

41. One method of slowing the growth of an insect population without using pesticides is to introduce into the population a number of sterile males that mate with fertile females but produce no offspring. If P represents the number of female insects in a population, S the number of sterile males introduced each generation, and r the population's natural growth rate, then the female population is related to time t by

$$t = \int \frac{P + S}{P[(r - 1)P - S]} dP$$

Suppose an insect population with 10,000 females grows at a rate of $r = 0.10$ and 900 sterile males are added. Evaluate the integral to give an equation relating the female population to time. (Note that the resulting equation can't be solved explicitly for P .)

42. The region under the curve

$$y = \frac{1}{x^2 + 3x + 2}$$

from $x = 0$ to $x = 1$ is rotated about the x -axis. Find the volume of the resulting solid.

- CAS** 43. (a) Use a computer algebra system to find the partial fraction decomposition of the function

$$f(x) = \frac{4x^3 - 27x^2 + 5x - 32}{30x^5 - 13x^4 + 50x^3 - 286x^2 - 299x - 70}$$

- (b) Use part (a) to find $\int f(x) dx$ (by hand) and compare with the result of using the CAS to integrate f directly. Comment on any discrepancy.

- CAS** 44. (a) Find the partial fraction decomposition of the function

$$f(x) = \frac{12x^5 - 7x^3 - 13x^2 + 8}{100x^6 - 80x^5 + 116x^4 - 80x^3 + 41x^2 - 20x + 4}$$

- (b) Use part (a) to find $\int f(x) dx$ and graph f and its indefinite integral on the same screen.

- (c) Use the graph of f to discover the main features of the graph of $\int f(x) dx$.

45. Suppose that F , G , and Q are polynomials and

$$\frac{F(x)}{Q(x)} = \frac{G(x)}{Q(x)}$$

for all x except when $Q(x) = 0$. Prove that $F(x) = G(x)$ for all x . [Hint: Use continuity.]

46. If f is a quadratic function such that $f(0) = 1$ and

$$\int \frac{f(x)}{x^2(x + 1)^3} dx$$

is a rational function, find the value of $f'(0)$.

H Polar Coordinates

Polar coordinates offer an alternative way of locating points in a plane. They are useful because, for certain types of regions and curves, polar coordinates provide very simple descriptions and equations. The principal applications of this idea occur in multivariable calculus: the evaluation of double integrals and the derivation of Kepler's laws of planetary motion.

H.1 Curves in Polar Coordinates

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. Usually we use Cartesian coordinates, which are directed distances from two perpendicular axes. Here we describe a coordinate system introduced by Newton, called the **polar coordinate system**, which is more convenient for many purposes.

We choose a point in the plane that is called the **pole** (or origin) and is labeled O . Then we draw a ray (half-line) starting at O called the **polar axis**. This axis is usually drawn horizontally to the right and corresponds to the positive x -axis in Cartesian coordinates.

If P is any other point in the plane, let r be the distance from O to P and let θ be the angle (usually measured in radians) between the polar axis and the line OP as in Figure 1. Then the point P is represented by the ordered pair (r, θ) and r, θ are called **polar coordinates** of P . We use the convention that an angle is positive if measured in the counterclock-

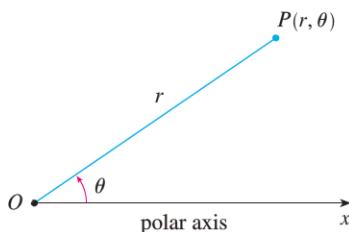


FIGURE 1

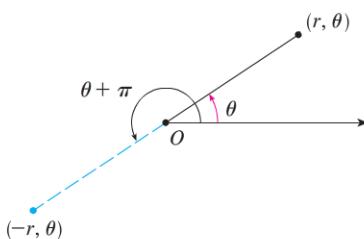


FIGURE 2

wise direction from the polar axis and negative in the clockwise direction. If $P = O$, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of θ .

We extend the meaning of polar coordinates (r, θ) to the case in which r is negative by agreeing that, as in Figure 2, the points $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance $|r|$ from O , but on opposite sides of O . If $r > 0$, the point (r, θ) lies in the same quadrant as θ ; if $r < 0$, it lies in the quadrant on the opposite side of the pole. Notice that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.

EXAMPLE 1 Plot the points whose polar coordinates are given.

- (a) $(1, 5\pi/4)$ (b) $(2, 3\pi)$ (c) $(2, -2\pi/3)$ (d) $(-3, 3\pi/4)$

SOLUTION The points are plotted in Figure 3. In part (d) the point $(-3, 3\pi/4)$ is located three units from the pole in the fourth quadrant because the angle $3\pi/4$ is in the second quadrant and $r = -3$ is negative.

