Solutions to Selected Problems

– from Madhow: Fundamentals of Digital Communication and from Johannesson & Zigangirov: Fundamentals of Convolutional Coding –

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Note: This material is prepared for the 2012 version of the Master course TSKS04 Digital Communication Continuation Course. For almost every task planned for tutorials you find either hints and answers or complete solutions. For the tasks where we give complete solutions, we have chosen to adjust the notations to concur with the notation that has been used in the lectures.
This document will evolve during the course as the tutorials go by.
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**Solutions to Selected Problems – from Madhow: Fundamentals of Digital Communication
and from Johannesson & Zigangirov: Fundamentals of Convolutional Coding**

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Tutorial 3

Task 2.15

Hints

Follow the instructions given in the task.

Task 2.17

Task

Consider a pulse $s(t) = \operatorname{sinc}(at)\operatorname{sinc}(bt)$ where $a \geq b$.

a. Sketch the spectrum $S(f)$ of the pulse.

b. Suppose that the pulse is to be used over an ideal real baseband channel with one-sided bandwidth 400 Hz. Determine a and b such that the pulse is Nyquist for 4-PAM at 1200 b/s and exactly fills the channel bandwidth.

c. Now, suppose that the pulse is to be used over a passband channel spanning the frequency band 2.4–2.42 GHz. Assuming that we use 64-QAM at 60 Mb/s, determine a and b so that the pulse is Nyquist and exactly fills the channel bandwidth.

d. Sketch an argument showing that the magnitude of the transmitted waveform in the preceding settings is always finite.

Solution

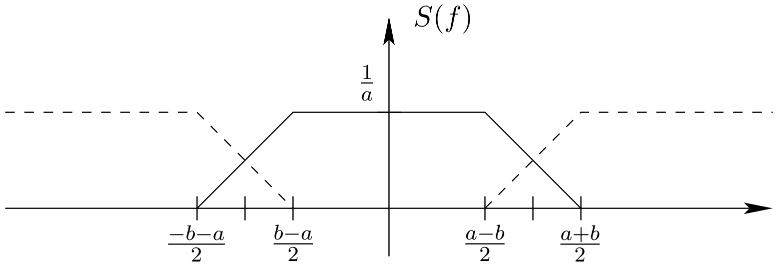
a. Define the signals

$$s_1(t) = \operatorname{sinc}(at),$$
$$s_2(t) = \operatorname{sinc}(bt).$$

Then we have

$$S_1(f) = \frac{1}{a} I_{\{|f|<a/2\}}(f),$$
$$S_2(f) = \frac{1}{b} I_{\{|f|<b/2\}}(f),$$
$$S(f) = (S_1 * S_2)(f).$$

The convolution results in the following graph, where also frequency-shifted versions according to Poisson are included.



b. We have the bit rate $R = 1200$ b/s and the alphabet size $M = 4$. The symbol rate $1/T$ is then given by

$$\frac{1}{T} = \frac{R}{\log_2(M)} = 600 \text{ symbols/s.}$$

This is our sampling frequency.

To fulfill the bandwidth demand, we have

$$\frac{a+b}{2} = 400 \text{ Hz}$$

To fulfill the Nyquist criterion (consider the frequency domain, eq. (2.80) in the text book), we must have

$$\frac{a+b}{2} + \frac{a-b}{2} = 600 \text{ Hz.}$$

The solution to these two equations is $a = 600$ Hz and $b = 200$ Hz.

c. We have a passband system with bandwidth 20 MHz, bit rate $R = 60$ Mb/s and alphabet size $M = 64$. The symbol rate $1/T$ is then given by

$$\frac{1}{T} = \frac{R}{\log_2(M)} = 10 \text{ Msymbols/s}$$

The one-sided bandwidth of the corresponding baseband system is 10 MHz. This time, we have

$$\frac{a+b}{2} = 10 \text{ MHz}$$
$$\frac{a+b}{2} + \frac{a-b}{2} = 10 \text{ MHz}$$

with solution $a = b = 10$ MHz.

d. Both $s_1(t)$ and $s_2(t)$ falls as $1/|t|$. Thus, $s(t)$ falls as $1/t^2$. The absolute value of the total signal is then bounded by the sum $\sum \frac{A}{(kT)^2}$, where A is proportional to the maximum symbol energy, and this sum is finite. So, the absolute value of the total signal is finite.

