①

Solution:

Q.2 a)

$$\mathrm{LPF}\{y_p(t)\cos 2\pi f_c t\}$$
$$= \mathrm{LPF}\{x^I(t)\cos^2 2\pi f_c t - \frac{x^Q(t)}{2}\sin 4\pi f_c t + n_p(t)\cos 2\pi f_c t\}$$
$$= \frac{x^I(t)}{2} + \underbrace{\mathrm{LPF}\{n_p(t)\cos 2\pi f_c t\}}_{\triangleq n^I(t)}.$$
$$= y^I(t),$$

and similarly

$$\mathrm{LPF}\{-y_p(t)\sin 2\pi f_c t\}$$
$$= \frac{x^Q(t)}{2} + \underbrace{\mathrm{LPF}\{n_p(t)\cdot -\sin 2\pi f_c t\}}_{\triangleq n^Q(t)}.$$
$$= y^Q(t).$$

~~Since $n_p(t)$ is a white Gaussian random process with power spectral density σ^2~~

$$n^I(t) = \int n_p(\tau)\cos 2\pi f_c \tau\, g(t-\tau)\, d\tau$$
$$n^Q(t) = -\int n_p(\tau)\cos 2\pi f_c \tau\, g(t-\tau)\, d\tau$$

where $g(t)$ is the impulse response of the LPF.

$$y^I[m] = y^I(mT) = \frac{x^I(mT)}{2} + n^I(mT)$$
$$= \frac{1}{2}\sum x^I[k]\,\mathrm{sinc}(m-k) + n^I(mT)$$
$$= \frac{1}{2}x^I[m] + n^I(mT).$$

Similarly,

$$y^Q[m] = \frac{1}{2} x^Q[m] + n^Q[mT].$$

where $x^I[m]$ and $x^Q[m]$ are the real and imaginary components of $x[m]$.

$$n^I(mT) = \int n_p(\tau) \cos 2\pi f_c \tau \, g(mT-\tau) \, d\tau.$$

$$n^Q(mT) = \int -n_p(\tau) \sin 2\pi f_c \tau \, g(mT-\tau) \, d\tau.$$

are both Gaussian random variables since $n_p(t)$ is a stationary white Gaussian random process.

Since $\mathbb{E}(n_p(t)) = 0$ for all t,

$\mathbb{E}(n^I(mT)) = \mathbb{E}(n^Q(mT)) = 0$, and.

$$\operatorname{Var}(n^I(mT)) = \mathbb{E}[n^{I^2}(mT)]$$

$$= \sigma^2 \int g^2(mT-\tau) \cos^2 2\pi f_c \tau \, d\tau$$

~~As~~ Here we use the result that, for any finite energy signal $s(t)$,

$\int s(t) n_p(t) dt$ is a Gaussian random variable, mean zero and variance $\sigma^2 \int s^2(t) dt$.

$$\int g^2(mT-\tau) \cos^2 2\pi f_c \tau \, d\tau$$

$$= \frac{1}{2}\int g^2(mT-\tau) \, (1 + \cos 4\pi f_c \tau) \, d\tau$$

$$= \frac{1}{2}\int g^2(mT-\tau) \, d\tau + \frac{1}{2}\int g^2(mT-\tau) \cos 4\pi f_c \tau \, d\tau$$

Using Parseval's theorem it can be shown that

$\int g^2(mT-\tau) \cos 4\pi f_c \tau \, d\tau = 0$ since $f_c > W$.

Parseval's theorem states that for any two Fourier transformable signals $u(t)$ and $v(t)$

~~$\int x(t) y^*(t) \, dt = \int X(f) \, Y^*(f) \, df$.~~

~~letting $x(\tau) = g^2(mT-\tau)$ and $y(\tau) = \cos 4\pi f_c \tau$, we know that~~

$$\int u(\tau) \, v^*(\tau) \, d\tau = \int U(f) V^*(f) \, df$$

letting $u(\tau) = g^2(mT-\tau)$ and

$v(\tau) = \cos 4\pi f_c \tau$,

we note that

$U(f) \neq 0$ only for $|f| < 2W$

and $V(f) \neq 0$ only for $f = 2f_c$ and $f = -2f_c$.

since $f_c > W$,

$2f_c \notin [-2W, 2W]$ and also for $-2f_c$.

$\therefore \int U(f)\, V^*(f) = 0$.

Similarly it can be shown that

$\int g^2(mT-\tau) \sin 2\pi f_c \tau \, d\tau = 0$.

$$\therefore \operatorname{var}(n^I(mT)) = \operatorname{var}(n^Q(mT))$$

$$= \frac{\sigma^2}{2} \int g^2(mT-\tau)\, d\tau$$

$$= \frac{\sigma^2}{2} \int g^2(\tau)\, d\tau.$$

$$= \frac{\sigma^2}{2} \int |G(f)|^2 df.$$

$$= \frac{\sigma^2}{2} \int |LPF(f)|^2 df$$

$$= \frac{\sigma^2}{2} \cdot (2W) = \sigma^2 W.$$

Another result that we know is that if $u(t)$ and $v(t)$ are orthogonal, i.e., $\int u(t) v(t)\, dt = 0$, then the Gaussian random variables $x = \int u(t)\, n_p(t)\, dt$ and $y = \int v(t)\, n_p(t)\, dt$ are uncorrelated and therefore independent.

i.e., $E[xy] = 0$.

In our case, $\cancel{\text{or}}$

$u(\tau) = g(mT-\tau)\cos 2\pi f_c\tau$ and

$v(\tau) = -g(mT-\tau)\sin 2\pi f_c\tau$ are orthogonal since

$$\int u(\tau)v(\tau)\,d\tau = -\int g^2(mT-\tau)\sin 2\pi f_c\tau \cos 2\pi f_c\tau\, d\tau$$

$$= -\frac{1}{2}\int g^2(mT-\tau)\sin 4\pi f_c\tau\, d\tau$$

$= 0$ (again using the passeval's theorem)

$\therefore \mathbb{E}[n^I(mT)\, n^Q(mT)] = 0$.

