18 Problems: Eigenvalues and Eigenvectors

1. Let $M = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Find all eigenvalues of M. Does M have two independent[3] eigenvectors? Can M be diagonalized?

[3]Independence of vectors is explained here.

2. Consider $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $L(x, y) = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$.

   (a) Write the matrix of L in the basis $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$.

   (b) When $\theta \neq 0$, explain how L acts on the plane. Draw a picture.

   (c) Do you expect L to have invariant directions?

   (d) Try to find real eigenvalues for L by solving the equation
$$L(v) = \lambda v.$$

   (e) Are there complex eigenvalues for L, assuming that $i = \sqrt{-1}$ exists?

3. Let L be the linear transformation $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $L(x, y, z) = (x + y, x + z, y + z)$. Let e_i be the vector with a one in the ith position and zeros in all other positions.

   (a) Find Le_i for each i.

   (b) Given a matrix $M = \begin{pmatrix} m_1^1 & m_2^1 & m_3^1 \\ m_1^2 & m_2^2 & m_3^2 \\ m_1^3 & m_2^3 & m_3^3 \end{pmatrix}$, what can you say about Me_i for each i?

   (c) Find a 3×3 matrix M representing L. Choose three nonzero vectors pointing in different directions and show that $Mv = Lv$ for each of your choices.

   (d) Find the eigenvectors and eigenvalues of M.