Concrete Bridges to Abstract Algebras

A Senior Project submitted to
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of
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Abstract

Starting with various modular groups, we construct concrete objects (Cayley digraphs and Cartesian graph products) to illustrate the underlying structure of these abstractions. We then use the edge relationships on these group graphs to construct and study the quotient rings $k[x_1, x_2, \ldots, x_n]/I(G)$, where $I(G)$ are edge ideals. In particular, We investigate which, if any, of the group characteristics are invariant throughout these various transformations and which have Cohen-Macaulay graphs.

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Dedication

To the future me I'm making possible today.

Acknowledgments

If not for the many Mathematics professors who have pushed me to push myself these last few years, this project would not have been possible. Thank You. Additionally, my mathematical contemporaries Anthony Perez and John Zoccoli were constant sources of motivation throughout this entire process–especially the last few weeks. Anthony's algorithms saved me time. John's graphs saved me pain.

1
Introduction

1.1 Project Motivation

The transition from computation to abstraction is intellectually painful for many aspiring mathematicians. It is a sad fact that when faced with the new and difficult challenges involved in advanced study, many number-liking students lose their desire to learn math. I sincerely believe this is a pedagogical problem that can be solved with a greater emphasis on studying and manipulating the concrete objects which motivate abstractions. In most advanced mathematics courses, unlike Calculus, visual and computational aides are often secondary in importance to proof-based theory. Countless texts and articles fail on this level, ignoring the fact that many students need visual aides and computational exercises for them to properly grasp abstract concepts. This project is the direct result of my desire to make the abstract more visible and thus understandable to myself and others.

The basic approach of this project is to build graphs from groups. Then after studying these group-based graphs, we take them and formulate quotient rings to test whether or not they are Cohen-Macaulay.

Chapter 2 presents the background and notation for the primary group and graph concepts this project studies. Section 2.1 focuses on graphs and introduces the group of *integers modulo* n, $\mathbb{Z}_n$ and the *symmetric group of degree* n, S_n. Section 2.2 then gives a general definition of a graph and the two main graphing approaches we use in this project: Cayley digraphs and Cartesian graph products.

Having clearly established the fundamental concepts necessary for this project, Chapter 3 begins the formal analysis of the integers modulo n. We start in Section 3.1 by formulating the Cayley digraphs for specific values of n. In general, we prove that for all $n \in \mathbb{Z}^+$, the undirected skeleton of a Cayley digraph (that is, a Cayley skeleton) is isomorphic to a cycle of length n. This completely classifies the Cayley digraphs of $\mathbb{Z}_n$ for all values of $n \in \mathbb{Z}^+$.

Building from these initial findings, Section 3.2 examines the direct product $\mathbb{Z}_m \times \mathbb{Z}_n$ for all possible values of $m, n \in \mathbb{Z}^+$. We analyze $\mathbb{Z}_m \times \mathbb{Z}_n$ when the $gcd(m, n) = 1$, when the $gcd(m, n) \neq 1$ and the special case when $m = n$. Throughout Section 3.2 we use the notion of Cayley digraphs and that of Cartesian graph products, finding several isomorphic relationships exist between the two in relation to this particular group.

Section 3.3 examines the Cayley Digraphs and Cartesian Graph Products associated with direct modular products of arbitrary length. We conclude with a general isomorphism proposition that shows the that the patterns observed hold true.

With a body of examples analyzed, this project then transitions to the ring theory part of this mathematical excursion. Chapter 4 introduces the background and notation for the major ring theory concepts needed: quotient rings, edges ideals, depth and dimension. We introduce the concept of Cohen-Macaulay rings and graphs and conditions associated with this concept.

In Chapter 5, we take graphs from Chapters 2 and 3 and construct the quotients rings using the corresponding vertex sets and edge ideals. We then test these quotient rings to see if they satisfy the Cohen-Macaulay conditions. We conclude with a theorem that classifies all values of m and n for which $\mathbb{Z}_m \times \mathbb{Z}_n$ is Cohen-Macaulay.

2

Groups and Graphs: Background and Notation

In terms of focus, this project partitions nicely into two distinct subjects. First we introduce groups and formulate graphical representations for analysis; Then we take these group-based graphs and construct their associated quotient rings. Since the basic building blocks of this project are groups and graphs, we do not require the ring theory material until after we fully explore the groups and graphs. Therefore, we present here only the group and graph concepts and notations.

2.1 Groups

A group, as the term is traditionally understood, consists of a set of persons, places or things, all of which have something in common. In mathematics, groups usually involve sets of numbers all having certain properties. We begin with the general definition before providing specific examples.

Definition 2.1.1. A **Group** is a set G that is closed under a binary operation, say $*$, and satisfies the following axioms:

1. There is an **identity element** e such that for every $a \in G$,

$$e * a = a * e = x.$$

2. There in an **inverse element** a' corresponding to each element a of G such that

$$a * a' = a' * a = e.$$

3. For all $a, b, c \in G$, the binary operation $*$ is associative. That is,

$$(a * b) * c = a * (b * c).$$

4. Additionally, if for all $a, b \in G$, the binary operation $*$ is commutative, that is, if

$$a + b = b + a,$$

we say that G is an abelian group.

There are two groups that we study in this project: $\mathbb{Z}_n$ and $\mathbb{S}_n$. After defining these two groups, we provide specific examples to illustrate.

Definition 2.1.2. In modular arithmetic, $\mathbb{Z}_n$ is an equivalence relation on $\mathbb{Z}$ that partitions $\mathbb{Z}$ into n classes of numbers having the same remainder when divided by n. For any $a \in \mathbb{Z}_n$, the inverse of a is $n - a$, written as a^{-1}. We call $\mathbb{Z}_n$ the group of **integers modulo** n.

Example 2.1.3. The group $\mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$, where $\overline{0} = \{3\mathbb{Z}\}, \overline{1} = \{3\mathbb{Z}+1\}$ and $\overline{2} = \{3\mathbb{Z}+2\}$. In this group, $\overline{0}$ is the identity element and $\overline{1}$ and $\overline{2}$ are inverses of each other.

Later in studying the graph constructs of this group family, we make use of the fact that $\mathbb{Z}_n$ is a cyclic group having generating elements. In particular, we establish edge relationships between the group elements generated by these generators. These concepts are defined bellow.

Definition 2.1.4. A group G is called **cyclic** if, for some $a \in G$, every $x \in G$ is of the form a^m, where $m \in \mathbb{Z}$. The element a is then called a **generator** of G, denoted as $\langle a \rangle$. Additionally, If $G = \langle a \rangle$, we say that G is a **cyclic group**.

Example 2.1.5. The group $\mathbb{Z}$ under addition is cyclic and the generators for $\mathbb{Z}$ are 1 and -1.

Definition 2.1.6. Let $S = \{1, 2, \ldots, n\}$. The set of all permutations of S is a one-to-one function from S to S called the **symmetric group of degree n**. We denote this set of $n!$ permutations by S_n.

Example 2.1.7. Let $A = \{1, 2, 3\}$. Then

$$S_A = \{(1), (1, 2), (1, 3), (2, 3), (1, 2, 3), (1, 3, 2)(2, 1, 3)\},$$

where the cycle notation (a, b, c) means that $a \mapsto b, b \mapsto c$ and $c \mapsto a$. Additionally, if a number is left out of this cycle notation, then that number remains unchanged in the new permutation. Letting $a = (1, 3)$ and $b = (1, 3, 2)$, we see that S_A is non-abelian since under permutation composition $a \circ b = (2, 3) \neq b \circ a = (1, 2, 3)$.

2.2 Graphs

Having established the group concepts, we can now define the types of graphs we use to study these groups as well as all relevant definitions and theorems. We begin with the basic definition of a graph. Then we detail exactly how it is adapted to the study of abstract groups.

Definition 2.2.1. A **graph** (V, E) consists of a nonempty set of **vertices** V and a set of **edges** E. Each edge is associated with an unordered pair of vertices that serve as its endpoints. If direction matters, then the edge associated with the ordered pair (u, v) is said to **start** at u and **end** at v and we call the graph a **directed graph** or **digraph**.

Example 2.2.2. Let G be the graph shown in Figure 2.2.1. For this graph, $V(G) = \{1, 2, \ldots, 7\}$ and $E(G) = \{a, b, \ldots, f\}$.

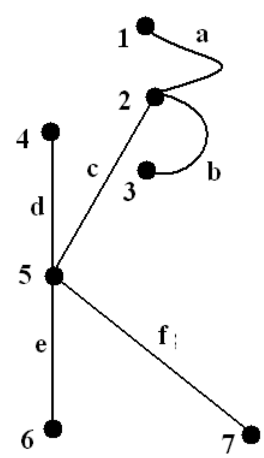

Figure 2.2.1. A simple graph with seven vertices and six edges.

Notice that this graph has no loops (edges that start and end at the same vertex) or multiple edges connecting the same two vertices. Such graphs are called simple and the same applies for digraphs, which are our main focus in this project.

There are two particular graph strategies this project uses to represent and study groups in productive study: (1) Arthur Cayley's notion of a group digraph and (2) The Cartesian graph product.

In 1878, Arthur Cayley established the practice of analyzing groups by way of graphical representations derived from a group's relations and sets of generators. This strategy is extremely useful because it allows mathematicians to visualize groups and it connects two important branches of mathematics: groups and graphs.

We begin first by defining the notion of a generating set and that of a Cayley digraph as well.

Definition 2.2.3. Let G be a group and let $a_i \in G$ for $i \in I$, where I is any set of indices. The smallest subgroup of G containing $\{a_i | i \in I\}$ is the **subgroup generated by** $\{a_i | i \in I\}$. If this subgroup is all of G, then $\{a_i | i \in I\}$ **generates** G and the a_i are **generators of** G.

When dealing with groups, Cayley observed that for each generating set S of a finite group G, there is a directed graph representing the group in terms of the generators in S.

Definition 2.2.4. In a digraph for a group G using a generating set S, we have one vertex, represented by a dot, for each element of G. We denote each generator $a \in S$ by a distinct type of edge. An edge joining $x, y \in G$ means in additive notation that $xa = y$. No indicated direction implies that two vertices are inverses of each other. This is the **Cayley digraph of** G, and for a particular group G we denote the Cayley digraph of G generated by $\langle a_1, a_2, \ldots, a_n\rangle$ as $C(G, \langle a_1, a_2, \ldots, a_n\rangle)$. Additionally, we denote the undirected, underlying skeleton of a Cayley digraph by $C^*(G, \langle a_1, a_2, \ldots, a_n\rangle)$ and call it a Cayley skeleton.