Task 2.18

Hints

Check the definitions of Nyquist and square-root Nyquist.

Answer

a. True (Because $p(kT) = \delta_{k_0}$)
b. False (For $p(t)$ to be square root Nyquist, we must have $\int_{-\infty}^{+\infty} p(t)p^*(t-kT)dt$ =0. It is easy to see that it is not true for $k = 1$).

Task 2.25

Hints

Write down the complex envelops $\tilde{s}_0$ and $\tilde{s}_1$ of s_0 and s_1 w.r.t frequency reference respectively. We know that

$$< s_0, s_1 >= Re\{< \tilde{s}_0, \tilde{s}_1 >\}$$

and compute the $Re\{< \tilde{s}_0, \tilde{s}_1 >\}$. Then, find the minimum frequency separation for $\phi_0 = \phi_1 = 0$ and for arbitrary ϕ_0 and ϕ_1.

Answer

a. $\|f_0 - f_1\| = 1/2T$
b. $\|f_0 - f_1\| = 1/T$

Task 2.26

Task

a. Specify the Walsh-Hadamard codes for 8-ary orthogonal signaling with non-coherent reception.

b. Plot the baseband waveforms corresponding to sending these codes using a square root raised cosine pulse with excess bandwidth of 50 %.

c. What is the fractional increase in the bandwidth efficiency if we use these waveforms as building blocks for biorthogonal signaling with coherent reception?

Solution

a. To make the matrices below more readable, let + denote 1 and let − denote −1. The Walch-Hadamard codes are given by the relations

$$H_0 = (+), \qquad H_m = \left(\begin{array}{c|c} H_{m-1} & H_{m-1} \\ \hline H_{m-1} & -H_{m-1} \end{array}\right), \quad \text{for } m > 0.$$

Thus, we have

$$H_1 = \left(\begin{array}{c|c} + & + \\ \hline + & - \end{array}\right), \qquad H_2 = \left(\begin{array}{cc|cc} + & + & + & + \\ + & - & + & - \\ \hline + & + & - & - \\ + & - & - & + \end{array}\right),$$

$$H_3 = \left(\begin{array}{cccc|cccc} + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ \hline + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + \\ + & + & - & - & - & - & + & + \\ + & - & - & + & - & + & + & - \end{array}\right)$$

The eight rows are then used as signals.

b. Omitted.

c. We note that the eight signals are orthogonal. Thus, we need $N = 8$ dimensions for octal signaling ($M = 8$) and the bandwidth efficiency is then

$$\frac{\log_2(M)}{N} = \frac{3}{8}$$

At bi-orthogonal signaling we have two antipodal signals per dimension. Thus, we have $M = 16$ signals. Using coherent detection, it is enough with $N = 8$ dimensions, and the bandwidth efficiency is then

$$\frac{\log_2(M)}{N} = \frac{4}{8}$$

The increase in percent is then given by

$$\frac{4/8 - 3/8}{3/8} = \frac{1}{3},$$

i.e. 33 %.

Task 2.27

Task

We wish to send at a rate of 10Mb/s over a passband channel. Assuming that an excess bandwidth of 50% is used, how much bandwidth is needed for each of the following schemes: QPSK, 64-QAM and 64-ary noncoherent orthogonal modulation using a Walsh-Hadamard code?

