# A New Look at Hadwiger's Conjecture

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# Abstract

The goal in this project is to make the rather intimidating particulars involved with Hadwiger's Conjecture understandable to a reader who is a novice to the complex field of graph theory. In a detailed exploration of the rules involved with proper vertex coloring, and explicitly defining the processes involved with creating a minor, this project proves the chromatic number of a large portion of known graphs, and establishes that Hadwiger's Conjecture holds through using unique proofs that do not rely on the Four Color Theorem. In addition, since Hadwiger's Conjecture has only been proven for graphs with a chromatic number of at most 6, this project attempts to expand the reach of Hadwiger's Conjecture. This attempt is made through creating two non-complete $n$-chromatic graphs unique to this project and establishing that, when the chromatic number exceeds 6, Hadwiger's Conjecture holds for these new graphs.

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# Dedication

This project is dedicated to those who helped provide a gentle breeze to fill the sails of my ship of progress, also, to all those who doubted me, for when I felt like giving up, their unbelief was the fuel that kept me pulling the load, and finally and most importantly to the futrure me that all this hard work is making possible.

# Acknowledgments

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# 1 Introduction

Since we know there exists graphs of arbitrarily high chromatic number, what, then can be said about the structure of such graphs. The conjecture proposed by Hadwiger in 1943 provides us a way to ascertain some special properties of graphs within the complicated niche of graph theory that involves proper vertex colorings. Hadwigger's Conjecture 2.0.3 can be stated as such: If $G$ is $n$-chromatic there exists a minor $H$ that is a complete graph and retains the same chromatic number of the original graph $G$ such that $H$ can be obtained from $G$ by deleting or contracting edges and merging vertices. Why should we care about such a compartmentalized aspect of graph theory. As in many aspects of mathematics, proving, or merely exploring a seemingly trivial piece of larger established work can yield unforeseen, and sometimes important results. Take for example the Four Color Theorem. While it becomes obvious that the Four Color Theorem is true when one simply colors any arrangement of shapes on a two dimensional surface (so long as the restrictions are met), proving the Four Color Theorem took years to figure out and the end result, which required the aid of computers, is not only quite long, but also dense and complicated.

As with many aspects of graph theory, Hadwiger's conjecture appears quite simple, for in many cases one can logically perform the task in question with simple examples. However, proving the case in general is exceedingly difficult, and some mathematicians believe it to be impossible [3]. Out of these fertile conditions sprouts a riddle that just begs to be solved, and even if not solved completely, at the very least explored. Through proving the chromatic number seeing if Hadwiger's Conjecture holds on a substantial group of known families of graphs in addition to a pair of $n$-chromatic non-complete graphs unique to this project, this project hopes open the door to the discovery of a new class of $n$ chromatic non-complete graphs that do not contain a proper $K_n$ subgraph. For one savvy in the field of graph theory, specifically in principles of coloring, Hadwiger's Conjecture when $n = 1$ and $n = 2$ is obvious; however, we will not dismiss these cases as trivial, as explicitly proving them will assist us as we move to the more complicated cases. The case when $n = 3$ can be easily verified because, as we will see, a 3-chromatic graph necessarily contains an odd cycle, and every odd cycle contains $K_3$ as a minor. Hadwiger (1943) settled the case when $n = 4$ [8]. This same case was strengthened by Dirac (1952) who established a slightly stronger theorem.

**Theorem 1.0.1** ([6])**.** *Every 4-chromatic graph contains a $K_4$-subdivision.*

This conjecture has also been verified for the case when $n = 5$ by Wagner (1964) who showed that when $n = 5$, Hadwiger's Conjecture is equivalent to the Four-Color Theorem; thus it is also true [11]. Robertson (1993) established that this conjecture is true when $n = 6$, and this proof, like Wagner's, also relies of the Four Color Theorem [10].

Despite all the minds that have struggled with to verify this conjecture, any proof with a chromatic number greater than 4, depends upon the Four Color Theorem, and proof of the general case remains elusive. Although in 1980 B. Bollobas, P.A. Catlin and P. Erdos posited in an article that "Hadwiger's Conjecture is true for almost every graph [2]," some mathematicians now believe that the Conjectuer may be false in the general case [3]. While Hadwiger's Conjecture has been proven for cases when $n \geq 6$ in this project we will offer original proofs which do not rely on the Four-Color Theorem, and explore the cases where $n > 6$ in two unique $n$-chromatic graphs.

In Chapter 2, we explicitly define Hadwiger's Conjecture, explaining the terms, definitions, and notation used throughout this project that are required to understand Hadwiger's Conjecture.

In Section 2.1, we define the various processes involved with modifying a graph building up to the process of forming a minor.

In Section 2.2, we define the definitions and notation involved with coloring a graph, as well as state and prove theorems involved with proper vertex coloring.

In Chapter 3, we explore and prove the chromatic number of the major families of graphs and locate their $K_n$ minors proving that they hold to Hadwiger's Conjecture. We proceed in a linear fashion beginning with simple graphs with low chromatic numbers progressing to more complex graphs with larger chromatic numbers. We do not treat the cases where $n \leq 3$ as trivial. Rather, we explore and prove these cases, as they help us to move to the more complex ones.

In Chapter 4, we introduce two $n$ chromatic non-complete graphs unique to this project.

In Section 4.1, we define the $n$-chromatic non-complete Butterfly graph and demonstrate its construction.

In Section 4.2, we define the $n$-chromatic non-complete Parachute Graph, which is constructed through a distinctive algorithm. We also modify this graph in a further attempt to craft a $n$-chromatic non-complete graph with out proper subgraphs.

In Chapter 5 Section 5.1, we explore how to color a graph with a function. This is called a chromatic polynomial and is useful in proving the chromatic number of complicated graphs.

In Section 5.2 we use the chromatic polynomial to locate a proper coloring for the parachute graph.

In Chapter 6 we state our findings and conclude by encouraging further research.

### 1.0.1 *History of Graph Theory*

Before we get into the meat of this project, let us take a step back to honor the man credited as the founder of graph theory[1] and explore the problem he solved thereby providing us with graph theory's first theorem. This interesting history of graph theory can be traced to the 18th century, specifically 1735, when the Swiss mathematician Leonhard Euler solved the Königsberg bridge problem.

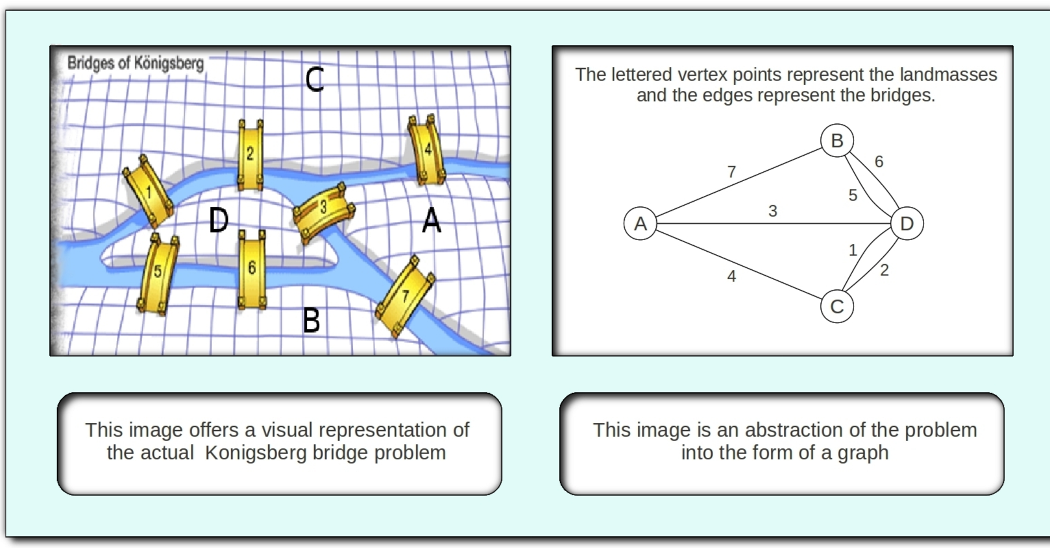


Figure 1.0.1. The Königsberg bridge diagram and corresponding graph

The Königsberg bridge problem is an old puzzle concerning the possibility of finding a path over every one of seven bridges that span a forked river flowing past an island, as

[1]For the novice reader, graph theory is a branch of mathematics concerned with networks of points connected by lines.

depicted in the leftmost illustration of Figure 1.0.1. However, there is a restricition: one can only cross each bridge once. Leonhard Euler was intrigued by the question of whether a route existed that would traverse each of the seven bridges exactly once. In disproving this, he laid the foundation for graph theory by simplifying the problem into an abstract graphical representation, as depicted in the rightmost illustration of Figure 1.0.1.

Euler's proof, through abstracting the problem into a graph, involved only references to the physical arrangement of the bridges. He not only solved the problem with his proof but, more importantly, he essentially proved the first theorem in graph theory, which years later was translated into the terminology of modern graph theory. The modern version the of his theorem can be restated as the following.

**Theorem 1.0.2** ([7]). *If there is a path along edges of a multigraph that traverses each edge once and only once, then there exist at most two vertices of odd degree; furthermore, if the path begins and ends at the same vertex, then no vertices will have odd degree.*

Although graph theory had its beginnings in recreational math problems, graphs are a powerful problem-solving tool. Graphs enable us to represent a complex situation both visually, and now, in modern times, with the aid of a computer. It is for these reasons that graph theory has grown into a significant area of mathematical research with applications in chemistry, operations research, social sciences, and computer science. The coloring of graphs in general, and the specifics of proper vertex coloring involved with this project, add yet another layer to graph theory allowing us to further abstract a graph and analyze its properties.

Now that we have gained a little knowledge of the origins of graph theory and the power of graphs, we can better appreciate the finer details of this field, such as the chromatic number of graphs and finding their $K_n$ *minors*, that this project explores.

# 2

# Background, Basic Definitions, and Notation

We begin this section by precisely stating Hadwiger's conjecture. Then we progress with definitions, notations, and explanations necessary to explicitly understand the implicit assertions within the conjecture. Hadwigger's conjecture can be stated as follows:

**Conjecture 2.0.3** (Hadwiger's Conjecture)**.** *If $G$ an is $n$-chromatic graph, then there exists a $K_n$ minor.*

This conjecture involves a number of concepts–mainly the coloring of a graph $G$, its $n$-chromatic number, and the notion of a $K_n$ *complete graph*, and the concept of a *minor*. However, in order to explore the coloring of graphs and locate a $K_n$ *minor* with the same chromatic number of the graph from which it is derived, we must define the basic terminology used in graph theory. We begin by defining a graph.

**Definition 2.0.4.** A **graph**, denoted by $G$ unless otherwise specified, is an ordered pair $(V(G),\ E(G))$ that consists of a non-empty finite set $V(G)$ of *vertices* and a set $E(G)$, of *edges*. △

Implicit within the definition of a graph are the components it is composed of. We will unpack this definition to define these components separately and then provide an example. Graphs, in a most basic sense, are made up of elements of the *vertex set* called *vertex points* that are usually represented geometrically as small circles or dots. If a *vertex set* contains multiple vertex points (often called *vertices*) they can stand alone or may be linked together as an unordered pair by an *edge*, which is normally represented as a continuous line or curve linking two vertex points. Although vertices and edges are usually represented as the geometric objects that form a graph, as in Figure 2.0.1, a strict definition requires us to define them as sets.

**Definition 2.0.5.** A **vertex set** is a collection of vertex points (vertices) within the graph $G$. The vertex set is denoted by $V(G)$, and vertices within the vertex set are denoted by $v_i$. $\triangle$

**Definition 2.0.6.** An **edge set** is a collection of edges within the graph $G$. The edge set is denoted by $E(G)$, and the edges of $E(G)$ represent unordered pairs of vertex points within $V(G)$. $\triangle$

The edge set can be empty in the case of a null graph. Such a graph consists of only vertex points. We will discuss such graphs in Section 1 of Chapter 3. If the edge set $E(G)$ is not empty, an edge represents an indicated function associating each edge of $G$ to an unordered (not necessarily distinct) pair of vertices in $V(G)$. If $e \in E(G)$ and $u$ and $v \in V(G)$, then $e$ is the unordered pair of vertices $u$ and $v$.

While some of the particular notation involved with defining graphs was stated in the previous definitions, there is additional notation used to identify specific elements of a graph. In dealing with vertices, $v_i$ will be used to denote a vertex point or vertice, and the index of $v_i$ denotes which vertice it is in the set. The notation, $|V(G)|$ will be used to denote the number of vertices in $G$. This brings us to edges. To denote an edge, we will use $e_i$ where the index of $e$ denotes which edge it is in the edge set. The notation, $|E(G)|$ denotes the number of edges in $G$. We now combine vertices and edges to define the function of an edge, as in what vertex points it connects to. This will be denoted by $(e_i) = \{v_i, v_j\}$, $i \neq j$.

In the preceding sections we will add additional notation specific to certain graphs and procedures as needed.

Now we will bring all of this information together with a concrete example.

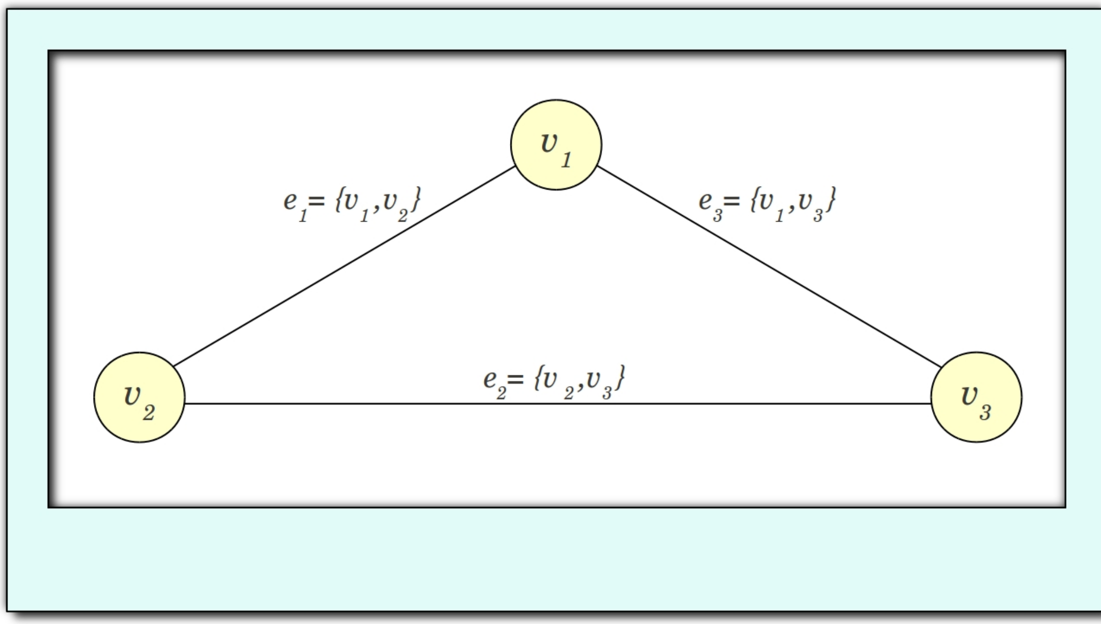


Figure 2.0.1. Illustration of a basic graph

**Example 2.0.7.** Let us consider the graph $G$ in Figure 2.0.1, and let $G$ represent this graph. We have that

$$V(G) = \{v_1, v_2, v_3\}$$

$$E(G) = \{e_1, e_2, e_3\},$$

where $e_1 = \{v_1, v_2\}$, $e_2 = \{v_2, v_3\}$, and $e_3 = \{v_1, v_3\}$. Thus, $|V(G)| = 3$, and $|E(G)| = 3$.

Now that we have an understanding of the components of a graph and the notation used to describe such, we now define some graphs that are implicit within some of the definitions and concepts contained in Hadwiger's Conjecture: the *simple graph*, the *complete graph*, and the *cycle graph*, which are used in the definitions of graphs that appear in later sections.

**Definition 2.0.8.** A **simple graph** is a graph that has no parallel edges (Two or more links with the same pair of vertices) or loops (edges with an identical ends). △

**Definition 2.0.9.** A **complete graph** is a simple graph in which any two vertices are adjacent, thus, every vertex point is connected to every other vertex point by a unique edge. Figures 2.0.1 and 5.1.4 are examples of complete graphs. Figure 2.0.1 is an example of the complete graph $K_3$, and Figure 5.1.4 is an example of the complete graph $K_6$. This type of graph is defined explicitly in Section 3.7 of Chapter 3. △

**Definition 2.0.10.** A **cycle graph** of order $n$ is denoted by $C_n$. This graph is a walk along an edge beginning at one vertex and continuing in a circuit to the next vertex along a different edge such that no vertex or edge is repeated in the progression. △

Taking the definitions and notation we have just defined, we now move to define the process of modifying a graph to obtain a *minor*.

## 2.1 Modifying Graphs

Having defined a graph, we now bring all of this information together to discuss how graphs are modified. This process is vital Hadwiger's conjecture–explicitly the definition of a *minor*; however, since this definition is dependent on a number of other graph theory concepts, we will begin with those to establish a more in depth understanding of our key concepts.

In defining the number of terms that provide us with a way to describe the processes that are involved with forming a *minor* through a series of vertex and edge contractions and edge deletions, we begin with *incidence*, and *adjacent*.

**Definition 2.1.1.** The ordered pair of vertex points that the edge represents are said to be **incident** with the said edge and the edge is incident to the vertices. △

Observe illustrations **A** and **B** in Figure 2.1.1. In illustration **A** the vertices $v$ and $w$ are incident to the edge $e$. In illustration **B** the vertices $v$ and $w$ are incident to the edge $a$ and or $a'$.

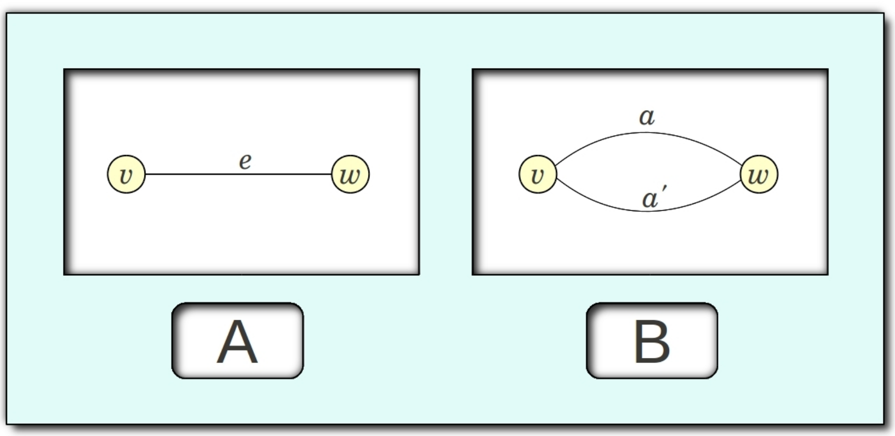


Figure 2.1.1. Illustration of vertices incident to an edge

**Definition 2.1.2.** Two vertices $v$ and $w$ of a graph $G$ elements of $V(G)$ are **adjacent** if there is an edge $e$ connecting them as an unordered pair $\{v, w\}$. $\triangle$

**Definition 2.1.3.** The **degree** of a vertex is the number of edges that are incident on $v$. The *total degree* of a graph is the sum of the degrees of all the vertices of $G$. $\triangle$

As we can see being incident is related to being adjacent. The relation between incident and adjacent can be explained by referring to illustration **A** of Figure 2.1.1. In this case, the vertices $v$ and $w$ are incident with the edge $e$. However, the vertices $v$ and $w$ are also adjacent to one another because they share a common edge. Thus, two vertices which are incident with a common edge are adjacent, as are two edges that are incident with a common vertex. We are interested in incidence and adjacency because they are vital to the understanding of coloring vertices.

Moving forward toward defining a minor, we now define how to obtain a *subgraph* $H$ from a graph $G$. A subgraph, like the name suggests, is a graph contained within a larger graph. Understanding the difference between a subgraph and a *minor* is important. A *minor* is similar to a subgraph, in that it is produced from a larger graph, but dissimilar because may not be contain within the larger graphs vertex set and edge set. However, although they are not the same, a working knowledge of the processes involved with forming a subgraph prepare us to understand the *minor*, which will be defined at the end of this section.

**Definition 2.1.4.** A graph $H$ is said to be a **subgraph** of $G$, if and only if, every vertex in $H$ is also a vertex in $G$, and every edge in $H$ is also an edge in $G$, and every edge in $H$ has the same end points as it has in $G$. A subgraph can be formed from $G$ by deleting elements of $V(G)$ and or elements of $E(G)$ to form a subgraph $H$. $\triangle$

Generally, if $X$ is any edge set in $G$, we denote the deletion of these edges by $G - X$. Thus, $G - X$ is the subgraph $H$ of $G$ obtained by deleting edges and or vertices in $G$.

Likewise, if $Y$ is any vertex set in $G$, we denote the deletion of these vertices by $G - Y$. Thus, $G - Y$ is the subgraph $H$ of $G$ obtained by deleting edges and or vertices in $G$.

In order to demonstrate how we can obtain subgraphs of a graph by deleting edges, or by deletion of vertices, let us consider an example for each case.

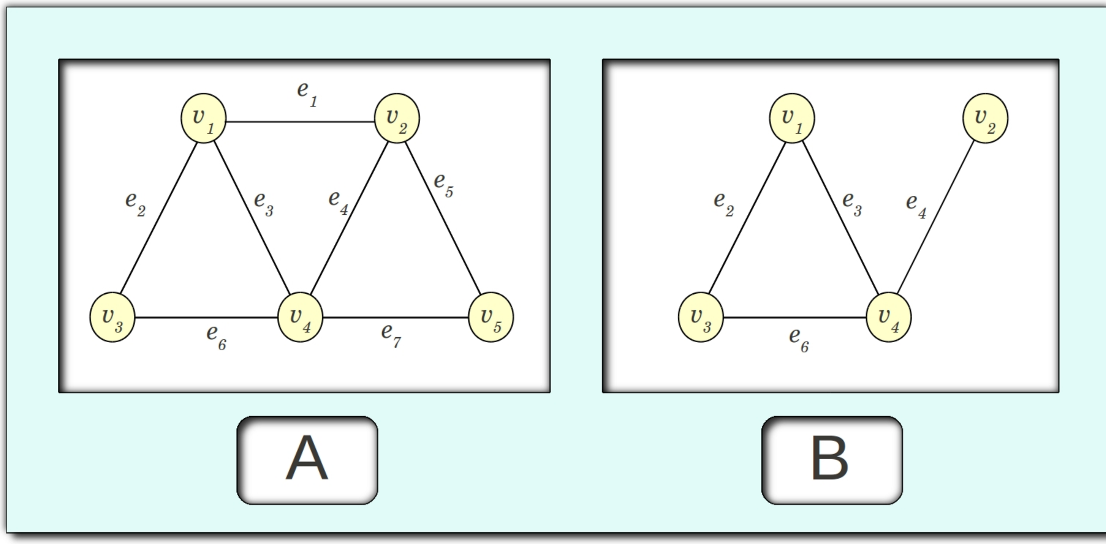


Figure 2.1.2. The formation of a subgraph from deletion of edges

**Example 2.1.5.** Let $G$ be the graph in illustration **A** of Figure 2.1.2. So $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, and $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. Now let us create a subgraph $H$ from $G$ by deleting edges $e_1, e_6$,and $e_7$ from $E(G)$, denoted by $G - \{e_1, e_6, e_7\}$. Let $X$ be the edge set $\{e_1, e_6, e_7\}$. Thus $H = G - X$, as depicted in illustration **B** of Figure 2.1.2 which is formed by these deletions. So $V(H) = \{v_1, v_2, v_3, v_4, \}$, and $E(H) = \{e_2, e_3, e_4, e_6\}$. Since $V(H) \subset V(G)$ and $E(H) \subset E(G)$, we see that $H$ is a subgraph of $G$.

**Example 2.1.6.** Let $G$ be the graph in illustration **A** of Figure 2.1.3. So $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, and $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. Now let us create a subgraph from $G$ by deleting vertices $v_2$,and $v_5$ from $V(G)$, denoted by $G - \{v_2, v_5\}$. Let $Y$ be the vertex

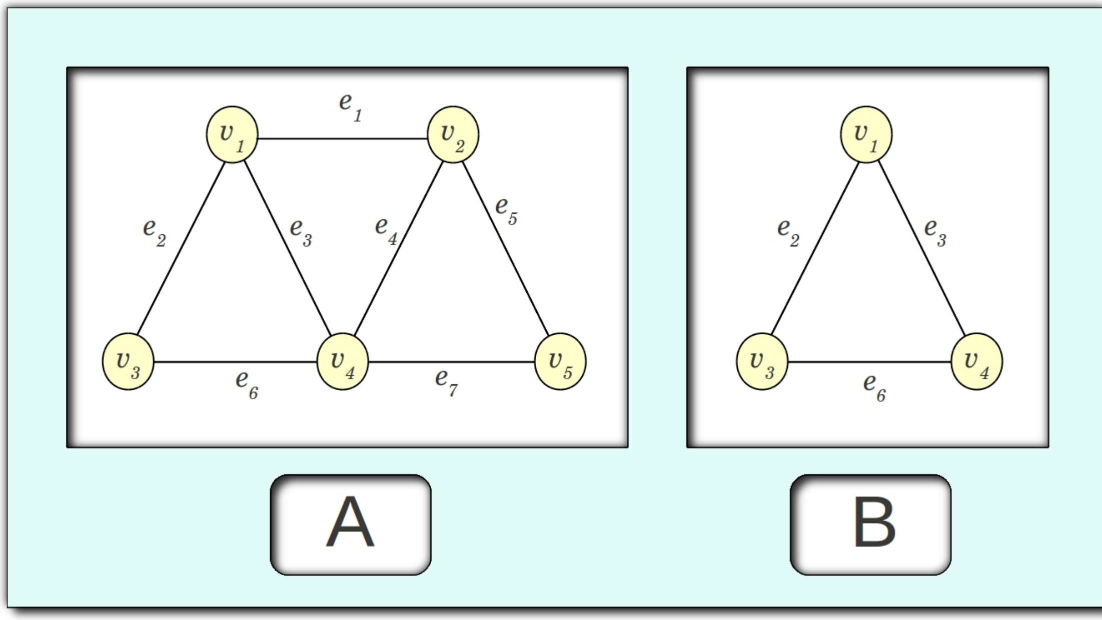


Figure 2.1.3. The formation of a subgraph from deletion of vertices

set $\{v_2, v_5\}$. Thus $H = G - Y$, as depicted in illustration **B** of Figure 2.1.3 which is formed by these deletions.

So, $V(H) = \{v_1, v_3, v_4, \}$, and $E(H) = \{e_2, e_3, e_6\}$. Since $V(H) \subset V(G)$ and $E(H) \subset E(G)$, we see that $H$ is a subgraph of $G$.

The deletion of vertices and edges is one way to modify a graph. However, there another process to accomplish this known as *contraction*. Now that we have defined how to obtain a subgraph, it is important for us to ascertain a solid understanding of both deletion and contraction, as they are central to this project.

**Definition 2.1.7.** [1] The process of **contraction** involves contracting an edge. We pick an edge $e_n$ and remove it identifying the unordered pair of vertex points it consists of, say vertex points $v$ and $w$. We now merge these two vertices such that the result is a single vertex point $(vw)$ that is adjacent with all the vertices $v$ and $w$ were originally adjacent with. Since we are interested only in simple graphs, if edges are duplicated in the process of merging vertices, one of the parallel edges is omitted. △

Observe the illustrations **A** and **B** of Figure 2.1.4. Illustration **A** depicts the process of contracting an edge, and illustration **B** depicts the omission of one of the parallel edges to form a simple graph from $G/e$.

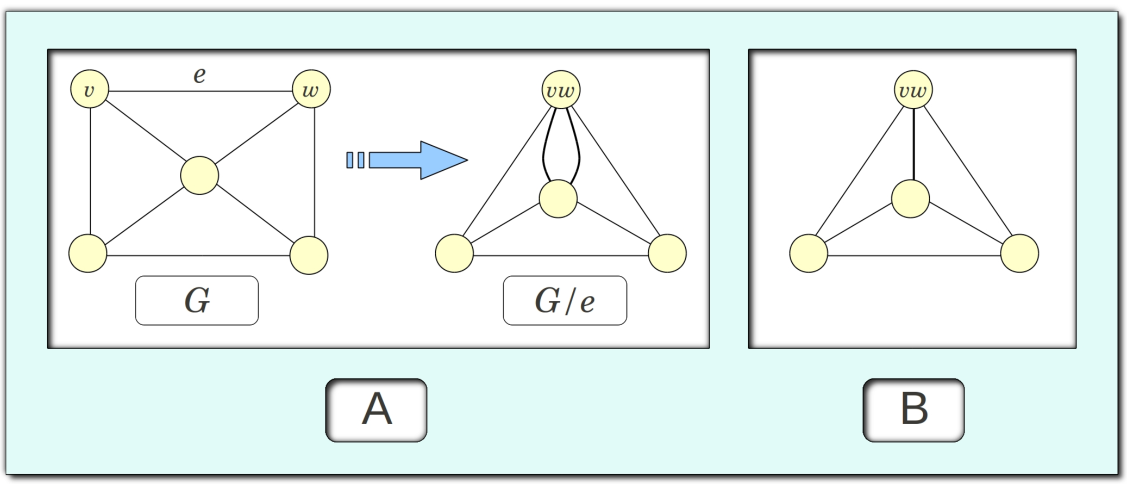


Figure 2.1.4. Illustration of the process of contraction

Now that we have an understanding of a the process of contraction, we can move to defining the one of the essential portion of Hadwiger's Conjecture: the *minor*.

[1] There is an inverse to the process of contraction and deleting edges, as we have just defined it. This process is called edge splitting and edge subdivision [3]. We will not go into detail about this process as it does not serve this projects objective. However, in Chapter 3 Section 3.3 we use this process to in induction to prove the chromatic number and construct even and odd cycles

**Definition 2.1.8.** A **minor** of a graph $G$ is any graph that can be obtained from $G$ by means of a sequence of the following operations:

1. edge deletion
2. vertex deletion
3. edge contraction.

