Math 500 - Topology and Geometry

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Introduction

Math 500 is a Masters-level first-course in Topology and Geometry. The course follows James Munkres' *Topology, 2ed.* and this set of notes is based on the Fall 2017 offering.

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Please email any corrections or suggestions to ianzach+notes@seas.upenn.edu.

Lecture 1 (2017-08-30)

What is Topology?

Definition. A **topology** is a set X with a collection of subsets $\mathcal{A} \subset \mathcal{P}(X)$ such that:

1 $\varnothing, X \in \mathcal{A}$

2 $\mathcal{A}$ is closed under finite intersection (the intersection of a finite subset of $\mathcal{A}$ is in $\mathcal{A}$)

3 $\mathcal{A}$ is closed under arbitrary union (the union of any (possibly infinite) subset of $\mathcal{A}$ is in $\mathcal{A}$)

The phrases "$\mathcal{A}$ is a topology on X", "X is a topological space with topology $\mathcal{A}$, and the notation $(X, \mathcal{A})$ all refer to the same concept.

Definition. The **standard topology** (also called the euclidean topology or metric topology) on $\mathbb{R}^n$ is the set of subsets $U \subset \mathbb{R}^n$ such that for every U, every point $x \in U$ is interior, meaning that there exists some radius $r > 0$ such that the ball of radius r centered at x is entirely contained in U.

Definition. A set is **open** in a topological space X if it belongs to the topology on X.

Example. The standard topology is a topology over $\mathbb{R}^n$:

1 Every point in the empty set is vacuously interior, and every point of $\mathbb{R}^n$ is trivially interior

2 If we take two open sets and intersect them, any point in the intersection must be an interior point in both constituent sets. The smaller of the two balls witnessing this must lie entirely within both constituent sets, and therefore entirely within the intersection. By induction, we have the finite intersection of open sets being open.

3 Intuitively, taking any union of open sets only creates a bigger set. The ball witnessing any point as interior to some open set clearly lies in any union including that open set.

We can see from this example why it's important to specify closure under *finite* intersection. Singleton sets are not open in the standard topology on $\mathbb{R}^n$, but the Nested Interval Theorem gives us a way to construct a singleton set from the countable intersection of open intervals.

Example. If X is our topological space, $\{\varnothing, X\}$ is a topology, called the **trivial topology**.

Example. Similarly, all of $\mathcal{P}(X)$ is a topology, called the **discrete topology**.

Example. The **Zariski topology** on $\mathbb{R}^n$ is a little more interesting. A set is open in the Zariski topology if it is the complement of the root set of some polynomial. Open sets in the one-dimensional case look like the real line minus a finite number of points. It gets a little more complicated in higher dimensions, as we can have zeroes along entire dimensions of a euclidean space. Let's verify that this is a topology:

1 The empty set is the complement of the root set of the zero function, and the entire space $\mathbb{R}^n$ is the complement of the root set of a polynomial which has no real roots, such as $f(\vec{x}) = 6$.

2 The intersection of two open sets, corresponding to polynomials P and Q is, by DeMorgan's Laws, $\mathbb{R}^n \setminus \{x \mid x \textit{ is a root of } P \textit{ or } Q\}$. Something is a root of P or Q, it must be a root of the product PQ. Since the finite product of polynomials is a polynomial, this set is still the complement of the root set of some polynomial, and is therefore open, and we have closure under finite intersection.

3 Again by DeMorgan's Laws, the union of two open sets corresponding to polynomials P and Q is the set $\mathbb{R}^n \setminus \{x \mid x\ is\ a\ root\ of\ P\ and\ Q\}$. The set of points which are roots of P and Q are the roots of the greatest common polynomial divisor of P and Q. Since this is also a polynomial, our set is the complement of the root set of a polynomial and is therefore open. Since the greatest common polynomial divisor of any set of polynomials has root set no greater than any of the constituent polynomials, we properly have closure under arbitrary union.

The Zariski topology is an object of importance in the area of algebraic geometry.

Definition. If X is a topological space with topology $\mathcal{A}$ and $Y \subset X$, then $\mathcal{B}$ is a topology on Y where a subset $V \subset Y$ is open in $\mathcal{B}$ if and only if there is a U open in $\mathcal{A}$ such that $V = U \cap Y$. This is called the **subset** or **subspace topology**.

