WEEK 1

SOLUTIONS

January 29, 2016
Calculus III
Weekly Assignment #1

Sec 9.1

1) Plane parallel to xy plane that passes through $(-4,5,-12)$
normal vector: $\langle 0,0,1 \rangle$

$0(x+4)+0(y-5)+1(z+12)=0$

$z+12$, $\boxed{z=-12}$

- Plane parallel to yz plane that passes through $(7,-2,-3)$
normal vector: $\langle 1,0,0 \rangle$

$1(x-7)+0(y+2)+0(z+3)=0$

$x-7=0$, $\boxed{x=7}$

- The sphere centered at the point $(2,1,3)$ and has the endpoint $(-1,0,-1)$ on it's surface.

$A=(2,1,3)$ $B=(-1,0,-1)$

$|AB|=\sqrt{(2+1)^2+(1-0)^2+(3+1)^2}$

$=\sqrt{9+1+16}$

$=\sqrt{26}$ = radius of sphere.

$\boxed{\text{Sphere}: (x-2)^2+(y-1)^2+(z-3)^2=(\sqrt{26})^2}$

- The sphere whose diameter has the endpoints $(-3,1,-5)$ and $(7,9,-1)$

Midpoint: $\frac{-3+7}{2}, \frac{1+9}{2}, \frac{-5-1}{2}=(2,5,-3)$ center.

Distance: $\sqrt{(-3-7)^2+(1-9)^2+(-5+1)^2}=\sqrt{180}/2=3\sqrt{5}$

$\boxed{(x-2)^2+(y-5)^2+(z+3)^2=(3\sqrt{5})^2}$

3)

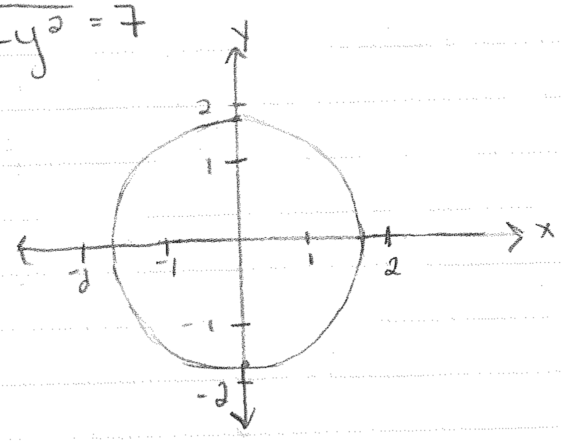
a) Domain: $(x,y) \in \mathbb{R}^2$ such that $x^2+y^2 \leq 4$

b) Range: Let $z \in \mathbb{R}$. The range would be $6 \leq z \leq 8$. This is because $\sqrt{4-x^2-y^2}$ will never be less than zero. Since $8-0$ is 8, then the highest number that the function can be is 8. The expression $4-x^2-y^2$ can never be greater than 4, and since $\sqrt{4}=2$, then the smallest output of the function is six.

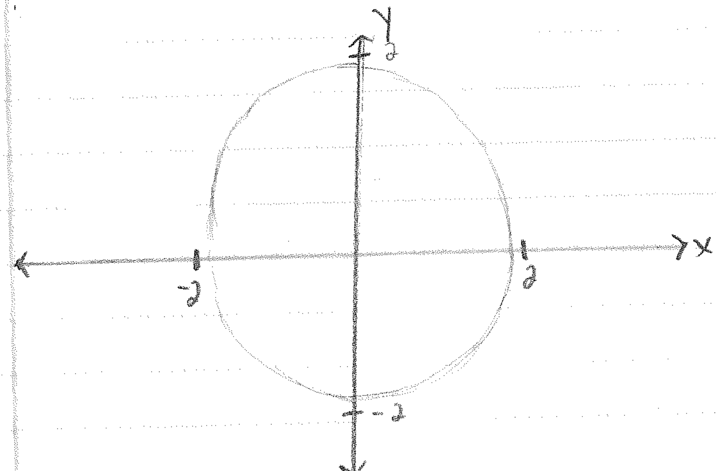
c) Choose 4 different values for the range of h and plot level curves

