RESEARCH OVERVIEW

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1. Introduction

My research interests are in the field of commutative algebra, the study of ideals and modules of commutative rings. Historically, commutative algebra has roots in many different strands of mathematics. Three major threads that are still applicable today are number theory, algebraic geometry (or the function theory of Riemann), and invariant theory.

The first non-trivial ring studied was the *Gaussian integers* $\mathbb{Z}[i]$, where $i^2 = -1$, and originated from Gauss in 1828. The fact this ring was a unique factorization domain (UFD) allowed Gauss to prove many different properties of the integers themselves. It did not take long for number theorists like Euler, Dirichlet and Kummer to adopt Gauss' ideas and extend $\mathbb{Z}$ to $\mathbb{Z}[\zeta]$, where ζ is a root of unity. As this ring "behaves like" $\mathbb{Z}$, Kummer was able to prove a special case of Fermat's last theorem, that is, he showed the unsolvability of the equation $x^n + y^n = z^n$ for $2 < n < 100$. Later, the works of Dedekind and Lasker made the following transitions:

$$\begin{array}{ccc}
z = p_1 p_2 \cdots p_n & & \text{Unique factorization of elements into prime elements} \\
\Big\downarrow{\scriptstyle \text{Dedekind, 1871}} & & \\
(z) = (p_1)(p_2)\cdots(p_n) & & \text{Unique factorization of ideals into prime ideals} \\
\Big\downarrow{\scriptstyle \text{Lasker, 1905}} & & \\
(z) = (q_1) \cap (q_2) \cap \cdots \cap (q_n) & & \text{Unique factorization of ideals into primary ideals}
\end{array}$$

In 1921, Emmy Noether[1] completely rewrote the theory of primary decomposition. She based her general theory on the ascending chain condition, that is, all increasing chains of ideals (with respect to inclusion) must stabilize. In her honor, rings with the ascending chain condition are called Noetherian rings and are one of the main objects of study in my research.

[1]Emmy Noether was not only a brilliant mathematician, but a very strong and accomplished woman. Her life is a great inspiration to me; so much so that my daughter is named after her.

The contributions of function theory can be traced to the coordinate geometry of Fermat and Descartes. Their work allowed us to represent plane curves as algebraic equations. With the introduction of Riemann's ideas in the 1860's, people were able to study meromorphic functions on a curve. At this time (1875-1882), mathematicians such as Kronecker, Weierstrass, Dedekind and Weber developed the arithmetic approach to function theory. That is, they took the newly developed methods for dealing with algebraic number fields and applied them to geometrically defined fields. This was the beginning of the relationship between commutative algebra and geometry.

Hilbert's results on invariant theory have been monumental in the development of commutative algebra. In the mid 1800's, people were interested in applying transformations to plane curves in order to determine what properties stayed invariant. Over time, it was realized that invariance under a coordinate change could be viewed as a group action. The fundamental problem of invariant theory is the following: let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k and consider a group G with an action that induces k-algebra automorphisms on R. The set of element fixed under this action is denoted R^G and is a k-subalgebra of R. When is R^G finitely generated as a k-algebra?

Between 1888 and 1893, Hilbert essentially solved the fundamental problem. His proof techniques were quite modern; so much so that Gordan[2] exclaimed, "This is not mathematics but theology!" The results that came out of Hilbert's papers are still used today. One such example is the basis theorem: if a ring R is Noetherian, then the polynomial ring $R[x]$ is also Noetherian. This theorem is used in much of commutative algebra today and especially in my own research.

2. Current Research

My current work stands on the shoulders of Dedekind, Hilbert and Noether and continues to expand the results they developed. In my PhD dissertation I focus on maximal Cohen-Macaulay modules and how countable Cohen-Macaulay representation type relates to finite Cohen-Macaulay representation type. Other research interests include resolutions of modules and the various homological properties associated to them, Hilbert functions, and Boij-Söderberg theory. Further, I also enjoy programing with the algebraic computer software *Macaulay2*.

2.1. **Maximal Cohen-Macaulay Modules.** The study of maximal Cohen-Macaulay modules can be traced back to the theory of integral representations of finite groups in the nineteenth century. In particular, it originated from the classification of crystallographic groups which is generalized in Hilbert's 18^{th} problem. My current research on maximal Cohen-Macaulay modules is motivated by the following question of Huneke and Leuschke [HL03]:

Question 1. *Let R be a complete local Cohen-Macaulay ring of countable Cohen-Macaulay representation type, and assume that R has an isolated singularity. Is R then necessarily of finite Cohen-Macaulay representation type?*

Here, a local Cohen-Macaulay ring is said to have finite (resp., countable) Cohen-Macaulay type if it has only finitely (resp., countably) many isomorphism classes of maximal Cohen-Macaulay modules.

