About

I took these notes during the Spring 2017 iteration of Professor Sampath Kannan's Theory of Computation (CIS511) course. The course followed Introduction to the Theory of Computation (3ed) by Michael Sipser. As taking notes in LaTeX on-the-fly is not an easy task, I am sure this document is full of typos, sloppy notation, and small mathematical errors. If you find such an error, please send me an email at {`ianzach+notes[at]seas.upenn.edu`} so I can correct it.

CIS 511: Theory of Computation **Spring 2017**

Lecture 1: Intro and Some Regular Languages

Professor Sampath Kannan *Zach Schutzman*

Introduction

Why theory?

1. Minimal approach to understanding the idea of **computation**.
2. What makes computation tick?
3. Theory anticipates technology.
4. Models of computation are interesting.

Mathematics!

Should know:

1. Sets
2. Functions
3. Relations
4. Logic
5. Proofs
6. Graphs

Definition 1.1 *An **alphabet** is a non-empty, finite set of characters.*

Definition 1.2 *A **string** s (over an alphabet Σ) is a finite ordered sequence of elements of Σ.*

Definition 1.3 *The **empty string**, ϵ, is the sequence of no symbols, and is in fact a valid string.*

Definition 1.4 *Let Σ^* be the set of all strings over Σ.*

Definition 1.5 *A **language** over Σ is any subset of Σ^*.*

The empty set is a language. This is not the same as the language only containing the empty string.

Finite State Machines: A First Model

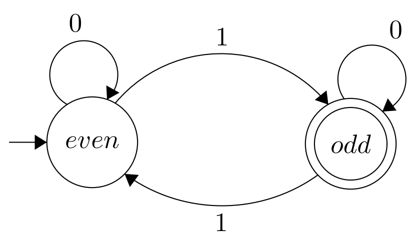
Scalability (asymptotics) is a requirement for any interesting model.

To compute on larger and larger inputs, a computer needs memory. What is the minimum amount of memory you need to do something interesting?

Definition 1.6 *A* ***finite state machine*** *will be a model with a constant amount of memory.*

The states of an FSM correspond to memory (a machine with k states can have 2^k 'bits' of memory).

Example: define the language $\mathcal{L} = \{s \in \Sigma^* | s \text{ has an odd number of } 1s\}$.



Definition 1.7 *A* ***deterministic finite automaton (DFA),*** M***, is a 5-tuple*** $(Q, \Sigma, \delta, q_0, F)$***.***

> Q*, the set of states.*
>
> Σ*, the alphabet*
>
> $\delta : Q \times \Sigma \to Q$*, the transition function*
>
> q_0*, the start state*
>
> F*, the accept states*

Definition 1.8 M ***accepts*** *a string $s = s_1 s_2 \dots s_k$ if there is a sequence of states in M starting with q_0 and ending in a final state $q_0 q_1 \dots q_k$ such that $\delta(q_i, s_{i+1}) = q_{i+1}$.*

Definition 1.9 *If $\mathcal{L} = \{s | M \text{ accepts } s\}$, the we say M* ***recognizes*** *$\mathcal{L}$.*

Definition 1.10 *If a language $\mathcal{L}$ is recognized by some DFA, then it is* ***regular****.*

Definition 1.11 *A* ***nondeterministic finite automaton (NFA),*** M***, is a 5-tuple*** $(Q, \Sigma, \delta, q_0, F)$***.***

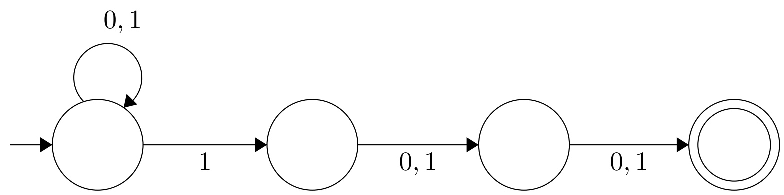
> Q*, the set of states.*
>
> Σ*, the alphabet*
>
> $\delta : Q \times \Sigma \cup \{\epsilon\} \to \mathcal{P}(Q)$*, the transition function*
>
> q_0*, the start state*
>
> F*, the accept states*

Now, δ maps the current state and input character or ϵ to some subset of states.

Definition 1.12 *A string s is* ***accepted*** *by and NFA M if there is some path for s from the start state to a final state.*

Definition 1.13 *M* ***recognizes*** *a language $\mathcal{L}$ consisting of all strings it accepts.*

Example: let $\mathcal{L} = \{s \mid$ the 3rd last character is a $1\}$.



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Lecture 2: More On Regular Languages

Professor Sampath Kannan *Zach Schutzman*

More Finite Automata

NFAs are Equivalent to DFAs

Recall: every DFA recognizes some (regular) language, a language is regular if there exists some DFA recognizing it. NFAs also recognize regular languages (every DFA is also an NFA).

Question: are NFAs more powerful than DFAs? They are certainly a broader class of machines, given that the set of all DFAs is a propers subset of the set of all NFAs.

Theorem 2.14 *NFAs recognize exactly the class of regular languages.*

Proof: *(A rough sketch)*

Consider an NFA M without ϵ-transitions. We can imagine a string's traversal as maintaining a set of possible states you are in, and the NFA accepts if and only that set contains a final state at the conclusion of reading the string. Let's consider the set of states at each step.

Create a DFA M' with states corresponding to subsets of states of the NFA. Now, add transitions in M' corresponding to the set of possible transitions in M. Formally, let $S \subseteq Q$, then $\delta'(S,a) = \bigcup\limits_{q \in S} \delta(q,a)$. These subsets S correspond to states in M'.

The start state of M', q_0' is the state corresponding to $\{q_0\}$. The final states of M', F' are the subsets of Q containing at least one element of F.

We can deal with ϵ-transitions by extending δ' to also include all states reachable from reading a and a following ϵ. Formally, define $E(q)$ as the set of states reachable from q without consuming an input character (i.e. 0 or more ϵ-transitions). Then make $\delta'(S,a) = \bigcup\limits_{p \in E(q)\ \forall\ q \in S} \delta(p,a)$.

■

Closure Properties

Theorem 2.15 *If L_1 and L_2 are regular languages, then so is $L_1 \cup L_2$.*

Proof:

Let M_1 and M_2 be DFAs that recognize L_1 and L_2, respectively. Add a new start state and add an ϵ-transition to the start states of M_1 and M_2. This NFA now accepts exactly the strings in either L_1 or L_2 (or both).

Since this is an NFA recognizing it, $L_1 \cup L_2$ is regular.

■

Theorem 2.16 *If L_1 and L_2 are regular languages, then so is their concatenation, denoted L_1L_2 or $L_1 \circ L_2$.*

Proof:

Let M_1 and M_2 be DFAs that recognize L_1 and L_2, respectively. Add ϵ-transitions from the final states of M_1 to the start state of M_2. The new start state is the original start state of M_1 and the final states are those from M_2. This NFA now non-deterministically tries to split the string into its L_1 and L_2 parts.

Since this is an NFA recognizing it, $L_1 \circ L_2$ is regular.

■

Theorem 2.17 *The Kleene Star of a regular language L, denoted L^*, is regular.*

The Kleene Star is a generalization of concatenation. L^* is the set of strings that are an arbitrary (finite) number of concatenations of L with itself (including zero, i.e. $\epsilon \in L^*$ for all L).

Proof:

Let M be a DFA recognizing L. Add a new start state which is also a final state, and add an ϵ-transition to the original start state. Then, add ϵ-transitions from all of the original final states to the new start state.

Since this is an NFA recognizing it, L^* is regular.

■

Theorem 2.18 *The complement of a regular language L, denoted $\bar{L}$ or L^c is regular.*

Proof: *Given a DFA M recognizing L, make all the accept states non-final and all non-final states final. This is now a DFA recognizing L^c.* ■

Theorem 2.19 *If L_1 and L_2 are regular languages, then so is $L_1 \cap L_2$.*

Proof: *This follows directly from the proofs of closure under union and complement and applying DeMorgan's Laws.*

■

Regular Expressions

We are going to define regular expressions inductively.

> *$\emptyset$ is a regular expression*
>
> *ϵ is a regular expression*
>
> *For each $a \in \Sigma$, a is a regular expression*
>
> *If r_1, r_2 are regular expressions, so are r_1r_2, $r_1 \cup r_2$, r_1^**

Theorem 2.20 *A language L is regular if and only if it is described by some regular expression.*

Proof:

The first direction is easy. We can make NFAs accepting nothing, ϵ, and any single character, and we have already shown the construction for NFAs for the three operations. Therefore, any regular expression can be converted into an NFA by this inductive construction.

To see that any DFA can be converted into a regular expression, we will use an inductive construction. We are going to transform the DFA by merging states and associating them with simpler regular expressions.

Let L be our regular language and M a DFA accepting it. Let's let $Q = [n]$, such that the states are numbered, in order to keep better track of them.

Define $R_{i,j}^{(k)} := \{x | x \text{ takes } M \text{ from } i \text{ to } j \text{ without passing through any state labelled greater than } k\}$

We know $R_{i,j}^{(0)} = \{a | \delta(i,a) = j\}$ *if* $i \neq j$. $R_{i,j}^{(0)} = \{a | \delta(i,a) = j\} \cup \{\epsilon\}$ *if* $i = j$. *Let's proceed inductively. Suppose we know $R_{i,j}^{(k)}$ for all i, j. To compute $R_{i,j}^{(k+1)}$, observe that this is a superset of $R_{i,j}^{(k)}$, and it also contains those strings passing through $(k+1)$ at least once. The strings that go from (i) to $(k+1)$ are captured by $R_{i,k+1}^{(k)}$, the strings that go from $(k+1)$ to another occurence of $(k+1)$ are captured by $(R_{k+1,k+1}^{(k)})^*$ and the strings from $(k+1)$ to (j) are captured by $R_{k+1,j}^{(k)}$. So $R_{i,j}^{(k+1)}$ is $R_{i,j}^{(k)} \cup R_{i,k+1}^{(k)} (R_{k+1,k+1}^{(k)})^* R_{k+1,j}^{(k)}$.*

This is a regular expression describing the language accepted by M.

$\blacksquare$

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Lecture 3: Regular Language Wrap-up and Intro to Turing Machines

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The rest of Regular Languages

Recall, regular languages are those recognized by DFAs, NFAs, and described by regular expressions

Are there languages that are not regular? Trivially, yes. The set of all languages is uncountable, but the set of all regular languages is countable.

How can we determine that a language is not regular?

Theorem 3.21 The Pumping Lemma *(for regular languages): If L is a regular language, then there exists a constant p, called its **pumping length**, such that if $x \in L$ and $|x| \geq p$, then x can be written as $x = uvw$, where v is non-empty, $|uv| \leq p$, and $uv^iw \in L$ for any $i \in \mathbb{N}$.*

Proof: *Suppose L is regular. Then there exists a DFA M that recognizes L. Say M has p states ($|Q| = p$). Suppose L contains some string x, $|x| > p$. Then by the Pigeonhole Principle, the path of x must pass through at least one state at least twice. Equivalently, the path of x contains some cycle.*

Let's think of x as broken into 3 parts: uvw, where u is the portion before the cycle, v is the cycle, and w is the portion after the cycle. We will allow u or w to be empty, but v must be non-empty. Note that $x \in L$ implies $uw \in L$, and more generally that $uv^iw \in L$, for any $i \in \mathbb{N}$.

We also note that $|uv| \leq p$, that is there will always be a cycle before processing p characters.

■

Corollary 3.22 *A language is not regular if there does not exist any way to break a string of length greater than p into a satisfactory uvw.*

Example:

Claim 3.23 *The language $L = \{0^n1^n | n \in \mathbb{N}\}$ is not regular.*

Proof:

Suppose, for the sake of contradiction, that L is regular with pumping length p. Consider the string 0^p1^p. Since we have $|uv| \leq p$, v is of the form 0^k, for some $1 \leq k \leq p$. The Pumping lemma asserts that uv^iw is in L for all $i \in \mathbb{N}$. But the string uv^2w has more 0s than 1s, and is not in L. This is a contradiction, therefore L is not a regular language.

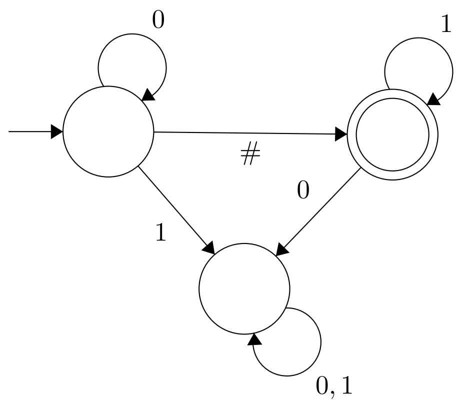
■

Example:

Claim 3.24 *The language $L = \{x\#y \mid |x| = |y|\}$ is not regular.*

Proof:

The class of regular languages is closed under intersection. Suppose, for the sake of contradiction, that L is regular. Let the language $K = \{0^*\#1^*\}$. K is regular, as it is recognized by the following machine:



Let's consider $K \cap L$. This language is $K \cap L = \{0^n\#1^n | n \in \mathbb{N}\}$. By a slight adjustment to the previous proof, we can say that $K \cap L$ is not regular. This contradicts the assumption that both K and L are regular, and since we know that K is regular, it must be the case that L is not.

■

Corollary 3.25 DFAs can't count!

