2 Stochastic Processes

2.1
First we determine the PSD as the Fourier transform of the ACF. We do that based on the hint from the Hints section or based on Page 19 in the booklet *Tables and Formulas for Signal Theory* used in the Signal Theory course.

$$R_X(f) = \sqrt{2\pi}e^{-2\pi^2 f^2}$$

The signal power P in the frequency interval $\mid f \mid < \frac{1}{4}$ is given by

$$P = \int_{-1/4}^{1/4} R_X(f)\, df = \int_{-1/4}^{1/4} \sqrt{2\pi}e^{-2\pi^2 f^2}\, df$$

Rewriting that using the variable substitution $\omega = 2\pi f$ gives us

$$P = \int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{2\pi}} e^{-\omega^2/2}\, d\omega$$

Comparing that to the definition of the Q-function,

$$Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt,$$

we see that the power P can be written as

$$P = Q(-\pi/2) - Q(\pi/2) = 1 - 2Q(\pi/2) \approx 1 - 2Q(1.57) \approx 1 - 2 \cdot 5.821 \cdot 10^{-2} \approx 0.883$$

We are supposed to compare that to the total power P_X, which we get as

$$P_X = r_X(0) = 1.$$

The fraction is

$$\frac{P}{P_X} \approx 0.883.$$

Answer:
88%

2.2
Let $h_1(t)$ be the total impulse response of the two cascaded filters. Then we have

$$h_1(t) = (h * h)(t) = \begin{cases} 4t, & 0 \le t < 3 \\ 24 - 4t, & 3 \le t < 6 \\ 0, & \text{elsewhere} \end{cases}$$

For the output $Y(t)$, we have

$$\begin{aligned} m_Y &= H_1(0) m_X = 0, \\ R_Y(f) &= \left|H_1(f)\right|^2 R_X(f) = \left|H_1(f)\right|^2, \end{aligned}$$

where $H_1(f)$ is the frequency response of the total filter. Since the mean is zero, we have

$$\sigma_Y^2 = P_Y = r_Y(0) = \int_{-\infty}^{\infty} R_Y(f)\, df = \int_{-\infty}^{\infty} \left|H_1(f)\right|^2 df$$

Using Parseval's relation, we get

$$\sigma_Y^2 = \int_{-\infty}^{\infty} \left|h_1(t)\right|^2 dt = 2 \int_0^3 16t^2\, dt = 2 \left[\frac{16t^3}{3}\right]_0^3 = 288$$

Answer:
288

4.3 Solution

Problem 1

a) The two hypotheses can be formulated as follows:

Hypothesis H_0: The signal is $s = 0$. The conditional distribution of X is $X = 0 + W \sim N(0, \sigma_W^2)$, with the conditional probability density function

$$f_{X|S}(x|0) = \frac{1}{\sqrt{2\pi\sigma_W^2}} e^{-\frac{x^2}{2\sigma_W^2}}.$$

Hypothesis H_1: The signal is $s = E$. The conditional distribution of X is $X = E + W \sim N(E, \sigma_W^2)$, with the conditional probability density function

$$f_{X|S}(x|E) = \frac{1}{\sqrt{2\pi\sigma_W^2}} e^{-\frac{(x-E)^2}{2\sigma_W^2}}.$$

b) The ML decision rule compares the conditional probability density functions. If we focus on Hypothesis H_0, we know that it is more likely than H_1 whenever

$$\begin{aligned} f_{X|S}(x|0) > f_{X|S}(x|E) \quad &\Leftrightarrow \quad \frac{1}{\sqrt{2\pi\sigma_W^2}} e^{-\frac{x^2}{2\sigma_W^2}} > \frac{1}{\sqrt{2\pi\sigma_W^2}} e^{-\frac{(x-E)^2}{2\sigma_W^2}} \\ &\Leftrightarrow \quad -\frac{x^2}{2\sigma_W^2} > -\frac{(x-E)^2}{2\sigma_W^2} \\ &\Leftrightarrow \quad -x^2 > -x^2 + 2Ex - E^2 \quad \Leftrightarrow \quad \frac{E}{2} > x. \end{aligned}$$

