into a set X and v_2 into a set Y such that our single edge is between v_1 in X and v_2 in Y. Also, we can observe that we can color this graph with two colors in such a way that the two adjacent vertices are not the same color: let v_1 be red and let v_2 be blue. Therefore, P_2 is bipartite and 2-colorable.

In addition, if we further analyze the picture of P_2, we observe that there is only one closed walk of length $2n$ from v_1 to v_1 because there is only one edge and one other vertex, so when you leave v_1 you can only travel along the single edge to v_2 and back to v_1. Secondly, we see that, by the same reasoning, there is only one walk of length $2n$ from v_2 to v_2. So, we see that the number of closed walks of length $2n$ is 2. □

With this argument we have a nice bridge between analytical data and combinatorial data as can be seen concretely in Equation 3.1.1.

3.1.2 Examination of P_3

Now, we examine the graph P_3. This example will be a tidbit more complicated than the previous. However, the fundamentals of deriving an equality remain the same. The first observation that the reader should make when observing the following picture of the graph P_3 is that it is bipartite, so we can apply Proposition 3.1.2 and analyze closed walks of even length only.



Proposition 3.1.4. *The number of closed walks of even length on P_3 is 2^{n+1}.*

Proof. The adjacency matrix for the graph P_3 is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

and its eigenvalues are

$$\lambda_0 = \sqrt{2}, \lambda_1 = 0, \text{ and } \lambda_2 = -\sqrt{2}$$

By Theorem 2.0.22 (The Combinatorial Trace Method), we get the following power sum:

$$(\sqrt{2})^{2n} + (0)^{2n} + (-\sqrt{2})^{2n} = (2)^n + (2)^n = 2^{n+1}.$$

□

Combinatorial Discussion

Note that according to Theorem 2.0.10, the number of closed walks from $v_1 \longrightarrow v_1$, $v_2 \longrightarrow v_2$, and $v_3 \longrightarrow v_3$ of length $2n$ are determined by the $a_{11}^{2(n)}$, $a_{22}^{2(n)}$, and $a_{33}^{2(n)}$ entries, respectively, in the n^{th} power of the adjacency matrix of P_3. In particular,

$$A^2 = A^{2(1)} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix};$$
$$A^4 = A^{2(2)} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix};$$
$$A^6 = A^{2(3)} = \begin{bmatrix} 4 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 4 \end{bmatrix};$$
$$A^8 = A^{2(4)} = \begin{bmatrix} 8 & 0 & 8 \\ 0 & 16 & 0 \\ 8 & 0 & 8 \end{bmatrix};$$
$$A^{10} = A^{2(5)} = \begin{bmatrix} 16 & 0 & 16 \\ 0 & 32 & 0 \\ 16 & 0 & 16 \end{bmatrix}.$$

Therefore if we let $v_1^{2(n)}$, $v_2^{2(n)}$, and $v_3^{2(n)}$ be the number of closed walks of length $2n$ from $v_1 \longrightarrow v_1$, $v_2 \longrightarrow v_2$, and $v_3 \longrightarrow v_3$ respectively, and we obtain

n	$v_1^{2(n)}$	$v_2^{2(n)}$	$v_3^{2(n)}$
1	1	2	1
2	2	4	2
3	4	8	4
4	8	16	8
5	16	32	16

The first thing that we notice is that certain vertices can be written in terms of other vertices. Namely, by the symmetry of P_3, we see that $v_1^{2(n)}{=}v_3^{2(n)}$. Further, since any closed walks of length $2n$ from v_1 contains a unique closed walk of length $2(n-1)$ from v_2, we have $v_2^{2(n-1)}{=}v_1^{2(n)}{=}v_3^{2(n)}$.

Conjecture 3.1.5. *For* $n{\in}\mathbb{Z}_{\geq 1}$, $v_1^{2(n)}{=}v_2^{2(n-1)}{=}v_3^{2(n)}{=}2^{n-1}$.

Assuming the conjecture, the number of closed walks of length n in P_3 is: $v_2^{2n} + v_2^{2n-2}{=}2^n + 2(2^{n-1}){=}2^n + 2^n{=}2^{n+1}$. Thus, our relationship between an expression and a power sum looks as follows

$$2^{n+1} = (\sqrt{2})^{2n} + (0)^{2n} + (-\sqrt{2})^{2n} = (2)^n + (2)^n.$$

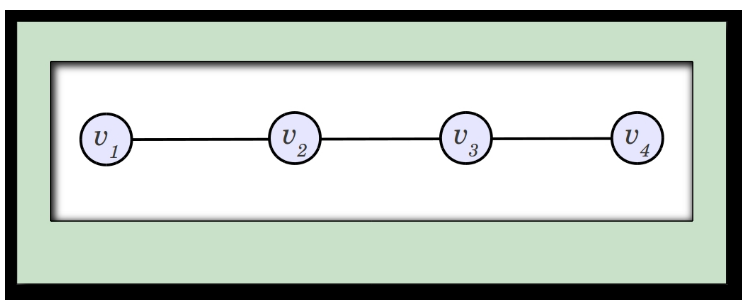
Now, remember that $\sum_{k=0}^{n}\binom{n}{k}=2^n$. So, let $n+1$ be treated as one variable, then we obtain

$$\sum_{k=0}^{n}\binom{n+1}{k} = 2^{n+1}.$$

Thus, We have constructed an equation where a combinatorial formula equals the power sum that was derived from our eigenvalues in accordance with Theorem 2.0.22.

3.1.3 Examination of P_4

Thus, we now examine P_4, which will be the last path that we examine before moving on to *Unclassified Graphs*. Take note that here, as with our other paths, that P_4 is bipartite. Therefore, we need only look at closed walks of even length. Nonetheless, we have the following picture of a graph P_4.



Proposition 3.1.6. *The number of closed walks of even length on P_4 is*

$$\frac{(3+\sqrt{5})^n + (3-\sqrt{5})^n}{2^{n-1}}.$$

Proof. The adjacency matrix for this graph P_4 is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

and the eigenvalues are

$$\lambda_0 = \sqrt{\frac{3+\sqrt{5}}{2}}, \lambda_1 = \sqrt{\frac{3-\sqrt{5}}{2}}, \lambda_2 = -\sqrt{\frac{3+\sqrt{5}}{2}}, \text{ and } \lambda_3 = -\sqrt{\frac{3-\sqrt{5}}{2}}.$$

By Theorem 2.0.22 (The Combinatorial Trace Method), we get the following power sum

$$\begin{aligned}
&\left(\sqrt{\frac{3+\sqrt{5}}{2}}\right)^{2n} + \left(\sqrt{\frac{3-\sqrt{5}}{2}}\right)^{2n} + \left(-\sqrt{\frac{3+\sqrt{5}}{2}}\right)^{2n} + \left(-\sqrt{\frac{3-\sqrt{5}}{2}}\right)^{2n} \\
&= \left(\frac{3+\sqrt{5}}{2}\right)^{n} + \left(\frac{3-\sqrt{5}}{2}\right)^{n} + \left(\frac{3+\sqrt{5}}{2}\right)^{n} + \left(\frac{3-\sqrt{5}}{2}\right)^{n} \\
&= 2\left(\frac{3+\sqrt{5}}{2}\right)^{n} + 2\left(\frac{3-\sqrt{5}}{2}\right)^{n} \\
&= \frac{(3+\sqrt{5})^n + (3-\sqrt{5})^n}{2^{n-1}}.
\end{aligned}$$

$\square$

Combinatorial Discussion

Here, we show that we can develop some formula and establish an equality with our above power sum. First, according to Theorem 2.0.10, as we have seen earlier, the number of closed walks from $v_1 \longrightarrow v_1$, $v_2 \longrightarrow v_2$, $v_3 \longrightarrow v_3$, and $v_4 \longrightarrow v_4$ of length $2n$ are determined by the $a_{11}^{2(n)}$, $a_{22}^{2(n)}$, $a_{33}^{2(n)}$, and $a_{44}^{2(n)}$ entries, respectively, in the n^{th} power of the adjacency matrix of P_4. Therefore we take the following powers of the adjacency matrix of graph P_4:

$$A^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix};$$

$$A^4 = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 0 & 5 & 0 & 3 \\ 3 & 0 & 5 & 0 \\ 0 & 3 & 0 & 2 \end{bmatrix};$$

$$A^6 = \begin{bmatrix} 5 & 0 & 8 & 0 \\ 0 & 13 & 0 & 8 \\ 8 & 0 & 13 & 0 \\ 0 & 8 & 0 & 5 \end{bmatrix};$$

$$A^8 = \begin{bmatrix} 13 & 0 & 21 & 0 \\ 0 & 34 & 0 & 21 \\ 21 & 0 & 34 & 0 \\ 0 & 21 & 0 & 13 \end{bmatrix};$$

$$A^{10} = \begin{bmatrix} 34 & 0 & 55 & 0 \\ 0 & 89 & 0 & 55 \\ 55 & 0 & 89 & 0 \\ 0 & 55 & 0 & 34 \end{bmatrix}.$$

So, if we let $v_1^{2(n)}$, $v_2^{2(n)}$, $v_3^{2(n)}$, and $v_4^{2(n)}$ be closed walks of length $2n$ from $v_1 \longrightarrow v_1$, $v_2 \longrightarrow v_2$, $v_3 \longrightarrow v_3$, and $v_4 \longrightarrow v_4$ respectively, we obtain:

n	$v_1^{2(n)}$	$v_2^{2(n)}$	$v_3^{2(n)}$	$v_4^{2(n)}$
1	1	2	2	1
2	2	5	5	2
3	5	13	13	5
4	13	34	34	13
5	34	89	89	34

Also, as before, we can observe that we can simplify our work by writing the sum of some closed walks in terms of others. Let us look at $v_1^{2(n)} = 1$, $v_2^{2(n)} = 2$, $v_3^{2(n)} = 2$, and $v_4^{2(n)} = 1$. We see that $v_1^{2(n)}=v_4^{2(n)}$ and that $v_2^{2(n)}=v_3^{2(n)}$. Also, we should take note of the Fibonacci numbers. Look at v_1^8 and v_2^8. Notice that 13 is the seventh Fibonacci number and that 34 is the ninth. Thus, we can construct the following pattern:

$$13 = F_7 = v_1^8 = v_4^8 \text{ and } 34 = F_9 = v_2^8 = v_3^8.$$

Conjecture 3.1.7. *For* $n\in\mathbb{Z}_{\geq 1}$, $F_{2(n+1)} = v_1^{2(n+1)} = v_2^{2(n)} = v_3^{2(n)} = v_3^{2(n+1)}$.