$\triangle$

Obtaining a minor from a graph is a process similar to what we have learned in the processes involved with contracting edges and vertices and creating subgraphs. However, since we add vertices and edges not originally in $G$ it cannot be a subgraph. Since the process of forming a minor is an integral part of this project, we will go through an example to bring everything together.

**Example 2.1.9.** Let us consider the graph $G$ in illustration **A** of Figure 2.1.5. To form a minor from $G$, we pick the edge $\{r, s\}$ and contract it. This then forms a new vertex $t$ such that $t$ is adjacent to all the verticies $r$ and $s$ were adjacent with, as depicted illustration **B** of Figure 2.1.5.

This gives us $H$, which is a minor of $G$. We want to continue with the process since the purpose of Hadwiger's Conjecture is to form a $K_n$ minor. Since the graph $G$ in illustration **A** of Figure 2.1.5 is 3-chromatic,[2] we want its minor to be $K_3$.

Continuing, we pick the edge $\{u, v\}$ to contract. Contracting this edge we form a new vertex $w$ such that $w$ is adjacent to all the vertices $u$ and $v$ were adjacent with, as in illustration **C** of Figure 2.1.5.

This gives us $I$, that is a minor to $G$ and to $H$, but we have not formed $K_3$ yet, so we

[2]This will be explained in detail in section 3

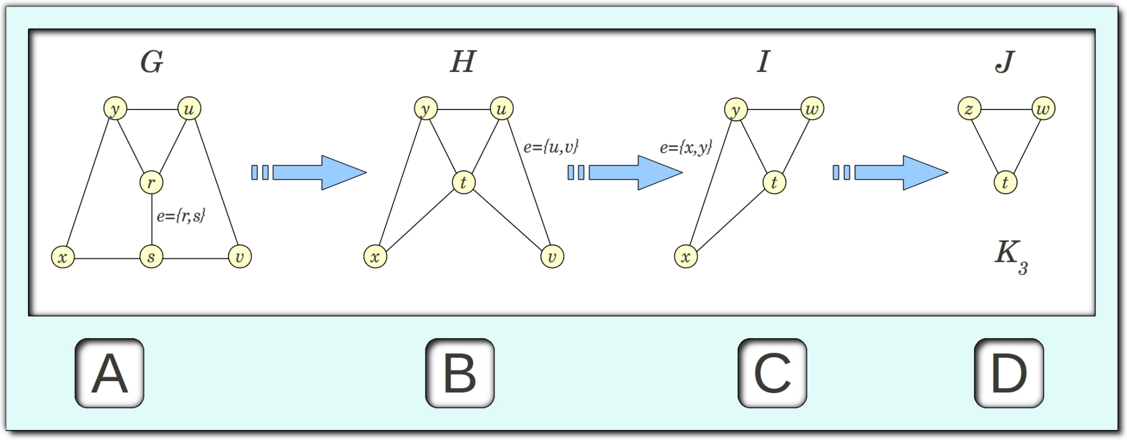


Figure 2.1.5. The formation of a minor

continue.

We can form a triangle ($K_3$) with one more edge deletion and contraction if we choose the last edge to be $\{x, y\}$, or $\{x, t\}$ and contract it. We choose the edge $\{x, y\}$. Contracting this edge forms a new vertex $z$ such that $z$ is adjacent to all the vertices $x$ and $y$ were adjacent with, as in illustration **D** of Figure 2.1.5. We now have a triangle ($K_3$), so we stop. Let this graph be $J$.

The resulting minor $J$ was formed from $I$, which was formed from $H$, which was formed from $G$. Thus $J$ through a sequence of vertex and edge deletions and edge contractions is a minor of $G$.

## 2.2 Graph Colorings

Since graph colorings–specifically vertex coloring–is the foundation of Hadwiger's conjecture, we will define the essential notation, definitions, and theorems involved with vertex coloring and how to color different types of graphs.

**Definition 2.2.1.** A **n-vertex coloring** of $G$ is an assignment of $n$ colors, $1, 2, 3, \ldots, n$ to the vertices of $G$. The coloring is *proper* if no two distinct adjacent vertices have the same color. Thus a proper $n$-vertex coloring of a loop-less graph $G$ is a partition $(v_1, v_2, \ldots, v_n)$ of $V$ into $n$ (possibly empty) independent sets. $G$ is *$n$-chromatic* if $G$ has a proper $n$-vertex coloring. △

We are also interested in the number of colors required to color a graph's vertices.

**Definition 2.2.2.** We say that a graph is ***n*-chromatic** if $n$ colors are the minimum number necessary for a proper coloring. This is denoted by $\chi(G) = n$, and $n$ is called the chromatic number. △

A *k-critical graph* is a classification of graphs that are necessary to comprehend some theorems involved with coloring.

**Definition 2.2.3.** A graph $G$ is **k-critical** if $\chi(G) = k$ and if the deletion of any vertex yields a graph with smaller chromatic number. We say that the graph $G$ is $k$-critical if $\chi(H) < \chi(G)$ for every proper subgraph $H$ of $G$. △

**Theorem 2.2.4** ([4, Theorem 8.1.1])**.** *If $G$ is $k$-critical , then $\delta \geq k - 1$ ($\delta$ represents the minimum degree of all vertices in $G$).*

*Proof.* By contradiction, let $G$ be a k-critical graph with $\delta < k - 1$, and let $v$ be a vertex of degree $\delta$ in $G$. Since $G$ is k-critical, $G - v$ is $(k-1)$-colorable. Let $(v_1, v_2, ..., v_{k-1})$ be a $(k-1)$-coloring of $G - v$. By definition, $v$ is adjacent in $G$ to $\delta < k-1$ vertices, and therefore $v$ must be nonadjacent in $G$ to every vertex of some $V_j$. But then $(v_1, v_2, ..., v_j \cup v, ..., v_{k-1})$ is a $(k-1)$-coloring of $G$, a contradiction. Thus $\delta \geq k - 1$. □

**Corollary 2.2.5.** *Every $k$-chromatic graph has at least $k$ vertices of degree at least $k - 1$, so $\delta \geq (k - 1)$*

*Proof.* Let $G$ be a k-chromatic graph, and let $H$ be a k-critical subgraph of $G$. By Theorem 2.2.4, each vertex of $H$ has a degree at least $(k-1)$ in $H$ and hence also in $G$. The corollary now follows since $H$, being $k$-chromatic, clearly has $k$ vertices. □

**Corollary 2.2.6.** *For any graph $G$, let $n$ be the chromatic number of the graph, and let $\delta$ be the maximum degree of the graph. Then*

$$n \leq \delta + 1$$

*Proof.* This is an immediate consequence of Corollary 2.2.5 □

If we know the degree of each vertex we can find the coloring of a graph.

**Theorem 2.2.7** ([12, Theorem 17.1])**.** *If $G$ is a simple graph with largest vertex-degree $\Delta$, then $G$ is $(\Delta - 1)$ -colorable.*

*Proof.* The proof is by induction on the number of vertices of $G$. Let $G$ be a simple graph with $n$ vertices. If we delete any vertex $v$ and its incident edges, then the graph that remains is a simple graph with $n-1$ vertices and the largest vertex-degree at most $\Delta$. By our induction hypothesis, this graph is $(\Delta-1)$-colorable. A $(\Delta-1)$-coloring is then obtained by coloring $v$ with a different color from the (at most $\Delta$) vertices adjacent to $v$ as depicted in Figure 2.2.1. □

Theorem 2.2.4, and Corollary 2.2.5 and 2.2.6 provides us a way to bound the chromatic number and obtain a coloring. However, there is little we can ascertain about the chromatic number of some arbitrary graph. All we can deduce is that if said graph has $n$ vertices, then its chromatic number cannot exceed $n$, and if the graph contains the complete graph $K_r$ as a subgraph, then its chromatic number cannot be less than $r$, but this is not very much information. Clearly, these results do not take us very far in general.

Theorem 2.2.7 is useful when most of the vertex degrees are approximately the same.

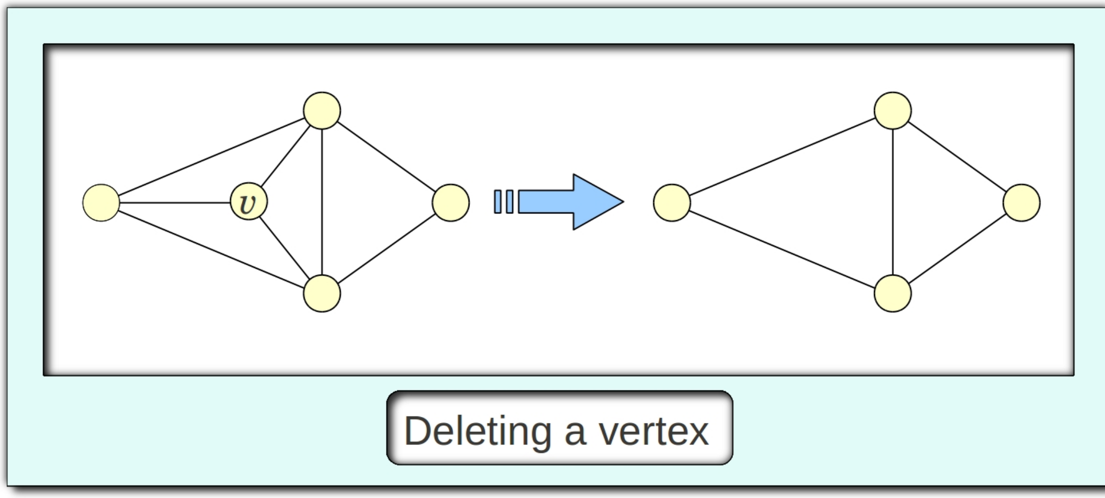


Figure 2.2.1. Illustration of Theorem 2.2.7

However, if the graph we are attempting to color has vertex degrees that vary greatly, say a few of high degree and the rest of low degree, then Theorem 2.2.7 tells us very little about a proper vertex coloring. We will address the problem of coloring graphs in the next section by proving the coloring of different families of graphs. Further, in Chapter 5 Section 5.1, we will introduce the chromatic polynomial to color graphs that do not easily conform to the theorems in this section, and have a chromatic number too difficult to prove otherwise.

# 3

# Coloring and finding the $K_n$ Minor of Classes of Graphs

Continuing our focused on Hadwiger's Conjecture 2.0.3, which states that every n-chromatic graph has a $K_n$-minor, and having defined the coloring of graphs in Section 2 Chapter 2, we will now explore and prove the proper coloring of graphs and find their corresponding minors.

We only deal with simple finite graphs where $E(G)$ and $V(G)$ are both finite sets. [1] Considering simple graphs helps us analyze larger $n$-colorable graphs since a graph is n-colorable if and only if its underlying simple graph is $n$-colorable. Therefore, by understanding the coloring of simple graphs, we can understand the coloring of the larger graphs containing these simple graphs. One way we can do this is by taking a simple graph where $\chi(G) = n$ and expanding its size (vertex set and edge set) while preserving its chromatic number.[2]

[1] Recall from definition 2.0.8 that Simple graphs are a special class of graphs with the property that no two edges have the same set of end points (vertices).

[2] This will be explained in detail later in Section 3.3.

There are many different types of simple finite graphs. We begin by defining different classes of graphs as well as analyzing their chromatic numbers. Finally, we show these graphs have corresponding $K_n$ minors.

Through the various sections in this chapter, we proceed in a linear fashion exploring the various families of graphs beginning with 1-chromatic graphs, then 2-chromatic, then 3-chromatic, and so on. We adopt a color by number system that will represent the actual colors of the vertices. Thus, a coloring will be visualized by a number inside of the vertex.

**Example 3.0.8.** Applying the number system to color a graph, consider the graph $G$ in Figure 3.0.1.

In the graph $G$, we have $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, \}$. Since $|V(G)| = 6$, we could

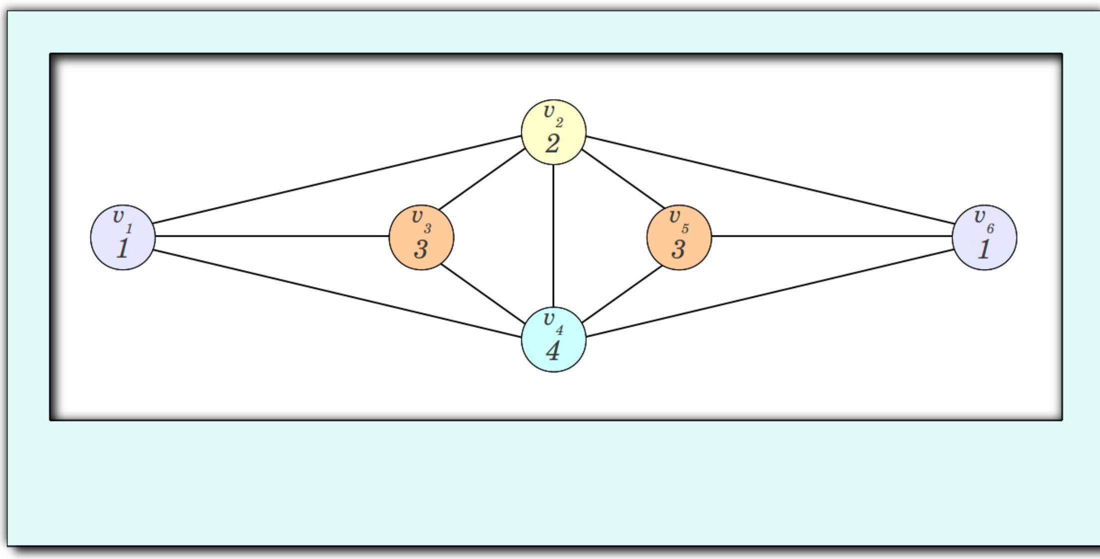


Figure 3.0.1. Color by number example

color the graphs verticies with 6 different colors such that each vertex had a unique color. However, in this project we are not so much concerned with a coloring of $G$. Rather, we are focused on the chromatic number, which is the minimum proper coloring for a graph

$G$. In this example $\chi(G) = 4$, thus we color the vertices in $V(G)$ with 4 distinct colors following the rules for a proper vertex coloring.[3]

The number below the vertex number indicates the coloring of said vertex. For instance, the vertices $v_1$, and $v_6$ are colored blue, so we let 1 be the color blue; the vertex $v_2$ is colored yellow, so we let 2 be the color yellow; the vertices $v_3$, and $v_5$ are colored orange, so we let 3 be the color orange; and lastly, the vertex $v_4$ is colored green, so we let 4 be the color green, as depicted in Figure 3.0.1.

## 3.1 1-chromatic Graphs

We begin by considering graphs that have no edge set, as these graphs are the easiest to color.

**Proposition 3.1.1.** *A simple graph is 1-chromatic if and only if its edge set is empty.*

*Proof.* Suppose $G$ is a simple graph with an empty edge set. Since there are two possibilities for the vertex set: $|V(G)| = 1$, or $|V(G)| > 1$, we consider $|V(G)| \geq 1$.

When $|V(G)| \geq 1$ and $E(G) = \{\ \}$, one color may be used for all vertices satisfying the definition of coloring.

For the converse, suppose $E(G)$ is not empty. Then there exists a pair of vertices that are adjacent. By Definition 2.2.1, this pair of adjacent vertices cannot have the same color. Thus, a graph with an edge set that is not empty is not 1-chromatic.

□

Let us consider two examples of graphs that fit within the general cases of the previous proof in order to get a visual representation.

[3] 4-chromatic graphs will be explored in detail in Section 3.6 of Chapter 3.

**Example 3.1.2.** Consider the graph in illustration **A** of Figure 3.1.1. Let this graph be $G$. Since this is a graph that has only one vertex point, $V(G) = \{v_1\}$ and $E(G) = \{\ \}$. Thus, since we have only one vertex point, we require only one color, so, $\chi(G) = 1$.

Now let us consider a case where a graph has multiple vertices, but still no edges, as represented in illustration **B** of Figure 3.1.1. Let this graph be $G'$. In the graph $G'$ we have $V(G') = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ and $E(G') = \{\ \}$. Although we have 10 vertices, since no two vertices are adjacent to one another, $\chi(G') = 1$.

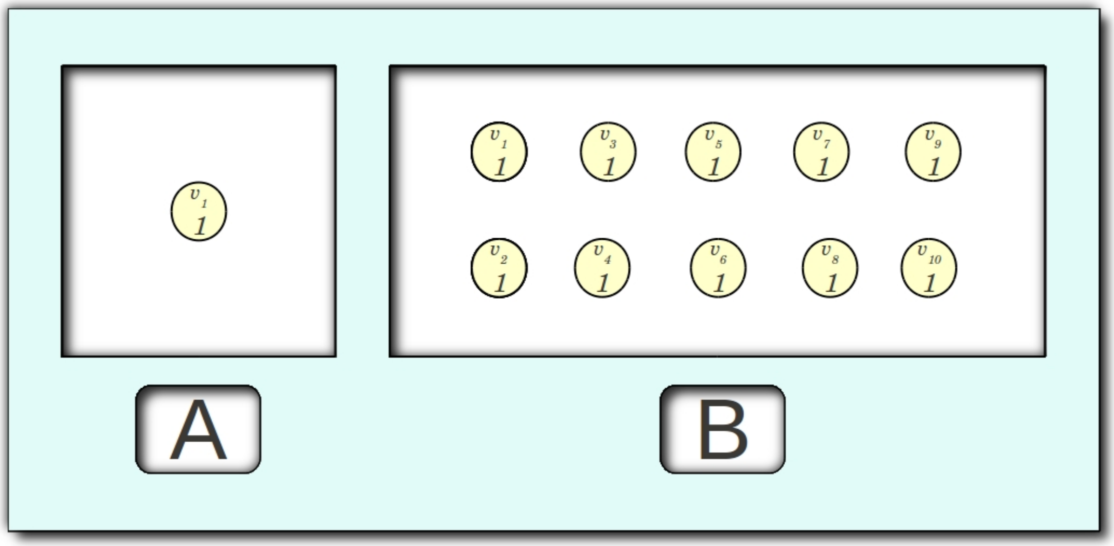


Figure 3.1.1. 1-chromatic graphs

Locating the minor of any 1-chromatic graph is trivial. Since all the vertices are the same color, we simply pick any vertex and delete the rest. This forms the $K_1$ minor. Since this exhausts all 1-chromatic graphs, we have an immediate corollary.

**Corollary 3.1.3.** *Hadwiger's Conjecture is true for 1-chromatic graphs.*

## 3.2 2-chromatic graphs

While there are many examples of 2-chromatic graphs, in this section we will prove that all of them can be defined as *bipartite*. This being the case, in order to establish which graphs are 2-chromatic, we will begin this section by proving the coloring of bipartite graphs.

**Definition 3.2.1.** A **bipartite graph** is graph whose vertex set can be partitioned into two subsets $X$ and $Y$ such that every edge has one vertex in $X$ and one in $Y$. Such a partition is called a bipartition of the graph and $X$ and $Y$ are called its parts. We denote a bipartite graph $G$ with a bipartition $(X, Y)$ by $G[X, Y]$. △

**Definition 3.2.2.** A graph is a **complete bipartite graph** if $G[X, Y]$ is simple and every vertex in $X$ is joined to every vertex in $Y$, then $G$ is called a *complete bipartite graph*. Complete bipartite graphs on $m$ and $n$ vertices are denoted by $K_{m,n}$ △

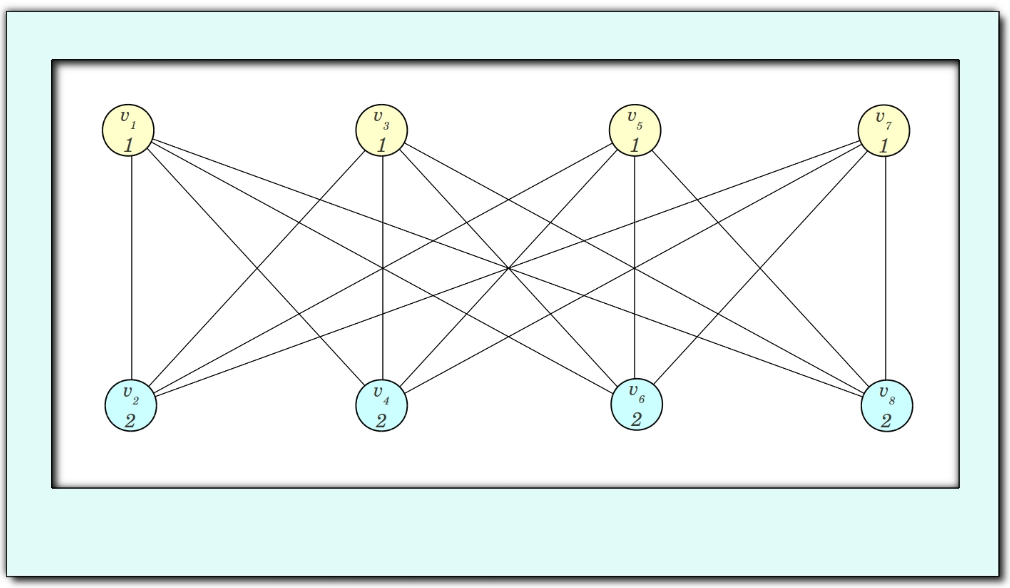


Figure 3.2.1. The complete bipartite graph, $K_{4,4}$

Before we prove the general case for bipartite graphs, let us consider an example.

**Example 3.2.3.** Observe the complete bipartite graph, $K_{4,4}$ illustrated in Figure 3.2.1. This graph $K_{4,4}$ has a vertex set that can be partitioned into two sets of 4 vertices. Let the first set $X$ consist of $\{v_1, v_3, v_5, v_7\}$, and let the second set $Y$ consist of $\{v_2, v_4, v_6, v_8\}$, such that $V(K_{4,4}) = X \cup Y$, and $X \cap Y = \{ \}$. Since there are no adjacent vertices in $X$, we color this set with color 1. Now since there there are no adjacent vertices in $Y$, but all the vertices in $Y$ are adjacent with the vertices in $X$ we color $Y$ with color 2. This is the minimum coloring, so $\chi(K_{4,4}) = 2$.

**Theorem 3.2.4.** *Let $G$ be a simple graph with a non-empty edge set, then $G$ is 2-chromatic if and only if it is bipartite.*

*Proof.* Suppose $G = (V, E)$ is 2-chromatic. Let $X$ be the set of vertices colored one color (say yellow), and let $Y$ be the set of vertices colored with the second color (say blue). Then $X$ and $Y$ are disjoint and $V(G) = X \cup Y$, with $X \cap Y = \{ \}$. Furthermore, every edge of $G$ consist of one vertex $v_i \in X$ and a vertex $v_j \in Y$ such that $e = \{v_i, v_j\}$. This is because $X \cap Y = \{ \}$. Therefore, since no two adjacent vertices are either in $X$ or both in $Y$, $G$ is bipartite.

Conversely, suppose that $G = (V, E)$ is a bipartite simple graph. Then $V(G) = X \cup Y$, where $X$ and $Y$ are disjoint sets and every edge $\{v_i, v_j\} \in E(G)$ consists of a vertex $v_i \in X$ and a vertex $v_j \in Y$. We can assign every vertex in $X$ one color and every vertex in $Y$ a second color such that no two adjacent vertices are colored with the same color.

This shows that $\chi(G) \geq 2$. By applying Proposition 3.1.1 we see that $\chi(G) = 2$. □

**Proposition 3.2.5.** *Hadwiger's Conjecture is true for all 2-chromatic graphs.*

*Proof.* Since for any graph $G$ with $\chi(G) = 2$ we know that $E(G) \neq \{ \}$. Thus, $E(G) \geq 1$. Therefore , we can form a $K_2$ minor by picking any edge $e = \{v_i, v_j\}$, $i \neq j$, and deleting the rest of the graph. □

In the proof of Proposition 3.2.5 we see that the minor $K_2$ is actually a subgraph of $G$. We stress this procedure of locating a $K_n$ subgraph within the $n$-chromatic graph $G$ because–when possible–this is the easiest way to locate the $K_n$ minor. We follow this same procedure to locate the $K_n$ minors of graphs with higher chromatic values where possible. However, in graphs where $\chi(G) > 2$, we may need to use the rules involved with contracting and deleting vertices and contracting and deleting edges, as defined in Section 2.1 of Chapter 2 where we modify graphs.

### *3.2.1 Paths*

We are interested in connected graphs with only one path between each pair of vertices. Such graphs in there simplest forms, are called paths.

**Definition 3.2.6.** A path, denoted as $P_n$ is a graph with $n-1$ edges whose vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise. △

Before we prove the general case for paths, let us consider some the examples to get a visual reference.

**Example 3.2.7.** Consider the path $P_2$, as depicted in illustration **A** of Figure 3.2.2. The graph $P_2$ is defined by $V(P_2) = \{v_1, v_2\}$, and $E(P_1) = \{\{v_1, v_2\}\}$. Since $|V(P_2)| = 2$, and $|E(P_2)| = 1$, we have the sequence from $v_1$ to $v_2$ along the singular edge $\{v_1, v_2\}$.

**Example 3.2.8.** Consider $P_7$, which is depicted in illustration **B** of Figure 3.2.2. The graph $P_7$ is defined by $V(P_7) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, \}$, and $E(P_7) = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_6, v_7\}\}$. Since $|V(P_7)| = 7$ and $|E(P_7)| = 6$, we have the linear sequence from $v_1$ to $v_2$ to $v_3$ to $v_4$ to $v_5$ to $v_6$ to $v_7$ along 6 edges.

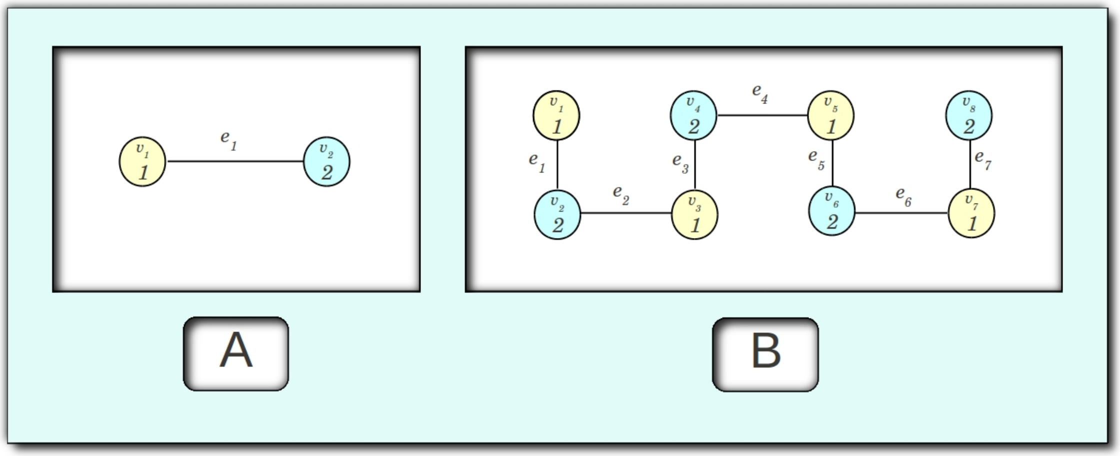


Figure 3.2.2. The path graphs, $P_2$, and $P_7$

Instead of proving the chromatic number of examples 3.2.7 and 3.2.8 individually, we will prove the chromatic number for paths in the general case.

**Proposition 3.2.9.** *Let $n \geq 2$. All paths $P_n$ are bipartite and therefore 2-chromatic.*

*Proof.* For $n \geq 2$ a path $P_n$ has a vertex set that can be partition into two subsets $X$ and $Y$. To see this let $X$ consist of the odd indexed vertices in $V(P_n)$, and we let $Y$ consist of the rest. By definition, No edge $e \in E(P_n)$ connects two elements of $X$, likewise for $Y$. Thus, edges in $E(P_n)$ only connect vertices in $X$ to vertices in $Y$. Therefore, $X$ and $Y$ are disjoint sets and every edge in $E(P_n)$ consists of a vertex $v_i \in X$ and a vertex $v_j \in Y$. Hence, $P_n$ is bipartite, and by Theorem 3.2.4 is 2-chromatic. $\square$

By Proposition 3.2.5, we know that since $P_n$ is 2-chromatic, we can form the $K_2$ minor by picking an edge and deleting the rest of the graph. Thus we have an immediate corollary.

**Corollary 3.2.10.** *Hadwiger's Conjecture holds for paths on $n \geq 2$ vertices.*

Now that we have seen examples of a path and proven that $\chi(P_n) = 2$, and that Hadwiger's Conjecture holds for paths we can define the notion of *connectivity*.

**Definition 3.2.11.** A graph $G$ is **connected** if and only if there is a path between each pair of vertices.

△

### *3.2.2 Fans*

With an understanding of the class of graphs called paths and the notion of connectivity, we move to the *fan graph*, as the fan graph will help us understand a *forest* and *trees*, which appear in the proceeding subsection.

**Definition 3.2.12.** The **fan graph**, denoted $F_n$, has $n + 1$ vertices and consists of a single hub vertex with $n$ spokes joining it to $n$ disjoint vertices. △

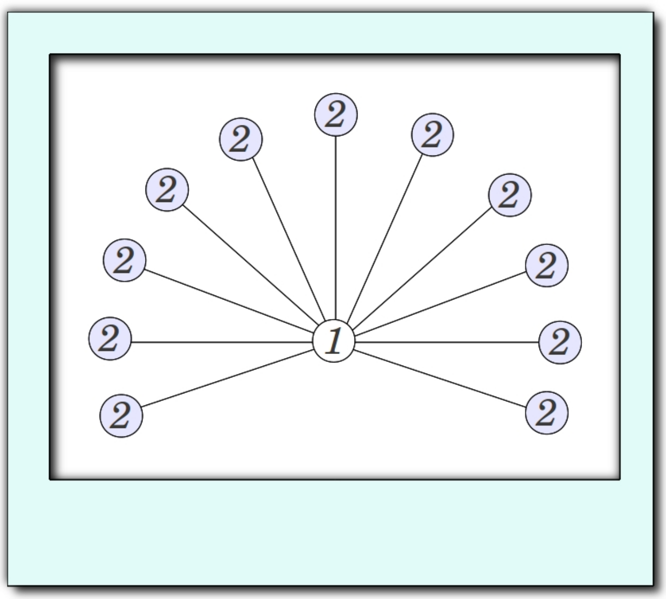


Figure 3.2.3. The fan graph, $F_{11}$

Let us consider an example of a fan before we prove the coloring of this class of graphs in the general case.