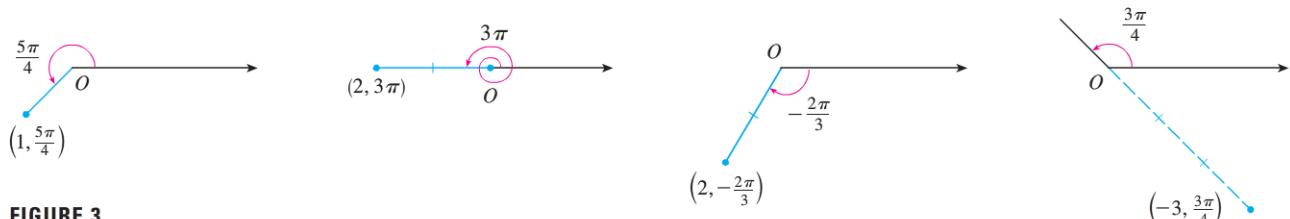


FIGURE 3



In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. For instance, the point $(1, 5\pi/4)$ in Example 1(a) could be written as $(1, -3\pi/4)$ or $(1, 13\pi/4)$ or $(-1, \pi/4)$. (See Figure 4.)

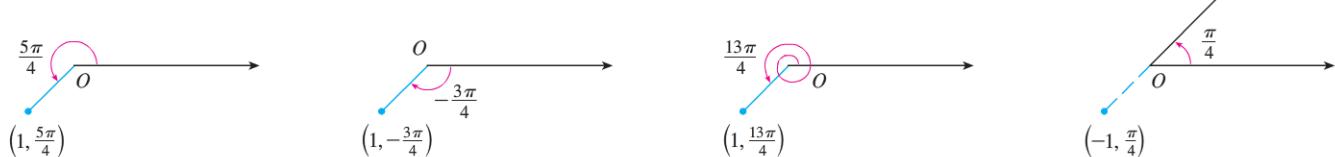


FIGURE 4

In fact, since a complete counterclockwise rotation is given by an angle 2π , the point represented by polar coordinates (r, θ) is also represented by

$$(r, \theta + 2n\pi) \quad \text{and} \quad (-r, \theta + (2n + 1)\pi)$$

where n is any integer.

The connection between polar and Cartesian coordinates can be seen from Figure 5, in which the pole corresponds to the origin and the polar axis coincides with the positive x -axis. If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ) , then, from the figure, we have

$$\cos \theta = \frac{x}{r} \quad \sin \theta = \frac{y}{r}$$

and so

1

$$x = r \cos \theta \quad y = r \sin \theta$$

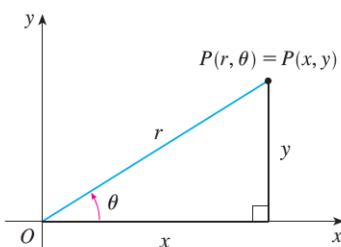


FIGURE 5

Although Equations 1 were deduced from Figure 5, which illustrates the case where $r > 0$ and $0 < \theta < \pi/2$, these equations are valid for all values of r and θ . (See the general definition of $\sin \theta$ and $\cos \theta$ in Appendix C.)

Equations 1 allow us to find the Cartesian coordinates of a point when the polar coordinates are known. To find r and θ when x and y are known, we use the equations

2

$$r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x}$$

which can be deduced from Equations 1 or simply read from Figure 5.

EXAMPLE 2 Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.

SOLUTION Since $r = 2$ and $\theta = \pi/3$, Equations 1 give

$$x = r \cos \theta = 2 \cos \frac{\pi}{3} = 2 \cdot \frac{1}{2} = 1$$

$$y = r \sin \theta = 2 \sin \frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

Therefore the point is $(1, \sqrt{3})$ in Cartesian coordinates. ■

EXAMPLE 3 Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

SOLUTION If we choose r to be positive, then Equations 2 give

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\tan \theta = \frac{y}{x} = -1$$

Since the point $(1, -1)$ lies in the fourth quadrant, we can choose $\theta = -\pi/4$ or $\theta = 7\pi/4$. Thus one possible answer is $(\sqrt{2}, -\pi/4)$; another is $(\sqrt{2}, 7\pi/4)$. ■

Note: Equations 2 do not uniquely determine θ when x and y are given because, as θ increases through the interval $0 \leq \theta < 2\pi$, each value of $\tan \theta$ occurs twice. Therefore, in converting from Cartesian to polar coordinates, it's not good enough just to find r and θ that satisfy Equations 2. As in Example 3, we must choose θ so that the point (r, θ) lies in the correct quadrant.

The **graph of a polar equation** $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

V EXAMPLE 4 What curve is represented by the polar equation $r = 2$?