Therefore, assuming the conjecture, the formula expressing our total number of walks is:

$$2F_{2(n)-1} + 2F_{2(n)+1},$$

and our equation representing a formula to express the number of closed walks equaling a power sum is

$$2(F_{2n-1} + F_{2n+1}) = \frac{(3+\sqrt{5})^n + (3-\sqrt{5})^n}{2^{n-1}}.$$

3.2 GRAPH: Unclassified

In this section, we deal with unclassified graphs, graphs on two vertices on which we add edges and loops. In addition, opposed to what we did in the previous section with the combinatorial explanation, our approach will be a little different. However, we will still come up with an equality where some combinatorial formula equals a power sum.

Before beginning, let us give a theorem and a corollary that will be pertinent to and will ameliorate our task of finding a combinatorial formula that we will need to build our equality.

Proposition 3.2.1. [6] *Let n be a positive integer. Then for all x and y,*

$$\begin{aligned}(x+y)^n &= x^n + \binom{n}{1} yx^{n-1} + \binom{n}{2} y^2x^{n-2} + ... + \binom{n}{n-1} y^{n-1}x + y^n \\ &= \sum_{k=0}^{n} \binom{n}{k} y^k x^{n-k}.\end{aligned}$$

Proof. Write $(x+y)^n$ as a product of n factors each equal to $x+y$. Suppose we expand this product until no parentheses remain. Since for each factor $(x+y)$ we can either choose x or y, there are 2^n terms that result and each can be arranged in the form $y^k x^{n-k}$ for some $k = 0, 1, ..., n$. We obtain the term $y^k x^{n-k}$ by choosing y in k of the factors and x in the remaining $n-k$ factors. Thus the number of times the term $y^k x^{n-k}$ occurs in the expanding product equals the number of K-combinations of the set of n factors, and this is $\binom{n}{k}$. Thus

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k x^{n-k}.$$

□

Corollary 3.2.2. *Let n be a positive integer. Then for all y,*

$$(1+y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k.$$

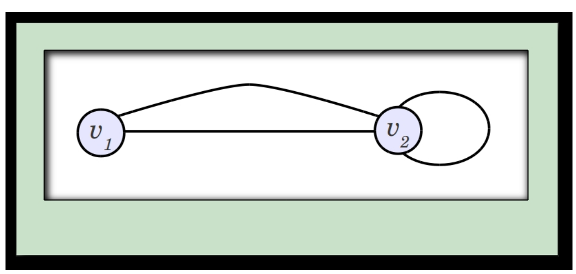
Proof. The proof follows from the preceding theorem:

$$(1+y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} y^k.$$

□

3.2.1 Examination of $Unclassified$ 2

In this first subsection, we deal with what we are considering to be the graph $Unclassified$ 2, a graph with two vertices, two edges, and a loop. We start with this graph because we have already dealt with what could be considered the graph $Unclassified$ 1–a graph with 2 vertices, an edge, and a loop–in Example 1.0.1. So, we begin here. Now, take a look at the following graph.



Proposition 3.2.3. *The number of closed walks of length n on the graph $Unclassified$ 2 is*

$$\frac{(1+\sqrt{17})^n + (1-\sqrt{17})^n}{2^n}.$$

Proof. The adjacency matrix for this graph is

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix},$$

and the eigenvalues are

$$\lambda_0 = \frac{1+\sqrt{17}}{2} \text{ and } \lambda_1 = \frac{1-\sqrt{17}}{2}.$$

Therefore, by Theorem 2.0.22, we obtain

$$\frac{(1+\sqrt{17})^n}{2^n} + \frac{(1-\sqrt{17})^n}{2^n} = \frac{(1+\sqrt{17})^n + (1-\sqrt{17})^n}{2^n}.$$

□

Proposition 3.2.4. *The number of closed walks of length n on the graph $Unclassified$ 2 is*

$$\frac{1}{2^{n-1}} \sum_{k=0}^{\lceil \frac{n}{2} \rceil} \binom{n}{2k} (\sqrt{17})^k.$$

Proof. Now, according to Corollary 3.2.2, we see that

$$(1+\sqrt{17})^n = \sum_{k=0}^{n} \binom{n}{k} (\sqrt{17})^k,$$

and that

$$(1-\sqrt{17})^n = \sum_{k=0}^{n} \binom{n}{k} (-\sqrt{17})^k.$$

So, we obtain that the number of closed walks of length n on the graph $Unclassified$ 2 are also

$$\frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (\sqrt{17})^k + \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (-\sqrt{17})^k.$$

However,

$$\begin{aligned}
\frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (\sqrt{17})^k + \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (-\sqrt{17})^k &= \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} [\sqrt{17}^k + (-1)^k \sqrt{17}^k] \\
&= \frac{1}{2^n} \sum_{k=0}^{\lceil \frac{n}{2} \rceil} \binom{n}{2k} \cdot 2(\sqrt{17})^k \\
&= \frac{1}{2^{n-1}} \sum_{k=0}^{\lceil \frac{n}{2} \rceil} \binom{n}{2k} (\sqrt{17})^k
\end{aligned}$$

□

Combining Proposition 3.2.3 and Proposition 3.2.4, we obtain the following equality

$$\frac{1}{2^{n-1}}\sum_{k=0}^{\lceil \frac{n}{2} \rceil}\binom{n}{2k}(\sqrt{17})^k = \frac{(1+\sqrt{17})^n+(1-\sqrt{17})^n}{2^n}.$$

Thus, we have an equality where a combinatorial expression equals some power sum. From here, we can also take traces of the adjacency matrix A^n and, according to Theorem 2.0.10 and Theorem 2.0.22, show that our equality works.

Example 3.2.5. Consider the $Unclassified$ 2 graph. According to Theorem 2.0.10, the number of closed walks from $v_1 \longrightarrow v_1$ and $v_2 \longrightarrow v_2$ of length n are determined by the $a_{11}^{(n)}$ and $a_{22}^{(n)}$ entries, respectively, in the n^{th} power of the adjacency matrix of our graph:

$$\begin{aligned}
A^2 &= \begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix}; \\
A^3 &= \begin{bmatrix} 4 & 10 \\ 10 & 9 \end{bmatrix}; \\
A^4 &= \begin{bmatrix} 20 & 18 \\ 18 & 29 \end{bmatrix}; \\
A^5 &= \begin{bmatrix} 36 & 58 \\ 58 & 65 \end{bmatrix}; \\
A^6 &= \begin{bmatrix} 116 & 130 \\ 130 & 181 \end{bmatrix}.
\end{aligned}$$

So, we will let $v_1^{(n)}$ and $v_2^{(n)}$ be closed walks of length n from $v_1 \longrightarrow v_1$ and $v_2 \longrightarrow v_2$ respectively. As a result,

n	$v_1^{(n)}$	$v_2^{(n)}$
0	1	1
1	0	1
2	4	5
3	4	9
4	20	29
5	36	65
6	116	181
7	260	441

Let us look at the closed walks of length 3. By taking the trace of the adjacency matrix A^3, we get

$$a_{11}^{(3)} + a_{22}^{(3)} = 4 + 9 = 13,$$

and, using the combinatorial formula in Proposition 3.2.4, we obtain

$$\begin{aligned}
\frac{1}{2^2}\left(\binom{3}{0}(\sqrt{17})^0 + \binom{3}{2}(\sqrt{17})^2\right) &= \frac{1}{2^2}\left(1 + \binom{3}{2}(17)\right) \\
&= \frac{1}{4}(1 + 3(17)) \\
&= \frac{51}{4} \\
&= 13.
\end{aligned}$$

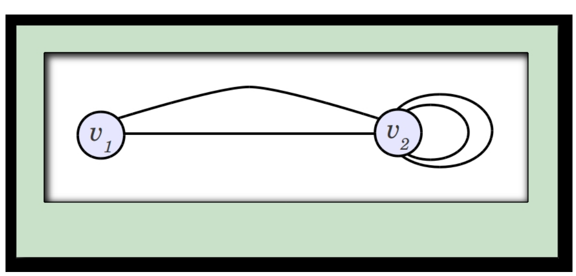
Now, using the power sum in Proposition 3.2.3, we obtain

$$\begin{aligned}
\frac{(1+\sqrt{17})^3 + (1-\sqrt{17})^3}{2^3} &= \frac{(1+\sqrt{17})(1+\sqrt{17})(1+\sqrt{17}) + (1-\sqrt{17})(1-\sqrt{17})(1-\sqrt{17})}{8} \\
&= \frac{(1 + 20\sqrt{17} + 51) + (1 - 20\sqrt{17} + 51)}{8} \\
&= \frac{104}{8} = 13.
\end{aligned}$$

Thus, we see that the trace, the power sum from Proposition 3.2.3, and the combinatorial formula from Proposition 3.2.4 all yield the same answer.

3.2.2 Examination of $Unclassified$ 3

In this section, the next graph we deal with is a graph that we are considering to be the $Unclassified$ 3 graph, a graph with two vertices, two edges, and two loops. Take a look at the following graph.