Turing Machines

We are moving from the simplest model of computation to the 'most'. We can think of a DFA as receiving an input on a tape, reading one character at a time, and only moving forward, and changing states according to the currently read character and the current state, via δ.

We can identify three principal limitations:

1. *The head doesn't pass the end of the tape*
2. *The tape is read-only*
3. *The head only moves forward*

For a Turing machine, we lift all three of these limitations. The head can move left and right, it can read and write, and it can move beyond the input.

We are going to expand the tape alphabet Γ to be a proper superset of Σ, and also containing a blank symbol $\sqcup$.

Claim 3.26 *Our language $L = \{0^n\#1^n | n \in \mathbb{N}\}$ can be recognized by a Turing Machine.*

Proof:

Consider an example input below. First, we can scan the string to validate it is of the correct form, if not, reject. Assume that it is.

0	*0*	$\cdots$	$\#$	*1*	$\cdots$	*1*	$\sqcup$	$\sqcup$	$\cdots$	$\cdots$

The following algorithm determines if the string is in L:

Replace the first zero on the tape with a special marker symbol.

Traverse the tape and replace the first one with a marker symbol.

Repeat until you either run out of zeros or ones.

If all symbols are marked, accept, otherwise, reject.

■

Definition 3.27 *A* ***Turing Machine*** *is a 7-tuple* $(Q, q_0, \Sigma, \Gamma, \delta, q_a, q_r)$.

Q *is the set of states*

q_0 *is the start state*

Σ *is the input alphabet*

Γ *is the tape alphabet, a proper superset of* Σ

$\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$ *is the transition function.*

q_a *is the accept state*

q_r *is the reject state.*

The transition function maps the current state and the currently read character to the next state, the character to write on that cell, and a LEFT or RIGHT move. The machine halts when it reaches q_a *or* q_r*, by definition.*

Definition 3.28 *Given a Turing machine* M*, the language* ***recognized*** *by* M *is the set of all strings that cause* M *to move to* q_a.

If M *recognizes* L *and* $x \notin L$*, there are two possibilities:*

1. M *reaches* q_r
2. M *computes infinitely and never reaches* q_a *or* q_r*. That is,* M *'loops forever'.*

In both of these cases, we say that M *doesn't recognize* x*, but in the second case, how do we know if* M *really won't halt on* x*, rather than just taking a really long time?*

Definition 3.29 *A Turing machine* M ***decides*** *a language* L *if for all* $x \in L$*,* M *reaches* q_a *and for all* $x \notin L$*,* M *reaches* q_r*. The set of Turing decidable languages is a subset of the Turing recognizable languages.*

Definition 3.30 *The* ***current configuration*** *of a Turing machine M is the current state, the tape contents, and the head position at some point in time.*

The notation for this is:

			q_c	a	$\cdots$					$\cdots$

to denote that we are in state q_c and the head is over a.

Example: *Let's construct a Turing machine to recognize the language of repeated words,*
$L = \{w\#w | w \in \Sigma^*\}$.

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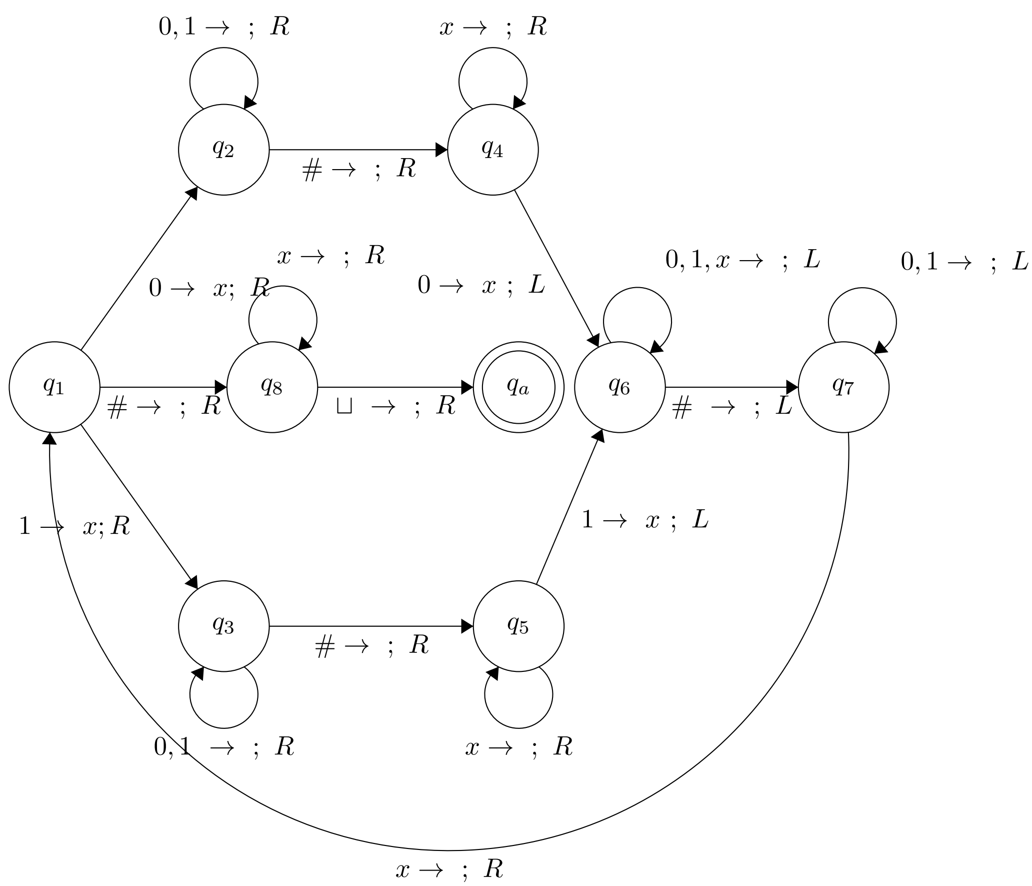
Lecture 4: More Turing Machines

Professor Sampath Kannan *Zach Schutzman*

Turing Machines

Let $B = \{w\#w | w \in \Sigma^*\}$. Here is a TM to accept this language:



Variations on the Turing Machine

How robust is the TM? Turns out, very!

Definition 4.31 *Define a **stay**-TM as one with move commands expanded to $\{L, R, S\}$, where S denotes no-move.*

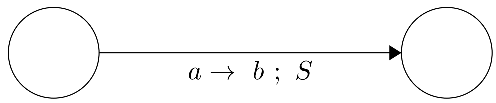
Claim 4.32 *A stay-TM is equal in power to the regular TM.*

Proof:

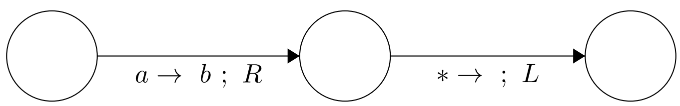
A regular TM is a stay-TM which does not use the S option, so clearly anything a TM can do, a stay-TM can as well.

Now, to see that a stay-TM does not recognize or decide any more languages than a regular TM, let M_s be a stay TM for some language. To convert it into a regular TM M', perform the following procedure.

Suppose M has a transition that uses the S option, of the form



for some $a, b \in \Gamma$. Replace this transition with an intermediate state and two transitions that read a character, perform the write, then move right, then move left without writing anything, as follows, where $*$ represents any character in Γ.



This pair of transitions performs the same modification to the state of M' as the original does to M, therefore stay-TMs are no more powerful than regular TMs.

$\blacksquare$

Definition 4.33 *A **multi-tape TM** is a Turing machine with a finite number (k) of tapes. Each transition in δ reads the character under each head and writes and moves independently on each tape. Formally, δ is now a function $\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R\}^k$.*

Claim 4.34 *A multi-tape TM is no more or less powerful than a regular TM.*

Proof:

One direction is trivial. A regular TM is a multi-tape TM with $k = 1$.

To see that a multi-tape TM is not more powerful than a regular TM, let M be a TM with k tapes. We will construct a single-tape TM S that simulates M.

Assume $\# \notin \Gamma_M$, which will be our delimiter symbol in S.

We will also expand Γ_S to include $a, \dot{a}$, for each $a \in \Gamma_M$.

Let S have a tape initialized to the concatenation of the contents of each of M's k tapes, separated by #.

We will use our dotted symbols to track where the head of each tape is, and since the head can only be in one place, there is exactly one dotted symbol in each of the k segments.

Let S sweep to the right, entering a state corresponding to the k-tuple of dotted symbols it reads and the current state of M. Then, S needs to execute the proper transition in M. We now know that the state space of S is at least $|Q_M| \times |\Gamma_M|^k$. On the way back to the left end, S should execute M's transitions.

We now have to deal with the issue of S overwriting the bounds of the tape segments. We give S a routine that pushes everything to the right in order to make space.

This machine simulates M, completing the proof.

■

Observe that if M has taken at most T steps, then it uses at most T cells on each of its tapes. Therefore, the maximum number of cells needed on S's single tape is $O(kT)$. We face no worse than quadratic slowdown, as each of the T steps of M take no more than T steps to simulate in S, leading to a slowdown on the order of T^2.

Is it possible for a multi-tape TM to have a truly quadratic speedup?

Claim 4.35 *Let $L = \{w\#w^R | w \in \Sigma^2\}$. This language gains a quadratic speedup on a 2-tape machine versus the naive implementation on a single tape machine.*

Proof:

First, copy the input to the second tape. Then, the head on the second tape can simultaneously match characters from right to left with the first tape going left to right.

How many steps does this take? If the input is of size m, we can copy the input in $O(m)$, move the first head back to start in $O(m)$, and step through the string in $O(m)$.

For the single tape machine, we need $O(m)$ steps to move back and forth to check each pair of characters, of which there are $\frac{m}{2}$, for $O(m)$ comparisons. Therefore, the 2-tape machine decides the language in $O(m)$ while the single tape needs $O(m^2)$.

■

Can we do better by being clever? No! If we try to batch the symbols and compress, we lose information which might lead to incorrect acceptance/rejection.

We can also show that the two-way infinite tape Turing Machine also has the same power as a regular TM.

Additionally, the TM with a k-dimensional infinite grid tape has the same power as a regular TM.

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Lecture 5: Nondeteminism in Turing Machines

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Nondeterministic Turing Machines

What if our TMs can explore a variety of possible transitions simultaneously.

Definition 5.36 *A* ***non-deterministic Turing Machine*** *(NTM) is a Turing machine M where $\delta_M : Q \times \Gamma \to 2^{Q\times\Gamma\times\{L,R\}}$.*

Why do we care?

Claim 5.37 *NTMs are equivalent in power to regular TMs.*

Theorem 5.38 *If M is a NTM recognizing language L, then there exists a deterministic TM D recognizing L.*

Proof:

The idea is to have D simulate the behavior of M. We can think of M's computation as a tree of configurations, branching when it makes non-deterministic choices. We might be tempted to fully explore each branch sequentially, but we have to be careful as there may be some branches that loop forever, and we don't want to get stuck on these.

Our solution will be to explore in a breadth-first manner. This way, if there is some accepting path, we will eventually find it without getting stuck in an infinite loop.

Formally, let D be a 3-tape Turing machine. The first tape will be a read-only input tape. The second will be a work tape that actually simulates M. The third tape will be used to address where we are in the configuration tree. We can upper-bound the number of children a configuration can have by B by observing that at each transition, there are no more than B possible resultant configurations. Tape 3 will store address values $x_1 \dots x_k$, which tell you, from the root, which child to explore. We will skip over invalid addresses.

Begin by copying the initial configuration from Tape 1 to Tape 2. Then look at Tape 3 and get x_1, then process that, then do it for x_2, until we reach x_k. When we reach the end, increment the value on Tape 3 and start over. If we reach q_r, we terminate that branch. If we reach q_a on some branch, we accept.

This runs slowly, but it deterministically simulates M, and therefore recognizes L.

■

Corollary 5.39 *Turing machines can simulate other Turing machines.*

The Turing machine was 'invented' by Alan Turing partly as a result of one of Hilbert's questions of finding solutions to Diophantine equations. Turing wanted to figure out how to show that no possible algorithm exists.

Meanwhile, Alonzo Church created the λ-calculus, which is a functional view of computability that is equivalent to Turing's algorithmic view, proved by Church and Turing. These are both equivalent to something in logic called partial recursive functions*.*

*Is there a stronger model of computability? The **Church-Turing Thesis** proposes that anything computable by any physical device is computable under the notions of Turing machines or λ-calculus. Presently, there has not been any evidence that there exists a physical device that is more powerful than a Turing machine.*

Turing Machine Computations = Algorithms

Imagine a Turing machine E hooked up to a printer (write-only output tape)

Definition 5.40 *Such a Turing machine E **enumerates** a language L if for each $x \in L$, E eventually outputs x and for each $y \notin L$, E never outputs y.*

Theorem 5.41 *A language L is recognizable by a TM if and only if there exists some TM that enumerates L.*

Proof:

Suppose L is recognized by a machine by a TM M. We want to build an enumerator E for L. Order the strings in Σ^* by length then by dictionary order (for example). In order, for each $x \in \Sigma*$, run M on x, and if M accepts, print x. We need to be careful about looping forever on some string. We will take a variable k and increment it as E runs. For $k = 1, 2 \ldots, \infty$, run M on each of the first k strings for k steps, printing those that M accepts on.