**Example 3.2.13.** Consider $F_{11}$, which is depicted in Figure 3.2.3. The graph $F_{11}$ is defined by $V(F_{11}) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$, and $E(F_{11}) = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_1, v_6\}, \{v_1, v_7\}, \{v_1, v_8\}, \{v_1, v_9\}, \{v_1, v_{10}\}, \{v_1, v_{11}\}, \{v_1, v_{12}\}\}$.

Instead of proving the chromatic number of the fan graph $F_{11}$, we move to prove the general case.

**Proposition 3.2.14.** *Let $n \geq 1$. Fan graphs $F_n$ are bipartite and therefore 2-chromatic.*

*Proof.* The the fan graph has a vertex set that can be partitioned into two disjoint subsets sets $X$ and $Y$ such that no edge connects two vertices in $X$ or $Y$. Let $X$ be the vertex that is the hub of the fan graph, and let $Y$ be all the $n$ disjoint vertices connected to this hub. Since $X$ consist of only one element it cannot have a edge containing two elements in $X$. Since $Y$ is composed of $n$ disjoint vertices, no edge contains two elements of $Y$. The only edges that exist contain the single element of $X$ and the $n$ elements of $Y$. Therefore it is bipartite, and thus, by Theorem 3.2.4 is 2-chromatic.

□

By Proposition 3.2.5, we know that since $F_n$ is 2-chromatic, we can form the $K_2$ minor by picking an edge and deleting the rest of the graph. Thus we have an immediate corollary.

**Corollary 3.2.15.** *Hadwiger's Conjecture holds for fan on $n \geq 1$ vertices.*

### *3.2.3 Trees and Forests*

The depiction of the fan graph in Figure 3.2.3 leads us to understand the *tree graph*. This is because, in a basic sense, we can think of a *tree graph* as multiple fan graphs connected such that no cycle is created. There is an additional graph that is composed of *trees* as separate *components* called a *forest*. Thus, since a *forest graph* is composed of *tree graphs*, we must put the *trees* before the *forests*. However, first we define *components* and establish a theorem necessary to understand trees.

**Definition 3.2.16.** The **components** of a graph are a partition of the graph $G$ into disjoint subsets not connected by a path. △

**Theorem 3.2.17** ([12, Theorem 5.2])**.** *Let $G$ be a simple graph on $n$ vertices. If $G$ has $k$ components, then the number of edges of $G$ satisfies*

$$n - k \leq m \leq (n - k)(n - k + 1)/2$$

**Lemma 3.2.18.** *Let $T$ be a graph with $n$ vertices. Then the following statements are equivalent:*

1. *$T$ is a tree;*
2. *$T$ contains no cycles, and has $n - 1$ edges;*
3. *$T$ is connected, and has $n - 1$ edges;*
4. *$T$ is connected, and each edge is a bridge;*
5. *any two vertices of $T$ are connected by exactly one path;*
6. *$T$ contains no cycles, but the addition of any new edge creates exactly one cycle.*

*Proof.* If $n = 1$, all six results are trivial; we therefore assume that $n \geq 2$.

(1) $\Rightarrow$ (2). Since $T$ contains no cycles, the removal of any edge must disconnect $T$ into two graphs, each of which is a tree. If follows by induction that the number of edges in each of these two trees is one fewer than the number of vertices. We deduce that the total number of edges of $T$ is $n - 1$.

(2) $\Rightarrow$ (3). If $T$ is disconnected, then each component of $T$ is a connected graph with no cycles and hence, by the previous part, the number of vertices in each component exceeds the number of edges by 1. It follows that the total number of vertices of $T$ exceeds the total number of edges by at least 2, contradicting the fact that $T$ has $n - 1$ edges.

(3) $\Rightarrow$ (4). The removal of any edge results in a graph with $n$ vertices and $n-2$ edges, which must be disconnected by Theorem 3.2.17.

(4) $\Rightarrow$ (5). Since $T$ is connected, each pair of vertices is connected by at least one path. Given a pair of vertices is connected by two paths, then they enclose a cycle, contradicting the fact that each edge is a bridge.

(5) $\Rightarrow$ (6). If $T$ contained a cycle, then any two vertices in the cycle would be connected by at least two paths, contradicting statement (5). If an edge $e$ is added to $T$, then, since the vertices incident with $e$ are already connected in $T$, a cycle is created.

(6) $\Rightarrow$ (1). Suppose that $T$ is disconnected. If we add to $T$ any edge joining a vertex of one component to a vertex in another, then no cycle is created. $\square$

**Definition 3.2.19.** A leaf of a tree graph $T_n$ is an end vertex of degree 1. $\triangle$

Thus, we see that a tree graph is essentially a path graph with leaves branching off at various vertex points. Although one may not readily recognize it as such, the tree graph is probably the most commonly used graph in every day life to easily relay information. Think of a family tree, a N.C.A.A. college basketball bracket, or even a depiction of a crime rings hierarchy, as so often seen in crime dramas, all of these are examples of trees.

In order to get a visual representation of a tree graph, let us consider an example of a tree graph.

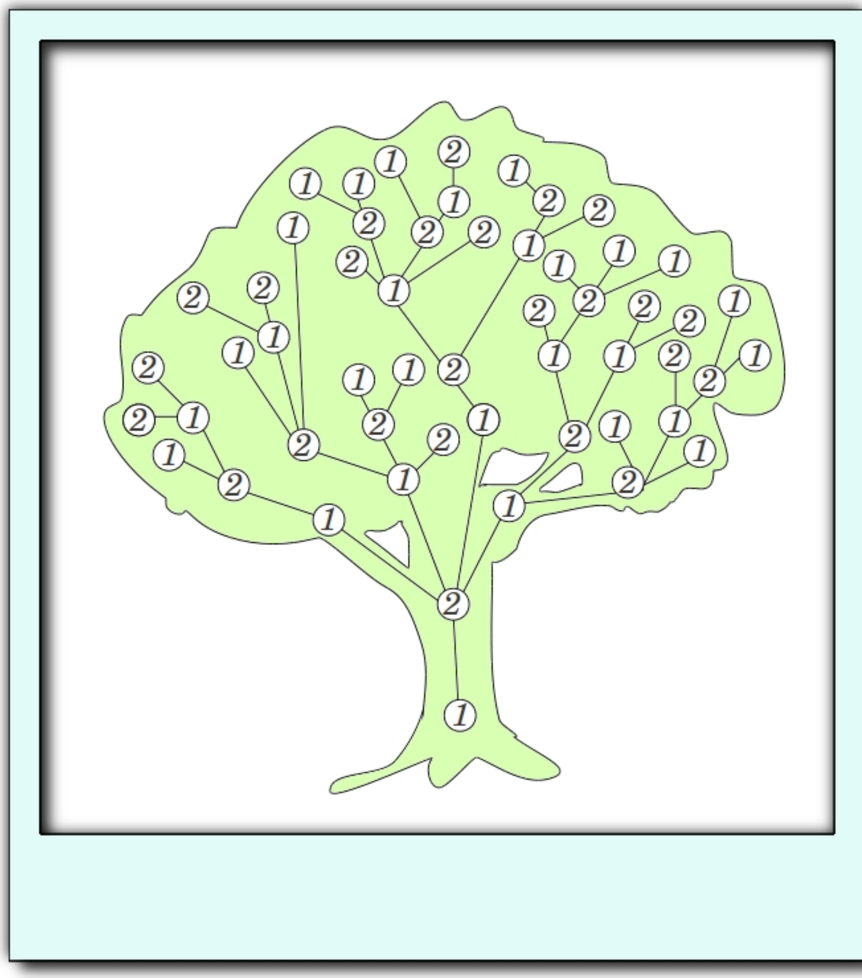

Figure 3.2.4. The tree graph, $T_{55}$

**Example 3.2.20.** Consider the graph $T_{55}$, which is depicted in Figure 3.2.4. The tree graph $T_{55}$ is defined by $|V(T_{55})| = 55$ and $|E(T_{55})| = 54$. Although it may be easy to see that the tree graph $T_{55}$ is 2-chromatic, we bypass proving this example choosing instead to prove the general case for all tree graphs.

**Proposition 3.2.21.** *All tree graphs are bipartite, and thereby 2-chromatic.*

*Proof.* We prove by induction on the number of vertices. Let $T_n$ be a tree.

<u>The base case $P(2)$</u>: When $n = 2$ we have $T_2$, which is equivalent to $P_2$, thus by Proposi-

tion 3.2.9 we are done.

We want to show $P(k) \Rightarrow P(k+1)$.

$\underline{P(k+1)}$: We have a tree on $k+1$ vertices. We delete one leaf $w$. Let the vertex that $w$ was adjacent to be $v$. Thus, by deletion of this leaf we have $T_k$, which is a tree on $k$ vertices. By induction hypothesis, $V(T_k)$ is bipartite. That is $V(T_k) = X \cup Y$ where $X \cap Y = \{\ \}$. Thus $T_k$ is partitioned into two disjoint sets such that if we have $\{u, v\} \in E(T_k)$, then $u \in X$ and $v \in Y$ or visa versa. Thus $T_k$ meets the criteria to be bipartite.

Now we add the leaf $w$ back to $V(T_k)$ giving us the edge $\{w, v\}$. If $v \in X$ we partition as follows:

$$V(T_{k+1}) = (X) \cup (Y \cup \{w\}).$$

If $v \in Y$ we partition as follows:

$$V(T_{k+1}) = (X \cup \{w\}) \cup (Y).$$

Therefore $T_{k+1}$ fits the criteria to be bipartite, and by Theorem 3.2.4 all trees are 2-chromatic. $\square$

By Proposition 3.2.5, we know that since $T_n$ is 2-chromatic, we form its $K_2$ minor by picking an edge and deleting the rest of the graph. Thus we have an immediate corollary.

**Corollary 3.2.22.** *Hadwiger's Conjecture holds for trees on $n \geq 2$ vertices.*

**Definition 3.2.23.** A **forest** is a graph that contains no cycles, and a connected forest is a *tree*. $\triangle$

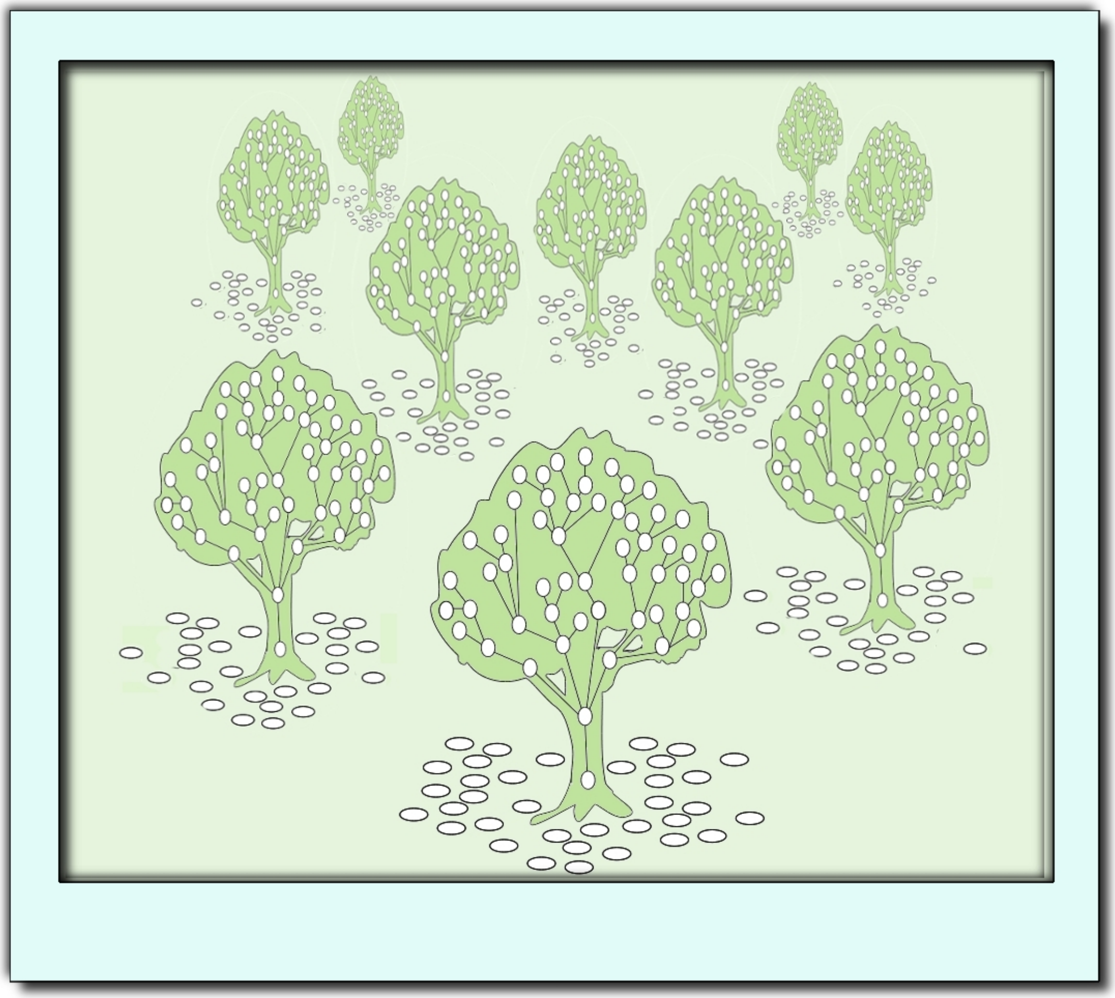

Figure 3.2.5. A forest composed of 10 separate $T_{55}$ tree graphs

Essentially, in terms of a graphical representation of this definition, trees are the components of the forest as in Figure 3.2.5.

Since, by definition, a forest is composed of disjoint trees, we have an immediate corollary from Propositions 3.2.21 and 3.2.5.

**Corollary 3.2.24.** *All forest are bipartite and thereby 2-chromatic and Hadwiger's Conjecture holds for all forest graphs.*

### *3.2.4 Boolean Lattice*

Now we will look at a rather complex graph called the *Boolean lattice*. After an example of this graph, we will prove that the Boolean lattice is 2-chromatic, and locate its $K_2$ minor to establish that Hadwiger's conjecture holds.

**Definition 3.2.25.** The *Boolean lattice*, denoted by $BL_n$ $(n \geq 1)$, is a graph whose vertex set is the set of all subsets of $\{1, 2, \ldots, n\}$, where two subsets $X$ and $Y$ are adjacent if there symmetric difference has precisely one element. $\triangle$

Before we prove the general case, to appreciate the complexity of the Boolean lattice, we now consider an example these graphs.

**Example 3.2.26.** Consider the Boolean lattice graph $BL_4$, as depicted in Figure 3.2.6. The graph $BL_4$ is composed of

$$\begin{aligned}
V(BL_4) =& \{\text{subsets} X_0, X_1, X_2, X_3, X_4\} \text{ such that:} \\
& X_0 = \{\},\ X_1 = \{\{1\}, \{2\}, \{3\}, \{4\}\}\},\ X_2 = \{12, 13, 14, 23, 24, 34\}, \\
& X_3 = \{123, 124, 134, 234\},\ X_4 = \{1234\},\ \text{and}
\end{aligned}$$

$$\begin{aligned}
E(G) =& \{\{\}\{1\}\}, \{\{\}\{2\}\}, \{\{\}\{3\}\}, \{\{\}\{4\}\}, \{\{1\}\{12\}\}, \{\{1\}\{13\}\}, \\
& \{\{1\}\{14\}\}, \{\{2\}\{12\}\}, \{\{2\}\{23\}\}, \{\{2\}\{24\}\}, \{\{3\}\{13\}\}, \\
& \{\{3\}\{23\}\}, \{\{3\}\{34\}\}, \{\{4\}\{14\}\}, \{\{4\}\{24\}\}, \{\{4\}\{34\}\}, \\
& \{\{12\}\{123\}\}, \{\{12\}\{124\}\}, \{\{13\}\{123\}\}, \{\{13\}\{134\}\}, \\
& \{\{14\}\{124\}\}, \{\{14\}\{134\}\}, \{\{123\}\{1234\}\}, \{\{124\}\{1234\}\}, \\
& \{\{134\}\{1234\}\}, \{\{234\}\{1234\}\}.
\end{aligned}$$

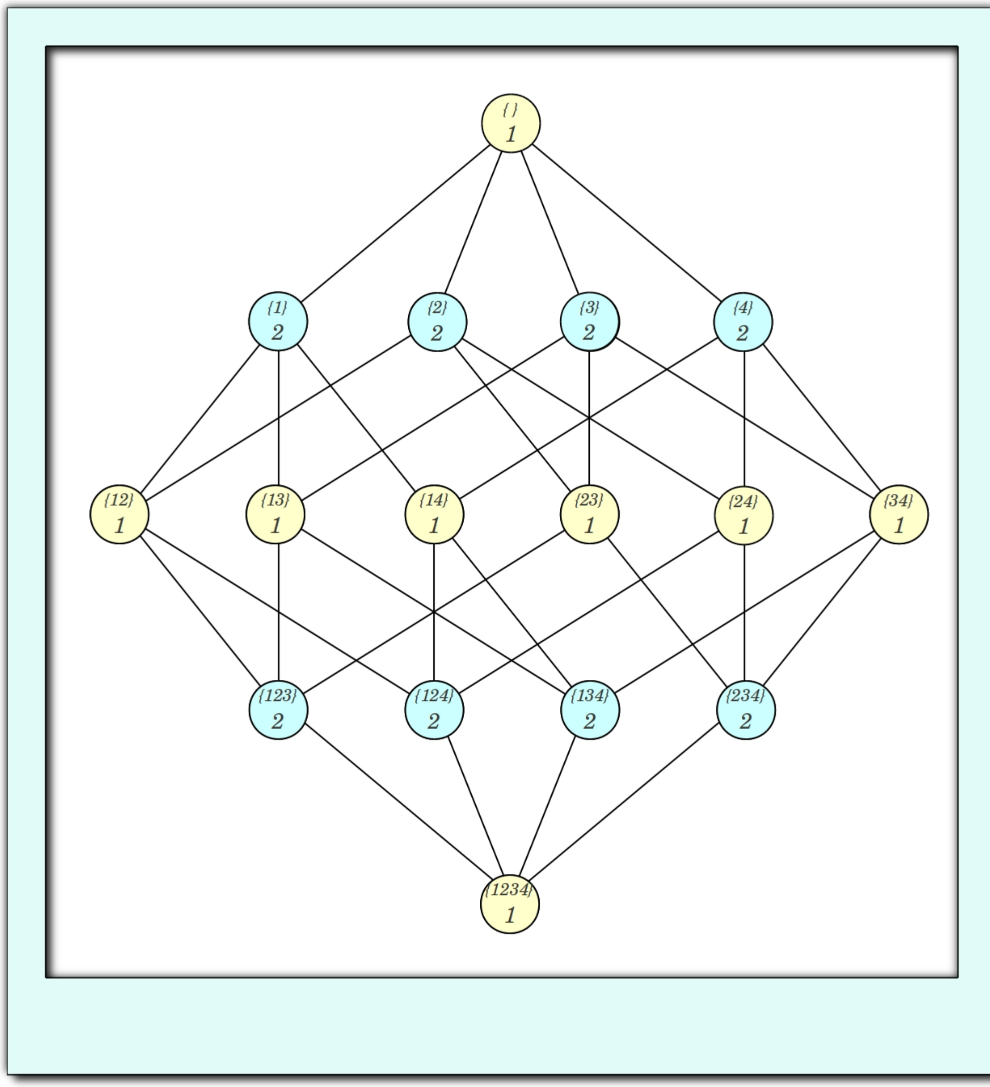


Figure 3.2.6. The Boolean lattice, $BL_4$

By observing the depiction of $BL_4$, it is clear that the subsets partition the vertex set into multiple subsets, effectively (when $n > 2$) creating a stacked arrangement of bipartite graphs.

As the above example demonstrates, the vertex and edge set of this graph can get complicated. Thus, to prove the general case we will introduce some new notation to make things easier. We let $[n] = \{1, 2, 3 \ldots, n\}$ , and $2^{[n]}$ = all subsets of $[n]$.

**Proposition 3.2.27.** *$BL_n$ is bipartite for all $n \geq 1$ thus, by Theorem 3.2.4 it is thereby 2-chromatic.*

*Proof.* We partition the subsets of $V(BL_n) = 2^{[n]}$ into two sets $X$ and $Y$ such that $X$ contains only odd numbered elements of $V(BL_n)$, and $Y$ contains only even numbered elements of $V(BL_n)$. By definition, the only edges that exist in $E(BL_n)$ contain elements that differ by precisely one element. Thus, there are no edges linking two elements in $X$ or two elements in $Y$. Therefore, $V(BL_n)$ meets the criteria to be bipartite and by Theorem 3.2.4, $V(BL_n)$ is 2-chromatic. □

By Proposition 3.2.5, we know that since $BL_n$ is 2-chromatic, its $K_2$ minor can be formed by picking an edge and deleting the rest of the graph. Thus we have an immediate corollary.

**Corollary 3.2.28.** *Hadwiger's Conjecture holds for all $BL_n$.*

### *3.2.5 n-Cube Graphs*

As we see from the previous example of the Boolean lattice, 2-chromatic graphs can get quite complex. We now explore another rather complicated example of a 2-chromatic graph called the *n-cubes.*

**Definition 3.2.29.** The $n$**-cube** is denoted as $Q_n$, and is defined as a graph whose vertex set is the set of all $n$-tuples of $0s$ and $1s$, where two $n$-tuples are adjacent if they differ in precisely one coordinate. △

Before we prove that the $n$-cube, $Q_n$ is 2-chromatic in the general case, let us look at an example to appreciate its complexity.

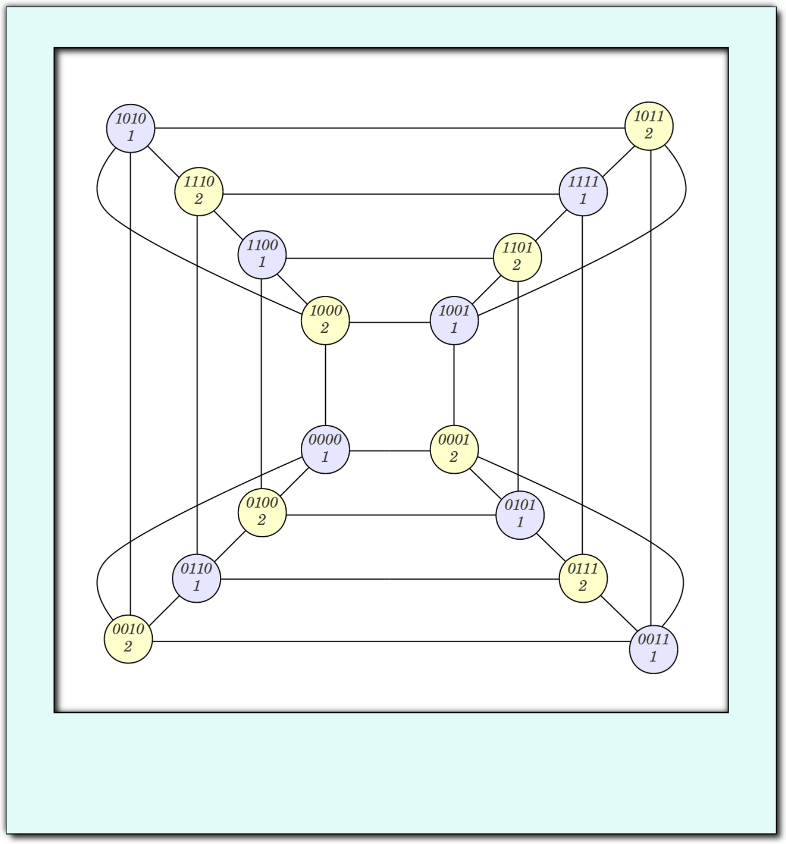


Figure 3.2.7. The $n$-cube, $Q_4$

**Example 3.2.30.** Consider the graph $Q_4$. The graph $Q_4$ is defined by

$$
\begin{aligned}
V(Q_4) =& \{0000, 0100, 0110, 0010, 1000, 1100, 1110, 1010, 1001, 1101, 1111, 1011, 0001, 1010, 0111, 0011\},\\
E(Q_4) =& \{0000, 0001\}, \{0000, 0100\}, \{0000, 1000\}, \{0000, 0010\}, \{0100, 0101\},\\
& \{0100, 1100\}, \{0100, 0110\}, \{0110, 0111\}, \{0110, 1110\}, \{0110, 0010\},\\
& \{0010, 0011\}, \{0010, 1010\}, \{1010, 1011\}, \{1010, 1000\}, \{1000, 1001\},\\
& \{1000, 1100\}, \{1100, 1101\}, \{1100, 1110\}, \{1110, 1111\}, \{1110, 1010\},\\
& \{1001, 1101\}, \{1001, 0001\}, \{1101, 0101\}, \{1101, 1111\}, \{1111, 1011\},\\
& \{1010, 1001\}, \{1011, 0011\}, \{0011, 0001\}, \{0001, 0101\}, \{0101, 0111\},\\
& \{1011, 1001\}, \{0011, 0111\}.
\end{aligned}
$$

By observing the depiction of $Q_4$, we can see that to have a proper vertex coloring, we require 2 colors. However, this is not sufficient to prove that $Q_n$ in 2-chromatic. Therefore, we now move to prove the general case establishing that the $n$-cube is 2-chromatic and therefore hold to Hadwiger's Conjecture having a $K_2$ minor.

**Proposition 3.2.31.** *The $n$-cube is bipartite and is thereby 2-chromatic.*

*Proof.* We want to partition the vertices of $Q_n$ into two sets $X$ and $Y$ such that $Q_n = X \cup Y$ where $X \cap Y = \{ \}$. First we pick a vertex $v$ consisting entirely of one element, say all 0's. Choosing all vertices that differ by an even number of elements $\geq 2$, we partition these elements into $X$. Now we choose all the elements that differ by an odd number of elements $\geq 1$ and place these elements into $Y$.

Since $X$ contains all elements that differ from our first choice by an even number $\geq 2$ and by definition, edges connect only vertices that differ by precisely one element we want to show that no two vertices in $X$ share an edge.

If we pick any vertex $\in X$, say $p$, and change it by one element, it now becomes $p'$, and $p' \in Y$ because it differs from our original vertex $p$ by an odd number $\geq 1$. Further, all vertices that are neighbors (adjacent) to $p$ are in $Y$. Thus there cannot be an edge $\in X$ consisting of two vertices $\in X$. The same holds for $Y$. Therefore, since the $n$-cube has a vertex set that can be partitioned into two disjoint subsets $X$ and $Y$ such that no edge exists that contains two elements in $X$ or two elements in $Y$, the $n$-cube is bipartite and thus, by Theorem 3.2.4, the $n$-cube is 2-chromatic. □

By Proposition 3.2.5, we know that since $Q_n$ is 2-chromatic, thus, we can form its $K_2$ minor by picking an edge and deleting the rest of the graph. Thus we have an immediate corollary.

**Corollary 3.2.32.** *Hadwiger's Conjecture holds for all $Q_n$.*

## 3.3 Cycle graphs

In this section, we will prove that in the family of graphs called *cycles*, the chromatic number is determined by the vertex set being even or odd. Recall from Definition 2.0.10 that a cycle graph of order $n$ is denoted by $C_n$ and is a walk along an edge beginning at one vertex and continuing in a circuit to the next vertex along a different edge such that no vertex or edge is repeated in the progression.

We will begin with a specific example proving the coloring of the smallest even cycle. Thus we begin with the 4-cycle denoted by $C_4$.

**Example 3.3.1.** We want to prove that a cycle on 4 vertices, $C_4$, is bipartite and therefore 2-chromatic.

*Proof.* The graph $C_4$, depicted in illustration **B** of Figure 3.3.1, is defined by $V(C_4) = \{v_1, v_2, v_3, v_4\}$, and $E(C_4) = \{e_1, e_2, e_3, e_4\}$, where $e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}, e_3 = \{v_3, v_4\}, e_4 = \{v_4, v_1\}$. Thus, $C_4$ is the closed walk $v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_1$. Since there is no edge connecting $v_1$ to $v_3$ and no edge connecting $v_2$ to $v_4$, we partition $V(C_4)$ into two disjoint sets $X = \{v_1, v_3\}$ and $Y = \{v_2, v_4\}$ such that no edge contains only elements of $X$ or $Y$. Thus, $C_4$ is bipartite, and, by Theorem 3.2.4, it is 2-chromatic.

Alternatively, if one desires a simpler version, note that each adjacent vertex in the 4-cycle can have a different color from the previous vertex using only 2 colors. Hence, $C_4$ is 2-chromatic. □

Next we look at the smallest odd cycle, which has 3 vertices and 3 edges connecting one vertex to another, denoted as $C_3$. The 3-cycle is the minimum amount for an odd cycle.

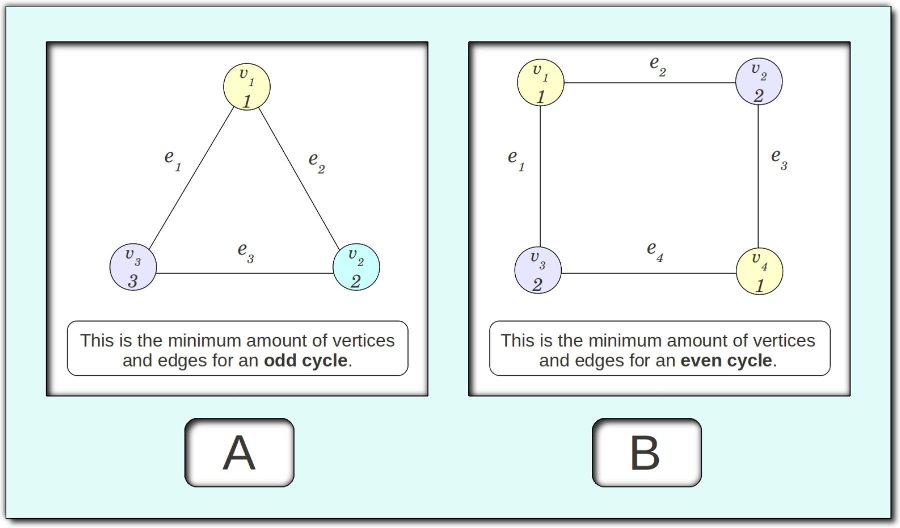


Figure 3.3.1. The cycle graphs, $C_4$, and $C_3$.