Proposition 3.2.6. *The number of closed walks on the graph* $Unclassified$ 3 *of length* n *is*

$$(1+\sqrt{5})^n+(1-\sqrt{5})^n.$$

Proof. The adjacency matrix for this graph is

$$A=\begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix},$$

and the eigenvalues are

$$\lambda_0=\frac{2+\sqrt{20}}{2} \text{ and } \lambda_1=\frac{2-\sqrt{20}}{2}.$$

Notice that $\frac{2+\sqrt{20}}{2}$ can be written as $\frac{2+\sqrt{4}\sqrt{5}}{2}=\frac{2+2\sqrt{5}}{2}$. We can factor out a 2 and obtain the following

$$\frac{2(1+\sqrt{5})}{2}=(1+\sqrt{5}).$$

The same can be done for $\frac{2-\sqrt{20}}{2}$. Therefore, by Theorem 2.0.22, we obtain that the number of closed walks of length n are

$$(1+\sqrt{5})^n+(1-\sqrt{5})^n.$$

□

Proposition 3.2.7. *The number of closed walks of length* n *on the graph* $Unclassified$ 2 *is*

$$2\sum_{k=0}^{\lceil \frac{n}{2} \rceil}\binom{n}{2k}(\sqrt{5})^k.$$

Proof. Now, according to Corollary 3.2.2, we see that

$$(1+\sqrt{5})^n=\sum_{k=0}^{n}\binom{n}{k}(\sqrt{5})^k,$$

and that

$$(1-\sqrt{5})^n=\sum_{k=0}^{n}\binom{n}{k}(-\sqrt{5})^k.$$

So, we obtain that the number of closed walks of length n on the graph $Unclassified$ 3 are also

$$\sum_{k=0}^{n}\binom{n}{k}(\sqrt{5})^k+\sum_{k=0}^{n}\binom{n}{k}(-\sqrt{5})^k.$$

However,

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(\sqrt{5})^k+\sum_{k=0}^{n}\binom{n}{k}(-\sqrt{5})^k &= \sum_{k=0}^{n}\binom{n}{k}[\sqrt{5}^k+(-1)^k\sqrt{5}^k] \\ &= \sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n}{2k}\cdot 2(\sqrt{5})^k \\ &= 2\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n}{2k}(\sqrt{5})^k\end{aligned}$$

□

Now, combining Proposition 3.2.6 and Proposition 3.2.7, we obtain the following equality

$$2\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n}{2k}(\sqrt{5})^k=(1+\sqrt{5})^n+(1-\sqrt{5})^n.$$

Thus, we now have an equality where a combinatorial expression equals some power sum. Moreover, as we have done in the preceding section with the $Unclassified$ 2 graph, we can also take traces of the adjacency matrix A^n and, according to Theorem 2.0.10 and Theorem 2.0.22, show that our equality works, which we will do in the following example.

Example 3.2.8. Consider the graph $Unclassified$ 3. According to theorem 2.0.10, the number of closed walks from $v_1 \longrightarrow v_1$ss and $v_2 \longrightarrow v_2$ of length n are determined by the a_{11}^n and a_{22}^n entries, respectively, in the n^{th} power of the adjacency matrix of our graph:

$$A^2 = \begin{bmatrix} 4 & 4 \\ 4 & 8 \end{bmatrix};$$
$$A^3 = \begin{bmatrix} 8 & 16 \\ 16 & 24 \end{bmatrix};$$
$$A^4 = \begin{bmatrix} 32 & 48 \\ 48 & 80 \end{bmatrix};$$
$$A^5 = \begin{bmatrix} 96 & 160 \\ 160 & 256 \end{bmatrix}.$$

So, we let $v_1^{(n)}$ and $v_2^{(n)}$ be closed walks of length n from $v_1 \longrightarrow v_1$ and $v_2 \longrightarrow v_2$ respectively. Now, we have

n	$v_1^{(n)}$	$v_2^{(n)}$
0	1	1
1	0	2
2	4	8
3	8	24
4	32	80
5	96	256

We will look at the closed walks of length 2. By taking the trace of the adjacency matrix A^2, we get

$$a_{11}^{(2)} + a_{22}^{(2)} = 4 + 8 = 12,$$

and, using the combinatorial formula from Proposition 3.2.7, we obtain

$$\begin{aligned} 2\left(\binom{2}{0}(\sqrt{5})^0 + \binom{2}{2}(\sqrt{5})^2\right) &= 2(1 + \binom{2}{2}(5)) \\ &= 2(1 + 1(5)) \\ &= 12. \end{aligned}$$

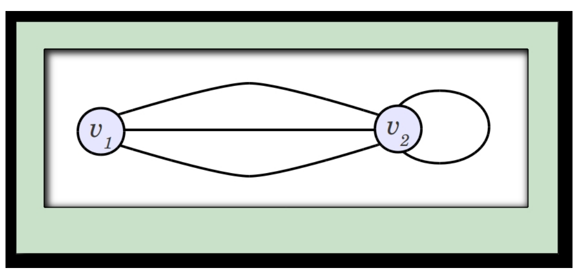
Now, using the power sum in Proposition 3.2.6, we obtain

$$
\begin{aligned}
(1+\sqrt{5})^2 + (1-\sqrt{5})^2 &= (1+\sqrt{5})(1+\sqrt{5}) + (1-\sqrt{5})(1-\sqrt{17}) \\
&= (1+\sqrt{5}+5) + (1-\sqrt{5}+5) \\
&= 1+5+1+5 = 12.
\end{aligned}
$$

Thus, we see that the trace, the power sum from Proposition 3.2.6, and the combinatorial formula from Proposition 3.2.7 all yield the same answer.

3.2.3 Examination of $Unclassified$ 4

In this section, as with the two previous sections, the next graph that we will deal with is a graph that we are considering to be the $Unclassified$ 4 graph, a graph with two vertices, three edges, and a loop.



Proposition 3.2.9. *The number of closed walks on the graph $Unclassified$ 4 of length n is*

$$\frac{(1+\sqrt{37})^n}{2^n} + \frac{(1-\sqrt{37})^n}{2^n}.$$

Proof. The adjacency matrix for this graph is:

$$A=\begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix},$$

and the eigenvalues are:

$$\lambda_0 = \frac{1+\sqrt{37}}{2} \text{ and } \lambda_1 = \frac{1-\sqrt{37}}{2}.$$

Therefore, by Theorem 2.0.22, we obtain that the number of closed walks of length n are

$$\frac{(1+\sqrt{37})^n}{2^n}+\frac{(1-\sqrt{37})^n}{2^n}.$$

$\square$

Proposition 3.2.10. *The number of closed walks of length n on the graph $Unclassified$ 4 is*

$$\frac{1}{2^{n-1}}\sum_{k=0}^{\lceil \frac{n}{2}\rceil}\binom{n}{2k}(\sqrt{37})^k.$$

Proof. Now, according to Corollary 3.2.2, we see that

$$(1+\sqrt{37})^n=\sum_{k=0}^{n}\binom{n}{k}(\sqrt{37})^k,$$

and that

$$(1-\sqrt{37})^n=\sum_{k=0}^{n}\binom{n}{k}(-\sqrt{37})^k.$$

So, we obtain that the number of closed walks of length n on the $Unclassified$ 4 graph are

$$\frac{1}{2^n}(\sum_{k=0}^{n}\binom{n}{k}(\sqrt{37})^k+\sum_{k=0}^{n}\binom{n}{k}(-\sqrt{37})^k).$$

However,

$$\begin{aligned}\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(\sqrt{37})^k+\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(-\sqrt{37})^k &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}[\sqrt{37}^k+(-1)^k\sqrt{37}^k]\\ &= \frac{1}{2^n}\sum_{k=0}^{\lceil \frac{n}{2}\rceil}\binom{n}{2k}\cdot 2(\sqrt{37})^k\\ &= \frac{1}{2^{n-1}}\sum_{k=0}^{\lceil \frac{n}{2}\rceil}\binom{n}{2k}(\sqrt{37})^k.\end{aligned}$$

$\square$

Therefore, combining Proposition 3.2.9 and Proposition 3.2.10, we obtain the following equality

$$\frac{1}{2^{n-1}} \sum_{k=0}^{\lceil \frac{n}{2} \rceil} \binom{n}{2k} (\sqrt{37})^k = \frac{(1+\sqrt{37})^n + (1-\sqrt{37})^n}{2^n}.$$

Thus, we have an equality where a combinatorial expression equals some power sum, and, as we have done in the preceding two sections with the graphs $Unclassified$ 2 and $Unclassified$ 3, we will also, in the following example, take traces of the adjacency matrix A^n and show that our equality works.

Example 3.2.11. Consider the $Unclassified$ 2 graph. According to theorem 2.0.10, the number of closed walks from $v_1 \longrightarrow v_1$ and $v_2 \longrightarrow v_2$ of length n are determined by the $a_{11}^{(n)}$ and $a_{22}^{(n)}$ entries, respectively, in the n^{th} power of the adjacency matrix of our graph:

$$A^2 = \begin{bmatrix} 9 & 3 \\ 3 & 10 \end{bmatrix};$$
$$A^3 = \begin{bmatrix} 9 & 30 \\ 30 & 19 \end{bmatrix};$$
$$A^4 = \begin{bmatrix} 90 & 57 \\ 57 & 109 \end{bmatrix};$$
$$A^5 = \begin{bmatrix} 171 & 327 \\ 327 & 280 \end{bmatrix}.$$

So, we will let $v_1^{(n)}$ and $v_2^{(n)}$ be closed walks of length n from $v_1 \longrightarrow v_1$ and $v_2 \longrightarrow v_2$ respectively. Then we obtain:

n	$v_1^{(n)}$	$v_2^{(n)}$
0	1	1
1	0	1
2	9	10
3	9	19
4	90	109
5	171	280

We will look at the closed walks of length 3. By taking the trace of the adjacency matrix A^3, we get

$$a_{11}^{(3)} + a_{22}^{(3)} = 9 + 19 = 28,$$

and, using the combinatorial formula in Proposition 3.2.10, we obtain

$$\begin{aligned}\frac{1}{2^2}\left(\binom{3}{0}(\sqrt{37})^0 + \binom{3}{2}(\sqrt{37})^2\right) &= \frac{1}{2^2}\left(1 + \binom{3}{2}(37))\right) \\ &= \frac{1}{4}(1 + 3(37)) \\ &= \frac{112}{4} \\ &= 28.\end{aligned}$$

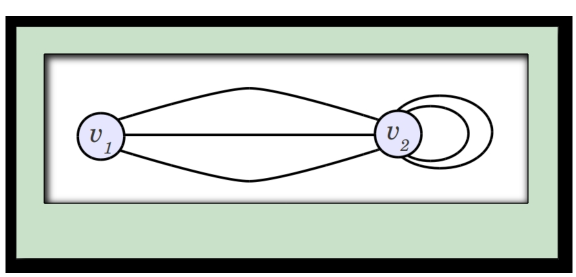
Now, using the power sum in Proposition 3.2.9, we obtain

$$\begin{aligned}\frac{(1+\sqrt{37})^3 + (1-\sqrt{37})^3}{2^3} &= \frac{(1+\sqrt{37})(1+\sqrt{37})(1+\sqrt{37}) + (1-\sqrt{37})(1-\sqrt{37})(1-\sqrt{37})}{8} \\ &= \frac{(1+20\sqrt{37}+111) + (1-20\sqrt{37}+111)}{8} \\ &= \frac{224}{8} = 28.\end{aligned}$$

Thus, we see that the trace, the power sum from Proposition 3.2.9, and the combinatorial formula from Proposition 3.2.10 all yield the same answer.

