TSKS01 Digital Communication

Solutions to Selected Problems from Problem Class 10

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6.3 Given: $\sum_{\bar{c} \in C} w_H(\bar{c}) = n \cdot 2^{k-1}$

The average weight among non-zero codewords is therefore given by

$$w = \frac{n \cdot 2^{k-1}}{2^k - 1} = \frac{15 \cdot 2^{4-1}}{2^4 - 1} = 8$$

The average weight among non-zero codewords is an upper bound on the smallest non-zero weight, and the smallest non-zero weight is also the minimum distance since the code is linear. Thus:

$$d \le w = 8$$

The error correction capability of the code:

$$t = \left\lfloor \frac{d-1}{2} \right\rfloor \le \left\lfloor \frac{w-1}{2} \right\rfloor = \left\lfloor \frac{8-1}{2} \right\rfloor = 3$$

6.7 a) We are supposed to construct a binary linear code with parameters $(n, k, d) = (5, 2, 3)$. For that we need a generator matrix with two rows. Each row is a non-zero codeword. Since the minimum distance is supposed to be 3, these rows must have weight at least 3. Apart from those two codewords, there is one more non-zero codeword, which is the sum of those two. That codeword must also have weight at least 3. The following generator matrix fulfills that:

$$G = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

The product code has length $N = n^2$, dimension $K = k^2$ and minimum distance $D = d^2$.

b) Error correction capability of the product code:

$$T = \left\lfloor \frac{D-1}{2} \right\rfloor = \left\lfloor \frac{d^2-1}{2} \right\rfloor = \left\lfloor \frac{3^2-1}{2} \right\rfloor = 4$$

c) Rate of the product code:

$$R = \frac{K}{N} = \frac{k^2}{n^2} = \frac{4}{25}$$

6.4 Perfect code: Place decoding-spheres of radius $\lfloor \frac{d-1}{2} \rfloor$ around each codeword. If the union of those decoding speres is the whole vector space, then the code is said to be perfect.
In this case with only two codewords, it is enough to check if vectors outside one decoding sphere is inside the other.

Notation: $\bar{0} = (0 \ldots 0)$, $\bar{1} = (1 \ldots 1)$

Observation: For an arbitrary vector $\bar{x}$, $d_H(\bar{x}, \bar{0})$ is the number of ones in $\bar{x}$ and $d_H(\bar{x}, \bar{1})$ is the number of zeros in $\bar{x}$. Thus, we have

$$d_H(\bar{x}, \bar{0}) + d_H(\bar{x}, \bar{1}) = n$$

Assume $d_H(\bar{x}, \bar{0}) > \frac{n-1}{2}$, i.e. $\bar{x}$ is not in the decoding sphere of $\bar{0}$. Then we have

$$d_H(\bar{x}, \bar{1}) = n - d_H(\bar{x}, \bar{0}) < n - \frac{n-1}{2} = \frac{n+1}{2}$$

These are all integers, and thus we have

$$d_H(\bar{x}, \bar{1}) \le \frac{n-1}{2}$$

i.e. $\bar{x}$ is in the decoding sphere of $\bar{1}$. Thus, there are no vectors outside the decoding spheres, and the code is perfect.

6.11 This is a generalization of the component code in 4.7. The following generator matrix fulfills the demands:

$$G = \begin{pmatrix} 1 \cdots 1 & 1 & 1 \cdots 1 & 0 & 0 \cdots 0 \\ \underbrace{0 \cdots 0}_{\substack{t \\ \text{columns}}} & 0 & \underbrace{1 \cdots 1}_{\substack{t \\ \text{columns}}} & 1 & \underbrace{1 \cdots 1}_{\substack{t \\ \text{columns}}} \end{pmatrix}$$

The two rows both have weight $2t+1$. The third non-zero codeword is

$$(\underbrace{1 \cdots 1}_{t} \; 1 \; \underbrace{0 \cdots 0}_{t} \; 1 \; \underbrace{1 \cdots 1}_{t})$$

This codeword has weight $2t+2$. Hence, this code has the correct parameters:

$n = 3t+2$	The number of columns in G.
$k = 2$	The number of rows in G.
$d = 2t+1$	The smallest non-zero weight.

6.12 Let c_1 and c_2 be vectors in a linear binary vectorspace and define $w_1 = w_{\mathrm{H}}(c_1)$, $w_2 = w_{\mathrm{H}}(c_2)$ and $w_3 = w_{\mathrm{H}}(c_1 + c_2)$. Then the following properties can easily be verified.

Property 1: If both w_1 and w_2 are even, then w_3 is also even.

Property 2: If both w_1 and w_2 are odd, then w_3 is even.

Property 3: If w_1 is even and w_2 is odd, or vice versa, then w_3 is odd.

Let G be a generator matrix of the code. We can identify two cases:

a. All rows of G have even weight.
Then all codewords are sums of a number of vectors of even weight. Thus, according to property 1, all codewords have even weight.

b. At least one row in G has odd weight.
Then we can create a new generator matrix of $\mathcal{C}$ in the following way. In a first step, we create the generator matrix G' by reordering the rows in G such that the last row in G' is one of the rows with odd weight. In the next step, we create the generator matrix G'' from G' by adding the last row to all other rows with odd weight. According to property 2, only the last row of G'' has odd weight. Consider the subcode $\mathcal{C}'$ that is generated by all rows in G'' except the last one. According to case **a**, $\mathcal{C}'$ only consists of codewords of even weight. Furthermore, the codewords in $\mathcal{C}'$ are half of the codewords in $\mathcal{C}$. We get the other half of the codewords in $\mathcal{C}$ by adding the last row in G'' to all codewords in $\mathcal{C}'$. According to property 3, all these codewords have odd weight.

Thus, we have shown that either $\mathcal{C}$ only consists of codewords of even weight (case **a**) *or* half the codewords in $\mathcal{C}$ have even weight and the other half have odd weight (case **b**).