Solution

Let R_b be the wanted data rate, i.e. $R_\mathrm{b} = 10$ Mb/s. The smallest possible bandwidth B_min is given by the Nyquist criterion as

$$B_\mathrm{min} = \frac{R_\mathrm{b}}{\eta_B},$$

where η_B is the bandwidth efficiency expressed in bits/symbol/dimension, and we are supposed to use 50% excess bandwidth. Our bandwidth is therefore

$$B = \frac{3}{2}B_\mathrm{min} = \frac{3R_\mathrm{b}}{2\eta_B}.$$

The bandwidth efficiency is defined as

$$\eta_B = \frac{\log_2(M)}{D},$$

where M is the number of signals in the constellation and D is the number of complex dimensions that are spanned by the signals. Totally, this gives us the bandwidth

$$B = \frac{3R_\mathrm{b}D}{2\log_2(M)}.$$

QPSK Here we have $M = 4$ and $D = 1$.

$$B = \frac{3R_\mathrm{b}D}{2\log_2(M)} = 7.5 \text{ MHz}.$$

64-QAM Here we have $M = 64$ and $D = 1$.

$$B = \frac{3R_\mathrm{b}D}{2\log_2(M)} = 2.5 \text{ MHz}.$$

64-Walsh-Hadamard Here we have $M = 64$. Non-coherent detection means that we need one complex dimension per orthogonal signal. Thus, we have $D = 64$.

$$B = \frac{3R_\mathrm{b}D}{2\log_2(M)} = 160 \text{ MHz}.$$

Task 2.29

Hints

Read up on DPSK. The information is represented by the phase differences.

a. According to Figure 2.22, the information bits 00, 10, 11, 01 corresponds to phase shift 0, $\pi/2$, π, $-\pi/2$, respectively. Let $\theta(n) \in \{\pm\pi/4, \pm3\pi/4\}$ (or alternatively $\theta(n) \in \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$.

Solution: $-\pi/4, 3\pi/4, \pi/4, 3\pi/4, -3\pi/4, 3\pi/4, -\pi/4, -\pi/4, -3\pi/4, \pi/4$

b. $r[2]r^*[1] = 1 + 3j$. Its argument is closest to a phase difference of $\pi/2$, which corresponds to information bits 10. This would be an error, since the sent bits are 11 at time instant $n = 2$.

Tutorial 4

Task 5.1

Task

Consider a digitally modulated system using QPSK at bit rate $2/T$. The transmit filter, channel and receive filter have the following impulse responses.

$$\begin{aligned} g_{\mathrm{TX}}(t) &= I_{\{0\le t<T/2\}}(t) - I_{\{T/2\le t<T\}}(t), \\ g_{\mathrm{C}}(t) &= \delta(t) - \frac{1}{2}\,\delta\!\left(t - \frac{T}{2}\right), \\ g_{\mathrm{RX}}(t) &= I_{\{0\le t<T/2\}}(t). \end{aligned}$$

Let $z[k]$ denote the receive filter output sampled ad time instance $kT_{\mathrm{s}} + \tau$, where T_{s} is a sampling interval to be chosen.

a. Show that ML sequence detection using the samples $\{z[k]\}$ is possible, given an appropriate choice of T_{s} and τ. Specify the corresponding choice of T_{s} and τ.

b. How many states are needed in the trellis for implementing ML sequence detection using the Viterbi algorithm?

Solution

Let $p(t)$ be the total impulse response of the cascade of the sender filter and the channel i.e.

$$p(t) = (g_{\mathrm{TX}} * g_{\mathrm{C}})(t) = I_{\left\{0\le t<\frac{T}{2}\right\}}(t) - \frac{3}{2} I_{\left\{\frac{T}{2}\le t<T\right\}}(t) + \frac{1}{2} I_{\left\{T\le t<\frac{3T}{2}\right\}}(t)$$

