(a) Determine $d_{\min}$ for $\widehat{\mathcal{B}}$.
(b) Give a parity-check matrix for the extended code $\widehat{\mathcal{B}}$.

1.16 Consider the (8,4) extended Hamming code.
(a) Give a parity-check matrix.
(b) Determine $d_{\min}$.
(c) Find an encoding matrix.
(d) Show how a decoder can detect that an odd number of errors has occurred.

1.17 The *Hamming sphere* of radius t with center at the N-tuple $\boldsymbol{x}$ is the set of all $\boldsymbol{y}$ in $\mathbb{F}_2^N$ such that $d_{\mathrm{H}}(\boldsymbol{x},\boldsymbol{y}) \leq t$. Thus, this Hamming sphere contains exactly

$$V_t = \sum_{i=0}^{t} \binom{N}{i}$$

distinct N-tuples.
Prove the *Hamming bound* for binary codes, that is,

$$V_{\left\lfloor \frac{d_{\min}-1}{2} \right\rfloor} \leq 2^{N(1-R)}$$

which is an implicit upper bound on $d_{\min}$ in terms of the block length N and rate R.

1.18 The systematic parity-check matrices for the binary Hamming codes can be written recursively as

$$H_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

and

$$H_m = \begin{pmatrix} H_{m-1} & H_{m-1} & \mathbf{0} \\ 1 \dots 1 & 0 \dots 0 & 1 \end{pmatrix}, \quad m \geq 3.$$

Find the parameters N, K, and $d_{\min}$ for the mth Hamming code.

1.19 A code for which the Hamming bound (see Problem 1.17) holds with equality is called a *perfect code*.
(a) Show that the *repetition code*, that is, the rate $R = 1/N$ binary linear code with generator matrix $G = (11\dots1)$, is a perfect code if and only if N is odd.
(b) Show that the Hamming codes of Problem 1.18 are perfect codes.
(c) Show that the Hamming bound admits the possibility that an $N = 23$ perfect binary code with $d_{\min} = 7$ might exist. What must K be?

Remark. The perfect code suggested in (c) was found by Golay in 1949 [Gol49]. There exist no perfect binary codes other than those mentioned in this problem.

1.20 Suppose that the block code $\mathcal{B}$ with parity-check matrix

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

is used for communication over a BSC with $0 < \epsilon < 1/2$.
(a) Find $d_{\min}$.
(b) How many errors can the code correct?
(c) How many errors can the code detect?
(d) For each syndrome give the error pattern $\widehat{\boldsymbol{e}}$ that corresponds to the error-correcting capability of the code.
(e) For $\boldsymbol{r} = 0111011$ find $\widehat{\boldsymbol{v}}$, the maximum-likelihood decision.
(f) For $\boldsymbol{r} = 0110111$ find $\widehat{\boldsymbol{v}}$, the maximum-likelihood decision.

1.21 Verify formula (1.66).
Hint: The (7,4) Hamming code has one codeword of weight 0, seven codewords of weight 3, seven codewords of weight 4, and one codeword of weight 7. The error probability is the same for all bits.

1.22 Given a Hamming code $\mathcal{B}$ with parity-check matrix H,
(a) Construct an *extended* code $\mathcal{B}_{\mathrm{ext}}$ with parity-check matrix

$$H_{\mathrm{ext}} = \begin{pmatrix} 0 & & & \\ \vdots & & H & \\ 0 & & & \\ 1 & 1 & \dots & 1 \end{pmatrix}$$

(b) Determine $d_{\min}$ for $\mathcal{B}_{\mathrm{ext}}$.
(c) Construct an *expurgated* code $\mathcal{B}_{\mathrm{exp}}$ with parity-check matrix

$$H_{\mathrm{exp}} = \begin{pmatrix} & H & \\ 1 & \dots & 1 \end{pmatrix}$$

(d) Determine $d_{\min}$ for $\mathcal{B}_{\mathrm{exp}}$.
(e) What is characteristic for the weights of the codewords of $\mathcal{B}_{\mathrm{exp}}$?

1.23 Consider the trellis given in Fig. 1.16.
(a) List all codewords.
(b) Find the ML estimate of the information sequence for the received sequence $\boldsymbol{r} =$ 01 10 01 10 11 on a BSC with $0 < \epsilon < 1/2$.

1.24 Consider the convolutional encoder shown in Fig. P1.24.
(a) Draw the trellis corresponding to four information digits and $m = 1$ dummy zero.
(b) Find the number of codewords M represented by the trellis in (a).
(c) Use the Viterbi algorithm to decode when the sequence $\boldsymbol{r} =$ 11 01 10 10 01 is received over a BSC with $0 < \epsilon < 1/2$.

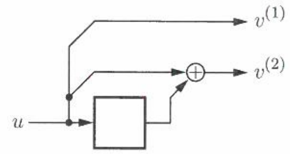

Figure P1.24 Convolutional encoder used in Problem 1.24.

