

Generating N -point spherical configurations with low mesh ratios using spherical area coordinates

Brian Hamilton

^aAcoustics & Audio Group, University of Edinburgh, 12 Nicolson Sq, Edinburgh, EH8 9DF, UK

Abstract

This short contribution presents a method for generating N -point spherical configurations with low mesh ratios. The method extends Caspar-Klug icosahedral point-grids to non-icosahedral nets through the use of planar barycentric coordinates, which are subsequently interpreted as spherical area coordinates for spherical point sets. The proposed procedure may be applied iteratively and is parameterised by a sequence of integer pairs. For well-chosen input parameters, the proposed method is able to generate point sets with mesh ratios that are lower than previously reported for $N < 10^6$.

Keywords:

spherical point set, mesh ratio, icosahedral nodes, uniform spherical grid

1. Introduction

Generating a set of N uniformly-distributed points on the unit sphere (\mathbb{S}^2) is a problem with applications in many fields [1]. There are many approaches to distributing N points on a sphere, including, e.g., Gaussian grids, uniform random distributions, spirals [2], t-designs [3], and subdivisions of regular polyhedra [4, 5, 6]; see [7] for a comprehensive review. There are many measures to evaluate the quality of such N -point spherical configuration, and sequences thereof, including, e.g., potential energies, separation and covering distances, and the mesh ratio [7]. Considering the mesh ratio (where lower is better), it is known that for a sequence of N -point spherical configurations the asymptotic lower bound is ≈ 0.618 [8]. By comparison, it has been shown that a sequence of equal-area icosahedral points sets has mesh ratios that tend to a value of ≈ 0.697 , which is the lowest reported thus far [7, 6].

In this short contribution, a method is presented to generate spherical point configurations with low mesh ratios using Caspar-Klug point sets interpreted as spherical area coordinates (SAC). The proposed SAC method is described in detail, subsequently evaluated with examples and compared to the state of the art, and demonstrates lower mesh ratios than previously reported in the literature. Accompanying code for the proposed method is available at <https://github.com/bsxfun/sac-method>.

2. Methods

2.1. Caspar-Klug point sets

Caspar and Klug provided a method for generating point sets on an icosahedron based on the unfolded icosahedral net of triangles superimposed on a triangular grid (hexagonal lattice) of points [4]. This is briefly described below.

Consider the unit vectors \mathbf{e}_1 and \mathbf{e}_2 in \mathbb{R}^2 :

$$\mathbf{e}_1 = (1, 0), \quad \mathbf{e}_2 = (1/2, \sqrt{3}/2) \quad (1)$$

The following set is a triangular lattice of points generated with the above vectors and the integer pairs (q_1, q_2) :

$$\mathbb{G} = \{\mathbf{r}_{q_1, q_2} = q_1 \mathbf{e}_1 + q_2 \mathbf{e}_2 : (q_1, q_2) \in \mathbb{Z}^2\} \quad (2)$$

Let $\Delta_{(m, n)}$ denote the triangle with vertices $\mathbf{0}$, $\mathbf{r}_{m, n}$, and $\mathbf{r}_{-n, m+n}$, and (m, n) is an integer-pair with $m \geq n$, $m > 0$, and $n \geq 0$. We define the point set $\mathbb{T}_{m, n}$ as:

$$\mathbb{T}_{(m, n)} = \{\mathbb{G} \cap \Delta_{(m, n)}\} \quad (3)$$

This is illustrated in Figure 1a.

Such triangular point grids can be generated over each of the 20 triangular faces of an icosahedron to give a point set on the surface of the icosahedron [4] (see Fig. 1b), and projecting those points onto the unit sphere gives a spherical point set with icosahedral symmetry (see Fig. 1c), also known as “radial icosahedral nodes” [6]. The cardinality N of such an icosahedral point set is given by:

$$N = 10\Gamma_{m, n} + 2 \quad (4)$$

with $\Gamma_{m, n} = m^2 + n^2 + mn$. This formula was given in [6] based on the unfolded net of the icosahedron overlaid on \mathbb{G} , and it was also noted that this

