TSKS04 Digital Communication, Continuation Course

Problems for Tutorial 1

1. Let the input to a digital modulator consist of independent, equally probable bits. Determine the power-spectral densities in the following cases.

   (a) On-Off Keying with $s_0(t) = 0$ and
   $$s_1(t) = \sqrt{\frac{2E}{T}}, \quad 0 \le t < T.$$

   (b) Bipolar signalling with
   $$s_0(t) = \sqrt{\frac{E}{T}}, \quad 0 \le t < T,$$
   and $s_1(t) = -s_0(t)$.

   (c) Orthogonal signalling with
   $$s_0(t) = \sqrt{\frac{E}{T}}, \quad 0 \le t < T,$$
   and
   $$s_1(t) = \begin{cases} \sqrt{\frac{E}{T}}, & 0 \le t < T/2, \\ -\sqrt{\frac{E}{T}}, & T/2 \le t < T. \end{cases}$$

2. Compare the results from Tasks 1a and 1b. What is the reason that there is an impulse in the PSD in one of the cases, but not in the other? Can you formulate that reason as a general rule?

3. A delay of a signal corresponds to a multiplication by a complex exponential in the transform domain. That means that the delay does not affect the absolute value of the transform of the signal. Thus, it can be convenient to first shift the signal so that the signal interval is from $-T/2$ to $T/2$ before calculating the transform. That especially holds if the signal has some symmetry around the center of the signal as in subtasks (a) and (b) below.

   Determine the energy spectra of the following basis functions:

   (a) $\phi_1(t) = \sqrt{\frac{1}{T}} \cdot \mathrm{rect}(t/T)$ (b) $\phi_2(t) = \sqrt{\frac{3}{T}} \cdot \mathrm{triangle}(2t/T)$

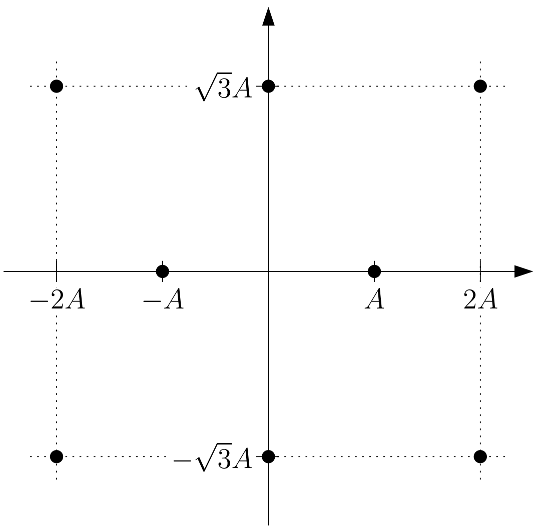
4. There are several ways to define the bandwidth of a signal. Those definitions try to capture to what extent our signal distorts other signals, to what extent other signals causes distortion to us, or to what extent we are sensitive to BP-filtering. Whatever bandwidth definition we use, we are usually interested in keeping the bandwidth small, given that all other parameters are fixed.

   Which of the two basis functions in Task 3 should we prefer based on the following bandwidth definitions?

   (a) The bandwidth is the smallest positive frequency where $|\Phi(f)|^2$ is zero.

   (b) The bandwidth is the smallest positive frequency B such that $|\Phi(f)|^2$ is at least 20 dB less than the maximum value of $|\Phi(f)|^2$ for all frequencies above B.

5. The eight signals in the following signal constellation are equally probable, and subsequent symbols are independent.



   Determine the power-spectral density if the two basis functions are

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t), \quad 0 \le t < T,$$

   and

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t), \quad 0 \le t < T,$$

   where $2f_cT$ is a positive integer.

Hints

1. (a) Interprete the input as a time-discrete process $A[n]$, consisting of independent, equally probable 0 and 1. Then PAM that using $s_1(t)$ as the pulse shape. The standard spectral relation for PAM can then be used:

$$R_S(f) = \frac{1}{T}\big|S_1(f)\big|^2 R_A[fT],$$

where $S_1(f)$ is the Fourier transform of $s_1(t)$, and where $S(t)$ is the output process.

   (b) The same as in a, but the input alphabet is ± 1, and the pulse shape is $s_0(t)$.

   (c) Notice that the signals are orthogonal. We have a two-dimensional situation, and the PSD is given by

$$R_S(f) \;=\; \frac{1}{T}\Big(S_1^*(f), S_2^*(f)\Big)\begin{pmatrix} R_{A_1,A_1}[fT] & R_{A_2,A_1}[fT] \\ R_{A_1,A_2}[fT] & R_{A_2,A_2}[fT] \end{pmatrix}\begin{pmatrix} S_1(f) \\ S_2(f) \end{pmatrix},$$

where $A_1[n]$ and $A_2[n]$ are the two component processes for the two dimensions. Identify the two component processes and how they are related, and determine the auto-correlation and cross-correlation functions.

2. Compare the formulas. Track the factor in front of the impulse through your calculations.

3. The energy spectrum of a signal $x(t)$ is $|X(f)|^2$, where $X(f)$ is the Fourier transform of $x(t)$.
The function triangle(at) is a convolution of two rect(at).

4. Basing the calculations on a sinc can be a bit complicated. Therefore simplify the situation and try to bound the bandwidths from above and from below. Therefore, consider $1/\pi x$ instead of $\big|\mathrm{sinc}(x)\big|$.

5. A two-dimensional situation. The PSD is given by

$$R_S(f) \;=\; \frac{1}{T}\Big(\Phi_1^*(f), \Phi_2^*(f)\Big)\begin{pmatrix} R_{S_1,S_1}[fT] & R_{S_2,S_1}[fT] \\ R_{S_1,S_2}[fT] & R_{S_2,S_2}[fT] \end{pmatrix}\begin{pmatrix} \Phi_1(f) \\ \Phi_2(f) \end{pmatrix},$$

where $S_1[n]$ and $S_2[n]$ are the two component processes for the two dimensions, and where $\Phi_1(f)$ and $\Phi_2(f)$ are the Fourier transforms of the two basis functions $\phi_1(t)$ and $\phi_2(t)$. Identify the two component processes and how they are related, and determine the auto-correlation and cross-correlation functions.

Answers

1. (a) $R_S(f) = \frac{E}{2}\left(1 + \frac{1}{T}\delta(f)\right)\operatorname{sinc}^2(fT)$
   (b) $R_S(f) = E \cdot \operatorname{sinc}^2(fT)$
   (c) $R_S(f) = \frac{E}{4}\left(\operatorname{sinc}^2(fT) + \frac{1}{T}\delta(f) + \operatorname{sinc}^2\left(\frac{fT}{2}\right)\sin^2\left(\pi\frac{fT}{2}\right) + \sum_{m\,\text{odd}} \frac{1}{(\pi m)^2 T}\delta\left(f - \frac{m}{T}\right)\right)$
2. It is the square of the mean of the resulting signal, which is a general rule.
3. (a) $|\Phi_1(f)|^2 = T \cdot \operatorname{sinc}^2(fT)$
   (b) $|\Phi_2(f)|^2 = \frac{3}{4}T \cdot \operatorname{sinc}^4\left(\frac{fT}{2}\right)$
4. (a) We prefer $\phi_1(t)$.
   (b) We prefer $\phi_2(t)$.
5. $R_S(f) = \frac{9}{4}A^2\left(\operatorname{sinc}^2\big((f + f_\mathrm{c})T\big) + \operatorname{sinc}^2\big((f - f_\mathrm{c})T\big)\right)$.