**Example 3.3.2.** We want to prove that the cycle graph on 3 vertices, $C_3$, is 3-chromatic.

*Proof.* The graph $C_3$, depicted in illustration **A** of Figure 3.3.1, is defined by $V(C_3) = \{v_1, v_2, v_, v_3, \}$, and $E(C_3) = \{e_1, e_2, e_3\}$, where $e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}, e_3 = \{v_3, v_1\}$. Thus we see that $C_3$ is the closed walk $v_1, e_1, v_2, e_2, v_3, e_3, v_1$. To color $C_3$, Suppose only 2 colors are needed. Let $v_0$ be an arbitrary initial vertex in $C_3$. Using only two colors, by definition of proper coloring, it is necessary to alternate colors as the graph is traversed in a clockwise direction. However, as the third vertex is reached, we see that it is adjacent to two vertices of different colors. Thus, a third color must be used to properly color $C_3$. Therefore, $\chi(C_3) = 3$. $\square$

Now, in order to prove that any even cycle is 2-chromatic, and that any odd cycle is 3-chromatic, we introduce the technique of *subdivision*.

**Definition 3.3.3.** **Subdivision** is the operation of taking an edge $e = \{u, v\}$ and replacing it by two edges and a new vertex $x$ of degree 2 such that the resulting two edges are $\{u, x\}$, and $\{v, x\}$. △

Since we use an even subdivision that adds 2 vertices, we expand the definition further.

**Definition 3.3.4.** **Even subdivision** is the operation of taking an edge $e = \{u, v\}$ and replacing it by three edges and two new vertices[4] $x$ and $y$ of degree 2 such that the resulting three edges are $\{u, x\}$, $\{x, y\}$, and $\{y, v\}$. △

**Lemma 3.3.5.** *An even subdivision along an edge preserves the chromatic number while increasing the vertex set.*

*Proof.* We begin by looking at a graph $G$ consisting of only a single edge such that $|V(G)| = 2$, where $V(G) = \{v_n, v_{n+1}\}$, and $|E(G)| = 1$, where $E(G) = \{v_n, v_{n+1}\}$. By definition of coloring, since $v_{n+1}$ is adjacent to $v_n$, $v_{n+1}$ must be one color, and $v_n$ is a different color, as depicted in illustration **A** of Figure 3.3.2.

Now let the even subdivision be represented by $S_1$. So, $|V(S_1)| = 2$, where $V(S_1) = \{s_1, s_2\}$, and $|E(S_1)| = 1$, where $E(S_1) = \{s_1, s_2\}$. We place this subdivision $S_1$ in $G$ along its single edge $\{v_n, v_{n+1}\}$. Let the resulting graph $G \cup S_1$ be $G_{s_1}$. Thus $G_{s_1}$ is defined by $|V(G_{s_1})| = 4$, where $V(G_{s_1}) = \{v_n, s_1, s_2, v_{n+1}\}$, and $|E(G_{s_1})| = 3$, where $E(G_{s_1}) = \{v_n, s_1\}, \{s_1, s_2\}\{s_2, v_{n+1}\}$, as depicted in illustration **B** and of Figure 3.3.2.

To find the proper vertex coloring of $G_{s_1}$, since $G_{s_1} = P_4$, by Proposition 3.2.9 $\chi(G_{s_1}) = 2$. Thus, we can properly color $G_{s_1}$ with the colors originally in $G$, as depicted in illustration **C** of Figure 3.3.2. Therefore, adding an even subdivision, to an edge will increase the vertex set by 2 but preserve the chromatic number. □

[4]We choose 2 because 2 is the minimum even number.

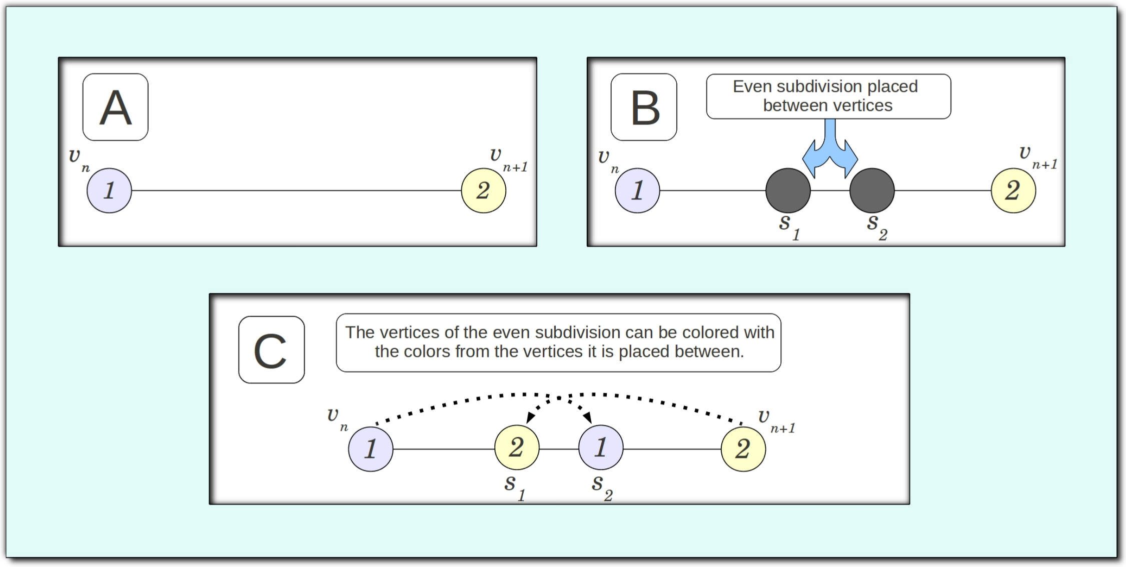


Figure 3.3.2. An even subdivision preserving the chromatic number

By adding an even subdivision along an edge of a cycle we want to show that we can generate any even or odd cycle from the smallest even and odd cycles while preserving their respective chromatic numbers. [5]

**Lemma 3.3.6.** *If a cycle is length $2n$, for $n \geq 2$, then the cycle is 2-chromatic.*

*Proof.* We prove this by induction. Let $P(n)$ be defined as the statement "$C_{2n}$ is 2-chromatic." We now prove the base case $P(2)$.

$\underline{P(2)}$: We have the graph $C_4$, which by Example 3.3.1 is 2-chromatic.

[5] The 2-cycle is actually the smallest even cycle, but a 2-cycle has parallel edges connecting the two vertex points thus disqualifying it from being a simple graph, and since we are only dealing with simple graphs in this project we will ignore the 2-cycle; however, Lemma 3.3.6 would work just as well in generating the even cycle and preserving the coloring if we added an even subdivision to one of the parallel edges of the 2-cycle. Also, a 1-cycle which consists of one vertex and a loop is actually the smallest odd cycle. However, the existence of a loop precludes us from looking at the 1-cycle because we are only dealing with simple graphs in this project. However, Lemma 3.3.7 would work just as well at generating the odd cycle if we added an even subdivision to the loop of the 1-cycle, and would establish a 3-coloring.

We want to show that $P(k) \Rightarrow P(k+1)$.

$\underline{P(k+1)}$: We have the graph $C_{2(k+1)} = C_{2k+2}$. See Figure 3.3.3.

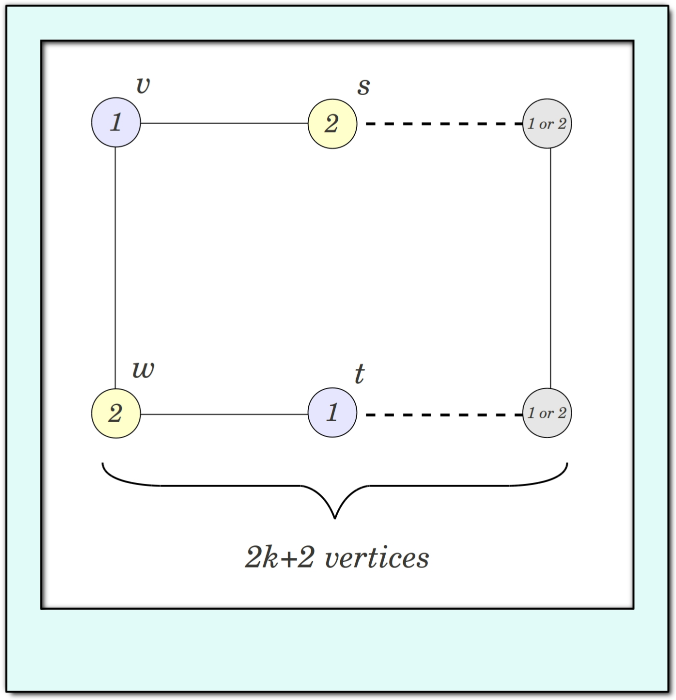


Figure 3.3.3. Proof of $C_{2n}$ by induction

We choose two adjacent vertices $v$ and $w$. Contract $w$ to $t$ and $v$ to $s$. We now have a cycle of length $2k$, as depicted in illustration **A** of Figure 3.3.4.

By the induction hypothesis, $C_{2k}$ is 2-chromatic. This implies that $s$ and $t$ are different colors. Let's now add $v$ and $w$ back to $C_{2k}$ by placing an even subdivision along $e = \{st\}$, giving us the original graph in Figure 3.3.3. We readily see that $v$ can be colored the same color as $s$, and that $w$ can be colored the same color as $t$. See illustration **B** of Figure 3.3.4.

Thus, we see that $C_{2k+2}$ is 2-chromatic, and $P(n)$ is true for $n \geq 2$. $\square$

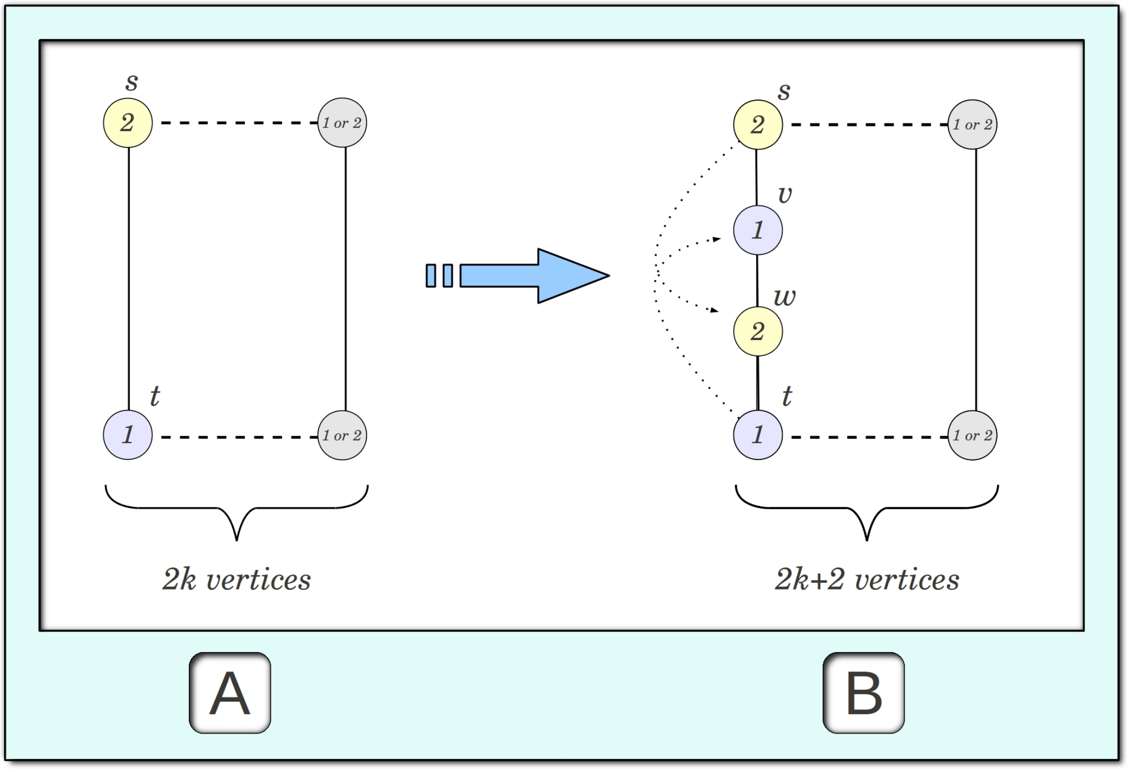


Figure 3.3.4. Contraction and subdivision of $C_{2n}$

**Lemma 3.3.7.** *If a cycle is length $2n+1$, for $n \geq 1$, then the cycle is 3-chromatic.*

*Proof.* We prove this by induction. Let $P(n)$ be defined as the statement "$C_{2n+1}$ is 3-chromatic." We now prove the base case $P(1)$.

$\underline{P(1)}$: We have the graph $C_3$, which by Example 3.3.2 is 3-chromatic.

We want to show that $P(k) \Rightarrow P(k+1)$.

$\underline{P(k+1)}$: We have the graph $C_{2(k+1)+1} = C_{2k+3}$. See Figure 3.3.5.

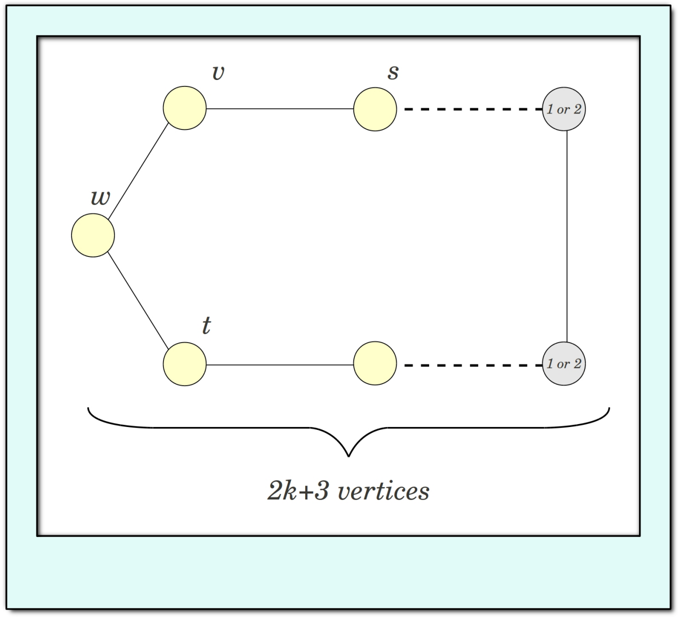


Figure 3.3.5. Proof of $C_{2n+1}$ by induction

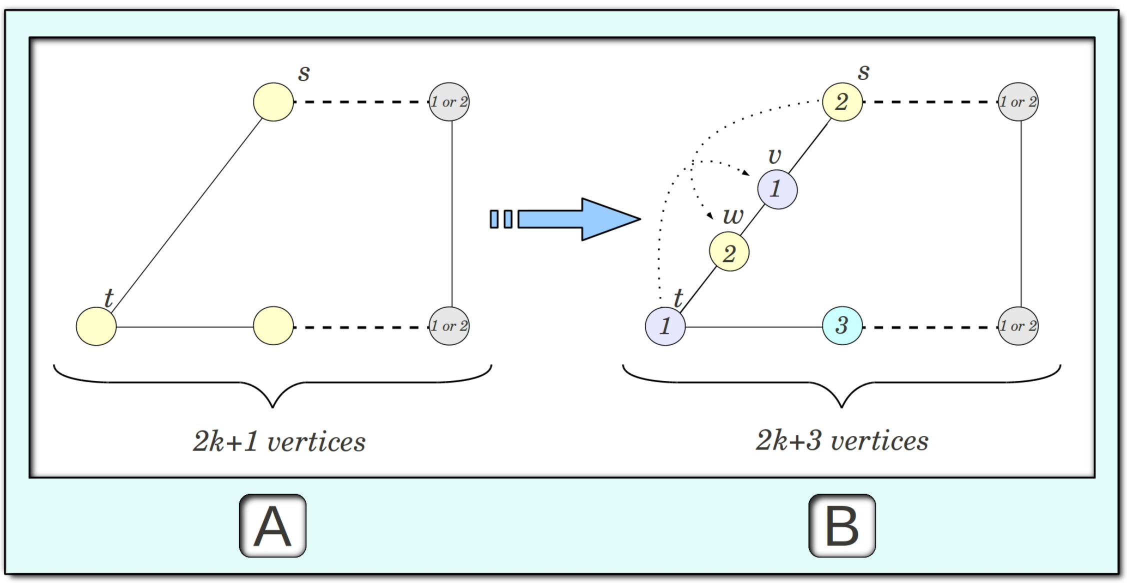


Figure 3.3.6. Contraction and subdivision of $C_{2n+1}$

We choose two adjacent vertices $v$ and $w$. Contract $w$ to $t$ and $v$ to $s$. We now have a cycle of length $2k+1$, as depicted in illustration **A** of Figure 3.3.6.

By the induction hypothesis, $C_{2k+1}$ is 3-chromatic. This implies that $s$ and $t$ are different colors. Let's now add $v$ and $w$ back to $C_{2k+1}$ by placing an even subdivision along $e = \{st\}$, giving us the original graph in Figure 3.3.5. We readily see that $v$ can be colored the same color as $s$, and that $w$ can be colored the same color as $t$, as depicted in illustration **B** of Figure 3.3.6.

Thus, we see that $C_{2k+3}$ is 3-chromatic, and $P(n)$ is true for all $n \geq 1$. □

As a consequence of Lemmas 3.3.6, and 3.3.7 we have an immediate corollary.

**Corollary 3.3.8.** *Placing an even subdivision of 2 along an edge of a cycle preserves the chromatic number.*

However, an even subdivision placed along an edge of any graph not a cycle does not necessary preserve the chromatic number.

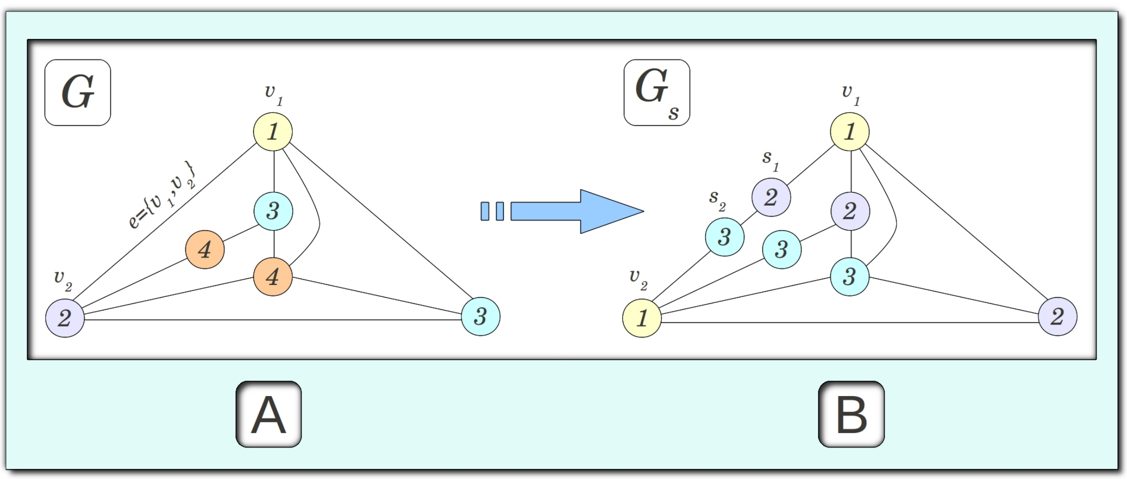


Figure 3.3.7. An even subdivision failing to preserve the chromatic number

**Example 3.3.9.** Consider the graph $G$ in illustration **A** of Figure 3.3.7. In this graph $\chi(G) = 4$. Now we create $G_s$ by adding an even subdivision of $\{s_1, s_2\}$ along the edge $\{v_1, v_2\}$. One may assume that by Lemma 3.3.5 the chromatic number of $G$ would be preserved by adding the even subdivision. However, this is false because $\chi(G_s) \neq 4$. Thus, we see that we cannot assume Lemma 3.3.5 true for any graph.

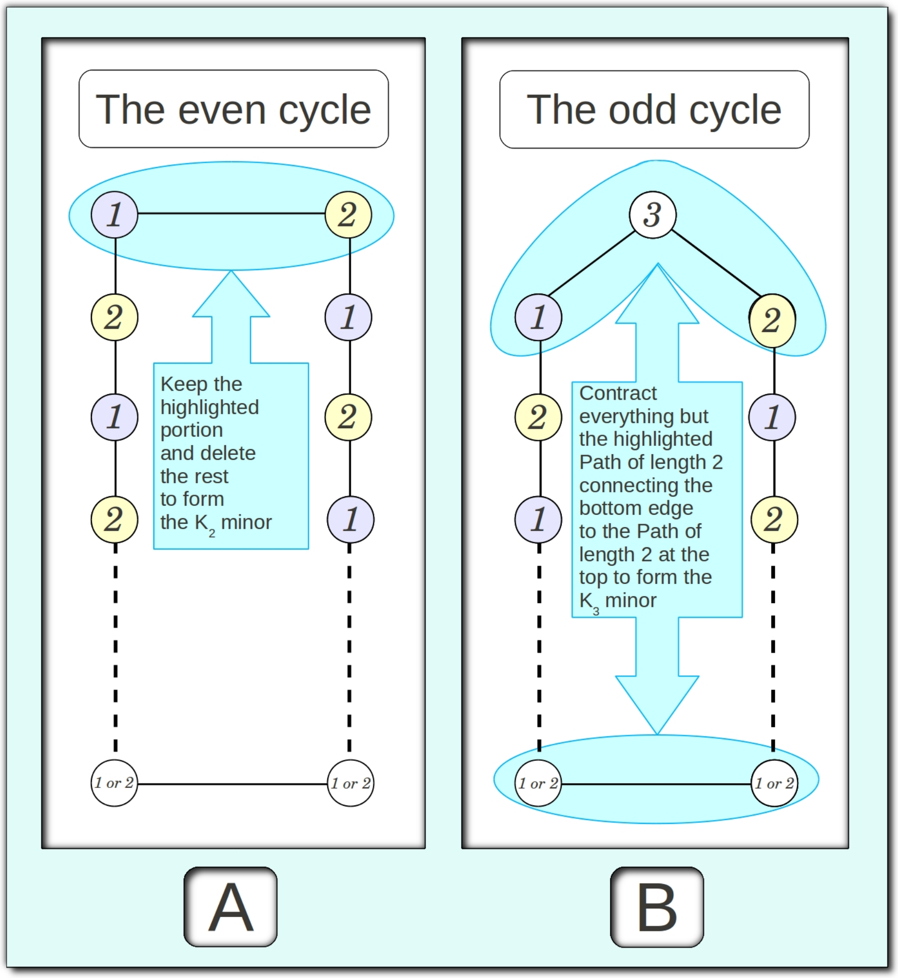


Figure 3.3.8. The $K_n$ minor for the even and odd cycle graphs

By Lemmas 3.3.6 we know that all even cycles are 2-chromatic, and by Lemma 3.3.7 we know that all odd cycles are 3-chromatic. We also know from Proposition 3.2.5, that since $C_{2n}$ is 2-chromatic, we can form the $K_2$ minor by picking an edge and deleting the rest of the graph,as shown in illustration **A** of Figure 3.3.8. Thus we have an immediate corollary.

**Corollary 3.3.10.** *Hadwiger's Conjecture holds for all even cycles $C_{2n}$.*

Since $K_3$ is also the smallest odd cycle ($C_3$), and all odd cycles ($C_{n+1}$) are 3-chromatic, we want to form a triangle out of $C_{n+1}$. Forming a $K_3$ minor from a 3-chromatic graph is more complicated than the 2-chromatic cases before it. This is because it may require a series of vertex and edge deletions and edge contractions, as shown in illustration **B** of Figure 3.3.8.

Now seeing if Hadwiger's Conjecture holds on any odd cycle, we pick an arbitrary path of length 2, and contract the rest until we are left with a triangle, as seen in illustration **B** of Figure 3.3.8. The result leaves us with $K_3$, which is the minimum vertex set ($V(K_3) = \{v_1, v_2, v_3\}$) and edge set ($E(K_3) = \{e_1, e_2, e_3\}$ to have a proper 3 coloring, see Figure 3.3.9.

Since the $K_3$ minor can be formed from any odd cycle, we have an immediate corollary.

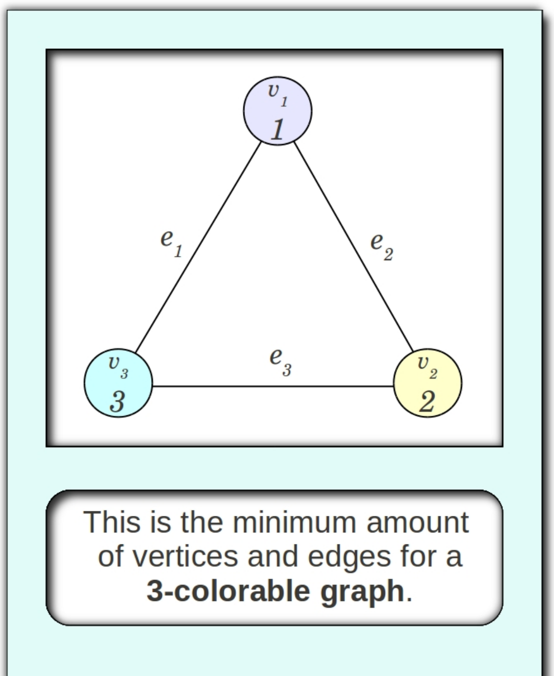


Figure 3.3.9. The complete graph, $K_3$

**Corollary 3.3.11.** *Hadwiger's Conjecture holds for all* $C_{2n+1}$.

## 3.4 The Peterson Graph

Up until this point in this project, we have seen two rather easy examples of 3-chromatic graphs, $C_{n+1}$ and the complete graph $K_3$. However, it is possible to have an almost infinite arrangement of more complicated 3-chromatic graphs. In this section we will consider the Peterson graph prove that it is 3-chromatic and from its $K_3$ minor.

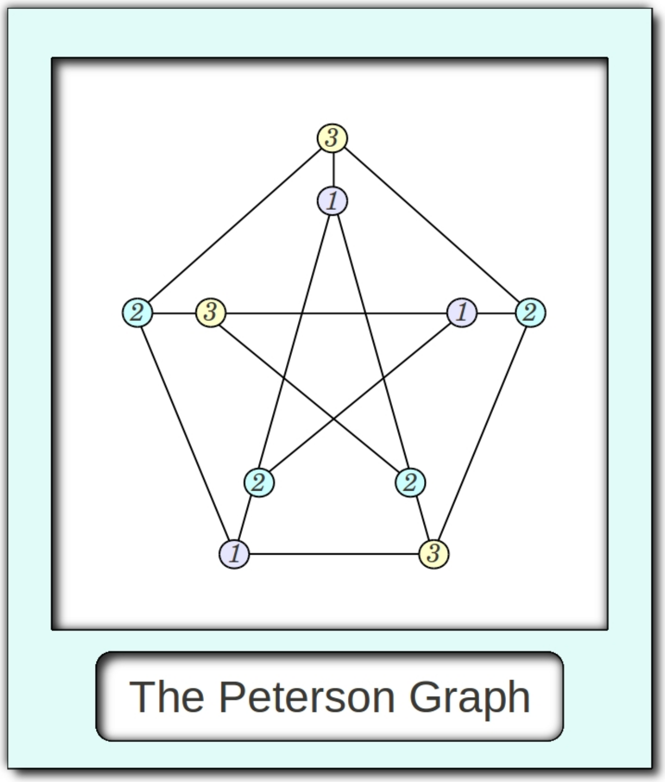


Figure 3.4.1. The Peterson graph

**Proposition 3.4.1.** *The Peterson graph is 3-chromatic.*

*Proof.* Since the Peterson graph is comprised of 2 odd cycles of length 5, by Lemma 3.3.7 the Peterson graph is not 2-chromatic, so we know at least 3 colors are required for a proper coloring of the Peterson graph. Since each vertex in the outer $C_5$ ring is linked to exactly one vertex in the inner $C_5$ cycle. We can properly color each 5-cycle by alternating 3 colors. Thus, the Peterson graph is 3-chromatic. $\square$

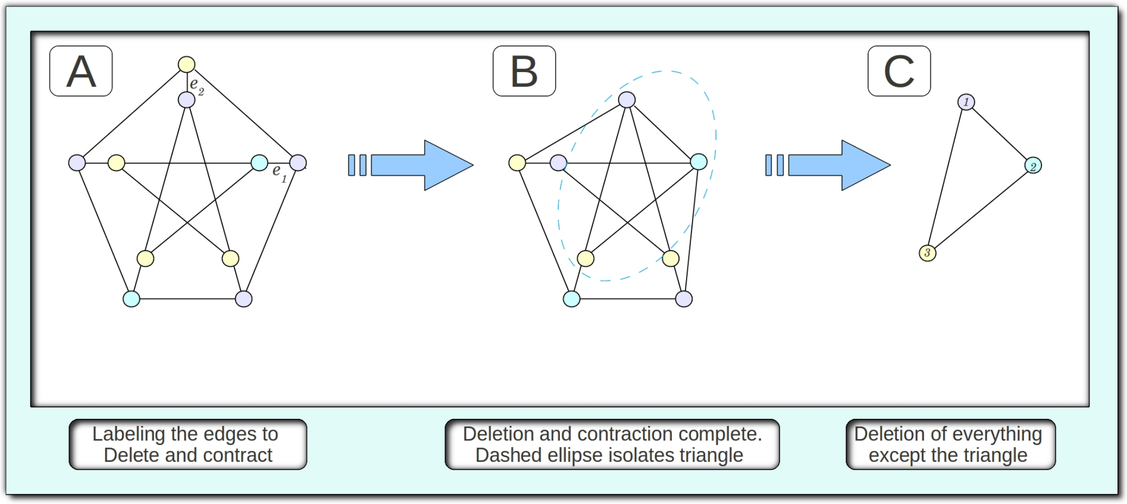


Figure 3.4.2. Forming the $K_3$ minor from the Peterson graph

To see if Hadwiger's Conjecture holds for the Peterson graph, we see if we can form a $K_3$ minor through a series of edge contractions.