To see that for every string $y \in L$, we know that y is the i^{th} string in language for some finite i, so for some $k = j$, M will run on y. Since M accepts y in a finite amount of steps, less than or equal to the $\max\{i, j\}$. Since this is finite, E eventually outputs y.

Now suppose L can be enumerated by an enumerator E. We want to build a machine M to recognize L. This is easy. M does the following: given a string x, every time E prints a string, compare it to x. If these values are equal, accept.

E never prints a string not in L, so M loops forever on strings not in L, which is fine.

*The proof is complete and we can say the class of languages recognized by Turing machines is exactly the **recursively enumerable languages**.*

■

Corollary 5.42 *A language is decidable if and only if the enumerator prints the strings in the lexicographic order.*

Decidable Languages

A language is decidable if there is a Turing machine that recognizes it and halts on all inputs.

Let's think about the problem of finding a minimum spanning tree in a weighted graph. We have an efficient algorithm that always terminates to do this. How do we translate a problem into a language?

Viewing this as a language, we can encode the graph G and tree T as strings and define the language as $L = \langle G, T \rangle$ such that T is the MST of G.

Alternatively, we can encode G and an integer K and define the language as $L = \langle G, K \rangle$ such that there exists a MST of G with weight at most K.

Using this second approach, we can do binary search (for example) to find the correct value of K. We can then find the tree itself by iteratively removing edges to find the edges in the tree by trial-and-error.

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Lecture 6: Decidability and Undecidability

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Decidable Languages

Definition 6.43 *Recall, a language is **decidable** if it can be recognized by a Turing Machine which always halts.*

Examples of Decidable languages:

- *The language of bipartite graphs:* $\{\langle G\rangle \mid g \text{ is bipartite}\}$
  A TM to decide if a graph is bipartite: use a second tape as a work tape, which will store two arrays. Begin by putting vertex v_1 on the 'left' side. Iteratively choose a vertex whose label we know, and add its neighbors to the other side. If we ever put a vertex on both sides, reject. If we successfully classify every vertex, accept.

- *The language of* $3SAT$*:* $\{\langle\Phi\rangle \mid \Phi \; is \; a \; boolean \; formula \; in \; 3\text{-}CNF \; with \; a \; satisfying \; assignment\}$
  This is our classic NP-Complete problem. CNF (conjunctive normal form) means that the formula is written as the AND of several clauses, each containing the OR of several literals (variables and/or their complements). 3-CNF is CNF where each clause has at most 3 literals. A TM to decide this language is to enumerate the 2^n possibilities for all assignments. If we find one that is true, accept. If we exhaust all possibilities, reject.

- *Every regular language (and context-free language) is decidable.*

Claim 6.44 *A Turing machine can be designed to simulate any other Turing machine. Essentially, we can describe a Turing machine as a string encoding its 7-tuple.*

Now, we can talk about Turing machines that take another TM as input and performs some computation with respect to that machine. A Turing machine T can take as input a a machine and a string $\langle M, w\rangle$, we can think of T like an interpreter and executer and M as a program with input w. T outputs $M(w)$ (if such a result can be obtained in finite time).

At a high level, T has a work tape that keeps the current configuration of M. It starts with the start state of M and the input string w. T consults the transition function of M and updates the work tape accordingly. At the end of execution, the work tape contains the output of M (with some extra configuration information).

Definition 6.45 *We call this T a **Universal Turing Machine**.*

Undecidable Languages

Is there even a language that is undecidable?

Claim 6.46 *Yes!*

Proof:

We use a counting argument to show the existence of an undecidable language (actually, there are uncountably many non-decidable languages). If Σ *is a finite alphabet,* Σ^* *is countably infinite, as it can be enumerated in lexicographic order, for example.*

The set of all Turing machines is also countable, as we can encode it as a string over some alphabet, say $\{0, 1, \#\}$*. Conversely, we can show* $\mathcal{P}(\Sigma^*)$*, the set of all subsets of* Σ^**, i.e. the set of all languages, is uncountable.*

By Cantor's Diagonal Drgument, we can show the set of binary functions on $\mathbb{N}$ *is uncountable. This set is in bijection with* $\mathcal{P}(\mathbb{N})$*. Since there is a bijection* $\Sigma^* \longleftrightarrow \mathbb{N}$*, and a bijection* $\mathcal{P}(\mathbb{N}) \longleftrightarrow \mathcal{P}(\Sigma^*)$ *and no bijection* $\mathcal{P}(\mathbb{N}) \longleftrightarrow \mathbb{N}$*, there is no bijection* $\mathcal{P}(\Sigma^*) \longleftrightarrow \mathbb{N}$*, because the composition of bijections is a bijection.*

To show the same thing, we can explicitly construct a diagonalization case by allowing the columns to correspond with each string in Σ^* *and each row is a binary function on* Σ^**, i.e. a language. By the Diagonal Argument directly, we see the set of languages is uncountable.*

Since the set of Turing machines is countable and the set of all languages is uncountable, there must be some (uncountable set of) languages not decidable (or even recognizable) by some Turing machine.

$\blacksquare$

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Lecture 7: Undecidability, Continued

Professor Sampath Kannan *Zach Schutzman*

An Undecidable Language

Recall, we showed last time that, by a cardinality argument, that there must exist some language that is not Turing decidable, or even recognizable.

Example: *Let the language $A_{TM} = \{\langle M, w\rangle | M(w) \text{ accepts}\}$, the language of machines and words such that M reaches its q_a on input w.*

Claim 7.47 *A_{TM} is not decidable.*

Proof: *Assume for the sake of contradiction that A_{TM} is decidable, and let H be the TM that decides it. We will show that H cannot exist.*

How does H work? H takes a pair $\langle M, w\rangle$ and accepts if M accepts w and rejects if M does not accept (reject or loop forever). Note that because of this 'loop forever' case, we can't just simulate M on w.

Create a new TM D that takes as input the description of a Turing machine $\langle M\rangle$. D calls H on $\langle M, \langle M\rangle\rangle$. Since H decides a language, it always terminates. Define the behavior of D to return the complement of the answer that H provides. That is, if H accepts on $\langle M, \langle M\rangle\rangle$, then D rejects. I.e., if H says that Machine #17 accepts on the input '17', then D rejects.

Now, how should D behave when given input $\langle D\rangle$? First, D calls H on $\langle D, \langle D\rangle\rangle$. Next, H decides if D accepts D. In the former case, it accepts, the latter, rejects. So, if H says that D accepts $\langle D\rangle$, then D rejects. Conversely, if H says that D does not accept $\langle D\rangle$, then D accepts. Both of these cases are paradoxical, hence no such decider H for A_{TM} exists, so A_{TM} is not a decidable language. ■

This proof is equivalent to Cantor's diagonal argument for uncountablility. Consider the rows indexed by Turing machines and the columns by string descriptions of Turing machines. The table values are either 'accept' or 'reject' when H receives input $\langle r, \langle c\rangle\rangle$. D then works by going down the diagonal and taking the complement of the entry at that point. If we think about 'accept' and 'reject' as being '1' and '0', we have exactly the same case as Cantor's diagonal argument that $\mathbb{R}$ is uncountable.

What about an unrecognizable language?

An Unrecognizable Language

Claim 7.48 *A language L is decidable if and only if L and L^C are both recognizable.*

Proof:

If L is decidable, then clearly it is recognizable. Additionally, L^C must be decidable, as it is decided by a machine that simulates a decider for L and outputs the complement of the result.

Corollary 7.49 *The class of decidable languages is closed under complement. Hence, L^C is recognizable.*

Now, assume that L and L^C are both recognizable. We will construct a decider for L. Let M and M_C be recognizers for L and L^C, respectively. Let M_D be a TM that works by alternating simulating M and M_C one step at a time. Because L and L^C are both recognizable, if a string x is in L, then M halts and accepts on x. If x is not in L, then M_C halts and accepts on x. Since M_D alternates steps of M and M_C, it halts in a finite amount of time, and outputs an answer corresponding to the submachine that accepted. M_D accepts if M accepts or M_C rejects. M_D rejects when M rejects or M_C accepts.

■

Is A_{TM} recognizable? Yes! The simple approach of simulating M on w will halt if M accepts. The issue came from us wanting a machine to halt in the case that w is not in $L(M)$.

Let's define $\overline{A_{TM}}$ as the language where M does not accept w. Is this language Turing recognizable?

Claim 7.50 *$\overline{A_{TM}}$ is not recognizable.*

Proof:

If $\overline{A_{TM}}$ is recognizable, then together with the proof that A_{TM} is recognizable, we know that A_{TM} would have to be decidable. We know this is false, so $\overline{A_{TM}}$ is not recognizable. ■

We found a language that is not recognizable!

Reductions

The idea of reductions is ubiquitous in mathematics and computer science. Fundamentally, it is the idea of solving a new problem by converting it to one you already know how to solve.

Suppose we have two languages A and B.

Definition 7.51 *A reduction from A to B is a method for using a procedure for solving/deciding/recognizing B to solve/decide/recognize A.*

We can think about a subroutine that can solve B being used to solve A.

The existence of a reduction from A to B in a sense means that A is no harder than B (B is at least as hard as A).

If we can reduce an undecidable language A to some language B, then B must be undecidable. We will denote "A reduces to B" as $A \preceq B$.

A reduction from A to B: given a TM S for solving B, construct a TM R that uses S as a subroutine to solve A.

Example: *Define $HALT_{TM} = \{\langle M, w\rangle \mid M \text{ halts on } w\}$. This is the set of pairs $\langle M, w\rangle$ such that M halts on w.*

Claim 7.52 *H_{TM} is undecidable*

Proof: *We prove this by reduction. We will show that, if we assume $HALT_{TM}$ is decidable, we can use it to decide $HALT_{TM}$. This is a reduction from A_{TM} to $HALT_{TM}$.*

Suppose S is a TM that decides $HALT_{TM}$. S takes as input $\langle M, w\rangle$ and accepts if M halts on w, rejects if M loops forever. Then a TM for A_{TM} works by taking input $\langle M, w\rangle$, asks S whether M halts on w, if S rejects, then reject, otherwise, simulate M on w and output the result.

This decides A_{TM}, which is a contradiction. Hence such an S cannot exist and $HALT_{TM}$ is not decidable.

■

Example: *Define $E_{TM} = \{\langle M\rangle \mid M \text{ recognizes } \emptyset\}$ be the language of TMs that accept no strings.*

Claim 7.53 *E_{TM} is undecidable*

Proof: *We proceed again by reduction from A_{TM} to E_{TM}.*

Let S be a decider for E_{TM}. S takes as input $\langle M\langle$ and accepts if M accepts no strings and rejects if M accepts at least one string. A machine for A_{TM} takes as input $\langle M, w\rangle$, a machine and a string. We will show that S can be used to build a decider for A_{TM}

Define a new machine M' such that M' on input $x \neq w$ rejects and on input $x = w$ simulates M on w. M' recognizes a language that is either $\emptyset$ or $\{w\}$. But M' accepts w only when M accepts w. We can then feed $\langle M'\rangle$ to S. If S accepts, then M does not accept w. Conversely, if S rejects, then M accepts w.

We can now decide A_{TM} as follows: on input $\langle M, w\rangle$, construct M' and pass it to S. Since S always halts, A_{TM} halts on all inputs, and is therefore decidable.

Because A_{TM} is not decidable, no such S can exist and E_{TM} must be undecidable.

■

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Lecture 8: More on Reducibility

Professor Sampath Kannan *Zach Schutzman*

Mapping Reducibility

Consider languages $A \subseteq \Sigma^*$, and $B \subseteq \Sigma^*$.

Recall the idea of a reduction is A reduces to B means that we can use B to solve A. If A is hard, then B must be (at least as) hard. If B is easy, then A must be easier.

Definition 8.54 *A Turing machine* ***computes*** *a function $f : \Sigma^* \to \Sigma^*$ if on all inputs x it halts with exactly $f(x)$ on its output tape.*

Definition 8.55 *A function f is* ***Turing computable*** *if there exists a Turing machine which computes it.*

Definition 8.56 *A* ***mapping reduction*** *is a Turing computable function $f : \Sigma^* \xrightarrow{TC} \Sigma^*$ such that if $x \in A$, then $f(x) \in B$. Similarly, if $y \notin A$, then $f(y) \notin B$.*

Recall last time we constructed a Turing reduction from $A_{TM} = \{\langle M, w\rangle | M \text{ accepts } w\}$ to $E_{TM}\{\langle M\rangle | L(M) = \emptyset\}$. Note that this was not a mapping reduction as we had to take the complement of the solver for E_{TM} at the end to properly complete the reduction.

Claim 8.57 *If we have a mapping reduction from A to B and A is not Turing recognizable, then B is not Turing recognizable.*

Proof: *If B is Turing recognizable, let $B = L(M)$. Then we construct a recognizer for A by using the mapping function on the input for A and running M on it.*

■

Example: *let $L_{AE} = \{\langle M\rangle | \epsilon \in L(M)\}$. We want to find a mapping reduction from A_{TM} to L_{AE}.*

Proof: *A typical input to A_{TM} is of the form $\langle M, w\rangle$. Our function f should map $\langle M, w\rangle$ to $\langle M'\rangle$. M', on any input, prints w on its input tape and runs M.*

We can see that M' either accepts everything or nothing, but it accepts if and only if $\langle M, w\rangle \in A_{TM}$. Because A_{TM} is undecidable, we must have L_{AE} undecidable as well. ■

Claim 8.58 *If $A \preceq_m B$, then $\bar{A} \preceq_m \bar{B}$.*

Proof:

We can use the same reduction as the original for the complement, because the function f preserves membership in sets.

