WEEK 2

SOLUTIONS

2-9-16

Week 2

Sec 9.4

THE SOLUTION SHOULD HAVE COMPLETE SENTENCES, BUT THE IDEA IS GOOD.

3. a) $\overrightarrow{PQ} = \langle -2, -4, 5 \rangle$

$\overrightarrow{PR} = \langle 1, 2, 1 \rangle$

b) $|\overrightarrow{PQ} \times \overrightarrow{PR}|$

$$\begin{vmatrix} i & j & k \\ -2 & -4 & 5 \\ 1 & 2 & 1 \end{vmatrix} = i \begin{vmatrix} -4 & 5 \\ 2 & 1 \end{vmatrix} - j \begin{vmatrix} -2 & 5 \\ 1 & 1 \end{vmatrix} + k \begin{vmatrix} -2 & -4 \\ 1 & 2 \end{vmatrix}$$

$= -14i + 7j + 0k \longrightarrow \langle -14, 7, 0 \rangle$

$= \sqrt{-14^2 + 7^2} =$

$= \sqrt{245}$

$$\boxed{Area_{\Delta} = \frac{\sqrt{245}}{2}}$$

c) $||n|| = \sqrt{245}$

Unit vector $= \dfrac{\vec{n}}{||n||} = \boxed{\left\langle -\frac{14}{\sqrt{245}}, \frac{7}{\sqrt{245}}, 0 \right\rangle}$

d) $\measuredangle PQR$

$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |\overrightarrow{PQ}| \cdot |\overrightarrow{PR}| \cdot \sin\theta$

$\sqrt{245} = \sqrt{45} \cdot \sqrt{6} \cdot \sin\theta$

$\boxed{\theta = 1.26}$

$3t = 2$

February 8, 2016

Sections 9.5

1) Let $P_1 = (1,2,-1)$ and $P_2 = (-2,1,-2)$, and let L be the line in $\mathbb{R}^3$ through P_1 and P_2.

a) What value of t makes $(x(t), y(t), z(t)) = P_1$? How about P_2?

$P_1 - P_2 = (1-(-2), 2-1, -1-(-2)) = \langle 3,1,1 \rangle$

$r(t) = (1,2,-1) + \langle 3,1,1 \rangle t$

$x(t) = 1+3t \quad y(t) = 2+t \quad z(t) = -1+t$

$\boxed{\text{When } t=0,\ (x(t), y(t), z(t)) = P_1}$

$-2 = 1+3t \quad 1 = 2+t \quad -2 = -1+t$

$\boxed{\text{When } t=-1,\ (x(t), y(t), z(t)) = P_2}$

b) What restrictions on the parameter t describes the line segment between the points?

When $t=0$, $r(t) = P_1$ and when $t=-1$ then $r(t) = P_2$

$\boxed{\text{Thus when } -1 \le t \le 0 \text{ describes the line segment between the points.}}$

c) What about the line segment (along the same line) from $(7,4,1)$ to $(-8,-1,-4)$?

$(7-(-8), 4-(-1), 1-(-4)) = \langle 15,5,5 \rangle$

$7 = 1+3t \quad 4 = 2+t \quad 1 = -1+t$

When $t=2$ $(x(t), y(t), z(t)) = (7,4,1)$

$-8 = 1+3t \quad -1 = 2+t \quad -4 = -1+t$

When $t=-3$ then $(x(t), y(t), z(t)) = (-8,-1,-4)$

$\boxed{\text{Thus } -3 \le t \le 2 \text{ describes the line segment between the points.}}$

d) Now, Consider a Segment that lies on a different line: parameterize the Segment that Connects $R=(4,-2,7)$ to $Q=(-11,4,27)$ S.t $t=0$ Corresponds to Q and $t=2$ to R.

$\vec{RQ} = (-11-4, 4-(-2), 27-7) = \langle -15, 6, 20 \rangle$

For $t=0$ to correspond to Q, then Q must be the point used in $r(t)$.

So $r(t) = (-11,4,27) + \langle -15, 6, 20 \rangle t$

$4 = -11 + 2x \qquad -2 = 4 + 2y \qquad 7 = 27 + 2z$

$x = 7.5 \qquad y = -3 \qquad -10$

$\boxed{r(t) = (-11,4,27) + \langle 7.5, -3, -10 \rangle t}$

Problem 2

(a) The equation for the first line is

$\vec{r}(s) = \langle x(s), y(s), z(s) \rangle = \langle 4-2s, -2+s, 1+3s \rangle$
$= \langle 4, -2, 1 \rangle + \langle -2s, s, 3s \rangle$
$= \langle 4, -2, 1 \rangle + s\langle -2, 1, 3 \rangle$

so the direction vector is $\langle -2, 1, 3 \rangle = \vec{u}$

(b) The second line passes through point $(-4, 2, 17)$ and has a direction vector $\vec{v} = \langle -2, 1, 5 \rangle$