Its matched filter is then

$$p_{\mathrm{MF}}(t) = p^*(-t) = I_{\left\{0<-t<\frac{T}{2}\right\}}(t) - \frac{3}{2} I_{\left\{\frac{T}{2}<-t<T\right\}}(t) + \frac{1}{2} I_{\left\{T<-t<\frac{3T}{2}\right\}}(t)$$

Furthermore, let $y(t)$ denote the output from the channel. We have the bit rate $2/T$. In QPSK, we have $M = 4$ signals, 2 bits ($\log_2(M)$) per symbol. Thus, we have the symbol rate $1/T$.

a. The optimal (ML) case is if we use $h_{\mathrm{MF}}(t)$ as our receiver filter. Let $z_0[n]$ be the output from that filter, sampled in the time instances nT. Then we have

$$z_0[n] = (y * p_{\mathrm{MF}})(nT) = \int_{-\infty}^{\infty} y(t) p_{\mathrm{MF}}(nT - t)\, dt.$$

The actual output from the receiver filter, sampled in the time instances $kT_{\mathrm{s}} + \tau$ for all integers k is given by

$$z[k] = (y * g_{\mathrm{RX}})(kT_{\mathrm{s}} + \tau) = \int_{-\infty}^{\infty} y(t) g_{\mathrm{RX}}(kT_{\mathrm{s}} + \tau - t)\, dt.$$

With $\tau = 0$ and $T_{\mathrm{s}} = T/2$ we get

$$z[k] = \int_{-\infty}^{\infty} y(t) g_{\mathrm{RX}}\left(k\frac{T}{2} - t\right)\, dt.$$

Furthermore, we have

$$h_{\mathrm{MF}}(t) = g_{\mathrm{RX}}\left(t + \frac{T}{2}\right) - \frac{3}{2}\, g_{\mathrm{RX}}(t+T) + \frac{1}{2}\, g_{\mathrm{RX}}\left(t + \frac{3T}{2}\right),$$

which can be rewritten as

$$\begin{aligned} &p_{\mathrm{MF}}(nT - t) = \\ &= g_{\mathrm{RX}}\left((2n+1)\frac{T}{2} - t\right) - \frac{3}{2}\, g_{\mathrm{RX}}\left((2n+2)\frac{T}{2} - t\right) + \frac{1}{2}\, g_{\mathrm{RX}}\left((2n+3)\frac{T}{2} - t\right) \end{aligned}$$

From the above, we draw the conclusion that

$$z_0[n] = z[2n+1] - \frac{3}{2} z[2n+2] + \frac{1}{2} z[2n+3]$$

holds.

b. The number of required states is M^L, where L is the memory length. The finite memory condition is

$$h[n] = 0, |n| > L.$$

It is clear that here we have

$$h[n] = 0, |n| > 1.$$

Thus, $L = 1$. The number of needed states is then

$$M^L = 4^1 = 4.$$

Task 5.2

Hints

a. The minimum required bandwidth $B_{\min} = R_b/\eta_B$, where $\eta_B = \log_2(M)/D$.

b. To have the minimum number of states, we need to sample in the time instances $\left(k + \frac{1}{2}\right) T$. In this case $L = 2$.

Answer

a. The minimum required bandwidth is 2 MHz

b. We need 64 states.

Task 5.3

Task

For BPSK-(± 1) signaling in the standard MLSE setting, suppose that the channel memory is $L = 1$, with $h[0] = 1$ and $h[1] = -0.3$.

a. What is the maximum pairwise error probability, as a function of the received E_b/N_0, for two bit sequences that differ only in the first two bits? Express your answer in terms of the Q function.

b. Plot the transfer function bound (log scale) as a function of E_b/N_0 (dB). Also plot the error probability of BPSK without ISI for comparison.

Solution

a. The pairwise error probability is according to page 233 given by

$$P_\mathrm{e} = Q\left(\frac{||s(b+2e) - s(b)||}{2\sigma}\right),$$

where $s(b)$ is the signal corresponding to the bit sequence b, i.e.

$$s(b) = \sum_n b[n]p(t - nT),$$

where $p(t)$ is the used pulse. Furthermore, $\sigma^2 = N_0/2$ is the PSD of the noise and e is a valid error sequence. According to the reasoning on page 233, we also have

$$\begin{aligned} P_\mathrm{e} &= Q\left(\frac{||s(e)||}{\sigma}\right) = Q\left(\sqrt{\frac{h[0]e^2[1] + h[0]e^2[2] + 2h[1]e[2]e[1]}{\sigma^2}}\right) \\ &= Q\left(\sqrt{\frac{2h[0] + 2h[1]e[2]e[1]}{\sigma^2}}\right), \end{aligned}$$

where we have used the fact that the error sequence is non-zero only in the first two positions. Since the Q function is strictly decreasing, the expression is maximized when the nominator inside the square root is minimized. We have four alternatives, namely

$$\begin{aligned} (e[1], e[2]) = (1, 1) &: \quad 2h[0] + 2h[1]e[2]e[1] = 2h[0] + 2h[1] = 1.4 \\ (e[1], e[2]) = (1, -1) &: \quad 2h[0] + 2h[1]e[2]e[1] = 2h[0] - 2h[1] = 2.6 \\ (e[1], e[2]) = (-1, 1) &: \quad 2h[0] + 2h[1]e[2]e[1] = 2h[0] - 2h[1] = 2.6 \\ (e[1], e[2]) = (-1, -1) &: \quad 2h[0] + 2h[1]e[2]e[1] = 2h[0] + 2h[1] = 1.4 \end{aligned}$$

Then we have

$$P_{\mathrm{e,max}} = Q\left(\sqrt{\frac{2h[0] + 2h[1]}{\sigma^2}}\right) = Q\left(\sqrt{4\frac{E_\mathrm{b}}{N_0}\left(1 + \frac{h[1]}{h[0]}\right)}\right) = Q\left(\sqrt{2.8\frac{E_\mathrm{b}}{N_0}}\right),$$

where we have used that we have the bit energy

$$E_\mathrm{b} = h[0]\mathrm{E}\{|b[n]|^2\} = h[0].$$

b. This task is almost identical to Example 5.8.2 on pages 238–240. We have the same coefficients as we have there, i.e.

$$\begin{aligned} a_0 &= e^{-\frac{h[0]}{2\sigma^2}} = e^{-\frac{E_\mathrm{b}}{N_0}}, \\ a_1 &= e^{-\frac{h[0]+2h[1]}{2\sigma^2}} = e^{-\frac{E_\mathrm{b}}{N_0}\left(1+\frac{2h[1]}{h[0]}\right)} = e^{-0.4\,\frac{E_\mathrm{b}}{N_0}}, \\ a_2 &= e^{-\frac{h[0]-2h[1]}{2\sigma^2}} = e^{-\frac{E_\mathrm{b}}{N_0}\left(1-\frac{2h[1]}{h[0]}\right)} = e^{-1.6\,\frac{E_\mathrm{b}}{N_0}}. \end{aligned}$$

We also have the same error probability expression as we have there,

$$P_\mathrm{e} \le \frac{\frac{1}{2}a_0}{\left[1-\frac{1}{2}(a_1+a_2)\right]^2} = \frac{\frac{1}{2}\,e^{-\frac{E_\mathrm{b}}{N_0}}}{\left[1-\frac{1}{2}\left(e^{-0.4\,\frac{E_\mathrm{b}}{N_0}}+e^{-1.6\,\frac{E_\mathrm{b}}{N_0}}\right)\right]^2}$$