Example 2.2.5. Let $S = \{1, 2\}$ be a generating set for $\mathbb{Z}_5$. Although S has more generators than needed (both 1 and 2 generate the whole group), we can still make a Cayley digraph for $\mathbb{Z}_5$ using S. Choosing $\rightarrow$ for the edges generated by 1 and $\dashrightarrow$ for the edges generated by 2 we have the $C(\mathbb{Z}_5, \langle 1, 2\rangle)$ of Figure 2.2.2.

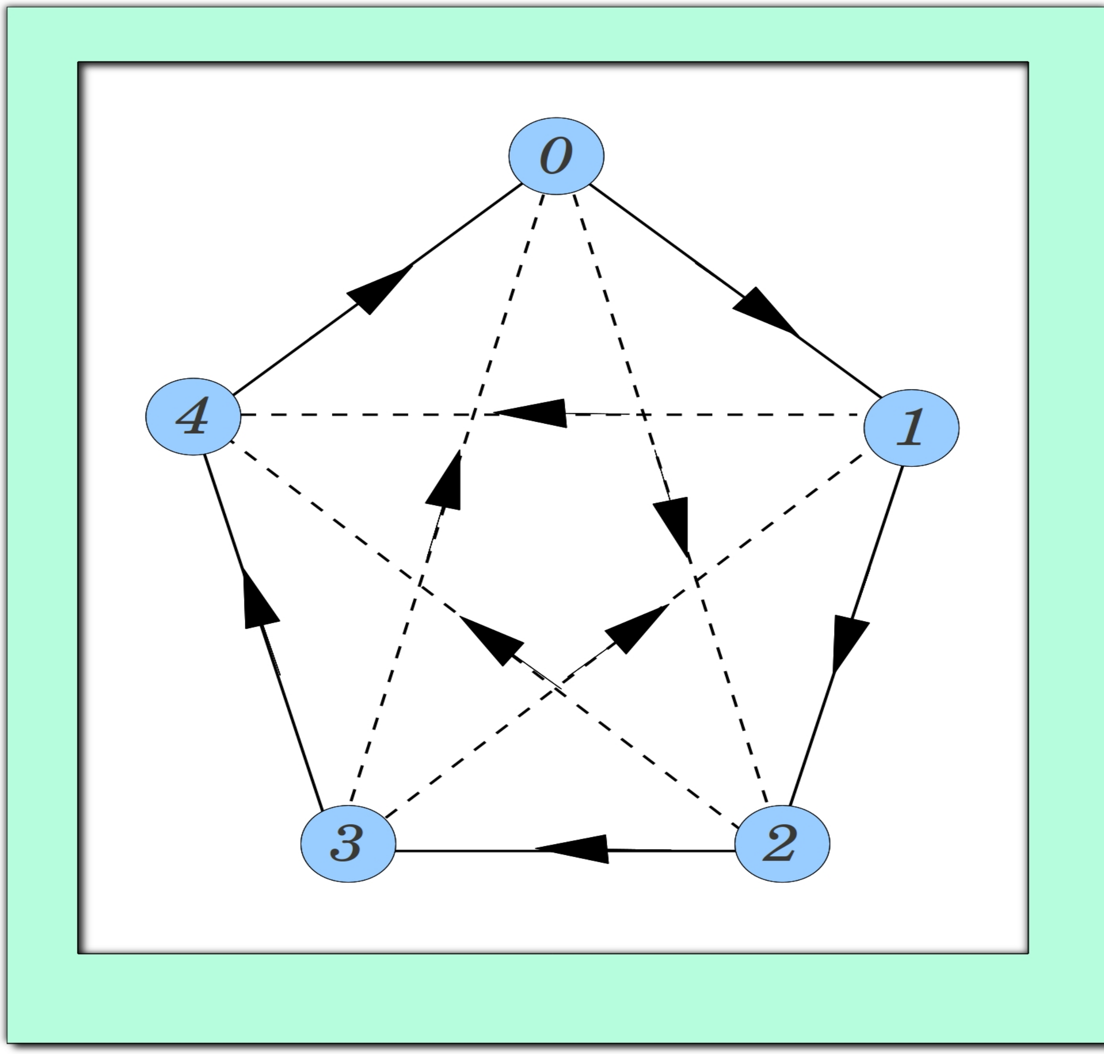

Figure 2.2.2. $C(\mathbb{Z}_5, \langle 1, 2\rangle)$

Notice from this Cayley digraph that $|1| = |2| = 5$, indicating that each generator yields a Cayley digraph that is a 5-cycle. Although we can learn much about a group from its Cayley digraph, we cannot learn everything. For example, it is not obvious that a graph is cyclic based on its Cayley digraph since a digraph of a cyclic group might be formed using a generating set of two or more elements, none of which generates the entire group.

Note also that the Cayley skeleton of the above graph is exactly the same but without the direction arrows.

In studying group-based graphs, it is not the appearance of a graph representation of a group that is important but rather the manner in which the vertices are connected, since this provides information about the group's structure. Each generating set, for example, uniquely determines the edge relations between each connected pair of vertices. We illustrate with an example.

Example 2.2.6. For S_3 with generating set $\langle (12), (123) \rangle$, the following two Cayley digraphs, though different in appearance, provide exactly the same information. See graphs A and B on Figure 2.2.3.

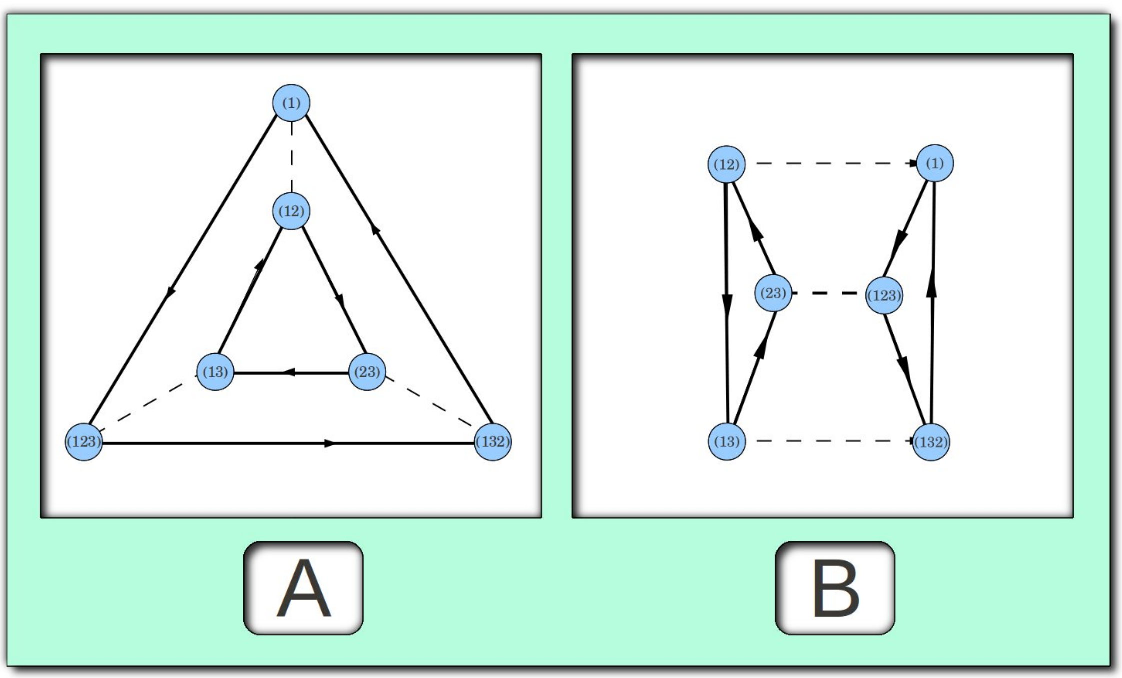

Figure 2.2.3. Two Graph Representations of $C(S_3, \langle (12), (123) \rangle)$

A second way this project represents and studies groups involves a variation of the familiar concept of the Cartesian product.

Definition 2.2.7. Let G and H be two graphs. The **Cartesian graph product** of G and H is a graph, denoted as $G \,\Box\, H$, whose vertex set is $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent precisely if $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$. Thus,

$$V(G \,\Box\, H) = \{(g, h) \mid g \in V(G) \textit{ and } h \in V(H)\},$$

$$E(G \,\Box\, H) = \{(g, h)(g', h') \mid g = g', hh' \in E(H), \text{or } gg' \in E(G), h = h'\}.$$

The graphs G and H are called **factors** of the product $G \,\Box\, H$.

In this project we take the Cartesian graph product of Cayley digraphs, finding in the case of modular groups that

$$C(\mathbb{Z}_n, \langle a \rangle) \simeq C_n$$

and in general that

$$C(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_k}, \langle e_1, e_2, \ldots, e_k \rangle) \simeq C_{n_1} \,\Box\, C_{n_2} \,\Box \cdots \Box\, C_{n_k}$$

where e_i is a basis vector with 1 in the i^{th} position and 0 otherwise.

