# 3 Signal & Systems (Repetition)

### 3.5

The hint and answers in the problem collection provides the full solution to this problem.

# 1 Stochastic Variables (Repetition)

### 1.1

**a**. Check that the product of probabilities are the probabilities of the intersections:

$$\begin{aligned}
\Pr\{A\}\cdot\Pr\{B\} &= \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6} = \Pr\{A\cap B\}\\
\Pr\{A\}\cdot\Pr\{C\} &= \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8} = \Pr\{A\cap C\}\\
\Pr\{B\}\cdot\Pr\{C\} &= \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12} = \Pr\{B\cap C\}\\
\Pr\{A\}\cdot\Pr\{B\}\cdot\Pr\{C\} &= \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4} = \frac{1}{24} \neq \Pr\{A\cap B\cap C\}
\end{aligned}$$

Thus, the events $A$, $B$ och $C$ are dependent, but they are pairwise independent.

**b**.

$$\Pr\{A\cup B\} = \Pr\{A\} + \Pr\{B\} - \Pr\{A\cap B\} = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3}$$

**c**.

$$\begin{aligned}
\Pr\{A\cup B\cup C\} &= \Pr\{A\} + \Pr\{B\} + \Pr\{B\}\\
&\quad - \Pr\{A\cap B\} - \Pr\{A\cap C\} - \Pr\{B\cap C\}\\
&\quad + \Pr\{A\cap B\cap C\}\\
&= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \frac{1}{12} + \frac{1}{16} = \frac{37}{48}
\end{aligned}$$

**d**.

$$\begin{aligned}
\Pr\{B \mid C\} &= \frac{\Pr\{B\cap C\}}{\Pr\{C\}} = \frac{1/12}{1/4} = \frac{1}{3}\\
\Pr\{A \mid B\cap C\} &= \frac{\Pr\{A\cap B\cap C\}}{\Pr\{B\cap C\}} = \frac{1/16}{1/12} = \frac{3}{4}
\end{aligned}$$

**Answer:**
**a.** The events $A$, $B$ och $C$ are dependent, but they are pairwise independent.
**b.** $\Pr\{A\cup B\} = \frac{2}{3}$ **c.** $\Pr\{A\cup B\cup C\} = \frac{37}{48}$ **d.** $\Pr\{B|C\} = \frac{1}{3}$, $\Pr\{A|B\cap C\} = \frac{3}{4}$

**1.2**
Let $M = (M_1M_2M_3)$ be the input. Then according the the problem description, we have

$$\Pr\{M = (000)\} = \Pr\{M = (110)\} = \Pr\{M = (101)\} = \Pr\{M = (011)\} = \frac{1}{4}$$
$$\Pr\{M = (111)\} = \Pr\{M = (001)\} = \Pr\{M = (010)\} = \Pr\{M = (100)\} = 0$$

Let $R = (R_1R_2R_3)$ be the output. Finally, let $B_i$ be the event $R_i = 0$.

**a**.

$$\begin{aligned}\Pr\{B_1\} &= \Pr\{M = (000)\} \cdot (1-p) + \Pr\{M = (011)\} \cdot (1-p) \\ &\quad + \Pr\{M = (110)\} \cdot p + \Pr\{M = (101)\} \cdot p \\ &= 2 \cdot \frac{1}{4}(1-p) + 2 \cdot \frac{1}{4}p = \frac{1}{2}\end{aligned}$$

Note: Obviously, we have $\Pr\{B_1\} = \Pr\{B_2\} = \Pr\{B_3\}$.

**b**.

$$\begin{aligned}\Pr\{R = (110)\} &= \Pr\{M = (000)\} \cdot p^2(1-p) + \Pr\{M = (011)\} \cdot p^2(1-p) \\ &\quad + \Pr\{M = (110)\} \cdot (1-p)^3 + \Pr\{M = (101)\} \cdot p^2(1-p) \\ &= \frac{3}{4}p^2(1-p) + \frac{1}{4}(1-p)^3 \approx 0.189\end{aligned}$$

**c**. Using Baye's rule, we get

$$\begin{aligned}\Pr\{M = (110) \mid R = (110)\} &= \frac{\Pr\{\Pr\{M = (110)\}\Pr\{R = (110) \mid M = (110)\}\}}{\Pr\{R = (110)\}} \\ &= \frac{\frac{1}{4}(1-p)^3}{0.189} \approx 0.9643\end{aligned}$$