Example. Let H^2 denote the closed upper-half plane in $\mathbb{R}^2$. That is, the set of points $(x, y) \in \mathbb{R}^2$ such that $y \geq 0$. Any set which was open in $\mathbb{R}^2$ and does not intersect the x-axis is still open in H^2. However, a set like an open half-disk against the x-axis together with the line segment where it rests up against the x-axis was not an open set in $\mathbb{R}^2$, as the boundary points are not interior, but it is open in H^2 with the subspace topology, as it is the intersection of an open disk in $\mathbb{R}^2$ with the upper half-plane.

Lecture 2 (2017-09-01)

Continuous Maps

Continuous maps are the standard morphisms in topology.

In Analysis, we have a definition of continuity which looks like:

Definition. A function $f : X \to Y$ is **continuous** at $x \in X$ if for any $\delta > 0$ there exists an $\epsilon > 0$ such that $\|x - y\| < \epsilon$ implies $\|f(x) - f(y)\| < \delta$.

The issue with this definition is that we have no natural notion of distance in topology. Instead, we use the definition:

Definition. A function $f : X \to Y$ is **continuous** if the inverse image of an open set in Y is open in X. Equivalently, the inverse image of closed sets are closed.

It turns out that in metric spaces like $\mathbb{R}^n$ with the standard topology, these definitions are equivalent.

Example. Let's consider two topological spaces: $(\mathbb{R}, std)$, the real numbers with the standard topology, and $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$, the real numbers with the discrete topology. The map $f : (\mathbb{R}, \mathcal{P}(\mathbb{R})) \to (\mathbb{R}, std)$, where $f(x) = x$ is continuous. Since every set is open in the discrete topology, the inverse image of any set, in particular any open set, is open. The map $g : (\mathbb{R}, std) \to (\mathbb{R}, \mathcal{P}(\mathbb{R}))$, where $g(x) = x$ is not continuous. To see this, take any set that is closed with respect to the standard topology. This set is open in the discrete topology, but its inverse image is closed.

This raises the question: is there any $g : (\mathbb{R}, std) \to (\mathbb{R}, \mathcal{P}(\mathbb{R}))$ which is continuous?

Theorem. *The only continuous functions $g : (\mathbb{R}, std) \to (\mathbb{R}, \mathcal{P}(\mathbb{R}))$ are the constant maps.*

Proof. First, it is easy to see that a constant map is continuous. Without loss of generality, we'll assume that $g(x) = 0$. Let V be open in the discrete topology. If V contains 0, then the inverse image of V is all of $\mathbb{R}$. If V does not contain zero, then the inverse image of V is the empty set. Since both of these are open in the standard topology, the inverse image of any open set is open, and the map is continuous.

To see that such a continuous map must be constant, first observe that $\mathbb{R}$ and $\varnothing$ are the only sets which are both closed and open with respect to the standard topology. Let $g : (\mathbb{R}, std) \to (\mathbb{R}, \mathcal{P}(\mathbb{R}))$ be a continuous map and pick some $x \in \mathbb{R}$. The set $\{g(x)\}$ is both closed and open in the discrete topology (as every set is closed and open), so its inverse image must be, in particular, open. But the inverse image cannot be empty, as we know for sure it contains x, and the only non-empty closed and open set in the standard topology is the entire space. Therefore, for any $x, y \in \mathbb{R}$, we have $g(x) = g(y)$, which is only true for constant maps.

$\square$

Definition. A **homeomorphism** is a continuous bijection between two topological spaces such that the inverse is also continuous.

Under a homeomorphism, we also have the property that the image of open sets is open. This induces a bijection between the open sets of the two topological spaces. In a sense, the existence of a homeomorphism means that two topological spaces are the same.

Example. The two spaces $(-1,1)$ and $(-2,2)$ with the standard topology are homeomorphic under the map $f(x)=2x$.

Example. The two spaces $(-1,1)$ and $\mathbb{R}$ with the standard topology on each are homeomorphic under the map $f(x)=tan(\frac{\pi}{2}x)$.

Example. Let $S^n=\mathbb{R}^n\cup\{\infty\}$. A set $U\subset S^n$ is open if:

$U=\varnothing$ or $U=S^n$

$U\subset\mathbb{R}^n$ and U is open with respect to the standard topology.

$\infty\in U$ and $U\cap\mathbb{R}^n$ is the complement of a compact subset of $\mathbb{R}^n$. That is, U looks like all of $\mathbb{R}^n$ with a closed and bounded chunk removed, and an additional point ∞.