3.2.4 Examination of $Unclassified$ 5

Lastly, in this section, we deal with the graph that we are considering to be the $Unclassified$ 5 graph, a graph with two vertices, three edges, and two loops.



Proposition 3.2.12. *The number of closed walks on the $Unclassified$ 5 graph of length n is*

$$(1+\sqrt{10})^n + (1-\sqrt{10})^n.$$

Proof. The adjacency matrix for this graph is:

$$A=\begin{bmatrix} 0 & 3 \\ 3 & 2 \end{bmatrix},$$

and the eigenvalues are:

$$\lambda_0 = \frac{2+\sqrt{40}}{2} \text{ and } \lambda_1 = \frac{2-\sqrt{40}}{2}.$$

Notice that $\frac{2+\sqrt{40}}{2}$ can be written as $\frac{2+\sqrt{4}\sqrt{10}}{2}=\frac{2+2\sqrt{10}}{2}$. We can factor out a 2 and we obtain the following

$$\frac{2(1+\sqrt{10})}{2} = (1+\sqrt{10}).$$

The same can be done for $\frac{2-\sqrt{20}}{2}$. Therefore, by Theorem 2.0.22, we obtain that the number of closed walks of length n are

$$(1+\sqrt{10})^n + (1-\sqrt{10})^n.$$

□

Proposition 3.2.13. *The number of closed walks of length n on the graph $Unclassified$ 2 is*

$$2\sum_{k=0}^{\lceil \frac{n}{2} \rceil} \binom{n}{2k} (\sqrt{10})^k.$$

Proof. Now, according to Corollary 3.2.2, we see that

$$(1+\sqrt{10})^n = \sum_{k=0}^{n} \binom{n}{k} (\sqrt{10})^k,$$

and that

$$(1-\sqrt{10})^n = \sum_{k=0}^{n} \binom{n}{k} (-\sqrt{10})^k.$$

Therefore, we obtain that the number of closed walks of length n on the $Unclassified$ 5 graph are also equal to the following expression

$$\sum_{k=0}^{n}\binom{n}{k}(\sqrt{10})^k+\sum_{k=0}^{n}\binom{n}{k}(-\sqrt{10})^k,$$

However,

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(\sqrt{10})^k+\sum_{k=0}^{n}\binom{n}{k}(-\sqrt{10})^k &= \sum_{k=0}^{n}\binom{n}{k}[\sqrt{10}^k+(-1)^k\sqrt{10}^k]\\ &= \sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n}{2k}\cdot 2(\sqrt{10})^k\\ &= 2\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n}{2k}(\sqrt{10})^k.\end{aligned}$$

$\square$

Now, combining Proposition 3.3.12 and Proposition 3.2.13, we obtain the following equality

$$2\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\binom{n}{2k}(\sqrt{10})^k=(1+\sqrt{10})^n+(1-\sqrt{10})^n.$$

Thus, we have an equality where a combinatorial expression equals some power sum, and, as we have done in the preceding sections, we will take traces of the adjacency matrix A^n and, according to Theorem 2.0.10 and Theorem 2.0.22, show that our equality works.

Example 3.2.14. Now, consider the $Unclassified$ 5 graph. According to Theorem 2.0.10, the number of closed walks from $v_1 \longrightarrow v_1$ and $v_2 \longrightarrow v_2$ of length n are determined by the $a_{11}^{(n)}$ and $a_{22}^{(n)}$ entries, respectively, in the n^{th} power of the adjacency matrix of our graph:

$$A^2 = \begin{bmatrix} 9 & 6 \\ 6 & 13 \end{bmatrix};$$
$$A^3 = \begin{bmatrix} 18 & 39 \\ 39 & 44 \end{bmatrix};$$
$$A^4 = \begin{bmatrix} 117 & 132 \\ 132 & 205 \end{bmatrix};$$
$$A^5 = \begin{bmatrix} 396 & 615 \\ 615 & 806 \end{bmatrix}.$$

So, we will let $v_1^{(n)}$, $v_2^{(n)}$ be closed walks of length n from $v_1 \longrightarrow v_1$ and $v_2 \longrightarrow v_2$ respectively. Then we obtain:

n	v_1^n	v_2^n
0	1	1
1	0	2
2	9	13
3	18	44
4	117	205
5	396	806

Looking at the closed walks of length 2 and taking the trace of the adjacency matrix A^2, we get

$$a_{11}^{(2)} + a_{22}^{(2)} = 4 + 8 = 22.$$

Now, using Proposition 3.2.13, we obtain

$$\begin{aligned} 2\left(\binom{2}{0}(\sqrt{10})^0 + \binom{2}{2}(\sqrt{10})^2\right) &= 2(1 + \binom{2}{2}(10)) \\ &= 2(1 + 1(10)) \\ &= 22, \end{aligned}$$

and, using Proposition 3.2.12, we obtain

$$\begin{aligned} (1+\sqrt{10})^2 + (1-\sqrt{10})^2 &= (1+\sqrt{10})(1+\sqrt{10}) + (1-\sqrt{10})(1-\sqrt{10}) \\ &= (1+\sqrt{10}+5) + (1-\sqrt{10}+5) \\ &= 1 + 10 + 1 + 10 = 12. \end{aligned}$$

Thus, we see that the trace, the power sum from Proposition 3.2.12, and the combinatorial expression from Proposition 3.2.13 all yield the same answer.

Now, we will move onto the graphs C_v, and see what interesting things we can find.

3.3 GRAPH: C_v (A Cycle Graph on v vertices)

In this section, in an attempt to deal with a more complicated situation, we will examine three graphs C_3, C_4, and, C_5.

Before moving on, we state a result as found in [4]. This will ameliorate our task, and we will use Definition 3.3.1 and Proposition 3.3.2 to find the eigenvalues of C_v graphs, as well as develop a general equality for the family of graphs C_v.

Definition 3.3.1. A matrix C is called circulant if it can be written in the form:

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \dots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \dots & c_0 \end{bmatrix}. \tag{3.3.1}$$

Proposition 3.3.2. *The adjacency matrix of C_v is circulant for all $v \geq 3$.*

Proof. The adjacency matrix of a C_v graph looks like

$$C_v = \begin{bmatrix} 0 & 1 & 0 & \dots & 1 \\ 1 & 0 & 1 & \dots & 0 \\ 0 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

If we look at a C_v graph's adjacency matrix, and take into account that a vertex is only adjacent to the vertex before and after it, we see that the ones lie in the adjacency matrix centered along the diagonal.(i.e., with the graphs C_v, v_n is adjacent to v_{n-1}). Thus, we see that the adjacency matrix of a graph C_v will be circulant. □

As a result, of all graphs C_v being circulant, the following proposition will be crucial to finding the eigenvalues for this family of graphs and establishing our general equalities for them. The following proposition is taken from [4] and is stated without proof.

Proposition 3.3.3. *The eigenvalues of the circulant matrix C given in equation (3.3.1) are:*

$$\chi_a = \sum_{j=0}^{n-1} c_j \xi^{aj},$$

where $\xi{=}e^{\frac{2\pi a}{n}}{=}\cos(\frac{2\pi a}{n}) + i\sin(\frac{2\pi a}{n})$ *and* $a = 0, 1, 2, ..., n-1$.

By Proposition 3.3.2, the eigenvalues of C_v are

$$\chi_a{=}\xi^a{+}\xi^{a(n-1)} {=}\cos(\tfrac{2\pi a}{n}){+}i\sin(\tfrac{2\pi a}{n}){+}\cos(\tfrac{2\pi a(n-1)}{n})i\sin(\tfrac{2\pi a(n-1)}{n}),$$

where $a{=}0, 1, 2, ..., n-1$. Note that

$$\cos(\frac{2\pi a(n-1)}{n}) = \cos(\frac{-2\pi a}{n}) = \cos(\frac{2\pi a}{n}).$$

Similarly,

$$\sin(\frac{2\pi a(n-1)}{n}) = -\sin(\frac{2\pi a}{n}).$$

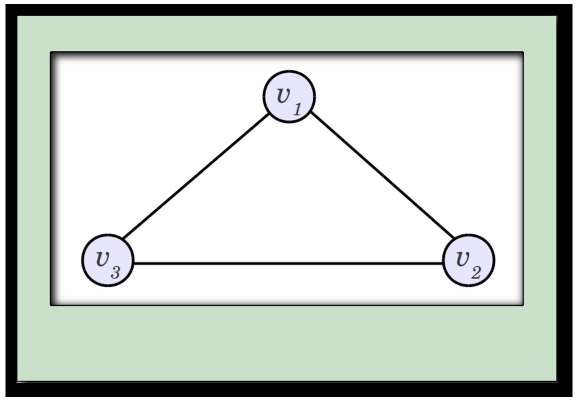
Therefore, we get

$$\chi_a = 2\cos\left(\frac{2\pi a}{n}\right).$$

Now, we will examine the first graph of this section: C_3.

3.3.1 Examination of C_3

We begin this section with the graph C_3, a cycle with three vertices and three edges as shown in the following picture.



Proposition 3.3.4. *The number of closed walks on C_3 of length n is*

$$2^n + (-1)^n \cdot 2.$$

Proof. The adjacency matrix for C_3 is

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

and its eigenvalues are

$$\lambda_0 = 2, \lambda_1 = -1, \text{ and } \lambda_2 = -1.$$

Therefore, by Theorem 2.0.22, we obtain

$$2^n + (-1)^n + (-1)^n = 2^n + 2(-1)^n.$$

$\square$

Combinatorial Discussion

As we seen with paths, according to Theorem 2.0.10, the number of closed walks from $v_1 \rightarrow v_1$, $v_2 \rightarrow v_2$, and $v_3 \rightarrow v_3$ of length n are determined by the $a_{11}^{(n)}$, $a_{22}^{(n)}$, and $a_{33}^{(n)}$ entries, respectively, in the n^{th} power of the adjacency matrix of a graph. Therefore, we take the following powers of the adjacency matrix of C_3.

