WEEK 3

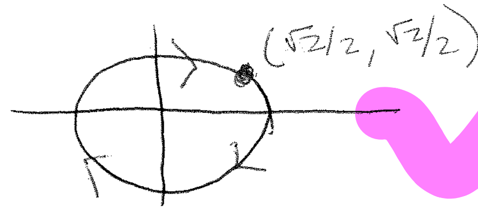
SOLUTIONS!

Weekly Assignment - February 17th, 2016

1) Section 9.6

(a)

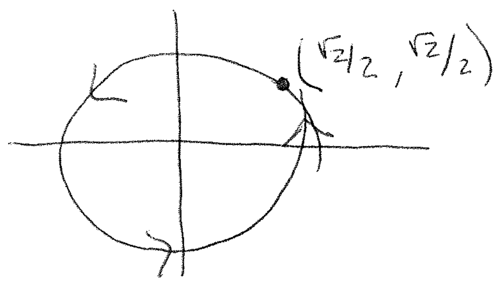
t	$r(t)$
0	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\pi/4$	$(1,0)$
$\pi/2$	$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$3\pi/4$	$(0,-1)$
π	$\left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$



$$\boxed{r(t) = \langle \sin(t + \pi/4), \cos(t + \pi/4) \rangle}$$

(b)

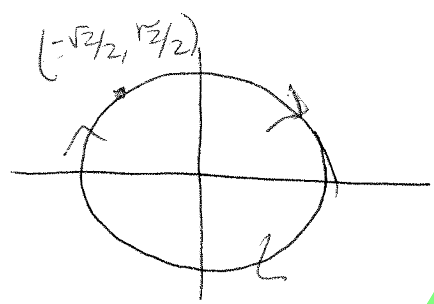
t	$r(t)$
0	$(\sqrt{2}/2, \sqrt{2}/2)$
$\pi/4$	$(0,1)$
$\pi/2$	$(-\sqrt{2}/2, \sqrt{2}/2)$
$3\pi/4$	$(-1,0)$



$$\boxed{r(t) = \langle \cos(t + \pi/4), \sin(t + \pi/4) \rangle}$$

(c)

t	$r(t)$
0	$(-\sqrt{2}/2, \sqrt{2}/2)$
$\pi/4$	$(0,1)$
$\pi/2$	$(\sqrt{2}/2, \sqrt{2}/2)$
$3\pi/4$	$(1,0)$
π	$(\sqrt{2}/2, -\sqrt{2}/2)$



$$\boxed{r(t) = \langle \cos(3\pi/4 - t), \sin(3\pi/4 - t) \rangle}$$

PLUG COMPONENTS INTO EQUATION AND SHOW EQUALITY HOLDS.

$$\frac{(a\cos t)^2}{a^2} + \frac{(b\sin t)^2}{b^2} = 1$$

a) Explain why the vector function defined by $r(t) = \langle a\cos t, b\sin t\rangle$, $0 \le t \le 2\pi$ is one parameterization of the ellipse $x^2/a^2 + y^2/b^2 = 1$

(h,k) is zero, therefore the ellipse would be centered at zero. The horizontal length is 2a, thus the horizontal "radius" would be a. Same applies for the vertical length and b. The given vector function results in a unit circle centered at zero. However, the horizontal radius is a and the vertical radius is b (since cos and sin are multiplied by a and b, respectively) Thus, the ellipse and vector function r(t) are the same.

b) Find a parameterization of the ellipse $x^2/4 + y^2/16 = 1$

$\boxed{r(t) = \langle 2\cos t, 4\sin t\rangle}$, $a^2 = 4$, $a = 2$. $b^2 = 16$, $b = 4$

c) Find a parameterization of the ellipse

$$\frac{(x+3)^2}{4} + \frac{(y-2)^2}{9} = 1.$$

Center = (-3, 2), a = 2, b = 3

$\boxed{r(t) = \langle 2\cos t - 3,\ 3\sin t + 2\rangle}$

d) Determine the x-y equation of the ellipse that is parameterized by $r(t) = \langle 3 + 4\sin(2t), 1 + 3\cos(2t)\rangle$.

Center (3, 1); vertical radius = 3 horizontal r = 4

$$\boxed{\frac{(x-3)^2}{16} + \frac{(y-1)^2}{9} = 1}$$

3.) Consider the two-variable functions $z = f(x,y)$
$= 3x^2 + 4y^2 - 2$.

a.) Determine a vector valued function r that para. the curve x=2 trace of $z = f(x,y)$. Do likewise for x = -2, -1, 0, and 1.