It can also be shown that

$$\mathbb{E}[n^I(mT)\, n^Q(kT)] = \mathbb{E}[n^I(kT)\, n^Q(mT)] = 0.$$

$\therefore$ Finally

$$\cancel{y[k]}\ y[m] \triangleq y^I[m] + j\,y^Q[m]$$

$$= \frac{1}{2}x[m] + \underbrace{(n^I[m] + j\,n^Q[m])}_{n[m]}.$$

where $n^I[m] = n^I(mT)$ and

$n^Q[m] = n^Q(mT)$.

solutions to

2 b)

i) ~~$x(f) = \sum x[k]$~~

Since $\{x[k]\}$ is an i.i.d. sequence of BPSK symbols, we have

$$\mathbb{E}\{x[m]\, x^*[k]\} = \mathbb{E}\{x[m]\}\, \mathbb{E}\{x^*[k]\}$$
$$= 0 \cdot 0$$
$$= 0$$

since $\mathbb{E}\{x[m]\} = 0$ for all m.

Hence the autocorrelation

$$R_x[k] \triangleq \mathbb{E}\{x[m]\, x^*[m-k]\}$$
$$= 0$$, depends only on the time lag k.

Since the mean is also independent of the time instance, $\{x[k]\}$ is a wide sense stationary process.

It can also be shown that $\{x[k]\}$ is ergodic in autocorrelation, ~~since~~

$$\cancel{\frac{1}{M+1}\sum_{m=0}^{M} x[m]\, x[m-k]}$$

$\therefore$ P.S.D of $x(t)$ is given by

$$R_x(f) = \begin{cases} T\, \mathbb{E}[|x[k]|^2] & \text{for } |f| < W \\ 0 & \text{otherwise} \end{cases}$$

where $W = 1/2T$.

and therefore

$$R_{x_p}(f) = \begin{cases} \frac{1}{2} R_x(f-f_c) + R_x(-f-f_c) \end{cases}$$

and hence,

$$R_{x_p}(f) = \begin{cases} \frac{T}{2} \mathbb{E}[|x[k]|^2] & f_c-W < f < f_c+W . \end{cases}$$

The ~~total~~ average transmitted power is therefore

$$\langle P \rangle = \int_{f_c-W}^{f_c+W} R_{x_p}(f)\, df = \frac{T}{2} \cdot \mathbb{E}[|x[k]|^2] \cdot 2W$$

$$= \frac{\mathbb{E}[|x[k]|^2]}{2} .$$

$$= \frac{E_b}{2} .$$

iii) ~~From part a) we have~~

$$\cancel{z^I[k] = \frac{1}{2} x^I[k] + n^I[k] ,}$$

$$\cancel{z^Q[k] = \frac{1}{2} x^Q[k] + n^Q[k] \qquad k=0 \ldots, M .}$$

~~$\therefore$ the MPE detector is given by.~~

$$\cancel{(\hat{x}[0], \hat{x}[1], \ldots, \hat{x}[M]) = \arg\min \text{Prob}(z^I[k], z^Q[k], k=0, M |}$$

$$\cancel{= \arg\min_{\substack{x^I[k] \in (-\sqrt{E_b}, +\sqrt{E_b}) \\ x^Q[k]=0}} \text{prob}(z^I[k], z^Q[k], k=0 \ldots M |}$$

ii) From part (a) we have

$$Z^I[k] = \frac{1}{2} X^I[k] + N^I[k],$$

$$Z^Q[k] = \frac{1}{2} X^Q[k] + N^Q[k], \quad k=0, \cdot\cdot M.$$

where $\{N^I[k]\}, \{N^Q[k]\}$ are sequences of independent Gaussian random variables with mean 0 and variance $\sigma^2 W$.

$X^I[k]$ is a random variable which is equal to $+\sqrt{E_b}$ with probability ½ and $-\sqrt{E_b}$ with probability ½.

$X^Q[k] = 0$ for all $k = 0, 1, 2, \cdot\cdot, M$.

given a deterministic received discrete-time sequence $\{z^I[k]\}, \{z^Q[k]\}$, the minimum probability of error detector is given by

$$(\hat{x^I[0]}, \ldots, \hat{x^I[M]}) = \arg \max_{\substack{x^I[k] \in (-\sqrt{E_b}, \sqrt{E_b}) \\ k=0,\cdot\cdot,M}} \text{Prob}\left(Z^I[k] = z^I[k], Z^Q[k] = z^Q[k], k=0,1,\cdot\cdot M \,\middle|\, X^I[k] = x^I[k], X^Q[k] = 0, k=0,\cdot\cdot,M\right)$$

8

$$= \arg \cancel{\min} \max_{\substack{x^I[k] \in (-\sqrt{E_b}, +\sqrt{E_b}) \\ k=0,..,M}} \prod_{k=0}^{M} \text{prob}\big(Z^I[k] = z^I[k],\ Z^Q[k] = z^Q[k] \,\big|\, X^I[k] = x^I[k],\ X^Q[k] = 0 \big)$$

since $N^I[k], N^Q[k]$ are both independent of $N^I[m], N^Q[m]$ for $m \neq k$.

Also since, $N^I[k]$ and $N^Q[k]$ are independent.

$$\text{prob}\big(Z^I[k] = z^I[k], Z^Q[k] = z^Q[k] \,\big|\, X^I[k] = x^I[k], X^Q[k] = 0 \big)$$

$$= \text{prob}\big(Z^I[k] = z^I[k] \,\big|\, X^I[k] = x^I[k] \big)\ \text{prob}\big(Z^Q[k] = z^Q[k] \,\big|\, X^Q[k] = 0 \big).$$

$$= \text{prob}\big(N^I[k] = z^I[k] - X^I[k] \,\big|\, X^I[k] = x^I[k] \big)\ \text{prob}\big(N^Q[k] = z^Q[k] - X^Q[k] \,\big|\, X^Q[k] = 0 \big)$$

$$= \text{prob}\big(N^I[k] = z^I[k] - x^I[k] \,\big|\, X^I[k] = x^I[k] \big)\ \text{prob}\big(N^Q[k] = z^Q[k] \,\big|\, X^Q[k] = 0 \big)$$

Since $N^I[k]$ is independent of $X^I[k]$ and $N^Q[k]$ is independent of $X^Q[k]$ we have:

$$= \text{prob}\big(N^I[k] = z^I[k] - \tfrac{1}{2} x^I[k] \big)\ \text{prob}\big(N^Q[k] = z^Q[k] \big)$$

$$= \frac{1}{(2\pi\sigma^2 W)} e^{-\frac{\left(z^I[k] - \frac{1}{2}x^I[k]\right)^2}{2\sigma^2 W}} \; e^{-\frac{z^Q[k]^2}{2\sigma^2 W}}$$