0.964

**d**. Check that the product of probabilities are the probabilities of the intersections:

$$\begin{aligned}\Pr\{B_1 \cap B_2\} &= \Pr\{M = (000)\} \cdot (1-p)^2 + \Pr\{M = (011)\} \cdot p(1-p) \\ &\quad + \Pr\{M = (110)\} \cdot p^2 + \Pr\{M = (101)\} \cdot p(1-p) = \frac{1}{4}\end{aligned}$$
$$\Pr\{B_1\}\Pr\{B_2\} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Similarily for $\Pr\{B_1 \cap B_3\}$ and $\Pr\{B_2 \cap B_3\}$. For all three events, we have

$$\begin{aligned}\Pr\{B_1 \cap B_2 \cap B_3\} &= \Pr\{M = (000)\} \cdot (1-p)^3 + \Pr\{M = (011)\} \cdot p^2(1-p) \\ &\quad + \Pr\{M = (110)\} \cdot p^2(1-p) + \Pr\{M = (101)\} \cdot p^2(1-p) \neq \frac{1}{8}\end{aligned}$$
$$\Pr\{B_1\}\Pr\{B_2\}\Pr\{B_3\} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

Thus, the events $B_1$, $B_2$, $B_3$ are dependent, but they are pairwise independent.

**Answer:**
**a.** $\Pr\{B_1\} = \frac{1}{2}$ **b.** $\Pr\{\bar{r} = (110)\} = 0.189$ **c.** 0.964
**d.** $B_1$, $B_2$, $B_3$ are dependent, but they are pairwise independent.

**1.3**

$$
\begin{aligned}
\Pr\{X < 0.99\} &= 1 - \Pr\{X \geq 0.99\} = 1 - Q\left(\frac{0.99 - m_X}{\sigma_X}\right) = 1 - Q(199),\\
\Pr\{X < -0.99\} &= 1 - \Pr\{X \geq -0.99\} = 1 - Q\left(\frac{-0.99 - m_X}{\sigma_X}\right) = 1 - Q(1),\\
\Pr\{X > -0.99\} &= Q\left(\frac{-0.99 - m_X}{\sigma_X}\right) = Q(1),\\
\Pr\{-1.3 < X < -1.03\} &= Q\left(\frac{-1.3 - m_X}{\sigma_X}\right) - Q\left(\frac{-1.03 - m_X}{\sigma_X}\right) = Q(-30) - Q(-3)\\
&= \big(1 - Q(30)\big) - \big(1 - Q(3)\big) = Q(3) - Q(30).
\end{aligned}
$$

**Answer:**

$\Pr\{X < 0.99\} = 1 - Q(199)$ $\quad$ $\Pr\{X < -0.99\} = 1 - Q(1)$

$\Pr\{X > -0.99\} = Q(1)$ $\quad$ $\Pr\{-1.3 < X < -1.03\} = Q(3) - Q(30)$

**1.4**
Sent signal: $X \in \{\pm 1\}$, equally probable.

Added noise: $N$, uniformly distributed on $[-2, 2)$.

Received signal: $Y = X + N$

We get conditional PDFs:

$$f_{Y|X}(y \mid -1) = \begin{cases} 1/4, & -3 \leq y < 1, \\ 0, & \text{elsewhere} \end{cases}$$
$$f_{Y|X}(y \mid 1) = \begin{cases} 1/4, & -1 \leq y < 3, \\ 0, & \text{elsewhere} \end{cases}$$

Error probability: $P_\mathrm{e} = \Pr\{\hat{X} \neq X\}$, where $\hat{X}$ is our estimate according to the chosen decision rule.

General expression:

$$\begin{aligned} P_\mathrm{e} &= \Pr\{\hat{X} \neq X\} = \Pr\{X = -1\}\Pr\{\hat{X} = 1 \mid X = -1\} + \Pr\{X = 1\}\Pr\{\hat{X} = -1 \mid X = 1\} \\ &= \frac{1}{2}\big(\Pr\{\hat{X} = 1 \mid X = -1\} + \Pr\{\hat{X} = -1 \mid X = 1\}\big) \end{aligned}$$

