TSKS01 Digital Communication

Solutions to Selected Problems from Problem Class 8

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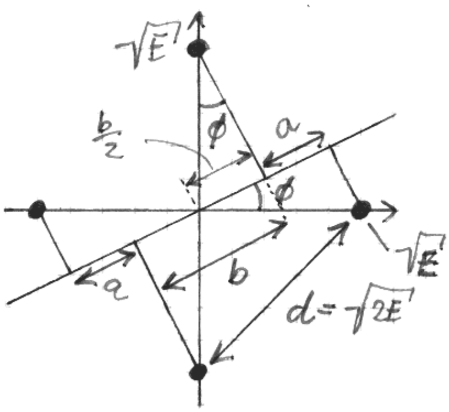
5.8 Given signal constellation and an arbitrarily chosen detection direction. This results in a projection of the signal points on the line. We introduce the two distances a and b between projection points. We have also identified the minimum distance d.



From the figure, we get

$$\frac{b}{2} = \sqrt{E}\cdot\sin(\phi) \qquad \text{and} \qquad a + \frac{b}{2} = \sqrt{E}\cos(\phi)$$

using elementary trigonometry. The smallest distance between projected points will determine the error probability. Therefore, we wish to maximize the smallest distance between projected points. This is accomplished by setting $a=b$. Study the quotient between the two equations above.

$$\tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)} = \frac{a/2}{3a/2} = \frac{1}{3}$$

This gives us

$$\phi = \arctan\left(\frac{1}{3}\right) \approx 0.322 \text{ rad} \approx 18^\circ$$

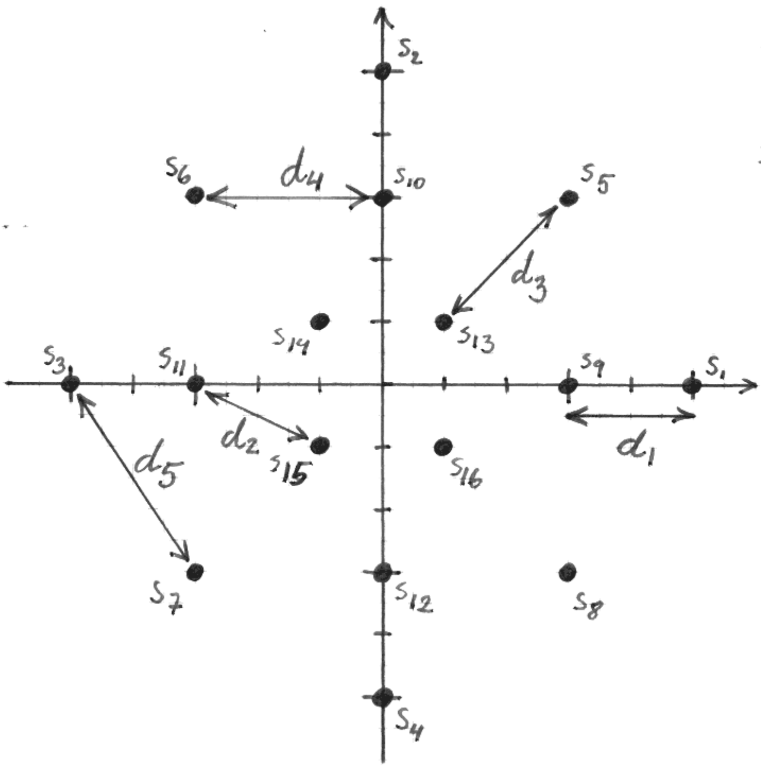
Compare a and d:

$$\frac{a}{d} = \frac{2\sqrt{E}\sin(\phi)}{\sqrt{2E}} = \sqrt{2}\sin\left(\arctan\left(\tfrac{1}{3}\right)\right) \approx 0.447$$

In dB:

$$20\cdot\log_{10}\left(\sqrt{2}\sin\left(\arctan\left(\tfrac{1}{3}\right)\right)\right) \approx -7.0\,\text{dB}$$

5.15a Given signal constellation:



$\frac{N_0}{2} = 7 \cdot 10^{-6}$ W/Hz

$E_{max} = 25 \cdot 10^{-4}$ J

We assume equally probable signals.

General error probability expression:

$$P_e = Pr\{\hat{S} \neq S\} = \sum_{i=1}^{16} Pr\{S = s_i\} \cdot Pr\{\hat{S} \neq s_i \mid S = s_i\},$$

where S is the sent signal, $\hat{S}$ is the estimated signal in the receiver. The symmetry in the constellation give us the following four similarity classes:

$$\{s_1, s_2, s_3, s_4\}, \{s_5, \ldots, s_8\}, \{s_9, \ldots, s_{12}\}, \{s_{13}, \ldots, s_{16}\}.$$

The probabilities $Pr\{\hat{S} \neq s_i \mid S = s_i\}$ are equal for all signals within the same class, based on symmetry. Therefore, it is enough to study one signal in each class.

The five smallest distances $d_1, \ldots, d_5$ are indicated in the figure above. We would like to use the nearest neighbour approximation of the error probability. The question is if that is possible.

We note that the signals in the first class above are those which have the largest energy. The first tick mark is therefore at $\sqrt{E_{max}}/5$.

5.15a continued.

The five smallest distances are

$$d_1 = \frac{2}{5}\sqrt{E_{max}} \qquad d_2 = \frac{\sqrt{5}}{5}\sqrt{E_{max}}$$

$$d_3 = \frac{\sqrt{8}}{5}\sqrt{E_{max}} \qquad d_4 = \frac{3}{5}\sqrt{E_{max}}$$

$$d_5 = \frac{\sqrt{13}}{5}\sqrt{E_{max}}$$

We evaluate the Q-function for those distances:

$$q_1 = Q\left(\frac{d_1}{\sqrt{2N_0}}\right) \approx Q(3.78) \approx 7.84\cdot 10^{-5}$$

$$q_2 = Q\left(\frac{d_2}{\sqrt{2N_0}}\right) \approx Q(4.23) \approx 1.17\cdot 10^{-5}$$

$$q_3 = Q\left(\frac{d_3}{\sqrt{2N_0}}\right) \approx Q(5.35) \approx 4.4\cdot 10^{-8} \ll q_1$$

$$q_4 = Q\left(\frac{d_4}{\sqrt{2N_0}}\right) \approx Q(5.67) \approx 7\cdot 10^{-9} \ll q_1$$

$$q_5 = Q\left(\frac{d_5}{\sqrt{2N_0}}\right) \approx Q(6.81) < 10^{-9} \ll q_1$$

From the above, we can see that distances d_1 and d_2 are significant for the error probability. Therefore, we consider a modified version of the nearest neighbour approx. where we consider the two smallest distances.

signal	# neighb. on dist d_1	# neighb. on dist d_2
s_1	1	0
s_5	0	0
s_9	1	2
s_{13}	2	2

$$\Rightarrow \Pr\{\hat{S}\neq s_1 \mid S=s_1\} \approx q_1$$

$$\Rightarrow \Pr\{\hat{S}\neq s_5 \mid S=s_5\} \approx 0$$

$$\Rightarrow \Pr\{\hat{S}\neq s_9 \mid S=s_9\} \approx q_1 + 2q_2$$

$$\Rightarrow \Pr\{\hat{S}\neq s_{13} \mid S=s_{13}\} \approx 2q_1 + 2q_2$$

Totally:

$$P_e \approx \frac{1}{4}\left(q_1 + 0 + q_1 + 2q_2 + 2q_1 + 2q_2\right) = q_1 + q_2 \approx 9.0\cdot 10^{-5}$$