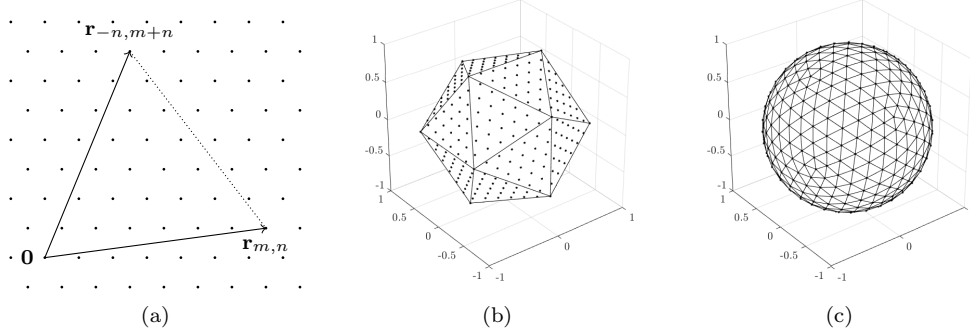


Figure 1: For the choice of $(m, n) = (6, 1)$: (a) Illustration of triangular grid, vectors $\mathbf{r}_{m,n}$ and $\mathbf{r}_{-n,m+n}$, triangle $\triangle_{(m,n)}$ and point set $\mathbb{T}_{m,n}$; (b) $\mathbb{T}_{m,n}$ on faces of icosahedron; (c) projection of icosahedral point set to unit sphere (triangulated with convex hull).

is independent of the planar unfolding of the icosahedron, which is ultimately due to the fact that $\triangle_{(m,n)}$ tiles the plane. As such, this result extends to any unfolded polyhedral net, provided it is made up of equilateral triangles. Thus, the basic operation of generating a point set on an equilateral triangle can be applied to faces of the regular tetrahedron ($V = 4$) or the regular octahedron ($V = 6$) to generate spherical point sets with the symmetries of those polyhedra. Analyzing the total area of the unfolded net, it can be shown that the cardinality of the resulting point set is

$$N = (V - 2)\Gamma_{m,n} + 2 \quad (5)$$

with $V \in \{4, 6, 12\}$.

2.2. Extension to non-equilateral triangles

This approach can be extended to the case of non-equilateral triangles by converting the points in $\mathbb{T}_{m,n}$ to their planar barycentric coordinates. Let $\mathbf{a} = \mathbf{r}_{m,n}$, $\mathbf{b} = \mathbf{r}_{-n,m+n}$, with $m_a = m$, $n_a = n$, $m_b = -n$, and $n_b = m + n$, such that:

$$\mathbf{a} = m_a \mathbf{e}_1 + n_a \mathbf{e}_2, \quad \mathbf{b} = m_b \mathbf{e}_1 + n_b \mathbf{e}_2 \quad (6)$$

and let $\triangle_{0ab} = \triangle_{(m,n)}$. Next, consider some point $\mathbf{p} = \mathbf{e}_{m_p, n_p} = m_p \mathbf{e}_1 + n_p \mathbf{e}_2$, such that $\mathbf{p} \in \triangle_{0ab}$ (i.e., $\mathbf{p} \in \mathbb{T}_{m,n}$). This can be written in terms of three barycentric coordinates $\lambda_{p,a}$, $\lambda_{p,b}$, and $\lambda_{p,0}$ as:

$$\mathbf{p} = \lambda_{p,0} \mathbf{0} + \lambda_{p,a} \mathbf{a} + \lambda_{p,b} \mathbf{b} \quad (7)$$

with

$$\lambda_{p,a} + \lambda_{p,b} + \lambda_{p,0} = 1 \quad (8)$$

and $0 \leq \lambda_{p,a}, \lambda_{p,b} \leq 1$. The coordinate $\lambda_{p,0}$ is not needed, and is only provided for convention. Given (m_p, n_p) one can then calculate $(\lambda_{p,a}, \lambda_{p,b})$ from:

$$\begin{pmatrix} \lambda_{p,a} \\ \lambda_{p,b} \end{pmatrix} = \begin{pmatrix} m_a & m_b \\ n_a & n_b \end{pmatrix}^{-1} \begin{pmatrix} m_p \\ n_p \end{pmatrix} = \frac{1}{\Gamma_{m,n}} \begin{pmatrix} m+n & n \\ -n & m \end{pmatrix} \begin{pmatrix} m_p \\ n_p \end{pmatrix} \quad (9)$$