**a**. Given rule: $\hat{X} = \begin{cases} -1, & Y \leq 0, \\ 1, & Y > 0. \end{cases}$

$$\Pr\{\hat{X} = 1 \mid X = -1\} = \Pr\{\hat{Y} > 0 \mid X = -1\} = \int_0^1 \frac{1}{4}\, dy = \frac{1}{4},$$
$$\Pr\{\hat{X} = -1 \mid X = 1\} = \Pr\{\hat{Y} \leq 0 \mid X = 1\} = \int_{-1}^0 \frac{1}{4}\, dy = \frac{1}{4}.$$

Plug that into the general expression, and we get

$$P_\mathrm{e} = \frac{1}{2}\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{4}$$

**b**. Given rule: $\hat{X} = \begin{cases} -1, & Y \leq -1, \\ 1, & Y > -1. \end{cases}$

$$\Pr\{\hat{X} = 1 \mid X = -1\} = \Pr\{\hat{Y} > -1 \mid X = -1\} = \int_{-1}^1 \frac{1}{4}\, dy = \frac{1}{2},$$
$$\Pr\{\hat{X} = -1 \mid X = 1\} = \Pr\{\hat{Y} \leq -1 \mid X = 1\} = 0.$$

Plug that into the general expression, and we get

$$P_{\mathrm{e}} = \frac{1}{2}\left(\frac{1}{2} + 0\right) = \frac{1}{4}$$

**c**. Given rule: $\hat{X} = \begin{cases} -1, & Y \leq -1, \\ 1, & Y > 1, \\ \pm 1, & \text{elsewhere.} \end{cases}$, where $\pm 1$ is chosen by the toss of a fair coin.

$$\begin{aligned}
\Pr\{\hat{X}=1 \mid X=-1\} &= \Pr\{\hat{Y}>1 \mid X=-1\} + \frac{1}{2}\Pr\{-1<Y\leq 1 \mid X=-1\} \\
&= 0 + \frac{1}{2}\int_{-1}^{1} \frac{1}{4}\, dy = \frac{1}{4}, \\
\Pr\{\hat{X}=-1 \mid X=1\} &= \Pr\{\hat{Y}\leq -1 \mid X=1\} + \frac{1}{2}\Pr\{-1<Y\leq 1 \mid X=1\} \\
&= 0 + \frac{1}{2}\int_{-1}^{1} \frac{1}{4}\, dy = \frac{1}{4}.
\end{aligned}$$

Plug that into the general expression, and we get

$$P_{\mathrm{e}} = \frac{1}{2}\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{4}$$

**Answer:**

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**1.5**

**a**. We have

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = 2\int_0^{\infty} e^{-cx}\,dx = 2\left[\frac{e^{-cx}}{-c}\right]_0^{\infty} = \frac{2}{c},$$

from which we get $c = 2$.

**b**. First we observe that all odd moments ($\mathrm{E}\{X^n\}$ for $n$ odd) are zero since $f_X(x)$ is even. For $n$ positive and even, we have

$$\mathrm{E}\{X^n\} = \int_{-\infty}^{\infty} x^n f_X(x)\,dx = \int_{-\infty}^{\infty} x^n e^{-2|x|}\,dx = 2\int_0^{\infty} x^n e^{-2x}\,dx.$$

We attack this expression using partial integration, where $x^n$ is one of the functions and $e^{-2x}$ is the other. We get

$$\mathrm{E}\{X^n\} = 2\left[x^n\cdot\frac{e^{-2x}}{-2}\right]_0^{\infty} - 2\int_0^{\infty} nx^{n-1}\cdot\frac{e^{-2x}}{-2}\,dx. = \frac{n}{2}\cdot 2\int_0^{\infty} x^{n-1}e^{-2x}\,dx.$$

another partial integration, and we get

$$\begin{aligned}\mathrm{E}\{X^n\} &= n\left[x^{n-1}\cdot\frac{e^{-2x}}{-2}\right]_0^{\infty} - \frac{n}{2}\cdot 2\int_0^{\infty}(n-1)x^{n-2}\cdot\frac{e^{-2x}}{-2}\,dx\\ &= \frac{n}{2}\cdot\frac{n-1}{2}\cdot 2\int_0^{\infty} x^{n-2}e^{-2x}\,dx = \frac{n}{2}\cdot\frac{n-1}{2}\cdot\mathrm{E}\{X^{n-2}\}.\end{aligned}$$

Doing this $n/2$ times, we end up with

$$\mathrm{E}\{X^n\} = \frac{n}{2}\cdot\frac{n-1}{2}\cdots\frac{2}{2}\cdot\frac{1}{2}\cdot\mathrm{E}\{X^0\}.$$