In my dissertation, I show a positive answer to Question 1 in the graded, non-Gorenstein case. In doing so, I have developed new techniques based on resolutions of modules and analysis of the syzygies. These results and techniques will be detailed in [Sto].

2.2. **Bounding Projective Resolutions.** Another major result that came out of Hilbert's research on invariant theory was the syzygy theorem: every finitely generated module over a polynomial ring has a free resolution of length n, where n is the number of variables. In recent years, Stillman [PS09, Problem 3.14] has asked an interesting question trying to better Hilbert's result:

Question 2. *Let k be a field. Does there exist a bound, independent of n, for the projective dimension of an arbitrary ideal $I \subset k[x_1, \ldots, x_n]$ which is generated by N forms of given degrees $d_1, \ldots, d_N$?*

[2]Paul Gordan is known in his time as the "king of invariants."

This question is open in all but low degree cases. In [McC11], McCullough produced a family of ideals whose projective dimensions far exceeded any known bound. In [BMnB+11], my co-authors and I generalize the family of ideals given in [McC11] to a larger family with much larger projective dimension. This family subsumes and improves upon constructions given by McCullough [McC11]. In particular, in the three-generated case, we produce a family of ideals with generators of degree d and projective dimension larger than $\sqrt{d}^{\sqrt{d}-1}$. This does not give an answer to Stillman's Question, but does impose larger lower bounds on any possible answer.

2.3. **Betti Table Decomposition Theory of Boij and Söderberg.** In 2008, Eisenbud and Schreyer proved the convex Betti decomposition conjectures due to Boij and Söderberg [BS08], [ES09]. These conjectures provided a decomposition theory for Betti diagrams of finitely generated graded Cohen-Macaulay modules over a standard graded polynomial ring. As opposed to on-the-nose structure theorems such as the Hilbert-Burch Theorem or the explicit description of minimal free resolutions for codimension three Gorenstein ideals, Boij and Söderberg focused only on the particular numerics of the minimal free resolutions without studying the differentials. Even more surprisingly, they envisioned a decomposition of Betti diagrams up to rational multiple.

The proof of these conjectures sparked a flurry of investigations, but there have been few results that use the theory to investigate module-theoretic properties. In fact, many such investigations uncover disturbing phenomena. In a polynomial ring S, the Boij-Söderberg decompositions of homogeneous ideal I and the quotient S/I may be wildly different, even though the minimal free resolutions of I and S/I are intimately related. However, this same theory proved the beautiful Multiplicity Conjecture of Herzog, Huneke, and Srinivasan [HS98], even with the apparent bad behavior.

In the summer of 2011, I attended the commutative algebra summer school at the Mathematical Science Research Institute in Berkeley, CA (MSRI). My research group[3] was asked to consider the following question:

Question 3. *Consider a complete intersection as a module over a graded polynomial ring. Do the degrees of the relations determine the rational coefficients in a Boij-Söderberg decomposition of its Betti table?*

We where able obtain a positive result for low codimensions [GJM+].

Proposition 1. *Question 3 has a positive result for codimension one, two, and three.*

While researching this topic, our group discovered an interesting calculus of the Betti tables based on an alternate decomposition. This concept needs further study and is one of my ongoing research projects.

2.4. **Packages in Macaulay2.** As stated on the *Macaulay2* [GS] website, "*Macaulay2* is a software system devoted to supporting research in algebraic geometry and commutative algebra, whose creation has been funded by the National Science Foundation since 1992." My work in *Macaulay2* is centered around the famous Quillen-Suslin theorem from K-theory.

In 1955, J.P. Serre posed the following question: Do there exist finitely generated projective $k[x_1, \ldots, x_n]$ modules, with k a field, which are not free? This question was known as "Serre's Problem" and the question in its full generality remained open for 21 years until it was resolved independently by Quillen and Suslin in 1976, resulting in the following well-known[4] theorem.