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix};$$
$$A^3 = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix};$$
$$A^4 = \begin{bmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{bmatrix};$$
$$A^5 = \begin{bmatrix} 10 & 11 & 11 \\ 11 & 10 & 11 \\ 11 & 11 & 10 \end{bmatrix};$$
$$A^6 = \begin{bmatrix} 22 & 21 & 21 \\ 21 & 22 & 21 \\ 21 & 21 & 22 \end{bmatrix};$$
$$A^7 = \begin{bmatrix} 42 & 43 & 43 \\ 43 & 42 & 43 \\ 43 & 43 & 42 \end{bmatrix}.$$

Therefore, we will let $v_1^{(n)}$, $v_2^{(n)}$, and $v_3^{(n)}$ be closed walks of length n from $v_1 \longrightarrow v_1$, $v_2 \longrightarrow v_2$,and $v_3 \longrightarrow v_3$ respectively. And, we obtain:

n	$v_1^{(n)}$	$v_2^{(n)}$	$v_3^{(n)}$
0	1	1	1
1	0	0	0
2	2	2	2
3	2	2	2
4	6	6	6
5	10	10	10
6	22	22	22
7	42	42	42

First we should make our work easier by checking to see if we can write any of the sums of walks in terms of any other. Upon observation, because of the symmetry of C_3, we see that we can; namely, $v_1^{(n)}=v_2^{(n)}=v_3^{(n)}$. Therefore, any formula that we find, we need only multiply it by three–the number of vertices of this graph–to obtain the total number of closed walks of C_3.

Here, to make our observations more concrete, I would like to introduce a few entries from Pascal's triangle. Before doing so, I would like the reader to take note of the following definition.

Definition 3.3.5.

$$\binom{n}{i} = 0$$

if $i < 0$ or $i > n$.

$$\binom{0}{0}$$

$$\binom{1}{0}\binom{1}{1}$$

$$\binom{2}{0}\binom{2}{1}\binom{2}{2}$$

$$\binom{3}{0}\binom{3}{1}\binom{3}{2}\binom{3}{3}$$

$$\binom{4}{0}\binom{4}{1}\binom{4}{2}\binom{4}{3}\binom{4}{4}$$

$$\binom{5}{0}\binom{5}{1}\binom{5}{2}\binom{5}{3}\binom{5}{4}\binom{5}{5}$$

$$\binom{6}{0}\binom{6}{1}\binom{6}{2}\binom{6}{3}\binom{6}{4}\binom{6}{5}\binom{6}{6}$$

$$\binom{7}{0}\binom{7}{1}\binom{7}{2}\binom{7}{3}\binom{7}{4}\binom{7}{5}\binom{7}{6}\binom{7}{7}$$

$$\binom{8}{0}\binom{8}{1}\binom{8}{2}\binom{8}{3}\binom{8}{4}\binom{8}{5}\binom{8}{6}\binom{8}{7}\binom{8}{8}$$

Now, let us look at $v_1^{(n)}$. Let n be the length of the closed walk. Also let $n = 0$ denote the row of Pascal's triangle with the entry $\binom{0}{0}$, $n = 1$ the line containing $\binom{1}{0}$ $\binom{1}{1}$, $n = 2$ the line containing $\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$, and so on. Therefore, we obtain the following:

$$v_1^{(2)} = \binom{2}{0} + \binom{2}{2} = 2$$
$$v_1^{(3)} = \binom{3}{0} + \binom{3}{3} = 2$$
$$v_1^{(4)} = \binom{4}{2} = 6$$
$$v_1^{(5)} = \binom{5}{1} + \binom{5}{4} = 10$$
$$v_1^{(6)} = \binom{6}{0} + \binom{6}{3} + \binom{6}{6} = 22$$
$$v_1^{(7)} = \binom{7}{2} + \binom{7}{5} = 42$$

Now, we have a pattern.

Conjecture 3.3.6. $v_1^{(n)}$ *can be defined recursively by:*

$$v_1^{(2)} = \binom{2}{0} + \binom{2}{2} = 2$$
$$v_1^{(3)} = \binom{3}{0} + \binom{3}{3} = 2.$$

For $n \geq 4$, *let*

$$\Lambda_n = \left\{ i \mid \binom{n-1}{i} \textit{ is summand of } v_1^{(n)} \right\}$$

and

$$J_n = \{ j \mid j = i - 1 \textit{ or } j = i + 2, \ i \in \Lambda_n \}.$$

So,

$$v_1^{(n)} = \sum_{j \in J_n}^{n} \binom{n}{j}.$$

Example 3.3.7. Let $n = 4$

$$\Lambda_4 = \{0, \ 3\},$$

and

$$J_4 = \{-1, \ 2, \ 2, \ 5\}.$$

Therefore, we obtain

$$v_1^4 = \binom{4}{-1} + \binom{4}{2} + \binom{4}{5}.$$

Recalling and using Definition 3.3.5, we have

$$\begin{aligned} v_1^4 &= 0 + \binom{4}{2} + 0 \\ &= \binom{4}{2} \\ &= 6, \end{aligned}$$

which is the same result we obtained above.

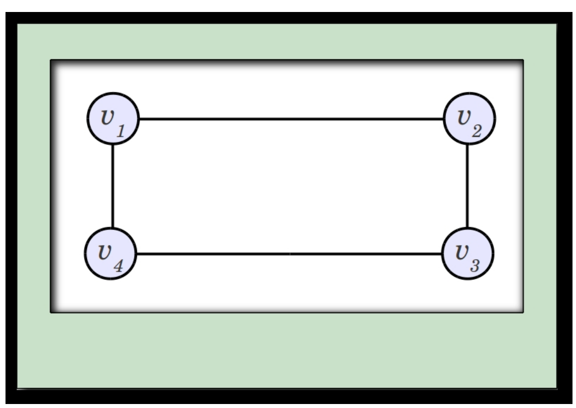
Now, assuming the conjecture, our equality is

$$3 \sum_{j \in J_n}^{n} \binom{n}{j} = 2^n + 2(-1)^n.$$

Thus, we have an equality where a combinatorial formula equals a power sum. From here, let us move on and show that we can find these same results for C_4.

3.3.2 Examination of C_4

Take note of the following C_4 graph.



The reader should immediately be aware that this graph is bipartite. If we look at C_4, we see that we can partition v_1 and v_3 into a set X and v_2 and v_4 into a set Y such that there is an edge that extends from v_1 and an edge that extends from v_3 in X to either v_2

or v_4 in Y. Also, we can observe that we can color this graph with two colors in such a way that the two adjacent vertices are not the same color: let v_1 be red, v_2 be blue, v_3 be red, and v_4 be blue. Therefore, C_4 is bipartite and 2-colorable. However, by Proposition 3.1.2, we need only look at closed walks of even length.

Proposition 3.3.8. *The number of closed walks of even length $2n$ on C_4 is $2 \cdot 4^n$.*

Proof. The adjacency matrix for this graph is

$$A=\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

and the eigenvalues are

$$\lambda_0 = 2, \lambda_1 = 0, \lambda_2 = 0, \text{ and } \lambda_3 = -2.$$

Therefore, by Theorem 2.0.22, we get

$$(2)^{2n} + (0)^{2n} + (0)^{2n} + (-2)^{2n} = 4^n + 4^n = 2 \cdot 4^n.$$

□

Combinatorial Discussion

We began by noting that according to Theorem 2.0.10, the number of closed walks from $v_1 \longrightarrow v_1$, $v_2 \longrightarrow v_2$, $v_3 \longrightarrow v_3$, and $v_4 \longrightarrow v_4$ of length n are determined by the $a_{11}^{2(n)}$, $a_{22}^{2(n)}$, $a_{33}^{2(n)}$, and $a_{44}^{2(n)}$ entries, respectively, in the n^{th} power of the adjacency matrix of a graph:

$$A^2 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix};$$

$$A^4 = \begin{bmatrix} 8 & 0 & 8 & 0 \\ 0 & 8 & 0 & 8 \\ 8 & 0 & 8 & 0 \\ 0 & 8 & 0 & 8 \end{bmatrix};$$

$$A^6 = \begin{bmatrix} 32 & 0 & 32 & 0 \\ 0 & 32 & 0 & 32 \\ 32 & 0 & 32 & 0 \\ 0 & 32 & 0 & 32 \end{bmatrix}.$$

The traces of A^1, A^3, and A^5 are zero, there are no closed walks in this graph for $n = 1, 3,$ and 5, which is another way of saying that there are no closed walks of odd length.

Nevertheless, let $v_1^{2(n)}$, $v_2^{2(n)}$, $v_3^{2(n)}$, and $v_4^{2(n)}$ be closed walks of length $2n$ from $v_1 \longrightarrow v_1$, $v_2 \longrightarrow v_2$, $v_3 \longrightarrow v_3$, and $v_4 \longrightarrow v_4$ respectively,and, as a result, we obtain

n	$v_1^{2(n)}$	$v_2^{2(n)}$	$v_3^{2(n)}$	$v_4^{2(n)}$
0	1	1	1	1
1	2	2	2	2
4	8	8	8	8
6	32	32	32	32

Notice that the diagonal entries of the powers of the adjacency matrix of C_4 are equal. As a result,$v_1^{2(n)}$=$v_2^{2(n)}$=$v_3^{2(n)}$=$v_4^{2(n)}$ for $n = 0, 1, 2, 3$. Therefore, let us look at $v_1^{2(n)}$. We can deduce here, similarly as we did with C_3, the following relation to Pascal's Triangle.

$$\begin{aligned} v_1^{2(1)} &= \binom{2}{1} = 2 \\ v_1^{2(2)} &= \binom{4}{0} + \binom{4}{2} + \binom{4}{4} = 8 \\ v_1^{2(3)} &= \binom{6}{1} + \binom{6}{3} + \binom{6}{5} = 32. \end{aligned}$$

As a result we obtain the following conjecture.

Conjecture 3.3.9. *The number of closed walks on the graph C_4 is*

$$\sum_{i=0}^{n} \binom{n}{2i}$$

when n is even, and

$$\sum_{i=0}^{n} \binom{2n}{2i-1}$$

when n is odd.