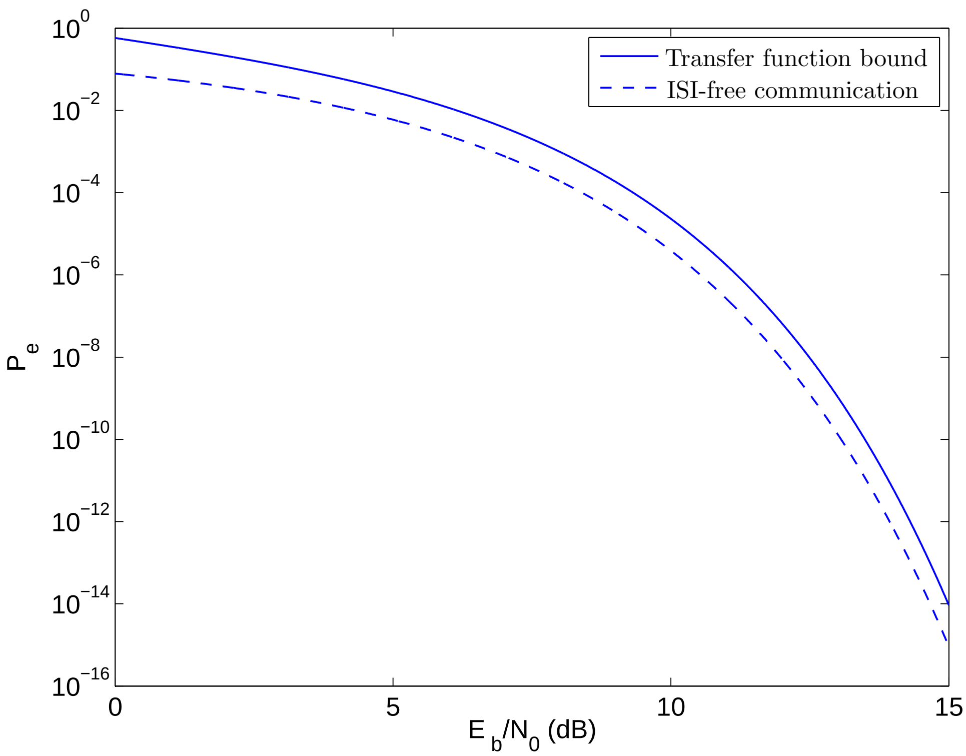
For ISI-free communication, $\vec{e} = (\pm 1, 0, \ldots)$

$$\|s(\vec{e})\|^2 = h[0]e^2[1] = h[0].$$

We have

$$\begin{aligned} P_\mathrm{e} &\le Q\left(\sqrt{\frac{h[0]}{\sigma^2}}\right) \\ &= Q\left(\sqrt{\frac{2E_\mathrm{b}}{N_0}}\right). \end{aligned}$$

A plot of these two:



Task 5.4

Hints

a. Simply check the Nyquist criterion.

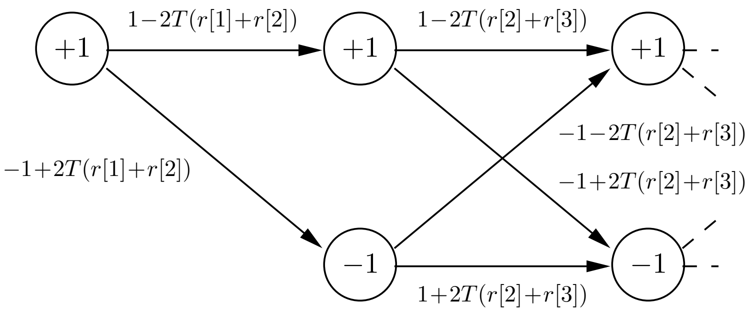
b. Be inspired by task 5.1.

c. Start with the usual metric in Equation 5.13 on page 206, which is to be maximized, and make use of the fact that all involved signals are real valued. Determine $h[0]$ and $h[1]$ for the given situation. ($h[n] = 0,\ \text{for}|n| > 1$) Simplify the metric, remove unnecessary constants and change its sign such that the maximization is replaced by minimization.

d. Use the definitions on pages 236–237.

Answer

a. No, the pulse is not Nyquist.
b. Choose $Ts = T$ and $\tau = T/2$. $Z_{\text{MF}}[n] = 1/2(r[n] + r[n+1])$
c. Metric: $m_n\big(b[n], s[n]\big) = b[n]\Big(b[n-1] - 2T\big(r[n] + r[n+1]\big)\Big)$



d. $\eta = 1$

Task 5.6

Task

We would like to develop a model for simulating the symbol rate sampled matched filter outputs for linear modulation through a dispersive channel. I.e. we wish to generate the samples $Z[k] = (Y * p_{\text{MF}})(kT)$ for $Y(t) = \sum_n B[n]p(t - nT) + W(t)$, where $p_{\text{MF}}(t)$ is matched to $p(t)$ and where $W(t)$ is complex WGN.

a. Show that the signal contribution Z_{s} to $Z[k]$ can be written as

$$Z_{\text{s}}[k] = \sum_n B[n]h[k-n] = (B * h)[k],$$

where we have $h[l] = (p * p_{\text{MF}})(lT)$.

b. *Not part of the plan* Show that the noise contribution $Z_W[k]$ to $Z[k]$ is WSS, proper complex Gaussian random process with zero mean and covariance function

$$C_{Z_W}[k] = \mathrm{E}\left\{Z_W[l]Z_W^*[k-l]\right\} = 2\sigma^2 h[k].$$

For real-valued symbols, signals and noise, $h[k]$ are real and

$$C_{Z_W}[k] = \mathrm{E}\left\{Z_W[l]Z_W[k-l]\right\} = \sigma^2 h[k].$$

c. Now specialize to the running example in Figure 5.1, with BPSK signaling ($B[n] \in \{\pm 1\}$). We can now restrict Y, p and W to be real-valued. Show that the result of **a** specializes to

$$Z_\mathrm{s} = \frac{3}{2}B[k] - \frac{1}{2}(B[k+1] + B[k-1]).$$

Show that the result of **b** specializes to

$$R_{Z_W}[z] = \sigma^2\left(\frac{3}{2} - \frac{1}{2}(z + z^{-1})\right), Task$$

where the PSD $R_{Z_W}[z]$ is the z-transform of $C_{Z_W}[k]$.

d. Suppose that $W[k]$ are i.i.d. $\mathrm{N}(0,1)$ random variables. Show that this discrete-time WGN sequence can be filtered to generate $Z_W[k]$ as follows:

$$Z_W[k] = g[0]W[k] + g[1]W[k-1].$$

Find the coefficients $g[0]$ and $g[1]$ such that $Z_W[k]$ has statistics as specified in **c**. **Hint**: Factorize $R_{Z_W}[z] = (a + bz)(a * + b * z^{-1})$ by finding the roots, and use one of the factors to specify the filter.

e. *Not part of the plan* Use the results in **a-d** to simulate the performance of MLSE for the running example. Compare the resulting BER (Bit-Error Rate) with that obtained using the transfer function bound.

Solution

a. Let $Y_s(t)$ be the contribution of the signal to $Y(t)$, i.e.

$$Y_s(t) = \sum_n B[n]p(t - nT).$$

Then $Z_s[k]$ can be rewritten as

$$\begin{aligned} Z_s[k] = (Y_s * p_{MF})(kT) &= \int_{-\infty}^{\infty} \sum_n B[n]p(t - nT)p_{MF}(kT - t)\,dt \\ &= \sum_n B[n] \int_{-\infty}^{\infty} p(t - nT)p_{MF}(kT - t)\,dt. \end{aligned}$$

The variable substitution $\tau = t - nT$ gives us

$$\begin{aligned} Z_s[k] &= \sum_n B[n] \int_{-\infty}^{\infty} p(\tau)p_{MF}((k - n)T - \tau)\,d\tau = \sum_n B[n](p * p_{MF})\big((k - n)T\big) \\ &= \sum_n B[n]h[k - n] = (B * h)[k], \end{aligned}$$

which is what was supposed to be shown.

b. A linear transformation on complex WGN results in a proper Gaussian random process (Chapter 4). Wide-sense stationarity is preserved as described in Appendix A.

c. For the example in Figure 5.1, we have (according to the box on page 205)

$$h[k] = \begin{cases} 3/2, & k = 0, \\ -1/2, & |k| = 1, \\ 0, & \text{else}, \end{cases}$$

Consequently, we have

$$\begin{aligned} Z_s &= \sum_n B[n]h[k - n] = h[1]B[k - 1] + h[0]B[k] + h[-1]B[k + 1] \\ &= \frac{3}{2}B[k] - \frac{1}{2}\big(B[k + 1] + B[k - 1]\big). \end{aligned}$$

We have real-valued entities, which gives us the covariance function

$$C_{Z_W}[k] = \sigma^2 h[k]$$

according to **b**. With our $h[k]$, this becomes

$$C_{Z_W}[k] = \begin{cases} \frac{3}{2}\sigma^2, & k = 0, \\ -\frac{1}{2}\sigma^2, & |k| = 1, \\ 0, & \text{else}, \end{cases}$$

Then we have the z-transform

$$R_{Z_W}[z] = \sum_k C_{Z_W}[k] z^{-k} = \sigma^2 \left(\frac{3}{2} - \frac{1}{2} \left(z + z^{-1} \right) \right).$$

which is what was supposed to be shown.

d. We want to factorize $R_{Z_W}[z]$ on the form

$$R_{Z_W}[z] = (a + bz)(a^* + b^* z^{-1}).$$

Then we can use one such factor as the transfer function of a filter. That filter has the correct output according to the super formula. The first factor corresponds to a non-causal filter, while the second factor corresponds to a causal filter. Of course, we choose the second one. We solve the related quadratic equation

$$-\frac{1}{2}x^2 + \frac{3}{2}x - \frac{1}{2} = 0,$$

which has the two roots

$$\begin{aligned} x_0 &= \frac{3 - \sqrt{5}}{2}, \\ x_1 &= \frac{3 + \sqrt{5}}{2} = x_0^{-1}, \end{aligned}$$

and we have the factorization

$$z^2 - 3z + 1 = (z - x_0^{-1})(z - x_0).$$

Then we have

$$\begin{aligned} R_{Z_W}[z] &= \sigma^2 \left(\frac{3}{2} - \frac{1}{2}(z + z^{-1}) \right) = -\frac{\sigma^2}{2} z^{-1}(z^2 - 3z + 1) \\ &= \frac{\sigma^2}{2}(x_0^{-1} - z)(1 - x_0 z^{-1}) = \frac{\sigma^2}{2x_0}(1 - x_0 z)(1 - x_0 z^{-1}) \\ &= \left(\frac{\sigma}{\sqrt{2x_0}} - \frac{\sigma}{2}\sqrt{2x_0}z \right) \left(\frac{\sigma}{\sqrt{2x_0}} - \frac{\sigma}{2}\sqrt{2x_0}z^{-1} \right). \end{aligned}$$

Thus, we get

$$\begin{aligned} a &= \frac{\sigma}{\sqrt{2x_0}} = \frac{\sigma}{\sqrt{3 - \sqrt{5}}} \\ b &= -\frac{\sigma}{2}\sqrt{3 - \sqrt{5}} \end{aligned}$$

The causal filter has the transfer function

$$G[z] = a^* + b^* z^{-1} = a + bz^{-1}$$

and impulse response

$$g[n] = a\delta[n] + b\delta[n-1] = \begin{cases} \frac{\sigma}{\sqrt{3-\sqrt{5}}}, & n = 0, \\ -\frac{\sigma}{2}\sqrt{3 - \sqrt{5}}, & n = 1, \\ 0, & \text{else.} \end{cases}$$

e. —