$\Rightarrow$ Its vector equation is

$\vec{q}(t) = \langle -4, 2, 17 \rangle + t\langle -2, 1, 5 \rangle$
$= \langle -4, 2, 17 \rangle + \langle -2t, t, 5t \rangle$
$= \langle -4-2t, 2+t, 17+5t \rangle$

$\Rightarrow$ The parametric equations are

$x'(t) = -4-2t, \ y'(t) = 2+t, \ z'(t) = 17+5t$

(c) The points on the first line are of the form $(x(s), y(s), z(s))$ and those on the second line are of the form $(x'(t), y'(t), z'(t))$

To find the common points, we need to solve

$$\begin{cases} x'(t) = x(s) \\ y'(t) = y(s) \\ z'(t) = z(s) \end{cases} \Leftrightarrow \begin{cases} 4-2s = -4-2t \\ -2+s = 2+t \\ 1+3s = 17+5t \end{cases} \Leftrightarrow \begin{cases} 2t-2s = -8 \\ s = 4+t \\ 1+3s = 17+5t \end{cases}$$

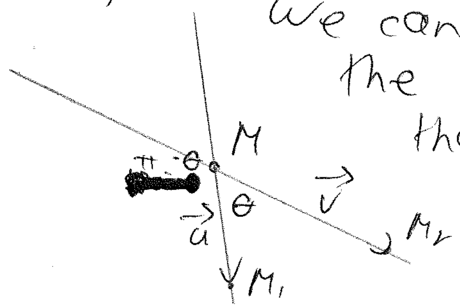
$$\Leftrightarrow \begin{cases} s = 4+t \\ s = 4+t \\ 1+3(4+t) = 17+5t \end{cases} \Leftrightarrow \begin{cases} s = 4+t \\ 1+12+3t = 17+5t \end{cases} \Leftrightarrow \begin{cases} s = 4+t \\ 2t = -4 \end{cases}$$

$$\Leftrightarrow \begin{cases} s = 4+(-2) \\ t = -2 \end{cases} \Leftrightarrow \begin{cases} s = 2 \\ t = -2 \end{cases}$$

The system of equations has a single solution $(s, t) = (2, -2)$ which implies the lines intersect at a single point p.6

(d) Let θ be the angle between the lines (we'll denote θ to be the smaller of the two complementary angles formed when the lines meet, which means that $0 \le \theta \le \frac{\pi}{2}$)

from (c) The lines meet at a single point - call it M. We can choose points M_1 and M_2 on the first and second line, respectively, such that $\overrightarrow{MM_1} = \vec{u}$ (the direction vector for the 1st line) and $\overrightarrow{MM_2} = \vec{v}$ (the direction vector for the 2nd line)



Clearly, the angle between $\vec{u}$ and $\vec{v}$ is either θ or $\pi - \theta$.

If $\measuredangle(\vec{u}, \vec{v}) = \pi - \theta$, then $\cos(\pi - \theta) = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$

and since $\cos(\pi - \theta) = -\cos\theta$

$\Rightarrow \cos\theta = -\dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$

If $\measuredangle(\vec{u}, \vec{v}) = \theta$, then $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$

So we always have that $\cos\theta = \left|\dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right|$

In our case, $\vec{u} = \langle -2, 1, 3 \rangle$ (from (a)) and $\vec{v} = \langle -2, 1, 5 \rangle$

$$\Rightarrow \cos\theta = \left|\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right| = \left|\frac{4+1+15}{\sqrt{4+1+9}\sqrt{4+1+25}}\right| = \left|\frac{20}{\sqrt{14}\cdot\sqrt{30}}\right|$$

$$= \frac{20}{2\sqrt{105}} = \frac{10}{\sqrt{105}}$$ and since $\theta \in [0, \frac{\pi}{2}]$

$$\Rightarrow \theta = \arccos(\cos\theta) = \arccos\frac{10}{\sqrt{105}}$$

$$\approx 0.220 \text{ rad}$$

SINCE THE LINES ARE IN THE PLANE,
YOU COULD USE ANY POINT ON EITHER LINE.

(e) A point on the plane is the point of intersection of the 2 lines.