SOLUTION The curve consists of all points (r, θ) with $r = 2$. Since r represents the distance from the point to the pole, the curve $r = 2$ represents the circle with center O and radius 2. In general, the equation $r = a$ represents a circle with center O and radius $|a|$. (See Figure 6.) ■

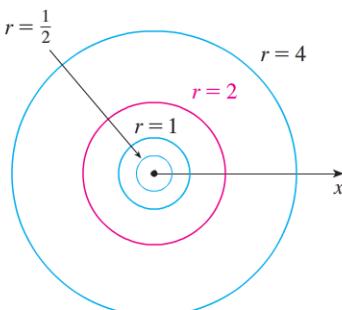


FIGURE 6

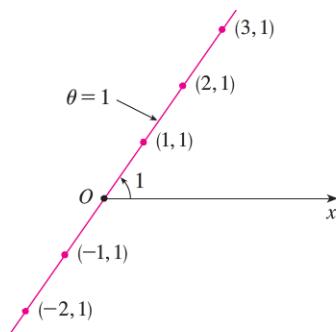


FIGURE 7

EXAMPLE 5 Sketch the polar curve $\theta = 1$.

SOLUTION This curve consists of all points (r, θ) such that the polar angle θ is 1 radian. It is the straight line that passes through O and makes an angle of 1 radian with the polar axis (see Figure 7). Notice that the points $(r, 1)$ on the line with $r > 0$ are in the first quadrant, whereas those with $r < 0$ are in the third quadrant.

EXAMPLE 6

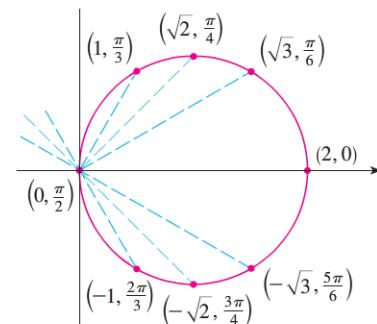
- Sketch the curve with polar equation $r = 2 \cos \theta$.
- Find a Cartesian equation for this curve.

SOLUTION

(a) In Figure 8 we find the values of r for some convenient values of θ and plot the corresponding points (r, θ) . Then we join these points to sketch the curve, which appears to be a circle. We have used only values of θ between 0 and π , since if we let θ increase beyond π , we obtain the same points again.

θ	$r = 2 \cos \theta$
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2

FIGURE 8
Table of values and graph of $r = 2 \cos \theta$



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(b) To convert the given equation into a Cartesian equation we use Equations 1 and 2. From $x = r \cos \theta$ we have $\cos \theta = x/r$, so the equation $r = 2 \cos \theta$ becomes $r = 2x/r$, which gives

$$2x = r^2 = x^2 + y^2 \quad \text{or} \quad x^2 + y^2 - 2x = 0$$

Completing the square, we obtain

$$(x - 1)^2 + y^2 = 1$$

which is an equation of a circle with center $(1, 0)$ and radius 1.

Figure 9 shows a geometrical illustration that the circle in Example 6 has the equation $r = 2 \cos \theta$. The angle OPQ is a right angle (Why?) and so $r/2 = \cos \theta$.

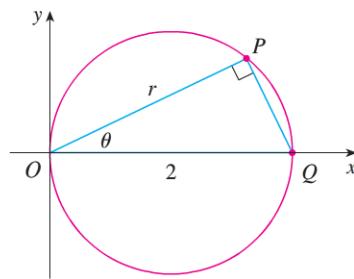


FIGURE 9

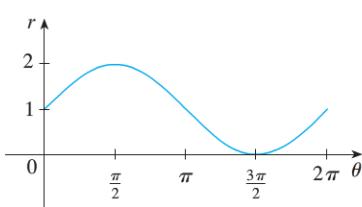


FIGURE 10
 $r = 1 + \sin \theta$ in Cartesian coordinates,
 $0 \leq \theta \leq 2\pi$

V EXAMPLE 7 Sketch the curve $r = 1 + \sin \theta$.

SOLUTION Instead of plotting points as in Example 6, we first sketch the graph of $r = 1 + \sin \theta$ in *Cartesian* coordinates in Figure 10 by shifting the sine curve up one unit. This enables us to read at a glance the values of r that correspond to increasing values of θ . For instance, we see that as θ increases from 0 to $\pi/2$, r (the distance from O) increases from 1 to 2, so we sketch the corresponding part of the polar curve in Figure 11(a). As θ increases from $\pi/2$ to π , Figure 10 shows that r decreases from 2 to 1, so we sketch the next part of the curve as in Figure 11(b). As θ increases from π to $3\pi/2$, r decreases from 1 to 0 as shown in part (c). Finally, as θ increases from $3\pi/2$ to 2π , r increases from 0 to 1 as shown in part (d). If we let θ increase beyond 2π or decrease beyond 0, we would simply retrace our path. Putting together the parts of the curve from Figure 11(a)–(d), we sketch the complete curve in part (e). It is called a **cardioid** because it's shaped like a heart.

