CIS 410/510 HW 2 - Brett Sumser

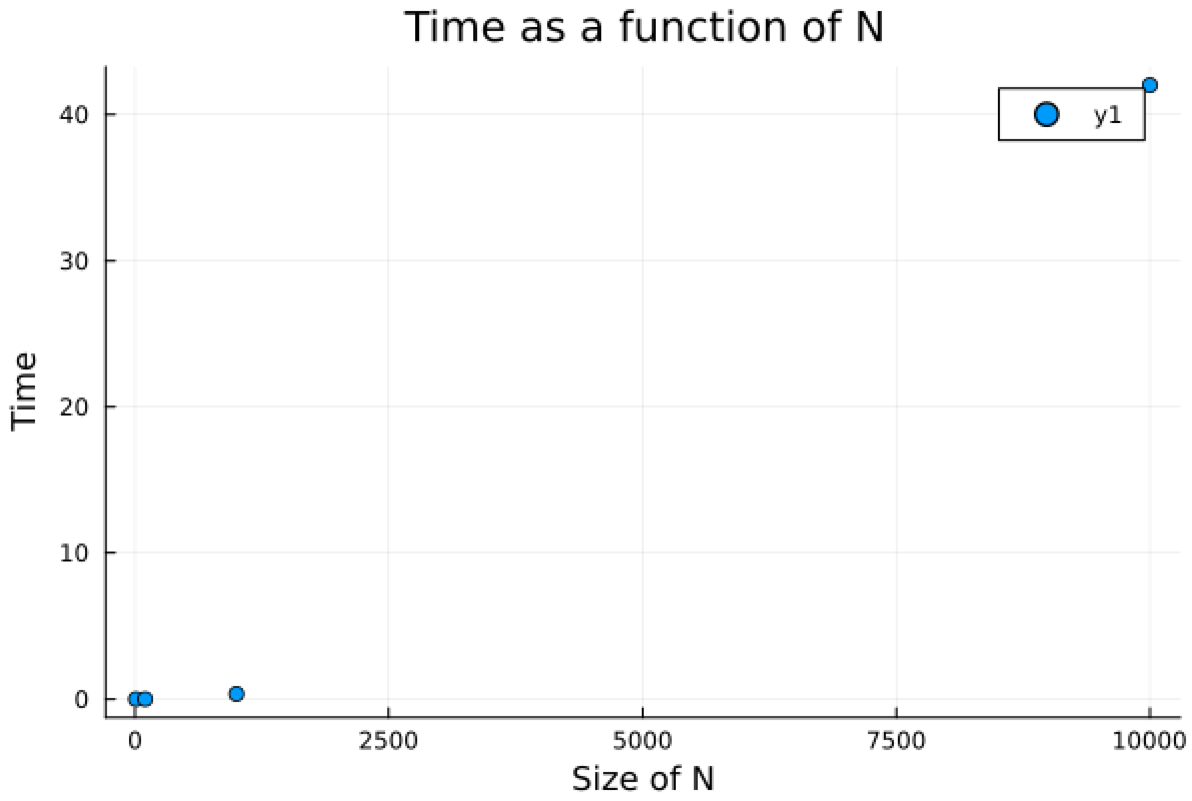
For the first part of this assignment, we have to implement the conjugate gradient descent algorithm. This algorithm is used to find a numerical solution for a system of linear equations where the matrix is positive definite. To check my implement for correctness, I implemented a test function that ran the algorithm on a small system where the solution was known.