Let $x=2$

$z = f(2,y) = 3(2^2) + 4y^2 - 2$
$= 10 + 4y$

Parameterize y to = t

$\boxed{\vec{r}(t) = \langle t, 10 + 4t^2 \rangle}$

$x(t) = t$

$y(t) = 10 + 4t^2$

Let $x = -2$

$z = f(-2,y) = 3(-2^2) + 4y^2 + 2$
$= 16 + 4y^2$

~~Para. y to = t~~

$\boxed{r(t) = \langle t, 16 + 4t^2 \rangle}$

$x(t) = t$

$y(t) = 16 + 4t^2$

Let $x = -1$

$z = f(-1,y) = 3(-1)^2 + 4y^2 - 2$
$= 4y^2 + 1$

let $y = t$

$\boxed{\vec{r}(t) = \langle t, 4t^2 + 1 \rangle}$

Let $x = 0$

$z = f(0,y) = 3(0)^2 + 4y^2 - 2$
$= 4y^2 - 2$

let $y = t$

$\boxed{\vec{r}(t) = \langle t, 4t^2 - 2 \rangle}$

(4)

3b) Determine a v.v.f. r that parameterizes the curve y=2 trace of $z=f(x,y)$. Do likewise for $y=-2,-1,0$, and 1.

let $y=2$

$z=f(x,2)=3x^2+4(2^2)-2$
$=3x^2+14$

let $x=t$

$\boxed{\vec{r}(t)=\langle 3t^2+14, t\rangle}$

let $y=1$

$z=f(x,1)=3x^2+4(1^2)-2$
$=3x^2+2$

let $x=t$

$\boxed{\vec{r}(t)=\langle 3t^2+2, t\rangle}$

let $y=0$

$z=f(x,0)=3x^2+0-2$
$=3x^2-2$

let $x=t$

$\boxed{\vec{r}(t)=\langle 3t^2-2, t\rangle}$

let $y=-1$

$z=f(x,-1)=3x^2+1-2$
$=3x^2-1$

$\boxed{\vec{r}(t)=\langle 3t^2-1, t\rangle}$

c) Determine a v.v.f. r that parameterizes the curve $z=2$ contour of $z=f(x,y)$. Do likewise for $z=-2,-1,0$, and 1.

let $z=2$

$2=3x^2+4y^2-2$

$\frac{3x^2}{4}+\frac{4y^2}{4}=\frac{4}{4}$

$\frac{x^2}{(4/3)}+\frac{y^2}{(1)}=1$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \xrightarrow{\text{parameterize}} r(t)=\langle a\cos t, b\sin t\rangle$

$\boxed{r(t)=\langle \sqrt{4/3}\cos(t), 1\sin(t)\rangle}$

· Above is an eq. of an ellipse centered at the origin.

let $z=1$

$1=3x^2+4y^2-2$

$3=3x^2+4y^2$

$1=x^2+\frac{y^2}{(3/4)}$

$\boxed{r(t)=\langle \sqrt{1}\cos(t), \sqrt{3/4}\sin(t)\rangle}$

let $z=0$

$$0 = 3x^2 + 4y^2 - 2$$

$$\frac{2}{2} = \frac{3x^2}{2} + \frac{4y^2}{2}$$

$$1 = \frac{x^2}{(2/3)} + \frac{y^2}{(1/2)}$$

$$\boxed{\left\langle \sqrt{2/3}\cos(t), \sqrt{1/2}\sin(t) \right\rangle}$$

let $z=-1$

$$-1 = 3x^2 + 4y^2 - 2$$

$$1 = 3x^2 + 4y^2$$

$$1 = \frac{x^2}{(1/3)} + \frac{y^2}{(1/4)}$$

$$\boxed{\left\langle \sqrt{1/3}\cos(t), \sqrt{1/4}\sin(t) \right\rangle}$$

let $z=-2$

$$-2 = 3x^2 + 4y - 2$$

$$0 = 3x^2 + 4y^2$$

radius of zero is indicative of a single point

d.

It will be a bowl. The edges go up infinitely.