Since $\Gamma_{m,n} > 0$, any valid integer pair (m_p, n_p) corresponding to point \mathbf{p} can be expressed in planar barycentric coordinates $(\lambda_{p,a}, \lambda_{p,b})$ with respect to \triangle_{0ab} . Having such a grid of barycentric coordinates, one can generate point sets on any triangle (not necessarily equilateral), simply using (7). Furthermore, one can generate a point set on the surface of any closed triangular mesh, and if that triangular mesh is the convex hull of its vertices, one can then project the generated point grid to the unit sphere to obtain a spherical point set. It follows that the cardinality of the resulting spherical point set is given by (5) for any $V \geq 4$.

Since any such spherical point set can be triangulated via its convex hull, one can also apply these operations *recursively*. Starting from a V_0 -point spherical configuration, recursive point-set generation by the above procedure results in a spherical point set with cardinality:

$$N = 2 + (V_0 - 2) \prod_{k=1}^K (m_k^2 + n_k^2 + m_k n_k) \quad (10)$$

where $((m_1, n_1), \dots, (m_K, n_K))$ is a sequence of $K > 1$ integer-pairs with $m_k \geq n_k$, $m_k > 0$, and $n_k \geq 0$.

2.3. Homogenizing with spherical area coordinates

In order to improve the uniformity of the generated point sets, we can re-interpret the planar barycentric coordinates as a form of *spherical* barycentric coordinates [9] known as *spherical area coordinates* [10] – a technique which is known to improve radial icosahedral point sets (with $K = 1$, $n_1 = 0$ in (10)) [9, 10].

For this, consider the planar triangle \triangle_{0ab} with point $\mathbf{p} \in \triangle_{0ab}$. It is well-known that for the point \mathbf{p} 's planar barycentric coordinates $\lambda_{p,0}, \lambda_{p,a}, \lambda_{p,b}$ (given by (9) and (8)), one has the properties that:

$$\frac{|\triangle_{0ap}|}{|\triangle_{0ab}|} = \lambda_{p,b}, \quad \frac{|\triangle_{pab}|}{|\triangle_{0ab}|} = \lambda_{p,0}, \quad \frac{|\triangle_{0pb}|}{|\triangle_{0ab}|} = \lambda_{p,a} \quad (11)$$

where $|\triangle_{0ab}|$ denotes the planar area of triangle \triangle_{0ab} .

Let $|\triangle_{0ab}|$ now denote the *spherical* area of the *spherical* triangle \triangle_{0ab} . In the spherical setting, we can use the term “spherical area coordinates” (after [10]) to describe $\lambda_{p,0}, \lambda_{p,a}, \lambda_{p,b}$ (under the constraints (11)). However, in this setting it is well-known that (7) does not hold, and we must instead solve the system of equations (11) to find a point \mathbf{p} (given its spherical area coordinates). Fortunately, an iterative solution to this problem was provided in [9]. Additionally, an analytic solution was recently presented in [10]. Formula for that analytic solution are left out for brevity (see [10]), but they are implemented in the accompanying code.

3. Evaluations

Having described the general method for generating spherical point sets from polyhedra with triangular faces, it remains to evaluate resulting N -point spherical configurations. As a quality measure for these point sets we focus on the mesh ratio, which is defined below. For a N -point spherical configuration ω_N , we first define the separation distance $\delta(\omega_N)$:

$$\delta(\omega_N) = \min_{\substack{x, y \in \omega_N \\ x \neq y}} |x - y| \quad (12)$$

and the covering radius $\eta(\omega_N)$:

$$\eta(\omega_N) = \max_{y \in \mathbb{S}^2} \min_{x \in \omega_N} |x - y| \quad (13)$$

The mesh ratio is then defined as:

$$\gamma(\omega_N) = \frac{\eta(\omega_N)}{\delta(\omega_N)} \quad (14)$$

For this definition of the mesh ratio, it is known that $\lim_{N \rightarrow \infty} \gamma(\omega_N) \geq \frac{\sec(\pi/5)}{2} \approx 0.618$ [8].