■

Example: *let* $EQ_{TM} = \{\langle M_1, M_2\rangle | L(M_1) = L(M_2)\}$*. We will show this is unrecognizable by a mapping reduction.*

Proof: *We will reduce from* $\overline{A_{TM}}$. $\overline{A_{TM}}$ *takes input of the form* $\langle M, w\rangle$*. We want to make a construction that builds two equivalent TMs if* M *accepts* w *and two inequivalent TMs if* M *does not accept* w*.*

Let $L(M_1) = \emptyset$ *and* M_2 *be a machine that on any input, runs* M *on* w*. This machine either accepts every string or no strings, depending on whether* M *accepts* w*.*

$M_1 \sim M_2$ *if and only if* M *rejects* w*, which is exactly what we wanted to show. Therefore,* EQ_{TM} *is unrecognizable.* ■

Example: *let's also prove that* $\overline{EQ_{TM}}$ *is also not recognizable, via mapping reduction.*

Proof:

If we can show a mapping reduction from A_{TM} *to* EQ_{TM} *then we know that* $\overline{A_{TM}}$ *has a mapping reduction to* $\overline{EQ_{TM}}$*, and* EQ_{TM} *is therefore unrecognizable.*

Let's make $L(M_1) = \Sigma^*$ *and* M_2 *works exactly as before. The same process shows that* $M_1 \sim M_2$ *if and only if* M *accepts* w*. Therefore, by examining the complements of these languages, we have a reduction from* $\overline{A_{TM}}$ *to* $\overline{EQ_{TM}}$*, so neither* EQ_{TM} *nor its complement are recognizable.* ■

Definition 8.59 *A* ***property*** *of Turing machines is a function from TM descriptions to* $\{0, 1\}$

Definition 8.60 *A property of Turing machines is a* ***language property*** *if only depends on the language recognized by the TM and not on the description of the TM itself. That is, the property is true for any machine* M *recognizing a language* L*.*

Let $L_P = \{\langle M\rangle | M \text{ has property } P\}$

Definition 8.61 *A property is* ***non-trivial*** *if there is some Turing machine that has the property and some that does not. That is,* L_P *is not the set of all TMs nor is it empty.*

Claim 8.62 *(Rice's Theorem)* L_P *for any non-trivial language property of Turing machines is undecidable.*

Proof:

Let P *be a non-trivial language property and* L_P *be the language of the property. We will show, by constructing a mapping reduction, that* L_P *is undecidable.*

Consider A_{TM}*. Assume that a Turing machine that accepts the empty language does not have property* P *(this is without loss of generality, because we can always consider* $\overline{P}$ *instead). On input* $\langle M, w\rangle$*, create a machine* M' *which, on any input* x*, first runs* M *on* w*. If* M *rejects, then* M' *rejects. Otherwise, then, runs some Turing machine* T *with property* P *on input* x*.*

If M *does not accept* w*, it may be because it rejects, or runs forever. In both cases,* M' *rejects. In the first, it explicitly does so. In the second, it never gets to the second step, and therefore rejects. So* $L(M') = \emptyset$*, which does not have property* P*.*

If M *accepts* w*, then* $L(M') = L(T)$*.* T *has property* P*, but since* P *is a language property,* M' *has the property as well. Therefore, the reduction results in* M' *having property* P *if and only if* M *accepts* w*. Since* A_{TM} *is undecidable,* L_P *must be as well.*

■

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Lecture 9: Finishing Decidability; Starting Complexity

Professor Sampath Kannan *Zach Schutzman*

Finishing Undecidablility

Recall Rice's Theorem states that any non-trivial language property of Turing machines is undecidable. We showed this via mapping reduction from A_{TM}.

Let's look at one more undecidable language. We'll think about randomness. Consider a Turing machine M on input x. At the end of computation, the tape has y on it.

Definition 9.63 *The **descriptive** or **Kolmogorov-Chaitin complexity** of a string $\mathcal{K}(y)$ is the length of the shortest description $\langle M\rangle, w$ such that M on input w halts with y on its tape.*

How do we go about writing the description of a Turing machine? One simple solution would be to duplicate each symbol in the representation, then terminate with a non-duplicated pair. For example, 10110101 *becomes* 110011110011001110*, where* 10 *is the terminating pair. Alternatively, we can write the length of the machine with the doubling, finished with a non-duplicate pair.*

For any x, $\mathcal{K}(x) \leq |x|+c$, for some constant c. To see this, just let M be the machine that immediately halts. It has some length c', which is fixed and should be fairly small. Using our first naive encoding, $2c' + |x| + 2$ is sufficient to encode x.

Definition 9.64 *A string x is **incompressible by** c if $\mathcal{K}(x) \geq |x| - c$. That is, x has no description that is c shorter than itself.*

Definition 9.65 *A string is **incompressible** if $\mathcal{K}(x) \geq |x|$.*

Claim 9.66 *There exist incompressible strings of every length.*

Proof: *Note that there are 2^n strings of length n (on a binary alphabet). The number of shorter descriptions is $\sum_{i=0}^{n-1} 2^i = 2^n - 1$. There are strictly fewer descriptions than strings, so there must be some string not representable by a shorter length, i.e. incompressible.*

■

How many strings of length n are compressible by 2? We want a description of length at most $n-3$. So $\sum_{i=0}^{n-3} = n^{n-2} - 1$. That is, at least three quarters of strings of any length are incompressible by 2.

If x, y are strings, and c a constant:

- $\mathcal{K}(xx) = \mathcal{K}(x) + c$

- $\mathcal{K}(xy) \leq \mathcal{K}(x) + \mathcal{K}(y) + 2\log_2(\mathcal{K}(x)) + c$, *because we need to encode a length for x at the beginning.*

Let $L = \{x | x$ is incompressible by $2\}$. Is L decidable?

Claim 9.67 *No.*

Proof:

Assume, for the sake of contradiction, that L is decidable, and has a decider D. D has description length c. Define a machine M, which generates each string of length N, then calls D on that string. M halts at the first string which D says is incompressible. M needs N as input, which has length $\log_2(N)$, plus it has D's description, of length c, plus some other constant piece to describe M with length c'. So we have input of string $2(c + c' + \log(N))$ which D says is incompressible. But this string of length $2(c + c' + \log(N))$ is shorter, therefore the string is compressible, which is a contradiction.

■

Time Complexity

Definition 9.68 *A **step** of a Turing machine is one execution of a transition.*

Suppose M is a deterministic TM to decide L.

Definition 9.69 *We say M **runs in time** $f(n)$ if for any input of length n, M terminates in at most $f(n)$ steps.*

Definition 9.70 *The complexity class **DTIME(**$f(n)$**)** is the set of all languages L for which a deterministic Turing machine takes $O(f(n))$ steps and decides L.*

Example: *DTIME(n)$\subset$ DTIME(n^2)*

Example: *Let $L = \{0^k 1^k | k \geq 0\}$. A TM to decide this can do our usual matching thing, where we move back and forth, matching characters. This requires $O(n^2)$ steps. We therefore have $L \in$DTIME(n^2). Can we do better? Yes. Cross off every other 0, then every other 1, and at each pass, check the parity of the number remaining characters. If this is ever odd, reject. If we ever cross out all of one character and have some of the others, reject. Otherwise, accept. To do this, we do $\log(n)$ rounds of crossing out, and we need 2 passes of n steps to do each round, for complexity of $n \log(n)$. We can therefore say that $L \in$DTIME($n \log n$).*

It turns out we can't do better than this. Any language that a single-tape deterministic TM decides in time $o(n \log n)$ is regular. We know L is not regular, so we can't do better than $O(n \log n)$.

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Lecture 10: Time Complexity, Continued

Professor Sampath Kannan *Zach Schutzman*

Asymptotics

Definition 10.71 *An algorithm with running time $t(n)$ takes no more than $O(t(n))$ steps on any input of size n. This is sometimes called **worst-case complexity**.*

Recall the class DTIME($t(n)$) is the set of languages L for which there exists a single-tape Turing machine that decides L in $O(t(n))$ time.

What about a multi-tape machine? We have seen that the language $L = \{ww^R | w \in \{0,1\}^*\}$ can be decided in $O(n)$ on a two-tape TM but is in $\Omega(n^2)$ on a single-tape machine. This language is in DTIME(n^2) because we consider only single-tape machines.

The Class P

Definition 10.72 *The class P is equal to $\bigcup_{k \geq 0} DTIME(n^k)$. That is, the class of languages decided in polynomial time on a single-tape deterministic Turing machine.*

Why is P an important class? First, a problem being in P implies there is a method of solving that problem without enumerating all possible solutions, which would typically require exponential time. Second, P is a robust class with a lot of nice compositional properties. The sum, product, and composition of polynomials is a polynomial. This means that if we have a polynomial-time subroutine, if we call it a polynomial number of times, our algorithm is still polynomial-time. Thirdly, we have the Extended Church-Turing Thesis, which states that every reasonable model of computation can simulate any other reasonable model of computation with only polynomial-time slowdown. This implies that the class P is invariant under models of computation (numbers of tapes, modern computers, quantum computers). We don't actually know that if is true, but there is no proof yet that it isn't correct.

Some languages we know are in P:

- $PATH = \{\langle G, s, t\rangle | G$ *a digraph* s, t *vertices, there is an* $s - t$ *path*$\}$.

  If G has m vertices, then there may be m^m paths to explore. But, we can run BFS from G starting at s. If there exists a path, we eventually reach t. As a side note, BFS requires m bits of space, keeping track of whether we have visited a node or not. Open question: can we do better?

- $UPATH$ *- the same, but for undirected paths.*

- *GCD: given $a, b \in \mathbb{N}$, find the greatest common divisor of a and b. The input size is $\log a + \log b$. The Euclidean algorithm is actually in P.*

The jury is still out on:

- $FACTORING = \{\langle N, K\rangle | N \text{ } has \text{ } a \text{ } factor \text{ } strictly \text{ } between \text{ } 1 \text{ } and \text{ } K\}$. *This is a sufficient subroutine to factor an integer, and if this is in P, then integer factorization is as well. Note that the language $PRIME$, which determines if a number is prime is actually in P, but is not sufficient to do factorization (as far as we know).*

The Class NP

Definition 10.73 *A non-deterministic Turing machine is called a **decider** if it halts on every branch.*

Definition 10.74 *The running time of a non-deterministic Turing machine on an input x is the maximum number of steps taken on any branch.*

Definition 10.75 *A language L is in $NTIME(t(n))$ if there is an NTM M that decides L in time $O(t(n))$ on all inputs of length n.*

Let M be an NTM that runs in time $f(n)$. Then on all of its branches, M halts in time at most $f(n)$ on all of its branches. We showed earlier how to make a deterministic TM that simulates an NTM. The branching tree has height at most $f(n)$, and the number of possible transitions at each step was some number B. Then the number of nodes in the tree is at most $b^{f(n)}$. This is no longer polynomial. To our knowledge, simulating a non-deterministic decider on a deterministic decider results in an exponential blowup in running time.

We therefore can say that $NTIME(f(n)) \subseteq DTIME(2^{O(f(n))})$

Definition 10.76 *The class NP is equal to $\bigcup_{k\geq 0} NTIME(n^k)$. That is, the class of languages decided in polynomial time on a non-deterministic Turing machine.*

Some languages in NP:

- $SAT = \{\phi | \phi \text{ } is \text{ } a \text{ } Boolean \text{ } formula \text{ } with \text{ } some \text{ } satisfying \text{ } assignment\}$
- *TSP: given a weighted graph, does there exist a tour with cost at most B? We can get around the branching factor being a function of the input by introducing a $\log n$ depth factor at each step to identify each vertex with a binary index.*

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Lecture 11: Even More Time Complexity

Professor Sampath Kannan *Zach Schutzman*

More About NP

Recall, a language is in NP if it can be decided by a non-deterministic Turing machine in polynomial time.

Recall the language $SAT = \{\phi | \phi \text{ is } satisfiable\}$. If ϕ is satisfiable, then there exists some satisfying assignment, and if an oracle gave you some satisfying assignment, you could very easily use this to verify that ϕ is satisfiable (just plug in).

Definition 11.77 *Alternatively, a language L is in NP if there is a polynomial time deterministic Turing machine V such that $L = \{x | \exists y : \ V \text{ accepts } (x, y)\}$. In English, V is a verifier and y a certificate. NP is the class of languages which, given a certificate, a verifier takes a language and a solution and decides whether x belongs to L. y is a 'proof' of x being in L.*

Some more example of languages in NP:

- *A clique is a set of vertices in a graph which are all pairwise adjacent.*

  The language $CLIQUE = \{\langle G, k\rangle | G \text{ has a } k\text{-clique}\}$ is in NP. We can see this alternatively by thinking about an NTM that guesses which nodes are in the clique or by considering a certificate, we can check whether the provided vertices actually form a clique by verifying that each pair of vertices are actually pairwise adjacent.

Claim 11.78 *The two definitions for NP are equivalent.*

Proof: *Let M be an NTM that recognizes L in polynomial time. Therefore, there is some accepting path in the branching tree, with path length polynomial in the input length. Specifying this path is an encoding of the address of the leaf at the end of the path. Since the height of the path is at most polynomial in the length of the input, we can encode this address in polynomial length. A verifier can then simulate M only on that branch, which decides in polynomial time.*

We can see the other direction by having the NTM non-deterministically guess the certificate. Since this is polynomial in length, it can be done in polynomial depth. ■

NP-Complete Languages

NP contains many problems that are important in practical optimization and decision problems. We would love to be able to solve problems in NP in polynomial time. The big question in computer science is "IS $P = NP$?"

