20 Problems: Diagonalization

1. Let $P_n(t)$ be the vector space of polynomials of degree n or less, and $\frac{d}{dt} : P_n(t) \mapsto P_{n-1}(t)$ be the derivative operator. Find the matrix of $\frac{d}{dt}$ in the bases $\{1, t, \ldots, t^n\}$ for $P_n(t)$ and $\{1, t, \ldots, t^{n-1}\}$ for $P_{n-1}(t)$.

2. When writing a matrix for a linear transformation, we have seen that the choice of basis matters. In fact, even the order of the basis matters!

   - Write all possible reorderings of the standard basis $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$.
   - Write each change of basis matrix between the standard basis $\{e_1, e_2, e_3\}$ and each of its reorderings. Make as many observations as you can about these matrices: what are their entries? Do you notice anything about how many of each type of entry appears in each row and column? What are their determinants? (Note: These matrices are known as *permutation matrices*.)
   - Given the linear transformation $L(x, y, z) = (2y - z, 3x, 2z + x + y)$, write the matrix M for L in the standard basis, and two other reorderings of the standard basis. How are these matrices related?

3. When is the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ diagonalizable? Include examples in your answer.

4. Show that similarity of matrices is an *equivalence relation*. (The definition of an equivalence relation is given in Homework 0.)

5. *Jordan form*

- Can the matrix $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ be diagonalized? Either diagonalize it or explain why this is impossible.
- Can the matrix $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$ be diagonalized? Either diagonalize it or explain why this is impossible.
- Can the $n \times n$ matrix $\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$ be diagonalized? Either diagonalize it or explain why this is impossible.

  Note: It turns out that every matrix is similar to a block matrix whose diagonal blocks look like diagonal matrices or the ones above and whose off-diagonal blocks are all zero. This is called the *Jordan form* of the matrix and a (maximal) block that look like

$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

  is called a *Jordan n-cell* or a *Jordan block* where n is the size of the block.