23 Problems: Diagonalizing Symmetric Matrices

1. (On Reality of Eigenvalues)

   (a) Suppose $z = x + iy$ where $x, y \in \mathbb{R}, i = \sqrt{-1}$, and $\overline{z} = x - iy$. Compute $z\overline{z}$ and $\overline{z}z$ in terms of x and y. What kind of numbers are $z\overline{z}$ and $\overline{z}z$? (The complex number $\overline{z}$ is called the *complex conjugate* of z).

   (b) Suppose that $\lambda = x + iy$ is a complex number with $x, y \in \mathbb{R}$, and that $\lambda = \overline{\lambda}$. Does this determine the value of x or y? What kind of number must λ be?

   (c) Let $x = \begin{pmatrix} z^1 \\ \vdots \\ z^n \end{pmatrix} \in \mathbb{C}^n$. Let $x^\dagger = \begin{pmatrix} \overline{z^1} & \cdots & \overline{z^n} \end{pmatrix} \in \mathbb{C}^n$ (a $1 \times n$ complex matrix or a row vector). Compute $x^\dagger x$. Using the result of part 1a, what can you say about the number $x^\dagger x$? (E.g., is it real, imaginary, positive, negative, etc.)

   (d) Suppose $M = M^T$ is an $n \times n$ symmetric matrix with real entries. Let λ be an eigenvalue of M with eigenvector x, so $Mx = \lambda x$. Compute:
   $$\frac{x^\dagger M x}{x^\dagger x}$$

   (e) Suppose Λ is a 1×1 matrix. What is Λ^T?

   (f) What is the size of the matrix $x^\dagger M x$?

   (g) For any matrix (or vector) N, we can compute $\overline{N}$ by applying complex conjugation to each entry of N. Compute $\overline{(x^\dagger)^T}$. Then compute $\overline{(x^\dagger M x)^T}$. Note that for matrices $\overline{AB + C} = \overline{A}\,\overline{B} + \overline{C}$.

   (h) Show that $\lambda = \overline{\lambda}$. Using the result of a previous part of this problem, what does this say about λ?

`Problem 1 hint`

2. Let $x_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, where $a^2 + b^2 + c^2 = 1$. Find vectors x_2 and x_3 such that $\{x_1, x_2, x_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

3. (Dimensions of Eigenspaces)

(a) Let $A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$. Find all eigenvalues of A.

(b) Find a basis for each eigenspace of A. What is the sum of the dimensions of the eigenspaces of A?

(c) Based on your answer to the previous part, guess a formula for the sum of the dimensions of the eigenspaces of a real $n \times n$ symmetric matrix. Explain why your formula must work for any real $n \times n$ symmetric matrix.

24 Problems: Kernel, Range, Nullity, Rank

1. Let $L\colon V \to W$ be a linear transformation. Show that $\ker L = \{0_V\}$ if and only if L is one-to-one:

   (a) First, suppose that $\ker L = \{0_V\}$. Show that L is one-to-one. Think about methods of proof–does a proof by contradiction, a proof by induction, or a direct proof seem most appropriate?

   (b) Now, suppose that L is one-to-one. Show that $\ker L = \{0_V\}$. That is, show that 0_V is in $\ker L$, and then show that there are no other vectors in $\ker L$.

Hint for 1