NP-Complete languages are in a sense the hardest languages in NP. If a language NP-Complete and there is a polynomial time algorithm for L, then $P = NP$.

Definition 11.79 *A language is called NP-Complete if it is in NP and there exists a polynomial time mapping reduction from any language in NP to it.*

To find NP-Complete languages, we need to identify one first. SAT was the first proven NP-Complete language (Cook-Levin Theorem). Then, if A is NP-Complete and A mapping reduces to B in polynomial time, then B is called NP-Hard. If we also have that B is in NP, then B is NP-Complete.

Definition 11.80 *A language B is **NP-Hard** if there is a polynomial time mapping reduction from a language $A \in NP$ to B*

Proof: *A sketch of the proof of Cook-Levin ($SAT \in NP$-Complete):*

We've already seen that SAT is in NP.

Let L be any language in NP and V be the polynomial time verifier for L. View the computation of V as a sequence of configurations (snapshots of tape and state after each transition). We want to construct a mapping reduction that transforms an input x to V with into a boolean formula ϕ that is satisfiable if and only if x is in L.

Call the certificate of L y. We don't know y, but we can think about its bits as unknown boolean variables $y_1, y_2 \dots$. The initial configuration of V looks like $q_o x \# y_1 y_2 \dots$. We can say that x is in L if we end up in q_{accept}.

We need to make sure that V can't be fooled. We need to be sure that w is a valid input and that if we are given a sequence of configurations, that sequence must be legal according to the transition function of V.

V runs in polynomial time, and its input is polynomial in length and for polynomially many steps. Bound the length of the input by l and the number of steps taken by l (one of these can be a loose upper bound). Let's make boolean variables $x_{ij\sigma}$ where $i, j \leq l$ and $\sigma \in \Gamma \cup Q$. Set $x_{ij\sigma}$ equal to 1 if cell i at step j contains σ and 0 otherwise.

We can say that $x \in L$ if and only if there exists an accepting table of V on w, where an accepting table refers to the legal sequence of tape configurations ending in an accept state. We want to create a formula which is satisfiable if and only if there is an accepting table of V on w.

We will build the corresponding ϕ:

These conditions define a legal beginning and end configuration:

$\bigwedge_{i,j} \bigvee_{\sigma} x_{ij\sigma}$ *- at least one symbol in each cell*

$\bigwedge_{\sigma_1,\sigma_2} \neg x_{ij\sigma_1} \vee \neg x_{ij\sigma_2}$ *- at most one symbol in each cell*

$\bigwedge(x_{11q_0}) \bigwedge(x_{12w_1}) \dots$ *- the input row contains q_0 and the correct value for the input w*

$\bigwedge(x_{l1q_{accept}})$ *- V takes exactly l steps and brings head to leftmost cell*

We need to make sure the steps are consistent with legal transitions:

We also AND some variables constructed from the transition function to ensure the value at cell i, j is either the same, or influenced by the ones just left or right at the previous step

If this formula is satisfiable, then we have a legal table, hence w is in L. If there is no table, then the formula cannot be satisfiable. ∎

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Lecture 12: NP-Completeness

Professor Sampath Kannan *Zach Schutzman*

Cook-Levin and NP-$Complete$ Languages

From last time, we sketched a proof of the Cook-Levin Theorem - any problem in NP can be converted into an instance of SAT in time and size polynomial in the original input.

Let's show that $3SAT = \{\phi | \phi \text{ is a satisfiable Boolean formula in 3-}CNF\}$ is NP-Complete.

Proof: *We will show that SAT reduces to $3SAT$ and $3SAT$ is in NP to show $3SAT$ is NP-Complete, because Cook-Levin gives us the reduction from any NP language to SAT, by the transitivity of composition of polynomial reductions.*

The reduction is as follows:

Take some instance ϕ of SAT. If we want $y \Leftrightarrow x_1 \wedge x_2$, then we can say $(\neg x_1 \vee \neg x_2 \vee y)$. We don't need to worry about the ORs. Taking the AND of all of these clauses gets us a logically equivalent $\phi' \in 3SAT$.

For the same reason SAT is in NP, $3SAT$ is as well, as it is easily verifiable.

■

Karp took Cook's proof and showed a number of problems are NP-Complete. Some of these are:

- *Independent Set - a subset of vertices, no two of which are adjacent (inputs are $\langle G, k\rangle$, for IS of size k). The reduction from $3SAT$ involves creating triangle for each clause and connecting negations of corresponding variables. Set k to the number of clauses.*
- *$CLIQUE$ is NP-Complete. Take a $\langle G, k\rangle$ instance of IS. Create G^C where two vertices are adjacent in G^C if and only if they are not adjacent in G. Now, the IS in G is exactly a k-clique in G^C.*
- *$VERTEX\ COVER$ is a vertex set of size k such that every edge is incident to some element in the set. The complement of an independent set is a vertex cover, by definition, so a $|V| - k$ independent set is a k vertex cover.*

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Lecture 13: Wrapping up NP and Beginning Space Complexity

Professor Sampath Kannan *Zach Schutzman*

More NP-$Complete$ness

Other problems are NP-Complete, the reduction for $3COLOR$ creates gadgets from each clause and each variable (see any Algorithms book).

There is an obvious reduction from $3COLOR$ to $4COLOR$ (and beyond). Add a new vertex, connect it to every vertex in your original graph.

$SUBSETSUM$, the question of whether a set of numbers contains a subset that sums to a particular value k is NP-Complete. The reduction is from $3SAT$. Given a formula ϕ with m clauses and n variables, we create $2n$ numbers with $n+m$ base 7 digits. Each variable gets a 1 in positions indicating the index of the variable and the clauses it appears in. ϕ is satisfiable if and only if there is a subset that sums to $11 \ldots 111$ in the first n positions, and $4 \ldots 44$ in the remaining m positions, by introducing some dummy numbers which are all zeros in the first n positions and 1 or 2 in the corresponding clause position.

We know that $P \subseteq NP$, NP-Complete $\subset NP$

Definition 13.81 *A language L is in Co-NP if its complement is in NP. Equivalently, given $x \notin L$, a verifier can check non-membership given the right certificate (easy to check that something is not a solution).*

Example: *$TAUT = \{\phi | \phi \ is \ satistfied \ by \ every \ assignment\}$ is in Co-NP. If a ϕ is not in L, then any non-satisfying assignment can be quickly checked.*

We don't know if $NP =$ Co-NP.

If $P = NP$, then $NP =$ Co-NP.

If $NP \neq$ Co-NP, then $P \neq NP$.

Space Complexity

Definition 13.82 Space complexity *refers to the number of cells of tape scanned by the head of a Turing machine in running an input.*

Definition 13.83 *$DSPACE(s(n))$ is the set of languages recongized by a deterministic TM in $O(s(n))$ space.*

Definition 13.84 *$NSPACE(s(n))$ is the set of languages recognized by a non-deterministic TM in $O(s(n))$ space.*

Let's look at $SAT \in NP$. What is its space complexity? Let's say n is the length of the input and k is the number of variables. If we don't care about time, we can represent the assignment as a k-bit number and *try one assignment at a time. We only need* $n + k + c$ *(where* c *is some small fixed workspace). Therefore,* $SAT \in DSPACE(n)$.

Let $L_{NANFA} = \{\langle M \rangle | M \textit{ is an } NFA,\ L(M) \neq \Sigma^*\}$. *We don't know if this is in* NP *because the naive approach for a verifier does not necessarily have a polynomial length certificate. We can show that it is in* $NSPACE(n)$.

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Lecture 14: More Space Complexity

Professor Sampath Kannan *Zach Schutzman*

$DSPACE$ v $NSPACE$

Recall, we showed last time that $SAT \in DSPACE(n)$.

We were thinking about the language $NA_{NFA} = \{\langle M\rangle | M$ is an NFA, $L(M) \neq \Sigma^*\}$. We showed in homework that the shortest string not accepted by an NFA can be exponential in the number of states of the machine. Recall also that for the DFA problem, the upper bound is simply the number of states in the machine. We can transform a k-state NFA into a DFA with at most 2^k states, so we can upper bound the length of the shortest non-accepted string by $2^k - 1$.

Our input is an NFA with description length n. The number of states is approximately equal to n (maybe n^2, we don't really care). A naive deterministic approach involves checking every string. Let's design a non-deterministic space algorithm to do the search. We non-deterministically guess one symbol at a time. The depth of the tree is $2^n - 1$, which is still exponential. But we don't really need to remember the whole path, just the current possible states and the counter of the depth. This count is length order n and the set of current states is also order n. We therefore have an $NSPACE(n)$ algorithm to solve this, hence $NA_{NFA} \in NSPACE(n)$.

Claim 14.85 *Savitch's Theorem: $NSPACE(f(n)) \subseteq DSPACE(f(n)^2)$. That is, we only get a quadratic increase by moving to a deterministic model.*

Proof:

(For space $f(n) = \Omega(n)$)

Given an $O(f(n))$-space NDTM M, we will design an $O(f(n)^2)$-space DTM D which recognizes the same language.

The input x determines the initial configuration of M, C_{init}. We are interested in whether M can get to some accepting configuration with the q_a state in the first cell and the rest of the tape blank (canonically). We have an upper bound of $2^{O(f(n))}$ steps needed to reach such a configuration.

Write $C_i \vDash_t^M C_j$ to say that there is some valid computation in M that moves from configuration C_i to configuration C_j on some branch in at most t steps. We want to know if $C_{init} \vDash^M_{2^{O(f(n))}} C_{accept}$. We can approach this with divide-and-conquer by asking whether $C_{init} \vDash^M_{\frac{2^{O(f(n))}}{2}} C_m \vDash^M_{\frac{2^{O(f(n))}}{2}} C_{accept}$ for some intermediate configuration C_m.

At each step, we need to check whether the start configuration can reach the end configuration. Since we can reuse space, we need to remember $\log 2^{O(f(n))} = O(f(n))$ intermediary configurations, each of length $O(f(n))$. This requires total space of $O(f(n)^2)$

■

Definition 14.86 *$PSPACE$ is defined as $\bigcup_k DSPACE(n^k)$. $NPSPACE$ is $\bigcup_k NSPACE(n^k)$.*

Corollary 14.87 *By Savitch's Theorem, $PSPACE = NPSPACE$. Nobody talks about $NPSPACE$ because it is a redundant name for $PSPACE$.*

Space vs Time Complexity

We have $DTIME(f(n)) \subseteq DSPACE(f(n))$, as if you can only take $f(n)$ steps, you are only using $f(n)$ cells (at most). The same holds for the non-deterministic case.

We also have that $DSPACE(f(n)) \subseteq DTIME(2^{O(f(n))})$, because we can enumerate all of the $2^{O(f(n))}$ configurations of the machine in $f(n)$ space, one at a time. To see that we only see each configuration at most once, observe that even in the non-deterministic setting, if two branches have the same configuration, the continuations of those branches must be identical, so we don't need to check it more than once. Therefore, without loss of generality, there is some accepting path that does not contain repeated configurations. We can keep a 'timer' that terminates a branch when it has exceeded $2^{O(f(n))}$ steps, thus ensuring that every branch terminates in at most $2^{O(f(n))}$ steps. The same holds for the non-deterministic case.

We now have $P \subseteq NP \subseteq PSPACE$. We don't know if $P \neq PSPACE$, but this should be easier to prove than $P \neq NP$.

The Class $PSPACE$-*Complete*

Definition 14.88 *A language L is $PSPACE$-Complete if L is in $PSPACE$ and for all $A \in PSPACE$, there exists a mapping reduction from A to L in polynomial time.*

Claim 14.89 *The language of True Quantified Boolean Formulas (TQBF) is $PSPACE$-Complete.*

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Lecture 15: PSPACE Completeness

Professor Sampath Kannan *Zach Schutzman*

The Class *PSPACE-Complete*

Definition 15.90 *A language L is PSPACE-Complete if L is in PSPACE and for all $A \in PSPACE$, there exists a mapping reduction from A to L in polynomial time.*

Definition 15.91 *A **quantified boolean formula** is a formula of the form $\forall x_1 \exists x_2 \forall x_3 \ldots \phi(x_1, x_2, \ldots)$, where ϕ is a boolean formula over the x_i.*

Claim 15.92 *The language of True Quantified Boolean Formulas (TQBF) is PSPACE-Complete.*

Proof:

We'll think of a machine that decides this language as taking a quantified boolean formula as input and accepting if it is true, rejecting if it is false.

This language is obviously in PSPACE, we can simply check each allowable assignment and accept if the formula is true.

To show that it is PSPACE-Complete, we need to show that every other PSPACE language can be reduced to it in polynomial time. Let L be a language in PSPACE. Without loss of generality, we can say that a decider for L takes $O(2^{n^k})$ time.

We saw in the Cook-Levin Theorem how to think of states of computation as boolean formulas, and we will do something similar here. Let C_{init} the initial configuration and C_{accept} as the canonical accepting configuration. We write $\Phi(C_a, C_b, t)$ to denote the quantified boolean formula corresponding to 'there exists a sequence of computation steps of length at most t such that the machine moves from configuration C_a to C_b. We want to find formula equivalent to $\Phi(C_{init}, C_{accept}, 2^{n^k})$ which is true if and only if the input x to the decider for L is in L.

