12 Problems: Elementary Matrices and Determinants

1. Let $M = \begin{pmatrix} m^1_1 & m^1_2 & m^1_3 \\ m^2_1 & m^2_2 & m^2_3 \\ m^3_1 & m^3_2 & m^3_3 \end{pmatrix}$. Use row operations to put M into *row echelon form*. For simplicity, assume that $m^1_1 \neq 0 \neq m^1_1 m^2_2 - m^2_1 m^1_2$.

   Prove that M is non-singular if and only if:

$$m^1_1 m^2_2 m^3_3 - m^1_1 m^2_3 m^3_2 + m^1_2 m^2_3 m^3_1 - m^1_2 m^2_1 m^3_3 + m^1_3 m^2_1 m^3_2 - m^1_3 m^2_2 m^3_1 \neq 0$$

2. (a) What does the matrix $E^1_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ do to $M = \begin{pmatrix} a & b \\ d & c \end{pmatrix}$ under left multiplication? What about right multiplication?

   (b) Find elementary matrices $R^1(\lambda)$ and $R^2(\lambda)$ that respectively multiply rows 1 and 2 of M by λ but otherwise leave M the same under left multiplication.

   (c) Find a matrix $S^1_2(\lambda)$ that adds a multiple λ of row 2 to row 1 under left multiplication.

3. Let M be a matrix and $S^i_j M$ the same matrix with rows i and j switched. Explain every line of the series of equations proving that $\det M = -\det(S^i_j M)$.