Example 3.3.10. Let $n = 2$, then n is even and we get

$$v_1^2 = \sum_{i=0}^{2} \binom{2}{2i},$$

which gives us

$$\begin{aligned} v_1^2 &= \binom{2}{0} + \binom{2}{2} + \binom{2}{4} \\ &(According\ Definition\ 3.3.5) \\ &= \binom{2}{0} + \binom{2}{2} + 0 \\ &= 1 + 1 + 0 \\ &= 2. \end{aligned}$$

Now, let $n = 3$, then n is odd and we get

$$\sum_{i=0}^{n} \binom{2n}{2i-1},$$

gives us

$$\begin{aligned} v_n^3 &= \sum_{i=0}^{3} \binom{6}{2i-1} \\ &= \binom{6}{-1} + \binom{6}{1} + \binom{6}{3} + \binom{6}{5} \\ &(According\ Definition\ 3.3.5) \\ &= 0 + \binom{6}{1} + \binom{6}{3} + \binom{6}{5} \\ &= 0 + 6 + 20 + 6 \\ &= 32. \end{aligned}$$

Thus, assuming the conjecture, an equality expressing the total number of closed walks of length n on C_4 is

$$4\sum_{i=0}^{n}\binom{n}{2i} = 2\cdot 4^n$$

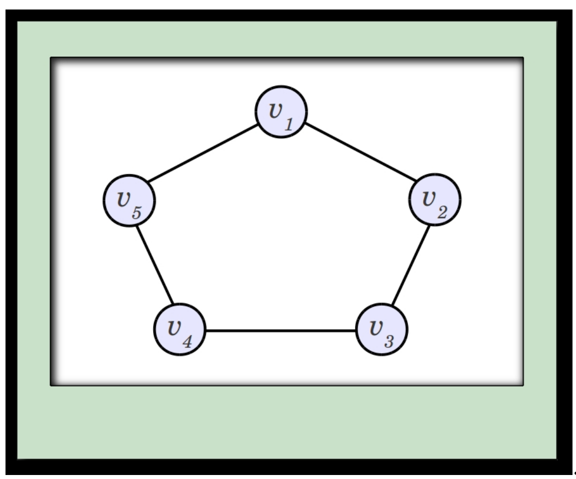
when n is even, and

$$4\sum_{i=0}^{n}\binom{2n}{2i-1} = 2\cdot 4^n$$

when n is odd. Thus, we have two equalities where a combinatorial formula equals a power sum. Therefore, let us move on to show that we can find these same results for the graph C_5.

3.3.3 Examination of C_5

Lastly, in this section, we will look at the graph C_5. Observe the following graph.



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Proposition 3.3.11. *The number of closed walks of length n on the graph C_5 is*

$$\frac{1}{2^{n-1}}(2^{2n-1} + (\sqrt{5}-1)^n + (-\sqrt{5}-1)^n).$$

Proof. The adjacency matrix for this graph is:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix},$$

and its eigenvalues are:

$$\lambda_0 = 2, \lambda_1 = \frac{\sqrt{5}-1}{2}, \lambda_2 = \frac{\sqrt{5}-1}{2}, \lambda_3 = \frac{-\sqrt{5}-1}{2} \text{ and } \lambda_4 = \frac{-\sqrt{5}-1}{2}.$$

Therefore, by Theorem 2.0.22, we get

$$\begin{aligned}
&2^n + \left(\frac{\sqrt{5}-1}{2}\right)^n + \left(\frac{\sqrt{5}-1}{2}\right)^n + \left(\frac{-\sqrt{5}-1}{2}\right)^n + \left(\frac{-\sqrt{5}-1}{2}\right)^n \\
&= 2^n + 2\left(\frac{\sqrt{5}-1}{2}\right)^n + 2\left(\frac{-\sqrt{5}-1}{2}\right)^n \\
&= \frac{1}{2^{n-1}}(2^{2n-1} + (\sqrt{5}-1)^n + (-\sqrt{5}-1)^n).
\end{aligned}$$

□

Combinatorial Discussion

We can find a formula other than the above power sum. According to Theorem 2.0.10, the number of closed walks from $v_1 \longrightarrow v_1$, $v_2 \longrightarrow v_2$, $v_3 \longrightarrow v_3$, $v_4 \longrightarrow v_4$, and $v_5 \longrightarrow v_5$ of length n are determined by the $a_{11}^{(n)}$, $a_{22}^{(n)}$, $a_{33}^{(n)}$, $a_{44}^{(n)}$, and $a_{55}^{(n)}$ entries, respectively, in the n^{th} power of the adjacency matrix of a graph:

$$A^2 = \begin{bmatrix} 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix};$$

$$A^3 = \begin{bmatrix} 0 & 3 & 1 & 1 & 3 \\ 3 & 0 & 3 & 1 & 1 \\ 1 & 3 & 0 & 3 & 1 \\ 1 & 1 & 3 & 0 & 3 \\ 3 & 1 & 1 & 3 & 0 \end{bmatrix};$$

$$A^4 = \begin{bmatrix} 6 & 1 & 4 & 4 & 1 \\ 1 & 6 & 1 & 4 & 4 \\ 4 & 1 & 6 & 1 & 4 \\ 4 & 4 & 6 & 1 & 4 \\ 1 & 4 & 4 & 1 & 6 \end{bmatrix};$$

$$A^5 = \begin{bmatrix} 2 & 10 & 5 & 5 & 10 \\ 10 & 2 & 10 & 5 & 5 \\ 5 & 10 & 2 & 10 & 5 \\ 5 & 5 & 10 & 2 & 10 \\ 10 & 5 & 5 & 10 & 2 \end{bmatrix};$$

$$A^6 = \begin{bmatrix} 20 & 7 & 15 & 15 & 7 \\ 7 & 20 & 7 & 15 & 15 \\ 15 & 7 & 20 & 7 & 15 \\ 15 & 15 & 7 & 20 & 7 \\ 7 & 15 & 15 & 7 & 20 \end{bmatrix};$$

$$A^7 = \begin{bmatrix} 14 & 35 & 22 & 22 & 35 \\ 35 & 14 & 35 & 22 & 22 \\ 22 & 35 & 14 & 35 & 22 \\ 22 & 22 & 35 & 14 & 35 \\ 35 & 22 & 22 & 35 & 14 \end{bmatrix}.$$

So, once more, we let $v_1^{(n)}$, $v_2^{(n)}$, $v_3^{(n)}$, $v_4^{(n)}$ and $v_5^{(n)}$ be closed walks of length n from $v_1 \longrightarrow v_1$, $v_2 \longrightarrow v_2$, $v_3 \longrightarrow v_3$, $v_4 \longrightarrow v_4$, and $v_5 \longrightarrow v_5$ respectively, and we obtain:

n	$v_1^{(n)}$	$v_2^{(n)}$	$v_3^{(n)}$	$v_4^{(n)}$	$v_5^{(n)}$
0	1	1	1	1	1
1	0	0	0	0	0
2	2	2	2	2	2
3	0	0	0	0	0
4	6	6	6	6	6
5	2	2	2	2	2
6	20	20	20	20	20
7	14	14	14	14	14

As with C_3 and C_4, The diagonal entries of the powers of the adjacency matrix for the graph C_5 are equal. As a result, $v_1^{(n)}=v_2^{(n)}=v_3^{(n)}=v_4^{(n)}=v_5^{(n)}$ for $n = 1, 2, 3, 4, 5, 6, 7$. Therefore, we will only look at $v_1^{(n)}$. Also, as with C_3 and C_4, we will deal with the following entries from Pascal's Triangle.

$$\begin{aligned}
v_1^{(2)} &= \binom{2}{1} = 2 \\
v_1^{(3)} &= 0 \\
v_1^{(4)} &= \binom{4}{2} = 6 \\
v_1^{(5)} &= \binom{5}{0} + \binom{5}{5} = 2 \\
v_1^{(6)} &= \binom{6}{3} = 20 \\
v_1^{(7)} &= \binom{7}{1} + \binom{7}{6} = 14
\end{aligned}$$

As a result of these entries we have a pattern, and we derive the following conjecture.

Conjecture 3.3.12. $v_1^{(n)}$ *can be defined recursively by:*

$$\begin{aligned}
v_1^{(2)} &= \binom{2}{1} = 2 \\
v_1^{(3)} &= 0.
\end{aligned}$$

For $n \geq 4$, let

$$\Lambda_n = \left\{ i \mid \binom{n-1}{i} \text{ is summand of } v_1^{(n)} \right\}$$

and

$$J_n = \{j \mid j = i - 2 \text{ or } j = i + 3, \ i \in \Lambda_n\}.$$

So,

$$v_1^{(n)} = \sum_{j \in J_n}^{n} \binom{n}{j}.$$

Example 3.3.13. Let $n = 5$

$$\Lambda_5 = \{2\},$$

and

$$J_4 = \{0,\ 5\}.$$

Therefore, we obtain

$$v_1^5 = \binom{5}{0} + \binom{5}{5}.$$

Thus, we have

$$\begin{aligned} v_1^5 &= 1 + 1 \\ &= 2, \end{aligned}$$

which is the same result we obtained above.

Assuming the conjecture, the equality expressing the total number of closed walks of length n on C_5 is

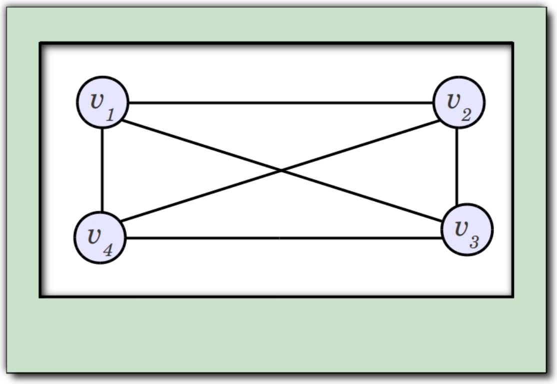
$$5 \sum_{j \in J_n}^{n} \binom{n}{j} = \frac{1}{2^{n-1}}(2^{2n-1} + (\sqrt{5} - 1)^n + (-\sqrt{5} - 1)^n).$$

3.4 GRAPH:K_v (Complete Graphs on v vertices)

In this section, we move to complete graphs, graphs where a vertex in the graph is connected to every other vertex in the graph by exactly one edge. Here we will attempt to do exactly as we have done in the previous three sections: derive an equality were some formula–preferably a combinatorial one–equals a power sum.

3.4.1 Examination of K_4

The first graph that we examine in this section is the graph K_4. The graph K_4 is a graph with 4 vertices where each vertex is connected to every other vertex by at most one edge. Let us analyze the following picture of a K_4 graph.