This forms a topology, and the set S^n is the surface of the n-dimensional sphere. If we think about S^2, there's a natural embedding in $\mathbb{R}^3$, but it turns out that $S^2\setminus\{(0,0,1)\}$ is homeomorphic to $\mathbb{R}^2$. If we use the (north polar) stereographic projection, which maps points in S^2 to the point in the $\mathbb{R}^2$ plane according to the straight line passing through the north pole and that point, we get a nice homeomorphism, and this is easy to see from the subspace topology that S^2 inherits from $\mathbb{R}^3$. If we then include that the north pole maps to our added point ∞, we get a map from all of S^2 to the set $\mathbb{R}^2\cup\{\infty\}$ which is a homeomorphism.

The Quotient Topology

Let $(X,\mathcal{A})$ be a topological space and $\sim$ an equivalence relation on X. Then $X/\sim$ inherits a topology, which is that $U'\subset X/\sim$ is open if and only if there is some open set $U\subset X$ such that $U'=U/\sim$.

Definition. This topology is called the **quotient topology**, or the **identification map**.

Example. Take $\mathbb{R}^2$ with the standard topology and define an equivalence relation $(x,y)\sim(x,-y)$. The quotient space looks like the closed (upper or lower) half-plane.

Example. $\mathbb{R}^2$ with the standard topology quotiented by the equivalence relation $(x,y)\sim(-x,-y)$ looks like a cone, and is actually homeomorphic to $\mathbb{R}^2$.

Example. $\mathbb{R}^2$ with the standard topology and the equivalence relation $(x,y)\sim(x+1,y)\sim(x,y+1)$ has a quotient space that looks like the unit square with opposite sides glued together. This is homeomorphic to a (genus 1) torus.

Lecture 3 (09-06-2017)

The Pullback Topology

Let $(X, \mathcal{A})$ be a topological space, and Y some set. Given a map $f : X \to Y$, Y inherits a topology from X where $V \subset Y$ is open if and only if $f^{(} - 1)(V) \subset X$ is open.

Definition. This topology on Y is called the **pullback topology**.

The pullback topology is the finest topology on Y which makes f a continuous map.

Example. Take $f : (-1, 1) \to \mathbb{R}$, with $f(x) = x$. The pullback topology on $\mathbb{R}$ has open sets $\varnothing$ and $\mathbb{R}$, any set in $\mathbb{R}$ which does not intersect the open interval $(-1, 1)$, any set which is open in $(-1, 1)$ in the standard topology, and any set whose intersection with $(-1, 1)$ is open in the standard topology.

Group Actions and Fundamental Regions

Let's think about $\mathbb{Z}^2$ as a group action on $\mathbb{R}^2$, where applying $(a, b) \in \mathbb{Z}^2$ to $(x, y) \in \mathbb{R}^2$ means shifting (x, y) right by a and up by b (left, down if a or b is negative, of course). We write this as $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by $(a, b) \cdot (x, y) = (x + a, y + b)$. This establishes an equivalence relation on $\mathbb{R}^2$: $(x_0, y_0) \sim (x_1, y_1)$ if there exists $(a, b) \in \mathbb{Z}^2$ such that $(a, b) \cdot (x_0, y_0) = (x_1, y_1)$.

This divides $\mathbb{R}^2$ into 1×1 squares, where each square is equivalent to any other, and we identify the left and right edges and the top and bottom edges, but no two points in the interior of any given square are equivalent. We call the squares fundamental regions.

Definition. A **fundamental region** of a group action and is the (closure of) largest region such that no two interior points are identified with respect to the induced equivalence relation.

Example. The fundamental region described above, the square with opposite edges identified, defines a torus.

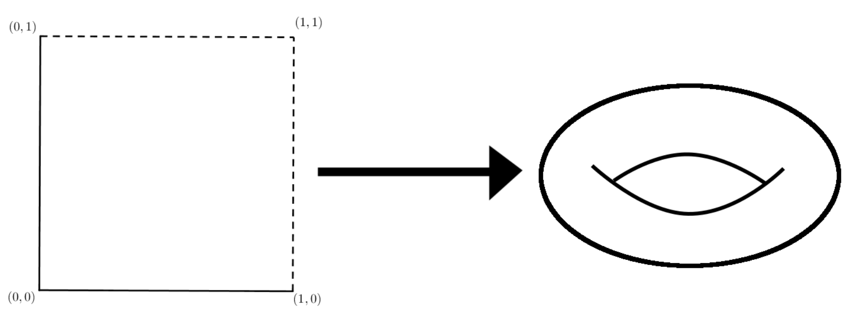

Figure 1:

Open sets in the torus $\mathbb{T}$ are $\varnothing$ and $\mathbb{T}$, and the intersection of any standard open set with the fundamental domain. This is the same as the subspace topology on $\mathbb{R}$.