1.25 Consider the convolutional encoder with generator matrix

$$G = \begin{pmatrix} 11 & 10 & 01 & 11 & & \\ & 11 & 10 & 01 & 11 & \\ & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

(a) Find the rate and the memory.
(b) Draw the encoder.
(c) Find the codeword $\boldsymbol{v}$ that corresponds to the information sequence $\boldsymbol{u} = 1100100\dots$.

1.26 Consider the code $\mathcal{C}$ with the encoding rule

$$\boldsymbol{v} = \boldsymbol{u}G + (11\,01\,11\,10\,11\dots)$$

"many" idea which we have exploited several times in this volume can be found in Fano's textbook. For convolutional codes it was first used in [CJZ84a].

Lower bounds on the error probability for block codes were derived by Shannon, Gallager, and Berlekamp [SGB67].

For convolutional codes the upper bounds on the error probability were derived by Yudkin in 1964 [Yud64] and the lower bounds by Viterbi in 1967 [Vit67]. The tightest upper bounds on the error probability were derived in [Zig85].

The ideas of constructing the convolutional coding exponent from the block coding exponent and vice versa, as well as the concept of critical length, go back to Forney [For74a].

For general, nonlinear trellis codes, Pinsker [Pin67] derived a lower bound on the error probability for decoding with finite back-search limit τ. His bound is similar to the sphere-packing bound for block codes of block length $N = \tau c$.

Tailbiting representations of block codes were introduced by Solomon and van Tilborg [Sov79]. See also [MaW86].

PROBLEMS

4.1 Consider the rate $R = 1/2$, memory $m = 1$, systematic encoding matrix $G(D) = (1 \quad 1+D)$.

(a) Draw the length $\ell = 5$ trellis.

(b) Suppose that the encoder is used to communicate over a BSC with crossover probability $0 < \epsilon < 1/2$. Use the Viterbi algorithm to decode the received sequence $\boldsymbol{r} = 10\,01\,11\,11\,11\,11$.

(c) How many channel errors have occurred if the optimal path corresponds to the transmitted sequence?

4.2 Consider the rate $R = 1/2$, memory $m = 2$ encoding matrix $G(D) = (1+D+D^2 \quad 1+D^2)$.

(a) Draw the length $\ell = 4$ trellis.

(b) Suppose that the encoder is used to communicate over a BSC with crossover probability $0 < \epsilon < 1/2$. Use the Viterbi algorithm to decode the received sequence $\boldsymbol{r} = 11\,11\,00\,11\,01\,11$.

(c) Suppose that the information sequence is $\boldsymbol{u} = 1011$. How many channel errors are corrected in (b)?

4.3 Repeat Problem 4.2 for $\boldsymbol{r} = 10\,11\,10\,10\,01\,00$ and $\boldsymbol{u} = 0100$.

4.4 Repeat Problem 4.2 for $\boldsymbol{r} = 00\,11\,10\,01\,00\,00$ and $\boldsymbol{u} = 0000$.

4.5 Consider a binary input, 8-ary output DMC with transition probabilities $P(r \mid v)$ given in Example 4.3.

(a) Suppose that the rate $R = 1/2$ encoder with encoding matrix $G(D) = (1+D+D^2 \quad 1+D^2)$ is used to communicate over the given channel. Use the Viterbi algorithm to decode $\boldsymbol{r} = 1_3 1_4 \; 1_3 0_4 \; 0_2 0_4 \; 0_1 1_3 \; 0_3 1_1 \; 0_4 1_1$.

(b) Connect the channel to a BSC by combining the soft-decision outputs $0_1, 0_2, 0_3, 0_4$ and $1_1, 1_2, 1_3, 1_4$ to hard-decision outputs 0 and 1, respectively. Use the Viterbi algorithm to decode the hard-decision version of the received sequence in (a).

4.6 Repeat Problem 4.5 for $\boldsymbol{r} = 0_4 0_3 \; 1_3 0_1 \; 0_1 1_1 \; 1_1 0_1 \; 0_1 1_2 \; 1_1 0_4$.

4.7 Repeat Problem 4.5 for $\boldsymbol{r} = 1_1 0_4 \; 0_1 1_2 \; 1_1 0_1 \; 0_1 1_1 \; 1_3 0_1 \; 0_4 0_3$.

4.8 Consider the binary input, 4-ary output DMC with transition probabilities given in Fig. P4.8.

(a) Suppose that the rate $R = 1/2$ encoder with encoding matrix $G(D) = (1+D+D^2 \quad 1+D^2)$ is used to communicate over the given channel. Use the Viterbi algorithm to decode $\boldsymbol{r} = 0_1 1_1 \; 1_2 0_1 \; 1_1 0_2 \; 0_1 1_1 \; 1_2 1_1 \; 1_1 1_2$.

(b) Connect the channel to a BSC by combining the soft-decision outputs $0_1, 0_2$ and $1_1, 1_2$ to hard-decision outputs 0 and 1, respectively. Use the Viterbi algorithm to decode the hard-decision version of the received sequence in (a).