Let $h=7$

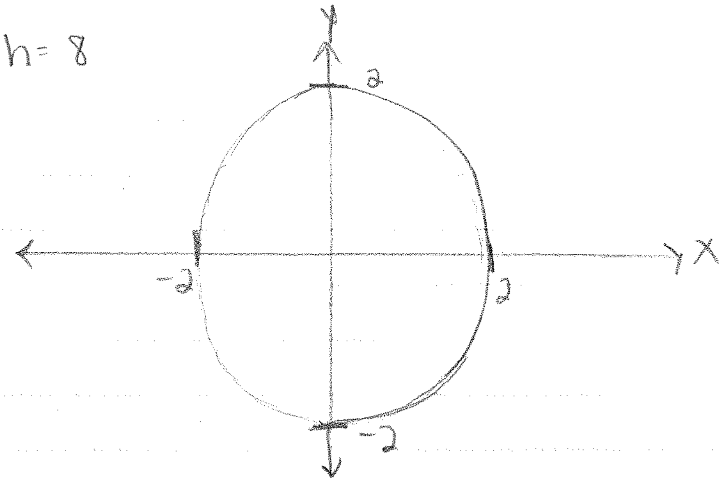
$8-\sqrt{4-x^2-y^2}=7$



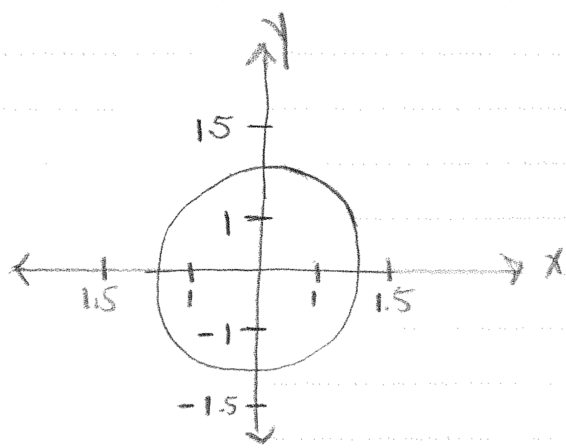
Let $h=7.5$



Let h= 8



Let h= 6.5

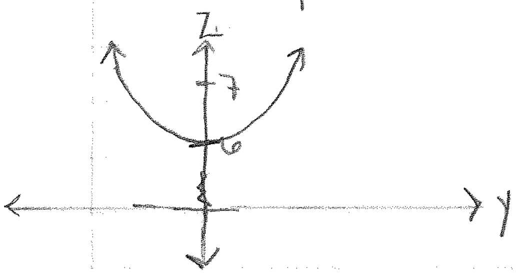


The shape of a typical level curve is a circle.

d) Choose 5 different values of x and Sketch the traces.

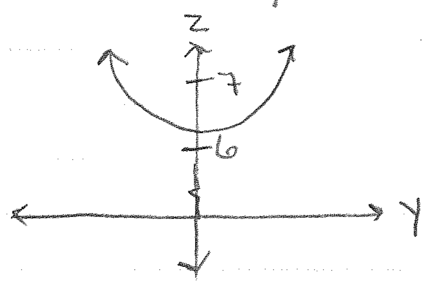
$x = 0$

$h = 8 - \sqrt{4 - y^2}$



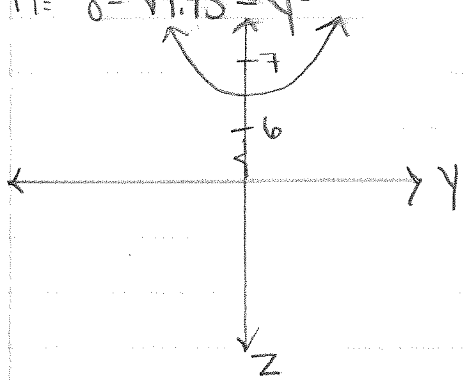
$x = -1$ and $x = 1$

$h = 8 - \sqrt{3 - y^2}$



$x = 1.5$ and $x = -1.5$

$h = 8 - \sqrt{1.75 - y^2}$

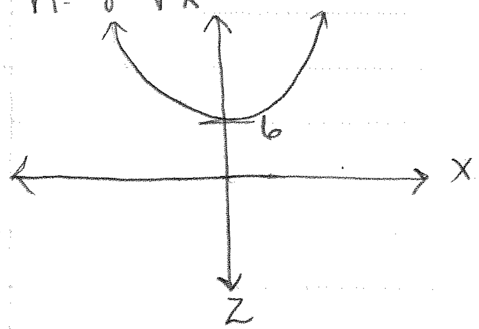


VERY NICE!