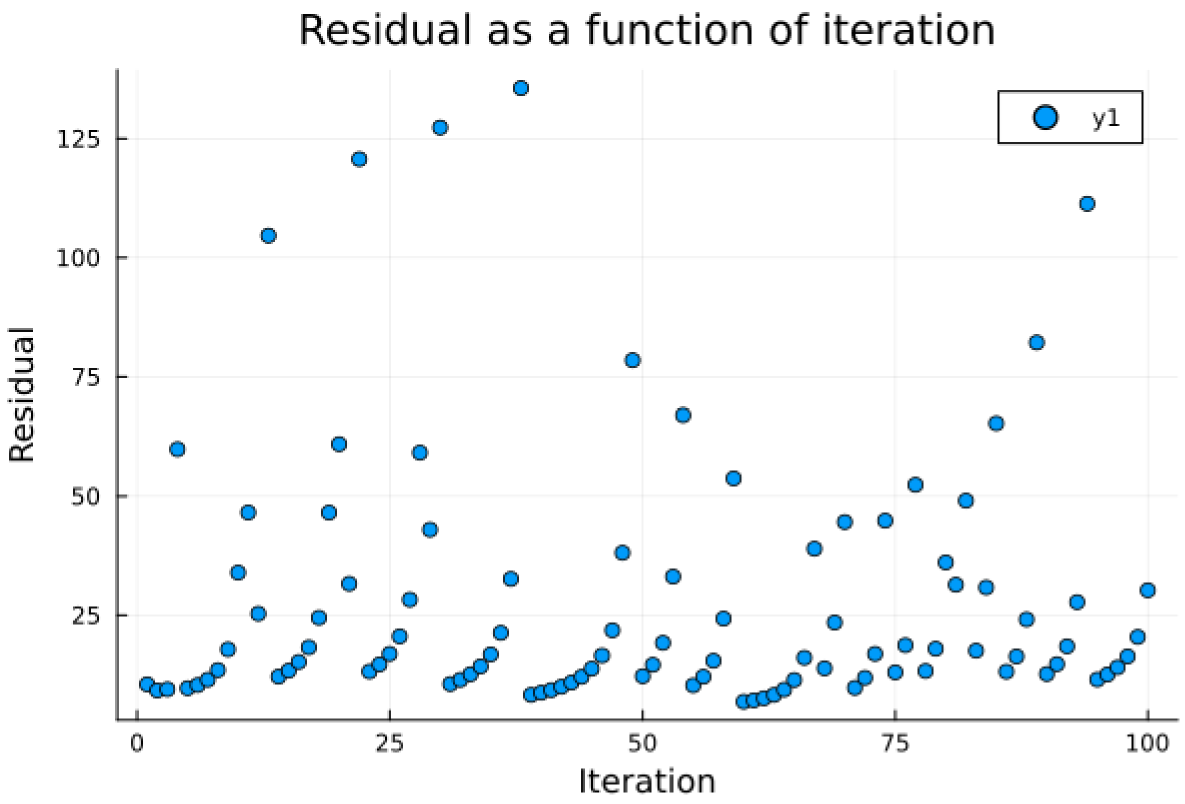
Here are the functions I was able to successfully implement:

- conj grad(A, x, b, ϵ, item max) - This is my implementation of the conjugate gradient algorithm. It takes a positive definite matrix A, an initial guess vector x, a tolerance $\epsilon andamaxiterationasarguments.$

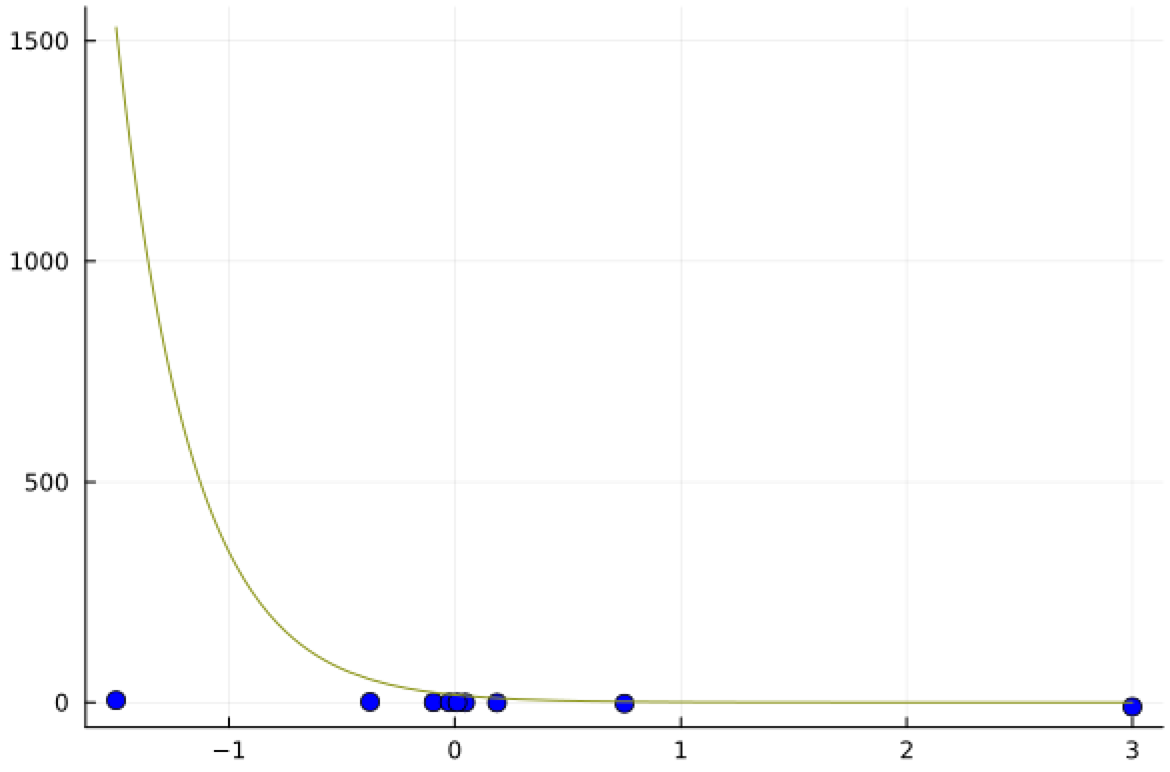
- conj grad test() - This is a test implementation of the conjugate gradient algorithm, but it solves a known system with a size N = 2. It calls the above function and compares the result against the known solution. This is how I checked that my implementation was working correctly.
- err R(A, x, b) - This function calculates the relative error, using a helper magnitude function to calculate: $err_R = \frac{||Ax-b||}{||x||}$
- var set(N) - This function sets the variables needed for the gradient algorithm, taking a size N and setting up the required vectors and matrix.
- magnitude(x) -This function takes a vector x and performs the 2-norm calculation.

From the below figure, it can be seen that up to $N = 1000$, the time is fairly consistent. But once you get up to $N = 10000$, the amount of time it takes for the algorithm to run increases by roughly 40x! I am not sure if I calculated my residuals correctly, but it seems like they grow cyclicly.





For the second part of the assignment, I struggled with modifying forwards euler into a backwards implmentation. Admittedly, I spent most of the time for the assignment working on the conjugate gradient, and might have overestimated how easy the second part would be. I was able to plot a line for the function $g(x) = 17 * e^(-3t)$, and it shoes that the values seem to converge appropriatly. I also was not sure how to plot a stable time step. I ended up using $= 0.5$.



```