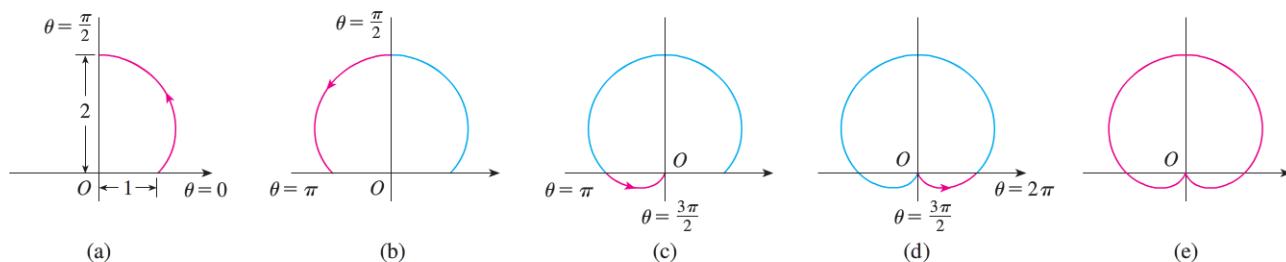


FIGURE 11 Stages in sketching the cardioid $r = 1 + \sin \theta$

EXAMPLE 8 Sketch the curve $r = \cos 2\theta$.

SOLUTION As in Example 7, we first sketch $r = \cos 2\theta$, $0 \leq \theta \leq 2\pi$, in *Cartesian* coordinates in Figure 12. As θ increases from 0 to $\pi/4$, Figure 12 shows that r decreases from 1 to 0 and so we draw the corresponding portion of the polar curve in Figure 13 (indicated by ①). As θ increases from $\pi/4$ to $\pi/2$, r goes from 0 to -1. This means that the distance from O increases from 0 to 1, but instead of being in the first quadrant this portion of the polar curve (indicated by ②) lies on the opposite side of the pole in the third quadrant. The remainder of the curve is drawn in a similar fashion, with the arrows and numbers indicating the order in which the portions are traced out. The resulting curve has four loops and is called a **four-leaved rose**.

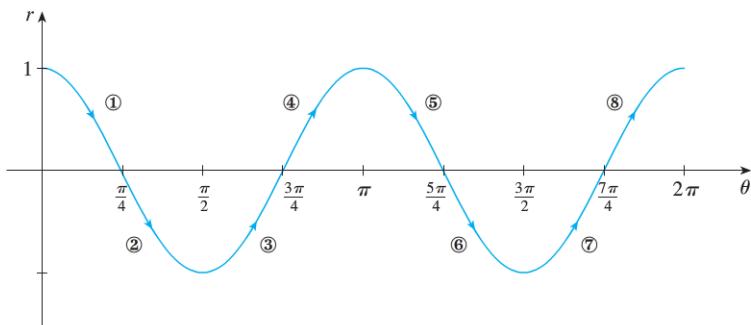


FIGURE 12
 $r = \cos 2\theta$ in Cartesian coordinates

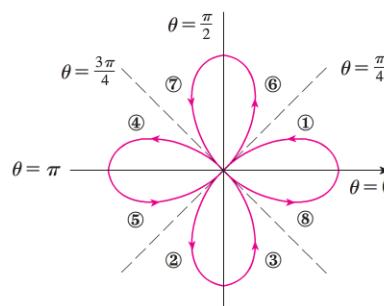


FIGURE 13
Four-leaved rose $r = \cos 2\theta$

When we sketch polar curves it is sometimes helpful to take advantage of symmetry. The following three rules are explained by Figure 14.

- If a polar equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about the polar axis.
- If the equation is unchanged when r is replaced by $-r$, or when θ is replaced by $\theta + \pi$, the curve is symmetric about the pole. (This means that the curve remains unchanged if we rotate it through 180° about the origin.)
- If the equation is unchanged when θ is replaced by $\pi - \theta$, the curve is symmetric about the vertical line $\theta = \pi/2$.