Choose 5 different values for y and draw the traces

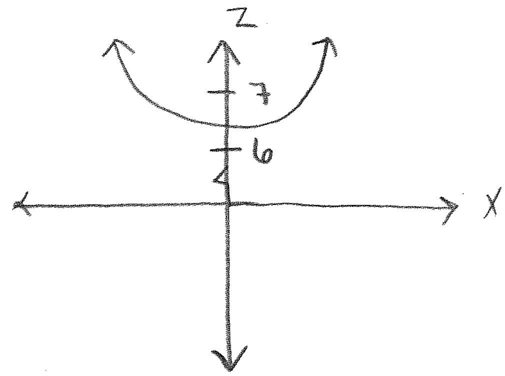
e) $y = 0$

$h = 8 - \sqrt{-x^2}$



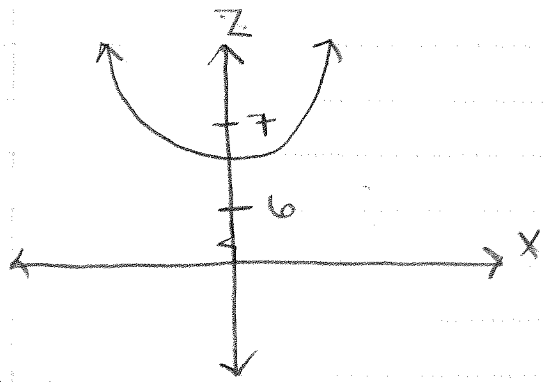
$y = 1$ and $y = -1$

$h = 8 - \sqrt{3 - x^2}$

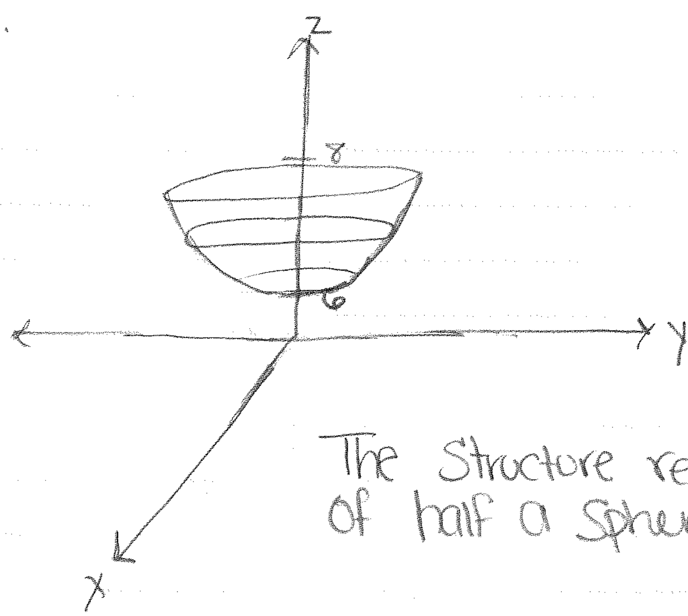


$y = 1.5$ and $y = -1.5$

$h = 8 - \sqrt{1.75 - x^2}$



f) Sketch an overall picture of the surface generated by h.



The structure reminds me of half a sphere.

Sec 9.2

1.

(c) $v = \langle 1,-2\rangle, \quad u = \langle 0,4\rangle \quad w = \langle -5,7\rangle$

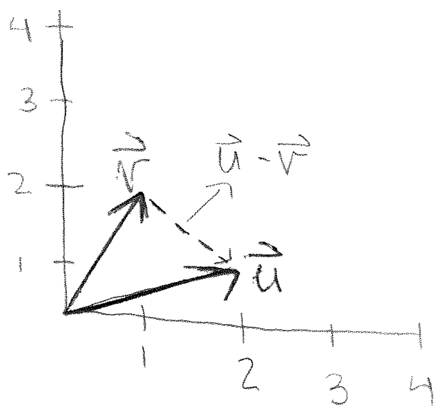
$v + 2u - 7w = \langle 1,-2\rangle + 2\langle 0,4\rangle - 7\langle -5,7\rangle$

$v + 2u - 7w = \langle 1,-2\rangle + \langle 0,8\rangle - \langle -35,49\rangle$

$\boxed{v + 2u - 7w = \langle 36,-43\rangle}$

2.