(4) a) Find a vector-valued function r that parameterizes the line through $(-2,1,4)$ in the direction of the vector $v=\langle 3,2,-5\rangle$

$r(t)=\langle -2,1,4\rangle+\langle 3,2,-5\rangle t$

$r(t)=\langle 3t-2,\ 2t+1,\ -5t+4\rangle$

b) Find a vector-valued function r that parameterizes the line of intersection of the planes $x+2y-z=4$ and $3x+y-2z=1$

normal vectors u and v, are $\langle 1,2,-1\rangle$ and $\langle 3,1,-2\rangle$

$$u \times v: \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 3 & 1 & -2 \end{vmatrix}$$

$$i\big(2(-2)-(-1)(1)\big)-j\big(1(-2)-(-1)(3)\big)+k\big(1(1)-3(2)\big)$$

$$i(-4+1)-j(-2+3)+k(1-6)$$

direction vector $= -3i-j-5k$

$r(t)=\langle 0,4,\tfrac{3}{2}\rangle+\langle -3,-1,-5\rangle t$

Point on line: $x+2y-z=4$ $\quad$ $3x+y-2z=1$

Let $x=0$

$2y-z=4$ $\quad$ $y-2z=1$

$2(1+2z)-z=4 \quad y=1+2z$

$2+4z-z=4$ $\quad$ $y=1+2(\tfrac{3}{2})$

$3z=2$ $\quad$ $y=1+3$

$z=3/2$

$y=4$

vector-valued function $r(t)$ for line of intersection of these two planes $=$

$\langle -3t,\ 4-t,\ \tfrac{3}{2}-5t\rangle$

Find Point of intersection:

c) $x = 2+3t$ $\quad x = 3+1s$

$y = 1-2t$ $\quad y = 3-2s$

$z = 4t$ $\quad z = 2s$

Point of intersection: $\langle 5, -1, 4 \rangle$

$3+1s = 2+3t$

$1+s = 3t$

$s = 3t-1$

$s = 3(1)-1$

$s = 2$

$1-2t = 3-2s$

$1-2t = 3-2(3t-1)$

$1-2t = 3-6t+2$

$1+4t = 5$

$4t = 4$

$t = 1$

$x = 2+3t$ $\quad x = 3+s$

$x = 2+3(1)$ $\quad x = 3+2$

$x = 5$ $\quad x = 5$

$y = 1-2(1)$ $\quad z = 4(1)$

$y = -1$ $\quad z = 4$

Vector valued function that parameterizes the line passing through this point & is perpendicular to both lines:

direction vector $= u \times v$

$= \langle 3, -2, 4 \rangle \times \langle 1, -2, 2 \rangle$

$= \begin{vmatrix} i & j & k \\ 3 & -2 & 4 \\ 1 & -2 & 2 \end{vmatrix}$

$= i(-2(2) - 4(-2)) - j(3(2) - 4(1)) + k(3(-2) - (-2)(1))$

$= i(4) - j(2) + k(-4)$

$r(w) = \langle 5+4w, -1-2w, 4-4w \rangle$

Section 9.7

2) Consider the two vector-valued functions given by

$r(t) = \langle t+1, \cos(\frac{\pi}{2}t), \frac{1}{1+t} \rangle$

$w(s) = \langle s^2, \sin(\frac{\pi}{2}s), s \rangle$

a) Determine the point of intersection of the curves generated by r(t) and w(s).

$x = t+1 = s^2, \ t = s^2 - 1$

$\frac{1}{1+(s^2-1)} = s \rightarrow \frac{1}{s^2} = s \rightarrow 1 = s^2 \cdot s \rightarrow 1 = s^3 \rightarrow s = 1$

$t = (1)^2 - 1 \rightarrow t = 0$

Check: $x = (0+1) = 1^2 = 1$ ✓, $\frac{1}{1+0} = 1 = 1$ ✓, $\cos(0) = \sin(\pi/2)$ ✓

$\boxed{\text{Point of intersection: } (1,1,1)}$

b) Use the value of a to find a vector form of the tangent line to $\vec{r}(t)$ at the point where $t=a$.

$$\vec{r}'(t) = \left\langle 1, -\frac{1}{2}\pi \sin\left(\frac{\pi t}{2}\right), -\frac{1}{(t+1)^2} \right\rangle$$

$\boxed{\vec{r}'(0) = \langle 1, 0, -1 \rangle}$

c) Use the value of b you determined to find a vector form of the tangent line to w(s) where s=b.

$$\vec{w}'(s) = \left\langle 2s, \frac{1}{2}\pi \cos\left(\frac{\pi s}{2}\right), 1 \right\rangle$$