Example 2.2.8. Let $G = K_2$, a complete graph on two vertices, and let $H = C_3$, a 3-cycle. Then $G \,\Box\, H = K_2 \,\Box\, C_3$ is a 3-prism consisting of two triangular faces, with rectangular sides and edges linking each corresponding pair of vertices. See Figure 2.2.4

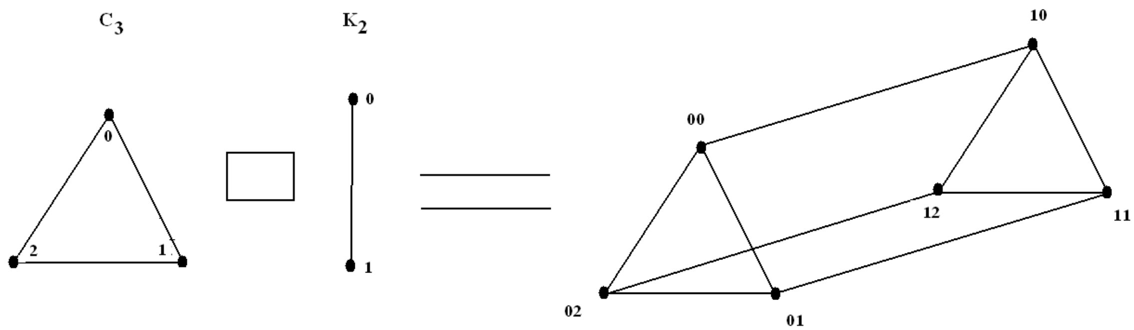

Figure 2.2.4. $K_2 \,\square\, C_3$

In this example

$$\begin{aligned}
V(C_3) &= \{0, 1, 2\}, \\
V(K_2) &= \{0, 1\}, \\
E(C_3) &= \{01, 12, 02\}, \\
E(K_2) &= \{01\}, \\
V(C_3 \,\square\, K_2) &= \{00, 10, 20, 01, 11, 21\}, \\
E(C_3 \,\square\, K_2) &= \{(00)(10), (10)(20), (00)(20), (01)(11), \\
&\qquad (11)(21), (01)(21), (00)(01), (10)(11), (20)(21)\}.
\end{aligned}$$

3

Modular Groups to Cayley Digraphs

3.1 $\mathbb{Z}_n \to C(\mathbb{Z}_n, \langle a \rangle)$

In this section we study the well-known abelian group $\mathbb{Z}_n$. The group $\mathbb{Z}_n$ has a as a generator for all a such that the $gcd(a, n) = 1$. The order of a is n, since n is the smallest possible integer such that $na = 0 \in \mathbb{Z}$. We define the group elements of $\mathbb{Z}_n$ as vertices of a graph G such that $b, c \in \mathbb{Z}_n$ are joined by an edge if and only if $b + a = c$, where a is a generator of $\mathbb{Z}_n$. This construction gives us the Cayley digraphs of $\mathbb{Z}_n$ generated by a. We denote the Cayley digraph generated by a by $C(\mathbb{Z}_n, \langle a \rangle)$. In this section, we prove that for every generator a of $\mathbb{Z}_n$, the $C^*(\mathbb{Z}_n, \langle a \rangle)$ is an n-cycle. We begin with an example and construct its Cayley digraph representation.

Example 3.1.1. The group $\mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$, where $\overline{0} = \{k \mid 3k \in Z\}$, $\overline{1} = \{k \mid 3k + 1 \in Z\}$ and $\overline{2} = \{k \mid 3k + 2 \in Z\}$. The single element generators are 1 and 2. Figure 3.1.1 shows the Cayley digraphs for each generator, namely two graphs whose skeletons are isomorphic 3-cycles.

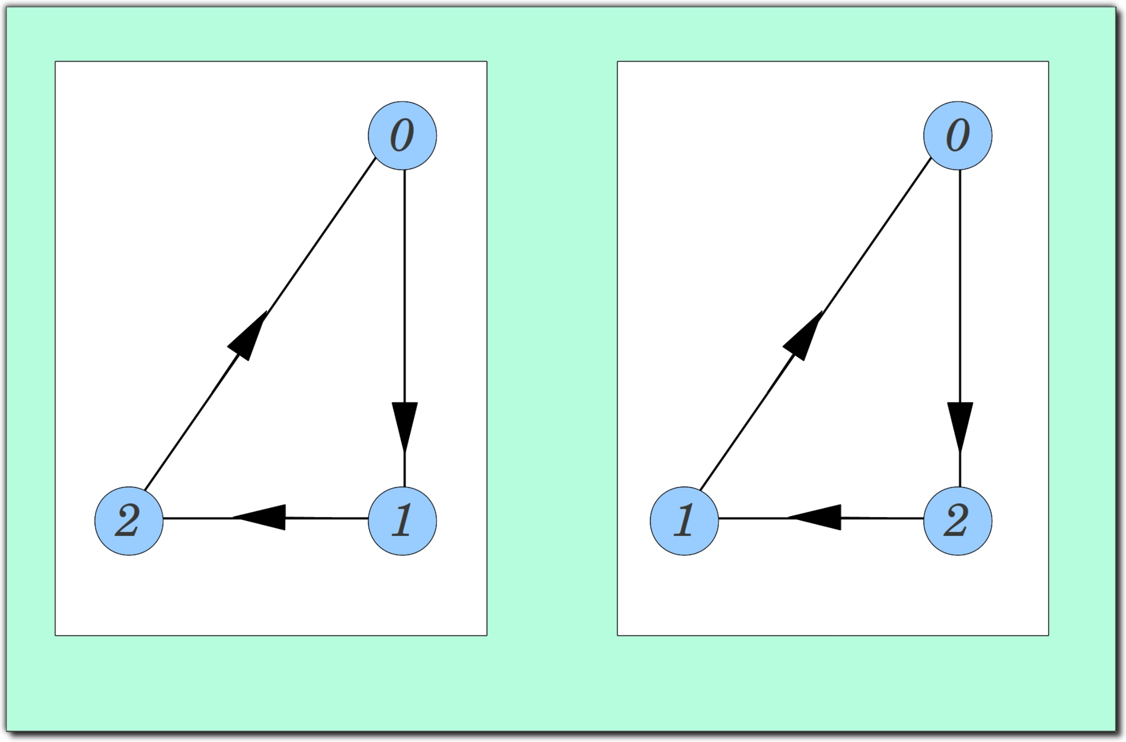

Figure 3.1.1. $C(\mathbb{Z}_3, \langle 1\rangle)$ and $C(\mathbb{Z}_3, \langle 2\rangle)$

As a proposition, we claim that the results of this example hold in general for $C^*(\mathbb{Z}_n, \langle a\rangle)$ whenever $n \geq 3$. Before proving this, however, it is necessary to first define three familiar concepts from graph theory: cycles, isomorphisms and adjacency matrices.

Definition 3.1.2. The **n-cycle** $C_n,\ n \geq 3$, consists of a vertex set consisting of n vertices $v_1, v_2, \ldots, v_n$ and an edge set consisting of the edges $\{v_1, v_2\}, \{v_2, v_2\}, \ldots, \{v_{n-1}, v_n\}$ and $\{v_n, v_1\}$.

Definition 3.1.3. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple (di)graphs. We say that G_1 and G_2 are **isomorphic** if there is a one-to-one and onto map f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all $a, b \in V_1$. We call the map f an **isomorphism**.

Definition 3.1.4. Let $G = (V, E)$ be a simple (di)graph where $|V| = n$. Suppose that the vertices of G are listed arbitrarily as $v_1, v_2, \ldots, v_n$. The **adjacency matrix A** (or $\mathbf{A}_G$) of G, with respect to this listing of the vertices, is the $n \times n$ zero-one matrix with 1 as its (i, j)th entry when v_i and v_j are adjacent, and 0 as its (i, j)th entry when they are not adjacent. In other words, if its adjacency matrix is $\mathbf{A} = [a_{ij}]$, then

$$A_G = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge in } G, \\ 0 & \textit{otherwise.} \end{cases}$$

Having defined the necessary concepts, we can formally state and prove the following proposition.

Proposition 3.1.5. *For any $n \geq 3$ and generator a of $\mathbb{Z}_n$,*

$$C^*(\mathbb{Z}_n, \langle a \rangle) \simeq C_n.$$

Proof. From the definition of a Cayley digraph, we know that in its skeleton graph the elements of the given group become vertices and that these vertices are joined by edges only when a is applied. Since $\mathbb{Z}_n = \{\overline{0 \cdot a}, \overline{1 \cdot a}, , \ldots, \overline{(n-1)a}\}$, we can rename these elements $a_0, a_1, \ldots, a_{n-1}$ where a_i still gives us all elements with remainder i in $\mathbb{Z}_n$. Because there are n elements in $\mathbb{Z}_n$, $C^*(\mathbb{Z}_n, \langle a \rangle)$ has $a_0, a_1, \ldots, a_{n-1}$ as its n vertices.

By definition, C_n consists of n vertices $v_0, v_1, \ldots, v_{n-1}$ and edges $\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{n-2}, v_{n-1}\}$, and $\{v_{n-1}, v_0\}$.

Since $C^*(\mathbb{Z}_n, \langle a \rangle)$ and C_n each have n vertices, the function f with

$$f(a_0) = v_0, f(a_1) = v_1, \ldots, f(a_{n-1}) = v_{n-1}$$

gives us a one-to-one correspondence between $V(C^*(\mathbb{Z}_n, \langle a \rangle))$ and $V(C_n)$.

Next we want to show that the corresponding adjacency matrices are element-wise equivalent.

Notice that the adjacency matrix for $C(\mathbb{Z}_n, \langle a \rangle)$ is $[A_{i,j}]$, with entries either 0 or 1 for the following conditions:

$$A_{C(\mathbb{Z}_n,\langle a \rangle)} = [A_{ij}] \begin{cases} 1 & \text{if } a_i - a_j \equiv a \pmod{n}, \\ 0 & \textit{otherwise}. \end{cases}$$

Now in $C^*(\mathbb{Z}_n, \langle a \rangle)$, $a_i - a_j \equiv a \pmod{n}$ if and only if $a_i - a_j = a$ our generator.

From the one-to-one function defined above and the definition of a cycle, it follows immediately that the adjacency matrix for C_n is the same with a_i replaced with v_i. Since we have adjacency matrices that preserve edges, we conclude that f is an isomorphism, so

$$C^*(\mathbb{Z}_n \langle a \rangle) \simeq C_n.$$

□

3.2 $\mathbb{Z}_m \times \mathbb{Z}_n \to C(\mathbb{Z}_m \times \mathbb{Z}_n, \langle (1,0), (0,1) \rangle)$

Having explored $\mathbb{Z}_n$, we can now extend the focus to include another familiar group from abstract algebra, namely, $\mathbb{Z}_m \times \mathbb{Z}_n = \{(a, b) \mid a \in \mathbb{Z}_m \text{ and } b \in \mathbb{Z}_n\}$.

From [3], we have the following useful theorem:

Theorem 3.2.1. *The group $\mathbb{Z}_m \times \mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{mn}$ if and only if m and n are relatively prime, that is, if and only if the gcd$(m, n) = 1$.*

Proof. Let H be the cyclic subgroup of $\mathbb{Z}_m \times \mathbb{Z}_n$ generated by (1,1). The order of H is the smallest power of (1,1) that gives us the identity (0,0). Adding (1,1) to itself repeatedly, we see that under addition by components, the first component $1 \in \mathbb{Z}_m$ yields 0 only after m summands, $2m$ summands, etc. Likewise, the second component $1 \in \mathbb{Z}_n$ yields 0 only after n summands, $2n$ summands, etc. In order for both components to yield 0 simultaneously, the number of summands has to be a multiple of both m and n. The number, mn, is the smallest such number if and only if $\gcd(m, n) = 1$. In this case, (1,1) generates a cyclic subgroup of order mn, which is the order of the entire group. Since $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic of order mn, we conclude that $\mathbb{Z}_m \times \mathbb{Z}_n \simeq \mathbb{Z}_{mn}$ if m and n are relatively prime.