We start by observing that if $x \in L$, then there must be some intermediate configuration C_m which the machine passes through in its computation. We can then write

$$\Phi(C_{init}, C_{accept}, 2^{n^k}) \iff \Phi(C_{init}, C_m, \frac{2^{n^k}}{2}) \wedge \Phi(C_m, C_{accept}, \frac{2^{n^k}}{2})$$

We can also consider quarter-way and eighth-way points, and so on and do the same thing, however, we get an exponentially large formula by the time we get down to step sizes of $t = 1$.

We get around this by introducing two new variables C_3, C_4 and universally quantifying them over our configuration pairs $(C_1, m), (m, C_2)$ such that we have

$$\Phi(C_1, C_2, t) = \exists m \forall (C_3, C_4) \in \{(C_1, m), (m, C_2)\}(\Phi(C_3, C_4, \frac{t}{2}))$$

We only need polynomial space for this, hence L is reducible to $TQBF$, and $TQBF$ is $PSPACE$-Complete. ■

Example: *The language Generalized Geography (GG) is $PSPACE$-Complete. GG is a two player game on a directed graph. One player picks a node, then the other picks a node for which there exists an arc from the previous player's choice into that node. A player loses if there are no nodes for her to legally choose (i.e. the previous player picked a node with out degree zero). Player One's first choice is a designated start node. The language GG is the decision problem of whether, given a directed graph with a start node, Player One can always win given perfect play.*

We can think of this as a $TQBF$ in the following way: Player One has a winning strategy if there exists a node with out-degree zero such that for any previous choice made by Player Two there exists a previous choice by Player One, and so on.

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Lecture 16: Space Complexity, Continued

Professor Sampath Kannan *Zach Schutzman*

Logarithmic Space

Recall, L and NL are the classes of problems recognizable in logarithmic space by a deterministic and non-deterministic Turing machine, respectively. These machines are modeled as having a read-only input tape and a read/write work tape. The space complexity is determined by the number of cells used on the work tape.

The language $PATH = \{\langle G, s, t\rangle | G \text{ is a digraph with a path from } s \text{ to } t\}$.

Claim 16.93 *PATH is in NL.*

Proof: *Nondeterministically guess a sequence of neighboring vertices, for at most n steps. This needs only* $O(log(n))$ *space to track the current vertex and a counter.* ■

It turns out, PATH is one of the 'hardest' languages in NL, that is, it is NL-Complete.

We need to show that PATH is in NL (done) and that any NL language can be reduced to it. What is our notion of reduction? Before, we had polynomial time reductions, but $NL \subset P$, *so a polynomial time (and hence polynomial space) could be powerful enough to solve our language. We therefore restrict our reductions to the class L, that is, a log-space reduction.*

Definition 16.94 *A* ***log-space transducer*** *is a machine with a read-only input, a read-write work tape, and a write-only output tape. The work tape has size* $O(log(n))$ *and the write tape can have polynomial length in n.*

Definition 16.95 *A language A* ***log-space reduces*** *to a language B (written* $A \preceq_m^L B$*) if there is a log-space transducer computing a function f such that* $x \in A \iff f(x) \in B$.

Claim 16.96 *PATH is NL-Complete.* **Proof:**

Let A be any language in NL. We show that there exists a log-space transducer reducing A to PATH.

We have some input $w \in \Sigma^*$ *to A and we want to construct a log-space transducer to convert it into a* $\langle G, s, t\rangle$. *The high level idea is to construct a graph with vertices corresponding to configurations of a machine recognizing A, s and t corresponding with initial and accept configurations, and edges for configurations which are one computation step apart.*

A configuration of the A machine is specified by the location of the head in the input tape, the current state, and the contents of the work tape. This requires only $O(log(n))$ *bits to specify, as we need* $O(log(n))$ *to keep track of the position in the input, plus* $O(log(n)))$ *to represent the work tape, plus some constant to track the current state. The number of possible configurations is* $2^{O(log(n))}$, *which is polynomial in n.*

The graph G has one node for each possible configuration, the node s will be the initial configuration, and the node t will be the accept configuration in canonical form. There will be an edge (c_i, c_j) *if the machine for A* *can go from configuration c_i to C_j in one step. There may be multiple edges from each node, corresponding to nondeterministic choice. There exists an s-t path in this graph if and only if $w \in A$.*

We now only need to show that a log-space transducer can compute this reduction. The transducer will first construct the vertices of G, then the edges.

First, the transducer knows the length of each configuration, so it uses its $O(log(n))$ space to iterate through the possible configurations and if it's a valid configuration c, writes c to its output tape. After this, all of the nodes in the graph are written to the output. Then, for the edges, check each pair of configurations to determine if the second is reachable from the first in 1 step.

This procedure only requires $O(log(n))$ work space and produces the appropriate G, s, t, hence is is a log-space reduction and the proof that $PATH$ is NL-Complete.

■

Note that any function computable by a log-space transducer can be computed in polynomial time, a transducer can never repeat a configuration (else it would not be a decider), and there are only polynomially many configurations.

Claim 16.97 $NL \subseteq P$

Proof: *We know that $PATH \in P$ (we can do breadth-first search). Therefore, given any $A \in NL$ and input w, we can, in polynomial time, log-space reduce A to an instance of $PATH$ with input $f(w)$ and use a polynomial time algorithm for $PATH$. This gives a polynomial time algorithm for A.* ■

The Class Co-NL

Definition 16.98 *A language A is in Co-NL if its complement is in NL.*

From before, a language B is in NL if there is a nondeterministic log-space Turing machine M such that given w in B, $M(w)$ accepts on some path. If $w \notin B$, all computation paths reject. Co-NL is the opposite. If $w \in C \in Co$-NL, then all paths accept. If $w \notin C$, there exists at least one rejecting path.

Claim 16.99 $NL = Co$-NL

Proof:

We have $PATH$ as an NL-Complete. Let's define the language $\overline{PATH} = \{\langle G, s, t\rangle | G \text{ is a digraph with no s-t path}\}$. The same reduction as before shows that $\overline{PATH}$ is Co-NL-Complete. We prove this theorem by showing $\overline{PATH} \in NL$. This implies $NL \subseteq Co$-NL, which by symmetry of complements gives us $NL = Co$-NL.

We want a nondeterministic log-space Turing machine M which, given $\langle G, s, t\rangle$ and a number c equal to the number of nodes reachable from s, accepts if there is no path from s to t in G. This machine will guess whether there is or is not a path from s to v_1. If there is one, using a log-space $PATH$ subroutine, continue guessing paths from s to v_2. If every path in this side rejects, we know there is no path from s to v_1, so we continue down the no branch. When we find a path from s to a vertex, we increment a counter. We continue checking for each vertex except t. If there is no s-t path, the counter will reach c before we finish exhausting vertices.

We now need to think about how to get c. Let c_i be the number of vertices reachable from s by paths of length at most i. We can, given c_i, compute c_{i+1}. We have $c_0 = 1$, so if we have a procedure to compute the increments, we can inductively find c_{n-1}. The idea will be to note that vertices reachable in at most $i + 1$

steps have an edge from something reachable in at most i steps.

■

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Lecture 17: Space Hierarchy

Professor Sampath Kannan *Zach Schutzman*

Finishing up $NL = Co\text{-}NL$

Claim 17.100 *$NL = Co\text{-}NL$ (continued from last time)*

Proof:

(Continued)

We were thinking about how to get the count of reachable vertices, c. Let c_i be the number of vertices reachable from s by paths of length at most i. We can, given c_i, compute c_{i+1}. We have $c_0 = 1$, so if we have a procedure to compute the increments, we can inductively find c_{n-1}. The idea will be to note that vertices reachable in at most $i + 1$ steps have an edge from something reachable in at most i steps.

Similarly, denote s_i the set of vertices reachable from s in i or fewer steps.

The following procedure will compute c_{i+1}: For each vertex v_j, $v_j \in s_{i+1}$ if $v_j \in s_i$ or there exists a vertex u such that $u \in s_i$ and (u, v_j) is an edge in G. We will nondeterministically guess whether each vertex v is in s_i. To confirm it, we use a NL procedure for $PATH$ limited to paths of length i. We'll keep a counter for the number of things we find in s_i and the number of things in s_{i+1}. For each vertex, if we find it in s_i, then it is in s_{i+1}, so we increment both of our counters. Else, if there is an edge from something in s_i to it, then we increment our counter for s_{i+1}. Otherwise, we move on to the next vertex. After checking all vertices we now have the number of things in s_{i+1} on the correct branch of the $PATH$ algorithm, which is the one that correctly computed the number of things in s_i.

Inductively, we now know our c, as we do this procedure to find $c = c_{n-1}$. Since we have c, we have an NL algorithm to decide $\overline{PATH}$, hence $L = Co\text{-}NL$.

■

We have $L \subseteq NL = Co\text{-}NL$. Whether L equals NL is unknown. The language $UPATH$, which is $PATH$ on undirected graphs is obviously in NL. It was proven about 10 years ago that $UPATH$ is actually in L.

We can say that Savitch's Theorem still applies. That is, $NSPACE(f(n)) \subseteq DSPACE(f(n)^2)$. Therefore, $NL \subseteq L^2$, deterministic log-squared space.

Space Hierarchy Theorem

We now are ready to show our first proper separation of time and space classes.

Definition 17.101 *A function $f(n)$ is 'nice' if it is **fully space-constructible**. That is, there is a Turing machine which given input, say 1^n, can compute $f(n)$ using no more than $f(n)$ space.*

Space-constructible functions include all the familiar ones, like monotonic polynomials, exponentials, logarithms, roots, etc.

Claim 17.102 (The Space Hierarchy Theorem) *Let $f(n)$ be a 'nice' function. Then there is a language L which can be decided by an $f(n)$-space deterministic Turing machine, but not by any deterministic Turing machine using $o(f(n))$ space. That is, for $f(n)$, there is a language requiring at least $O(f(n))$ space to decide by a deterministic Turing machine.*

Proof:

We prove this by constructing such a language. via diagonalization. Define a deterministic Turing machine M which uses $f(n)$ space and $L(M)$ cannot be decided by any Turing machine using $o(f(n))$ space. We'll think of M as being different from every machine which uses $o(f(n))$ space.

M on input $\langle x\rangle$ first checks if $\langle x\rangle$ is a description of a Turing machine. If not, reject. If it is, M should run $\langle x\rangle$ x. If x finishes in $f(n)$ space, then M outputs the opposite answer as x. M stops and rejects if $\langle x\rangle$ exceeds $f(n)$ space.

We also need to count the steps of x to make sure we don't exceed $2^{O(f(n))}$ time.

We also have an issue because $O(f(n))$ and $o(f(n))$ are asymptotic notions, so we may get something wrong for relatively small n.

We'll modify the behavior of M slightly. On any input, it checks to see if it is the description of a Turing machine, followed by 01^*. If not, reject. If so, simulate x on the whole input. As before, if x properly terminates, we output the complement of its answer. Now, we have that if x uses $g(n)$ space, we know there exists an n_0 such that for all $n > n_0$, $g(n) \leq f(n)$, because there is a sufficiently long input string such that x requires no more space than M.

■

We know $L \subsetneq PSPACE$, and by Savitch's Theorem, $NL \subsetneq PSPACE$. We also have, if $r_1 < r_2 \in \mathbb{R}^+$, then $DSPACE(n^{r_1}) \subsetneq DSPACE(n^{r_2})$.

Similar results hold for $NSPACE$.

Time Hierarchy (briefly)

We want to say that if $g(n) \in o(f(n))$, for time-constructible function $f(n)$, there is a language in $DTIME(f(n))$ that is not in $DTIME(g(n))$, but this isn't necessarily true. If we try the same idea as for Space Hierarchy, we run M on x for $f(n)$ steps, but the problem is that we need to somehow count the steps, which requires keeping a counter of size $O(log(f(n)))$, which may all need to be altered after a step. Additionally, we need access to the transition function of x, which we can keep in space, but in time, we need to track back and forth for lookups, which again adds a quadratic blowup. We can keep this description along with us, but the counter is still a problem to update.

We therefore need to rephrase the statement of the theorem to be

Claim 17.103 *If there is a language in $g(n) \in o(f(n))/log(f(n))$, then there is a language in $DTIME(f(n))$ not in $DTIME(g(n))$.*

The proof follows the same structure as the Space Hierarchy Theorem, including $log(f(n))$ additional steps to update the counter.

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Lecture 18: Randomized Complexity

Professor Sampath Kannan *Zach Schutzman*

Randomized Algorithms

Definition 18.104 *A* **randomized algorithm** *is one in which we can toss random coins.*

We don't worry about how this randomness is produced, we just take it as given that we can make these queries from an idealized source of randomness.