$\vec{w}'(1) = \langle 2(1), \frac{1}{2}\pi \cos(\frac{\pi}{2}), 1 \rangle$

$\boxed{\vec{w}'(1) = \langle 2, 0, 1 \rangle}$

d) Use your preceding work to determine the equations of the tangent plane.

$$\begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 2 & 0 & 1 \end{vmatrix} = i(0) - j(3) + k(0)$$

normal vector: $\langle 0, -3, 0 \rangle$ (to plane)

$\boxed{\text{Tangent plane } 0(x-1) - 3(y-1) + 0(z-1) = 0}$

3) In this excercise, we determine the equation of a plane tangent to the surface defined by $f(x,y)=\sqrt{x^2+y^2}$ at the point $(3,4,5)$.

a) Find a parameterization for the $x=3$ trace of f.

$\boxed{\vec{r}(t)=\langle 3, t, \sqrt{9+t^2}\rangle}$

What is the direction vector for the line tangent to this trace at point $(3,4,5)$?

$\vec{r}'(t)=\langle 0, 1, t/\sqrt{t^2+9}\rangle$

$\vec{r}'(4)=\langle 0, 1, 4/\sqrt{16+9}\rangle = \boxed{\langle 0, 1, 4/5\rangle = \text{direction vector}}$

b) Find a parameterization for the $y=4$ trace of f. Find direction vector for point $(3,4,5)$

$\boxed{\vec{r}(t)=\langle t, 4, \sqrt{t^2+16}\rangle}$

$\vec{r}'(t)=\langle 1, 0, t/\sqrt{t^2+16}\rangle$

$\vec{r}'(3)=\langle 1, 0, 3/\sqrt{25}\rangle = \boxed{\langle 1, 0, 3/5\rangle = \text{direction vector}}$

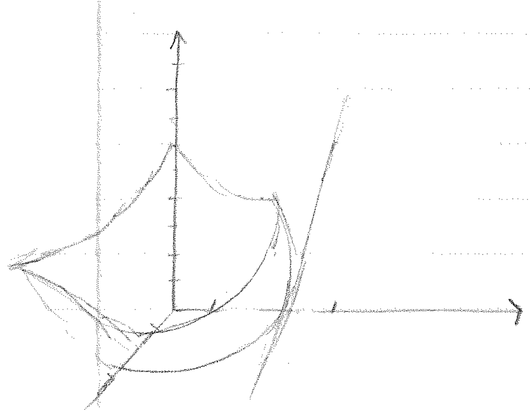
c) What is a normal vector for the plane containing $(3,4,5)$ with direction vectors a and b.

$$\begin{array}{ccc} i & j & k \\ 0 & 1 & 4/5 \\ 1 & 0 & 3/5 \end{array} = i(3/5) - j(4/5) + k(-1)$$

$\boxed{\text{normal vector}: \langle 3/5, 4/5, -1\rangle}$

d) Write an equation for the tangent plane. Graph f and the plane on the same axes.

$\boxed{3/5(x-3)+4/5(y-4)-1(z-5)=0}$



5) In this excercise, we develop the formula for the position function of a projectile that has been launched at an initial speed $|V_0|$ and a launch angle of θ. Recall that $a(t) = \langle 0, -g \rangle$ is the constant acceleration.

a) Find all velocity vectors for the given acceleration vector a.

$\vec{v}(t) = \int a(t)\,dt = \boxed{\langle C, -gt + D \rangle = \vec{v}(t)}$

b) Use the given information to find V_0, the initial velocity of the projectile.

So $\vec{v}(0) = \langle C, D \rangle = V_0$

$x = \vec{v}_0 \cos\theta$, $y = \vec{v}_0 \sin\theta$

c) Find the specific velocity vector function $v(t)$ for the projectile. That is, combine your work from part a and b.

$\boxed{\vec{v}(t) = \langle |V_0| \cos\theta, |V_0| \sin\theta - gt \rangle}$

d) Find all possible position vectors for the velocity vector.

$\boxed{\vec{r}(t) = \int \vec{v}(t) = \langle |V_0| \cos\theta\, t + C, |V_0| \sin\theta\, t - \frac{gt^2}{2} + D \rangle}$

Section 9.8

3) Consider the single variable function $y=4x^2-x^3$

a) Find a parameterization of the form $r(t)=\langle x(t),y(t)\rangle$ that traces the curve $y=4x^2-x^3$.