For the converse, suppose that the $\gcd(m,n) = d > 1$. Then mn/d is divisible by both m and n. Consequently, for any $(r,s) \in \mathbb{Z}_m \times \mathbb{Z}_n$, we have

$$\underbrace{(r,s) + (r,s) + \ldots + (r,s)}_{\text{mn/d Summands}} = (0,0).$$

Hence no element $(r,s) \in \mathbb{Z}_m \times \mathbb{Z}_n$ can generate the entire group. This shows that $\mathbb{Z}_m \times \mathbb{Z}_n$ is not cyclic and therefore not isomorphic to $\mathbb{Z}_{mn}$. □

With relative ease, the above assertion can be extended to any length chain of Cartesian products.

Corollary 3.2.2. *The group $\prod_{i=1}^{n} \mathbb{Z}_{m_i}$ is cyclic and isomorphic to $\mathbb{Z}_{m_1 m_2 \cdots m_n}$ if and only if m_i for $i = 1, \cdots, n$ are such that the gcd of any two of them is 1.*

In terms of Cayley skeletons, Theorem 3.2.1 and Corollary 3.2.2 provide us with the following Proposition.

Proposition 3.2.3. *If the $\gcd(m,n) = 1$, then $C^*(\mathbb{Z}_m \times \mathbb{Z}_n, \langle(1,1)\rangle) \simeq C^*(\mathbb{Z}_{mn}, \langle 1\rangle) \simeq C_{mn}$. In other words, $C^*(\mathbb{Z}_m \times \mathbb{Z}_n, \langle(1,1)\rangle)$ and $C^*(\mathbb{Z}_{mn}, \langle 1\rangle)$ are both cycles of length mn generated by a single generator.*

Proof. If m and n are relatively prime, Theorem 3.2.1 tells us that $\mathbb{Z}_m \times \mathbb{Z}_n \simeq \mathbb{Z}_{mn}$, where (1,1)$\rightarrow$ 1. However, by Proposition 3.1.2, we know that $C^*(\mathbb{Z}_{mn}, \langle a\rangle) \simeq C_{mn}$, for any generator a of $\mathbb{Z}_{mn}$. Therefore,

$$C^*(\mathbb{Z}_m \times \mathbb{Z}_n, \langle(1,1)\rangle) \simeq C^*(Z_{mn}, \langle 1\rangle) \simeq C_{mn}.$$

□

Similarly, we can easily generalize Proposition 3.2.3 with a corresponding Corollary.

Corollary 3.2.4. *The graph* $C^*(\prod_{i=1}^{n}\mathbb{Z}_{m_i},\langle(1,1)\rangle)$ *is isomorphic to* $C^*(\mathbb{Z}_{m_1m_2\cdots m_n},\langle 1\rangle)$ *if and only if* m_i *for* $i=1,\cdots,n$ *are such that the gcd of any two of them is 1.*

Proof. See proof for Proposition 3.2.3. □

Example 3.2.5. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_3 = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\}$. The $gcd(3,2)=1$, so $\mathbb{Z}_2\times\mathbb{Z}_3 \simeq \mathbb{Z}_6$. This means $\mathbb{Z}_2\times\mathbb{Z}_3$ can be generated with a single generator, e.g., $(1,1)$. Starting at (1,1) and adding (1,1) repeatedly, we see that the $C(\mathbb{Z}_2\times\mathbb{Z}_3,\langle(1,1)\rangle)$ is a cycle of length 6. See Figure 3.2.1.

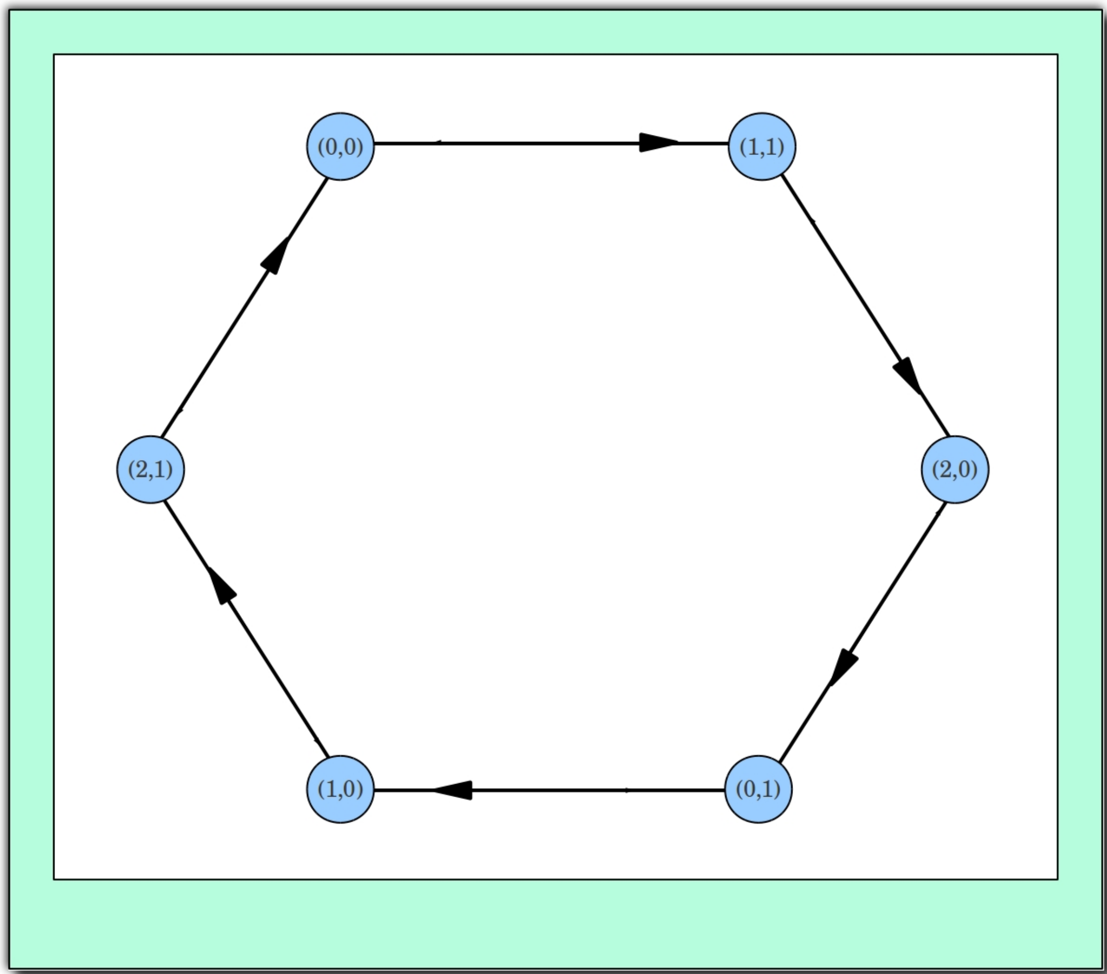

Figure 3.2.1. $C(\mathbb{Z}_2 \times \mathbb{Z}_3, \langle(1,1)\rangle)$

It is important to realize that $C^*(\mathbb{Z}_2 \times \mathbb{Z}_3, \langle(1,1)\rangle) \neq C^*(\mathbb{Z}_2, \langle a\rangle) \,\square\, C^*(\mathbb{Z}_3, \langle b\rangle)$. In the first case, $C^*(\mathbb{Z}_2 \times \mathbb{Z}_3, \langle(1,1)\rangle) \simeq C_6$, a 6-cycle as in Figure 3.2.1. Whereas $C^*(\mathbb{Z}_2, \langle a\rangle) \,\square\, C^*(\mathbb{Z}_3, \langle b\rangle) \simeq K_2 \,\square\, C_3$ can be represented graphically by a prism with triangular bases and rectangular sides. See Figure 2.2.4. Moreover, in the special case $\mathbb{Z}_n \times \mathbb{Z}_n$, experimentation suggests that if the group is $\mathbb{Z}_n \times \mathbb{Z}_n$, then $C^*(\mathbb{Z}_n \times \mathbb{Z}_n, \langle(1,0),(0,1)\rangle)$ approximates an n-torus as $n \longrightarrow \infty$.

That takes care of the case when the $\gcd(m,n) = 1$. If m and n are not relatively prime, then two generators are necessary to generate the group $\mathbb{Z}_m \times \mathbb{Z}_n$ and its Cayley digraphs.

Proposition 3.2.6. *Suppose $G = \mathbb{Z}_m \times \mathbb{Z}_n$. Then for all $m, n \in \mathbb{Z}^+$ such that the gcd$(m,n) \neq 1$, $\langle(1,0)\rangle$ and $\langle(0,1)\rangle$ are generators for $C(G)$.*

Proof. Let (m,n) be an arbitrary element of $\mathbb{Z}_m \times \mathbb{Z}_n$. The identity element of $\mathbb{Z}_m \times \mathbb{Z}_n$ is $(0,0)$, so starting from this ordered pair we can add m-factors of $(1,0)$ and n-factors of $(0,1)$ to obtain (m,n). Similarly we can use various additive combinations of these two factors to obtain all other ordered pairs in $\mathbb{Z}_m \times \mathbb{Z}_n$. This shows that $\langle(1,0),(0,1)\rangle$ are generators for all m and n.

$\square$

Example 3.2.7. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_4$. The *gcd*$(2,4) = 2 \neq 1$, so $\mathbb{Z}_2 \times \mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_8$ and must be generated with at least two generators. Unless otherwise stated, $(1,0)$ and $(0,1)$ will be the generators used whenever possible. Figure 3.2.2 shows $C^*(\mathbb{Z}_2 \times \mathbb{Z}_4, \langle(1,0),(0,1)\rangle)$.

