14 Problems: Properties of the Determinant

1. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show:

$$\det M = \frac{1}{2}(\operatorname{tr} M)^2 - \frac{1}{2}\operatorname{tr}(M^2)$$

   Suppose M is a 3×3 matrix. Find and verify a similar formula for $\det M$ in terms of $\operatorname{tr}(M^3)$, $(\operatorname{tr} M)(\operatorname{tr}(M^2))$, and $(\operatorname{tr} M)^3$.

2. Suppose $M = LU$ is an LU decomposition. Explain how you would efficiently compute $\det M$ in this case.

3. In computer science, the *complexity* of an algorithm is computed (roughly) by counting the number of times a given operation is performed. Suppose adding or subtracting any two numbers takes a seconds, and multiplying two numbers takes m seconds. Then, for example, computing $2 \cdot 6 - 5$ would take $a + m$ seconds.

   (a) How many additions and multiplications does it take to compute the determinant of a general 2×2 matrix?

   (b) Write a formula for the number of additions and multiplications it takes to compute the determinant of a general $n \times n$ matrix using the definition of the determinant. Assume that finding and multiplying by the sign of a permutation is free.

   (c) How many additions and multiplications does it take to compute the determinant of a general 3×3 matrix using expansion by minors? Assuming $m = 2a$, is this faster than computing the determinant from the definition?

`Problem 3 hint`