Additionally, for visualization purposes, we consider a triangle quality measure defined here as the ratio of smallest and largest edge lengths in a triangle (henceforth, “the edge ratio”). This edge ratio is non-negative and bounded by one, and the bound one is achieved with an equilateral triangle (thus, higher is better). The appearance of higher quality triangles (locally) does not necessarily translate to a lower mesh ratio (a global measure), but

viewing the overall distribution of triangle qualities helps to provide some insight into the overall mesh ratio. This triangle quality measure, which is just one of many possible [11], is chosen for its simplicity to compute.

For brevity, we consider only icosahedral point sets, wherein the base spherical configuration are $V_0 = 12$ points from a regular icosahedron. Sequences of integer-pairs are herein indicated as $((m_1, n_1), \dots, (m_K, n_K))$, where a superscript integer indicates repeated entries (e.g., $((1, 1), (2, 0), (2, 0))$ can be written $((1, 1), (2, 0)^2)$. The SAC method presented here is compared to the “Equal-area Icosahedral” (EQA) configurations from [6] (computed with the accompanying Matlab code to [6]). Note that the SAC method with $K = 1$ and $n_1 = 0$ is that of [10].

To start, we consider the configurations with $N = 252$ illustrated in Figs. 2a–2c. subtle differences between the EQA method (Fig. 2a) and the SAC method (Fig. 2b) may be observed for the integer-pair $(5, 0)$. In particular the EQA method provides a point set whose triangulation has higher-quality triangles concentrated at centers of faces of the original icosahedron. On the other hand, the SAC configuration, which provides a lower mesh ratio (higher quality point configuration) and tends to have a more even distribution of edge ratios throughout its triangulation.

Fig. 2c shows the first application of the general SAC method presented here (e.g., with $K > 1$). For the same $N = 252$, this SAC approach provides a mesh ratio that is a significant improvement over Figs. 2a and 2b, and a triangulation that shows a more even distribution of edge ratios.

For further evaluations, consider the configurations in Figs. 2d–2i with N chosen larger (in the range 7000–8000). It can be seen comparing Fig. 2d and Fig. 2e that the EQA approach tends to have higher-quality triangles, yet overall the SAC method achieves a lower mesh ratio (recall, the mesh ratio is a global quantity) for the same N . Fig. 2f shows that for a slightly higher value of N this SAC method is able to achieve a lower mesh ratio with well-chosen input parameters (note: mesh ratios tend to increase with N). Furthermore, for $N = 7682$, Figs. 2g and 2h demonstrate alternative, improved, choices of integer-pair sequences. Finally, Fig. 2i shows a slight variation and improvement to that in Fig. 2f. It is clear from Figs. 2f–2i that a lower mesh ratio is achieved through a more uniform distribution of triangle qualities, rather than the appearance of high-quality triangles in local region.

For the final evaluation of this approach in this study, we investigate the behavior of *sequences* of N -point spherical configurations that may be generated with this SAC approach. Shown in Fig. 3 are calculated mesh ratios

for N -point spherical configurations with N limited to 10^6 . The sequences of spherical configurations featured here are given by parameterised integer-sequences with one integer parameter $l > 0$. The first sequence, labeled “EQA $((l, 0))$ ”, is a sequence from the Icosahedral Equal-Area method featured in [6] with the lowest reported limiting mesh ratio for any sequence in the literature (to the author’s knowledge). The following sequences use the general SAC approach. It can be seen that for $N < 60,000$, the SAC sequence “ $((l, 0))$ ” outperforms the EQA approach in terms of calculated mesh ratios. Although, a better choice for $N < 200,000$, is the SAC sequence “ $((1, 1), (l, 0))$ ”. It is important to note that limiting mesh ratios for those SAC sequences do not appear to be bounded for large N , so they lack the property of *quasi-uniformity* [6].