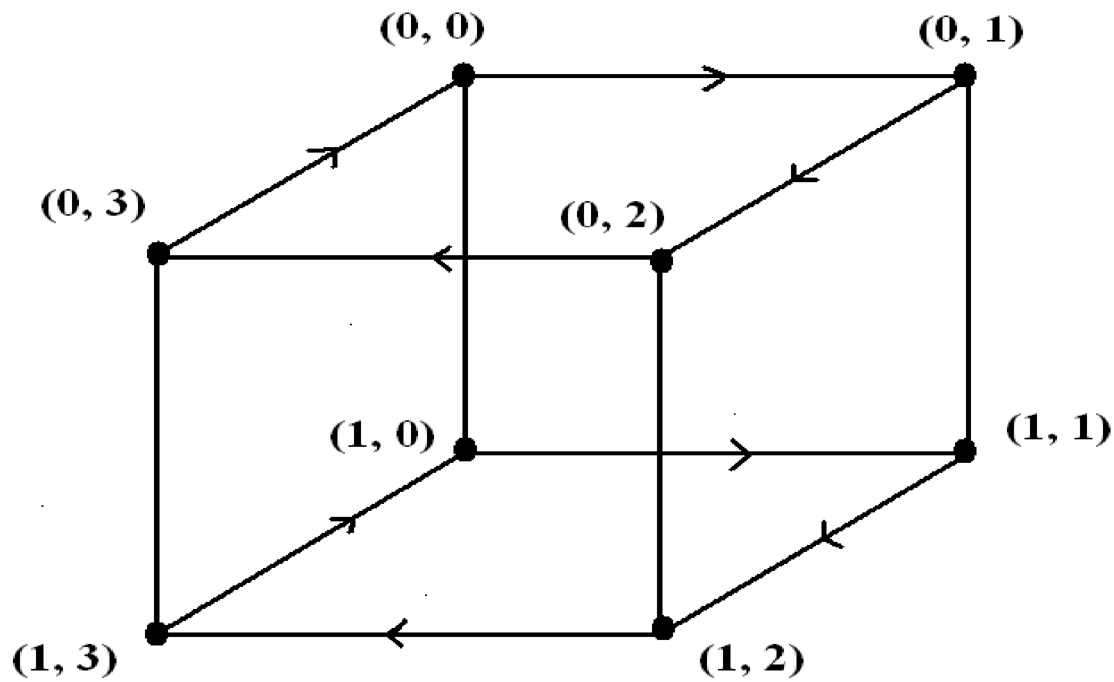

Figure 3.2.2. $C(\mathbb{Z}_2 \times \mathbb{Z}_4, \langle (1,0),(0,1) \rangle)$

Notice that this graph's Cayley skeleton is isomorphic to the Cartesian graph product $C_2 \,\Box\, C_4$. In general, we have the following proposition.

Proposition 3.2.8. *For any $m, n \in \mathbb{Z}^+$ with generating set $\langle e_1, e_2 \rangle$,*

$$C^*(\mathbb{Z}_m \times \mathbb{Z}_n, \langle e_1, e_2 \rangle) \simeq C_m \,\Box\, C_n.$$

Since this is but a particular case for a general proposition proven in the next section, we omit the proof here to minimize redundancy.

3.3 $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_k} \to C(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_k}, \langle e_1, e_2, \ldots, e_k \rangle)$

Having constructed, studied and classified the Cayley digraphs associated with modular products of one and two factors, we use this section to do the same for an arbitrarily extended set of modular factors. First, we introduce the necessary notation for a Cartesian graph product with an arbitrary number of factors.

Definition 3.3.1. Let $G_1, G_2, \ldots, G_k$ be graphs. Then their **Cartesian graph product** is the graph

$$G_1 \square G_2 \square \cdots \square G_k = \square_{i=1}^{k} G_i$$

with vertex set $\{(x_1, x_2, \ldots, x_k) | x_i \in V(G_i)\}$, and for which two vertices $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$ are adjacent whenever $x_i y_i \in E(G_i)$ for exactly one index $1 \leq i \leq k$, and $x_j = y_j$ for each index $j \neq i$. Additionally, we denote the k^{th} power of G with respect to the Cartesian product as $G^{\square,k} = \square_{i=1}^{k} G$.

Based on what the analysis of the one and two factor cases, we formulate the following proposition:

Proposition 3.3.2. *For any $n_1, n_2, \ldots, n_k \in \mathbb{Z}^+$ with generating set $\langle e_1, e_2, \ldots, e_k \rangle$,*

$$C^*(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdot \times \mathbb{Z}_{n_k}, \langle e_1, e_2, \ldots, e_k \rangle) \simeq C_{n_1} \square C_{n_2} \square \cdots \square C_{n_k}.$$

Proof. Let $G = C^*(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}, \langle e_1, e_2, \ldots, e_k \rangle)$ and let $H = C_{n_1} \square C_{n_2} \square \cdots \square C_{n_k}$.

We know

$$V(G) = \{(a_{1j}, a_{2j}, \ldots, a_{ij}, \ldots, a_{kj}) |\ a_{1j} \in \mathbb{Z}_{n_1}, a_{2j} \in \mathbb{Z}_{n_2}, \ldots, a_{kj} \in \mathbb{Z}_{n_k}\}$$

where $0 \leq j \leq n_i - 1$, and

$$E(G) = \{(a_{1i}, a_{2i}, \ldots, a_{ki})(a_{1j}, a_{2j}, \ldots, a_{kj}) |\ (a_{1i}, a_{2i}, \ldots, a_{ki}) = e_m + (a_{1j}, a_{2j}, \ldots, a_{kj})\}$$

for some $m = 1, 2, \ldots, k$.

We also know that each C_{n_i} has n_i elements, namely the set $\{\overline{0}, \overline{1}, \ldots, \overline{n_i - 1}\}$. For $a_{ij} \in \mathbb{Z}_{n_k}$, i tells us which n_i-cycle we are dealing with and j the $(j-1)^{th}$ vertex of this n_i-cycle. Clearly, $V(C_{n_i})$ has n_i vertices. Therefore, $V(H)$ consists of $n_1 n_2 \cdots n_k$ vertices also.

Having defined the vertex and edge sets, we establish the following function:

$$f : V(H) \to V(G)$$

where

$$f(a_i) = (a_i).$$

This function maintains a one-to-one correspondence between both vertex and edge sets.

Next we want to show that the corresponding adjacency matrices are element-wise equivalent. First we order the $n_1 n_2 \cdots n_k$ vertices of G as $\{a_0, a_1, \ldots, a_{n_1 n_2 \cdots n_k}\}$. Then the adjacency matrix for G is $[A_{i,j}]$, with entries either 0 or 1 for the following conditions:

$$A_G = \begin{cases} 1 & \text{if } a_i - a_j = e_l \text{ for some } l, \\ 0 & otherwise. \end{cases}$$

Likewise, the adjacency matrix for H is $[A_{ij}]$, with

$$A_H = \begin{cases} 1 & \text{if } f(a_i) - f(a_j) = e_l \text{ for some l}, \\ 0 & otherwise. \end{cases}$$

From the one-to-one function defined above and our earlier isomorphism proofs, it follows immediately that $A_G = A_H$ with the vertices of G simply mapped onto H. Since we have adjacency matrices here that preserve edges, we conclude that f is an isomorphism, so $G \simeq H$ as desired.

$\square$

Although creating graphical representations of Cayley digraphs and Cartesian graph products becomes exponentially more difficult as the number of modular factors involved increases, we can still illustrate with an example.

Example 3.3.3. Let $G = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Then $C^*(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \langle e_1, e_2, e_3 \rangle) \simeq C_3 \,\Box\, C_3 \,\Box\, C_3$ as shown in Figure 3.3.1.

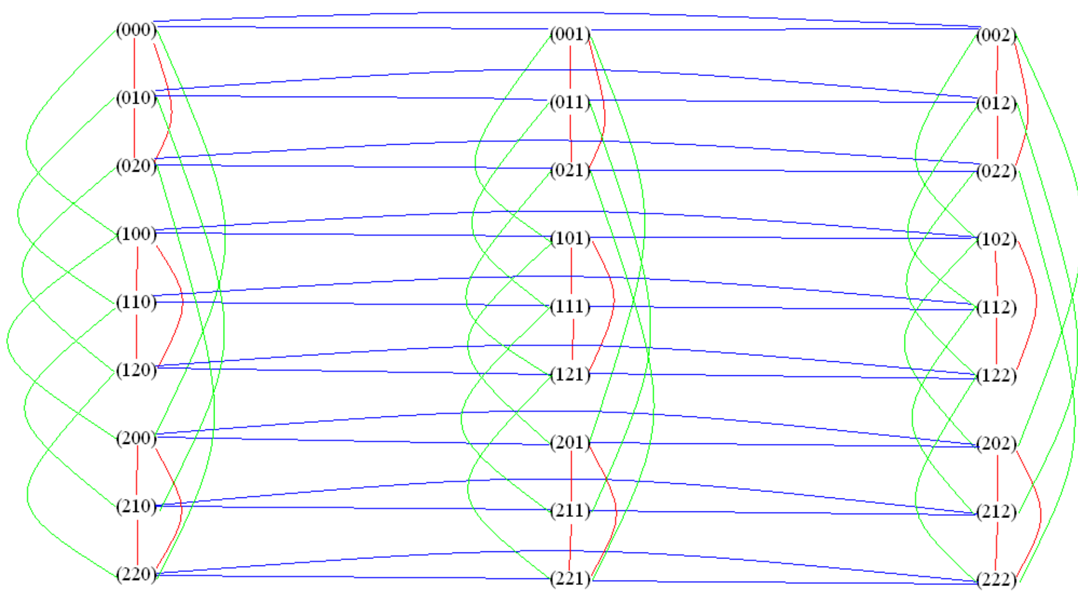

Figure 3.3.1. $C^*(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \langle e_1, e_2, e_3 \rangle) \simeq C_3 \,\Box\, C_3 \,\Box\, C_3$

4

Rings: Background and Notation

Having constructed and analyzed the Cayley digraphs and Cartesian graph products for all cases of the groups $\mathbb{Z}_n$ and $\mathbb{Z}_m \times \mathbb{Z}_n$, as well as the arbitrarily extended case $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_k}$, we can now use these graphs to study some applicable concepts of ring theory. In particular, we define, construct and study the quotient rings $k[x_1, x_2, \ldots, x_n]/I(G)$, where $k[x_1, x_2, \ldots, x_n]$ is a polynomial ring in indeterminants $x_1, x_2, \ldots, x_n$ with coefficients over the field k and $I(G)$ an edge ideal of a graph G consisting of the edge set of G connecting the vertices in G, namely $x_1, x_2 \ldots, x_n$. Our classified graphs from previous chapters will be the main mathematical ingredients for these constructed quotient rings.

Now since there are a number of concepts involved in the above quotient, we take the time here to deconstruct this mathematical entity in order to define its various parts.

4.1 Rings

The concept of a ring builds from the notion of a group in that it is a non-empty set which satisfies the conditions of a group under the binary operation of addition. Unlike groups, however, rings must also satisfy properties for the binary operation of multiplication. Below is the formal definition.