Substituting these into the MPE detector expression we have

$$\left(\hat{x}^I[0], \ldots, \hat{x}^I[M]\right) = \underset{\substack{x^I[k] \in (-\sqrt{E_b}, +\sqrt{E_b}) \\ k=0,1,\ldots,M}}{\arg\,\cancel{\min}\,\max} \frac{1}{(2\pi\sigma^2 W)^{M+1}} e^{-\frac{\sum_{k=0}^{M} z^Q[k]^2}{2\sigma^2 W}} \; e^{-\frac{\sum_{k=0}^{M}\left(z^I[k] - \frac{x^I[k]}{2}\right)^2}{2\sigma^2 W}}$$

$$= \underset{\substack{x^I[k] \\ \in (-\sqrt{E_b}, +\sqrt{E_b}) \\ k=0,1,\ldots,M}}{\arg\min} \sum_{k=0}^{M} \left(z^I[k] - \frac{x^I[k]}{2}\right)^2$$

From this it follows that

$$\hat{x}^I[k] = \underset{x^I[k] \in (-\sqrt{E_b}, +\sqrt{E_b})}{\arg\min} \left(z^I[k] - \frac{x^I[k]}{2}\right)^2$$

the joint minimization over $(x^I[0], \ldots, x^I[M])$ is separable into separate minimization for each $x^I[k]$. This is because there is no ISI at the receiver.

iii) Prob of Error Expression for each bit.

For the k-th bit we have

$$Z^I[k] = \frac{1}{2} X^I[k] + N^I[k]$$

↑ mean 0 and variance $\sigma^2 w$.

The MPE detector is

$$\hat{X^I}[k] = \arg\min_{x^I[k] \in (-\sqrt{E_b}, +\sqrt{E_b})} \left(z^I[k] - \frac{x^I[k]}{2} \right)^2$$

where $z^I[k]$ is the deterministic received sample.

$$\text{Prob of error} = \text{Prob}\left(X^I[k] \neq \hat{X^I}[k] \right)$$

$$= \text{prob}\left(X^I[k] \neq \hat{X^I}[k] \text{ and } X^I[k] = +\sqrt{E_b} \right)$$

$$+ \text{Prob}\left(X^I[k] \neq \hat{X^I}[k] \text{ and } X^I[k] = -\sqrt{E_b} \right)$$

Consider

$$\text{prob}\left(\hat{X^I}[k] \neq X^I[k] \text{ and } X^I[k] = +\sqrt{E_b} \right)$$

$$= \text{prob}\left(\hat{X^I}[k] \neq +\sqrt{E_b} \text{ and } X^I[k] = +\sqrt{E_b} \right)$$

$$= \text{prob}\left(\hat{X^I}[k] \neq +\sqrt{E_b} \,\middle|\, X^I[k] = +\sqrt{E_b} \right) \text{prob}\left(X^I[k] = +\sqrt{E_b} \right)$$

$$= \frac{1}{2} \text{prob}\left(\hat{X^I}[k] \neq +\sqrt{E_b} \,\middle|\, X^I[k] = +\sqrt{E_b} \right)$$

Since $x^I[k] = +\sqrt{E_b}$,

$$z^I[k] = \frac{1}{2} + \sqrt{E_b} + N^I[k].$$

the MPE detector is in error

i.e., $\hat{x}^I[k] = -\sqrt{E_b}$, if and only if

$$\left(z^I[k] - \left(-\frac{\sqrt{E_b}}{2}\right)\right)^2 < \left(z^I[k] - \frac{\sqrt{E_b}}{2}\right)^2$$

using the fact that

$z^I[k] = +\frac{\sqrt{E_b}}{2} + N^I[k]$, in the above inequality we have

$$\left(\sqrt{E_b} + N^I[k]\right)^2 < \left(N^I[k]\right)^2$$

or $$E_b + 2\sqrt{E_b}\, N^I[k] < 0$$

or $$N^I[k] < \frac{-\sqrt{E_b}}{2}.$$

$\therefore$ prob $\left(\hat{x}^I[k] \neq +\sqrt{E_b} \mid x^I[k] = +\sqrt{E_b}\right)$

$$= \text{prob}\left(N^I[k] < \frac{-\sqrt{E_b}}{2}\right)$$

$$= \int_{-\infty}^{-\sqrt{E_b}/2} \frac{1}{\sqrt{2\pi\sigma^2 W}}\, e^{-\frac{t^2}{2\sigma^2 W}}\, dt$$

$$= Q\left(\sqrt{\frac{E_b}{4\sigma^2 W}}\right)$$

where $$Q(x) \triangleq \int_x^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\, dt.$$

$\therefore$ prob $\left(\hat{x}^I[k] \neq x^I[k] \text{ and } x^I[k] = +\sqrt{E_b}\right)$

$$= \frac{1}{2} Q\left(\sqrt{\frac{E_b}{4\sigma^2 W}}\right)$$

$\therefore$ similarly

$$\text{Prob}\left(\hat{X}^{I}[k] \neq X^{I}[k],\ X^{I}[k] = -\sqrt{E_b}\right)$$

$$= \frac{1}{2} Q\left(\sqrt{\frac{E_b}{4\sigma^2 W}}\right),$$

$\therefore$ Prob of error $= \cancel{Q\left(\sqrt{\frac{E_b}{\sigma^2 W}}\right)}\ Q\left(\sqrt{\frac{E_b}{4\sigma^2 W}}\right)$

$$= Q\left(\sqrt{\frac{\cancel{2}\langle P\rangle}{2\sigma^2 W}}\right) = Q\left(\sqrt{\frac{\langle P\rangle}{2\sigma^2 W}}\right)$$

Since the duration of each bit $= T$ seconds

$= 1/2W$ seconds.

energy of each bit $= \langle P\rangle T$

$$= \frac{\langle P\rangle}{2W}.$$

$\therefore$ Prob of Error $= Q\left(\sqrt{\frac{\cancel{2}(\text{energy per bit})}{\sigma^2}}\right)$

$$= Q\left(\sqrt{\frac{2(\text{energy per bit})}{\sigma'^2}}\right)$$

where

$\sigma'^2 = 2\sigma^2$ is

the one sided P.S.D of

$n_p(t)$.