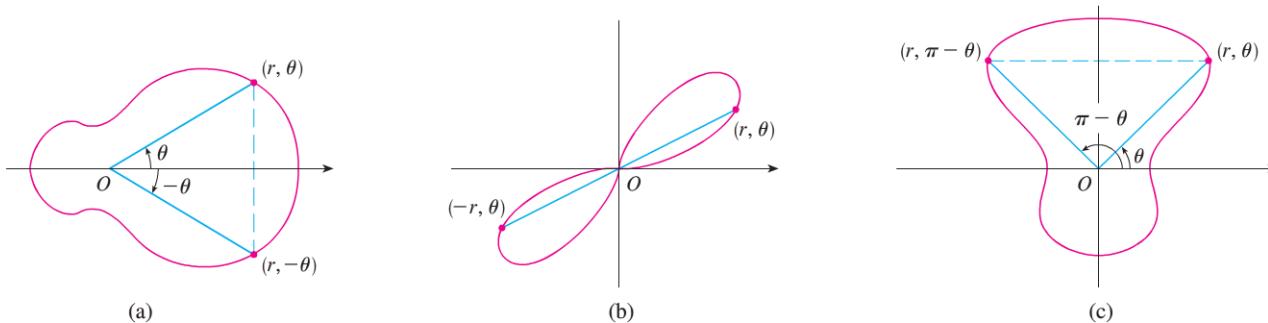


FIGURE 14

The curves sketched in Examples 6 and 8 are symmetric about the polar axis, since $\cos(-\theta) = \cos \theta$. The curves in Examples 7 and 8 are symmetric about $\theta = \pi/2$ because $\sin(\pi - \theta) = \sin \theta$ and $\cos 2(\pi - \theta) = \cos 2\theta$. The four-leaved rose is also symmetric about the pole. These symmetry properties could have been used in sketching the curves. For instance, in Example 6 we need only have plotted points for $0 \leq \theta \leq \pi/2$ and then reflected about the polar axis to obtain the complete circle.

Tangents to Polar Curves

To find a tangent line to a polar curve $r = f(\theta)$ we regard θ as a parameter and write its parametric equations as

$$x = r \cos \theta = f(\theta) \cos \theta \quad y = r \sin \theta = f(\theta) \sin \theta$$

Then, using the method for finding slopes of parametric curves (Equation 3.4.7) and the Product Rule, we have

$$\boxed{3} \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}$$

We locate horizontal tangents by finding the points where $dy/d\theta = 0$ (provided that $dx/d\theta \neq 0$). Likewise, we locate vertical tangents at the points where $dx/d\theta = 0$ (provided that $dy/d\theta \neq 0$).

Notice that if we are looking for tangent lines at the pole, then $r = 0$ and Equation 3 simplifies to

$$\frac{dy}{dx} = \tan \theta \quad \text{if } \frac{dr}{d\theta} \neq 0$$

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For instance, in Example 8 we found that $r = \cos 2\theta = 0$ when $\theta = \pi/4$ or $3\pi/4$. This means that the lines $\theta = \pi/4$ and $\theta = 3\pi/4$ (or $y = x$ and $y = -x$) are tangent lines to $r = \cos 2\theta$ at the origin.

EXAMPLE 9

- (a) For the cardioid $r = 1 + \sin \theta$ of Example 7, find the slope of the tangent line when $\theta = \pi/3$.
- (b) Find the points on the cardioid where the tangent line is horizontal or vertical.

SOLUTION Using Equation 3 with $r = 1 + \sin \theta$, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} = \frac{\cos \theta \sin \theta + (1 + \sin \theta) \cos \theta}{\cos \theta \cos \theta - (1 + \sin \theta) \sin \theta} \\ &= \frac{\cos \theta (1 + 2 \sin \theta)}{1 - 2 \sin^2 \theta - \sin \theta} = \frac{\cos \theta (1 + 2 \sin \theta)}{(1 + \sin \theta)(1 - 2 \sin \theta)} \end{aligned}$$

- (a) The slope of the tangent at the point where $\theta = \pi/3$ is

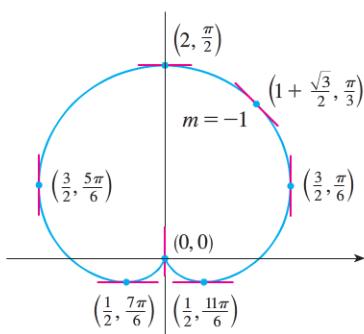
$$\begin{aligned} \left. \frac{dy}{dx} \right|_{\theta=\pi/3} &= \frac{\cos(\pi/3)(1 + 2 \sin(\pi/3))}{(1 + \sin(\pi/3))(1 - 2 \sin(\pi/3))} = \frac{\frac{1}{2}(1 + \sqrt{3})}{(1 + \sqrt{3}/2)(1 - \sqrt{3})} \\ &= \frac{1 + \sqrt{3}}{(2 + \sqrt{3})(1 - \sqrt{3})} = \frac{1 + \sqrt{3}}{-1 - \sqrt{3}} = -1 \end{aligned}$$

- (b) Observe that

$$\frac{dy}{d\theta} = \cos \theta (1 + 2 \sin \theta) = 0 \quad \text{when } \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\frac{dx}{d\theta} = (1 + \sin \theta)(1 - 2 \sin \theta) = 0 \quad \text{when } \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