Theorem 2 (Quillen-Suslin). *Let $S = k[x_1, \ldots, x_n]$, with k a field. Then every finitely generated projective S-module is free.*

However, the proofs given were not entirely constructive, and it was not until the early 1990's that mathematicians began giving fully constructive versions of the proof. In 1992, Logar and Sturmfels [LS92] published the algorithmic proof of the Quillen-Suslin Theorem that forms the basis for the

[3]The research group I worked with at MSRI comprised of C. Gibbons, J. Jeffries, S. Mayes, C. Raiciu, B. White and myself

[4]This theorem is well-known to those who know it.

methods in our *Macaulay2* package *QuillenSuslin*. In their paper, Logar and Sturmfels describe, via the technique of completion of unimodular rows, how to construct a free generating set for a projective module over $\mathbb{C}[x_1, \ldots, x_n]$. One can extend these constructive techniques to work over more general coefficient rings such as $\mathbb{Q}, \mathbb{Z}$, and $\mathbb{Z}/p\mathbb{Z}$, for p a prime integer. In joint work with Brett Barwick, we have implemented these algorithms, with some modifications, in our *QuillenSuslin* package for *Macaulay2*. For an overview of the package see [BS11].

3. Undergraduate Research Projects

While a graduate student at the University of Kansas, and recently as a Visiting Professor at Bard College, I have had several opportunities to work with undergraduate students on research projects. Below is a sample of these undertakings.

3.1. **Bidding Tic-Tac-Toe.** While a GK-12 Fellow, part of my duties was to advise undergraduate math education majors in a research methods course. I advised many semester long projects during this time, but my favorite one was the investigation of a bidding version of tic-tac-toe. This is an interesting game that was created by Craig Huneke. Basically, the game is tic-tac-toe with a gambling component; you have to bid to play. Here are the rules:

(1) Each person has a set amount of money, say \$1000 each;
(2) The players then bid for the next move;
(3) The loser of the bidding gets to make the first bid on the next move;
(4) The game is over when one player has three in a row.

An example of how the game might be played is as follows:

- Player 1 bids \$500 and player 2 concedes.
- Thus player 1 wins the turn and places an X on the center square.
- Since player 2 lost the last bid, player 2 gets to make the initial bid on the second move.
- Player 2 bids \$100.
- Player 1 counters with \$200 and player 2 concedes again, allowing player 1 to place another X on the board.
- Note, at this point player 1 has \$300 an two X's on the board while player 2 has \$1000.
- Player 2 now bids \$301 three times in a row to win the game.

As one can see, a bidding system added to ordinary tic-tac-toe makes the game a little more dynamic. It was the students research project to determine the best strategy for wining. Through the analysis of many examples, the student was able to assert and prove the following.

Proposition 3. *If player 1 spent more than \$556 on the first two moves, then player 2 could win regardless of how player 1 bid after that. Further, if player 1 spent less than \$512 on the first two moves, then regardless of how player 2 bid, player 1 could win.*

3.2. **Applications of Graph Theory to Chaotic Systems.** Part of my duties working for the Bard Prison Initiative is to be an adviser senior thesis projects. I am currently advising two students who are investigating different aspects of chaotic dynamical systems. One of the projects is using Graph Theory as a tool to investigate chaotic systems. In particular, the following question is being studied.

Question 4. *Given a chaotic system, one can generate a corresponding combinatorial graph and then analyze this graph with tools from Graph Theory. However, in most instances, a generated complex network is not unique. Is it possible to develop tools that preserve more information during the translation process?*

The student has been able to take a chaotic system and translate it to a time series via a bifurcation diagram. From here, various algorithms were developed in order to obtain a combinatorial graph of the chaotic system. It is an on going process to find what characteristics of the chaotic system remain invariant under this transformation.

3.3. **Algebraic Structures and Boij-Söderberg Theory.** As mentioned above, Boij-Söderberg theory is relatively new on the scene of mathematics. As such, not much is really known about the subject. I am currently working on a senior thesis project with a Bard student on the following problem.

Question 5. *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of R-modules. Given the Boij-Söderberg decomposition of the Betti tables of A and C, can we determine the decomposition of the Betti table of B?

What makes this question interesting is that a positive answer would give an algebraic interpretation to the numerics of the decomposition. Currently my student has shown (on her own) a positive answer to short exact sequences of the from

$$0 \longrightarrow A \longrightarrow A \oplus C \longrightarrow C \longrightarrow 0.$$

While this result is easy to see for an expert, it is quite challenging for an undergraduate student. I am very happy with the current progress of the project and hope the student will have a class of cyclic modules that give a positive answer to the question by the end of the Spring semester.

3.4. **Future Projects.** All of the projects mentioned above have many different avenues to explore and can be picked up by future students. I also have ideas rooted in graph theory and its relation to commutative algebra. I would also be interested in seeing students programing in *Macaulay2*. There are many different packages that need developing and are very accessible to an undergraduate student who is interested in computer algorithms. In fact, my student who is working on the Boij-Söderberg project has already experimented with computations with *Macaulay2*.

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