using Plots # add Plots.jl from the package manager if you have not already done so.
using Printf # for formatting text output

# HW 1 starting script (if you want): contains function computeLU() - to compute an LU
# matrix A, namely, A = LU, where L and U are lower and upper triangular matrices.

#-------------------------HOMEWORK 2 FUNCTIONS--------------------------------------------
#-----------------------------------------------------------------------------------------
"""
    conj_grad(A, x, b,   , iter_max)

Function that performs conjugate gradient algorithm given:
    A - positive definite matrix
    x_0 - initial guess
       - tolerance
    iter_max - maximum number of iterations

Returns approximate solution to Ax = b, where relative err_R <=
"""
function conj_grad(A, x, b,   , iter_max)
    res_i = []
    N = size(A, 1)
    r = Matrix{Float64}(undef, N, 1)
    r .= 0
    r = r[:]

       = Matrix{Float64}(undef, N, 1)
       .= 0
       =   [:]

       = Matrix{Float64}(undef, N, 1)
       .= 0
       =    [:]

    p = Matrix{Float64}(undef, N, 1)
    p .= 0
    p = p[:]

    new_r = Matrix{Float64}(undef, 2, 1)
    new_r .= 0
    new_r = r[:]

    new_x = 0

    new_p = Matrix{Float64}(undef, 2, 1)
    new_p .= 0
    new_p = r[:]

    r = b - A * x
    p .= r

    for i = 1:iter_max # march across columns
           = (r'*r) ./ (p'*A*p)
        new_x = x +    .* p
        x = new_x
        new_r .= r -    .* A * p
           = (new_r' * new_r) ./ (r' * r)
        if (err_R(A,x,b) <=   )
            display("converged")
            return new_x
        end
        push!(res_i, (magnitude(A * x - b)))
        r = new_r
        new_p .= new_r +    .* p
        p = new_p
    end

    return (new_x, res_i)
end

"""
    conj_grad_test()

This is a test function for the conjugate gradient algorithm.