**Proposition 3.4.2.** *Hadwiger's Conjecture holds for the Peterson graph.*

*Proof.* By Proposition 3.4.1, we know that the Peterson graph is 3-chromatic. To locate the $K_3$ minor, we use the process of contraction and deletion, as shown in Figure 3.4.2. We choose two separate edges that each link vertex in the outer $C_5$ ring to exactly one vertex in the inner $C_5$ cycle. Let these edges be $e_1$, and $e_2$, as depicted in illustration **A**. We delete these edges to form a triangle with in the Peterson graph, as seen in illustration **B**. We now delete everything except for the triangle (which we formed by contracting $e_1$, and $e_2$), as seen in illustration **C**. We are left with our $K_3$ minor. $\square$

## 3.5 Wheel Graphs

In this section, similar to the section on cycles, we show that a vertex set being even or odd determines the chromatic number for the family of graphs called *wheels*, which we will prove to be 3 or 4 chromatic.

**Definition 3.5.1.** A **wheel graph** can be obtained from a cycle $C_{n-1}$ by joining each vertex in $C_{n-1}$ by a edge called a spoke to a new vertex $v$. A wheel graph with $n$ spokes is denoted by $W_n$, $\triangle$

**Remark 3.5.2.** As a consequence of the definition of wheel graphs, we see that the subscript $n$ refers to the edges (spokes) emanating from the central hub. This being the case, $n$ does not define the total amount of vertices. Rather, $n$ defines all the vetrices connected to the central hub. Thus, $n + 1$ would give us $|V(W_n)|$, and $n = |E(W_n)|$.

We can think of the definition in terms of an actual wheel, where a cycle is the tire and the vertex $v$ in the center of a cycle as a hub with spokes emanating out linking the center vertex $v$ to every vertex in the cycle.

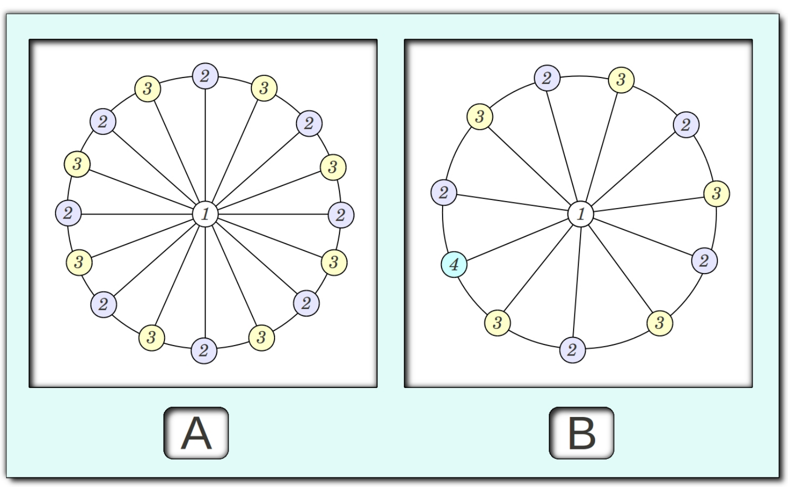


Figure 3.5.1. The wheel graphs, $W_{16}$ and $W_{11}$

**Example 3.5.3.** Consider the Wheel graph $W_{16}$ in illustration **A** of Figure 3.5.1. The graph $W_{16}$ is defined by $|V(W_{16}| = 17$ and $|E(W_{16}| = 16$. Thus, we see that it has an odd vertex set, in the following proposition we will prove that by virtue of having an odd vertex set that $\chi(W_16) = 3$.

**Example 3.5.4.** Consider the Wheel graph $W_{11}$ in illustration **B** of Figure 3.5.1. The graph $W_{11}$ is defined by $|V(W_{11}| = 12$ and $|E(W_{11}| = 11$. Thus, we see that it has an even vertex set, in the following proposition we will prove that by virtue of having an even vertex set that $\chi(W_11) = 4$.

We now establish that, in the general case, the vertex set being odd or even defines if the wheel is 3 or 4 chromatic.

**Proposition 3.5.5.** *Wheel Graphs are 3-chromatic when the vertex set is odd, and 4-chromatic when the vertex set is even.*

*Proof.* We consider two cases.

Case 1: Wheels whose vertex sets are odd.

In this case the hub vertex is adjacent to the $2n$ vertices of an even cycle. By Lemma 3.3.6 we know that even cycles are 2-chromatic. Therefore, the hub vertex, in this case, is always adjacent to at least 2 different colors. Thus, by Definition 2.2.1, the hub must be a third distinct color. Therefore, every wheel whose vertex set is odd (when $n$ is even) is 3-chromatic.

Case 2: Wheel graphs graphs whose vertex set is even.

From the definition of a wheel, we know that these graphs are odd cycles with an additional hub vertex that is adjacent to each cycles $2n+1$ vertices. By Lemma 3.3.7 we know that every odd cycle is 3-chromatic. This means that the hub vertex is always adjacent to at least 3 different colors. Thus, by Definition 2.2.1, the hub must be a fourth distinct color. Therefore, every wheel whose vertex set is even (when $n$ is odd) is 4-chromatic.

□

With the above Proposition in mind, we revisit the examples of wheel graphs in Figure 3.5.1. Illustration **A**, $W_{16}$ has an odd vertex, and is 3-chromatic. Illustration **B**, $W_{11}$ has an even vertex set and is 4-chromatic.

By Proposition 3.5.5, we know that when $|V(W_n)|$ is odd the graph is 3-chromatic. The $K_3$ minor can be formed formed by picking a triangle that consists of the hub vertex and any two adjacent vertices. We now deleting the rest of the graph thereby forming the $K_3$ minor. Since this minor can be formed from $W_n$ when $|V(W_n)|$ is odd, we have an immediate corollary.

**Corollary 3.5.6.** *Hadwiger's Conjecture holds for all $W_n$ with an odd vertex set.*

To see if Hadwiger's Conjecture holds on the 4-chromatic wheel graph whose vertex set is even, we consider the following proposition.

**Proposition 3.5.7.** *Hadwiger's Conjecture is true for all wheel graphs with an even vertex set.*

*Proof.* Since $\chi(W_{2n+1}) = 4$, we want to show that we can form a $K_4$ minor through a series of vertex deletions, and edge contractions and deletions. Pick an arbitrary vertex in the outer cycle to begin with. Let this vertex be $v$. Take its neighbor to the right, let this be $w$, and $v$'s neighbor to the left, let this be $x$. We now contract all the edges between the vertices $w$ and $x$ but not the edges connected to the hub vertex such that $w$ and $x$ are connected by an edge, as depicted in Figure 3.5.2. Thus we are left with a minor that consists of 4 vertices such that the three vertices $v, w$, and $x$ are, by definition of a wheel,

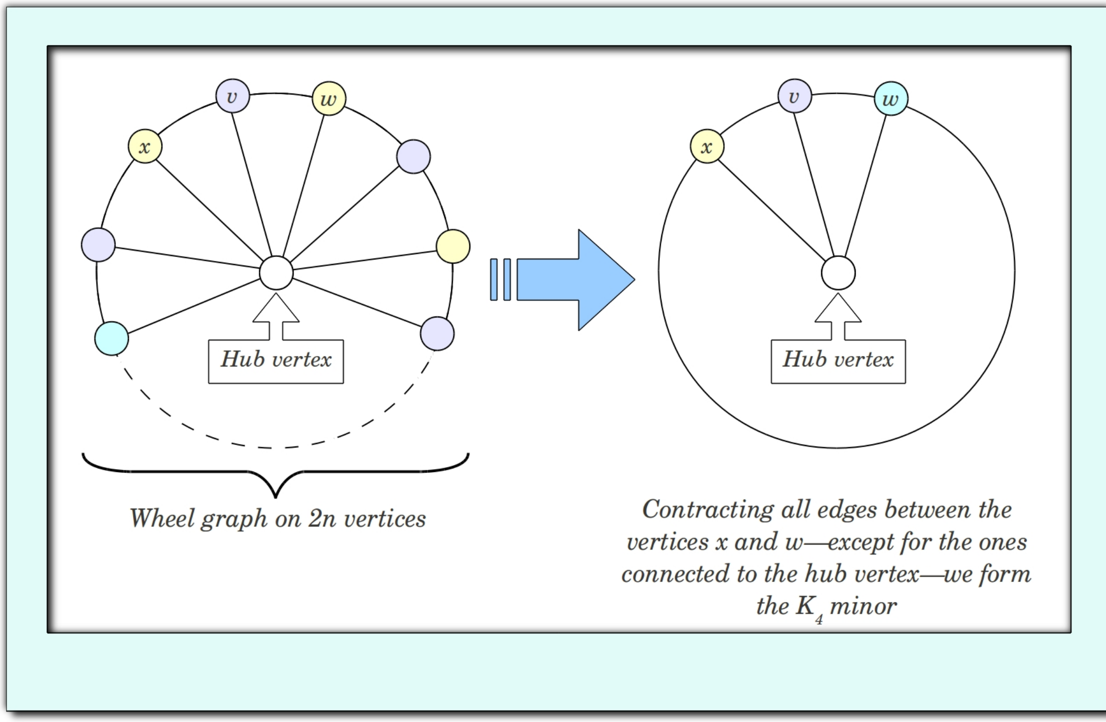


Figure 3.5.2. Forming the $K_4$ minor for the wheel graph $W_{2n+1}$

connected to a central hub vertex thereby forming $K_4$, as seen in illustration **B** of Figure 3.5.2. □

## 3.6 4-chromatic graphs

There are many types of graph that are 4-chromatic, for instance, in the previous section, a wheel with an even vertex set. While there maybe countless examples of 4-chromatic graphs that can get quite complex, within the host of 4-chromatic graphs is family of graphs that has many applications to mapping called *planar graphs*.

**Definition 3.6.1. Planar graphs** are a class of graphs that can be drawn in the plane in such a way that the edges meet only at points corresponding to there common ends. The edges of a planar graph never cross one another and only meet at vertex points. △

We will begin with a simple example of a 4-chromatic graph.

**Example 3.6.2.** Let us consider the graph $K_4$ that is defined by $V(K_4) = \{v_1, v_2, v_3, v_4\}$ and $E(K_4) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, where $e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}, e_3 = \{, v_3, v_4\}, e_4 = \{v_4, v_1\}, e_5\{v_4, v_2\}$, and $e_6\{v_4, v_3\}$. In this graph each vertex is adjacent to every other vertex in the graph. This being the case, we cannot properly color it with any thing less than 4 colors. Thus, $\chi(K_4) = 4$.

One way to represent this graph geometrically is in illustration **A** of Figure 3.6.1. This graph is not a *planar graph* because it has edges that cross. However, we could make the graph $K_4$ in illustration **A** of Figure 3.6.1 a planar graph by arcing one of the middle edges around the outside such that it connects to the same vertex points but does not cross any edge of $K_4$ as in illustration **B** in Figure 3.6.1.

**Theorem 3.6.3** ([12, Theorem 17.5 (Appel and Haken, 1976.)])**.** *Every simple planar graph is at most 4-chromatic.*

*Proof.* There proof, which took them several years and a substantial anount of computer time, ultimately depends on a complicated extension of the ideas in the proof of the five color theorem. [12] □

Let us consider an example of a more complicated planar graph to apply Theorem 3.6.3

**Example 3.6.4.** Consider the graph $G$ in illustration **C** of Figure 3.6.1. This graph consists of $|V(G)| = 8$ and $|E(G)| = 18$. Although we could use a different color for every vertex point in $G$, we want to find a proper coloring. We know by Theorem 3.6.3, for every planar graph 4 colors or less are required for a proper coloring. Thus, we have a upper bound to properly color $G$. Since $G$ contains $K_4$ as a proper subgraph (see higlighted ellipse

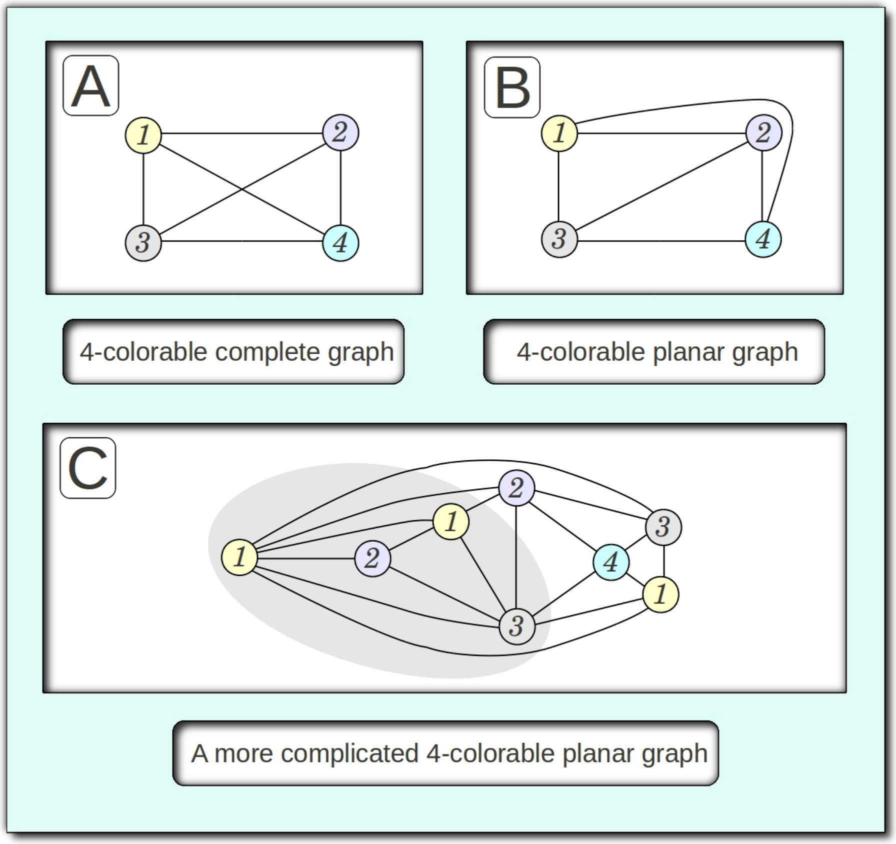


Figure 3.6.1. Illustrations of 4-chromatic graphs

in illustration **C** of Figure 3.6.1) and in Example 3.6.2 we established that $\chi(K_4) = 4$, we have that $\chi(G) = 4$.

## 3.7 Complete Graphs

While we have seen examples of complete graphs up to this point in the project, we have yet to establish a proposition concerning their chromatic number and how Hadwiger's Conjecture holds on such graphs. Recall that complete graphs, denoted as $K_n$ are simple graphs in which any two vertices are adjacent, thus, every vertex point is connected to every other vertex point by a unique edge. (Observe Figure 3.3.9 that depicts a simple 3-chromatic graph, illustrations **A** and **B** of Figure 3.6.1 are also examples of complete graphs, as is Figure 5.1.4).

**Proposition 3.7.1.** *The chromatic number of a complete graph, $K_n$, is $n$.*

*Proof.* Proving the chromatic number of any complete graph is trivial. Since every vertex is adjacent to every other vertex (by definition) the chromatic number is determined by the amount of vertices in the vertex set. Thus, $\chi(K_n) = n$.

□

Now that we have establish a proposition for complete graphs, we will look at an example of a 6-chromatic complete graph.

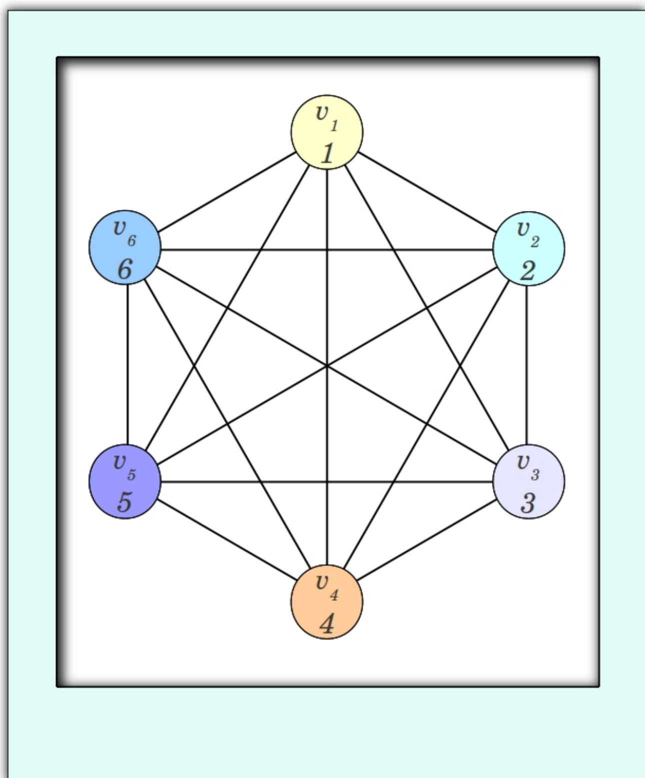


Figure 3.7.1. The complete graph $K_6$

**Example 3.7.2.** Consider the graph $K_6$ that is depicted in Figure 3.7.1. The graph $K_6$ is defined by $|V(K_6)| = 6$, and $|E(K_6)| = 15$.

Since each vertex is connected to every other vertex, 6 is the minimum coloring for the graph $K_6$. Thus, $\chi(K_6) = 6$. Because $K_6$ is its own minor with the same chromatic number, Hadwiger's Conjecture holds for the graph $K_6$.

In showing Hadwiger's conjecture is true for complete graphs we see that forming the graphs $K_n$ minor is trivial because every complete graph is its own minor. This brings us to an immediate corrolary.

**Corollary 3.7.3.** *Hadwiger's conjecture is true for all complete graphs.*

# 4

# Non-complete $n$-chromatic Graphs

Hadwiger's Conjecture has yet to be proven for graph with $\chi(G) > 6$ [3]. Since non-complete graphs with a chromatic number of 6 or greater are difficult to classify and the chromatic number for such graphs is difficult to prove. In this chapter we construct two examples of non-complete $n$-chromatic graphs, and show Hadwiger's Conjecture holds. Our motivation is the desire to find a graph with a large chromatic number $n$ that does not contain a $K_n$ subgraph. Unfortunately, we were unable to accomplish this task, but in the midst of our investigation, we were able to create two different approaches in constructing $n$-chromatic non-complete graphs.

## 4.1 The Butterfly Graph

Our first $n$-chromatic non-complete graph is the *butterfly graph*. Since this graphs definition involves the process of *gluing* vertices of graphs together, we must define this action.

**Definition 4.1.1.** To **glue** $n$ vertices together is to merge these $n$ vertices into one vertex. $\triangle$

**Definition 4.1.2.** The **butterfly graph**, denoted $BZ_{n,m}$ is constructed by associating four copies of $K_n$ in the following stages.

1. Pair up the four graphs and for each pair glue $m$ vertices together;
2. Pick one vertex for each pair and glue together;
3. Delete parallel edges. △

Since the metamorphosis of the butterfly graph from its $K_n$ larva is quite complex, we will go through the stages in an example prove the coloring and then show Hadwiger's Conjecture holds for an example of the butterfly graph.

**Example 4.1.3.** We will construct the butterfly graph $BZ_{9,2}$. Thus, we begin the first stage with the complete graph $K_9$ composed of $|V(K_9)| = 9$, and $E(K_9) = 72$, as depicted in illustration **A** of Figure 4.1.1.

We now copy $K_9$ into four graphs as in illustration **B** of Figure 4.1.1. Finally we glue

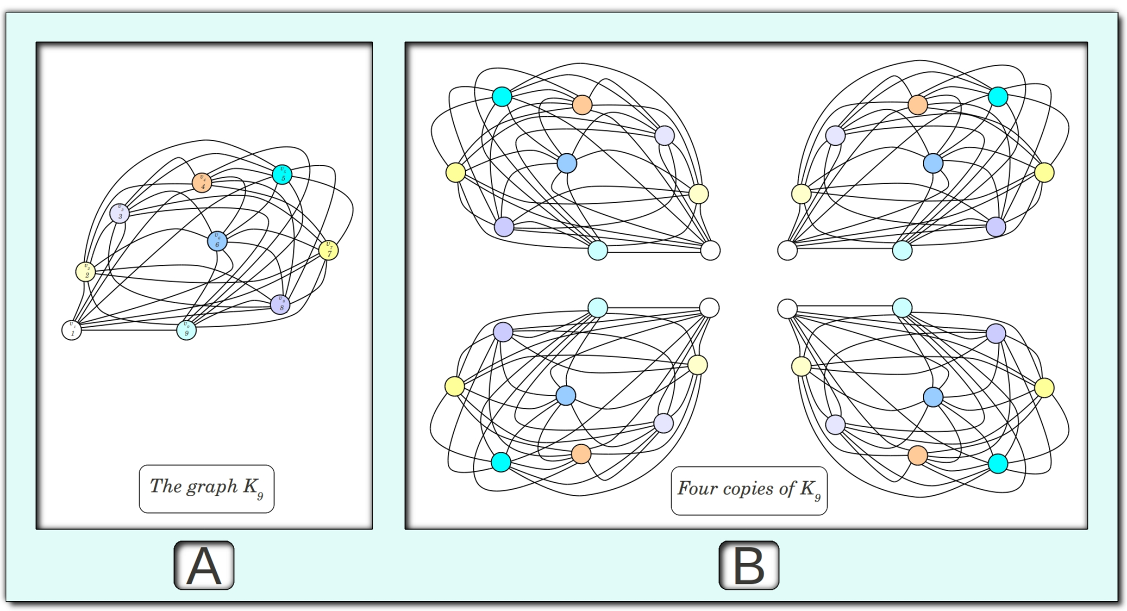


Figure 4.1.1. Construction of the butterfly graph, $BZ_{9,2}$

$m$ vertices of the pairs together, in this case $m = 2$, as depicted in illustration **A** of Figure

4.1.2. The resulting pairs $G_1$ and $G_2$ have a vertex sets composed of

$$|V(G_1)| = 2(9) - 2, \quad |V(G_2)| = 2(9) - 2$$

and an edge set composed of

$$|E(G)| = 72 - 1 \quad |E(G_2)| = 72 - 1$$

since two of the vertices of $K_9$ are shared in the paired composites $G_1$, and $G_2$.

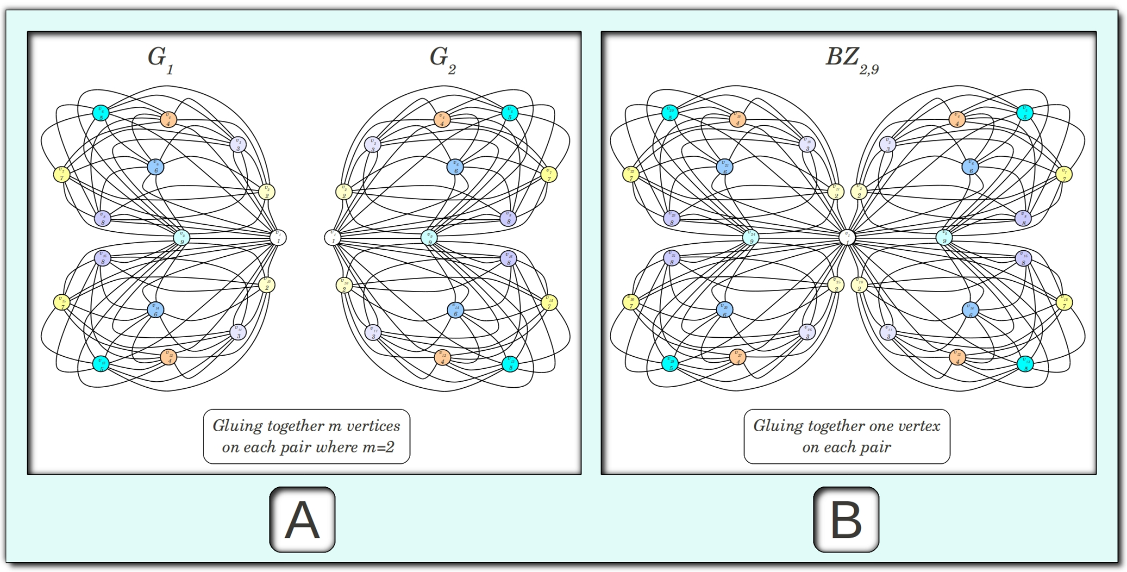


Figure 4.1.2. Completion of the butterfly graph, $BZ_{9,2}$

We now glue the two graphs $G_1$ and $G_2$ (composed of 16 vertices) together on one vertex, as depicted in illustration **B** of Figure 4.1.2. The resulting graph is $BZ_{9,2}$ and is composed of

$$|V(BZ_{9,2})| = 2(16) - 1 = 31,$$

because one vertex of $G_1$ is shared with $G_2$ after the gluing, and

$$|E(BZ_{9,2})| = 2(72 - 1) = 142,$$

because no edges are shared in the gluing of the graphs $G_1$ and $G_2$ on one vertex.

In stage 1 we began with $K_9$ and made four copies gluing them into pairs on $m = 2$

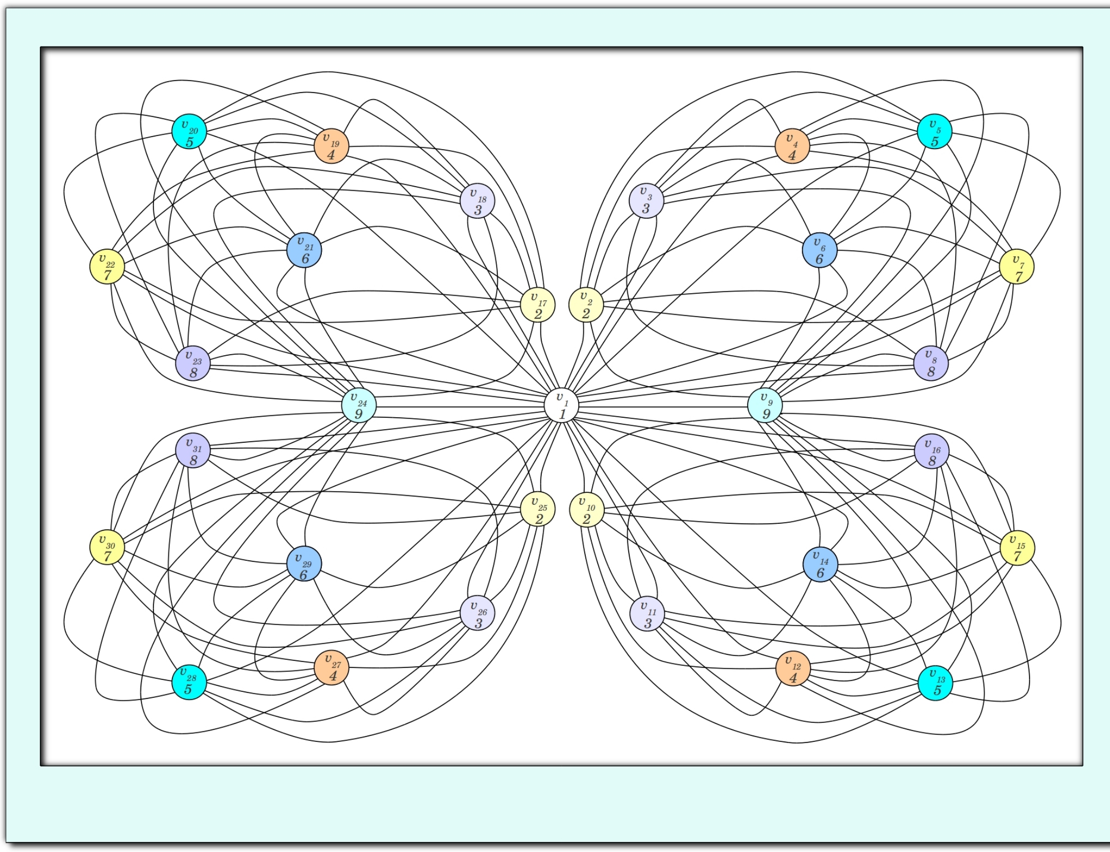

Figure 4.1.3. The butterfly graph, $BZ_{9,2}$

vertices, according to the definition. We know by Proposition 3.7.1 we have that $\chi(K_9) = 9$ , thus we have that $\chi(BZ_{9,2}) \geq 9$. Since the gluing process merges two vertices we have a graph with two $K_9$ subgraphs that can have a proper coloring with 9 colors. Thus,

$\chi(BZ_{9,2}) = 9$.

In order for Hadwiger's Conjecture to hold we need to either form a $K_9$ minor or simply locate a $K_9$ subgraph. Locating the $K_9$ minor in the butterfly graph is simply a matter of isolating the $K_9$ graph we started with. Since $K_9 \subset BZ_{9,2}$, and $BZ_{9,2}$ is 9-chromatic, we simply delete everything except for $K_9$. Thus Hadwiger's Conjecture holds for the graph $BZ_{9,2}$.

**Proposition 4.1.4.** *The butterfly graph's chromatic number is $n$, where $n$ is the amount of vertices of the complete graph chosen to copy.*

*Proof.* At the first stage, the Butterfly graph is composed four copies of a complete graph $K_n$ paired up into groups of two. Since each pair shares $m$ vertices after the gluing process, its coloring is equal to the complete graph that was glued to. Thus, its coloring does not have to increase because all of the vertices except for the ones glued together are disconnected from the original, so we can color the unglued vertices with the remaining colors from $K_n$. Therefore, the chromatic number at stage one stays the same.