Therefore there are horizontal tangents at the points $(2, \pi/2)$, $(\frac{1}{2}, 7\pi/6)$, $(\frac{1}{2}, 11\pi/6)$ and vertical tangents at $(\frac{3}{2}, \pi/6)$ and $(\frac{3}{2}, 5\pi/6)$. When $\theta = 3\pi/2$, both $dy/d\theta$ and $dx/d\theta$ are 0, so we must be careful. Using l'Hospital's Rule, we have



By symmetry,

$$\begin{aligned} \lim_{\theta \rightarrow (3\pi/2)^-} \frac{dy}{dx} &= \left(\lim_{\theta \rightarrow (3\pi/2)^-} \frac{1 + 2 \sin \theta}{1 - 2 \sin \theta} \right) \left(\lim_{\theta \rightarrow (3\pi/2)^-} \frac{\cos \theta}{1 + \sin \theta} \right) \\ &= -\frac{1}{3} \lim_{\theta \rightarrow (3\pi/2)^-} \frac{\cos \theta}{1 + \sin \theta} = -\frac{1}{3} \lim_{\theta \rightarrow (3\pi/2)^-} \frac{-\sin \theta}{\cos \theta} = \infty \end{aligned}$$

FIGURE 15

Tangent lines for $r = 1 + \sin \theta$

Thus there is a vertical tangent line at the pole (see Figure 15).

Note: Instead of having to remember Equation 3, we could employ the method used to derive it. For instance, in Example 9 we could have written

$$x = r \cos \theta = (1 + \sin \theta) \cos \theta = \cos \theta + \frac{1}{2} \sin 2\theta$$

$$y = r \sin \theta = (1 + \sin \theta) \sin \theta = \sin \theta + \sin^2 \theta$$

Then we have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos \theta + 2 \sin \theta \cos \theta}{-\sin \theta + \cos 2\theta} = \frac{\cos \theta + \sin 2\theta}{-\sin \theta + \cos 2\theta}$$

which is equivalent to our previous expression.

Graphing Polar Curves with Graphing Devices

Although it's useful to be able to sketch simple polar curves by hand, we need to use a graphing calculator or computer when we are faced with a curve as complicated as the ones shown in Figures 16 and 17.

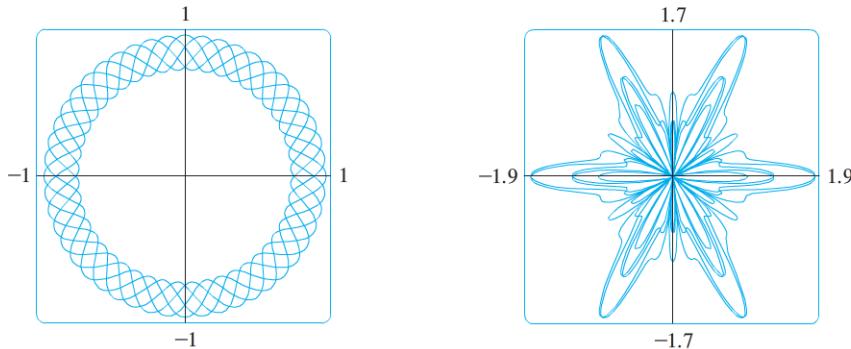


FIGURE 16
 $r = \sin^2(2.4\theta) + \cos^4(2.4\theta)$

FIGURE 17
 $r = \sin^2(1.2\theta) + \cos^3(6\theta)$

Some graphing devices have commands that enable us to graph polar curves directly. With other machines we need to convert to parametric equations first. In this case we take the polar equation $r = f(\theta)$ and write its parametric equations as

$$x = r \cos \theta = f(\theta) \cos \theta \quad y = r \sin \theta = f(\theta) \sin \theta$$

Some machines require that the parameter be called t rather than θ .

EXAMPLE 10 Graph the curve $r = \sin(8\theta/5)$.

SOLUTION Let's assume that our graphing device doesn't have a built-in polar graphing command. In this case we need to work with the corresponding parametric equations, which are

$$x = r \cos \theta = \sin(8\theta/5) \cos \theta \quad y = r \sin \theta = \sin(8\theta/5) \sin \theta$$

In any case we need to determine the domain for θ . So we ask ourselves: How many complete rotations are required until the curve starts to repeat itself? If the answer is n , then

$$\sin \frac{8(\theta + 2n\pi)}{5} = \sin \left(\frac{8\theta}{5} + \frac{16n\pi}{5} \right) = \sin \frac{8\theta}{5}$$