Example. If we consider $\mathbb{R}^2/\mathbb{Z}$, where $a \in \mathbb{Z}$ acts on $\mathbb{R}^2$ by $a \cdot (x, y) = (x + a, y)$. A fundamental domain of this action is a vertical strip of unit width.

Again, this inherits a subspace topology from $\mathbb{R}^2$.

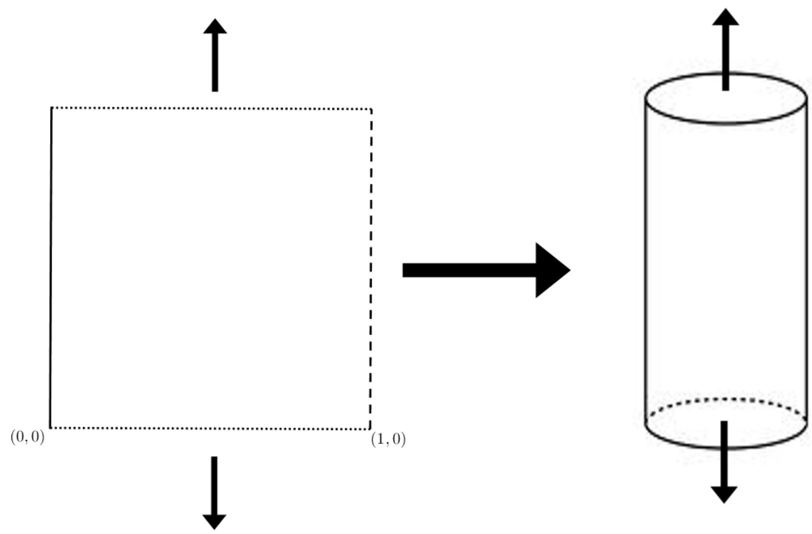

Figure 2:

Example. Consider again $\mathbb{R}^2/\mathbb{Z}$, but this time with the group action $a \cdot (x, y) = (x + a, (-1)^a y)$. The fundamental region is still a strip of unit width, but this time instead of identifying points on the boundary with their horizontal translation, we identify them with their horizontal translation composed with reflection about the x-axis. This space is homeomorphic to an infinite Moebius strip, which is difficult to draw.

Example. Consider the equivalence relation on $\mathbb{R}^2$ described by $(x, y) \sim (x, y)$, $(0, y) \sim (1, y)$, and $(x, 0) \sim (1 - x, 1)$. The fundamental region again is a square with the left and right edges identified by simple translation, but the top and bottom edges are now identified by translation plus a flip across the square's vertical axis of symmetry. This is homeomorphic to the Klein bottle, which is, again, hard to draw.

Example. The previous example where we also identify the left and right edges by translation and a flip is called the real projective plane, denoted $\mathbb{R}P^2$. Both vertical and horizontal strips of this space look like Moebius strips. This is, once again, not easy to draw.

Example. This one we can draw! Take the unit square as the fundamental region, but identify the top and left edge with each other by symmetry about the corner where they intersect, and do the same for the bottom and right edge. This space is homeomporhic to the 2-sphere $\mathbb{S}^2$.

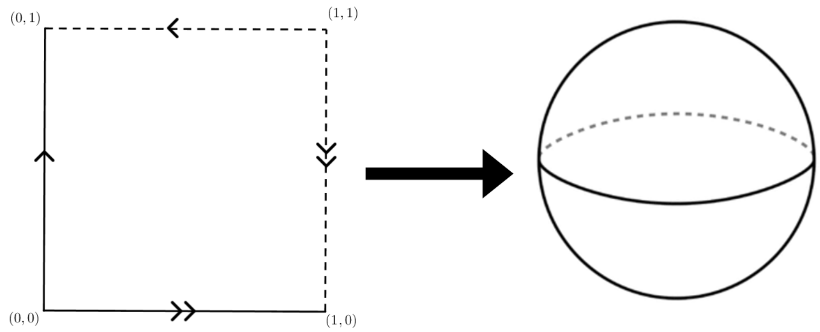

Figure 3: