TSKS01 Digital Communication

Solutions to Selected Problems from Tutorial 5

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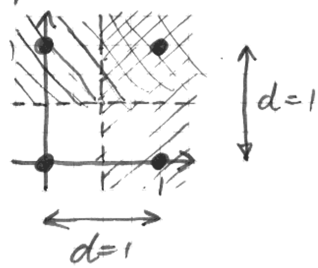
4.4 We start by considering a 2-D version of the problem with 4 signals in 2D.

Preparation.



The situation is symmetric. Each signal point sees the same situation. Thus, the error probability given any of the four sent signals is the same. It is therefore enough to analyze one of the signals.

Assume that the signal in the origin is sent. Then $Q\left(\frac{d}{\sqrt{2N_0}}\right)$ is the probability to end up in . Also, $Q\left(\frac{d}{\sqrt{2N_0}}\right)$ is the probability to end up in . Thus $2Q\left(\frac{d}{\sqrt{2N_0}}\right)$ is an overestimation of the error probability, since has been taken twice. We need to subtract the probability of that event. Orthogonal noise components are independent, and the decision borders intersect orthogonally. Therefore $Q^2\left(\frac{d}{\sqrt{2N_0}}\right)$ is the probability to end up in . We thus have

$$P_e = 2Q\left(\frac{1}{\sqrt{2N_0}}\right) - Q^2\left(\frac{1}{\sqrt{2N_0}}\right)$$

Alternatively, using orthogonal decision borders, we can write

$$P_e = 1 - Pr\{\text{Correct}\} = 1 - \left(1 - Q\left(\frac{1}{\sqrt{2N_0}}\right)\right)^2$$

Union bound takes all distances into account and we have 2 signals on distance 1 from the sent signal and 1 signal on distance $\sqrt{2}$.

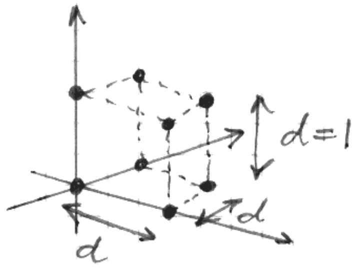
$$P_e \leq 2Q\left(\frac{1}{\sqrt{2N_0}}\right) + Q\left(\frac{\sqrt{2}}{\sqrt{2N_0}}\right)$$

Nearest neighbour only deals with nearest neighbours. Each signal point has 2 neighbours on distance 1.

$$P_e \approx 2Q\left(\frac{1}{\sqrt{2N_0}}\right)$$

4.4 Real solution: The signal vectors are corners in a cube with side 1 in one quadrant and with one corner in the origin

Cont'd.



Optimal decision borders are at 0.5 in each direction.

This is an immediate generalization of the 2-D reasoning on the previous page. Many details are left out.

a) Orth. decision borders, orth. noise components are indep:

$$P_e = 1-\left(1-Q\left(\frac{d}{\sqrt{2N_0}}\right)\right)^3 = 3Q\left(\frac{1}{\sqrt{2N_0}}\right)-3Q^2\left(\frac{1}{\sqrt{2N_0}}\right)+Q^3\left(\frac{1}{\sqrt{2N_0}}\right)$$

b) Each signal has 3 signals on distance 1, 3 signals on distance $\sqrt{2}$ and one signal on dist. $\sqrt{3}$:

$$P_e \leq 3Q\left(\frac{1}{\sqrt{2N_0}}\right)+3Q\left(\frac{\sqrt{2}}{\sqrt{2N_0}}\right)+Q\left(\frac{\sqrt{3}}{\sqrt{2N_0}}\right)$$

c) Each signal has 3 nearest neighbours on distance 1:

$$P_e \approx 3Q\left(\frac{1}{\sqrt{2N_0}}\right)$$

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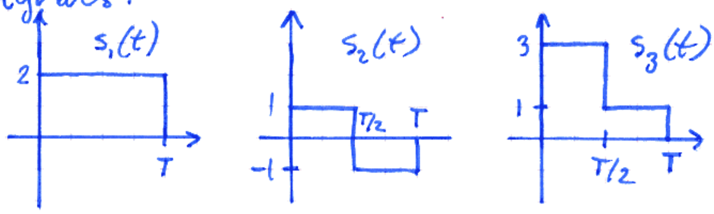
Signals:



Energies:

$$E_1 = \int_0^T s_1^2(t)\,dt = 2^2 \cdot T = 4T \Rightarrow \|s_1\| = 2\sqrt{T} \overset{T=4}{=} 4$$

$$E_2 = \int_0^T s_2^2(t)\,dt = (\pm 1)^2 \cdot T = T \Rightarrow \|s_2\| = \sqrt{T} = 2$$

