Tutorial - 5

Solution to problem 1)

a) For a given received $(y[1], y[2], y[3])$ the generalized likelihood detector is given by (due to AWGN)

$$(\hat{x}[1], \hat{x}[2], \hat{x}[3])$$

$$= \arg \min_{\substack{a>0, \\ \theta \in [-\pi, \pi) \\ \underline{x} = (x[1], x[2], x[3]) \in \mathcal{X}}} \sum_{k=1}^{3} |y[k] - a e^{j\theta} x[k]|^2$$

$$= \arg \min_{\substack{\underline{x} \in \mathcal{X}, \\ a>0,\ \theta \in [-\pi, \pi)}} a^2 \sum_{k=1}^{3} |x[k]|^2 - 2a \operatorname{Re}\left\{ e^{-j\theta} \sum_{k=1}^{3} x^*[k]\, y[k] \right\}$$

(since $\sum_{k=1}^{3} |y[k]|^2$ does not depend on any of the optimization variables)

Fixing $\underline{x}$ and a, and optimizing over θ, we have

$$(\hat{x}[1], \hat{x}[2], \hat{x}[3])$$

$$= \arg \min_{\substack{\underline{x} \in \mathcal{X} \\ a>0}} \left[\min_{\theta \in [-\pi, \pi)} a^2 \sum_{k=1}^{3} |x[k]|^2 - 2a \operatorname{Re}\left\{ e^{-j\theta} \sum_{k=1}^{3} x^*[k]\, y[k] \right\} \right]$$

$\therefore \; (\hat{x}[1], \hat{x}[2], \hat{x}[3])$

$$= \arg\min_{\substack{\underline{x} \in \mathcal{X} \\ a > 0}} \; a^2 \sum_{k=1}^{3} |x[k]|^2 - 2a \left| \sum_{k=1}^{3} x^*[k]\, y[k] \right|$$

$$= \arg\min_{\underline{x} \in \mathcal{X}} \left[\min_{a>0} \; a^2 \sum_{k=1}^{3} |x[k]|^2 - 2a \left| \sum_{k=1}^{3} x^*[k]\, y[k] \right| \right]$$

$$= \arg\max_{\underline{x} \in \mathcal{X}} \frac{\left| \sum_{k=1}^{3} x^*[k]\, y[k] \right|^2}{\sum_{k=1}^{3} |x[k]|^2}$$

b) For the given set $\mathcal{X}$, $|x[k]|^2 = 1$ and therefore $\sum_{k=1}^{3} |x[k]|^2 = 3$ for any $\underline{x} \in \mathcal{X}$.

$$\therefore \; (\hat{x}[1], \hat{x}[2], \hat{x}[3]) = \arg\max_{\underline{x} \in \mathcal{X}} \left| \sum_{k=1}^{3} x^*[k]\, y[k] \right|^2$$

for $\underline{x} = (1, 1, 1)$

$\left| \sum_{k=1}^{3} x^*[k]\, y[k] \right|^2$

$= \left| \sum_{k=1}^{3} y[k] \right|^2 = |2 - j|^2 = 5,$

for $\underline{x} = (1, 1, -1)$

$$\left| \sum_{k=1}^{3} x^*[k]\, y[k] \right|^2$$

$$= \left| y[1] + y[2] - y[3] \right|^2$$

$$= \cancel{\left|2+j+2-2j\right|} \left| 4+j \right|^2 = 17.$$

for $\underline{x} = (1, -1, 1)$

$$\left| \sum_{k=1}^{3} x^*[k]\, y[k] \right|^2$$

$$= \left| y[1] - y[2] + y[3] \right|^2$$

$$= \left| j - 2j \right|^2 = 4.$$

and for $\underline{x} = (1, -1, -1)$

$$\left| \sum_{k=1}^{3} x^*[k]\, y[k] \right|^2 = \left| y[1] - y[2] - y[3] \right|^2$$

$$= \left| j + j - 2 \right|^2 = 8.$$

$$\therefore \left(\hat{x}[1], \hat{x}[2], \hat{x}[3] \right) = (1, 1, -1) \in \mathcal{X}.$$

c) Extending the set $\mathcal{X}$ to include $(-1, \pm 1, \pm 1)$ will lead to "ambiguity" at the receiver since

for $(1, b_1, b_2)$, $\left| \sum_{k=1}^{3} x^*[k]\, y[k] \right|$

$$= \left| y[1] + b_1 y[2] + b_2 y[3] \right|$$

and for $(-1, -b_1, -b_2)$, $\left| \sum_{k=1}^{3} x^*[k]\, y[k] \right|$

$$= \left| -y[1] - b_1 y[2] - b_2 y[3] \right|$$

$$= \left| y[1] + b_1 y[2] + b_2 y[3] \right|.$$

④

-tion to Q.2.

a) Since there are 4 messages, the number of information bits communicated is $\log_2 4 = 2$.

The total time for communication
$$= 4 \cdot T = 4 \cdot \frac{1}{2W}$$

$\therefore$ bit-rate $= \dfrac{2}{(4/2W)} = W$ bits/second.

bandwidth used $= 2W$ ($[f_c - W, f_c + W]$)

$\therefore$

η (spectral efficiency) $= \dfrac{\text{bit-rate}}{\text{bandwidth used}}$

$$= \frac{W}{2W}$$

$$= \frac{1}{2} \text{ bits/sec/Hz.}$$

~~We also have the rate that~~

~~P (avg. Tx power) = bit-rate × energy per inf. bit.~~

For each of the 4 messages, the total transmitted energy
$$= \int_{-\infty}^{+\infty} x_p^2(t)\, dt = \int_{-\infty}^{+\infty} |x(t)|^2\, dt$$

$$= \int_0^{4T} \sum_{k_1=1}^{4} \sum_{k_2=1}^{4} x^*[k_1]\, x[k_2] \int_{-\infty}^{\infty} 2W \operatorname{sinc}(2Wt-k_1) \operatorname{sinc}(2Wt-k_2)\, dt$$

$$= \sum_{k=1}^{4} |x[k]|^2$$

$$= 11 \text{ joules}$$

$\therefore$ Since 2 information bits are transmitted, the energy per information bit is $E_b = \frac{11}{2} = 5.5$ joules.

From the relation,

avg. power transmitted

$$P = \text{bit-rate} \times E_b$$

$$= 5.5 \text{ joules} \times W \text{ bits/sec}$$

$$= 5.5W \text{ watts.}$$

b) From shannon formula for AWGN channel capacity we know that

$$\frac{E_b}{\sigma^2} \geq \frac{2^{\eta}-1}{\eta}$$

for $\eta = \frac{1}{2}$ b/s/Hz (as for the given channel coding scheme)

$$\frac{E_b}{\sigma^2} \geq 0.8284$$