$\boxed{r(t)=\langle t, 4t^2-t^3\rangle}$

b) Write a definite integral which, if evaluated, gives the exact length of the given curve from $x=-3$ to $x=3$. Why is the integral difficult to evaluate?

$$\boxed{L=\int_{-3}^{3}|r'(t)| = \int_{-3}^{3}|\langle 1, 8t-3t^2\rangle|}$$

I'm assuming the integral would be difficult to evaluate exactly because of the radical involved with finding the length.

c) Determine the curvature, $K(t)$, of the parameterized curve.

$$T(t)=\frac{r'(t)}{|r'(t)|}=\frac{\langle 1, 8t-3t\rangle}{\sqrt{(1)^2+(8t-3t)^2}}=\frac{\langle 1, 8t-3t\rangle}{\sqrt{25t^2+1}}$$

Curve $=\langle t, 4t^2-t^3\rangle$

$r'(t)=\langle 1, 8t-3t^2\rangle$ $\quad r''(t)=\langle 0, 8-6t\rangle$

$$K(t)=\frac{\|r'(t)\times r''(t)\|}{\|r'(t)\|^3}$$

$$=\frac{\|\langle 1, 8t-3t^2\rangle\times\langle 0, 8-6t\rangle\|}{\|\langle 1, 8t-3t^2\rangle\|^3}=\frac{\|\langle 8-6t\rangle\|}{\|\langle 1, 8t-3t^2\rangle\|^3}$$

$$=\frac{\sqrt{(8-6t)^2}}{\left(\sqrt{1+(8t-3t^2)}\right)^3}=\boxed{\frac{2\cdot|3t-4|}{(-3t^2+8t+1)^{3/2}}=K(t)}$$

d) Compare absolute maximum and minimum.

$y=4x^2-x^3$ max $=(2.6, 9.4)$ min $(0,0)$

$K(t)$ max $=(0,8)$, min $(1, .136)$

4) Consider the standard helix parameterized by $r(t) = \cos(t)\,i + \sin(t)\,j + t\,k$.

a) Find $T(t)$.

$$r'(t) = -\sin t\, i + \cos t\, j + 1k$$

$$\|r'(t)\| = \sqrt{-\sin^2 t + \cos^2 t + 1^2}$$

$$= \sqrt{2\cos(t)^2}$$

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \boxed{\frac{-\sin t\, i + \cos t\, j + k}{\sqrt{2\cos(t)^2}} = T(t)}$$

b) Explain why the fact that $|T(t)| = 1$ implies that T and T' are orthogonal vectors.

$$T \cdot T = T^2 = 1$$

$$d/dt\,[T \cdot T] = T' * T + T * T' = 0$$

$$T \cdot T' = 0$$

c) For the given function $r(t)$ with unit tangent vector $T(t)$, determine $N(t) = \frac{1}{|T'(t)|} T'(t)$.

$$T'(t) = \tfrac{1}{2}\sec^2 t(-\sin(t) - \cos(t)) + \tan(t)\sec^2(t)(-\sin t + \cos t + 1$$

$$N(t) = \boxed{\frac{T'(t)}{|T'(t)|} = \frac{\tfrac{1}{2}\sec^2 t(-\sin t - \cos t) + \tan t \sec^2(t)(-\sin t + \cos t + 1)}{\sqrt{(\tfrac{1}{2}\sec^2 t(-\sin t - \cos t))^2 + (\tan t \sec^2(t)(-\sin t + \cos t + 1))^2}}}$$

d) What geometric properties does $N(t)$ have?

I'm not sure what properties it has. Finding the length of that vector would be very difficult.

e) Let $B(t) = T(t) \times N(t)$ and compute $B(t)$.

$$B(t) = \left(\frac{\tfrac{1}{2}\sec^2 t(-\sin(t) - \cos(t)) + \tan(t)\sec^2 t(-\sin t + \cos t + 1)}{\sqrt{\tfrac{1}{2}\sec^2 t(-\sin(t) - \cos(t))^2 + (\tan(t)\sec^2 t(-\sin t + \cos t + 1))^2}}\right)$$

$$\times \left(\frac{-\sin t\, i + \cos t\, j + k}{\sqrt{2\cos(t)^2}}\right) = B(t)$$

f) What geometric properties does $B(t)$ have?

I would assume this vector is longer than $N(t)$ and $B(t)$ because it is the product of them.

g) Sketch the helix.