Definition 18.105 *A* **randomized** *or* **probabilistic Turing machine** *may either move deterministically or reach a state where it can flip a coin and move according to the random output. A probabilistic Turing machine M is said to accept a string $x \in L$ with probability equal to the sum of the probabilities of the leaves where M produces the correct answer.*

Definition 18.106 *The class* **Bounded Probabilistic Polynomial time (BPP)** *is the class of languages recognizable by a randomized Turing machine which accept on inputs $x \in L$ with probability at least $\frac{2}{3}$.*

Claim 18.107 *We can equivalently define BPP as having error $0 \leq \epsilon < \frac{1}{2}$.*

Proof: *Suppose we have a machine M with error $\epsilon < \frac{1}{2}$. Then we show there is an M' with error $\frac{1}{2^n}$ on inputs of size n. We simply have M' run M $2k$ times on input x and outputs the majority answer, where k is constructed as follows:*

To see this, consider our sequence of M's outputs, e.g. $cccwwcwcwccwc\ldots$. A sequence with more than k ws will cause an error in M'. We want this to occur with probability at most $\frac{1}{2^n}$. Any particular sequence with some number of cs and ws is $\epsilon^{\#w}(1-\epsilon)^{\#c}$. So M' makes a mistake when $\#w \geq k$. But this is less than $\epsilon^k(1-\epsilon)^k$. There are at most 2^{2k} 'bad' sequences (this is all of them!). Thus, the total probability of all bad sequences is less than or equal to $2^{2k}\epsilon^k(1-\epsilon)^k = (4\epsilon(1-\epsilon))^k$. This is strictly less than one whenever $\epsilon < \frac{1}{2}$.

We can then choose k to make the lower bound as low as we want for an equivalent definition of BPP. ■

What is the space complexity of probabilistic Turing machines?

Consider the language $UPATH = \{\langle G, s, t\rangle |$ there is an st path in $G\}$. This is in randomized log-space (RL). We can do this by randomly choosing a neighbor at each step, not exceeding n^3 steps on any branch.

Polynomial Identity

The problem of polynomial identity testing asks whether two multivariate polynomials f and g on n variables are exactly identical. We have two black boxes, one for f and one for g. We want to know if $f = g$.

Denote $L_{non-id} = \{\langle f, g\rangle | f \neq g\}$ for polynomials f and g. If f and g are univariate of degree less than or equal to d, we can know if $f = g$ just by testing $d + 1$ different values.

Lemma 18.108 *(Schwartz-Zippel) Let h be a non-zero polynomial on n variables of degree d. If values $x_1 \dots x_n$ are chosen uniformly and independently at random from a set S of size n, then the probability that $h(x) = 0$ is less than or equal to $\frac{d}{n}$.*

If we think of $h = f - g$, then we have the 'bad' event of getting unlucky of picking $d+1$ points where $f(x) = g(x)$ even when $f \neq g$ is sufficiently small.

Now we can come up with a randomized algorithm to recognize L_{not-id}. We choose a set S of $3d$ points from the domain of the variables. For each x_i pick uniformly at random some $r_i \in S$. Now, evaluate $f(r)$ and $g(r)$. If we get different answers, accept. Otherwise, reject. If f and g are identical, then the algorithm always correctly rejects. If f and g are different, the probability we picked a 'bad' point and make a mistake is at most $\frac{1}{3}$, because we have $\frac{d}{3d}$ probability by the lemma.

Definition 18.109 *The language L is in **Probabilistic Polytime (PP)**if there is a probabilistic Turing machine that gets the answer wrong with probability less than $\frac{1}{2}$.*

This class is not really interesting. A PP algorithm for SAT picks a random assignment. If it satisfies the formula, we accept. If not, flip a coin with probability marginally less than $\frac{1}{2}$ to come up heads. If we get heads, reject, otherwise, accept.

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Lecture 19: Randomized Complexity, Continued

Professor Sampath Kannan *Zach Schutzman*

Randomized Polynomial (RP)

Last time we defined BPP as the class of algorithms with probabilistic polynomial time Turing machines which correctly decided membership in a language with probability at least $\frac{1}{3}$ (equivalently, strictly bounded from $\frac{1}{2}$). We showed last time that polynomial identity testing is in BPP (equivalently, deciding whether a polynomial is identically zero).

We also have the class PP, which is not very interesting. PP allows error as high as $\frac{1}{2} - \frac{1}{2^n}$. Things like SAT and beyond are in PP.

Definition 19.110 *A language L is in **Randomized Polynomial** (RP) if there is a probabilistic polynomial time Turing machine M such that given $x \in L$, M accepts with probability greater than or equal to $\frac{1}{2}$, and if $x \notin L$, M always rejects.*

We say RP has one-sided error, because we only make mistakes on strings in the language. If the algorithm accepts, we know for certain that the string is in the language. If it rejects, we cannot say for sure.

Let's look at polynomial non-identity testing again. Our algorithm from before always accepts when a polynomial is identically zero, but sometimes makes mistakes if the polynomial is not, but we got unlucky.

We know that $P \subseteq RP$ and $P \subseteq BPP$.

We can also do amplification on RP algorithms. We just run the algorithm k times and accept if any execution excepts. Therefore, $RP \subseteq BPP$ because we can run our RP algorithm twice to bring the error probability below $\frac{1}{3}$.

We also have $RP \subseteq NP$. To see this, consider $L \in RP$. If $x \in L$, then our RP machine M accepts x with probability at least $\frac{1}{2}$. There exists a sequence of 'coin flips' that cause M to accept on x. This serves as our certificate with which we can verify membership by deterministically simulating M on x with the specific sequence of random choices. Similarly, Co-$RP \subseteq$ Co-NP. Equality and/or proper containment is unknown.

Let's think about the problem of matrix multiplication checking. The language $L_{mmult} = \{\langle A, B, C\rangle | A, B, C \text{ are } n\times n \text{ matrices and } AB \neq C\}$. Matrix multiplication is already known to be in P ($O(n^{2.34\cdots})$). Therefore, verification is also in P. But, we can use randomization to verify in $O(n^2)$ steps.

Let's suppose $AB = D$ and we want to know if $D = C$. We can randomly sample cells in D to check if they match cells of C. We can find and compare a pair of cells in n steps. But we want to think of this in a worst-case or adversarial setting, where maybe C and D only differ at one entry. Then we need to check n^2 entries for n steps each, which isn't any better than solving the whole thing.

What we will do is, instead of randomly sampling, we just take a random vector v and check if $(AB)v = Cv$. This at least has the property that if AB equals C then we accept. We can compute $A(Bv)$ in $n^2 + n^2$ steps, then do Cv in n^2 steps, so the process takes $O(n^2)$ steps. What is the probability of error? Suppose $D \neq C$. What is the probability that $Dv = Cv$?. C and D differ in at least one entry, call it $D_{ij} \neq C_{ij}$. If v is a binary vector chosen uniformly at random, suppose every value except v_j chosen ahead of time, which we will *think of as a variable. When we multiply the ith row of D by v, we get some constant $c_1 + D_{ij}v_j$. Similarly, for the ith row of C, we get $c_2 + C_{ij}v_j$. We now want to think about choosing v_j. If $c_1 = c_2$, then the choice of $v_j = 1$ gives different answers. This occurs with probability $\frac{1}{2}$. On the other hand, if $c_1 \neq c_2$, then choosing $v_j = 0$ shows the difference, which also occurs with probability $\frac{1}{2}$. Hence this algorithm is in RP. We can improve this error probability by letting v have arbitrary, rather than binary entries.*

Communication Complexity

Let's think about a communication problem, originally called The Man on the Moon Problem. Let x be a string on a spaceship, of which Ground Control has a copy y. Before executing x, the spaceship wants to verify that their copy has not been corrupted, so it wants to communicate with Ground Control to confirm $x = y$. We are concerned about the communication complexity of this problem; the number of bits we need to communicate in order to accomplish this task. We will assume that communication is two-way, reliable, and interactive.

We'll use cryptographic convention and call the spaceship player Alice (A) and the Earth player Bob (B).

We will think of corruption of x as adversarial. If the length of x and y is n, to do this deterministically, we need to exchange at least n bits to verify the whole string. To see this, suppose $x_1 \neq x_2$ are two n-bit strings.

Claim 19.111 *We claim that any communication transcript on input (x_1, x_1) must be different from the transcript on (x_2, x_2).*

Proof:

Suppose the input is actually (x_1, x_2) and we assume for the sake of contradiction that the transcripts for (x_1, x_1) and (x_2, x_2) are the same.

Without loss of generality, assume that Alice transmits the first bit, assuming the protocol for (x_1, x_1). When Bob receives this, it's consistent with the transcript for (x_2, x_2), so Bob sends the next bit assuming the input is (x_2, x_2). This repeats and neither side has any information at any point to contradict their assumptions that they are working on (x_1, x_1) and (x_2, x_2), respectively.

We therefore need 2^n different transcripts to correspond to the 2^n possible n-bit strings they have. By the pigeonhole principle, at least one of these must have length 2^n, because the number of binary strings of length strictly less than 2^n is $2^n - 1$.

$\blacksquare$

Let's think of a randomized protocol that uses fewer bits. Consider $S = \{2, 3, 5, 7 \dots\}$ the set of the first $2n$ primes. Bob picks a random prime p from S and sends $y \mod p$ and p. Alice computes $x \mod p$ and accepts if $x \mod p = y \mod p$. Since the $2n^{th}$ prime is around $2n\ln(2n)$, so we need $O(log(n))$ bits to send it. Additionally, $y \mod p$ is no bigger than p, so it is also $O(log(n))$ in size. We have one sided correctness; if $x = y$, Alice always accepts. But what happens when $x \neq y$?

Assume $x \neq y$ are both n-bit strings. A bad situation would be our protocol failing because $x \mod p = y \mod p$. We chose p randomly, so we want to know for how many primes in S do we have this bad case? If $x \mod p_i = y \mod p_i$ and $x \mod p_j = y \mod p_j$, then p_i and p_j divide $x - y$. So p_ip_j divides $x - y$ (this holds for cases with more than two factors). The product of any n primes in S is at least 2^n. If $x \neq y$, then $x - y \neq 0$. We also have $x - y \leq 2^n$. If x and y agree modulo n different primes, then $x \cong y \mod \prod p_k$, but this product is greater than 2^n, which is impossible, hence x must be exactly equal to y.

We therefore know that x and y can be congruent modulo at most n primes in S. Since we pick a prime *from S at random, our probability of picking a bad one is at most $\frac{1}{2}$, which is what we wanted to show.*

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Lecture 20: Interactive Proof

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Interactive Proof

NP is the class of languages for which an infinitely powerful prover can convince a polynomial time verifier of string membership by sending one message (the certificate). This is a non-interactive case.

*We need to show that such proofs are both **sound** and **complete**. That is, if $x \notin L$, no message from the prover will convince the verifier, and for all $x \in L$, there is some message from the prover which will convince the verifier.*

We'll denote P the prover, V the verifier, and $\tilde{P}$ a prover which may be adversarial. Completeness means one good P exists. Soundness means no $\tilde{P}$ can falsely convince V of membership.

*What if we allow for interaction? Are there languages for which there are **interactive proofs** that are not in NP? Well, if the prover is all-powerful, because the verifier is deterministic, the prover can simulate the interaction, and can produce as a proof a transcript of the potential interaction. Therefore, interaction alone only lets us prove things in NP.*

The problem of Graph Non-isomorphism L_{GNI} asks whether two input graphs G, H are not isomorphic , where two graphs are isomorphic if there exists a bijection f from $V(G)$ to $V(H)$ such that $(u, v) \in E(G)$ if and only if $(f(u), f(v)) \in E(H)$. Graph Isomorphism is in NP (the certificate is the bijection f). L_{GNI} is not known to be in NP.

We do have an interactive proof for L_{GNI}.

Proof:

Suppose we have two graphs G, H and we want to try to prove $G \not\simeq H$. V will pick one of G, H at random, call it K. V then permutes the adjacency matrix of K to form K' which is, by definition, isomorphic to K. V then sends K' to P, and P must figure out whether $K' \simeq G$ or $K' \simeq H$.

Suppose $G \not\simeq H$. An honest prover will try to find an isomorphism between K' and G and K' and H, and will find isomorphism for exactly one of them. If G, H are actually isomorphic, then the prover will have no idea whether K' was derived from G or H, and it is equally likely to have been generated by V from either. In this case, P will have to guess and will get it wrong half of the time. Therefore, a prover cannot convince a verifier that two truly isomorphic graphs are not isomorphic, so we have soundness and completeness.

■

Definition 20.112 *The class **interactive proof** IP is the class of languages L such that there exists a prover P and a probabilistic polynomial time verifier V and a communication protocol such that P can convince V that some $x \in L$ with probability at least $\frac{2}{3}$. For any $x \notin L$ and any prover P', V is convinced $x \in L$ with probability at most $\frac{1}{3}$.*

Claim 20.113 *As it turns out, $IP = PSPACE$.*

Proof:

First, we will show $IP \subseteq PSPACE$. Let's think about the prover as a tree, where it chooses one of 2^l messages of length l and the verifier does the same. A leaf of the tree corresponds to the verifier accepting or rejecting, with probability 0 or 1. At a prover node, we assign it a probability equal to the maximum across its children, because the prover wants to convince the verifier. At an internal verifier node, we assign it the average probability of its children. We just need to answer if the probability assigned to the root is more or less than $\frac{2}{3}$. If greater, $x \in L$, else $x \notin L$. Since the tree has polynomial breadth and depth, we can use depth-first traversal and only store polynomially many answers at a time, therefore, this is in $PSPACE$.

Now, to see $PSPACE \subseteq IP$, we will start by showing $\#SAT(\phi)$, the number of satisfying assignments of ϕ, is in IP. We want to show that P can convince V that ϕ has exactly k satisfying assignments (we'll assume ϕ is in 3-CNF).