Solution to part c)

2c)

for a given sequence $\underline{x} = \{x[m]\}_{m=0}^{M}$

i) $$\langle P \rangle_{\underline{x}} = \frac{1}{MT} \int_0^{MT} \left(\sum_{m=0}^{M} x[m] \operatorname{sinc}\left(\frac{t}{T} - m\right)\right)^2 dt$$

$$= \frac{1}{MT} \sum_{m_1=0}^{M} \sum_{m_2=0}^{M} x[m_1]\, x[m_2] \int_0^{MT} \operatorname{sinc}\left(\frac{t}{T} - m_1\right) \operatorname{sinc}\left(\frac{t}{T} - m_2\right) dt$$

for $m_1 \neq m_2$

$$\int_{-\infty}^{\infty} \operatorname{sinc}\left(\frac{t}{T} - m_1\right) \operatorname{sinc}\left(\frac{t}{T} - m_2\right) dt$$

$$= T \operatorname{sinc}(m_1 - m_2)$$

$$= 0$$

$$\therefore \langle P \rangle_{\underline{x}} \approx \frac{1}{MT} \sum_{m=0}^{M} x^2[m] \int_0^{MT} \operatorname{sinc}^2(t/T - m)\, dt$$

$$\approx \frac{1}{M} \sum_{m=0}^{M} x^2[m]$$

$$\approx \frac{1}{M} \sum_{k=0}^{M/2} \left(x^2[2k] + x^2[2k+1]\right)$$

Assuming that the sequence of two-dimensional symbols $\{(x[2k], x[2k+1])\}_{k=0}^{M/2}$ are i.i.d.

$$\lim_{M \to \infty} \frac{1}{M} \sum_{k=0}^{M/2} \left(x^2[2k] + x^2[2k+1]\right)$$

$$= \frac{1}{2} \mathbb{E}\left(x^2[2k] + x^2[2k+1]\right)$$

Further assuming that all the three possibilities i.e.,

$\left(0, +\sqrt{2E_b}\right)$, $\left(-\sqrt{\frac{3E_b}{2}}, -\sqrt{\frac{E_b}{2}}\right)$ and $\left(\sqrt{\frac{3E_b}{2}}, -\sqrt{\frac{E_b}{2}}\right)$ are equally likely we have

$$\mathbb{E}\left(x^2[2k] + x^2[2k+1]\right) = \frac{1}{3}\left[2E_b + 2E_b + 2E_b\right]$$

$$= 2E_b .$$

$$\lim_{M \to \infty} \langle P \rangle_{\underline{x}} = E_b .$$

$\therefore$ Since

$$\frac{1}{MT}\int_0^{MT} x_p^2(t)\, dt \approx \frac{1}{MT} \cdot \frac{1}{2} \int_0^{MT} x^2(t)\, dt$$

the average transmitted power $= \frac{E_b}{2}$,

which is the same as for the simple BPSK transmission scheme.

ii) The spectral efficiency of the simple BPSK scheme is given by

$$\eta_{BPSK} = \frac{\text{bit-rate}}{\text{total bandwidth occupied}}$$

bit-rate $= 1/T$ (1 bit every T second).

total bandwidth occupied $= 2W = 1/T$

$\therefore \eta_{BPSK} = 1$ bps/Hz

For the scheme in part c, each 2-dimensional symbol ~~Ø~~ can take 3 possible possibles

$\therefore$ In time MT, the total number of possible messages $= 3^{M/2}$

$\therefore$ no. of bits communicated in MT seconds

$$= \log_2 3^{M/2}$$

$$= \frac{M}{2} \log_2 3$$

$$\therefore \text{bit-rate} = \frac{\frac{M}{2}\log_2 3}{MT} = \frac{\log_2 3}{2T}.$$

$$\therefore \eta_{c\,scheme} = \frac{\log_2 3/2T}{2W} = \frac{\log_2 3}{2}$$

$$\approx 0.8 \text{ bps/Hz}.$$

iii) We derive a tight upper bound on the error probability of the new scheme,

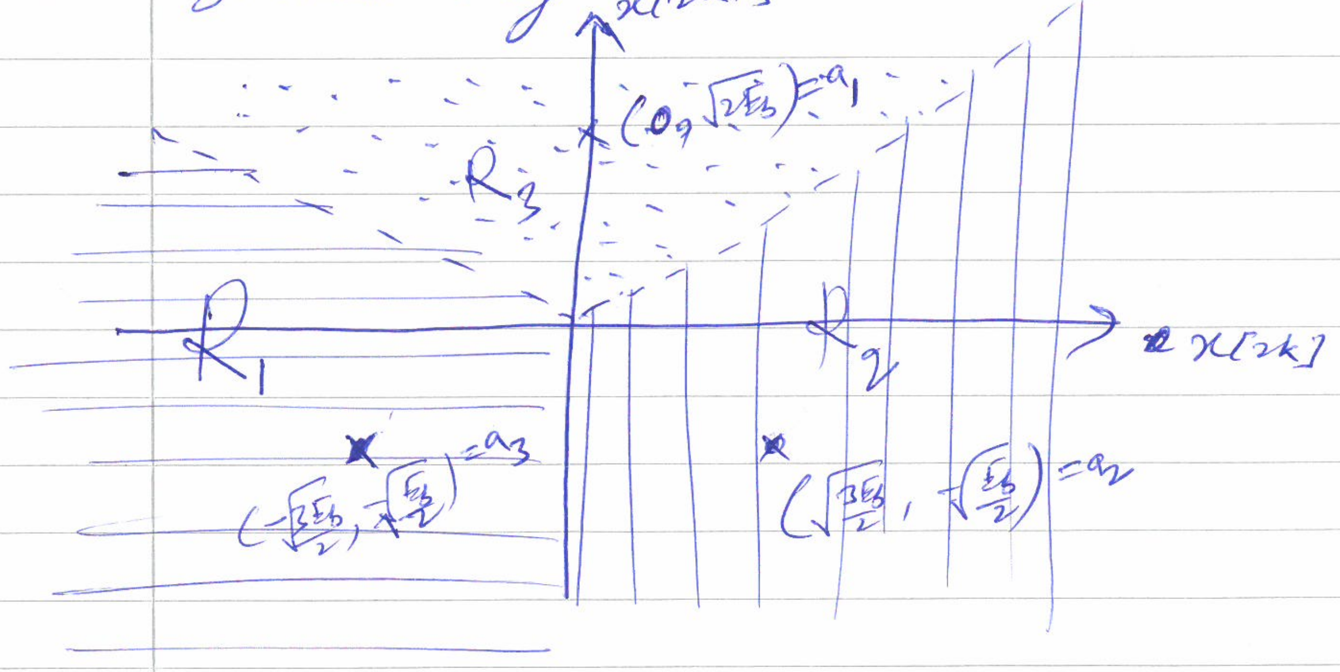
The MPE detector is given by

$$\left(\hat{x}[2k], \hat{x}[2k+1]\right) = \underset{(x[2k], x[2k+1]) \in A}{\arg\min} \; \left(z[2k+1] - x[2k+1]\right)^2 + \left(z[2k] - x[2k]\right)^2$$