From (c), the point is $(x(2), y(2), z(2)) = (4-4, -2+2, 1+6) = (0, 0, 7)$

The direction vectors for the two lines are, respectively, $\vec{u} = \langle -2, 1, 3 \rangle$ and $\vec{v} = \langle -2, 1, 5 \rangle$

Then, $\vec{u}$ and $\vec{v}$ line on the plane and $\vec{n} = \vec{u} \times \vec{v}$ is orthogonal to the plane

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 3 \\ -2 & 1 & 5 \end{vmatrix} = \hat{i}(5-3) - \hat{j}(-10+6) + \hat{k}(0) = 2\hat{i} + 4\hat{j}$$

$\Rightarrow \vec{n} = 2\hat{i} + 4\hat{j}$ and the equation of the plane

is $2(x-0) + 4(y-0) + 0 = 0$

$\Leftrightarrow 2x + 4y = 0$

$\Leftrightarrow x + 2y = 0$

(3) Plane 1: $4x - 5y + z = -2$

Plane 2: Passes through $(1,1,1)$, $(0,1,-1)$, and $(4,2,-1)$

a) vector normal to first plane: $\langle 4, -5, 1 \rangle$

b) scalar equation of second plane:

Let $P=(1,1,1)$, $Q=(0,1,-1)$ and $R=(4,2,-1)$

$\overrightarrow{QR}=\langle 4-0, 2-1, -1-(-1)\rangle=\langle 4,1,0\rangle$

$\overrightarrow{QP}=\langle 1-0, 1-1, 1-(-1)\rangle=\langle 1,0,2\rangle$

n = normal vector $\quad n_{plane}=\overrightarrow{QR}\times\overrightarrow{QP}$

$$=\begin{vmatrix} i & j & k \\ 4 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

$$=i(1(2)-0(0))-j(4(2)-0(1))+k(4(0)-1(1))$$

$$=2i-8j-k$$

$$=\langle 2,-8,-1\rangle$$

Scalar eq. of plane : $2(x-0)-8(y-1)-1(z+1)=0$

$$2x-8y+8-z-1=0$$

$$2x-8y-z=-7$$

c) $u\cdot v=|u||v|\cos\theta$

u = normal vector of plane 1

v = normal vector of plane 2

$$\langle 4,-5,1\rangle\cdot\langle 2,-8,-1\rangle=\left(\sqrt{4^2+(-5)^2+1^2}\right)\left(\sqrt{2^2+(-8)^2+(-1)^2}\right)\cos\theta$$

$$8+40-1=(\sqrt{42})(\sqrt{69})\cos\theta$$

$$47=\sqrt{2898}\cos\theta$$

$$\theta=29.2^\circ$$ OR 0.5 RAD

NOTICE DIFFERENT CONSTRUCTIONS CAN GIVE $\theta=150^\circ$ OR 2.63 RAD
THIS JUST MEANS ONE OF THE NORMAL VECTORS IS MULTIPLIED BY -1.

d) Plane 1 = $4x-5y+z=-2$

Plane 2 = $2x-8y-z=-7$

Set $x=0$ and solve:

$-5y+z=-2$

$z=-2+5y$

$-8y-z=-7$

$-8y+2-5y=-7$

$-4y=-7$

$y=\frac{7}{11}$

$z=-2-5\left(\frac{7}{11}\right)$

$=-2-\frac{35}{11}$

$=-\frac{22}{11}-\frac{35}{11}$

$=\frac{-57}{11}$

Point $\left(0,\frac{7}{11},-\frac{57}{11}\right)$ is on both planes

e) direction vector of plane 1 X direction vector plane 2 = direction vector of line

$\langle 4,-5,1\rangle\times\langle 2,-8,-1\rangle$

$$\begin{vmatrix} i & j & k \\ 4 & -5 & 1 \\ 2 & -8 & -1 \end{vmatrix}$$

$i(5+8)-j(-4-2)+k(-32+10)$

$$\boxed{13i+6j-22k}$$

f) Line of intersection of two planes : $r(t) = \langle 0, \frac{7}{11}, \frac{-57}{11} \rangle + \langle 13, 6, -22 \rangle t$

Parametric equations

$x(t) = 13t$

$y(t) = \frac{7}{11} + 6t$

$z(t) = -\frac{57}{11} - 22t$

4.) Let p be the plane with equation $z = -4x + 3y + 4$, and $Q = (4, -1, 8)$.

a.) Prove or disprove that Q lies on z.

$z = -4x + 3y + 4$, $Q = (4,-1,8)$

$8 = -4(4) + 3(-1) + 4$

$8 = -16 - 3 + 4$

$8 \neq -15$

No Q does not lie on the z plane.

b.) Find a normal vector to the plane p.