The final three recursive sequences (EQA $((1, 1), (2, 0)^l)$, etc.) show the lowest mesh ratios of all, but are sparser in terms of N values. Mesh ratios for those recursive sequences do appear to be bounded as N increases, which hints at the property of quasi-uniformity. In particular, the sequence given by $((1, 1), (4, 0)^l)$ has a mesh ratio that appears to be bounded by 0.630, which is significantly lower than the best EQA sequence from [6].

4. Conclusions

A method for generating N -point spherical configurations was presented in this short article. The method features Caspar-Klug triangular point grids extended to non-equilateral spherical triangles using spherical area coordinates (SAC). This SAC method can be applied recursively and is parameterised by a sequence of integer pairs. The method was evaluated numerically and demonstrated the ability to return N -point spherical configurations with mesh ratios lower than previously reported in the literature.

The method presented herein is rather general, but may require careful selection of input parameters (integer-pair sequences) for good results. Some favorable examples are given here, but it is not always clear which parameters are optimal for a given N . Future work could investigate more choices of integer-pair sequences with this approach, as well as evaluate such configurations with other measures (e.g., potential energies). Only icosahedral configurations were investigated for brevity, but the method works well with octahedral point sets. Comparisons with existing octahedral configurations is left for future work.

Python code to generate spherical configurations with this method is provided at <https://github.com/bsxfun/sac-method>.

References

- [1] E. B. Saff, A. B. Kuijlaars, Distributing many points on a sphere, *The Mathematical Intelligencer* 19 (1) (1997) 5–11.
- [2] C. G. Koay, Analytically exact spiral scheme for generating uniformly distributed points on the unit sphere, *Journal of Computational Wcience* 2 (1) (2011) 88–91.
- [3] R. S. Womersley, Efficient spherical designs with good geometric properties, in: *Contemporary computational mathematics-A celebration of the 80th birthday of Ian Sloan*, Springer, 2018, pp. 1243–1285.
- [4] D. L. Caspar, A. Klug, Physical principles in the construction of regular viruses, in: *Cold Spring Harbor symposia on quantitative biology*, Vol. 27, Cold Spring Harbor Laboratory Press, 1962, pp. 1–24.
- [5] A. Holhoş, D. Roşca, An octahedral equal area partition of the sphere and near optimal configurations of points, *Computers & Mathematics with Applications* 67 (5) (2014) 1092–1107.
- [6] T. Michaels, Equidistributed icosahedral configurations on the sphere, *Computers & Mathematics with Applications* 74 (4) (2017) 605–612.
- [7] D. P. Hardin, T. Michaels, E. B. Saff, A comparison of popular point configurations on S^2 , arXiv preprint arXiv:1607.04590 (2016).
- [8] A. Bondarenko, D. P. Hardin, E. B. Saff, Mesh ratios for best-packing and limits of minimal energy configurations, *Acta Mathematica Hungarica* 142 (1) (2014) 118–131.
- [9] R. Bauer, Distribution of points on a sphere with application to star catalogs, *Journal of Guidance, Control, and Dynamics* 23 (1) (2000) 130–137.
- [10] K. Lei, D. Qi, X. Tian, A new coordinate system for constructing spherical grid systems, *Applied Sciences* 10 (2) (2020) 655.
- [11] P. Pébay, T. Baker, Analysis of triangle quality measures, *Mathematics of computation* 72 (244) (2003) 1817–1839.