where A is the 2-dimensional alphabet set

$$A = \Big\{ \underbrace{\left(0, \sqrt{2E_b}\right)}_{a_1}, \underbrace{\left(-\sqrt{\tfrac{3E_b}{2}}, -\sqrt{\tfrac{E_b}{2}}\right)}_{a_2}, \underbrace{\left(\sqrt{\tfrac{3E_b}{2}}, -\sqrt{\tfrac{E_b}{2}}\right)}_{a_3} \Big\}$$

Since the symbols are two-dimensional, it helps to visualize the signals geometrically



The MPE detector, given the received 2-dimensional signal $(z[2k], z[2k+1]) = \underline{z}[k]$ it ~~looks for~~ decides in favour of the transmitted symbol (~~....~~ in $\mathcal{A}$), which is closest to $(z[2k], z[2k+1])$ in Euclidean distance.

Therefore the whole R^2 can be divided into decision regions R_1, R_2 and R_3.

If the received vector ~~is in~~ lies in R_1 the detected transmit symbol is $\left(-\sqrt{\frac{3E_b}{2}}, \sqrt{\frac{E_b}{2}}\right)$. Similarly if the received vector lies in R_2, the detected transmit symbol is $\left(\sqrt{\frac{3E_b}{2}}, -\sqrt{\frac{E_b}{2}}\right)$.

Prob of Error

$$= \sum_{i=1}^{3} \text{Prob}\left((\hat{x}[2k], \hat{x}[2k+1]) \neq a_i \,\middle|\, (x[2k], x[2k+1]) = a_i\right) \text{Prob}\left((x[2k], x[2k+1]) = a_i\right)$$

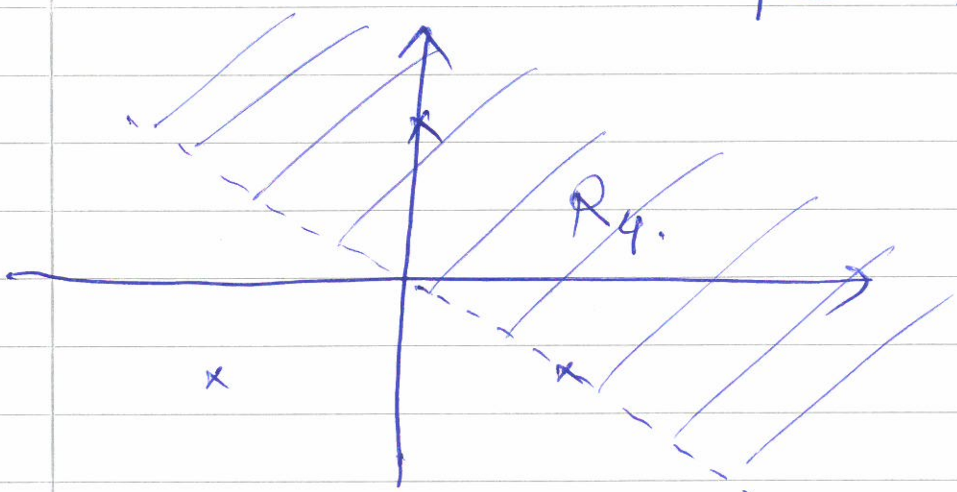
Due to symmetry all the three terms in the above summation are the same therefore

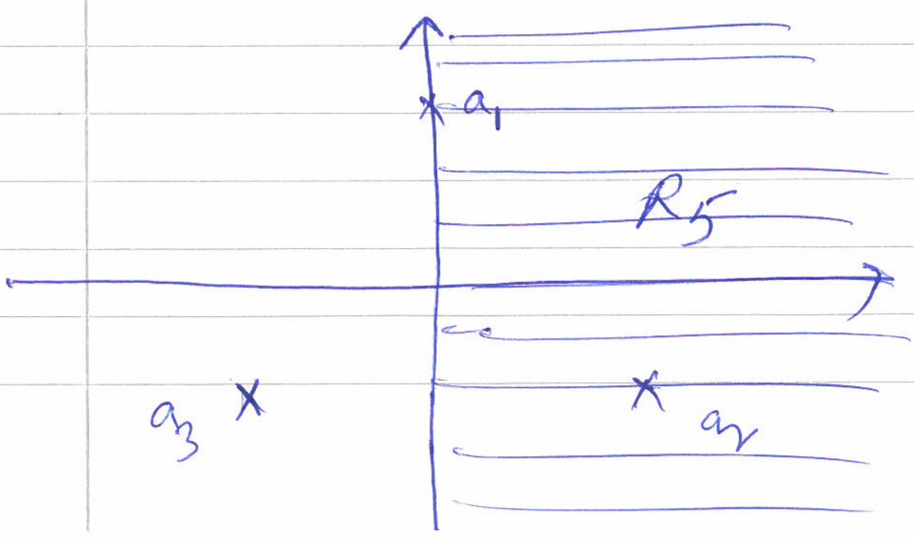
$$\text{prob of Error} = \text{prob}\left((\hat{x}[2k], \hat{x}[2k+1]) \neq a_3 \;\middle|\; (x[2k], x[2k+1]) = a_3 \right)$$

$$= \text{prob}\left((\hat{x}[2k], \hat{x}[2k+1]) = a_1 \text{ or } (\hat{x}[2k], \hat{x}[2k+1]) = a_2 \;\middle|\; (x[2k], x[2k+1]) = a_3 \right)$$

$$\cancel{= \text{prob}\left((N^I[2k], N^I[2k+1]) \notin R_1 \right.}$$

$$= \text{prob}\left((Z^I[2k], Z^I[2k+1]) \in R_2 \cup R_3 \;\middle|\; (X[2k], X[2k+1]) = a_3 \right)$$





We observe that

$$R_2 \cup R_3 \quad \cancel{\quad} \quad R_4 \cup R_5$$

Therefore

$$\text{Prob of Error} = \text{prob}\left((z^I[2k], z^I[2k+1]) \in R_4 \cup R_5 \;\middle|\; (x^I[2k], x^I[2k+1]) = a_3 \right)$$

$$\leq \text{prob}\left((z^I[2k], z^I[2k+1]) \in R_4 \;\middle|\; (x^I[2k], x^I[2k+1]) = a_3 \right)$$

$$+ \text{prob}\left((z^I[2k], z^I[2k+1]) \in R_5 \;\middle|\; (x^I[2k], x^I[2k+1]) = a_3 \right)$$

Note that $\text{prob}\left((z^I[2k], z^I[2k+1]) \in R_4 \;\middle|\; (x^I[2k], x^I[2k+1]) = a_3 \right)$

is the pairwise error probability between a_3 and a_1 given that a_3 is transmitted. This is because R_4 is exactly the region of vectors which are closer to a_1 as compared to a_3.