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Proposition 3.4.1. *The number of closed walks of length n on K_4 are*

$$3^n + (-1)^n \cdot 3.$$

Proof. The adjacency matrix for this graph is

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

and its eigenvalues are

$$\lambda_0 = 3, \lambda_1 = -1, \lambda_1 = -1, \text{ and } \lambda_1 = -1.$$

Therefore, by Theorem 2.0.22, we obtain

$$3^n + (-1)^n + (-1)^n + (-1)^n = 3^n + (-1)^n \cdot 3.$$

□

Combinatorial Discussion

According to Theorem 2.0.10, the number of closed walks from $v_1 \longrightarrow v_1$, $v_2 \longrightarrow v_2$, $v_3 \longrightarrow v_3$, and $v_4 \longrightarrow v_4$ of length n are determined by the $a_{11}^{(n)}$, $a_{22}^{(n)}$, $a_{33}^{(n)}$, and $a_{44}^{(n)}$ entries, respectively, in the n^{th} power of the adjacency matrix of a graph:

$$A^2 = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix};$$

$$A^3 = \begin{bmatrix} 6 & 7 & 7 & 7 \\ 7 & 6 & 7 & 7 \\ 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 6 \end{bmatrix};$$

$$A^4 = \begin{bmatrix} 21 & 20 & 20 & 20 \\ 20 & 21 & 20 & 20 \\ 20 & 20 & 21 & 20 \\ 20 & 20 & 20 & 21 \end{bmatrix};$$

$$A^5 = \begin{bmatrix} 60 & 61 & 61 & 61 \\ 61 & 60 & 61 & 61 \\ 61 & 61 & 60 & 61 \\ 61 & 61 & 61 & 60 \end{bmatrix};$$

$$A^6 = \begin{bmatrix} 183 & 182 & 182 & 182 \\ 182 & 183 & 182 & 182 \\ 182 & 182 & 183 & 182 \\ 183 & 182 & 182 & 183 \end{bmatrix}.$$

So here, as before with the graphs C_v, we let $v_1^{(n)}$, $v_2^{(n)}$, $v_3^{(n)}$, and $v_4^{(n)}$ be closed walks of length n from $v_1 \longrightarrow v_1$, $v_2 \longrightarrow v_2$, $v_3 \longrightarrow v_3$, and $v_4 \longrightarrow v_4$ respectively, and we obtain:

n	$v_1^{(n)}$	$v_2^{(n)}$	$v_3^{(n)}$	$v_4^{(n)}$
0	1	1	1	1
1	0	0	0	0
2	3	3	3	3
3	6	6	6	6
4	21	21	21	21
5	60	60	60	60
6	183	183	183	183

Note that the adjacency matrix of K_4 is symmetric; therefore, we see that $v_1^{(n)}=v_2^{(n)}=v_3^{(n)}=v_4^{(n)}$. As a result, we need only analyze the closed walks at one vertex and translate our findings to the other vertices. We will look at $v_1^{(n)}$. From $v_1^{(n)}$, we obtain the following observation:

$$
\begin{aligned}
v_1^2 &= 3 = 3 \\
v_1^3 &= 3^2 - 3 = 6 \\
v_1^4 &= 3^3 - 3^2 + 3 = 21 \\
v_1^5 &= 3^4 - 3^3 + 3^3 - 3 = 60 \\
v_1^6 &= 3^5 - 3^4 + 3^3 - 3^2 + 3 = 183.
\end{aligned}
$$

Subsequently, we obtain the following conjecture

Conjecture 3.4.2. *For all n, the number of closed walks on K_4 of length n at a single vertex is*

$$\sum_{k=1}^{n-1}(-1)^{n+k-1}3^k.$$

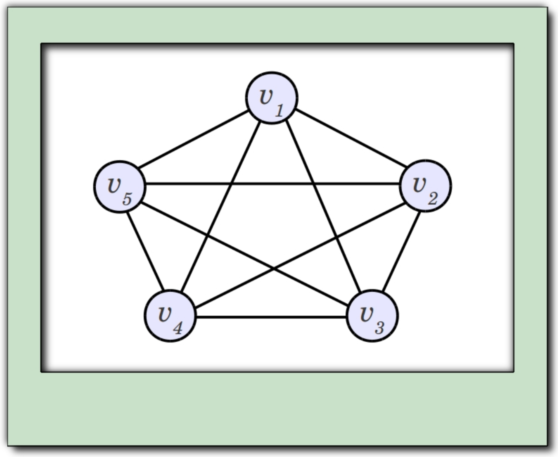
So, assuming the conjecture, we have the following equality where a combinatorial formula equals a power sum:

$$\sum_{k=1}^{n-1}(-1)^{n+k-1}3^k = 3^n + (-1)^n \cdot 3.$$

As of right now, we will hold onto this observation; however, we will move onto K_5 and see if we notice any other observation and whether the preceding pattern carries over.

3.4.2 Examination of K_5

First notice the following picture.



This is a picture of a K_5 graph, a graph with 5 vertices where each vertex is connected to every other vertex just like in K_4.

Proposition 3.4.3. *The number of closed walks of length n on the graph K_5 is*

$$4^n + (-1)^n \cdot 4.$$

Proof. The adjacency matrix for this graph is

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix},$$

and its eigenvalues are

$$\lambda_0 = 4, \lambda_1 = -1, \lambda_2 = -1, \lambda_3 = -1 \text{ and } \lambda_4 = -1.$$

Therefore, by Theorem 2.0.22, we obtain

$$4^n + (-1)^n + (-1)^n + (-1)^n + (-1)^n = 4^n + (-1)^n \cdot 4.$$

□

Combinatorial Discussion

Here, we will find another formula. According to Theorem 2.0.10, the number of closed walks from $v_1 \longrightarrow v_1$, $v_2 \longrightarrow v_2$, $v_3 \longrightarrow v_3$, $v_4 \longrightarrow v_4$, and $v_5 \longrightarrow v_5$ of length n are determined by the $a_{11}^{(n)}$, $a_{22}^{(n)}$, $a_{33}^{(n)}$, $a_{44}^{(n)}$, and $a_{55}^{(n)}$ entries, respectively, in the n^{th} power of the adjacency matrix of a graph:

$$A^2 = \begin{bmatrix} 4 & 3 & 3 & 3 & 3 \\ 3 & 4 & 3 & 3 & 3 \\ 3 & 3 & 4 & 3 & 3 \\ 3 & 3 & 3 & 4 & 3 \\ 3 & 3 & 3 & 3 & 4 \end{bmatrix};$$

$$A^3 = \begin{bmatrix} 12 & 13 & 13 & 13 & 13 \\ 13 & 12 & 13 & 13 & 13 \\ 13 & 13 & 12 & 13 & 13 \\ 13 & 13 & 13 & 12 & 13 \\ 13 & 13 & 13 & 13 & 12 \end{bmatrix};$$

$$A^4 = \begin{bmatrix} 52 & 51 & 51 & 51 & 51 \\ 51 & 52 & 51 & 51 & 51 \\ 51 & 51 & 52 & 51 & 51 \\ 51 & 51 & 51 & 52 & 51 \\ 51 & 51 & 51 & 51 & 52 \end{bmatrix};$$

$$A^5 = \begin{bmatrix} 204 & 205 & 205 & 205 & 205 \\ 205 & 204 & 205 & 205 & 205 \\ 205 & 205 & 204 & 205 & 205 \\ 205 & 205 & 205 & 204 & 205 \\ 205 & 205 & 205 & 205 & 204 \end{bmatrix};$$

$$A^6 = \begin{bmatrix} 820 & 819 & 819 & 819 & 819 \\ 819 & 820 & 819 & 819 & 819 \\ 819 & 819 & 820 & 819 & 819 \\ 819 & 819 & 819 & 820 & 819 \\ 819 & 819 & 819 & 819 & 820 \end{bmatrix}.$$

So, as with K_4, we let $v_1^{(n)}$, $v_2^{(n)}$, $v_3^{(n)}$, $v_4^{(n)}$ and $v_5^{(n)}$ be closed walks of length n from $v_1 \longrightarrow v_1$, $v_2 \longrightarrow v_2$, $v_3 \longrightarrow v_3$, $v_4 \longrightarrow v_4$, and $v_5 \longrightarrow v_5$ respectively, and we obtain:

n	v_1^n	v_2^n	v_3^n	v_4^n	v_5^n
0	1	1	1	1	1
1	0	0	0	0	0
2	4	4	4	4	4
3	12	12	12	12	12
4	52	52	52	52	52
5	204	204	204	204	204
6	820	820	820	820	820

Notice that K_5 is symmetric and $v_1^{(n)}{=}v_2^{(n)}{=}v_3^{(n)}{=}v_4^{(n)}{=}v_5^{(n)}$. As a result, we need only analyze the closed walks at one vertex also: $v_1^{(n)}$. From $v_1^{(n)}$, we obtain the following observation:

$$v_1^2 = 4 = 4$$

$$v_1^3 = 4^2 - 4 = 12$$

$$v_1^4 = 4^3 - 4^2 + 4 = 52$$

$$v_1^5 = 4^4 - 4^3 + 4^3 - 4 = 204$$

$$v_1^6 = 4^5 - 4^4 + 4^3 - 4^2 + 4 = 820.$$

As with the K_4 graphs, we derive the following conjecture.

Conjecture 3.4.4. *For all n, the number of closed walks on k_5 of length n at a single vertex is*

$$\sum_{k=1}^{n-1}(-1)^{n+k-1}4^k.$$

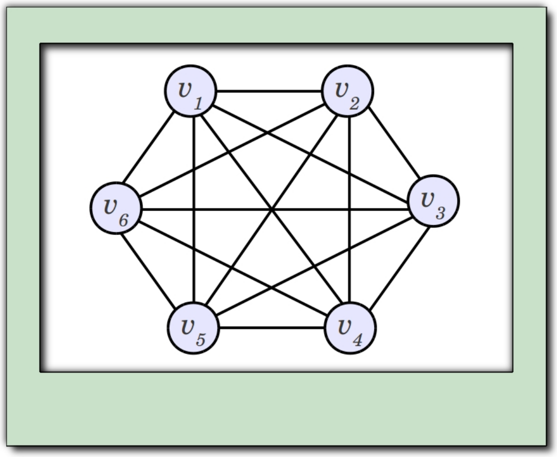
Thus, if we assume the conjecture, we now have the following equality where a combinatorial formula equals a power sum:

$$\sum_{k=1}^{n-1}(-1)^{n+k-1}4^k = 4^n + 4(-1)^n$$

As of right now, we will move onto K_6 and see if the preceding pattern carries over. If so, we will try and determine if we can conjecture some general formula for K_v.