The ML decision rule is therefore

$$\hat{H}_{\mathrm{ML}}(x) = \begin{cases} H_0 & x < \frac{E}{2}, \\ H_1 & x \geq \frac{E}{2}, \end{cases}$$

where the point $x = \frac{E}{2}$ can be mapped to any of the hypotheses.

c) The error probability is

$$\begin{aligned} P_e &= \Pr\{S=0\} \Pr\left\{x > \frac{E}{2} \Big| S = 0\right\} + \Pr\{S=E\} \Pr\left\{x < \frac{E}{2} \Big| S = E\right\} \\ &= \frac{1}{2}\int_{\frac{E}{2}}^{\infty} f_{X|S}(x|0)dx + \frac{1}{2}\int_{-\infty}^{\frac{E}{2}} f_{X|S}(x|E)dx, \end{aligned}$$

where the first part is the probability of selecting H_1 when H_0 is true and the second part is the probability of selecting H_0 when H_1 is true. Since the noise distribution and noise variance is the same in both cases, we have a symmetry that implies that

$$\int_{\frac{E}{2}}^{\infty} f_{X|S}(x|0)dx = \int_{-\infty}^{\frac{E}{2}} f_{X|S}(x|E)dx$$

so we only need to compute one of them. Notice that

$$\int_{\frac{E}{2}}^{\infty} f_{X|S}(x|0)dx = Q\left(\frac{\frac{E}{2}-0}{\sigma_W}\right)$$

by using properties of the Q-function. In summary, the error probability is

$$P_e = Q\left(\frac{E}{2\sigma_W}\right).$$

d) The error probability was $Q\left(\frac{1}{\sigma_W}\right)$ in the "motivational example". In this problem we achieve the same error probability for $E = 2$. This value corresponds to the distance between the points 0 and $E = 2$, the two possible values of the signal. This is the same distance as between -1 and $+1$ in the "motivational example". Hence, the error probability only depends on the distance between the two signal points and not on their exact values.

Problem 2

a) The two hypotheses can be formulated as follows:

Hypothesis H_0: The signal is $s = 0$. The conditional distribution of X is $X = 0 + W \sim \text{Laplace}(0, \sigma_W/2)$, with the conditional probability density function

$$f_{X|S}(x|0) = \frac{1}{\sqrt{2\sigma_W^2}} e^{-\frac{\sqrt{2}}{\sigma_W}|x|}.$$

Hypothesis H_1: The signal is $s = E$. The conditional distribution of X is $X = E + W \sim \text{Laplace}(E, \sigma_W/2)$, with the conditional probability density function

$$f_{X|S}(x|E) = \frac{1}{\sqrt{2\sigma_W^2}} e^{-\frac{\sqrt{2}}{\sigma_W}|x-E|}.$$

b) The ML decision rule compares the conditional probability density functions. If we focus on Hypothesis H_0, we know that it is more likely than H_1 whenever

$$\begin{aligned} f_{X|S}(x|0) > f_{X|S}(x|E) \quad &\Leftrightarrow \quad \frac{1}{\sqrt{2\sigma_W^2}} e^{-\frac{\sqrt{2}}{\sigma_W}|x|} > \frac{1}{\sqrt{2\sigma_W^2}} e^{-\frac{\sqrt{2}}{\sigma_W}|x-E|} \\ &\Leftrightarrow \quad -\frac{\sqrt{2}}{\sigma_W}|x| > -\frac{\sqrt{2}}{\sigma_W}|x-E| \\ &\Leftrightarrow \quad -|x| > -|x-E| \quad \Leftrightarrow \quad |x-E| > |x|. \end{aligned}$$

To get rid of the absolute values we need to consider three intervals:
1) $x \geq E$: $|x-E| > |x|$ becomes $x - E > x$ or equivalently $0 > E$, which is never satisfied.
2) $0 < x < E$: $|x-E| > |x|$ becomes $-(x-E) > x$ or equivalently $\frac{E}{2} > x$, which makes H_0 most likely for $0 < x < \frac{E}{2}$.
3) $x \leq 0$: $|x-E| > |x|$ becomes $-(x-E) > -x$ or equivalently $0 < E$, which is always satisfied and makes H_0 most likely in this interval.