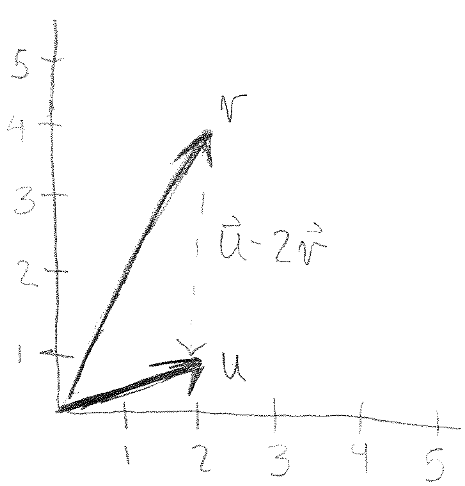
(c) $u = \langle 2,1\rangle \quad v = \langle 1,2\rangle$



$\vec{u} - \vec{v} = d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

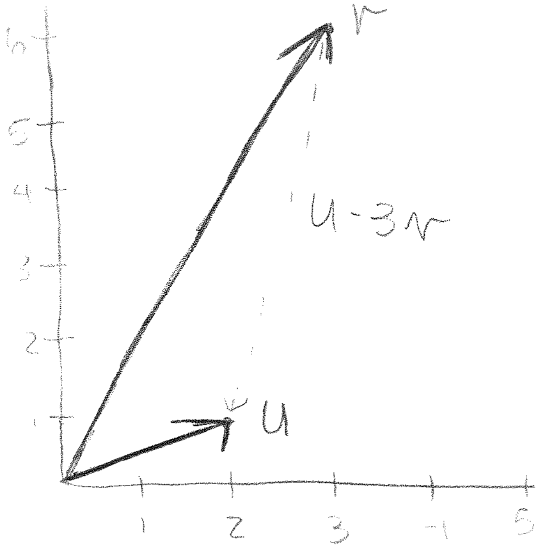
$\vec{u} - \vec{v} = d = \sqrt{1+1} = \boxed{\sqrt{2}}$

$\vec{u} - 2\vec{v} \Rightarrow u = \langle 2,1\rangle \quad v = \langle 2,4\rangle$



$\vec{u} - 2\vec{v} = d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

$u - 2\vec{v} = d = \sqrt{0+9} = \sqrt{9} = \boxed{3}$

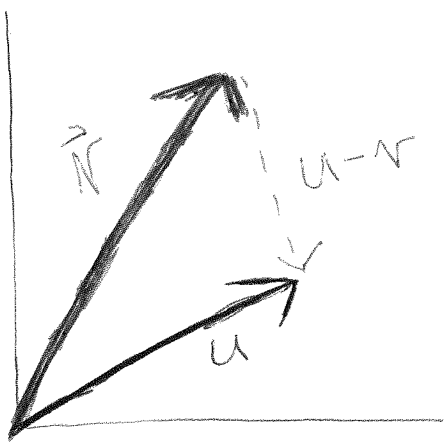


$u - 3v \Rightarrow u = \langle 2, 1 \rangle \quad v = \langle 3, 6 \rangle$

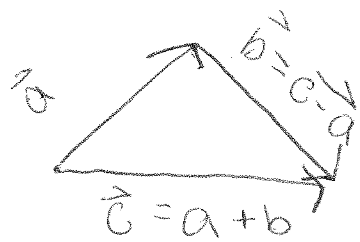
$u - 3v = d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

$u - 3v = d = \sqrt{1 + 25} = \boxed{\sqrt{26}}$

(d)



We want to add vectors. A diagram would be



We know that to add vectors, we need to find the resultant by using the tip to toe method and finding the distance. [a+b=c]. By manipulating the formula, we can get [b=c-a] To explain it in words, it is as if one person is going to the $\vec{a}$ direction and another person wants to go to the c direction. To meet up, they need to travel along the b direction. Always, the arrow points in the direction of what the direction is that is being ~~&~~ subtracted from. In this case, it is $\vec{u}$.

3a) $V = \langle 3,4 \rangle$ Determine $|v|$ as well as $u = \frac{1}{|v|}$ and u

$|v|$ = distance formula $= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$= \sqrt{(0-3)^2+(0-4)^2}$

$= \sqrt{25}$

$= 5$

Scalar multiplicaction

$U = \frac{1}{5} V$

$= \langle 3 \cdot \frac{1}{5}, 4 \cdot \frac{1}{5} \rangle$

$= \langle \frac{3}{5}, \frac{4}{5} \rangle$

$|u| = \sqrt{(0-3/5)^2+(0-4/5)^2}$

$= 9/25 + 16/25$

$= 25/25$

$= 1$

The direction of both U and V are the same. The only change that occurs is in V's magnitude. Because $U = \frac{1}{5}V$ its magnitude is also $1/5$ that of V.

b.) Let $w = 3i - 3j$. Determine a unit vector u in the same direction as w.