It solves a simple Ax = b using matrix and vectors of size 2 with a known solution
"""
function conj_grad_test()
    N = 2
    B = rand(N,N)

    b = rand(N,1)
    b = b[:]

    A = Matrix{Float64}(undef, 2, 2)
    A .= [4 1;1 3]
    display(A)

    x = Matrix{Float64}(undef, 2, 1)
    x .= 0
    x = x[:]
    x = [2 1]
    x = vec(x)
    display(x)

    b = Matrix{Float64}(undef, 2, 1)
    b .= 0
    b = [:]
    b = [1 2]
    b = vec(b)
    display(b)

    r = Matrix{Float64}(undef, 2, 1)
    r .= 0
    r = r[:]
    display(r)

    p = Matrix{Float64}(undef, 2, 1)
    p .= 0

       = 10^-6
    iter_max = 2

    x = conj_grad(A, x, b,  , iter_max)
    return x

end

"""
    err_R(A, x, b)

Function that calculates the relative error based on formula in assignment
"""
function err_R(A, x, b)
    err_r = magnitude(A * x - b) / magnitude(x)
    return err_r
end

"""
    var_set(N)

Function that prepares variables for conjugate gradient algorithm.
Takes parameter N for size of matrix/vectors. Returns:
    A - random matrix of size N made using A = I +B^T B, where I is the identity matri
    b - random vector of size N
    x - 0 vector for initial guess
"""
function var_set(N)
       = 10^-4
    iter_max = 100
    A = Matrix{Float64}(undef, N, N)

    B = rand(N,N)
    b = rand(N,1)
    b = b[:]

    x = rand(N,1)
    x .= 0
    x = x[:]
    I = Matrix{Float64}(undef, N, N)
    I .= 0

    for i = 1:N
        I[i,i] = 1
    end
    A .= I .+ B'B

    return (A, b, x,   , iter_max)
end

"""
    magnitude(x)

Performs magnitude calculation of vector
"""
function magnitude(x)
    N = size(x, 1)
    sum = 0
    mag = 0
    for i = 1:N
        sum += (x[i])^2
    end
    mag = sqrt(sum)
    return mag
end
#-------------------------END HOMEWORK 2 FUNCTIONS-------------------------------------
#------------------------------------------------------------------------------------

#-------------------------HOMEWORK 1 FUNCTIONS-----------------------------------------
#------------------------------------------------------------------------------------
"""
    computeLU(A)
Compute and return LU factorization `LU = A` of square matrix `A`.
Might not work on all matrices, since no pivoting is done!
# Examples (don't need examples, but fine to include)
'''
julia> A = [6 -2 2;12 -8 6;3 -13 3]
3 3  Array{Int64,2}:
  6   -2  2
 12   -8  6
  3  -13  3
julia> (L, U) = computeLU(A)
([1.0 0.0 0.0; 2.0 1.0 0.0; 0.5 3.0 1.0], [6.0 -2.0 2.0; 0.0 -4.0 2.0; 0.0 0.0 -4.0])
julia> norm(A - L*U)
0.0
'''
"""
function computeLU(A)

    N = size(A)[1]

    #Id = Matrix{Float64}(I, N, N) # N x N identity matrix
    Id = create_identity(N)

    L = copy(Id)   # initialize
    U = copy(Id)   # initialize
     A    = copy(A) # initialize. A   corresponds to A as it goes under elimination sta
    for k = 1:N-1 # march across columns

        (Lk, Lk_inv) = compute_Lk(A  , k)

         A   .= Lk * A
        L .= L * Lk_inv

    end

    U .= A

    return (L, U)

end


"""
    compute_Lk(A, k)
Compute Lk and its inverse from A, assuming first k-1 columns have undergone eliminati
"""
function compute_Lk(A, k)


    N = size(A)[1]

    Lk = create_identity(N) # Matrix{Float64}(I, N, N)       # initialize as identity
    Lk_inv = create_identity(N)# Matrix{Float64}(I, N, N)   # initialize as identity m

    # now modify column k, strictly below diagonal (i = k+1:N)
    for i = k+1:N
        Lk[i,k] = -A[i,k] / A[k,k]    # fill me in (compute elimination factors)
        Lk_inv[i,k] = A[i,k] / A[k,k]  # fill me in (compute elimination factors)
    end

    return (Lk, Lk_inv)

end

"""
    create_identity(N)
Given integer N, constructs a square identity matrix of size N.