Definition 4.1.1. A **ring** is a non-empty set R with the binary operations addition $(+)$ and multiplication $(\cdot)$ such that for any $a, b, c \in R$, the following properties are satisfied:

1. $\langle R, +\rangle$ is an abelian group under addition.

2. Multiplication is associative.

3. For every $a, b, c \in R$, the $left\ distributive\ law$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and the $right\ distributive\ law$ $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ hold.

Example 4.1.2. Consider the cyclic group $(\mathbb{Z}_n, +)$. Given any $a, b \in \mathbb{Z}_n$, we define the product ab as the remainder of the usual product of the integers a and b when divided by n. Then $(\mathbb{Z}_n, +, \cdot)$ is a ring. Thus, in $\mathbb{Z}_5$, we have (2)(3)=1. We call this operation on $\mathbb{Z}_n$, **multiplication modulo n**.

As the concept that acts as a bridge between the graphs studied in earlier chapters and the quotient rings we study in this chapter, *ideals* are particularly significant. Informally, ring ideals are just subgroups of rings made up of cosets, that is, left or right multiplicative or additive subsets of the ring elements. Formally, we have the following definition.

Definition 4.1.3. An **ideal** is a subgroup I of a ring R such that

$$aI \subseteq I \text{ and } Ib \subseteq I$$

for all $a, b \in R$.

Example 4.1.4. In the ring $\mathbb{Z}$, we see that $3\mathbb{Z}$ is an ideal since it is a subgroup, and $a(3k) = (3k)a = 3(ka) \in 3\mathbb{Z}$ for all $a \in \mathbb{Z}$.

This concept of an ideal has been adapted to graphs by using the edges of a graph as the generators.

Definition 4.1.5. Given a graph G with vertices $x_1, \ldots, x_n$, we define the **edge ideal** of G, denoted $I(G)$, as the ideal of the polynomial ring $k[x_1, \ldots, x_n]$ with generators specified as follows: $x_i x_j$ is a generator of $I(G)$ if and only if x_i is connected to x_j in G.

In short, this establishes edge ideals for any graph whose vertices are joined by edges.

Example 4.1.6. Letting $G = C(\mathbb{Z}_2 \times \mathbb{Z}_4, \langle (1,0), (0,1) \rangle)$ as in Figure 4.1.1, we see that its edge ideal is

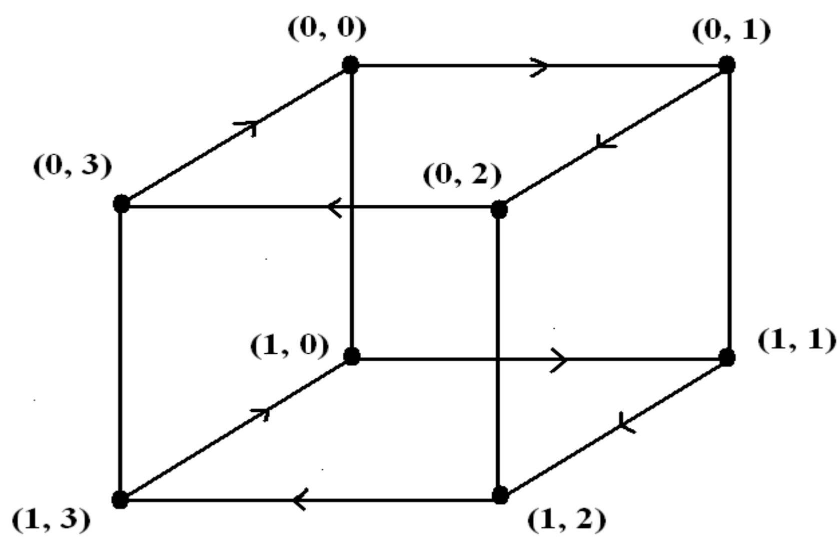

Figure 4.1.1. $C(\mathbb{Z}_2 \times \mathbb{Z}_4, \langle (1,0), (0,1) \rangle) \simeq C_2 \,\square\, C_4$

$$\begin{aligned} I(G) = \quad & \{(0,0)(0,1),(0,1)(0,2),(0,2)(0,3),(0,3)(0,0),(1,0)(1,1),(1,1)(1,2), \\ & (1,2)(1,3),(1,3)(1,0),(0,0)(1,0),(0,1)(1,1),(0,2)(1,2),(0,3)(1,3)\}. \end{aligned}$$

Definition 4.1.7. Let I be an ideal of a ring R. Then R/I is the ring formed by the additive cosets of I with binary operations defined as follows:

$$(a+I)+(b+I)=(a+b)+I$$

and

$$(a+I)(b+I)=ab+I.$$

We call the ring R/I the **quotient ring** (or factor ring) **of** R **by** I.

Example 4.1.8. It is immediate that $n\mathbb{Z}$ is an ideal of $\mathbb{Z}$ for all $n \in \mathbb{Z}$. Moreover, these are the only ideals of $\mathbb{Z}$, for they are the only subgroups of $\mathbb{Z}$. For $n = 12$ in the quotient ring $\mathbb{Z}/12\mathbb{Z}$, the coset elements are $\overline{0}, \overline{1}, \ldots, \overline{10}, \overline{11}$.

To add (or multiply) in $\mathbb{Z}/12\mathbb{Z}$, simply select any representatives for the two coset elements, add (or multiply) them in the integers $\mathbb{Z}$, and take the resultant coset that contains their sum (or product). For example, $\overline{7}+\overline{8}=\overline{15}=\overline{3}$, so $\overline{7}+\overline{8}=\overline{3}$ in $\mathbb{Z}/12\mathbb{Z}$. Likewise, $\overline{78}=\overline{56}=\overline{8}$ in $\mathbb{Z}/12\mathbb{Z}$.

In this project we study the quotient ring R/I, where $R = k[x_1, x_2, \ldots, x_n]$, a polynomial ring defined as follows:

Definition 4.1.9. Let k be a field. A **polynomial** $f(x)$ **with coefficients** in k is an infinite sum $\displaystyle\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x^1 + \ldots + a_n x^n + \ldots$, where $a_i \in k$ and $a_i = 0$ for all but a finite number of values of i. The a_i are the **coefficients of** $f(x)$. Addition and multiplication of polynomials with coefficients in a ring $R = k[x]$ are defined as follows: If

$$f(x) = a_0 + a_1x + \ldots + a_nx^n + \ldots$$

and

$$g(x) = b_0 + b_1x + \ldots + b_nx^n + \ldots,$$

then for polynomial addition, we have

$$f(x) + g(x) = c_0 + c_1x + \ldots + c_nx^n + \ldots$$

where $c_n = a_n + b_n$.

Likewise, for polynomial multiplication, we have

$$f(x)g(x) = d_0 + d_1x + \ldots + d_nx^n + \ldots +$$

where $d_n = \sum_{i=0}^{n} a_ib_{n-i}$.

In relation to the quotient ring R/I, where $R = k[x_1, x_2, \ldots, x_n]$ and $I(G)$ are defined as above, algebraists have interest in determining which of them satisfy the conditions defined below as Cohen-Macaulay.

Definition 4.1.10. The quotient ring R/I is called Cohen-Macaulay (or CM for short) if

$$depth(R/I) = dim(R/I)$$

and the ideal I is said to be CM if R/I is CM.

Although dimension and depth are not explicitly studied in this project, we present them here for the sake of self-sufficient reading, using Villarreal's definitions in [4].

Definition 4.1.11. Given a ring R, the **Krull dimension** (or simply the dimension) **of** R, denoted by $\dim(R)$ is the supremum of the length of all chains of prime ideals in R. The dimension of R is said to be infinite if R has arbitrary long chains of distinct prime ideals.

Definition 4.1.12. Let R be a polynomial ring over a field k with irrelevant maximal ideal α and I a graded ideal of R. The **depth** of R/I, denoted by $depth(R/I)$, is the largest integer r such that there is a regular sequence $x_i, \ldots, x_r$ in α that is not a zero-divisor of $R/(I, x_1, \ldots, x_{i-1})$ for all $1 \leq i \leq r$, with $f_0 = 0$.

For each group-based graph, it is possible to formulate a corresponding polynomial ring. We do so by letting G be a graph with vertices $v_1, v_2, \ldots, v_n$ and $R = k[x_1, x_2, \ldots, x_n]$ a polynomial ring over a field k, such that there is one variable x_i for each vertex v_i. We identify the vertex v_i with the variable x_i.

In terms of the Cohen-Macaulay relationship between a graph G and its associated quotient ring $R/I(G)$, there is the following definition:

Definition 4.1.13. The graph G is said to be Cohen-Macaulay over the field k (CM for short) if $R/I(G)$ is a Cohen-Macaulay ring.

Analyzing a quotient ring's depth and dimension can be quite complicated. Ideally, we would like to have a quick and easy test for determining which of them are Cohen-Macaulay.

Unfortunately, no formula has been discovered yet that gives both necessary and sufficient conditions. Thanks to the work of R.H. Villarreal in [4], however, we do have an easy test for a necessary condition all graphs must satisfy in order to be Cohen-Macaulay

Since we are dealing with related concepts from two distinct branches of mathematics, we define each separately. First, we define what it means for a graph to be unmixed. Then we explain what it means for an ideal generated by variables to be a minimal prime and introduce the notion of a minimal primary decomposition, which is a useful tool to use with Macaulay2 software in determining whether or not a particular graph happens to be unmixed. Finally, we move on to define the graph theory concept of a vertex cover, providing examples using graphs from earlier in the project.

Definition 4.1.14. An edge ideal $I(G)$ is **unmixed of dimension** r if all of its minimal prime ideals are generated by n-r variables. A graph is unmixed of dimension r if its edge ideal is unmixed of dimension r.

Definition 4.1.15. A **prime ideal** is an ideal $I \neq R$ in a commutative ring R such that if $ab \in I$, then either $a \in I$ or $b \in I$ for $a, b \in R$.

Definition 4.1.16. A **minimal prime ideal** is a prime ideal containing $I(G)$ and is minimal with respect to inclusion.

Definition 4.1.17. To say that $I(G)$ is a **minimal primary decomposition** is equivalent to saying it is the intersection of its minimal prime ideals.

This last concept is extremely powerful in that Macaulay2 has a "primaryDecomposition" command that outputs a list of all minimal prime ideals for each inputed edge ideal of a given graph. Once we have this list, a simple comparison of the lengths of these minimal prime ideals tells us whether or not we have an unmixed and therefore potentially Cohen-Macaulay graph.