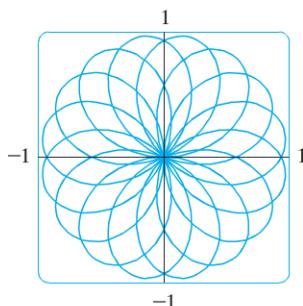


FIGURE 18
 $r = \sin(8\theta/5)$

In Exercise 47 you are asked to prove analytically what we have discovered from the graphs in Figure 19.

and so we require that $16n\pi/5$ be an even multiple of π . This will first occur when $n = 5$. Therefore we will graph the entire curve if we specify that $0 \leq \theta \leq 10\pi$. Switching from θ to t , we have the equations

$$x = \sin(8t/5) \cos t \quad y = \sin(8t/5) \sin t \quad 0 \leq t \leq 10\pi$$

and Figure 18 shows the resulting curve. Notice that this rose has 16 loops. ■

V EXAMPLE 11 Investigate the family of polar curves given by $r = 1 + c \sin \theta$. How does the shape change as c changes? (These curves are called **limaçons**, after a French word for snail, because of the shape of the curves for certain values of c .)

SOLUTION Figure 19 shows computer-drawn graphs for various values of c . For $c > 1$ there is a loop that decreases in size as c decreases. When $c = 1$ the loop disappears and the curve becomes the cardioid that we sketched in Example 7. For c between 1 and $\frac{1}{2}$ the cardioid's cusp is smoothed out and becomes a “dimple.” When c decreases from $\frac{1}{2}$ to 0, the limaçon is shaped like an oval. This oval becomes more circular as $c \rightarrow 0$, and when $c = 0$ the curve is just the circle $r = 1$.

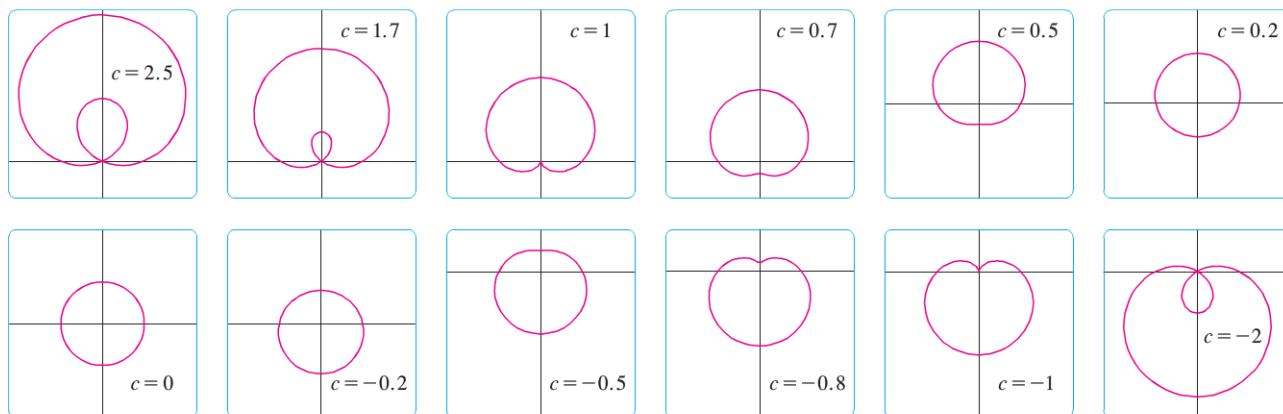


FIGURE 19
Members of the family of limaçons $r = 1 + c \sin \theta$

The remaining parts of Figure 19 show that as c becomes negative, the shapes change in reverse order. In fact, these curves are reflections about the horizontal axis of the corresponding curves with positive c . ■

H.1 Exercises

1–2 Plot the point whose polar coordinates are given. Then find two other pairs of polar coordinates of this point, one with $r > 0$ and one with $r < 0$.

1. (a) $(2, \pi/3)$ (b) $(1, -3\pi/4)$ (c) $(-1, \pi/2)$
2. (a) $(1, 7\pi/4)$ (b) $(-3, \pi/6)$ (c) $(1, -1)$

3–4 Plot the point whose polar coordinates are given. Then find the Cartesian coordinates of the point.

3. (a) $(1, \pi)$ (b) $(2, -2\pi/3)$ (c) $(-2, 3\pi/4)$

4. (a) $(-\sqrt{2}, 5\pi/4)$ (b) $(1, 5\pi/2)$ (c) $(2, -7\pi/6)$

5–6 The Cartesian coordinates of a point are given.

- (i) Find polar coordinates (r, θ) of the point, where $r > 0$ and $0 \leq \theta < 2\pi$.
- (ii) Find polar coordinates (r, θ) of the point, where $r < 0$ and $0 \leq \theta < 2\pi$.

5. (a) $(2, -2)$ (b) $(-1, \sqrt{3})$
6. (a) $(3\sqrt{3}, 3)$ (b) $(1, -2)$

A64 APPENDIX H POLAR COORDINATES

7–12 Sketch the region in the plane consisting of points whose polar coordinates satisfy the given conditions.