The ML decision rule is therefore

$$\hat{H}_{\mathrm{ML}}(x) = \begin{cases} H_0 & x < \frac{E}{2}, \\ H_1 & x \geq \frac{E}{2}, \end{cases}$$

where the point $x = \frac{E}{2}$ can be mapped to any of the hypotheses.

c) The error probability is

$$\begin{aligned} P_e &= \Pr\{S=0\} \Pr\left\{x > \frac{E}{2} \Big| S = 0\right\} + \Pr\{S=E\} \Pr\left\{x < \frac{E}{2} \Big| S = E\right\} \\ &= \frac{1}{2}\int_{\frac{E}{2}}^{\infty} f_{X|S}(x|0)dx + \frac{1}{2}\int_{-\infty}^{\frac{E}{2}} f_{X|S}(x|E)dx, \end{aligned}$$

where the first part is the probability of selecting H_1 when H_0 is true and the second part is the probability of selecting H_0 when H_1 is true. Since the noise distribution and noise variance are the same in both hypotheses, we have a symmetry that implies that

$$\int_{\frac{E}{2}}^{\infty} f_{X|S}(x|0)dx = \int_{-\infty}^{\frac{E}{2}} f_{X|S}(x|E)dx$$

so we only need to compute one of them. Notice that

$$\begin{aligned} \int_{\frac{E}{2}}^{\infty} f_{X|S}(x|0)dx &= \int_{\frac{E}{2}}^{\infty} \frac{1}{\sqrt{2\sigma_W^2}} e^{-\frac{\sqrt{2}}{\sigma_W}|x|} dx = \int_{\frac{E}{2}}^{\infty} \frac{1}{\sqrt{2\sigma_W^2}} e^{-\frac{\sqrt{2}}{\sigma_W}x} dx = \left[-\frac{1}{2} e^{-\frac{\sqrt{2}}{\sigma_W}x}\right]_{x=E/2}^{\infty} \\ &= \frac{1}{2} e^{-\frac{\sqrt{2}}{\sigma_W}\frac{E}{2}} = \frac{1}{2} e^{-\frac{E}{\sqrt{2}\sigma_W}} \end{aligned}$$