Finally, we have the graph theory concept of a vertex cover that gives us extremely useful information associated with a group's algebraic structure.

Definition 4.1.18. Suppose G is a graph with vertex set $V = \{x_1, \ldots, x_n\}$. A subset $A \subseteq V$ is a **vertex cover** of G if every edge in G is incident to some vertex in A. A vertex cover A is *minimal* if no proper subset of A is a vertex cover.

Since from [2], we know that $I(G)$ is the intersection of minimal primes (that is, of minimal vertex covers), we can test the lengths of a graph's vertex covers to see if the graph is mixed or unmixed.

Proposition 4.1.19. *If $I(G)$ is Cohen-Macaulay, then $I(G)$ is unmixed.*

Additionally from [2] we have another proposition of Villarreal's that links a combinatoric aspect of a graph with a corresponding algebraic aspect:

Proposition 4.1.20. *Let G be a graph with vertices $x_1, \ldots, x_n$. Then, an ideal generated by variables is a minimal prime of $I(G)$ if and only if the corresponding vertices are a minimal vertex cover of G.*

For a proof, see [2].

In short, it turns out that if any two minimal vertex covers of a graph are the same length, then the graph is unmixed. While this does not, by itself, determine if the graph is Cohen-Macaulay, it does increase the probability of it being so, thus justifying further investigative efforts. Moreover, if there exists variance in the lengths of a graph's minimal vertex covers, we can see this immediately and conclude definitively that it is not Cohen-Macaulay.

In particular, we frequently use the converse of Proposition 4.1.19. as a test for which graphs are not Cohen-Macaulay: If $I(G)$ is mixed, the graph is not Cohen-Macaulay. When $I(G)$ is unmixed, we test it further using Macaulay2 to determine if depth(R/I) =dim(R/I). This strategy makes it possible to formulate a theorem that categorizes the values of n for which $C(\mathbb{Z}_n, \langle a \rangle)$ is Cohen-Macaulay.

To illustrate the correspondence between the primary decomposition of a graph's edge ideal and its minimal vertex covers, we provide an example that demonstrates the methodology this project uses in its general investigations.

Example 4.1.21. Let $G = C(\mathbb{Z}_2 \times \mathbb{Z}_3, \langle (1,0), (0,1) \rangle)$. Defining $I(G)$ in Macaulay2 and using the "primaryDecomposition" command yields the following set of minimal prime ideals whose intersection is $I(G) : \{(x_{00}, x_{01}, x_{10}, x_{12}), (x_{00}, x_{01}, x_{11}, x_{12}),$
$(x_{00}, x_{02}, x_{10}, x_{11}), (x_{00}, x_{02}, x_{11}, x_{12}), (x_{01}, x_{02}, x_{10}, x_{11}), (x_{01}, x_{02}, x_{10}, x_{12})\}$.

From Figure 4.1.2, it is clear that each of these six sets of vertices produces a minimal vertex cover of G. (In this graph, edges without an arrow indicate that both of its vertices are in the vertex cover). Additionally, we know that since all of these minimal vertex covers are unmixed, that this graph is potentially Cohen-Macaulay. And calculating the depth(G) and dim(G) with Macaulay2, we find that this graph is in fact Cohen-Macaulay.

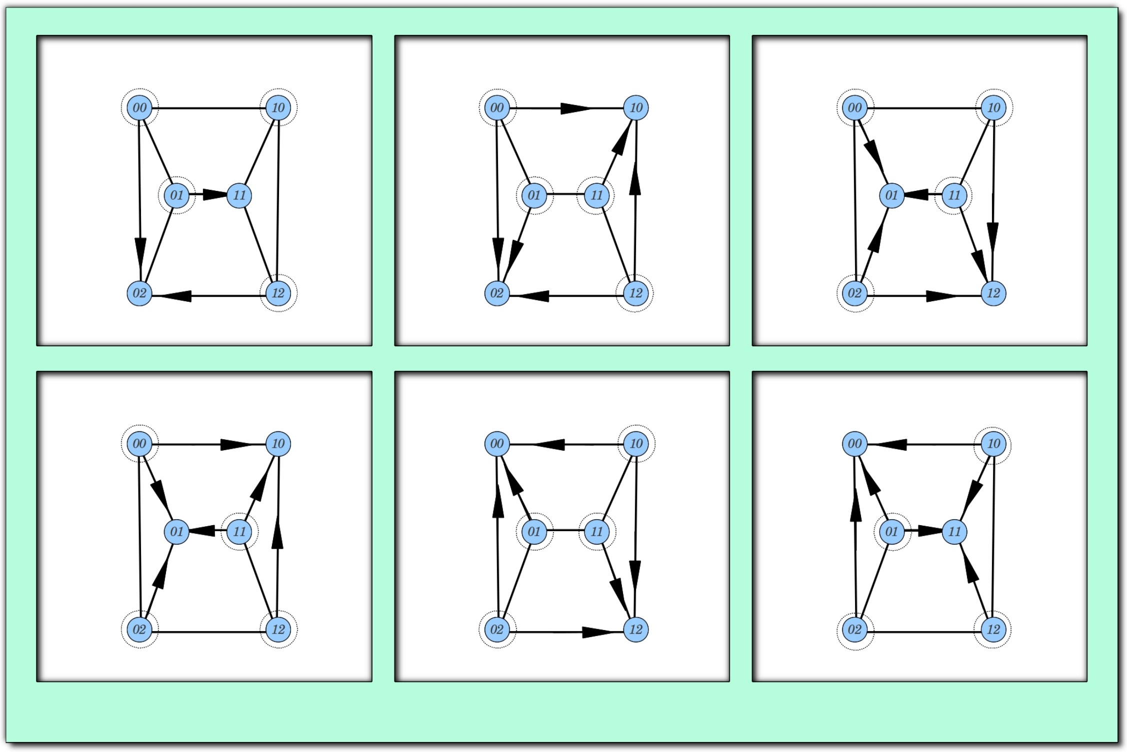

Figure 4.1.2. The Six Minimal Vertex Covers of $C(\mathbb{Z}_2 \times \mathbb{Z}_3, \langle(1,0),(0,1)\rangle)$

5
Graphs to Quotient Rings

Having established the key concepts necessary for our intended purposes, we now take some of the group-based graphs studied already and construct their corresponding quotient rings for further investigations. After formulating these quotient rings, we then investigate their minimal primary decompositions, depths and dimensions using Macaulay2, determining when possible whether or not they are Cohen-Macaulay. Due to the exponential growth in the complexity of calculating the specific values involved in some of our examples, even the advanced power of a computer produces an extremely limited amount of data. Nevertheless, we are still able to make some observations and formulate a conjecture on these observed patterns.

As was the case earlier, we begin with the least complicated group $\mathbb{Z}_n$. For each value of $n \in \mathbb{Z}^+$, we have a polynomial ring $R = k[x_1, x_2, \ldots, x_n]$ over a field k whose indeterminants are in a one-to-one correspondence with the vertex set of $C(\mathbb{Z}_n, \langle a \rangle)$. Furthermore, for each graph $G = C(\mathbb{Z}_n, \langle a \rangle)$, the edge ideal $I(G)$ is simply the edges of an *n-cycle*. In this section we determine some particular polynomial rings and edge ideals. With the information obtained, we then construct and study their quotient rings using Macaulay2 software.

5.1 Quotient Rings for $C(\mathbb{Z}_n, \langle a\rangle)$

Example 5.1.1. Let $G = C(\mathbb{Z}_3, \langle 2\rangle)$. Since $\mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$, the corresponding polynomial ring is $R = k[x_0, x_1, x_2]$. From Proposition 3.1.5., we know $C^*(\mathbb{Z}_3, \langle a\rangle) \simeq C_3$, a 3-cycle. Thus

$$I(G) = \{x_0x_1, x_1x_2, x_2x_0\}.$$

Having established these two basic components, we see that G has the following quotient ring:

$$H = \frac{k[x_0, x_1, x_2]}{\{x_0x_1, x_1x_2, x_2x_0\}}.$$

Using Macaulay2, the first thing we look at is the primary decomposition of H to see if it is unmixed or mixed. Applying the "primaryDecomposition" command, we find that the minimal vertex covers of H are all of length 2: $\{x_0, x_1\}, \{x_0, x_2\}$, and $\{x_1, x_2\}$.

Since H is unmixed, we continue the investigation by checking the depth and dimension of H. Doing so, we find that $depth(H) = 1$ and $dim(H) = 1$. Since $depth(H) = dim(H)$, we see that H is Cohen-Macaulay.

Following these steps for $C^*(\mathbb{Z}_n, \langle a\rangle)$, where $2 \leq n \leq 20$, Macaulay2 yields the collection of data in the table of Figure 5.1.1.

As seen by the data of Figure 5.1.1, only 5 of these first 19 cases are unmixed; of these 5 unmixed cases, only 3 prove to be Cohen-Macaulay. Moreover, Figure 5.1.1 reveals two particularly significant patterns: (1) the $dim(H)$ increases at a greater rate than the

$G = Z_n$	$C^*(G, \langle a \rangle)$	*Unmixed*	*Dim*(R/I)	*Depth*(R/I)	*C-M*
$G = Z_2$	K_2	yes	1	1	yes
$G = Z_3$	C_3	yes	1	1	yes
$G = Z_4$	C_4	yes	2	1	no
$G = Z_5$	C_5	yes	2	2	yes
$G = Z_6$	C_6	no	3	2	no
$G = Z_7$	C_7	yes	3	2	no
$G = Z_8$	C_8	no	4	3	no
$G = Z_9$	C_9	no	4	3	no
$G = Z_{10}$	C_{10}	no	5	3	no
$G = Z_{11}$	C_{11}	no	5	4	no
$G = Z_{12}$	C_{12}	no	6	4	no
$G = Z_{13}$	C_{13}	no	6	4	no
$G = Z_{14}$	C_{14}	no	7	5	no
$G = Z_{15}$	C_{15}	no	7	5	no
$G = Z_{16}$	C_{16}	no	8	5	no
$G = Z_{17}$	C_{17}	no	8	6	no
$G = Z_{18}$	C_{18}	no	9	6	no
$G = Z_{19}$	C_{19}	no	9	6	no
$G = Z_{20}$	C_{20}	no	10	7	no

Figure 5.1.1. Data on $\mathbb{Z}_n$, $2 \leq n \leq 20$

depth(H), and (2) for $n \geq 7$, $C(G, \langle a \rangle)$ is mixed. For the first pattern, we make the following conjecture:

Conjecture 5.1.2. *Given the quotient ring $H = R/I(G)$ associated to the graph $G = C^*(\mathbb{Z}_n, \langle a \rangle)$ for values of $n \geq 6$, the dim$(H) >$ depth(H).*

As for the second observed pattern, it seems that H is Cohen-Macaulay only when $n = 2, 3$ or 5. We state this as a theorem and provide a proof.

