6 Problems: Vector Spaces

1. Check that $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\} = \mathbb{R}^2$ with the usual addition and scalar multiplication is a vector space.

2. Check that the complex numbers $\mathbb{C} = \{x + iy | x, y \in \mathbb{R}\}$ form a vector space over $\mathbb{C}$. Make sure you state carefully what your rules for vector addition and scalar multiplication are. Also, explain what would happen if you used $\mathbb{R}$ as the base field (try comparing to problem 1).

3. (a) Consider the set of convergent sequences, with the same addition and scalar multiplication that we defined for the space of sequences:

$$V = \left\{ f | f \colon \mathbb{N} \to \mathbb{R}, \lim_{n \to \infty} f \in \mathbb{R} \right\}$$

Is this still a vector space? Explain why or why not.

(b) Now consider the set of divergent sequences, with the same addition and scalar multiplication as before:

$$V = \left\{ f | f \colon \mathbb{N} \to \mathbb{R}, \lim_{n \to \infty} f \text{ does not exist or is } \pm\infty \right\}$$

Is this a vector space? Explain why or why not.

4. Consider the set of 2×4 matrices:

$$V = \left\{ \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} | a, b, c, d, e, f, g, h \in \mathbb{C} \right\}$$

Propose definitions for addition and scalar multiplication in V. Identify the zero vector in V, and check that every matrix has an additive inverse.

5. Let $P_3^{\mathbb{R}}$ be the set of polynomials with real coefficients of degree three or less.

- Propose a definition of addition and scalar multiplication to make $P_3^{\mathbb{R}}$ a vector space.
- Identify the zero vector, and find the additive inverse for the vector $-3 - 2x + x^2$.
- Show that $P_3^{\mathbb{R}}$ is not a vector space over $\mathbb{C}$. Propose a small change to the definition of $P_3^{\mathbb{R}}$ to make it a vector space over $\mathbb{C}$.

Problem 5 hint