Since the second stage glues together the pairs, which were $n$-chromatic, on one vertex, we have $\chi(BZ_{n,m}) \leq n$ because there exists a coloring from $K_n$. However, since $K_n \subset BZ_{n,m}$ we also have $\chi(BZ_{n,m}) \geq n$. Therefore, $\chi(BZ_{n,m}) = n$. □

Locating the $K_n$ minor of the Butterfly graph is simple. Since it is composed of a complete graph copied four times and glued according to the definition, we simply isolate the $K_n$ graph we began with and delete the rest. This brings us to an immediate corollary.

**Corollary 4.1.5.** *Hadwiger's conjecture holds for all butterfly graphs.*

## 4.2 The Parachute Graph

The butterfly graph, although creative, is a rather simple way to generate a non-complete $n$-chromatic graph; however, it always contains a $K_n$ subgraph. We are interested in the behavior of more complex $n$-chromatic non-complete graphs that do not have a $K_n$ subgraph. In hopes of achieving this end, we created the *parachute graph*.

**Definition 4.2.1.** We define the **parachute graph**, denoted as $\mathfrak{P}_0$, to be the graph illustrated in Figure 4.2.1. △

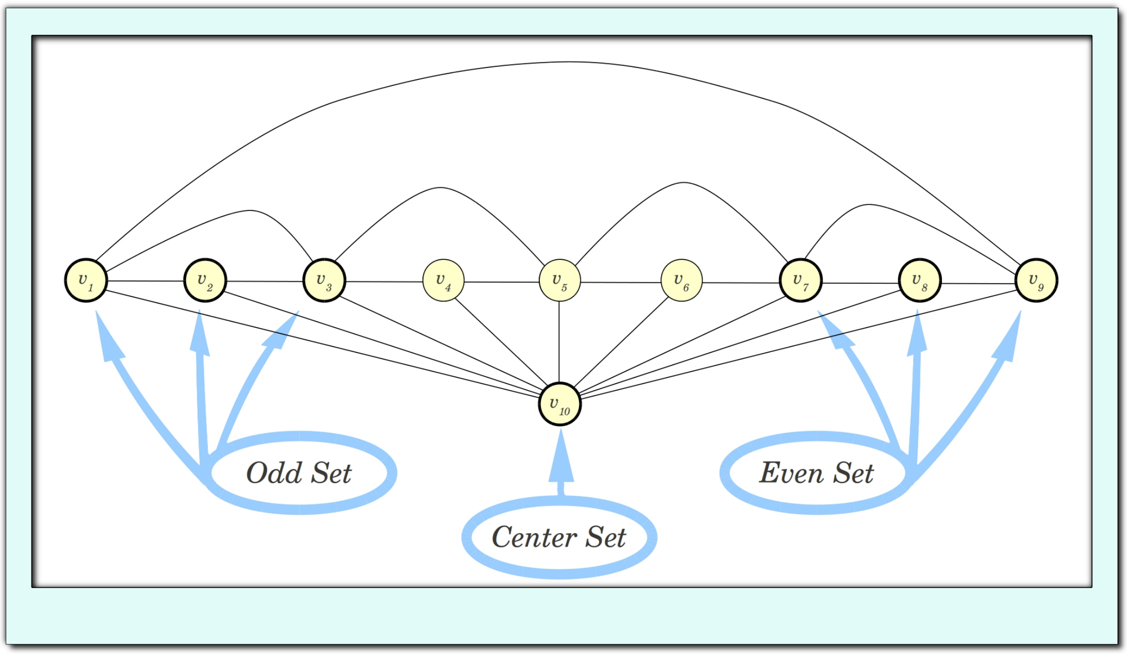


Figure 4.2.1. Illustration of the parachute graph, $\mathfrak{P}_0$

**Proposition 4.2.2.** *The parachute graph $\mathfrak{P}_0$ is 4-chromatic.*

*Proof.* We color the graph $\mathfrak{P}_0$ with 4 colors such that the center set vertex is color 1, the odd set are colors 2, 3, and 4, the even set are colors 2, 3, and 4. Now we have 3 vertices left to color between the odd and even set and above the center. We can color these colors 2, 3, and 4. Since we have a 4 coloring, this implies $\chi(\mathfrak{P}_0) \leq 4$. There is a $K_4$ subgraph in $\mathfrak{P}_0$, as depicted in Figure 4.2.2. We know by Proposition 3.7.1 that this subgraph is 4-chromatic, thus $\chi\mathfrak{P}_0 \leq 4$. However, since we established a 4-coloring $\chi(\mathfrak{P}_0) = 4$. $\square$

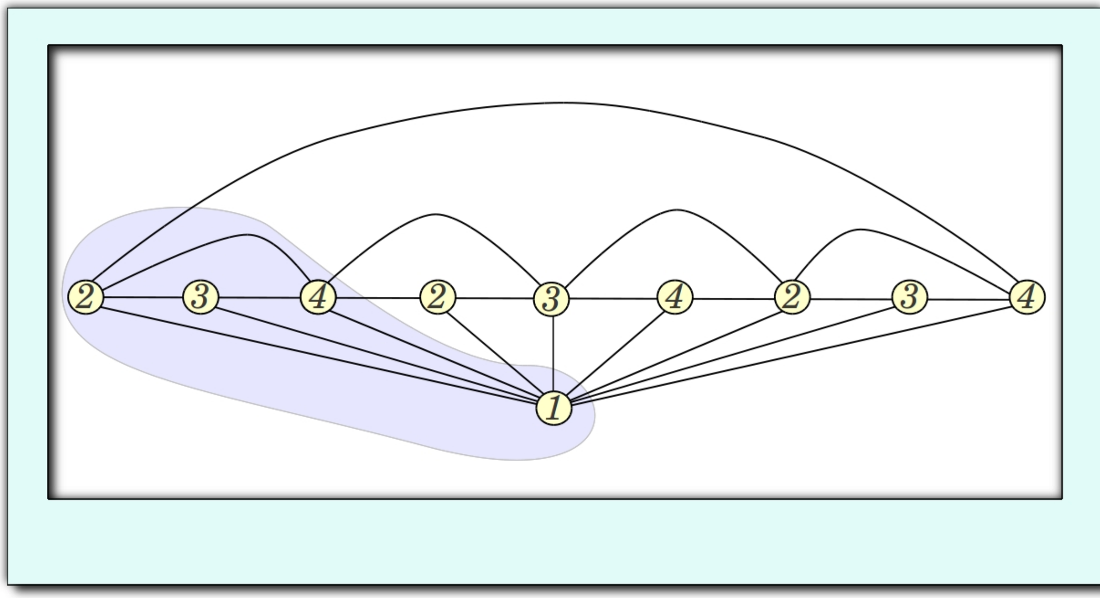


Figure 4.2.2. The 4-chromatic parachute graph $\mathfrak{P}_0$ with highlighted $K_4$ minor

Since in the base form of the parachute graph $\chi(\mathfrak{P}_0) = 4$, to see if Hadwiger's conjecture holds we want to form a $K_4$ minor from this graph. The graph $\mathfrak{P}_0$ contains the graph $K_4$ as a subgraph, as seen in the highlighted portion of Figure 4.2.2. Thus, we isolate this subgraph by deleting the rest of the graph to form the $K_4$ minor. This provides us with an immediate corollary.

**Corollary 4.2.3.** *Hadwiger's Conjecture holds for the parachute graph* $\mathfrak{P}_0$.

In order to generalize the definition and explain the algorithm that allows us to add vertices to the parachute graph, we need to define the terms, *odd set*, *even set*, and *center set*.

**Definition 4.2.4.** The **odd set** is the 3 leftmost vertices of the graph $\mathfrak{P}_0$, as labeled in Figure 4.2.1, $|odd\ set| = 3$. △

**Definition 4.2.5.** The **even set** is the 3 rightmost vertices of the graph $\mathfrak{P}_0$, as labeled in Figure 4.2.1, $|even\ set| = 3$. △

**Definition 4.2.6.** The **center set** is bottommost vertex of the graph $\mathfrak{P}_0$, as labeled in Figure 4.2.1, and any added vertices, $|center\ set| = 1 + n$, where $n$ represents the added vertices. △

Since we desire to create an arbitrarily large parachute graph with $i$ vertices, we define a new action, *center cone over* that defines connecting a new vertex to the center set.

**Definition 4.2.7.** Let $G$ be a graph with a subset called the center set. Let $S \subset V(G)$. To **center cone over** $S$ is to add a new vertex $v$ to the set of center elements in the center set of $G$, and then create an edge between any vertex in $S$ and the new vertex $v$. △

With these new definitions at hand we now define a general definition for the parachute graph.

**Definition 4.2.8.** We define the parachute graph $\mathfrak{P}_i, \quad i > 0$ recursively as follows:

$$\mathfrak{P}_i = \begin{cases} \text{center cone over odd and center set, if } i \text{ is odd.} \\ \text{center cone over even and center set, if } i \text{ is even.} \end{cases} \tag{4.2.1}$$

△

In order to get a visual representation of the parachute graphs algorithm (as defined above) in action, observe the illustrations **A,B,C,D,E,** and **F** in Figure 4.2.3, where $i$ is the number on each new vertex.

Now that we see how the parachute graph is constructed, we want to demonstrate that, as we add vertices to the parachute graph according to the prescribed algorithm in

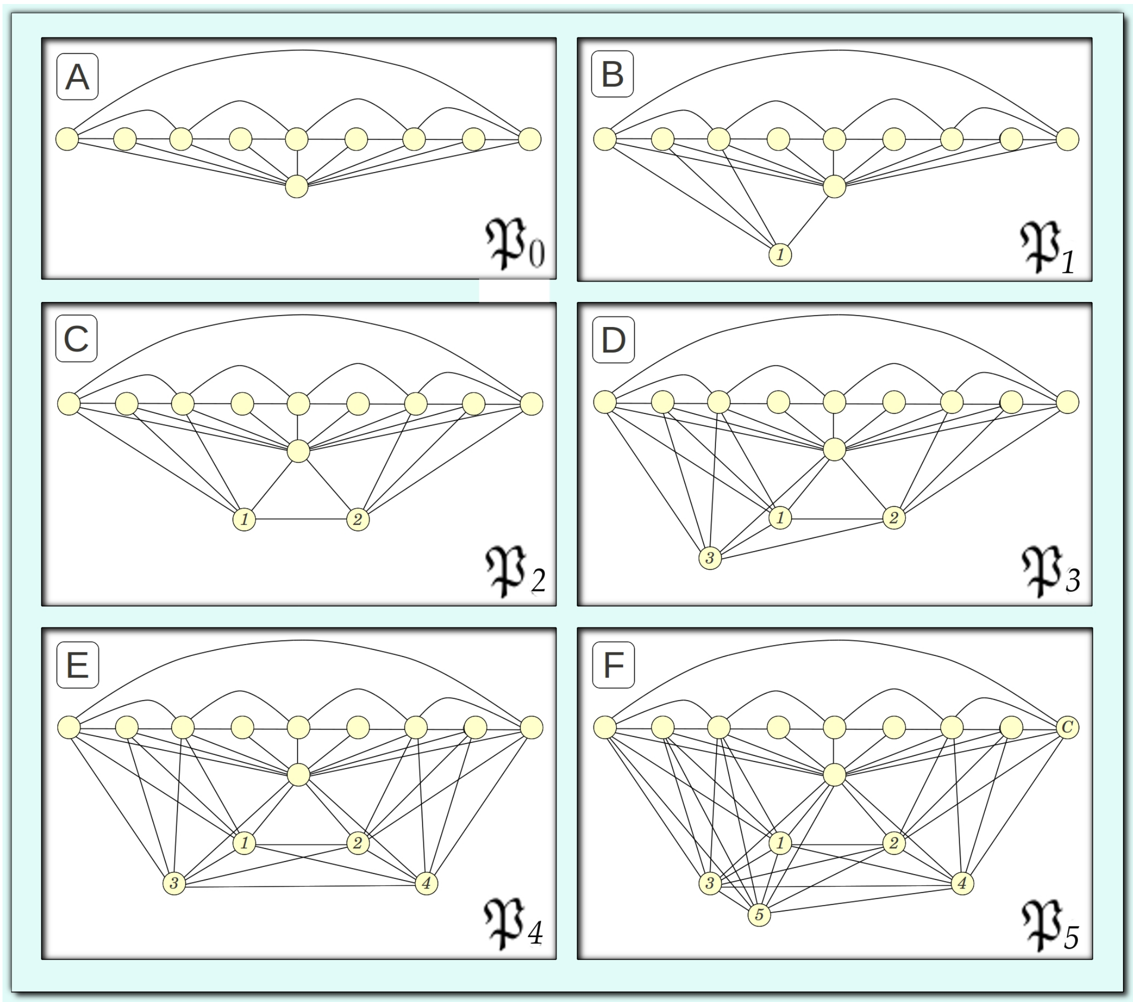


Figure 4.2.3. Illustration of parachute graph algorithm

Definition 4.2.8, its chromatic number increases. We will go through a series of examples where we add two vertices one at a time to observe how the chromatic number behaves, and then show that Hadwiger's Conjecture holds for new graphs. Although the series of examples may seem repetitive, the process is necessary to observe a unique property of this graph that when $n \leq 7$ the chromatic number is $4 + \lceil \frac{n}{2} \rceil$, but when $n > 7$, the chromatic number increases with each additional vertex.

**Example 4.2.9.** We will look at the parachute graph $\mathfrak{P}_1$, and $\mathfrak{P}_2$. First consider the parachute graph $\mathfrak{P}_1$. Since we add one vertex and connect it according to the algorithm we create a graph that contains a $K_5$ subgraph, as depicted in the highlighted region of illustration **A** of Figure 4.2.4. By Proposition 3.7.1 the chromatic number of this subgraph is 5. So $\chi(\mathfrak{P}_1) \geq 5$. Since in the previous graph $\chi(\mathfrak{P}_0) = 4$, we require one additional color for a proper coloring of the graph $\mathfrak{P}_1$. Thus, the chromatic number is increased by one, so $\chi(\mathfrak{P}_1) = 5$.

Now we add an additional vertice to $\mathfrak{P}_1$, to create the graph $\mathfrak{P}_2$. Since according to the

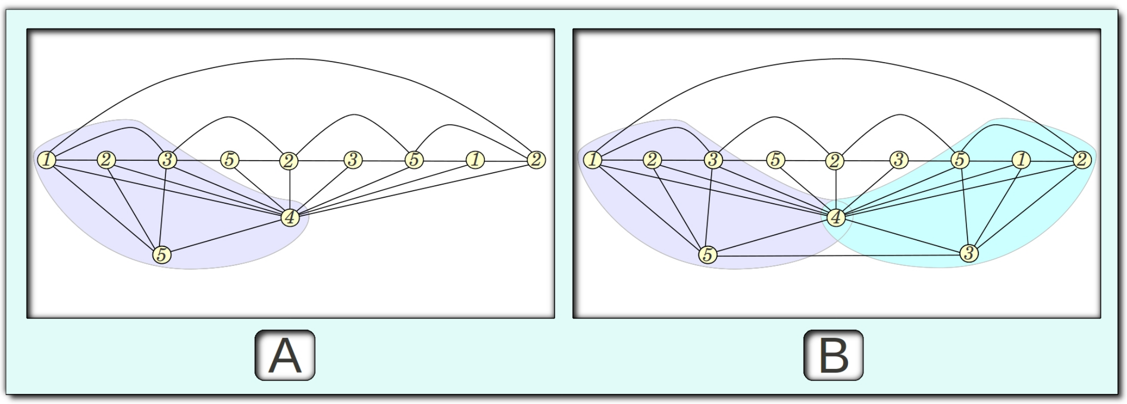


Figure 4.2.4. The 5-chromatic parachute graphs $\mathfrak{P}_1$, and $\mathfrak{P}_2$

algorithm this new vertex (an even number) is connected to the even set and the center set we form another $K_5$ subgraph, as depicted in the highlighted region of illustration **B** of Figure 4.2.4, so by Proposition 3.7.1 we know this subgraph is 5-chromatic. The fact that it must also be connected to the previously added odd vertex from $\mathfrak{P}_1$ does not increase the chromatic number in the new graph $\mathfrak{P}_2$. This is because since both subgraphs share the first vertex in the center set, (see Figure 4.2.4) and the new vertex is the only vertex in the second $K_5$ subgraph adjacent to any other vertex in the $K_5$ subgraph formed in $\mathfrak{P}_1$, we are left with 4 choices to color the remaining vertices of the second $K_5$ subgraph. Therefore, we can properly color $\mathfrak{P}_2$ with 5 colors. Thus, $\chi(\mathfrak{P}_2) = 5$.

Since both $\mathfrak{P}_1$, and $\mathfrak{P}_2$ are 5-chromatic and contain a $K_5$ subgraph, we can form a $K_5$ minor by deleting all the vertices except for the $K_5$ subgraph. Therefore Hadwigers conjecture holds for the parachute graph when $i \leq 2$.

We now look at another example of adding 2 additional vertices one at a time to observe how the chromatic number behaves, and then see if Hadwiger's Conjecture holds on the resulting new graphs.

**Example 4.2.10.** Now we add an additional vertex to $\mathfrak{P}_2$, to create the graph $\mathfrak{P}_3$. This new vertex (an odd number) must be connected to the odd set and the center set. Effectively we add another vertex to a $K_5$ subgraph forming a $K_6$ subgraph, as depicted in the highlighted region of illustration **B** of Figure 4.2.5, so by Proposition 3.7.1 we know this subgraph is 6-chromatic and $\chi(\mathfrak{P}_3) \geq 6$. Since the previous graph had a chromatic number of 5 and we now have added an additional vertex such that it forms a $K_6$ subgraph, we require at least 6 colors for a proper coloring. We color additional vertex with the 6th color and we have a proper coloring for the graph $\mathfrak{P}_3$ with 6 colors. Thus, $\chi(\mathfrak{P}_3) = 6$.

Now we consider the graph $\mathfrak{P}_4$ created by adding an additional vertex to $\mathfrak{P}_3$. Since

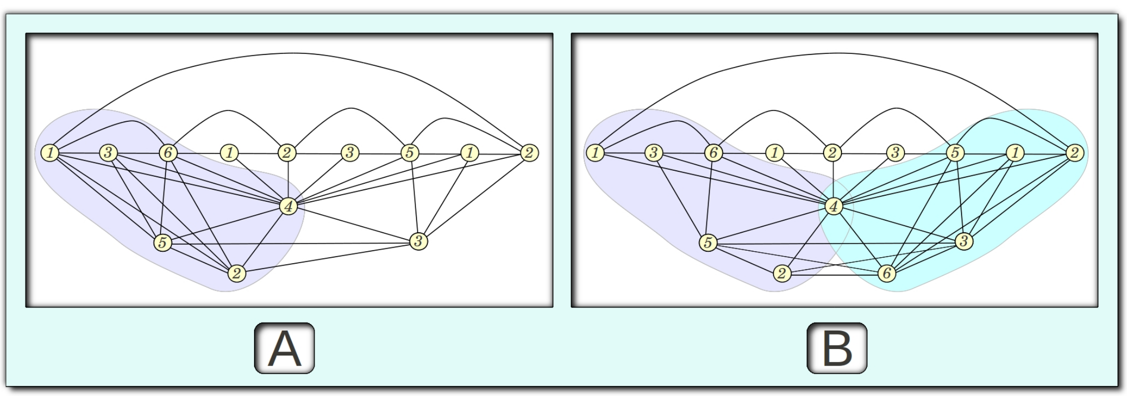


Figure 4.2.5. The 6-chromatic parachute graphs, $\mathfrak{P}_3$, and $\mathfrak{P}_4$

according to the algorithm this new vertex (an even number) connected to the even set and the center set, forms a $K_6$ subgraph, as depicted in the highlighted region of illustration

**B** of Figure 4.2.5, so by Proposition 3.7.1, we know this subgraph is 6-chromatic and $\chi(\mathfrak{P}_2) \geq 6$. The fact that the new vertex connects to 4 shared vertices in the center set does not increase the chromatic number. This is because since the $K_6$ subgraph formed in $\mathfrak{P}_3$ and the new $K_6$ subgraph share the central vertex (the first vertex in the center set), and the newly added vertex is only adjacent to 2 of the vertices in the first $K_6$ subgraph (formed in $\mathfrak{P}_3$), we are left with 3 choices to color the remaining 5 vertices of the second $K_6$ subgraph. By alternating the colors with the even set, we can properly color $\mathfrak{P}_4$ with 6 colors. Thus, $\chi(\mathfrak{P}_4) = 6$.

Since both $\mathfrak{P}_3$, and $\mathfrak{P}_4$ are 6-chromatic and contain a $K_6$ subgraph, we can form a $K_6$ minor. Therefore Hadwigers conjecture holds for the parachute graph when $i \leq 4$.

We now look at another example of adding 2 additional vertices one at a time to observe how the chromatic number behaves, and then see if Hadwiger's Conjecture holds on the resulting new graphs

**Example 4.2.11.** Consider the parachute graph $\mathfrak{P}_5$. To form $\mathfrak{P}_5$ we add an additional vertex to $\mathfrak{P}_4$. Since according to the algorithm this new vertex (an odd number) must be connected to the odd set and the center set that formed a $K_6$ subgraph, adding an additional vertex forms a $K_7$ subgraph, as depicted in the highlighted region of illustration **B** of Figure 4.2.6, so by Proposition 3.7.1 we know this subgraph is 7-chromatic, and $\chi(\mathfrak{P}_5) \geq 7$. Since the previous graph had a chromatic number of 6 and we now have added an additional vertex such that it forms a $K_7$ subgraph, we require at least 7 colors for a proper coloring. We color additional vertex with the 7th color and we have a proper coloring for the graph $\mathfrak{P}_5$ with 7 colors. Thus, $\chi(\mathfrak{P}_5) = 7$.

We now consider the graph $\mathfrak{P}_6$, which is created by adding an additional vertex to $\mathfrak{P}_5$. Since according to the algorithm this new vertex (an even number) connects to the even and center sets, we form a $K_7$ subgraph with in the center set, as depicted in the

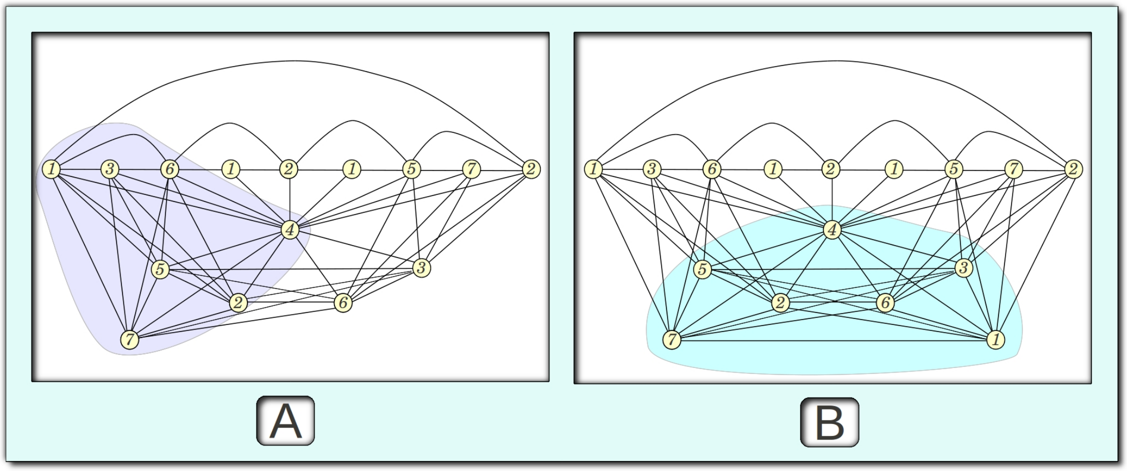


Figure 4.2.6. The 7-chromatic parachute graphs, $\mathfrak{P}_5$, and $\mathfrak{P}_6$

highlighted region of illustration **B** of Figure 4.2.6, so by Proposition 3.7.1 we know this subgraph is 7-chromatic, and $\chi(\mathfrak{P}_6) \geq 7$. Since we can color this graph with 7 colors by alternating the colors in the even and odd sets, we have $\chi(\mathfrak{P}_6) \leq 6$. Therefore, we can properly color $\mathfrak{P}_6$ with 7 colors. Thus, $\chi(\mathfrak{P}_6) = 7$.

Since both $\mathfrak{P}_5$, and $\mathfrak{P}_6$ are 7-chromatic and contain a $K_7$ subgraph, we can form a $K_7$ minor by deleting all the vertices not in the $K_7$ subgraph. Therefore Hadwiger's conjecture holds for the parachute graph when $i \leq 6$.

We now come to the example where the behavior of the chromatic number will shift from going up every other time we add a new vertex to going up each time we add a new vertex.

**Example 4.2.12.** Consider the graphs $\mathfrak{P}_7$, and $\mathfrak{P}_8$. We begin with $\mathfrak{P}_7$, which adds one vertex to $\mathfrak{P}_6$ connected to the odd set and the center set. Since we had a subgraph of $K_7$, as highlighted in illustration **A** of Figure 4.2.7, adding another vertex, will create a $K_8$ subgraph in the center set Thus, the chromatic number (which was $\chi(\mathfrak{P}_6) = 7$ in the previous example), must also increase by one. Therefore, $\chi(\mathfrak{P}_7) = 8$.

Now we consider the graph $\mathfrak{P}_8$, which adds another vertex to $\mathfrak{P}_7$. By virtue of the

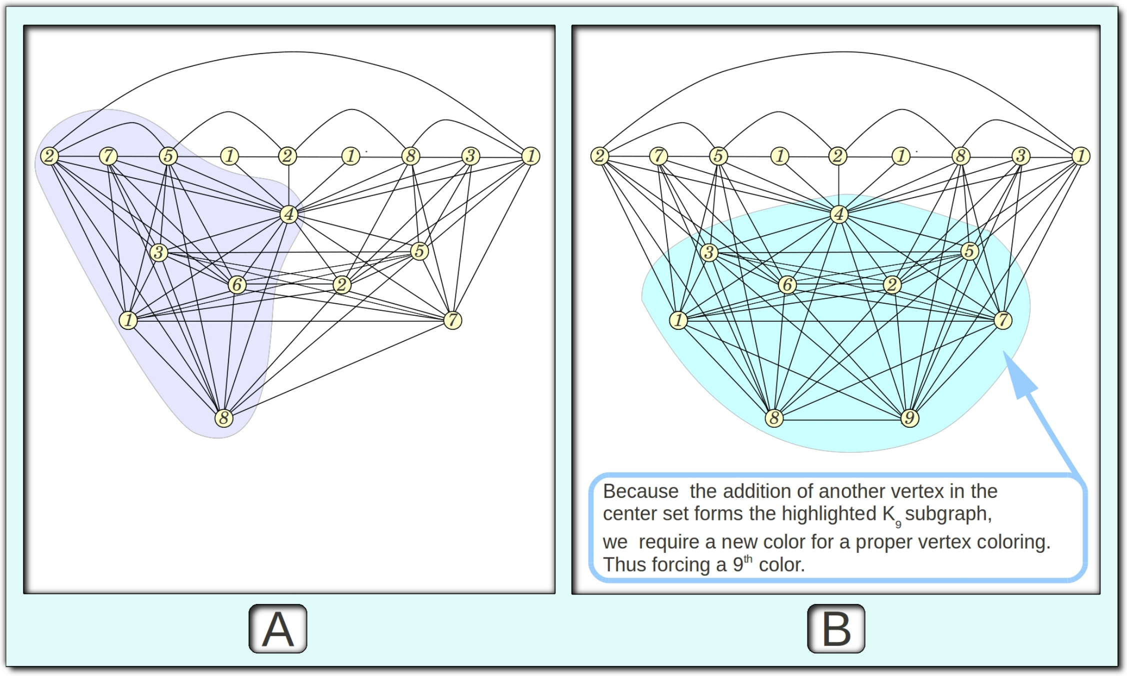


Figure 4.2.7. The parachute graphs $\mathfrak{P}_7$, and $\mathfrak{P}_8$

algorithms progression, in $\mathfrak{P}_7$ we have formed a $K_8$ subgraph in the center set. Adding another vertex to create $\mathfrak{P}_8$ adds one more to the center set. Thus, we form a $K_9$ subgraph, as depicted by in illustration **B** in Figure 4.2.7. Since by Proposition 3.7.1 $\chi(K_9) = 9$, we require a new color for the additional vertex in order to have a proper coloring. Therefore, $\chi(\mathfrak{P}_8) = 9$. This marks a point where the behavior of this graphs chromatic number shifts to going up with each additional vertex instead of with the addition of every other vertex.

Since $\mathfrak{P}_7$ contains a $K_8$ subgraph, and $\mathfrak{P}_8$ contains a $K_9$ subgraph, we can form there respective $K_8$, and $K_9$ minors by isolating these subgraphs and deleting the rest of the vertices. Therefore, Hadwiger's conjecture holds for the parachute graph when $i \leq 8$.

We will proceed with another example adding 2 vertices to see if this behavior continues.

**Example 4.2.13.** Consider the graphs $\mathfrak{P}_9$, and $\mathfrak{P}_{10}$. We must take note that the accumulation of the vertex points added beyond the base has (at $\mathfrak{P}_8$) formed a $K_9$ subgraph in the center set, as depicted in illustration **A** of Figure 4.2.8. Since we are adding another vertex (an odd number) it is connected to the odd set, and the center set, this additional vertex forms a $K_{10}$ subgraph, which we know has a chromatic number of 10. Thus, because we cannot properly color our new graph with 9 colors, we have to color this vertex with a new 10th color. Thus, $\chi(\mathfrak{P}_9) = 10$.