Obviously, we have

$$\mathrm{E}\{X^0\} = \mathrm{E}\{1\} = 1.$$

Totally, this gives us

$$\mathrm{E}\{X^n\} = \begin{cases} 0, & n \text{ odd},\\ \frac{n!}{2^n}, & n \text{ even}.\end{cases}$$

**c**. We want to determine $f_Z(z)$ for $Z = X + Y$, where $X$ and $Y$ are independent with $f_X(x) = f_Y(x) = e^{-2|x|}$. That is a line integral of $f_{X,Y}(x,y)$ along the line $x + y = z$, i.e.

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\,dx$$

$X$ and $Y$ are independent, which gives us

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx = (f_X * f_Y)(z)$$

To calculate this convolution, we need to consider the two cases $z > 0$ and $z \le 0$. For $z > 0$, we have

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{0} e^{2x} e^{-2(z-x)}\, dx + \int_{0}^{z} e^{-2x} e^{-2(z-x)}\, dx + \int_{z}^{\infty} e^{-2x} e^{2(z-x)}\, dx \\ &= e^{-2z} \int_{-\infty}^{0} e^{4x}\, dx + = e^{-2z} \int_{0}^{z} dx + e^{2z} \int_{z}^{\infty} e^{-4x}\, dx \\ &= e^{-2z} \frac{1}{4} + = e^{-2z} \cdot z + e^{2z} \frac{1}{4} = e^{-2z} \left( \frac{1}{2} + z \right) \end{aligned}$$

Similarily for $z \le 0$, we have

$$f_Z(z) = e^{2z} \left( \frac{1}{2} - z \right).$$

Totally, we can write this as

$$f_Z(z) = e^{-2|z|} \left( \frac{1}{2} + |z| \right)$$

for all $z$.

**d**. To determine $\mathrm{E}\{Z^2\}$, we use the results from above including the independence:

$$\begin{aligned} \mathrm{E}\{Z^2\} &= \mathrm{E}\{(X+Y)^2\} = \mathrm{E}\{X^2\} + 2\mathrm{E}\{XY\} + \mathrm{E}\{Y^2\} \\ &= 2\mathrm{E}\{X^2\} + 2\mathrm{E}\{XY\} = 2\frac{2!}{2^2} + 2 \cdot 0 = 1. \end{aligned}$$

**Answer:**

**a.** $c = 2$ **b.** $\mathrm{E}\{X^n\} = \begin{cases} 0, & n \text{ odd} \\ \frac{n!}{2^n}, & n \text{ even} \end{cases}$ **c.** $f_Z(z) = e^{-2|z|} \left( \frac{1}{2} + |z| \right)$

**d.** $\mathrm{E}\{Z^2\} = 1$

**1.6**

**a**. The area of a circle of radius 1 is $\pi$. The integral of the PDF is always 1. Thus, we have $B = \frac{1}{\pi}$.

**b**. For $|x| > 1$, we have $f_{XY}(x, y) = 0$ for all $y$, and we obviously have

$$f_X(x) = 0.$$

For $|x| \leq 1$, we have $f_{XY}(x, y) = 1/\pi$ for all $|y| < \sqrt{1 - x^2}$ and zero elsewhere. Thus, we have

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f_{XY}(x, y)\, dy = \frac{2}{\pi}\sqrt{1 - x^2}$$

Totally, we have

$$f_X(x) = \begin{cases} \frac{2}{\pi}\sqrt{1 - x^2}, & |x| \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

**c**. When we have $X = 0$, then $Y$ is uniformly distributed between $\pm 1$. Thus, we have

$$f_{Y|X}(y|x = 0) = \begin{cases} \frac{1}{2}, & |y| \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

When we have $X = 0.9$, then $Y$ is uniformly distributed between $\pm\sqrt{1 - 0.9^2} = \pm\sqrt{0.19}$. Thus, we have

$$f_{Y|X}(y|x = 0.9) = \begin{cases} \frac{1}{2\sqrt{0.19}}, & |y| \leq \sqrt{0.19}, \\ 0, & \text{elsewhere.} \end{cases}$$