We next find this pairwise error probability between a_1 and a_3 given that a_3 is transmitted.

Since a_3 is transmitted,

$$Z^I[2k] = -\left(\sqrt{\frac{3E_b}{2}}\right)\frac{1}{2} + N^I[2k],$$

$$Z^Q[2k+1] = \left(-\sqrt{\frac{E_b}{2}}\right)\frac{1}{2} + N^I[2k+1].$$

For the pairwise error to happen

$$\left(Z^I[2k] - \frac{a_3(1)}{2}\right)^2 + \left(Z^I[2k+1] - \frac{a_3(2)}{2}\right)^2 \geq \left(Z^I[2k] - \frac{a_1(1)}{2}\right)^2 + \left(Z^I[2k+1] - \frac{a_1(2)}{2}\right)^2$$

where $a_3(1)$ and $a_3(2)$ are the first and second components of a_3.

Combining the above two equations, we have

$$N^{I^2}[2k] + N^{I^2}[2k+1] \geq \left(N^I[2k] - \frac{1}{2}\sqrt{\frac{3E_b}{2}}\right)^2 + \left(N^I[2k+1] - \frac{1}{2}\left(\sqrt{2} + \frac{1}{\sqrt{2}}\right)\sqrt{E_b}\right)^2$$

i.e.,

$$\left(\sqrt{6E_b}\, N^I[2k] + 3\sqrt{2E_b}\, N^I[2k+1]\right) \geq 6E_b.$$

or $\left(\sqrt{6}\, N^I[2k] + 3\sqrt{2}\, N^I[2k+1]\right) \geq 6\sqrt{E_b}$

(22)

i.e.,

$$\left(\sqrt{\frac{3}{2}}\, N^I[2k] + 3\sqrt{\frac{1}{2}}\, N^I[2k+1]\right) \geq \frac{6\sqrt{E_b}}{4},$$

Since $N^I[2k]$ and $N^I[2k+1]$ are i.i.d. Gaussian r.v.,

Variance of $\left(\sqrt{\frac{3}{2}}\, N^I[2k] + \frac{3}{\sqrt{2}} N^I[2k+1]\right)$

$$= \sigma^2 W \left(3/2 + 9/2\right)$$

$$= 6\sigma^2 W.$$

$\therefore$ Prob of pairwise Error between a_1 and a_3 given that a_3 is txmt.

$$= Q\left(\sqrt{\frac{36 E_b}{16 \times 6 \sigma^2 W}}\right)$$

$$= Q\left(\sqrt{\frac{36 E_b}{96 \sigma^2 W}}\right) = Q\left(\sqrt{\frac{3 E_b}{8\sigma^2 W}}\right)$$

~~For the simple BPSK scheme with same average transmitted power as that for this 2-dimensional scheme, the prob of error~~

Similarly the pairwise Error probability between a_2 and a_3 given that a_3 is txmt is also

$$Q\left(\sqrt{\frac{3 E_b}{8\sigma^2 W}}\right).$$

$\therefore$ Prob of Error

$$\leq \text{Prob}\left((z^I[2k], z^I[2k+1]) \in R_4 \mid a_3 \text{ is transmitted} \right)$$

$$+ \text{Prob}\left((z^I[2k], z^I[2k+1]) \in R_5 \mid a_3 \text{ is transmitted} \right)$$

$$= 2\,Q\left(\sqrt{\frac{\cancel{3}\,E_b}{8\sigma^2 W}}\right).$$

For the simple BPSK scheme with the same average transmitted power as this 2-dimensional scheme, we know that the error probability is $Q\left(\sqrt{\frac{E_b}{4\sigma^2 W}}\right)$.

At high signal to noise ratio, i.e., high E_b/σ^2,

$$p_{2\text{-dim}}(E_b) = 2\,Q\left(\sqrt{\frac{3E_b}{8\sigma^2 W}}\right) \cancel{\approx 2e^{-\frac{3E_b}{16\sigma^2 W}}}$$

and $p_{bpsk}(E_b) = Q\left(\sqrt{\frac{E_b}{4\sigma^2 W}}\right) \cancel{\approx e^{-\frac{E_b}{8\sigma^2 W}}}$

~~where we have used the approximation that~~

Since $Q(x) \leq e^{-x^2/2}$ for $x >> 1$.

at high $E_b/\sigma^2 W$, Infact $Q(x)$ goes

~~$p_{bpsk} \leq$~~ exponentially down to zero

as $x \to \infty$. Therefore for sufficiently large $\frac{E_b}{\sigma^2 W}$,

$$p_{2\text{-dim}} < p_{bpsk}$$ (for same $\langle P \rangle$, i.e., $\langle P \rangle_{bpsk} = \langle P \rangle_{2\text{-dim}}$)

On the other hand for the same prob of error, i.e.,

if $p_{2\text{-dim}}(E_{b_{2\text{-dim}}}) = p_{bpsk}(E_{b_{bpsk}})$

or $$2\,Q\left(\sqrt{\frac{3E_{b_{2\text{-dim}}}}{8\sigma^2 W}}\right) = Q\left(\sqrt{\frac{E_{b_{bpsk}}}{4\sigma^2 W}}\right)$$

since $Q(x) \to 0$ exponentially as $x \to \infty$, for small error probabilities,

$$\frac{3E_{b_{2\text{-dim}}}}{8\sigma^2 W} = \frac{E_{b_{bpsk}}}{4\sigma^2 W}$$

or $$E_{b_{2\text{-dim}}} = \frac{2}{3} E_{b_{bpsk}}$$, i.e.,

the 2-dimensional scheme is more power efficient than the bpsk scheme by $10\log_{10}(3/2) = 1.7$ dB.

solution

~~LECTURE-8 (OFDM)~~

Q.3 Consider an I.S.I channel with additive noise.