Consider the graph $\mathfrak{P}_{10}$. Similar to above, we are connecting a new vertex to a graph

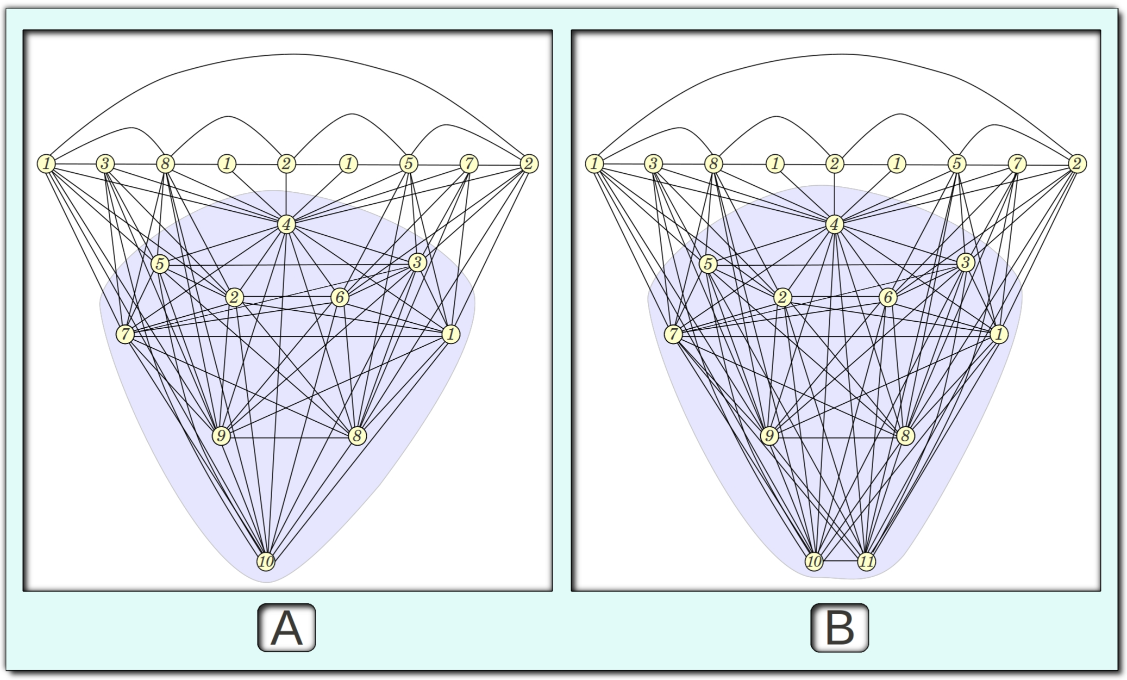


Figure 4.2.8. The parachute graphs $\mathfrak{P}_9$, and $\mathfrak{P}_{10}$

with a $K_{10}$ subgraph such that it is adjacent to every vertex in the $K_{10}$ subgraph. The chromatic number was previously 10, but it must now increase to 11 because we formed a $K_{11}$ subgraph, as depicted in illustration **B** of Figure 4.2.8, and by Proposition 3.7.1 $\chi(K_{11}) = 11$. Therefore, $\chi(\mathfrak{P}_{10}) = 11$.

Since $\mathfrak{P}_9$ contains a $K_{10}$ subgraph, and $\mathfrak{P}_{10}$ contains a $K_{11}$ subgraph, we can form there respective $K_{10}$, and $K_{11}$ minors by isolating these subgraphs and deleting the rest of the vertices. Therefore, Hadwiger's conjecture holds for the parachute graph when $i \leq 10$.

Based on the previous examples, we can make a conjecture about the chromatic properties of the parachute graph.

**Conjecture 4.2.14.** *The chromatic number of the parachute graph is:*

$$\chi(\mathfrak{P}_i) = \begin{cases} 4 + \lceil \frac{i}{2} \rceil, \text{ when } i \leq 7 \\ i + 1, \text{ when } i > 7. \end{cases} \tag{4.2.2}$$

Since it is apparent form the previous examples that each parachute graph contains a subgraph of the same chromatic number, and we can form a $K_n$ minor from deleting around a subgraph, we can make a general conjecture about the parachute graph.

**Conjecture 4.2.15.** *Hadwiger's Conjecture holds for all parachute graphs* $\mathfrak{P}_i$, $i \geq 0$.

## 4.3 Modified Parachute Graph

As we can see, the parachute graph contains a proper subgraph of the same chromatic number. Since we are attempting to create a graph $G$ that is $n$-chromatic, non-complete, and does not contain a proper subgraph of the same chromatic number as $G$, we modify the parachute graph $\mathfrak{P}_0$, by adding 2 additional edges from the endmost vertex of the odd and even sets to the center vertex above the center set, as depicted in Figure 4.2.1 in attempt to achieve our desired result.

**Definition 4.3.1.** We define the **modified parachute graph**, denoted as $\mathfrak{MP}_0$, to be the graph illustrated in Figure 4.3.1. △

**Proposition 4.3.2.** $\chi(\mathfrak{MP}_0) = 4$.

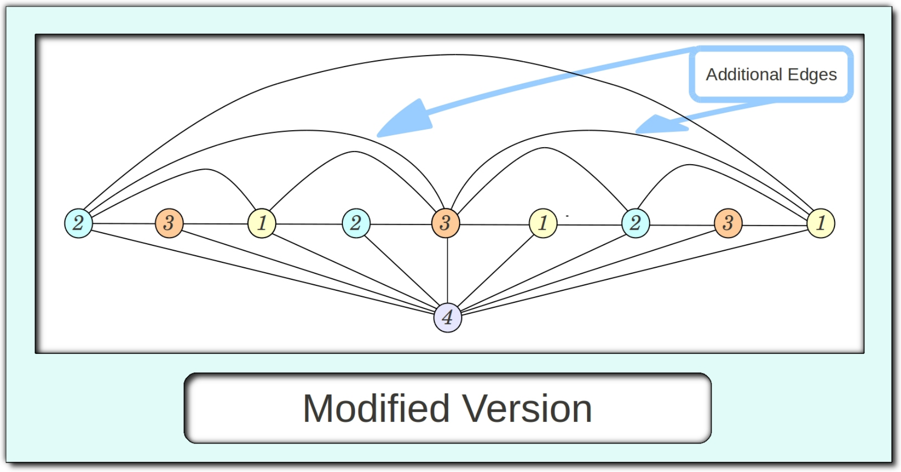


Figure 4.3.1. Modified construction of parachute graph

*Proof.* We color the graph $\mathfrak{MP}_0$ with 4 colors such that the center set vertex is color 4, the odd set are colors 2, 3, and 1, the even set are colors 2, 3, and 1. Now we have 3 vertices left to color between the odd and even set and above the center. We can color these colors 2, 3, and 1. Since we have a 4 coloring, this implies $\chi(\mathfrak{MP}_0) \leq 4$. There is a $K_4$ subgraph in $\mathfrak{MP}_0$, as depicted in Figure 4.3.2. We know by Proposition 3.7.1 that this subgraph is 4-chromatic, thus $\chi(\mathfrak{MP}_0) \geq 4$. However, since we established a 4-coloring $\chi(\mathfrak{MP}_0) = 4$. $\square$

Since in the base form of the modified parachute graph $\chi(\mathfrak{MP}_0) = 4$, to see if Hadwiger's Conjecture holds we want to form a $K_4$ minor from this graph. The graph $\mathfrak{MP}_0$ contains the graph $K_4$ as a subgraph, as seen in the highlighted portion of Figure 4.3.2. Thus, we isolate this subgraph by deleting the rest of the graph to form the $K_4$ minor. This provides us with an immediate corollary.

**Corollary 4.3.3.** *Hadwiger's Conjecture holds for the graph* $\mathfrak{MP}_0$.

As previously, in order to generalize the definition and explain the algorithm that allows us to add vertices to the modified parachute graph, we recall from the previous definitions

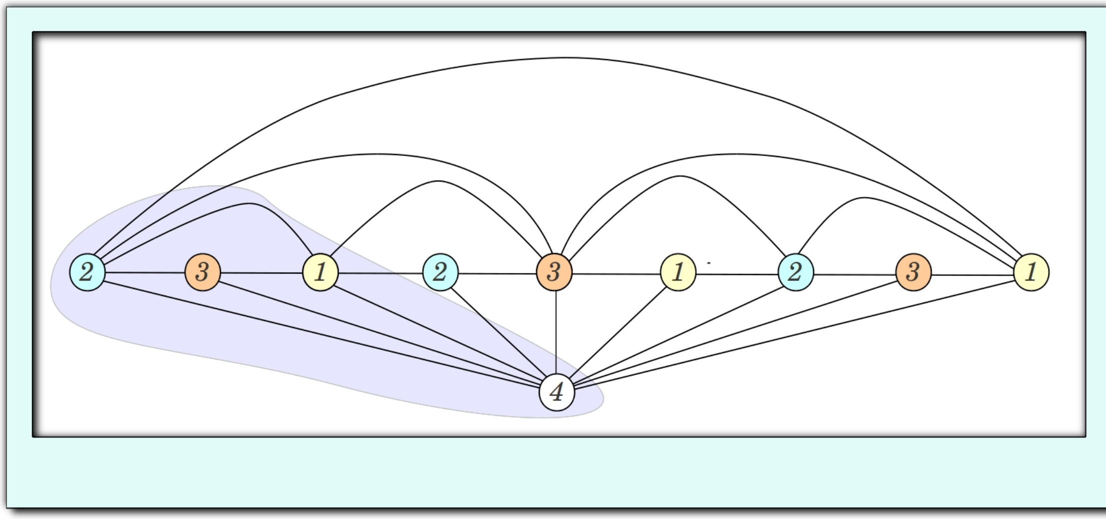


Figure 4.3.2. The highlighted $K_4$ subgraph of the modified parachute graph $\mathfrak{MP}_0$

the following terms odd set, even set, center set, and center core over.

With these definitions at hand we now define a general definition for the modified parachute graph.

**Definition 4.3.4.** We define the parachute graph $\mathfrak{MP}_i, \;\; i \geq 0$ recursively as follows:

$$\mathfrak{MP}_i = \begin{cases} \text{center core over odd and center set, if } i \text{ is odd.} \\ \text{center core over even and center set, if } i \text{ is even.} \end{cases} \tag{4.3.1}$$

$\triangle$

Now let us consider an example where $i < 7$ that in the unmodified parachute graph would have had a chromatic number of $4 + \lceil \frac{n}{2} \rceil$ to see if the modification increases the chromatic number.

**Example 4.3.5.** Consider the graphs $\mathfrak{MP}_1$, and $\mathfrak{MP}_2$, as depicted in Figure 4.3.3. In the graph $\mathfrak{MP}_1$ we have that a odd numbered vertex is added. Thus, it is connected to the odd set and the center set. Since this new vertex connects to every vertex in a $K_4$ subgraph within the graph $\mathfrak{MP}_0$, we form a $K_5$ subgraph, thus we must increase the chromatic number, so $\chi(\mathfrak{MP}_1) = 5$, as depicted in illustration **A** of Figure 4.3.3.

Now consider the graph $\mathfrak{MP}_2$, upon inspection it may appear as though we are forced

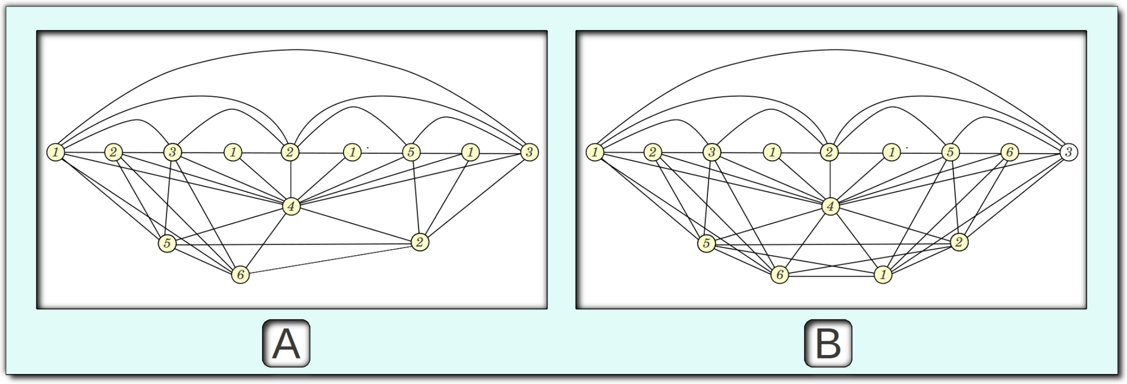


Figure 4.3.3. The modified parachute graph $\mathfrak{MP}_1$, and $\mathfrak{MP}_2$

to use a new color. If this were true, this would be exactly what we are seeking to do: have the chromatic number go up so that the graph $\mathfrak{MP}_2$ is 6-chromatic but does not a $K_6$ subgraph. However, we can color the graph $\mathfrak{MP}_2$ with 5 colors by alternating the colors in the top row that contains the odd and even sets. Thus, we have $\chi(\mathfrak{MP}_1) \leq 5$, and since it contains a $K_5$ subgraph, we have that $\chi(\mathfrak{MP}_2) \geq 5$. Therefore, $\chi(\mathfrak{MP}_1) = 5$, as depicted in illustration **B** of Figure 4.3.3.

Since both $\mathfrak{MP}_1$, and $\mathfrak{MP}_2$ are 5-chromatic and contain a $K_5$ subgraph, we can form a $K_5$ minor by deleting all the vertices except for the $K_5$ subgraph. Therefore Hadwigers conjecture holds for the parachute graph when $i \leq 2$

Thus, we see that even with the modification of two additional edges, the behavior of the chromatic number remains the same for the modified parachute graph when two vertices are added as it is for the original parachute graph when two vertices are added. This being the case we will test adding additional vertices until we reach the point where the original parachute graphs chromatic number shifts from going up with every other vertex addition to going up with each new vertex addition.

**Example 4.3.6.** Consider the graphs $\mathfrak{MP}_3$, and $\mathfrak{MP}_4$ as depicted in Figure 4.3.4. Since the previous graph had a chromatic number of 5 and we now have added an additional vertex such that it forms a $K_6$ subgraph, we require at least 6 colors for a proper coloring. We color additional vertex with the 6th color and we have a proper coloring for the graph $\mathfrak{MP}_3$ with 6 colors. Thus, $\chi(\mathfrak{MP}_3) = 6$.

By alternating the colors with the even set, we can color $\mathfrak{MP}_4$ with 6 colors. Thus,

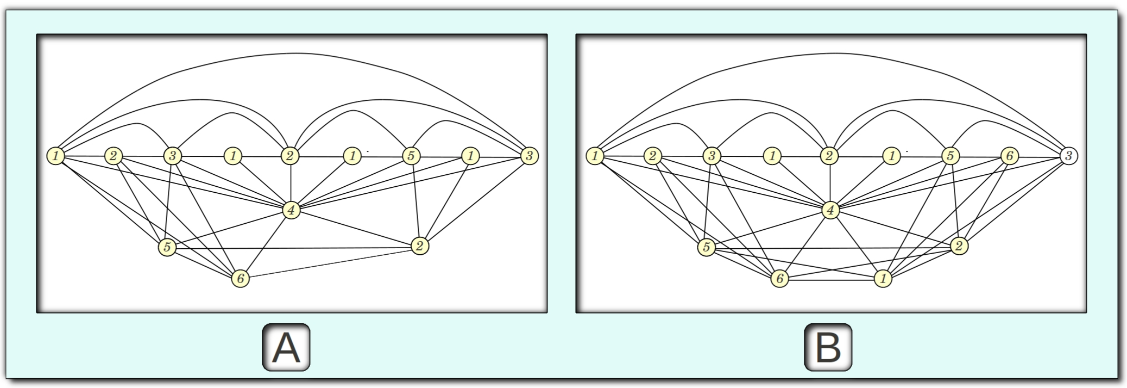


Figure 4.3.4. The modified parachute graphs $\mathfrak{MP}_3$, and $\mathfrak{MP}_4$

$\chi(\mathfrak{MP}_4) \leq 6$. Since the graph $\mathfrak{MP}_4$ contains a $K_6$ subgraph, we know $\chi(\mathfrak{MP}_4) \geq 6$. Therefore, $\chi(\mathfrak{MP}_4) = 6$.

Since both $\mathfrak{P}_3$, and $\mathfrak{P}_4$ are 6-chromatic and contain a $K_6$ subgraph, we can form a $K_6$ minor. Therefore Hadwigers conjecture holds for the parachute graph when $i \leq 4$.

We still have the exact same behavior for the chromatic number n the modified parachute graph as in the original parachute graph. We will go through two more examples in hopes of finding that the chromatic number is not equal to the proper subgraph within our graph.

**Example 4.3.7.** Consider the graphs $\mathfrak{MP}_5$, and $\mathfrak{MP}_6$, as depicted in Figure 4.3.5. Since the previous graph, $\mathfrak{MP}_4$, had a chromatic number of 6, and we now have added an additional vertex such that it forms a $K_7$ subgraph, we require at least 7 colors for a proper coloring. We color additional vertex with the 7th color and we have a proper coloring for the graph $\mathfrak{MP}_5$ with 7 colors. Thus, $\chi(\mathfrak{MP}_5) = 7$.

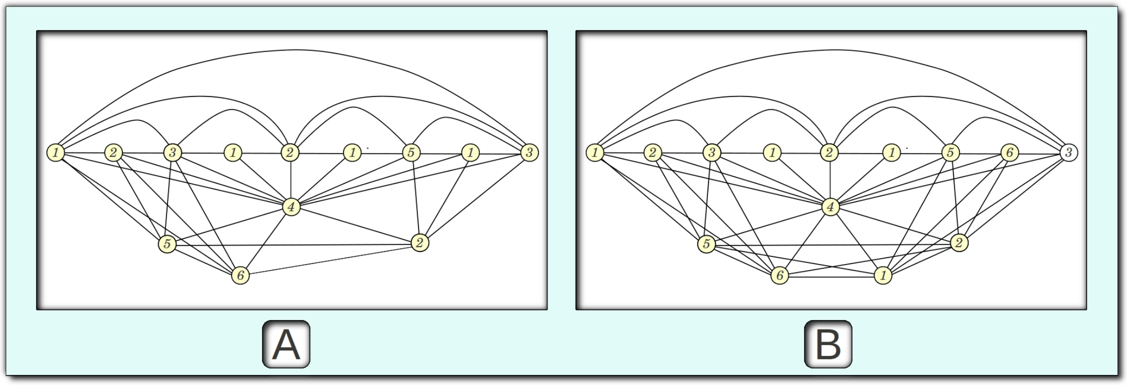


Figure 4.3.5. The parachute graphs $\mathfrak{MP}_5$, and $\mathfrak{MP}_6$

Since we can color this graph with 7 colors by alternating the colors in the even and odd sets, we have $\chi(\mathfrak{MP}_6) \leq 6$. We also know that $K_7$ is a subgraph of the graph $\mathfrak{MP}_6$, so $\chi(\mathfrak{MP}_6) \geq 7$ Therefore, $\chi(\mathfrak{MP}_6) = 7$.

Since both $\mathfrak{MP}_5$, and $\mathfrak{MP}_6$ are 7-chromatic and contain a $K_7$ subgraph, we can form a $K_7$ minor by deleting all the vertices not in the $K_7$ subgraph. Therefore Hadwiger's conjecture holds for the parachute graph when $i \leq 6$.

Again the behavior of the chromatic number of the modified parachute graph runs parallel to that of the original parachute graph. If the addition of one more vertex creates a $K_8$ subgraph in the center set, we will have established that the behavior of the modified version of the parachute graph mirrors that of the original.

**Example 4.3.8.** Consider the graph $\mathfrak{MP}_7$ as depicted in illustration **A** of Figure 4.3.6. Since we had a subgraph of $K_7$, in the graph $\mathfrak{MP}_6$, adding another vertex, will create a $K_8$ subgraph in the center set Thus, the chromatic number must increase by one. Therefore,

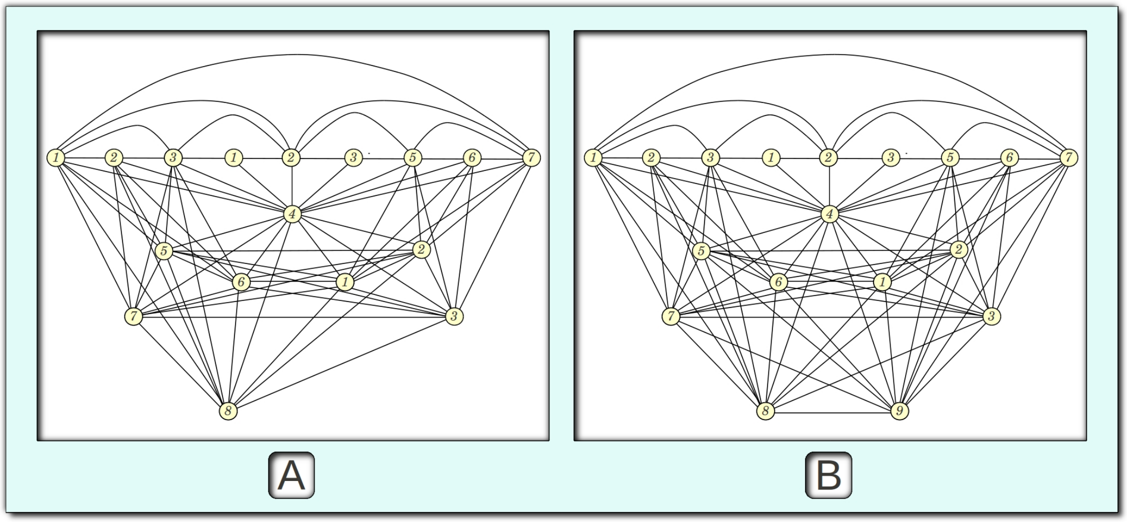


Figure 4.3.6. The parachute graphs $\mathfrak{MP}_7$, and $\mathfrak{MP}_8$

$\chi(\mathfrak{MP}_7) = 8$.

Any additional vertices as in the graph $\mathfrak{MP}_8$, depicted in illustration **B** of Figure 4.3.6, increase the complete subgraph in the center set by one, thus $\chi(\mathfrak{MP}_8) = 9$.

Since $\mathfrak{MP}_7$ contains a $K_8$ subgraph, and $\mathfrak{MP}_8$ contains a $K_9$ subgraph, we can form there respective $K_8$, and $K_9$ minors by isolating these subgraphs and deleting the rest of the vertices. Therefore, Hadwiger's conjecture holds for the modified parachute graph when $i \leq 8$.

It is apparent that the graph we were seeking to create did not manifest through the modification of the parachute graph. We were unable to have the chromatic number exceed the proper subgraph. However, after observing the parallel nature of the chromatic numbers behavior in the modified parachute graph to that of the original version, we can make conjecture about the chromatic properties of the modified parachute graph.

**Conjecture 4.3.9.** *The chromatic number of the modified parachute graph is:*

$$\chi(\mathfrak{MP}_i) = \begin{cases} 4 + \lceil \frac{i}{2} \rceil, \text{ when } i \leq 7 \\ i + 1, \text{ when } i > 7. \end{cases} \tag{4.3.2}$$

Since it is apparent form the previous examples that each modified parachute graph contains a subgraph of the same chromatic number, and we can form a $K_n$ minor from deleting around a subgraph, we can make a general conjecture about the modified version of the parachute graph.

**Conjecture 4.3.10.** *Hadwiger's Conjecture holds for all modified parachute graphs* $\mathfrak{MP}_i$, $i \geq 0$.

# 5

# Further Direction Concerning Hadwiger's Conjecture

Taking the knowledge gained up to this point in the project we see that Hadwiger's Conjecture can be proven for specific families of graphs, and that it holds on examples of graphs with chromatic numbers that exceed the limit for which it has been proven. We have proven such cases without the use of the Four Color Theorem. Thus, it may be possible that proving the general case for Hadwiger's Conjecture may not need to rely on the Four Color Theorem, or possibly, that proving Hadwiger's Conjecture in the general case would provide an alternative proof to the Four Color Theorem. Of course this is only conjecture, but one never knows what fruit a different approach may yield. To possibly push research in a different direction we provide a different way to establish a proper coloring with the chromatic polynomial.

## 5.1 Chromatic Polynomials

As we have seen throughout this project, it can often be confusing to figure out the proper vertex coloring of a complicated graph. This being the case, one can apply an algebraic method in the form of assigning a function to the graph that will tell us the coloring and chromatic number of said graph. This method is called a chromatic polynomial and it is useful in proving the chromatic number of complicated graphs.

In order to understand this, we will have to define some simple notation concerned with chromatic polynomials. We let $G$ be a simple graph, and we let $P_G(k)$ denote the number of ways that we can color the vertices of $G$ with $k$ colors such that no two adjacent verticies have that same color (a proper vertex coloring). We will call $P_G$ the chromatic function of $G$.

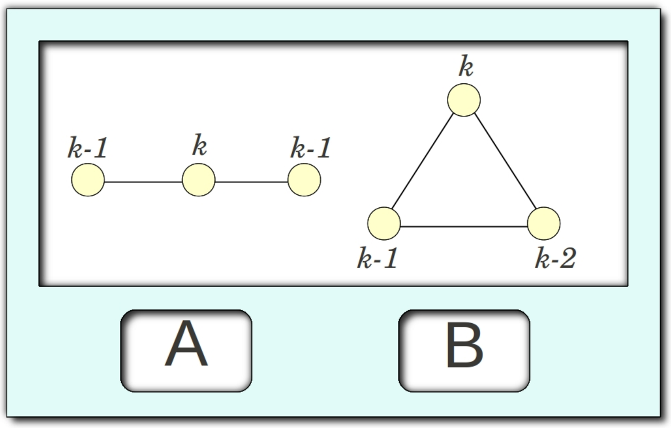


Figure 5.1.1. Applying $k$ colors to graphs

**Example 5.1.1.** Consider the graph $P_3$ in illustration **A** of Figure 5.1.1. This is a path of length 2, which contains 3 vertices and 2 edges. Since every path is bipartite (see Proposition 3.2.9), we know by Theorem 3.2.4 it is 2-chromatic. Thus, the middle vertex can be colored in $k$ ways, and the end vertices can be colored in $(k-1)$ ways. So $P_G(k) = k(k-1)^2$. Observe that (k-1) is squared. This is because we use this color twice.

This result can be extended to a general case to prove that if $G$ is any tree or path with $n$ vertices, then $P_G(k) = k(k-1)^{n-1}$

Similar to the above application we can apply this procedure to complete graphs.

**Example 5.1.2.** Consider the graph $K_3$ in illustration **B** of Figure 5.1.1. We know from Proposition 3.7.1 that the number of vertices a complete graph has is also its chromatic number. Thus, for $K_3$, $P_G(k) = k(k-1)(k-2)$. This can be extended to any simple graph as thus: $P_G(k) = k(k-1)(k-2)...(k-n+1)$ if $G$ is the graph $K_n$.

It is clear that if $k, \chi(G)$, then $P_G(k) = 0$ and that if $k \geq \chi(G)$, then $P_G(k) > 0$. Note that the four color theorem is equivalent to the statement : if $G$ is a simple planar graph, then $P_G(4) > 0$.

If we are given an arbitrary simple graph, it is usually difficult to obtain its chromatic function merely by inspection. Thankfully we have Theorem 5.1.3 and Corollary 5.1.4 that provide us with a systematic method for obtaining the chromatic function of a simple graph in terms of the chromatic function of null graphs, or, as we will see, graphs that have a known low chromatic number, such as trees, paths, triangles, and even or odd cycles.

**Theorem 5.1.3.** *Let $G$ be a simple graph, and let $G$-$e$ and $G/e$ be the graphs obtained from $G$ by deleting an edge $e$ and then contracting this edge $e$ respectively. Then*

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k)$$

*Proof.* Let $e = vw$. The number of k-colorings of $G - e$ in which $v$ and $w$ have different colors is unchanged if the edge $e$ is drawn joining $v$ and $w$, and is therefore equal to $P_G(k)$. Similarly, the number of k-colorings of $G - e$ in which $v$ and $w$ have the same color is unchanged if $v$ and $w$ are identified, and is therefore equal to $P_{G/e}(k)$. The total number $P_{G-e}(k)$ of k-colorings of $G - e$ is therefore $P_G(k) + P_{G/e}(k)$, as required. □

To get a visual representation of Theorem 5.1.3, consider the series of graphs in Figure 5.1.2. Let $G$ be the representation of a 4-chromatic graph, and observe the corresponding

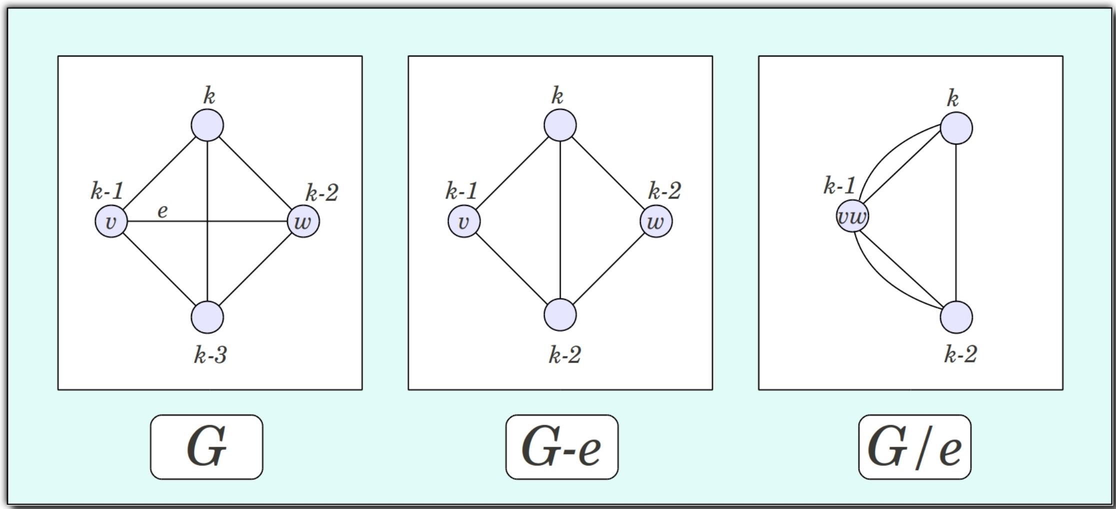


Figure 5.1.2. The contraction of an edge, and merging of vertices

graphs $G-e$ and $G/e$ appearing to the right of $G$. With this information we apply Theorem 5.1.3 to state that:

$$k(k-1)(k-2)(k-3) = [k(k-1)(k-2)^2] - [k(k-1)(k-2)]. \tag{5.1.1}$$

This algebraic representation of the chromatic number establishes a useful corollary.