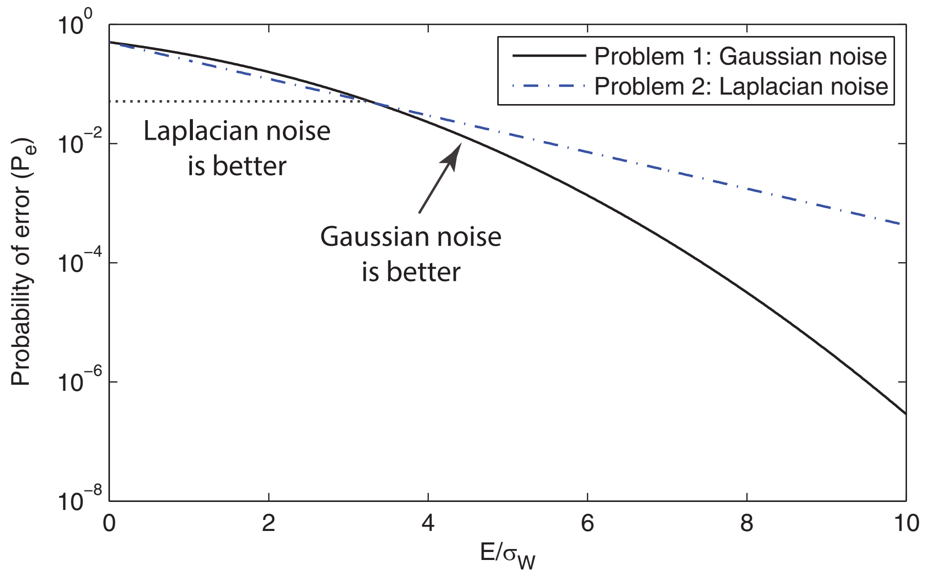

Figure 3: Comparison of error probabilities from Problem 2 b)

by using that $|x| = x$ when integrating over positive values. In summary, the error probability is

$$P_e = \frac{1}{2} e^{-\frac{E}{\sqrt{2}\sigma_W}}.$$

d) We should compare $P_e = \frac{1}{2} e^{-\frac{E}{\sqrt{2\sigma_W^2}}}$ with $P_e = Q\left(\frac{E}{2\sigma_W}\right)$ from Problem 1.

One way of doing this is to consider the approximation $Q(x) \approx \frac{1}{x\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ which is tight when x is large. This leads to $Q\left(\frac{E}{2\sigma_W}\right) \approx \frac{2\sigma_W}{E\sqrt{2\pi}} e^{-\frac{E^2}{8\sigma_W^2}}$, which shows that Gaussian noise leads to an error probability that goes to zero much faster when E/σ_W increases than in the case of Laplacian noise. However, this only reveals the behavior when E/σ_W is large.

To get the complete knowledge, one can try different values of E/σ_W and plot the probability of error, as is done in Figure 3. This shows that Laplacian noise is better for small E/σ_W, since the majority of probability mass lies closer to the mean value than with Gaussian noise. However, for large values of E/σ_W the probability of error is much better with Gaussian noise, because Laplacian noise has larger "tails" in the distribution—the probability of having very large noise realizations is larger than with Gaussian noise.

Problem 3

Hypothesis H_0 has the conditional probability density function

$$f_{X|H_0}(x) = \begin{cases} 1/2 & |x| \leq 1, \\ 0 & |x| > 1, \end{cases}$$

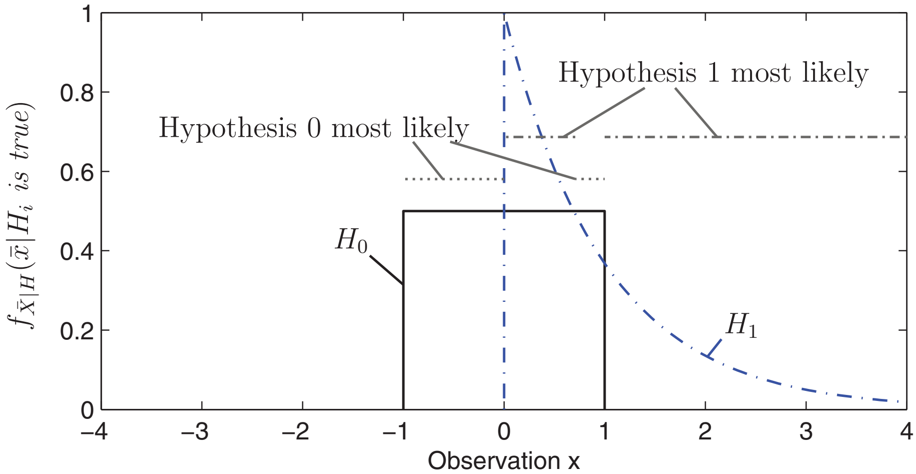

Figure 4: Sketch of the ML decision boundaries in Problem 3 a)

and hypothesis H_1 has the conditional probability density function

$$f_{X|H_1}(x) = e^{-x}u(x).$$

a) The ML decision rule compares the conditional probability density functions. If we focus on Hypothesis H_0, we know that it is more likely than H_1 whenever