Consider the following linear modulation scheme, where $(0 \leq t \leq T_0)$

$$x_p(t) = \sum_{k=-M'}^{M'} \left(x^I[k] \cos 2\pi\left(f_c + \frac{k}{T_0}\right)t - x^Q[k] \sin 2\pi\left(f_c + k/T_0\right)t \right)$$

where $x[k]$ are the complex information bearing symbols.

These type of schemes are called Multicarrier schemes, since there are carriers of different frequencies $\left[\left(f_c + \frac{k}{T_0}\right) \quad k = -M', \ldots, M'\right.$, in the Example above$]$

Since in this example ~~on~~ above the different carriers are almost orthogonal, i.e., for $m \neq k$

$$\int_0^{T_0} \cos 2\pi\left(f_c + \frac{k}{T_0}\right)t \cos 2\pi\left(f_c + \frac{m}{T_0}\right)t \, dt$$

$$= \frac{1}{2} \int_0^{T_0} \left[\cos 2\pi \frac{(m-k)}{T_0}t + \cos 2\pi\left(2f_c + \frac{m+k}{T_0}\right)t\right] dt$$

$$= \frac{1}{2} \left[\sin 2\pi\left(2f_c + \frac{m+k}{T_0}\right)t\right]_0^{T_0} \times \frac{1}{2\pi\left(2f_c + \frac{m+k}{T_0}\right)}$$

$$= \frac{1}{4\pi}\left(\sin 4\pi f_c T_0\right) \times \frac{1}{\left(2f_c + \frac{m+k}{T_0}\right)} \simeq 0 \text{ (in practice)}$$

Due to the orthogonality of the carriers, these scheme is referred to as orthogonal frequency division multiplexing. (OFDM).

let $h_p(t)$ be the impulse response of the ISI channel.
Then the output / received signal is given by

$$y_p(t) = h_p(t) * x_p(t) + n_p(t)$$

$$= \sum_{k=-M}^{M-1} \Big[x^I[k] \int h_p(\tau) \cos 2\pi (f_c + \tfrac{k}{T_0})(t-\tau)\, d\tau$$

$$- x^Q[k] \int h_p(\tau) \sin 2\pi (f_c + \tfrac{k}{T_0})(t-\tau)\, d\tau \Big]$$

$$+ n_p(t)$$

Note that

$$x^I[k] \int h_p(\tau) \cos 2\pi (f_c + \tfrac{k}{T_0})(t-\tau)\, d\tau$$

$$- x^Q[k] \int h_p(\tau) \sin 2\pi (f_c + \tfrac{k}{T_0})(t-\tau)\, d\tau$$

$$= Re\left\{ (x^I[k] + j x^Q[k]) \int h_p(\tau) e^{j 2\pi (f_c + \frac{k}{T_0})(t-\tau)} d\tau \right\}$$

$$= Re\left\{ (x^I[k] + j x^Q[k]) e^{j 2\pi (f_c + \frac{k}{T_0}) t} \int h_p(\tau) e^{-j 2\pi (f_c + \frac{k}{T_0})\tau} d\tau \right\}$$

$$\approx Re\left\{ (x^I[k] + j x^Q[k]) e^{j 2\pi (f_c + k/T_0) t} H_p(f = f_c + k/T_0) \right\}$$

where

$$H_p(f) \triangleq \int h_p(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

is the Fourier transform of $h_p(t)$.

$$\cancel{y_p(t) = \sum_{k=-M'}^{M'} [x^I[k]\, H_p(f_c}$$

~~Similarly~~,

$$y_p(t) = \sum_{k=-M'}^{M'} \Big[x^I[k]\, H_p(f_c + \tfrac{k}{T_0}) \cos 2\pi (f_c + \tfrac{k}{T_0}) - x^Q[k]\, H_p(f_c + \tfrac{k}{T_0}) \sin 2\pi (f_c + \tfrac{k}{T_0}) \Big] + n_p(t).$$

Since $\left\{ \cos 2\pi (f_c + k/T_0), -\sin 2\pi (f_c + k/T_0) \right\}_{k=-M'}^{M'}$

is a sequence of orthogonal waveforms.

$$y^I[m] \triangleq \int_0^{T_0} y_p(t) \cos 2\pi (f_c + \tfrac{m}{T_0}) t \, dt$$

$$\approx \frac{T_0}{2} x^I[m]\, H_p(f_c + \tfrac{m}{T_0}) + \int_0^{T_0} n_p(t) \cos 2\pi (f_c + \tfrac{m}{T_0}) t \, dt.$$

and

$$y^Q[m] \triangleq \int_0^{T_0} -y_p(t) \sin 2\pi (f_c + \tfrac{m}{T_0}) t \, dt$$

$$\approx \frac{T_0}{2} x^Q[m]\, H_p(f_c + m/T_0) + \int_0^{T_0} n_p(t) - \sin 2\pi (f_c + \tfrac{m}{T_0}) t \, dt.$$

therefore due to the orthogonality of the ~~carriers~~ multiple carriers

we have been able to separate the different information symbols on each carrier.

Also notice that now there is no ISI due to the channel.

$$\therefore\ y[m] \triangleq y^{I}[m] + y^{Q}[m] \qquad m = -M', \ldots, M'$$

$$= \frac{T_0}{2}\left(x^{I}[m] + j x^{Q}[m]\right) H_p\left(f_c + \frac{m}{T_0}\right)$$

$$+ \left(N^{I}[m] + j N^{Q}[m]\right)$$

where $N^{I}[m] \triangleq \int n_p(t) \cos 2\pi\left(f_c + \frac{m}{T_0}\right)t\, dt$

and $N^{Q}[m] \triangleq \int n_p(t) - \sin 2\pi\left(f_c + \frac{m}{T_0}\right)t\, dt$

are i.i.d. Gaussian random variables.

$$\mathrm{Var}\left(N^{I}[m]\right) = \mathrm{Var}\left(N^{Q}[m]\right)$$

$$= \sigma^2 \int \cos^2 2\pi\left(f_c + \frac{m}{T_0}\right)t\, dt$$

$$= \frac{\sigma^2 T_0}{2}.$$

Implementation:

~~Note that~~,

$$x_p(t) = \sum_{k=-M'}^{M'} \Big\{ x^I[k] \cos 2\pi\left(f_c + \frac{k}{T_0}\right)t - x^Q[k] \sin 2\pi\left(f_c + \frac{k}{T_0}\right)t \Big\}$$

$$= Re\left\{ \left(\sum_{k=-M'}^{M'} x[k] e^{-j 2\pi \frac{kt}{T_0}} \right) e^{j 2\pi f_c t} \right\}$$

The equivalent complex baseband transmit signal is therefore

$$x(t) = \sum_{k=-M'}^{M'} x[k] e^{-j 2\pi k t / T_0}$$

where $x[k] = x^I[k] + j x^Q[k]$.