**Corollary 5.1.4.** *The chromatic function of a simple graph is a polynomial.*

*Proof.* We continue the above procedure by choosing edges in $G-e$ and $G/e$ and deleting and contracting them. We then repeat the procedure for these four new graphs, and so on. The process terminates when no edges remain, or in other words, when each remaining graph is a null graph. Since the chromatic function of a null graph is a polynomial $(k)^r$, where $r$ is the number of vertices. It follows by repeated application of Theorem 5.1.3 that the chromatic function of the graph $G$ must be a sum of polynomials and therefore must itself be a polynomial. □

We can proceed by drawing and labeling the main graph's vertices with the corresponding $k$ ways of coloring and then draw the "decomposition graphs," obtained by the

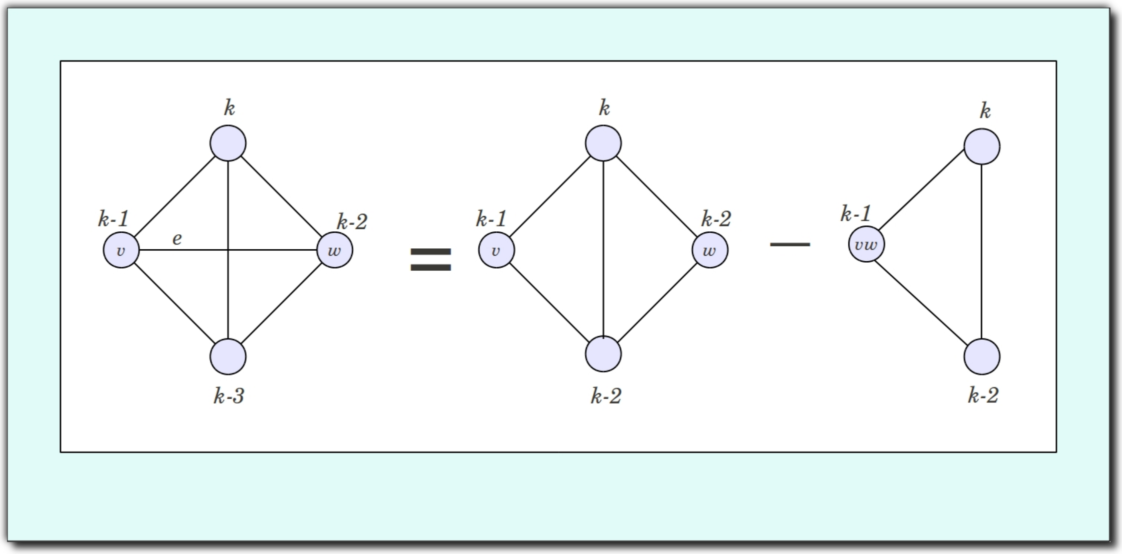


Figure 5.1.3. The recursion formula

process of deleting and contracting edges, as an equation. Once this process has reduced the graph sufficiently into the corresponding smaller graphs (trees, or triangles ideally) we can then label the separate graphs vertices with the $k$ ways to color it, and then write it as an equation. A basic visual representation of this procedure is provided in Figure 5.1.3, which utilizes the same graphs as in Figure 5.1.2 for convenience; however, it is important to note that in this process parallel edges are omitted. From the resulting process using Corollary 5.1.4 for the graph in 5.1.3 we find the resulting equation:

$$k(k-1)(k-2)(k-3) = [k(k-1)(k-2)^2] - [k(k-1)(k-2)]$$

Since this process can quickly become quite tedious, it is important to note that if one knows the chromatic function of a graph such as a tree or even or odd cycle, or a triangle, it is enough to reduce each graph to one of these. The above procedure can be very time consuming and complicated, as will be shown in the following example. Thus, one can save some time and effort by reducing to known graphs, as opposed to reducing all the way down to the null graph.

In order to get an enhanced understanding of how the procedure implementing Theorem 5.1.3 works on a more complicated graph, we proceed with an example using the complete graph $K_6$. We choose $K_6$ because we know by Proposition 3.7.1 that $\chi(K_6) = 6$. Thus, the chromatic polynomial should yield the same result.

**Example 5.1.5.** For purpose of this example we consider the graph $K_6$, as depicted in Figure 5.1.4.

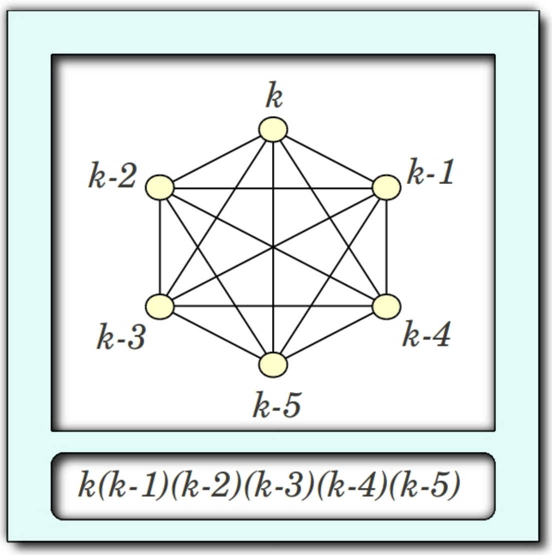


Figure 5.1.4. A $k$ coloring of the complete graph $K_6$

We determine the chromatic polynomial of $K_6$ as

$$\begin{aligned} P_{K_6}(k) &= k(k-1)(k-2)(k-3)(k-4)(k-5) \\ &= k^6 - ak^5 + bk^4 - ck^3 + dk^2 - ek, \end{aligned} \tag{5.1.2}$$

where $a,\ b,\ c,\ d$ and $e$ are positive constants. We can deduce the value of $a$ by a property of chromatic polynomials. That is, we know that $a = \binom{6}{2} = \frac{5 \cdot 6}{2} = 15$, which is just the number of edges in our graph. Although the chromatic polynomial of certain graphs, such as $K_6$, are easy to attain, there are cases in which ambiguity arises. In these cases, we must use the recursion formula to derive the polynomial. We proceed to use $K_6$ to show the process of "decomposing" a graph into smaller, more convenient ones in order to determine the polynomial.

What proceeds is the decomposition of $K_6$ through the process of constructing a chromatic polynomial.[1]

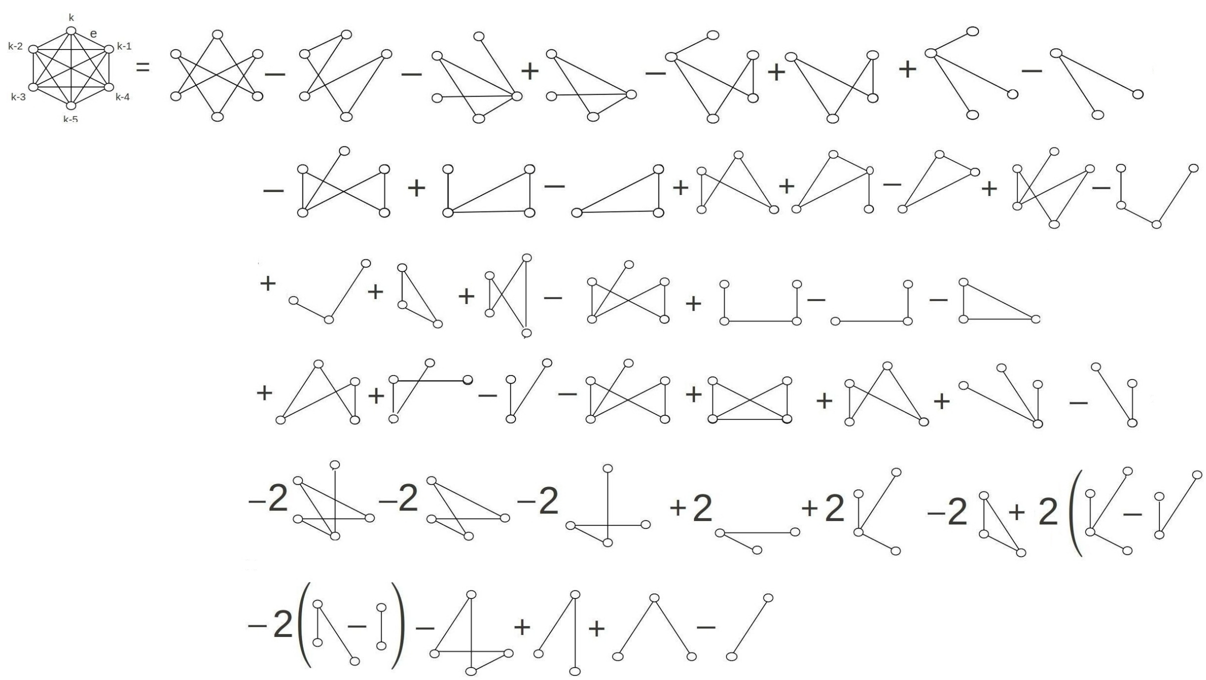


Figure 5.1.5. Edited recursion formula results for $K_6$

[1]The entire decomposition process to obtain the chromatic polynomial can be found in Appendix A.

Note that what was originally a problem of determining a chromatic polynomial has now turned into the problem of keeping track of many smaller edge contractions/deletions (see Figure 5.1.5). Although the subsequent graphs have simpler chromatic polynomials, we nevertheless had to tediously expend much time and effort into getting these smaller graphs in the first place.

By Theorem 5.1.3, the sum of the chromatic polynomials of these smaller graphs must add up to $P_{K_6}(k)$. Therefore, we have

$$k(k-1)(k-2)(k-3)(k-4)(k-5) = k^6 - 15k^5 + 85k^4 - 225k^3 + 274k^2 - 120k.$$

Thus, we have the form we were looking for in equation 5.1.2 establishing that for example $K_6$. Note that $a = 15$.

Using $P_{K_6}(k)$, we see that $\chi(K_6) = 6$ because this is the smallest number $k$ such that $P_{K_6}(k) \neq 0$.

## 5.2 Applying the chromatic polynomial the parachute graphs

**Example 5.2.1.** We apply the process of finding a chromatic polynomial and Theorem 5.1.3 to prove that the parachute graph $\mathfrak{P}_0$ is 4-chromatic. The function of its chromatic polynomial is $P_{\mathfrak{P}_0}(k) = k(k-1)(k-2)^4(k-3)^4$, as depicted in Figure 5.2.1. Since 4 is the smallest number such that $P_{\mathfrak{P}_0}(k) \neq 0$, $\chi(\mathfrak{P}_0) = 4$.

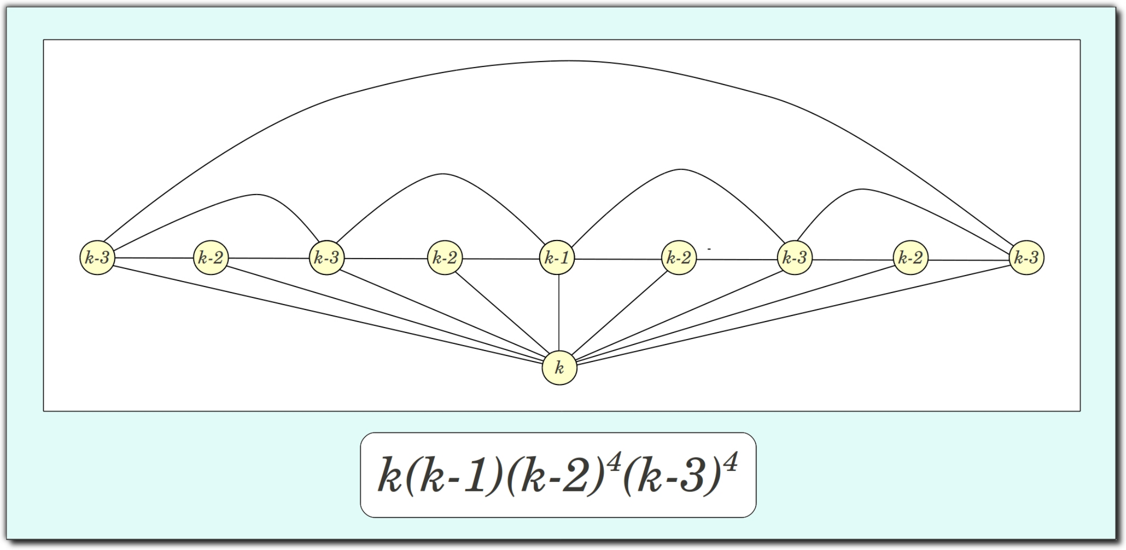


Figure 5.2.1. Applying a $k$ coloring to the parachute graph $\mathfrak{P}_0$

As we can see, the chromatic polynomial is a powerful tool to locate the chromatic number of any graph. Since Hadwiger's Conjecture relies on having the chromatic n of the graph in question, the chromatic polynomial may provide a manner in which to take Hadwiger's conjecture beyond the limit for which it has been proven (graphs with $\chi(G) \leq 6$).

# 6
# Conclusion

In this project we have explored Hadwiger's Conjecture in great detail and provided original proofs that did not rely on the Four Color Theorem. Hadwiger's Conjecture is proven for graphs with $\chi(G) \leq 6$, assuming the Four Color Theorem [10]. Graphs with a chromatic number of 6 or greater (that are not complete graphs) are difficult to find. This being the case, since we wanted to expand the reach of Hadwiger's conjecture in this project, we created two new graphs, established the behavior of their chromatic numbers as vertices are added, and saw that, as their chromatic number exceeded 6, Hadwiger's Conjecture held on every example of these new graphs.

The butterfly graph was constructed so that we could create an $n$-chromatic non-complete graph whose chromatic number increased as we added vertices. We achieved this goal with the butterfly graph.

The goal in the construction of the parachute graph was to craft a graph that had a chromatic number that increased as additional vertices were added such that it did not form a proper subgraph of the same chromatic number. The existence of such a graph is not known, and would have been a substantial discovery. Despite the unyielding effort and determination put to meet this end through the creation of the parachute graph and even a modification of the parachute graph, we were unsuccessful. The construction of a graph with our desired properties remained elusive. However, the effort was by no means fruitless, for we created another unique $n$-chromatic non-complete graph whose chromatic number increases as we add vertices.

This graph may be discovered with further research, and the reader is encouraged to further the efforts in this project with such a discovery.

# Appendix A

## Entire recursion formula for $K_6$

The following is the total recursion formula to reduce the graph $K_6$ into trees, cycles or triangles.

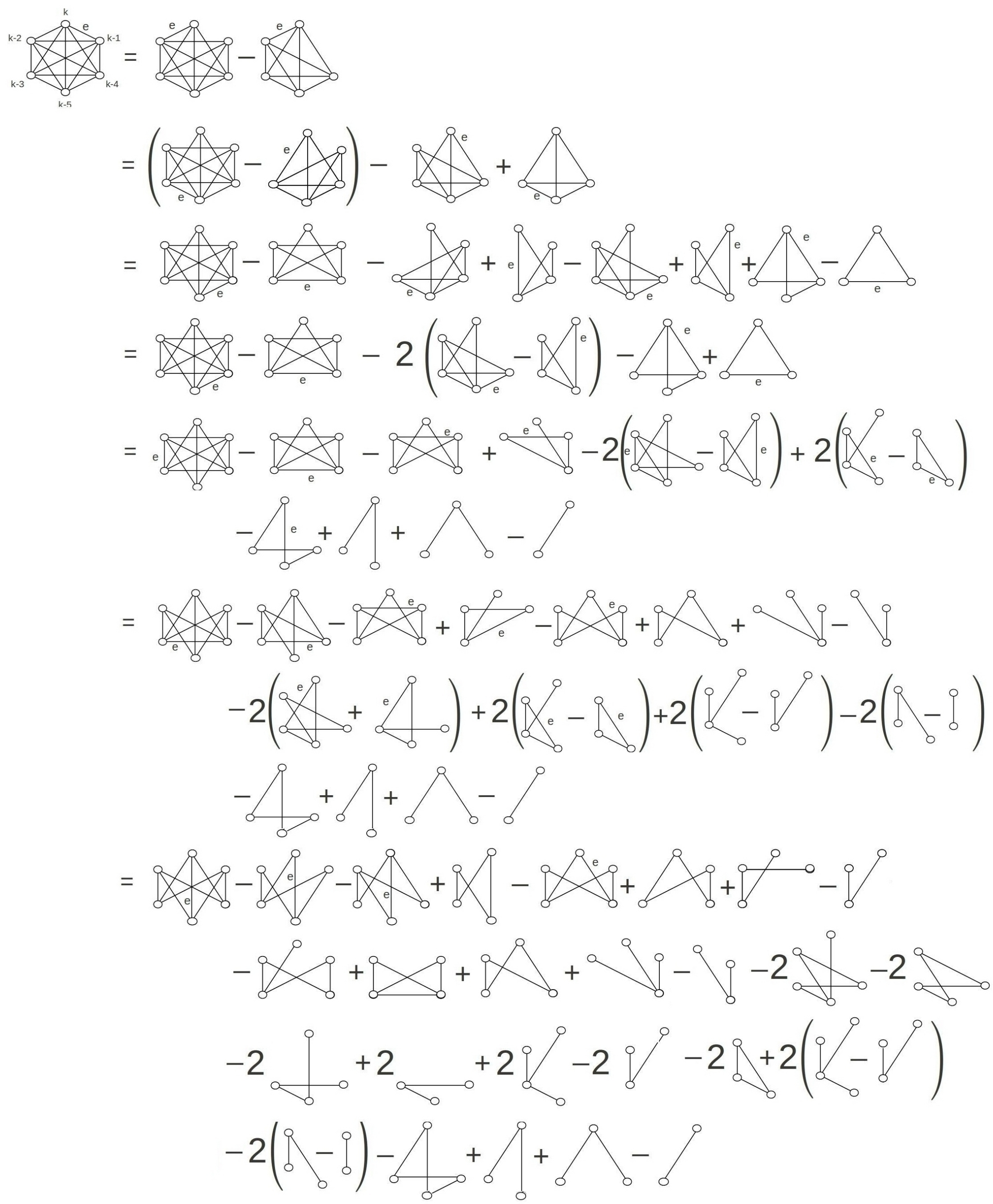

Figure A.0.1.

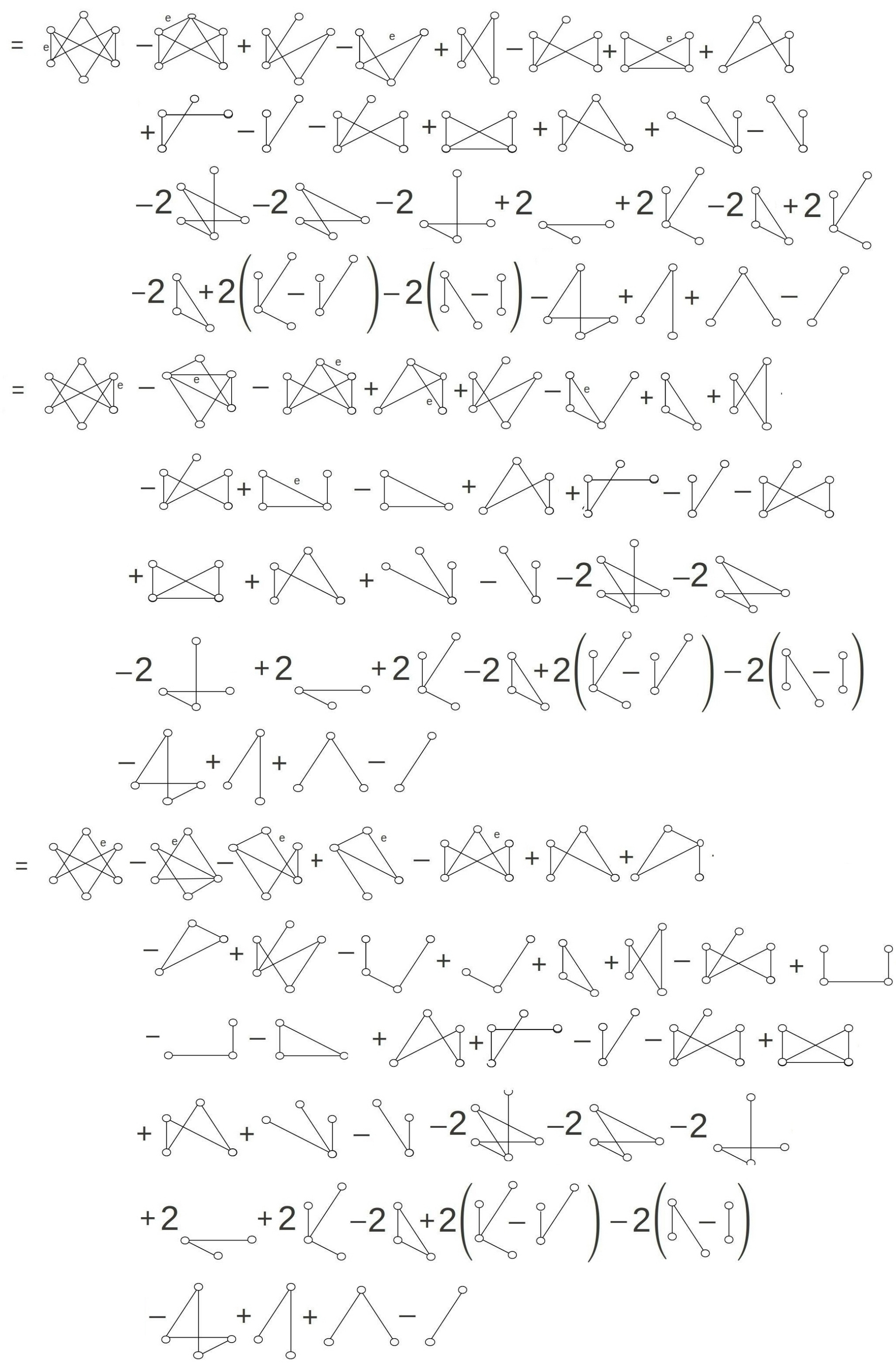

Figure A.0.2.

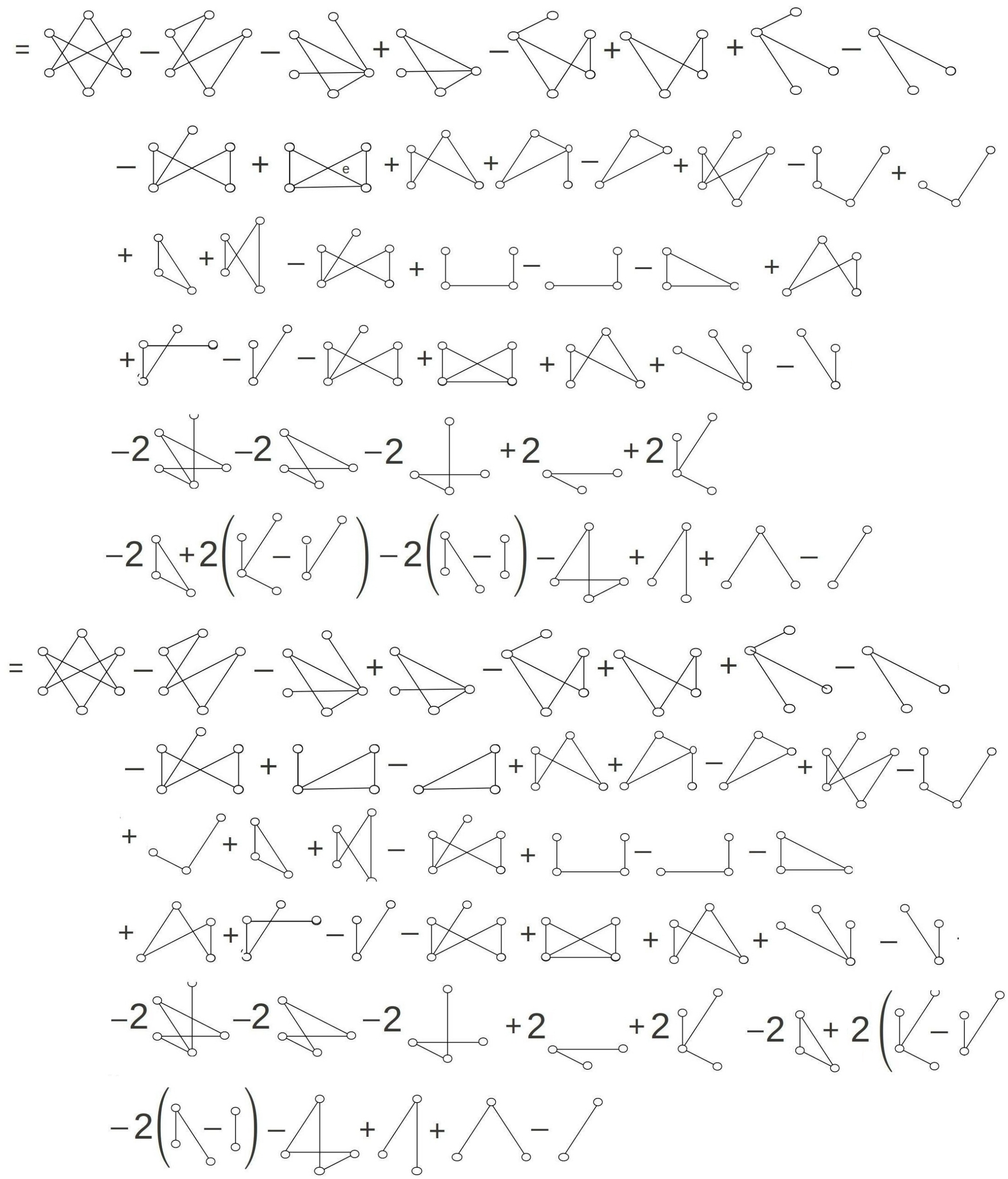

Figure A.0.3.

# Appendix B
## Mathematical Artwork

The following illustration is a product of my low level mastery of the graphics software used to create the graphics in this project. This illustration is used as the background in for my project poster. It was inspired by my fascination with the Golden ratio and the Fibonacci sequence. The image is built on a blueprint composed of rotating four overlapped images of a Fibonacci tiling with logarithmic spiral that corresponds to the Golden ratio inside. My motivation was to give the illusion of light passing through a stained glass window. The light represents the beauty of pure mathematics that pervades our known universe, and the window represents humanity's attempts to define and understand this beauty.

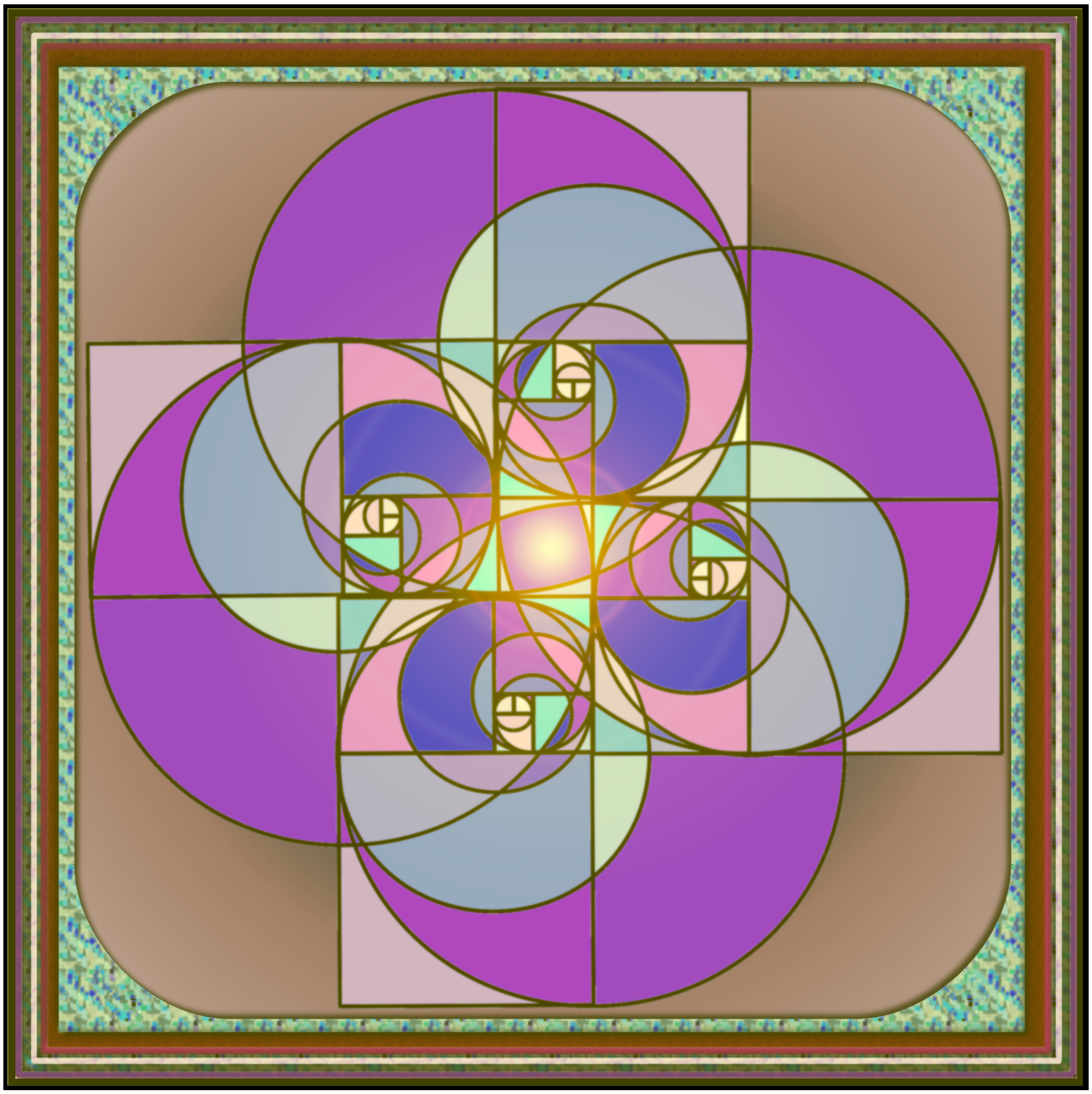

Figure B.0.1.

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