19 Problems: Eigenvalues and Eigenvectors II

1. Explain why the characteristic polynomial of an $n \times n$ matrix has degree n. Make your explanation easy to read by starting with some simple examples, and then use properties of the determinant to give a *general* explanation.

2. Compute the characteristic polynomial $P_M(\lambda)$ of the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Now, since we can evaluate polynomials on square matrices, we can plug M into its characteristic polynomial and find the *matrix* $P_M(M)$. What do you find from this computation? Does something similar hold for 3×3 matrices? What about $n \times n$ matrices?

3. *Discrete dynamical system.* Let M be the matrix given by

$$M = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}.$$

Given any vector $v(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$, we can create an infinite sequence of vectors $v(1), v(2), v(3)$, and so on using the rule

$$v(t+1) = Mv(t) \text{ for all natural numbers } t.$$

(This is known as a *discrete dynamical system* whose *initial condition* is $v(0)$.)

(a) Find all eigenvectors and eigenvalues of M.

(b) Find all vectors $v(0)$ such that

$$v(0) = v(1) = v(2) = v(3) = \cdots$$

(Such a vector is known as a *fixed point* of the dynamical system.)

(c) Find all vectors $v(0)$ such that $v(0), v(1), v(2), v(3), \ldots$ all point in the same direction. (Any such vector describes an *invariant curve* of the dynamical system.)