8 Problems: Matrices

1. Compute the following matrix products

$$\begin{pmatrix} 1 & 2 & 1 \\ 4 & 5 & 2 \\ 7 & 8 & 2 \end{pmatrix} \begin{pmatrix} -2 & \frac{4}{3} & -\frac{1}{3} \\ 2 & -\frac{5}{3} & \frac{2}{3} \\ -1 & 2 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix},$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 1 \\ 4 & 5 & 2 \\ 7 & 8 & 2 \end{pmatrix} \begin{pmatrix} -2 & \frac{4}{3} & -\frac{1}{3} \\ 2 & -\frac{5}{3} & \frac{2}{3} \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 4 & 5 & 2 \\ 7 & 8 & 2 \end{pmatrix},$$

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \begin{pmatrix} 2 & 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -2 & \frac{4}{3} & -\frac{1}{3} \\ 2 & -\frac{5}{3} & \frac{2}{3} \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 4 & \frac{2}{3} & -\frac{2}{3} \\ 6 & \frac{5}{3} & -\frac{2}{3} \\ 12 & -\frac{16}{3} & \frac{10}{3} \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 4 & 5 & 2 \\ 7 & 8 & 2 \end{pmatrix}.$$

2. Let's prove the theorem $(MN)^T = N^T M^T$.

   Note: the following is a common technique for proving matrix identities.

   (a) Let $M = (m^i_j)$ and let $N = (n^i_j)$. Write out a few of the entries of each matrix in the form given at the beginning of this chapter.

   (b) Multiply out MN and write out a few of its entries in the same form as in part a. In terms of the entries of M and the entries of N, what is the entry in row i and column j of MN?

   (c) Take the transpose $(MN)^T$ and write out a few of its entries in the same form as in part a. In terms of the entries of M and the entries of N, what is the entry in row i and column j of $(MN)^T$?

   (d) Take the transposes N^T and M^T and write out a few of their entries in the same form as in part a.

   (e) Multiply out $N^T M^T$ and write out a few of its entries in the same form as in part a. In terms of the entries of M and the entries of N, what is the entry in row i and column j of $N^T M^T$?

   (f) Show that the answers you got in parts c and e are the same.

3. Let M be any $m \times n$ matrix. Show that $M^T M$ and MM^T are symmetric. (Hint: use the result of the previous problem.) What are their sizes?

4. Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ be column vectors. Show that the dot product $x \bullet y = x^T \ \mathbb{1} \ y$.

5. Above, we showed that *left* multiplication by an $r \times s$ matrix N was a linear transformation $M^s_k \xrightarrow{N} M^r_k$. Show that *right* multiplication by a $k \times m$ matrix R is a linear transformation $M^s_k \xrightarrow{R} M^s_m$. In other words, show that right matrix multiplication obeys linearity.

6. Explain what happens to a matrix when:

   (a) You multiply it on the left by a diagonal matrix.

   (b) You multiply it on the right by a diagonal matrix.

   Give a few simple examples before you start explaining.