TSKS01 Digital Communication

Solutions to Selected Problems from Tutorial 4

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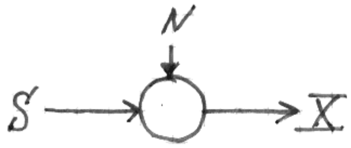
4.1



S & N indep.

$S = \pm 2$, eq. prob

$N \sim N(0, \sqrt{0.37})$

$\Rightarrow$ Optimal decision border : 0.

$$\hat{S} = \begin{cases} 2, & X > 0 \\ -2, & X < 0 \end{cases}$$

Error probability:

$$P_e = Pr\{\hat{S} \neq S\} = Pr\{S=2\} \cdot Pr\{\hat{S}=-2 \mid S=2\} + Pr\{S=-2\} \cdot Pr\{\hat{S}=2 \mid S=-2\}$$

$$\overset{\text{eq. prob}}{=} \frac{1}{2} Pr\{X<0 \mid S=2\} + \frac{1}{2} Pr\{X>0 \mid S=-2\}$$

$$= \frac{1}{2} Pr\{N<-2\} + \frac{1}{2} Pr\{N>2\} = 2 \cdot \frac{1}{2} Q\left(\frac{2}{\sqrt{0.37}}\right)$$

$$\approx Q(3.29) \approx 5.0 \cdot 10^{-4}$$

4.2

a $\quad P_e = Q\left(\frac{d/2}{\sigma_N}\right) = 0.01 \Rightarrow \frac{2}{\sigma_N} = Q^{-1}(0.01) \overset{\text{Q-table}}{=} 2.32 \Rightarrow \sigma_N^2 \approx 0.74$

b. Same channel, same distances. Set $q = 0.01$, which is the probability that the noise exceeds 2.

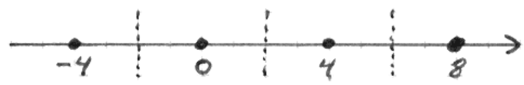
Eq. prob $\Rightarrow$

optimal decision borders (ML)

$$P_e = Pr\{S=-4\} Pr\{X>-2 \mid S=-4\} + Pr\{S=0\} Pr\{|X|>2 \mid S=0\} + Pr\{S=4\} \cdot Pr\{X<2 \mid S=4\}$$

$$= \frac{1}{3}(q + 2q + q) = \frac{4}{3} q = \frac{0.04}{3} \approx 0.0133$$

c. similarly for



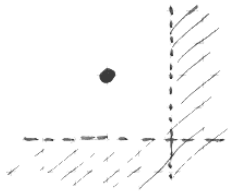
$$P_e = \frac{1}{4}(q + 2q + 2q + q) = \frac{6}{4} q = \frac{0.06}{4} = 0.015$$

d. No change. The decision borders should be shifted equally much.

4.7 $q = Pr\left\{ \boxed{W} > \frac{d}{2} \right\}$

The angle θ and the measure l are irrelevant. All that matter are distances.

The corner points see the following situation:

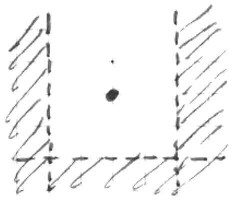


The error probability given such a point sent is

$p_1 = 1 - (1-q)^2 = 2q - q^2$

There are 4 such points.

The points in the middle of the sides see the following situation:

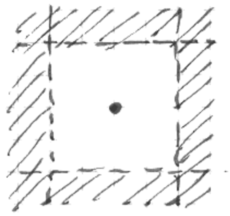


The error prob. given such a point is

$p_2 = 1 - (1-q)(1-2q) = 3q - 2q^2$

There are 4 such points.

The point in the middle of the constellation sees the following situation:



Error prob. given that point:

$p_3 = 1 - (1-2q)^2 = 4q - 4q^2$

There is one such point.

The total error probability is then given by

$$P_e = \frac{1}{9}(4p_1 + 4p_2 + p_3) = \frac{1}{9}(24q - 16q)$$

4.11 We have $s_0(t) = \begin{cases} \frac{A}{T} \cdot t, & 0 \le t < T \\ 0, & \text{elsewh.} \end{cases}$ and $s_1(t) = \begin{cases} A\left(\frac{t}{T} - 1\right), & 0 \le t < T \\ 0, & \text{elsew.} \end{cases}$

We are supposed to determine α and β such that $s_2(t) = \alpha s_0(t) + \beta s_1(t)$ minimizes the error prob. and such that all signals have the same energy.

AWGN channel, eq. prob. signals and ML det.

First the energies:

$$E_0 = E_1 = \int_0^T s_0^2(t)\,dt = \int_0^T \frac{A^2}{T^2} t^2\,dt = \left[\frac{A^2 t^3}{3T^2}\right]_0^T = \frac{A^2 T}{3}$$

We wish to determine the angle between the signals $s_0(t)$ and $s_1(t)$. Therefore we determine

$$(s_0, s_1) = \int_0^T s_0(t)\, s_1(t)\,dt = \int_0^T \frac{A}{T} t \left(A\frac{t}{T} - A\right) dt =$$

$$= \int_0^T \left(\frac{A^2}{T^2} t^2 - \frac{A^2}{T} t\right) dt = \left[\frac{A^2 t^3}{3T^2} - \frac{A^2 t^2}{2T}\right]_0^T$$

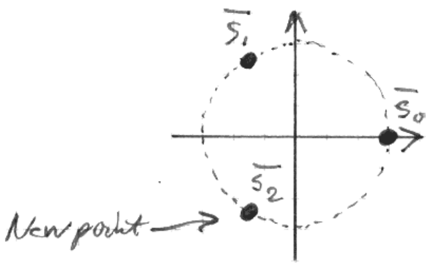
$$= A^2 T \left(\frac{1}{3} - \frac{1}{2}\right) = -\frac{A^2 T}{6}$$

The angle is then given by

$$\cos\alpha_{01} = \frac{(s_0, s_1)}{\sqrt{E_0 E_1}} = \frac{-A^2T/6}{A^2T/3} = -\frac{1}{2}$$

$$\Rightarrow \quad \alpha_{01} = \frac{2\pi}{3}$$

The signal $s_2(t)$ has to be on the same radius as $s_0(t)$ and $s_1(t)$. We therefore get



The vector $\bar{s}_2$ should be as far away as possible from the two original signals.

We get from elementary geometry:

$\bar{s}_2 = -s_1 - s_2$

From this we get the signal

$$s_2(t) = -s_1(t) - s_2(t),$$

and we have $\alpha = \beta = -1$.