"""
function create_identity(N)

    I = Matrix{Float64}(undef, N, N)
    I .= 0

    for i = 1:N
        I[i, i] = 1
    end

    return I
end

"""
    find_pivot(A, k)

Given matrix A and column k, find largest element in that column. Uses built in method
where A[] is the proper slice of the matrix for the column we need. findmax() returns
```

```
and a Cartesian Coordinate pair for the index of the element.

julia> A .= [6 -2 2;12 -8 6;3 -13 3]
3 3  Matrix{Float64}:
  6.0   -2.0  2.0
 12.0   -8.0  6.0
  3.0  -13.0  3.0

julia> A[:,1:1]
3 1  Matrix{Float64}:
  6.0
 12.0
  3.0

julia> A[:,2:2]
3 1  Matrix{Float64}:
  -2.0
  -8.0
 -13.0
"""
function find_pivot(A, k)
    return (value, index) = findmax(A[:,k:k])
end

"""
    swap(L, j, k)

Function that swaps rows j and k in all columns from 1:k-1 in matrix L by constructing
permutation matrix.
"""
function swap(L, j, k)
    print("swap called")
    N = size(L)[1]
    I = Matrix{Float64}(undef, N, N)
    I .= 0

    for i = 1:N
        I[i,i] = 1
    end

    for i = 1:N
        I[j,i], I[k,i] = I[k,i], I[j,i]
    end

    L .= I * L

    return L
end

"""
    luDoolittleDecomp(A, N)

Function that performs an LU decomposition using the doolittle algorithm.
"""
function luDoolittleDecomp(A,N)
    U = Matrix{Float64}(undef, N, N)
    U .= 0

    L = Matrix{Float64}(undef, N, N)
    L .= 0

    for i = 1:N
        for k = i:N
            sum = 0
            for j = 1:i
                sum += (L[i,j] * U[j,k])

            end
            U[i,k] = A[i,k] - sum
        end
        for k = i:N
            if (i == k)
                L[i,i] = 1
            else
                sum = 0
                for j = 1:i
                    sum += (L[k,j] * U[j,i])
                end
                L[k,i] = (A[k,i] - sum) / U[i,i]
            end
        end
    end
    #display(U)
    #display(L)
    return(L,U)
end

"""
    LUPsolve(A)

Function that solves Ax=b by computing LUP-factorization and performs forward/backward
"""
function LUPsolve(A, b)
    # test matrix to check for accuracy in solving
    #L = Matrix{Float64}(undef, 3, 3)
    #L .= [1 0 0;4 1 0;4 0.5 1]
    #U = Matrix{Float64}(undef, 3, 3)
    #U .= [1 2 2;0 -4 -6;0 0 -1]

    #size of matrix working
    N = size(A)[1]

    (L,U) = luDoolittleDecomp(A,N)
    #display(U)
    #display(L)

    y = forward_sub(L, b)
    x = backward_sub(U, y)

    #display(x)
    return x
end

"""
    forward_sub(L, b)

Give lower triangular matrix L and vector b, perform the forward substitution to solve
"""
```

```
function forward_sub(L, b)
    N = size(L)[1]
    x = similar(L)
    x .= 0

    for i = 1:N
        temp = b[i]
        for j = 1:i-1
            temp -= L[i,j] * x[j]
        end
        x[i] = temp / L[i,i]
    end
    return x
end


"""
    backward_sub(U, b)

Give upper triangular matrix U and vector b, perform the forward substitution to solve
(Backward version of forward substitution)
"""
function backward_sub(U, b)
    N = size(U)[1]
    x = similar(U)
    x .= 0

    for i = N:-1:1
        temp = b[i]
        for j = i+1:N
            temp -= U[i,j] * x[j]
        end
        x[i] = temp / U[i,i]
    end
    return x
end

#-------------------------END HOMEWORK 1 FUNCTIONS-------------------------------------
#--------------------------------------------------------------------------------------

