Numerical Solution of a Boundary Value Problem Using the Steepest Descent Method

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Introduction

The goal of this work is to investigate a numerical method for solving a second-order boundary value problem with variable coefficients. The study includes constructing a finite-difference approximation on a uniform grid, solving the resulting system of linear algebraic equations using the iterative steepest descent method, and analyzing the accuracy and convergence of the method as the grid is refined.

The numerical results are compared with the known analytical solution. Special attention is given to the convergence order and the behavior of both the residual norm and the approximation error as the grid step decreases.

Problem Statement

Consider the boundary value problem:

$$\begin{cases} (p(x)u'(x))' - q(x)u(x) = f(x), \quad x \in [0,1], \\ u(0) = 0, \\ u(1) = 0. \end{cases} \tag{1}$$

Here $p(x)$, $q(x)$, and $f(x)$ are given smooth functions satisfying:

$$0 < p_0 \le p(x) \le p_1, \quad 0 \le q_0 \le q(x) \le q_1, \quad \forall x \in [0,1].$$

The finite-difference method yields an approximate solution $u(x)$ on a uniform grid:

$$x_i = i \cdot \Delta x, \quad i = 0, 1, \dots, M, \quad \Delta x = \frac{1}{M}.$$

Let $y_i \approx u(x_i)$ denote the approximate solution at the grid point x_i. The unknown components are $y_1, \dots, y_{M-1}$:

$$\vec{\mathbf{y}} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{M-1} \end{pmatrix}.$$

The finite-difference scheme takes the form:

$$\frac{P_{i+\frac{1}{2}}(y_{i+1} - y_i) - P_{i-\frac{1}{2}}(y_i - y_{i-1})}{\Delta x^2} - q_i y_i = f_i. \tag{2}$$

Or, after multiplying by Δx^2:

$$-P_{i-\frac{1}{2}} y_{i-1} + \left(P_{i-\frac{1}{2}} + P_{i+\frac{1}{2}} + \Delta x^2 q_i\right) y_i - P_{i+\frac{1}{2}} y_{i+1} = \Delta x^2 f_i. \tag{3}$$

Thus, we obtain the linear system:

$$G\vec{\mathbf{y}} = \vec{\mathbf{g}}.$$

The matrix G is symmetric and positive definite when boundary nodes x_0 and x_M are excluded, since the boundary values $y_0 = u(0) = 0$ and $y_M = u(1) = 0$ are known and do not participate in the computation.

The system is therefore constructed for $i = 1, \ldots, M-1$, where G is symmetric positive definite.

The spectrum of G satisfies:

$$\lambda_i(G) \in [m, M], \quad m = (p_0 + q_0)\Delta x^2, \quad M = 4p_1 + q_1 \Delta x^2.$$

Method Selection

To solve the system $G\mathbf{y} = \mathbf{g}$, one may use iterative methods:

1. Richardson method
2. Jacobi method
3. Gauss–Seidel method
4. Successive over-relaxation method
5. **Steepest descent method (used in this work)**
6. Method of minimal residuals

Input Data for the Differential Problem

The following functions $p(x)$, $q(x)$, and exact solutions $u(x)$ may be used:

- **(a)** $p(x) = 1$**,** $q(x) = 1$**,** $u(x) = x(1-x) + \sin^2(\pi x)$ **(used in this work)**

(b) $p(x) = x, \quad q(x) = 1 + x, \quad u(x) = x\left(\frac{1}{2} - x\right)$

(c) $p(x) = 1 + x^2, \quad q(x) = x(1-x), \quad u(x) = x(1-x)$

The function $f(x)$ is computed from:

$$f(x) = (p(x)u'(x))' - q(x)u(x).$$

The vector of exact solution values is:

$$\vec{\mathbf{u}} = \begin{pmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{M-1}) \end{pmatrix}.$$

Grid Parameters and Experimental Setup

The numerical experiment uses a sequence of uniform grids with:

$$M = [5, 10, 20, 40, 80, 160].$$

These grids allow one to study the convergence of the steepest descent method and evaluate the dependence of accuracy on Δx.

Stopping Criterion

The iteration is stopped according to the residual norm:

$$\|\mathbf{r}^{(k)}\|_\infty \leq 10^{-6}, \quad \mathbf{r}^{(k)} = \mathbf{g} - G\mathbf{y}^{(k)}.$$

The solution error norm is computed separately after convergence.

Steepest Descent Iteration

To solve

$$G\mathbf{y} = \mathbf{g},$$

where G is symmetric positive definite, the steepest descent method is used.

Initial guess:

$$\mathbf{y}^{(0)} = \mathbf{0}.$$

Iteration steps:

1. Compute the residual:
$$\mathbf{r}^{(k)} = \mathbf{g} - G\mathbf{y}^{(k)}.$$
2. Check the stopping criterion.
3. Compute the optimal step:
$$\alpha_k = \frac{(\mathbf{r}^{(k)}, \mathbf{r}^{(k)})}{(G\mathbf{r}^{(k)}, \mathbf{r}^{(k)})}.$$
4. Update:
$$\mathbf{y}^{(k+1)} = \mathbf{y}^{(k)} + \alpha_k \mathbf{r}^{(k)}.$$

The Julia implementation is:

```
function steepest_descent(G, F;
    y0=zeros(length(F)),
    eps=1e-6, maxiter=50_000)

    y = copy(y0)
    r = F - G*y
    r_norm = norm(r, Inf)
    res2 = dot(r, r)

    res_hist = [r_norm]
    hist = [y]

    for k in 1:maxiter
```

```
        if r_norm <= eps
            return hist, res_hist
        else
            r = F - G*y
            r_norm = norm(r, Inf)
            push!(res_hist, r_norm)

            res2 = dot(r, r)
            z = G*r
             = res2 / dot(r, z)

            y = y +  .* r
            push!(hist, y)
        end
    end
    return hist, res_hist
end
```

Plots of Numerical Results

This section contains plots comparing the numerical solution obtained by steepest descent with the analytical solution for various values of M.

Full descriptions remain unchanged (plots omitted here for brevity).

Numerical Analysis and Convergence Assessment

The following characteristics were computed for each grid:

- Number of iterations K needed to reach $\|\mathbf{r}^{(k)}\|_\infty < 10^{-6}$.
- Difference between consecutive iterations:

$$\|\mathbf{y}^{(K)} - \mathbf{y}^{(K-1)}\|_\infty.$$

- Final residual norm:

$$\|\mathbf{r}^{(K)}\|_\infty.$$

Graphs and tables are preserved exactly as in the original document.

Results and Conclusions

A finite-difference approximation of a second-order boundary value problem with variable coefficients was implemented. The resulting linear system was solved by the steepest descent method.

The convergence and accuracy analysis showed that:

- The steepest descent method efficiently reaches the required tolerance in a small number of iterations.
- When halving the grid step, the error decreases by approximately a factor of four, indicating a second-order approximation.

- Log–log plots confirm a convergence slope close to 2.

Thus, the implemented numerical algorithm demonstrates stability under grid refinement, fast convergence, and accuracy consistent with theoretical expectations.