TSKS04 – Extra problems for Tutorial 1
Solutions.

Q.1 The relation between a real passband signal and its corresponding complex envelope is given by

$$x_p(t) = Re\left\{ x(t)\exp(j2\pi f_c t)\right\}$$

when $x_p(t)$: passband signal
$x(t)$: convex envelope of $x_p(t)$

Similarly

$$y_p(t) = Re\left\{ y(t)\exp(j2\pi f_c t)\right\}$$

for some $a \in \mathbb{C}$ it holds $Re(a) = \frac{1}{2}(a + a^*)$

where $(\cdot)^*$ corresponds to the operation of complex conjugation.

Hence

$$x_p(t) = \frac{1}{2}\left\{x(t)\exp(j2\pi f_c t) + x^*(t)\exp(-j2\pi f_c t)\right\}$$

$$y_p(t) = \frac{1}{2}\left\{y(t)\exp(j2\pi f_c t) + y^*(t)\exp(-j2\pi f_c t)\right\}$$

By substituting we can show

$$\int_{-\infty}^{+\infty} x_p(t)\, y_p(t)\, dt =$$

$$= \frac{1}{4}\int_{-\infty}^{+\infty}\left[x(t)\exp(j2\pi f_c t) + x^*(t)\exp(-j2\pi f_c t)\right]\cdot$$

$$\cdot\left[y(t)\exp(j2\pi f_c t) + y^*(t)\exp(-j2\pi f_c t)\right] dt$$

$$= \frac{1}{4}\int_{-\infty}^{+\infty} \left(x(t)y^*(t) + x^*(t)y(t)\right) dt$$

$$+ \frac{1}{4}\int_{-\infty}^{+\infty} \left[x(t)y(t)\exp(j4\pi f_c t) + x^*(t)y^*(t)\exp(-j4\pi f_c t)\right] dt$$

$$= \frac{1}{2}\,\mathrm{Re}\left[\int_{-\infty}^{+\infty} x(t)y^*(t)\,dt\right] + \frac{1}{2}\,\mathrm{Re}\left[\int_{-\infty}^{+\infty} y(t)\exp(j4\pi f_c t)(x^*(t))^*\,dt\right] \quad (*)$$

By the properties of the Fourier transform we have

$$y(t)\exp(j4\pi f_c t) \longleftrightarrow Y(f-2f_c)$$

$$x^*(t) \longleftrightarrow X^*(-f)$$

By the Parseval's relation we have that

$$\int_{-\infty}^{+\infty} y(t)\exp(j4\pi f_c t)(x^*(t))^*\,dt = \int_{-\infty}^{+\infty} Y(f-2f_c)(X^*(-f))^*\,df$$

Since both signals are bandlimited and $f_c \gg W$, the spectra of $X^*(-f)$ and $Y(f-2f_c)$ are centered around 0 and $2f_c$ respectively and do not overlap. Hence

$$\int_{-\infty}^{+\infty} Y(f-2f_c)(X^*(-f))^*\,df = 0$$

and the second integral in $(*)$ is 0.

Finally

$$\int_{-\infty}^{+\infty} x_p(t)y_p(t)\,dt = \frac{1}{2}\,\mathrm{Re}\left[\int_{-\infty}^{+\infty} x(t)y^*(t)\,dt\right]$$

Q.2 a) Since $f_c >> W$ and $\frac{vT}{c} << \frac{1}{2W}$, the narrowband baseband signals $x^I(t)$, $x^Q(t)$ do not change appreciably after a time interval of $\frac{vt}{c}$, $t \in [0, T]$.

Hence
$$x^I(t) \approx x^I\left(t - \frac{vt}{c}\right)$$
$$x^Q(t) \approx x^Q\left(t - \frac{vt}{c}\right)$$

b) $y_p(t) = \mathrm{Re}\left(y(t)\exp(j2\pi f_c t)\right)$. Also

$$
\begin{aligned}
y_p(t) &= x_p\left(t - \frac{vt}{c}\right) \\
&= x^I\left(t-\frac{vt}{c}\right)\cos\left(2\pi f_c\left(t-\frac{vt}{c}\right)\right) \\
&\quad - x^Q\left(t-\frac{vt}{c}\right)\sin\left(2\pi f_c\left(t-\frac{vt}{c}\right)\right) \\
&\approx x^I\left(t-\frac{vt}{c}\right)\left[\cos(2\pi f_c t)\cos\left(2\pi f_c \frac{v}{c} t\right)\right. \\
&\quad \left. + \sin(2\pi f_c t)\sin\left(2\pi f_c \frac{v}{c} t\right)\right] \\
&\quad - x^Q\left(t-\frac{vt}{c}\right)\left[\sin(2\pi f_c t)\cos\left(2\pi f_c \frac{v}{c} t\right)\right. \\
&\quad \left. - \cos(2\pi f_c t)\sin\left(2\pi f_c \frac{v}{c} t\right)\right] \\
&\approx \left[x^I(t)\cos\left(2\pi f_c \frac{v}{c} t\right) + x^Q(t)\sin\left(2\pi f_c \frac{v}{c} t\right)\right] \cdot \\
&\quad \cdot \cos(2\pi f_c t) \\
&\quad - \left[x^Q(t)\cos\left(2\pi f_c \frac{v}{c} t\right) - x^I(t)\sin\left(2\pi f_c \frac{v}{c} t\right)\right] \cdot \\
&\quad \cdot \sin(2\pi f_c t)
\end{aligned}
$$

$$y_p(t) \approx \mathrm{Re}\left(x(t)\exp\left(-j2\pi f_c \frac{v}{c} t\right)\right)\cos(2\pi f_c t)$$

$$- \mathrm{Im}\left(x(t)\exp\left(-j2\pi f_c \frac{v}{c} t\right)\right)\sin(2\pi f_c t)$$

$$\approx \mathrm{Re}\left(\left(x(t)\exp\left(-j2\pi f_c \frac{v}{c} t\right)\right)\exp(j2\pi f_c t)\right)$$

Since $\quad y_p(t) = \mathrm{Re}(y(t)\exp(j2\pi f_c t))$

$$y(t) \approx x(t)\exp\left(-j2\pi f_c \frac{v}{c} t\right)$$

By the properties of the fourier transform

$$Y(f) \approx X\left(f + f_c \frac{v}{c}\right)$$

This implies that the spectrum of the received signal $y(t)$ is a shifted version of the transmitted signal $x(t)$. The frequency shift $\Delta f_D = f_c \frac{v}{c}$ is called Doppler frequency shift.

Application: $f_c = 2\,\mathrm{GHz}$, $v = 30\,\mathrm{m/s}$, $c = 3\cdot 10^8\,\mathrm{m/s}$

$$\Delta f_D = 2\cdot 10^9 \frac{30}{3\cdot 10^8} = 200\,\mathrm{Hz}.$$