Let $f_i(a_1, a_2 \dots a_i)$ be the number of satisfying assignments of ϕ restricted to $x_1 = a_1, x_2 = a_2 \dots$. Here is a proof that's too long, but it will be useful conceptually. Suppose the prover sends $f_0()$, and V checks that $f_0() = k$, then $f_1(0), f_1(1)$. The sum $f_1(0)+f_1(1)$ should equal $f_0()$, and V checks that this is true. We continue this procedure sending all partial assignments, and the verifier checks appropriate sums ($f_2(11)+f_2(10) = f_1(1)$). At the final round, we check all full assignments, and the verifier knows what values to expect in the last round. If the prover tries to lie it must cascade through the tree structure and the verifier can catch it in the last round.

The problem, of course, is that we send 2^m things in the last step. We can embed this structure in a richer algebraic space by arithmetizing our boolean formula. Given a boolean formula ψ, we can call its arithmetization Ψ. If the formula has one variable, its arithmetization is the single variable. If the formula is the AND of two formulas, the arithmetization is the product, NOTs become $(1 - \Psi)$, and ORs become $(1 - \Psi_1)(1 - \Psi_2)$. We can see that $\{0, 1\}$ assignments $\vec{a}$ of ψ agree with $\Psi(\vec{a})$. Once we arithmetize our formula $\phi \to \Phi$. Φ is a function of the same variables, but we can plug in integer/rational/real/complex values instead of binary.

Let $F_1(z)$ for some variable z is the number of satisfying assignments for Φ when the first variable is set to z and the rest are boolean variables. We know $F_1(0) = f_1(0)$ and $F_1(1) = f_1(1)$. $F_1(z)$ is a univariate polynomial in z of degree at most n. The prover sends $f_0()$ then $F_1(z)$. The verifier then asks the prover to prove that for some point r that $F_1(z) = F_1(r)$

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Lecture 21: Interactive Proof, Continued

Professor Sampath Kannan *Zach Schutzman*

Interactive Proof (continued)

We were in the middle of proving that there is an interactive proof for $\#SAT$. **Proof:** *Recall ϕ was our original formula over $x_1, \ldots, x_m$ and its arithmetization Φ, where $\phi(\vec{x}) = \Phi(\vec{x})$ if $\vec{x}$ is a binary vector/boolean assignment. We didn't say this last time, but the polynomial verifier can do this arithmetization.*

The prover wants to convince the verifier that ϕ has k satisfying assignments. Equivalently, $\sum_{i=1}^{m} \sum_{v_i \in \{0,1\}} \phi(x_1, \ldots, x_k) = k$. If we replace ϕ with Φ, we should have the same thing, and the verifier can do just that.

Let's decompose the sum. We have $\sum_{v_2 \in \{0,1\}} \sum_{v_3 \in \{0,1\}} \cdots \sum_{v_m \in \{0,1\}} \Phi(z, x_2, x_3, \ldots, x_m) = F_1(z)$. There are 2^{m-1} possible assignments for the remaining $m-1$ variables. We therefore have a polynomial over our finite field $\mathcal{F}$ in z of degree no more than the number of times x_1 occurs, which is no more than the length of the input, n.

The prover sends $F_1(z)$ to the verifier. $F_1(0) + F_1(1)$ should equal k. The verifier can plug these in for z and check that the sum is equal to k. If not, the verifier rejects. The verifier then picks an element $r_1 \in \mathcal{F}$ at random, and asks the prover to prove that $F_1(r_1)$ has the value it is supposed to have.

The prover then looks at $F_2(r_1, z)$, which is the summation over $x_3 \ldots x_m \in \{0,1\}$ of $\Phi(r_1, z, x_3 \ldots)$. The prover sends the coefficients of $F_2(r_1, z)$ to the verifier. The verifier checks that $F_2(r_1, 0) + F_2(r_1, 1) = F_1(r_1)$. Then, it picks a random $r_2 \in \mathcal{F}$, and asks the prover to prove that $F_2(r_1, r_2)$ has the value it should. This procedure repeats until we get to F_m.

We now need to verify correctness. Suppose the prover is honest and $\#\phi = k$, then the prover will convince the verifier of this fact. Suppose the prover may be dishonest. We will show that with high probability, the verifier will catch a lie. If the prover $\tilde{P}$ claims $\#\phi\tilde{k} \neq k$, let $\tilde{F}_1(z)$ be the polynomial that $\tilde{P}$ sends the verifier in the first round. The correct polynomial is $F_1(z)$. Could $\tilde{F}_1(z) = F_1(z)$? No! Because then $\tilde{F}_1(0) + \tilde{F}_1(1) = F_1(0) + F_1(1) = k$. $\tilde{F}_1(z)$ must be different, otherwise $\tilde{P}$ would be caught immediately.

Then, $\tilde{F}_1(z) - F_1(z) \not\equiv 0$. This difference polynomial has degree less than or equal to n. The verifier picks r_1 at random. Then, at most n values for r_1 will be a root of $\tilde{F}_1(z) - F_1(z)$. Therefore, the probability that r_1 is a root is at most $\frac{n}{|\mathcal{F}|}$. If r_1 is not a root of this difference, then $\tilde{F}_1(r_1) \neq F_1(r_1)$, and the prover is still trying to prove something false. At any round, the prover is only off the hook if the verifier happens to pick a root of this difference for r_i.

At the end, $\tilde{P}$ claims $\tilde{F}_m(r_1, r_2, \ldots, r_m) = \tilde{k}$, but this is equal to $\Phi(r_1, r_2, \ldots, r_m)$, which the verifier can check itself, catching a lie. What is the probability that a lying prover does not get caught? The probability that a prover escapes detection is no more than $\frac{n}{|\mathcal{F}|}$. Therefore, the probability that the verifier catches a dishonest prover is $1 - \frac{n}{|\mathcal{F}|}$.

As long as we pick a field of size at least $3n^2$, we have a valid interactive proof.

■

To complete the proof that $IP = PSPACE$, we need to show $PSPACE \subseteq IP$ by taking a PSPACE-Complete language and showing it has an interactive proof. We can do this for TQBF. The structure of the proof will take a formula $\psi = q_1x_1q_2x_2 \ldots q_mx_m\phi(x_1, x_2 \ldots x_m)$ and perform arithmetization to strip variables off and reduce the problem. For the quantifiers, we can replace the $FOR\ ALLs$ by ANDing two formulas and the $EXISTS$ by ORing two formulas. The degrees of this polynomial will grow quickly by this process, so it may take exponential space to write the polynomial down. Therefore, the verifier can't compute the arithmetization. We fix this by replacing high degrees of variables with lower degrees, because they will still agree on $x = 0, 1$. This is subtle and a little tricky, but doable.

Zero Knowledge Proofs

Interactive proof is important in cryptography. We are interested in cases where Alice uses a 'secret' to perform a task to convince Bob that the she knows the secret without revealing it to Bob.

We want to show that zero-knowledge proofs are sound (a dishonest prover gets caught), complete (if both parties are honest, the prover can convince the verifier), and zero-knowledge (an honest verifier doesn't reveal the secret).

Let's suppose we have a graph $G = (V, E)$. The prover wants to convince V in zero knowledge that G is 3-colorable, which it knows. If we didn't have the zero-knowledge requirement, we could just treat this as an NP language and be done in one step.

Here's the interactive proof. At each round, the prover creates a random graph H isomorphic to G and permutes the colors. The prover correctly 3-colors H and hides the colors and the edges. The prover could be lying in a couple of ways: H could not be isomorphic to G or the coloring it picked is not a valid 3-coloring. The verifier knows the original graph G and the three colors that should be used.

The verifier is allowed to make one of two challenges at each round. It can ask the prover to prove that the presented graph is isomorphic to G. This can be done by revealing the edges and giving a short proof of isomorphism. The second challenge is that the verifier can pick any two vertices and ask the prover to show that they are properly colored. This can be done by revealing at most the edge between them and the respective coloring. If the prover succeeds in showing something valid, then we move to the next round. Otherwise, the verifier catches a lie and rejects.

What is the probability the cheating prover gets caught? If the graph is not isomorphic, then the prover gets caught with probability $\frac{1}{2}$. If the prover is lying about the colorabilty of the graph, then there is some pair of vertices colored illegally. The verifier will catch this with probability $O(\frac{1}{n^2})$. Therefore, in n^3 rounds, the probability of failing to catch a lying prover is $O(e^{-n})$.

Where does zero-knowledge come in? An IP is zero-knowledge if the transcript could have been generated without the verifier actually looking at the graph. Since challenges and vertex pairs are chosen at random, it could commit, a priori, to a (secret) order of challenges, so the verifier doesn't really learn anything from the prover. Since the verifier can simulate the protocol, it is zero-knowledge.

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Lecture 22: Cryptography

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Cryptography Basics

Historically, cryptography has been about sending encrypted messages, such that only the intended recipient can decipher the contents.

In general, we have some plaintext m and ciphertext c and a function $f(m) = c$. The message cannot be sent as m, so instead the first party, Alice, computes $f(m)$ and sends the second party, Bob, c. Bob should be the only one who can compute $f^{-1}(c) = m$.

Easy examples include rotation ciphers (shift letters forward/backwards some number of places) and total replacement. These are both really easy to decode, so in modern settings, they are not practical. We also don't think that you can get better security by making the function more complex (i.e. composing a bunch of bad methods in a convoluted way). We want methods that are provably difficult for an eavesdropper, Eve, to invert.

*The **one-time pad (private-key encrpytion)** is a secure scheme. Suppose we have a key k with $|k| = |m|$. Alice takes m and k and send $c = m \oplus k$, where $\oplus$ is the bitwise exclusive-or operation. Bob knows k and computes $m = c \oplus k$ (exclusive-or is self-inverse). If k is drawn uniformly from all strings of that length, the cyphertext is random with respect to the plaintext. This is a **symmetric key encoding scheme**, because Alice and Bob both have the same information, and we use a key to perform the encryption/decryption. This whole scheme relies on keeping k secret.*

One drawback is that each pair of parties needs a separate key, agreed upon beforehand. If there are n users, we need $O(n^2)$ keys of length $O(2^m)$. This is a lot. Can we do better?

***Public-key cryptography** fixes some of these drawbacks.*

Cryptography and Complexity

For cryptography to work, we need functions that are easy to compute but hard to compute.

***Definition 22.114** A function $f : \{0,1\}^* \to \{0,1\}^*$ is **length-preserving** if $|f(w) = |w|$ for all $w \in \{0,1\}^*$.*

***Definition 22.115** A function f is a **permutation** if it is bijective on $\{0,1\}^n$ for all n (bijective and length-preserving).*

A probabilistic Turing machine M on input x, we can talk about the probability that M outputs y. By definition, $\sum_y (M \text{ outputs } y) \leq 1$.

Definition 22.116 *A length-preserving function f is a* **one-way function** *if there is a poly-time deterministic Turing machine M that on input w outputs $f(w)$ and for n large enough, if w is chosen uniformly at random from $\{0,1\}^n$, and k is any constant, and D is any probabilistic poly-time Turing machine, the probability that D on input $f(w)$ computes y, such that $f(y) = f(w)$ is less than or equal to $\frac{1}{n^k}$.*

Basically, the function is easily computed but a decoder can find an inverse with extremely low probability.

One-way functions are good for passwords. The user computes the encryption of the password and sends that ciphertext to the system. At login, the system only needs to check that the ciphertexts match, and you never reveal the plaintext of the password.

Definition 22.117 *A* **trapdoor function** *is a one-way function which is easy to invert given that you know a certain 'secret'.*

Example: *let $N = pq$, where p, q are n-bit primes. Then, the set Z_N^* is the set of all numbers less that N and relatively prime to N. (E.g. $Z_{10}^* = \{1,3,7,9\}$, $Z_7^* = \{1,2,3,4,5,6\}$). We have a function f which maps $x \in Z_N^*$ to $x^2 \in Z_N^*$. f is a trapdoor function. This is believed to be one-way because if we could do it quickly, we could factor integers.*

Proof: *We want to show that being able to compute the discrete square root in Z_n^* efficiently gives us a fast algorithm for factoring. Suppose we have a black box that computes discrete square roots. If we take a random r and compute $a = r^2$ and feed a to the black box to get a result b. Then, check if $r+b$ or $r-b$ share a factor with N (via the Euclidean algorithm). If so, we have found one of the factors of N and therefore can get the other.*

We claim this succeeds with probability $\frac{1}{2}$ and is therefore a probabilistic polynomial algorithm. When we pick r, it has some residue mod p and mod q. Since $r^2 = a$, we have the squares of the same residues. The black box returns a square root b. Half of the time, we get back the same residues (plus or minus), and half the time we do not. If we got back the same ones, we failed as we gained no new information. The other half of the time, we get back some multiple of either p or q, and then can compute the greatest common divisor with N. ■

We can do encryption with this by doing the following: Bob picks primes p, q and publishes $N = pq$, but not p or q. When Alice wants to send a message m, she computes $c = m^2 \mod N$ and sends c to Bob. Since Bob can factor N, he can easily invert the computation and recover one of the four square roots of m^2. If Alice makes sure her message is shorter than $\frac{N}{2}$, then we reduce to only two possible roots, and the Jacobi symbol can further reduce it to one. Bob can then read m.

This is a form of public-key cryptography called the ***Rabin cryptosystem****, and is algorithmically equivalent to factoring. The drawback is the four-to-one mapping, and we don't like this potential for collision.*