7. $1 \leq r \leq 2$
8. $r \geq 0, \pi/3 \leq \theta \leq 2\pi/3$
9. $0 \leq r < 4, -\pi/2 \leq \theta < \pi/6$
10. $2 < r \leq 5, 3\pi/4 < \theta < 5\pi/4$
11. $2 < r < 3, 5\pi/3 \leq \theta \leq 7\pi/3$
12. $r \geq 1, \pi \leq \theta \leq 2\pi$

13–16 Identify the curve by finding a Cartesian equation for the curve.

- | | |
|-------------------------|---|
| 13. $r = 3 \sin \theta$ | 14. $r = 2 \sin \theta + 2 \cos \theta$ |
| 15. $r = \csc \theta$ | 16. $r = \tan \theta \sec \theta$ |

17–20 Find a polar equation for the curve represented by the given Cartesian equation.

- | | |
|-----------------------|-----------------|
| 17. $x = -y^2$ | 18. $x + y = 9$ |
| 19. $x^2 + y^2 = 2cx$ | 20. $xy = 4$ |

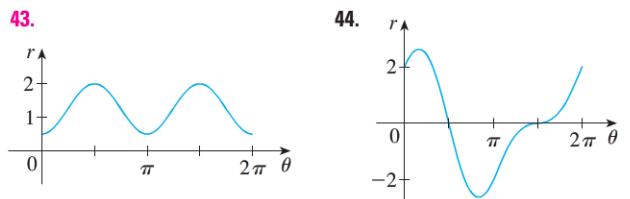
21–22 For each of the described curves, decide if the curve would be more easily given by a polar equation or a Cartesian equation. Then write an equation for the curve.

- | | |
|--|--|
| 21. (a) A line through the origin that makes an angle of $\pi/6$ with the positive x -axis | |
| (b) A vertical line through the point $(3, 3)$ | |
| 22. (a) A circle with radius 5 and center $(2, 3)$ | |
| (b) A circle centered at the origin with radius 4 | |

23–42 Sketch the curve with the given polar equation.

- | | |
|---|-------------------------------------|
| 23. $\theta = -\pi/6$ | 24. $r^2 - 3r + 2 = 0$ |
| 25. $r = \sin \theta$ | 26. $r = -3 \cos \theta$ |
| 27. $r = 2(1 - \sin \theta), \theta \geq 0$ | 28. $r = 1 - 3 \cos \theta$ |
| 29. $r = \theta, \theta \geq 0$ | 30. $r = \ln \theta, \theta \geq 1$ |
| 31. $r = 4 \sin 3\theta$ | 32. $r = \cos 5\theta$ |
| 33. $r = 2 \cos 4\theta$ | 34. $r = 3 \cos 6\theta$ |
| 35. $r = 1 - 2 \sin \theta$ | 36. $r = 2 + \sin \theta$ |
| 37. $r^2 = 9 \sin 2\theta$ | 38. $r^2 = \cos 4\theta$ |
| 39. $r = 2 \cos(3\theta/2)$ | 40. $r^2\theta = 1$ |
| 41. $r = 1 + 2 \cos 2\theta$ | 42. $r = 1 + 2 \cos(\theta/2)$ |

43–44 The figure shows a graph of r as a function of θ in Cartesian coordinates. Use it to sketch the corresponding polar curve.



45. Show that the polar curve $r = 4 + 2 \sec \theta$ (called a **conchoid**) has the line $x = 2$ as a vertical asymptote by showing that $\lim_{r \rightarrow \pm\infty} x = 2$. Use this fact to help sketch the conchoid.
46. Show that the curve $r = \sin \theta \tan \theta$ (called a **cissoid of Diocles**) has the line $x = 1$ as a vertical asymptote. Show also that the curve lies entirely within the vertical strip $0 \leq x < 1$. Use these facts to help sketch the cissoid.
47. (a) In Example 11 the graphs suggest that the limacon $r = 1 + c \sin \theta$ has an inner loop when $|c| > 1$. Prove that this is true, and find the values of θ that correspond to the inner loop.
 (b) From Figure 19 it appears that the limacon loses its dimple when $c = \frac{1}{2}$. Prove this.
48. Match the polar equations with the graphs labeled I–VI. Give reasons for your choices. (Don't use a graphing device.)

(a) $r = \sqrt{\theta}, 0 \leq \theta \leq 16\pi$	(b) $r = \theta^2, 0 \leq \theta \leq 16\pi$
(c) $r = \cos(\theta/3)$	(d) $r = 1 + 2 \cos \theta$
(e) $r = 2 + \sin 3\theta$	(f) $r = 1 + 2 \sin 3\theta$