**Answer:**

**a.** $B = \frac{1}{\pi}$ **b.** $f_X(x) = 2B\sqrt{1 - x^2}$

**c.** $f_{Y|X}(y|x = 0) = \begin{cases} \frac{1}{2}, & |y| \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$ $f_{Y|X}(y|x = 0) = \begin{cases} \frac{1}{2\sqrt{0.19}}, & |y| \leq \sqrt{0.19}, \\ 0, & \text{elsewhere.} \end{cases}$

**1.7**

From the problem formulation, we find $Y_1 = X_1 - X_2$. The expectation is linear, and therefore we have

$$m_{Y_1} = m_{X_1} - m_{X_2} = 1 - 2 = -1.$$

Let $A$ be the matrix relating $Y$ to $X$. The simplest way to calculate the second moments is probably this:

$$C_Y = AC_XA^{\mathrm{T}} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -1 & 7 \end{pmatrix}.$$

We have thus found that we have $\sigma_{Y_1}^2 = 3$. Alternatively, we can do that using the definition of variance:

$$\begin{aligned} \sigma_{Y_1}^2 &= \mathrm{E}\{(Y_1 - m_{Y_1})^2\} = \mathrm{E}\{(X_1 - m_{X_1} - X_2 + m_{X_2})^2\} \\ &= \mathrm{E}\{(X_1 - m_{X_1})^2\} + \mathrm{E}\{(X_2 - m_{X_2})^2\} - 2\mathrm{E}\{(X_1 - m_{X_1})(X_2 - m_{X_2})\} \\ &= \sigma_{X_1}^2 + \sigma_{X_2}^2 - 2\lambda_{X_1,X_2} = 2 + 3 - 2 \cdot 1 = 3 \end{aligned}$$

**Answer:**

$\mathrm{E}\{Y_1\} = \mu_1 - \mu_2 = -1$ and $\mathrm{Var}\{Y_1\} = 3$

**1.8**

From problem Problem 1.6, we have

$$f_{XY}(x,y) = \begin{cases} B, & x^2+y^2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

**a**. Based on symmetry, we have $m_X = m_Y = 0$. What is left is then to show that we have $\mathrm{E}\{XY\} = m_X m_Y = 0$. We have

$$\mathrm{E}\{XY\} = \iint\limits_{-\infty}^{\infty} xy \cdot f_{XY}(x,y)\,dx\,dy = \iint\limits_{x^2+y^2\le 1} xy \cdot \frac{1}{\pi}\,dx\,dy = \frac{1}{\pi}\int\limits_{-1}^{1} x \int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y\,dx\,dy = 0$$

OK, uncorrelated!

**b**. We want to show that we have $f_{XY}(x,y) \ne f_X(x)f_Y(y)$, at least for some pair $(x,y)$. From the solution to Problem 1.6, we have

$$f_X(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2}, & |x| \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

and obviously from the symmetry of $f_{XY}(x,y)$, we also have $f_Y(x) = f_X(x)$. We have

$$f_X(x)f_Y(y) = \begin{cases} \frac{4}{\pi^2}\sqrt{1-x^2}\sqrt{1-y^2}, & |x| \le 1 \text{ and } |y| \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Obviously, we have

$$f_{XY}(x,y) \ne f_X(x)f_Y(y),$$

and the two variables are definitely dependent.

**Answer:**

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**1.9**
The stochastic variable $\Theta$ is uniformly distributed over $[0, 2\pi)$. Thus, it has the PDF

$$f_\Theta(\theta) = \begin{cases} 1/2\pi, & 0 \le \theta < 2\pi, \\ 0, & \text{elsewhere.} \end{cases}$$

**a.**

$$\mathrm{E}\{\cos(\Theta)\} = \int_{-\infty}^{\infty} \cos(\theta) \cdot f_\Theta(\theta)\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \cos(\theta)\, d\theta = 0$$

**b.**

$$\mathrm{E}\{\cos^2(\Theta)\} = \int_{-\infty}^{\infty} \cos^2(\theta) \cdot f_\Theta(\theta)\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2}\big(1 + \cos(2\theta)\big)\, d\theta = \frac{1}{2}$$

**c.**

$$\mathrm{E}\{\cos(\Theta)\sin(\Theta)\} = \int_{-\infty}^{\infty} \cos(\theta)\sin(\theta) \cdot f_\Theta(\theta)\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2}\sin(2\theta)\, d\theta = 0$$

If you cannot remember these trigonometric relations, then Euler can be used to quickly prove them.

**Answer:**
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