$s_1(t)$ and $s_2(t)$ are orthogonal since

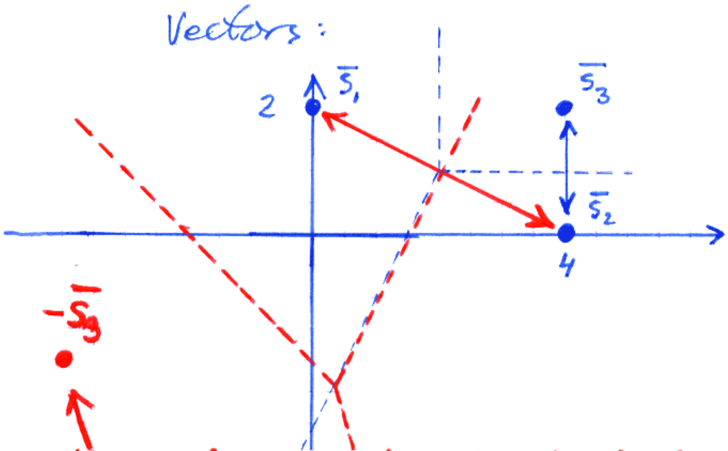
$$(s_1, s_2) = \int_0^T s_1(t) s_2(t)\,dt = \int_0^{T/2} 2\,dt + \int_{T/2}^T (-2)\,dt = 0$$

Furthermore, we obviously have.

$$s_3(t) = s_1(t) + s_2(t).$$

$$\Rightarrow \bar{s}_3 = \bar{s}_1 + \bar{s}_2$$

Vectors:



Min. dist: 2

$$Pr\{\hat{\bar{s}} \neq \bar{s}_3 \mid \bar{s} = \bar{s}_3\}$$
$$= Q\left(\frac{2}{\sqrt{2N_0}}\right) + Q\left(\frac{4}{\cdots}\right) - Q\left(\frac{2}{\cdots}\right) \cdot Q\left(\frac{4}{\cdots}\right)$$
$$= Q(1) + Q(2) - Q(1)Q(2)$$
$$\approx 0.1778$$

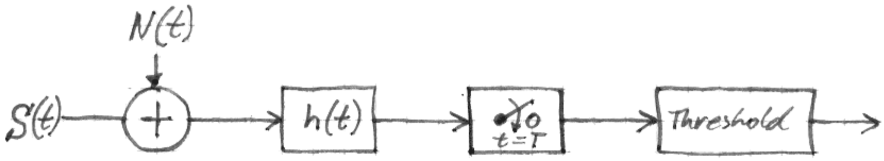
New signal point instead of $\bar{s}_3$ with the same energy as $\bar{s}_3$. We get the new minimum distance

$$\sqrt{2^2 + 4^2} = 2\sqrt{5}$$

Red borders for the new signal constellation.

Blue borders for the old signal constellation.

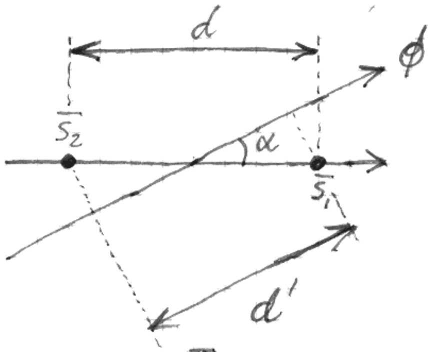
4.12



$$s_1(t) = A\cdot\sin\left(\frac{\pi t}{T}\right), \quad 0\le t<T$$
$$s_2(t) = -s_1(t)$$

$$h(t) = \frac{1}{\sqrt{T}}, \quad 0\le t<T$$

$h(t)$ is matched to $\phi(t) = h(T-t) = \frac{1}{\sqrt{T}}, \quad 0\le t<T.$



$$E_1 = \|s_1\|^2 = \int_0^T A^2\sin^2\left(\frac{\pi t}{T}\right)dt = \frac{A^2T}{2}$$
$$\Rightarrow s_{11} = -s_{21} = A\sqrt{\frac{T}{2}}$$
$$\|\phi\|^2 = \int_0^T \frac{1}{T}\,dt = 1$$

$$(s_1,\phi) = \int_0^T \frac{A}{\sqrt{T}}\sin\left(\frac{\pi t}{T}\right)dt = \left[\frac{A}{\sqrt{T}}\cdot\frac{-\cos(\pi t/T)}{\pi/T}\right]_0^T$$
$$= \frac{A\sqrt{T}}{\pi}\cdot\left(-\cos(\pi)-(-\cos(0))\right) = \frac{2}{\pi}A\sqrt{T}$$

$$\cos(\alpha) = \frac{(s_1,\phi)}{\|s_1\|\cdot\|\phi\|} = \frac{\frac{2}{\pi}A\sqrt{T}}{A\sqrt{\frac{T}{2}}\cdot 1} = \frac{\sqrt{8}}{\pi}$$

We have $d' = d\cdot\cos(\alpha)$

Thus, if we replace the filter by one that is matched to $s_1(t)$, we can reduce the amplitude by the factor $\cos(\alpha)$ and thus keep the error probability.

In dB:

$$20\cdot\log_{10}\frac{\sqrt{8}}{\pi} \approx 0.91 \text{ dB}.$$