$$f_{X|H_0}(x) > f_{X|H_1}(x).$$

None of the hypotheses can produce $x < -1$, while only H_0 has a non-zero probability of giving x between -1 and 0. Only H_1 has a non-zero probability of giving a $x > 1$. Finally, we have the range $0 \leq x \leq 1$ in which

$$f_{X|H_0}(x) > f_{X|H_1}(x) \quad \Leftrightarrow \quad 1/2 > e^{-x} \quad \Leftrightarrow \quad x > \ln(2).$$

In summary, the decision rule is

$$\hat{H}_{\mathrm{ML}}(x) = \begin{cases} H_0 & -1 \leq x < 0, \ln(2) < x \leq 1, \\ H_1 & 0 \leq x \leq \ln(2), x > 1, \\ \text{none} & x < -1. \end{cases}$$

Note that any of hypotheses can be selected at the point $x = \ln(2) \approx 0.6931$.

These decision boundaries are illustrated in Figure 4.

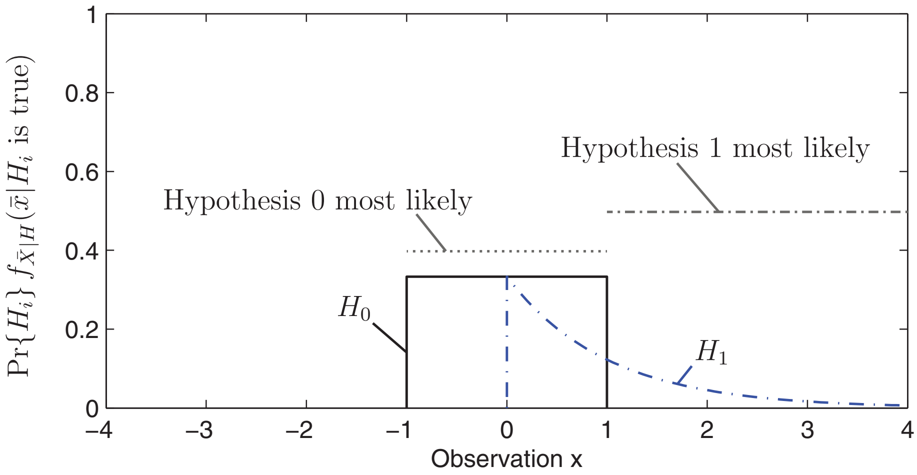

Figure 5: Sketch of the MAP decision boundaries in Problem 3 b)

b) The MAP decision rule weighs the conditional probability density functions with probability of each hypothesis. If we focus on Hypothesis H_0, we know x was most likely generated by this hypothesis whenever

$$\Pr\{H_0\}f_{X|H_0}(x) > \Pr\{H_1\}f_{X|H_1}(x).$$

H_0 is obviously the most likely hypothesis in $-1 < x < 0$ and H_1 is most likely for $x > 1$. In the interesting interval $0 \le x \le 1$ we notice that $\Pr\{H_0\}f_{X|H_0}(x) = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$, while $\Pr\{H_1\}f_{X|H_1}(x) = \frac{1}{3}e^{-x} \le \frac{1}{3}$ since e^{-x} is smaller or equal to one. Hence, H_0 is most likely also in the interval $0 \le x \le 1$.

In summary, the decision rule is

$$\hat{H}_{\text{MAP}}(x) = \begin{cases} H_0 & -1 \le x \le 1, \\ H_1 & x > 1, \\ \text{none} & x < -1. \end{cases}$$

These decision boundaries are illustrated in Figure 5.

c) We can be entirely sure about our decision in intervals where one hypothesis has zero probability density and the other has a positive probability density. If we observe x between -1 and 0, we can be completely sure that H_0 is true. If we observe x greater than 1, we can be completely sure that H_1 is true.