Note that the bandwidth of $x(t)$ is ~~between~~ approximately between $[-W, W]$ where $W = \frac{M'}{T_0}$.

Therefore by the sampling theorem,

$$x(t) = \sum_{r=-M'}^{M'} x\left(\frac{r}{2W}\right) \operatorname{sinc}(2Wt - r)$$

and therefore only the samples ~~are~~ $x\left(\frac{r}{2W}\right)$ are enough.

$$x\left(\frac{r}{2W}\right) = \sum_{k=-M'}^{M'} x[k] e^{-j \frac{2\pi k r}{2W T_0}}$$

$$= \sum_{k=-M'}^{M'} x[k] e^{-j \frac{2\pi r k}{2M'}}$$

which is nothing but the Discrete Fourier transform of the sequence $\{x[k]\}$

Similarly receiver operation can also be reduced to an inverse fourier transform.

Demerits of OFDM :-

- high peak to average power ratio of the transmitted signal, since multicarrier transmission.

- since

$$Y[m] = \frac{T_0}{2} (x[m]) H_p \left(f_c + \frac{m}{T_0}\right) + N[m].$$

the multiplicative factor of $H_p\left(f_c + \frac{m}{T_0}\right)$ can ~~be~~ degrade the performance if it has a very small value. therefore in real world channels, information symbols across the carriers are coded together.

Multicarrier modulation

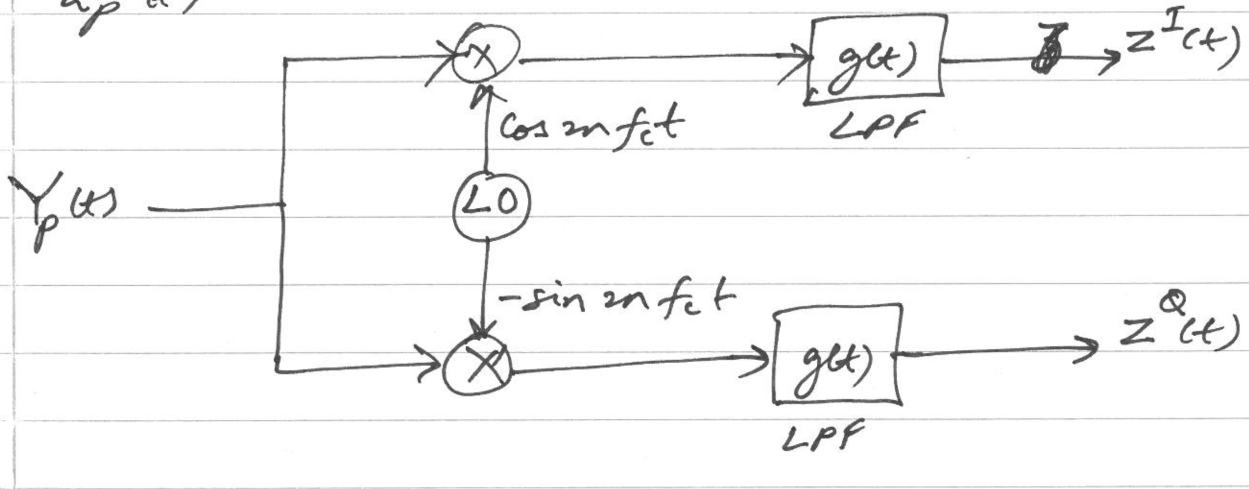
Receiver processing:

We receive

$$Y_p(t) = h_p(t) * \left\{ \sum_{k=-M'}^{M'} Re\left[X[k] e^{j2\pi(f_c + \frac{k}{T_0})t} \right] \right\} + N_p(t)$$

$$= \sum_{k=-M'}^{M'} Re\left\{ X[k] H_p(f_c + \tfrac{k}{T_0}) e^{j2\pi(f_c + \frac{k}{T_0})t} \right\} + N_p(t)$$

where $H_p(f) \triangleq \int h_p(t) e^{-j2\pi ft} dt$ is the Fourier transform of ~~$H_p(f)$~~ $h_p(t)$.



$g(t)$ is the impulse response of the low pass filter (LPF) and has the Fourier transform

$$G(f) = \begin{cases} 1 & \text{if } |f| < \frac{M'}{T_0} \\ 0 & \text{otherwise} \end{cases}$$

It follows that

$$Z(t) \triangleq Z^{I}(t) + j Z^{Q}(t)$$

$$= \frac{1}{2} \sum_{k=-M'}^{M'} X[k]\, e^{j2\pi \frac{kt}{T_0}}\, H_p\left(f_c + \frac{k}{T_0}\right)$$

$$+ N(t)$$

Since $Z(t)$ is band limited to $\left[-\frac{M'}{T_0}, \frac{M'}{T_0}\right]$ it suffices to sample $Z(t)$ at $\frac{2M'}{T_0}$ samples per second.

$$Z[r] \triangleq Z\left(\frac{r}{2M'/T_0}\right)$$

$$= \frac{1}{2} \sum_{k=-M'}^{M'} X[k]\, e^{j2\pi \frac{kr}{2M'}}\, H_p\left(f_c + \frac{k}{T_0}\right)$$

$$+ \underbrace{N[r]}_{\triangleq N\left(\frac{r}{2M'/T_0}\right)}$$

Since the input is time-limited to $[0, T_0]$ it suffices to sample the output also for a time interval of length approximately T_0, say

$r = 0$ to $r = \frac{2M'}{T_0}$.

Then recovering $\{X[k],\ k = -M', \ldots, M'\}$ from $\{Z[r]\}$ is equivalent to a discrete Fourier transform (DFT).