$z = -4x + 3y + 4$

$= -4x + 3y + 4 - z$

$-4 \qquad -4$

$-4 = -4x + 3y - z$

$4 = 4x - 3y + z$

$\boxed{\vec{n} = \langle 4, -3, 1 \rangle}$

THIS WORKS, BUT YOU COULD JUST DO WHAT WE DID IN CLASS. THAT IS LET $x = y = 0$ & SOLVE FOR z. SO $P = (0,0,4)$.

c.) Find a point on the plane.

$4x - 3y + z - 4 = 0$

$\vec{n} = \langle 4, -3, 1 \rangle$

$r(t) = O + tv$

$= \langle 0,0,0 \rangle + t\langle 4, -3, 1 \rangle$

P.E

$x = 4t = 8/13$

$y = -3t = -6/13$

$z = 1t = 2/13$

$4(4t) - 3(-3t) + (1t) - 4 = 0$

$16t + 9t + 1t - 4 = 0$

$26t - 4 = 0$

$26t = 4$

$t = 2/13$

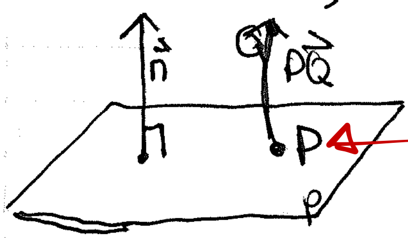
$\boxed{\text{Point} = (8/13, -6/13, 2/13)}$

NOTE: YOU WILL BE ASKED ABOUT THIS SETUP ON THE EXAM.

c.) Find the component $\vec{PQ}$. Draw what we've found thus far.

$P = \left(8/13, -6/13, 2/13\right)$ $Q = (4, -1, 8)$

$\vec{PQ}: \langle 44/13, 7/13, 102/13 \rangle$



JUST NOTE THAT $\vec{PQ}$ MAY NOT BE NORMAL TO THE PLANE.

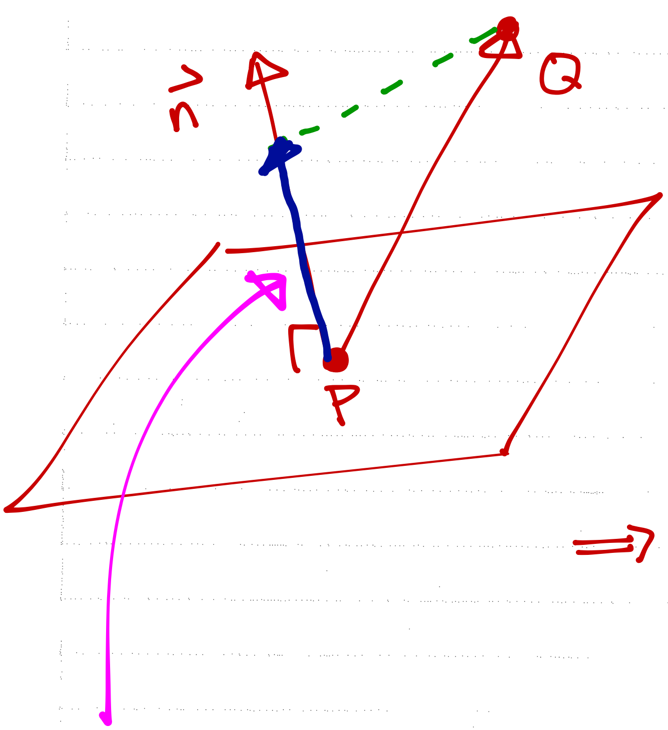
d) Calculate the distance from Q to p.

$4x - 3y + z - 4 = 0$

THE PURPOSE OF THIS PROBLEM IS TO SHOW DIST = |Comp|

FORMULAS DON'T HELP HERE.

$$d = \frac{|4(4) \cancel{-3(-1)} + (8) - 4|}{\sqrt{4^2 + \cancel{3^2} + 1^2}} = \frac{23}{\sqrt{26}} \approx \boxed{4.5}$$



$$\mathrm{Proj}_{\vec{n}} \vec{PQ} = \frac{\vec{PQ} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \cdot \vec{n}$$

$$= \frac{\frac{1}{13}(176 + 21 + 102)}{16 + 9 + 1} \cdot \langle 4, -3, 1 \rangle$$

$$= \frac{23}{26} \cdot \langle 4, -3, 1 \rangle$$

$$\Rightarrow \mathrm{DIST} = |\mathrm{Proj}| = \frac{23}{26} \cdot \sqrt{26} = |\mathrm{Comp}|$$

$$\approx 4.51$$

THE BLUE VECTOR IS $\mathrm{Proj}_{\vec{n}} \vec{PQ}$ TO FIND THE DISTANCE WE JUST FIND THE MAGNITUDE OF THIS.