Theorem 5.1.3. *$C^*(\mathbb{Z}_n, \langle a \rangle)$ is Cohen-Macaulay if and only if $n = 2, 3,$ or 5.*

Proof. From Proposition 3.1.5., we know that for any $n \geq 3$ and generator a, ${}^*C(\mathbb{Z}_n, \langle a \rangle) \simeq C_n$. Thus we have that $C^*\mathbb{Z}_3, \langle a \rangle) \simeq C_3$ and $C^*(\mathbb{Z}_5, \langle a \rangle) \simeq C_5$. Now in [4], Villarreal proves that C_3 and C_5 are the only Cohen-Macaulay cycles; therefore, $C^*(\mathbb{Z}_3, \langle a \rangle)$ and $C^*(\mathbb{Z}_5, \langle a \rangle)$ are the only Cohen-Macaulay graphs for $n \geq 3$.

For $n = 2$, $C^*(\mathbb{Z}_2, \langle 1 \rangle) \simeq K_2$, so neither Corollary 5.0.17 from [4] nor Proposition 3.1.5. provide any information as to whether or not it is Cohen-Macaulay. However, the data in Figure 5.1.1 obtained using Macaulay2, verifies that dimension equals depth in this case. This shows that $C^*(\mathbb{Z}_2, \langle a \rangle)$ is also Cohen-Macaulay.

Thus, we have shown that $C^*(\mathbb{Z}_n, \langle a \rangle)$ is Cohen-Macaulay if and only if $n = 2, 3$, or 5.

□

5.2 Quotient Rings for $\mathbb{Z}_m \times \mathbb{Z}_n, \langle (1,0), (0,1) \rangle)$

In this section, we proceed in the same fashion as in Section 5.1. However, due to the combinatorial complexity involved in determining the depths, dimensions and primary decompositions of the quotient rings associated with Cayley skeletons with increasingly large vertex sets, the data we are able to obtain is extremely limited (even with the power of Macaulay2!). We begin with an analysis of $C^*(\mathbb{Z}_2 \times \mathbb{Z}_n, \langle (1,0), (0,1) \rangle), 2 \leq n \leq 20$.

Example 5.2.1. Let $G = C^*(\mathbb{Z}_2 \times \mathbb{Z}_3, \langle (1,0), (0,1))$. Since

$$\mathbb{Z}_2 \times \mathbb{Z}_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\},$$

the corresponding polynomial ring is $R = k[x_{00}, x_{01}, x_{02}, x_{10}, x_{11}, x_{12}]$. Proposition 3.2.9. tells us that $G \simeq C_2 \,\Box\, C_3$, and from Example 2.2.8. we know that

$$I(G) = \{x_{00}x_{01}, x_{01}x_{02}, x_{02}x_{00}, x_{10}x_{11}, x_{11}x_{12}, x_{12}x_{10}, x_{00}x_{10}, x_{01}x_{11}, x_{02}x_{12}\}.$$

Hence we have the quotient ring

$$H = \frac{k[x_{00}, x_{01}, x_{02}, x_{10}, x_{11}, x_{12}]}{\{x_{00}x_{01}, x_{01}x_{02}, x_{02}x_{00}, x_{10}x_{11}, x_{11}x_{12}, x_{12}x_{10}, x_{00}x_{10}, x_{01}x_{11}, x_{02}x_{12}\}}.$$

Using Macaulay2, we find that H is unmixed and that $dim(H) = depth(H) = 2$. Since this shows that H is a Cohen-Macaulay ring, we conclude that $C^*(\mathbb{Z}_2 \times \mathbb{Z}_3, \langle(1,0),(0,1)\rangle)$ is Cohen-Macaulay over the field k. This graph is represented in the Appendix of [4], which lists all Cohen-Macaulay graphs on up to six vertices.

Although the data obtained is limited even for relatively small values of n, Macaulay2 still yields the following useful table of Figure 5.2.1.

$G = Z_m \times Z_n$	$C^*(G, \langle(1,0),(0,1)\rangle)$	*Unmixed*	*Dim*(R/I)	*Depth*(R/I)	*C-M*
$G = Z_2 \times Z_2$	$C_2 \square C_2$	yes	2	1	no
$G = Z_2 \times Z_3$	$C_2 \square C_3$	yes	2	2	yes
$G = Z_2 \times Z_4$	$C_2 \square C_4$	no	4	2	no
$G = Z_2 \times Z_5$	$C_2 \square C_5$	yes	4	2	no
$G = Z_2 \times Z_6$	$C_2 \square C_6$	no	6	3	no
$G = Z_2 \times Z_7$	$C_2 \square C_7$	no	6	4	no
$G = Z_2 \times Z_8$	$C_2 \square C_8$	no	8	4	no
$G = Z_2 \times Z_9$	$C_2 \square C_9$	no	8	4	no
$G = Z_2 \times Z_{10}$	$C_2 \square C_{10}$	no	10	5	no
$G = Z_2 \times Z_{11}$	$C_2 \square C_{11}$	no	11	6	no
$G = Z_2 \times Z_{12}$	$C_2 \square C_{12}$	no	?	?	no
$G = Z_2 \times Z_{13}$	$C_2 \square C_{13}$	no	?	?	no
$G = Z_2 \times Z_{14}$	$C_2 \square C_{14}$	no	?	?	no
$G = Z_2 \times Z_{15}$	$C_2 \square C_{15}$	no	?	?	no
$G = Z_2 \times Z_{16}$	$C_2 \square C_{16}$	no	?	?	no
$G = Z_2 \times Z_{17}$	$C_2 \square C_{17}$	no	?	?	no
$G = Z_2 \times Z_{18}$	$C_2 \square C_{18}$	no	?	?	no
$G = Z_2 \times Z_{19}$	$C_2 \square C_{19}$	no	?	?	no
$G = Z_2 \times Z_{20}$	$C_2 \square C_{20}$	no	?	?	no

Figure 5.2.1. Data on $\mathbb{Z}_2 \times \mathbb{Z}_n,\ 2 \leq n \leq 20$

From the table, we see that $dim(R/I)$ is at least 2 values greater than $depth(R/I)$. We make the following conjecture based upon this increasing difference between the depths and dimensions of these quotient rings:

Conjecture 5.2.2. *Given the group $G = \mathbb{Z}_2 \times \mathbb{Z}_n$, the quotient ring associated with its Cayley skeleton is Cohen-Macaulay if and only if $n = 3$.*

Finally, Macaulay2 gives the following data for $G = C^*(\mathbb{Z}_3 \times \mathbb{Z}_n, \langle(1,0),(0,1)\rangle)$ for $3 \leq n \leq 10$. See the table of Figure 5.2.2.

$G = Z_m \times Z_n$	$C^*(G, \langle(1,0),(0,1)\rangle)$	*Unmixed*	*Dim*(R/I)	*Depth*(R/I)	$C-M$
$G = Z_3 \times Z_3$	$C_3 \square C_3$	yes	3	2	no
$G = Z_3 \times Z_4$	$C_3 \square C_4$	yes	4	2	no
$G = Z_3 \times Z_5$	$C_3 \square C_5$	yes	5	4	no
$G = Z_3 \times Z_6$	$C_3 \square C_6$	yes	?	?	?
$G = Z_3 \times Z_7$	$C_3 \square C_7$	yes	?	?	?
$G = Z_3 \times Z_8$	$C_3 \square C_8$	yes	?	?	?
$G = Z_3 \times Z_9$	$C_3 \square C_9$	yes	?	?	?
$G = Z_3 \times Z_{10}$	$C_3 \square C_{10}$	yes	?	?	?

Figure 5.2.2. Data on $\mathbb{Z}_3 \times \mathbb{Z}_n,\ 3 \leq n \leq 10$

Interestingly, it seems that if $\mathbb{Z}_3$ is one of the Cayley skeleton's modular factors, then its edge ideal is unmixed for all $\mathbb{Z}_n$. We state this as another conjecture.

Conjecture 5.2.3. *Let $G = \mathbb{Z}_3 \times \mathbb{Z}_n$. Then its corresponding Cayley skeleton is unmixed for all values of n.*

Although at present, we are unable to verify which, if any, of these graphs are Cohen-Macaulay, the fact that they are unmixed suggests that future research would be justified. It could be that fixing one modular factor in $\mathbb{Z}_m \times \mathbb{Z}_n$ with particular values forces its Cayley skeletons to be unmixed. Likewise, in cases of more than two modular factors, one could try fixing the values of whole sets of factors to see if that forces the graphs to be unmixed.

6
Conclusion

Having completely studied the Cayley digraphs, Cayley skeletons, Cartesian graph products and quotient rings for several types of modular groups, we restate some of the results found in the process. We found and proved that the Cayley skeletons of the group $\mathbb{Z}_n$ are n-cycles. Moreover, for any arbitrary number of modular factors, we have firmly established several isomorphic relationships between the group's Cayley skeletons and its Cartesian graph products.

Taking the group-based graphs studied in Chapters 3, we constructed their associated quotient rings and using Macaulay2 software investigated which were Cohen-Macaulay. Then combining these findings with the data obtained from Macaulay2 and a Corollary established by Villarreal, we were able to prove a general thereom completely classifying the Cohen-Macualyness of an entire family of Cayley digraphs. From there, we devoted the rest of the project to analyzing the data obtained from Macaulay2.

At the start of this project, my goal was to gain a better understanding of abstract math by way of using graph representations. Undoubtedly I've accomplished that goal. Above all, I learned how rewarding it can be to completely classify something, turning my own observations into propositions with proofs. In the process, however, I became painfully aware of just how little I know in the seemingly infinite world of mathematics. Chapter 5 perfectly illustrates this point in that I do not have a complete understanding of most of the concepts I used. For that reason, I view Chapter 5 as one of my more significant accomplishments. With minimal understanding of the mathematics, I successfully used definitions, computer software and the powerful math skill of pattern recognition to find several potentially fruit-yielding paths to future research projects. In the future, as I follow these paths, I will at least know that I have it in me to open my own doors. In terms of the great realm of mathematical unknowns, this project has replaced an inexperienced fear with a more mature mathematical faith.

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