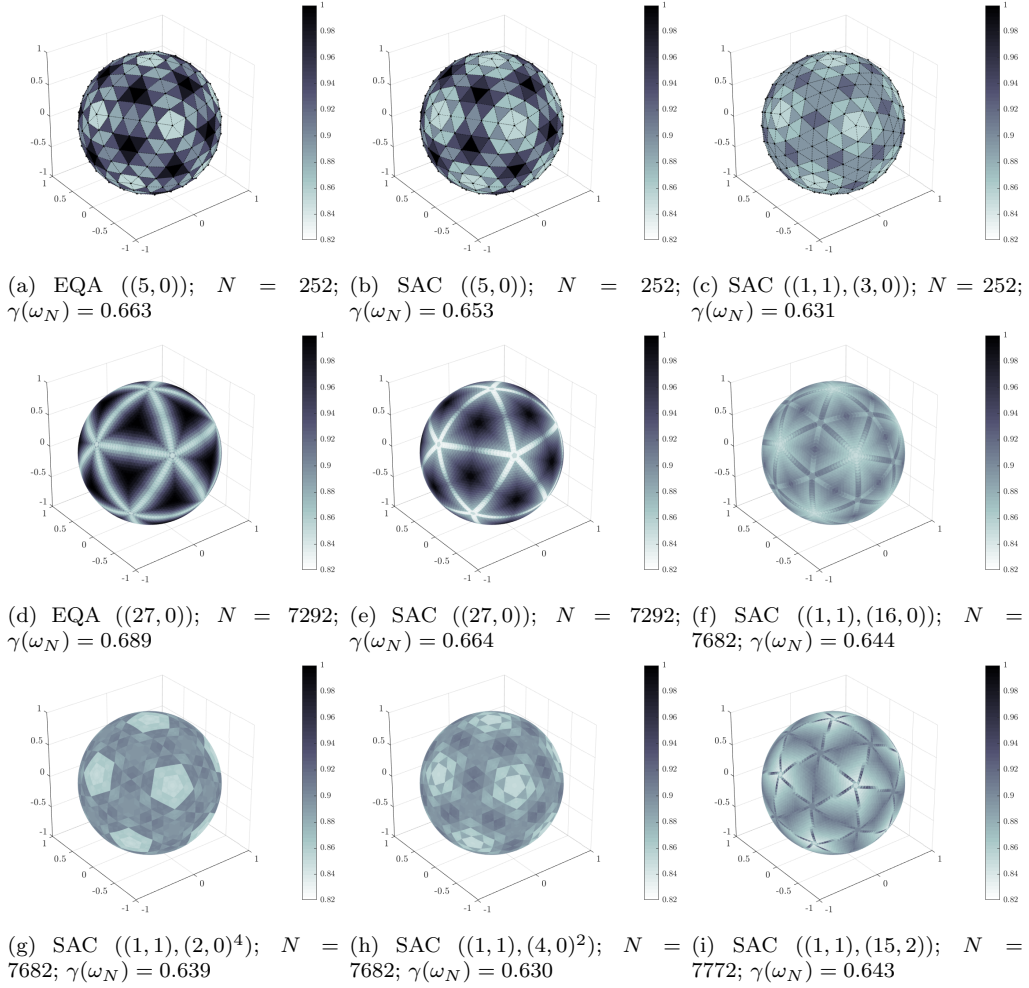


Figure 2: Illustrations of N -point icosahedral spherical configurations with equal-area method (EQA) and spherical-area-coordinate (SAC) method for sequence of (m_k, n_k) integer-pairs. Calculated mesh ratios are as indicated, and triangles are colored by edge ratios. For ease of viewing, triangle edges and vertices are not displayed for denser point sets in Figs. 2d–2i.

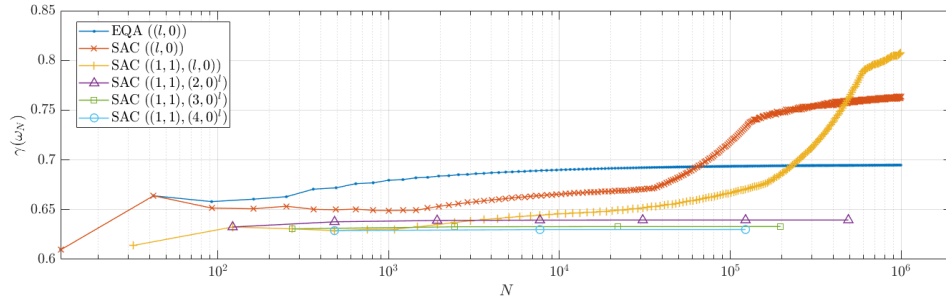


Figure 3: Mesh ratios for sequences of N -point spherical configuration with $N < 10^6$ (with integer $l > 0$ increasing within each sequence).