$i = \langle 1,0 \rangle$
$j = \langle 0,1 \rangle$

$w = 3\langle 1,0 \rangle + -3\langle 0,1 \rangle$
$= \langle 3,0 \rangle + \langle 0,-3 \rangle$
$\boxed{w = \langle 3,-3 \rangle}$

• $\langle 4,-4 \rangle$ would be in the same direction as w.

c.) Let $v = \langle 2,3,5 \rangle$. Find $|v|$ and determine the components of the vector $u = \frac{1}{|v|} v$. What is $|u|$? Does its direction differ from v?

$v = \langle 2,3,5 \rangle$
$|v| = \sqrt{(0-2)^2 + (0-3)^2 + (0-5)^2}$
$= \sqrt{4 + 9 + 25}$
$\boxed{|v| = \sqrt{38}}$

$u = \frac{1}{\sqrt{38}} \langle 2,3,5 \rangle$

$\boxed{u = \left\langle \frac{2}{\sqrt{38}}, \frac{3}{\sqrt{38}}, \frac{5}{\sqrt{38}} \right\rangle}$

$|u| = \sqrt{(0 - 2/\sqrt{38})^2 + (0 - 3/\sqrt{38})^2 + (0 - 5/\sqrt{38})^2}$
$= 4/38 + 9/38 + 25/38$
$= 38/38$
$\boxed{|u| = 1}$

• The direction of v and u are the same.

D) Let V be any arbitrary vector in R3. Write a general formula for a vector parallel to v.

$V = \langle 1, 2, 3 \rangle$

$\vec{v} = \langle x, y, z \rangle$, $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$.

~~that~~ $n \cdot V$ where n is any real number is parallel to V, because all multiples of a vector are parallel to said vector.

Section 9.3

① (c) $<2,-1,-2> \cdot \vec{y} = <2,-1,-2> \cdot <0,3,-2>$
$= 2\cdot 0 + (-1)\cdot 3 + (-2)(-2)$
$= -3+4 = 1$

so $<2,-1,-2>\cdot\vec{y} \neq 0$ which implies the two vectors are not perpendicular

(d) Let $\vec{w} = <a,b,c>$ be a unit vector perpendicular to $\vec{y}$

Since $\vec{w}$ is a unit vector, $|\vec{w}|=1$, $\Leftrightarrow \sqrt{a^2+b^2+c^2}=1$
$\Leftrightarrow a^2+b^2+c^2=1$

Since $\vec{w}\perp\vec{y} \Leftrightarrow \vec{w}\cdot\vec{y}=0$
$\Rightarrow 0a+3b-2c=0$
$\Rightarrow 3b-2c=0$

So the conditions $\begin{cases} a^2+b^2+c^2=1 \\ 3b-2c=0 \end{cases}$ are enough to guarantee all wanted properties

$\begin{cases} a^2+b^2+c^2=1 \\ 3b-2c=0 \end{cases} \Leftrightarrow \begin{cases} a^2+\frac{4c^2}{9}+c^2=1 \\ b=\frac{2c}{3} \end{cases} \Leftrightarrow \begin{cases} a^2+\frac{13c^2}{9}=1 \\ b=\frac{2c}{3} \end{cases}$

We can see the first ~~equa~~ equation ~~real~~ can have infinitely many solutions (it's an equation of an ellipse)

$\Rightarrow$ There are infinitely many unit vectors, perpendicular to $\vec{y}$. One such vector is when $c=\frac{3}{\sqrt{13}} \Rightarrow b=\frac{2}{\sqrt{13}}$ and $a=0$

So $\vec{w}=<0,\frac{2}{\sqrt{13}},\frac{3}{\sqrt{13}}>$ is such a vector

p. 8

(e) Let $\vec{r}$ be a unit vector perpendicular to both $\vec{y}$ and $\vec{z}$. Let $\vec{r} = \langle m, n, k \rangle$