"""
    main()

Main function to perform conjugate gradient at sizes N = 10, 100, 1000, 10000
Also outputs a plot with time as a function of N
"""
function main()
    y_time = []
    testSizes = [10, 100, 1000]
    @printf("starting loop for values")
    for i = 1:size(testSizes,1)
        (A, b, x,    , iter_max) = var_set(testSizes[i])
        temp = @timed (x, res_i) = conj_grad(A, x, b,    , iter_max)
        push!(y_time, temp[2])
        @printf("done at N = %d\n", testSizes[i])
    end
    @printf("done\n")
    res_i = []
    step = rand(100,1)
    step .= 0
    step = step[:]
    for i = 1:100
        step[i] = i
    end
    (A, b, x,   , iter_max) = var_set(1000)
    (x, res_i) = conj_grad(A, x, b,   , iter_max)
    scatter(step, res_i, xlabel="Iteration", ylabel="Residual", title = "Residual as a
    savefig("testPlotNew.png")
end

main()

#b = rand(3, 1)
#
#(L, U) = computeLU(A)
#@assert A*x[iter_max]     b

using Plots

# write forward Euler for the IVP system y' = f(t, y)
# where y is a vector in R^n (i.e. has n components)

function f(t,y)
    return -3 * y
end

"""
    my_forward_euler(t0, Tf,  t , y0, f)

Function to perform the forwards euler algorithm.
"""
function my_forward_euler(t0, Tf,  t , y0, f)

    # y0 has N components
    N = size(y0)

    M = Integer(Tf/ t )  # M+1 total temporal nodes

    t = Vector{Float64}(undef, M+1)
    y = Matrix{Float64}(undef, N, M+1)

    # fill in the initial condition:
    t[1] = t0
    y[:, 1] = y0

    for n = 1:M # take N time steps
        y[:, n+1] = y[:, n] +  t *f(t[n],y[:, n])
        t[n+1] = t[n] +  t
    end

    return (t, y)
end


"""
    my_backward_euler(t0, Tf,  t , y0, f)

Function to perform backwards euler algorithm.
```

```
"""
function my_backward_euler(t0, Tf,  t , y0, f)
    # y0 has N components
    N = size(y0)

    M = Integer(Tf/ t )  # M+1 total temporal nodes

    t = Vector{Float64}(undef, M+1)
    y = Matrix{Float64}(undef, N, M+1)

    # fill in the initial condition:
    t[1] = t0
    y[:, 1] = y0

    for n = 1:M # take N time steps
        y[:, n+1] = y[:, n] +  t *f(t[n+1],y[:, n+1])
        t[n+1] = t[n] +  t
    end
    scatter(t, y, xlabel="time", ylabel="Y-value", title = "Y values as function of ti
    savefig("testPlotEuler.png")

    return (t, y)
end


# Write forward Euler to solve the linear system IVP:
# y' = Ay + b on 0     t     Tf
# with initial y0

# Do this on particular example, where A = [-3 13; -5 -1]
# y0 = [3; -10]

t0 = 0
y0 = [3; -10]

#A = [-3 13;-5 -1]

Tf = 4
 t  = 0.5

N = Integer(Tf/ t ) # N+1 total temporal nodes


y = Matrix{Float64}(undef, 2, N+1)
t = Vector{Float64}(undef, N+1)
t[1] = 0
# fill in initial condition:
y[:, 1] = y0


for n = 1:N  # take N time steps
    y[:, n+1] = y[:, n] +  t *-3*y[:, n]  # Forward Euler
    t[n+1] = t[n] +  t
end


#plot(t, y[1, :])  # plot first component of solution vector
#plot!(t, y[2, :])  # plot second component
```

```
# plot initial condition:
p = plot([y[1, 1]], [y[2, 1]], marker=(:circle,5), color = :blue, legend = false)

g(x) = 17*2.71828^(-3x)

for n = 2:N+1
    p = plot!([y[1, n]], [y[2, n]], marker=(:circle,5), color = :blue, legend = false)
    display(p)
    sleep(1)
end
plot!(g)
savefig("testPlotEulerForward.png")

my_backward_euler(t0, Tf,  t , y0, f)
```