3.4.3 Examination of K_6

Lastly, we will analyze the graph K_6, and attempt to reasonably conjecture some general formula for the graphs K_v.



Proposition 3.4.5. *The number of closed walks of length n on the graph K_6 is*

$$5^n + (-1)^n \cdot 5.$$

Proof. The adjacency matrix for this graph is

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix},$$

and the eigenvalues are

$$\lambda_0 = 5, \lambda_1 = -1, \lambda_2 = -1, \lambda_3 = -1, \lambda_4 = -1, \text{ and } \lambda_5 = -1.$$

Therefore, by Theorem 2.0.22, we obtain

$$5^n + (-1)^n + (-1)^n + (-1)^n + (-1)^n + (-1)^n = 5^n + (-1)^n \cdot 5.$$

$\square$

Combinatorial Discussion

We note that according to Theorem 2.0.10, the number of closed walks from $v_1 \longrightarrow v_1$, $v_2 \longrightarrow v_2$, $v_3 \longrightarrow v_3$, $v_4 \longrightarrow v_4$, $v_5 \longrightarrow v_5$, and $v_6 \longrightarrow v_6$ of length n are determined by the $a_{11}^{(n)}$, $a_{22}^{(n)}$, $a_{33}^{(n)}$, $a_{44}^{(n)}$, $a_{55}^{(n)}$, and $a_{66}^{(n)}$ entries, respectively, in the n^{th} power of the adjacency matrix of a graph:

$$A^2 = \begin{bmatrix} 5 & 4 & 4 & 4 & 4 & 4 \\ 4 & 5 & 4 & 4 & 4 & 4 \\ 4 & 4 & 5 & 4 & 4 & 4 \\ 4 & 4 & 4 & 5 & 4 & 4 \\ 4 & 4 & 4 & 4 & 5 & 4 \\ 4 & 4 & 4 & 4 & 4 & 5 \end{bmatrix};$$

$$A^3 = \begin{bmatrix} 20 & 21 & 21 & 21 & 21 & 21 \\ 21 & 20 & 21 & 21 & 21 & 21 \\ 21 & 21 & 20 & 21 & 21 & 21 \\ 21 & 21 & 21 & 20 & 21 & 21 \\ 21 & 21 & 21 & 21 & 20 & 21 \\ 21 & 21 & 21 & 21 & 21 & 20 \end{bmatrix};$$

$$A^4 = \begin{bmatrix} 105 & 104 & 104 & 104 & 104 & 104 \\ 104 & 105 & 104 & 104 & 104 & 104 \\ 104 & 104 & 105 & 104 & 104 & 104 \\ 104 & 104 & 104 & 105 & 104 & 104 \\ 104 & 104 & 104 & 104 & 105 & 104 \\ 104 & 104 & 104 & 104 & 104 & 105 \end{bmatrix};$$

$$A^5 = \begin{bmatrix} 520 & 521 & 521 & 521 & 521 & 521 \\ 521 & 520 & 521 & 521 & 521 & 521 \\ 521 & 521 & 520 & 521 & 521 & 521 \\ 521 & 521 & 521 & 520 & 521 & 521 \\ 512 & 521 & 521 & 521 & 520 & 521 \\ 521 & 521 & 521 & 521 & 521 & 520 \end{bmatrix};$$

$$A^6 = \begin{bmatrix} 2605 & 2604 & 2604 & 2604 & 2604 & 2604 \\ 2604 & 2605 & 2604 & 2604 & 2604 & 2604 \\ 2604 & 2604 & 2605 & 2604 & 2604 & 2604 \\ 2604 & 2604 & 2604 & 2605 & 2604 & 2604 \\ 2604 & 2604 & 2604 & 2604 & 2605 & 2604 \\ 2604 & 2604 & 2604 & 2604 & 2604 & 2605 \end{bmatrix}.$$

So, we let $v_1^{(n)}$, $v_2^{(n)}$, $v_3^{(n)}$, $v_4^{(n)}$, $v_5^{(n)}$ and $v_6^{(n)}$ be closed walks of length n from $v_1 \longrightarrow v_1$, $v_2 \longrightarrow v_2$, $v_3 \longrightarrow v_3$, $v_4 \longrightarrow v_4$, $v_5 \longrightarrow v_5$, and $v_6 \longrightarrow v_6$ respectively, and we obtain:

n	v_1^n	v_2^n	v_3^n	v_4^n	v_5^n	v_6^n
0	1	1	1	1	1	1
1	0	0	0	0	0	0
2	5	5	5	5	5	5
3	20	20	20	20	20	20
4	105	105	105	105	105	105
5	520	520	520	520	520	520
6	2605	2605	2605	2605	2605	2605

Notice that, as with K_4 and K_5, the adjacency matrix of K_6 is symmetric and $v_1^{(n)}=v_2^{(n)}=v_3^{(n)}=v_4^{(n)}=v_5^{(n)}=v_5^{(n)}$. As a result, we need only analyze the closed walks at one vertex also: $v_1^{(n)}$. From $v_1^{(n)}$, we obtain the following observation:

$$v_1^2 = 5 = 5$$

$$v_1^3 = 5^2 - 5 = 20$$

$$v_1^4 = 5^3 - 5^2 + 5 = 105$$

$$v_1^5 = 5^4 - 5^3 + 5^3 - 5 = 520$$

$$v_1^6 = 5^5 - 5^4 + 5^3 - 5^2 + 5 = 2605.$$

Thus we obtain, as we did with the graphs K_4 and K_5, the following conjecture.

Conjecture 3.4.6. *For all n, the number of closed walks on k_5 of length n at a single vertex is*

$$\sum_{k=1}^{n-1}(-1)^{n+k-1}5^k.$$

So, assuming the conjecture, we have the following equality where a combinatorial formula equals a power sum:

$$\sum_{k=1}^{n-1}(-1)^{n+k-1}5^k = 5^n + 5(-1)^n.$$

Thus, we have a similar pattern for all three graphs. Therefore, we will now generalize our conjecture. However, before we do, note the following proposition.

Proposition 3.4.7. *The adjacency matrix of K_v is circulant for all $v \geq 4$.*

Proof. The adjacency matrix of a K_v graph looks like

$$C_v = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix}.$$

If we look at a K_v graph's adjacency matrix, and take into account that every vertex is adjacent to every other vertex, we see that the zeros lie along the diagonal of the adjacency matrix with ones every where else. Thus, we see that the adjacency matrix of a graph K_v will be circulant. □

Because the adjacency matrices of the family of graphs K_v are circulant, we start by borrowing from Definition 3.3.1 and Proposition 3.3.3 and letting v be the number of vertices in the complete graph, then we obtain the following conjecture, which gives an equation in which a formula equals a power sum for every K_v graph.

Conjecture 3.4.8. *Let $\xi = e^{\frac{2\pi a}{n}} = \cos(\frac{2\pi a}{n}) + i\sin(\frac{2\pi a}{n})$ and $a = 0, 1, 2, ..., n-1$, then*

$$\sum_{K=1}^{n-1} (-1)^{n+k-1}(v-1)^k = \sum_{j=0}^{n-1} c_j \xi^{aj}.$$

3.4.4 Conclusion

Thus, in Chapter 3, we have accomplished what we have set out to do. That is, we have taken several graphs G, computed their eigenvalues, developed a formula to count the number of closed walks on G, and developed equalities where some formula equals a power sum. We have even went one step further and conjectured a generalization for the family of graphs K_v. Thus, we conclude the main work of this thesis. However we proceed with given an application and detailing some future work we would like to accomplish in this area of mathematics.

4

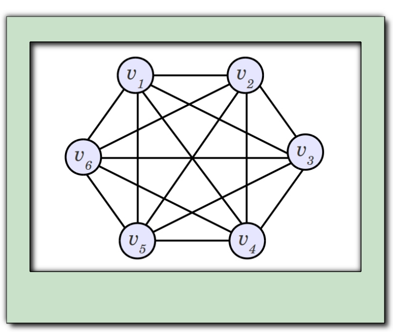
Application and Future Directions

4.0.5 *Application*

We will explore an application dealing with computers and communication.

Example 4.0.9. We can regard a graph as a communications network. We can let the vertices represent entities that we wish to have communicate with one another, such as computers. We can let the edges represent connections between them, such as fiber optic cables. Two computers can communicate directly with one another if and only if there is an edge–a direct line–that runs between them (i.e., if the two vertices are adjacent).

Suppose we have six computers that are all connected to each other by at most one fiber optic cables each. We obtain a picture that looks similar to the graph K_6.



Now, suppose we wanted to know how many ways in which a message can be passed between these computers with the message traveling between a specific number of these computers and with the condition that it must be returned to the originating computer. To solve such a problem, we use the work that we have done with K_6 graphs. All we need to do is let the "how many ways" be counted as the number of closed walks and letting the "specific number of computers" be the length n of the closed walks. Now, suppose we let the specific number of computers be 5, and we want to know how many ways can this message be passed between them and returned to the original computer. All we need to do is employ the following equality.

$$\sum_{k=1}^{4}(-1)^{5+k-1}5^k = 5^5 + 5(-1)^5,$$

and, the "how many" ways are

$$5^4 - 5^3 + 5^2 - 5 = 520.$$

This is just one application of this type of work among many. We will leave off here because this application should suffice to give the reader an idea of the type of applications that the work done in this thesis can be applied.

4.0.6 Future Directions

In the future, I would like to take a graph G for which the eigenvalues are difficult to compute explicitly, and use the number of walks to obtain information about the eigenvalues. Because of its use in computer science, I would like to look at d-regular graphs– whose largest eigenvalue is d– and analyze its second largest eigenvalue. In addition, I would like to extend the work of this project, in conjunction with what was just stated, with the more ambitious goal of finding a formula to compute the second largest eigenvalue of d-regular graphs for an infinite family of graphs, similar to what we did with the K_v graphs.

In addition, provided more time, I would attempt to prove every conjecture that has been used in this thesis, especially those conjectures in the Unclassified Graphs' section. Namely, for those graphs whose power sums (the n^{th} power of the eigenvalues) are of the form

$$\frac{(1+\sqrt{x})^n}{2^n}+\frac{(1-\sqrt{x})^n}{2^n},$$

I would like to understand the relationship between the x's in them: how is it changing.

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