Since ~~r is a~~ $\vec{r}$ is a unit vector

$\Rightarrow |\vec{r}| = \sqrt{m^2+n^2+k^2} = 1 \iff m^2+n^2+k^2=1$

Since $\vec{r} \perp \vec{y} \iff \vec{r}\cdot\vec{y}=0$

$\Rightarrow 0m + 3(n) - 2(k) = 3n-2k = 0$

Since $\vec{r} \perp \vec{z} \iff \vec{r}\cdot\vec{z} = 0$

$\Rightarrow 2m + n + 0(k) = 0$

$\iff 2m+n=0$

So we have

$$\begin{cases} m^2+n^2+k^2=1 \\ 3n-2k=0 \\ 2m+n=0 \end{cases} \iff \begin{cases} m^2+n^2+k^2=1 \\ k=\frac{3n}{2} \\ m=-\frac{1}{2}n \end{cases}$$

$$\iff \begin{cases} \frac{1}{4}n^2+n^2+\frac{9}{4}n^2=1 \\ k=\frac{3n}{2} \\ m=-\frac{1}{2}n \end{cases} \iff \begin{cases} \frac{14}{4}n^2=1 \\ k=\frac{3n}{2} \\ m=-\frac{1}{2}n \end{cases}$$

$\Rightarrow n = \sqrt{\frac{4}{14}} = \frac{2}{\sqrt{14}} \Rightarrow k = \frac{3}{\sqrt{14}}, m = -\frac{1}{\sqrt{14}}$

The equations show us they uniquely determine the vector $\vec{r}$ so there's only one such vector. It is

$\vec{r} = \langle -\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \rangle$

② (a) Let ~~O be the~~ point O be the origin

Then, $\vec{PQ} = \vec{OQ} - \vec{OP} = \langle 1,-2,4\rangle - \langle 3,2,-1\rangle$

$= \langle -2, -4, 5 \rangle$

and $\vec{PR} = \vec{OR} - \vec{OP} = \langle 4,4,0\rangle - \langle 3,2,-1\rangle$

$= \langle 1, 2, 1 \rangle$

$|\vec{PQ}| = \sqrt{4+16+25} = \sqrt{45} = 3\sqrt{5}$

$|\vec{PR}| = \sqrt{1+4+1} = \sqrt{6}$

$$\overrightarrow{PQ} \cdot \overrightarrow{PR} = \langle -2, -4, 5 \rangle \cdot \langle 1, 2, 1 \rangle$$
$$= -2 \cdot 1 - 4 \cdot 2 + 5 \cdot 1 = -2 - 8 + 5 = -5$$

On the other hand,
$$\overrightarrow{PQ} \cdot \overrightarrow{PR} = |\overrightarrow{PQ}||\overrightarrow{PR}| \cdot \cos \measuredangle RPQ$$

$$\Rightarrow \cos \measuredangle RPQ = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{|\overrightarrow{PQ}| \cdot |\overrightarrow{PR}|} = \frac{-5}{3\sqrt{5} \cdot \sqrt{6}} = -\frac{5}{3\sqrt{30}}$$

$$\Rightarrow \measuredangle RPQ = \arccos \frac{-5}{3\sqrt{30}} \approx 107.72^\circ$$

$$\overrightarrow{QP} = -\overrightarrow{PQ} = -\langle -2, -4, 5 \rangle = \langle 2, 4, -5 \rangle$$

$$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = \langle 4, 4, 0 \rangle - \langle 1, -2, 4 \rangle = \langle 3, 6, -4 \rangle$$

$$\Rightarrow \overrightarrow{QP} \cdot \overrightarrow{QR} = \langle 2, 4, -5 \rangle \cdot \langle 3, 6, -4 \rangle$$

$$= 2 \cdot 3 + 4 \cdot 6 - 5(-4) = 6 + 24 + 20 = 50$$

$$|\overrightarrow{QP}| = \sqrt{4+16+25} = \sqrt{45} = 3\sqrt{5}$$
$$|\overrightarrow{QR}| = \sqrt{9+36+16} = \sqrt{61}$$

$\Rightarrow$ On the other hand,

$$\cancel{|\overrightarrow{QP}}\ \overrightarrow{QP} \cdot \overrightarrow{QR} = |\overrightarrow{QP}||\overrightarrow{QR}| \cos \measuredangle PQR$$

$$\Rightarrow \cos \measuredangle PQR = \frac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{|\overrightarrow{QP}| \cdot |\overrightarrow{QR}|} = \frac{50}{3\sqrt{5} \cdot \sqrt{61}} = \frac{50}{3\sqrt{305}}$$

$$\Rightarrow \measuredangle PQR = \arccos \frac{50}{3\sqrt{305}} \approx 17.38^\circ$$

$$\Rightarrow \measuredangle PRQ = 180^\circ - (\measuredangle PQR + \measuredangle RPQ)$$
$$= 180^\circ - 125.1^\circ = 54.90^\circ$$