Capacity of band-limited AWGN channels

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Consider the following set of complex orthogonal waveforms which are approximately time limited to $[0, T_0]$ and approximately band-limited to $[-W, W]$

$$\mathcal{G} = \{ \phi_1(t), \ldots, \phi_M(t) \}, \quad t \in [0, T_0].$$

$$\int_0^{T_0} \phi_i(t)\, \phi_k^*(t)\, dt = \begin{cases} 0 & \text{if } i \neq k \\ 1 & i = k \end{cases}$$

~~One~~, and also that

$$\Phi_m(f) = \int_0^{T_0} \phi_m(t)\, e^{-j2\pi f t}\, dt = 0 \text{ for } |f| > W. \quad (1)$$

One example of such an orthonormal set is

$$\mathcal{G} = \left\{ \phi_m(t) = \frac{1}{\sqrt{T_0}} e^{j2\pi \frac{mt}{T_0}}, \; m = -WT_0, \ldots, WT_0 \right\}$$

We consider linear modulation of the form

$$x(t) = \sqrt{2} \sum_{k=1}^{M} x[k]\, \phi_k(t), \quad x[k] \text{ are complex.} \quad (2)$$

Let the corresponding passband signal be given by

$$x_p(t) = \mathrm{Re}\left(x(t)\, e^{j2\pi f_c t} \right). \quad (3)$$

where f_c is the carrier frequency.

Note that the information is conveyed /communicated using the complex symbols $X[k]$, $k = 1, 2, \ldots, M$.

We know that the number of complex orthonormal waveforms of approximate bandwidth occupancy $[-W, W]$ and time-limited to $[0, T_0]$ is roughly $2WT_0$. Therefore let us choose

$$M = 2WT_0 .$$

There is usually an average power constraint on the ~~transmitted~~ ~~signal~~ passband signal $x_p(t)$ given by.

$$\frac{1}{T_0}\int_0^{T_0} x_p^2(t)\,dt \le P \, , \quad (5)$$

~~which is same as~~

We know that

$$\int_0^{T_0} x_p^2(t)\,dt = \frac{1}{2}\int_0^{T_0} |x(t)|^2 dt$$

$$= \frac{1}{2}\int_0^{T_0} \left|\left\{\sqrt{2}\sum_{k=1}^{M} X[k]\,\phi_k(t)\right\}\right|^2 dt$$

(using equation (2))

$$= \sum_{k_1=1}^{M}\sum_{k_2=1}^{M}\left\{\int_0^{T_0}\phi_{k_1}(t)\,\phi_{k_2}^*(t)\,dt\right\} X[k_1]\,X^*[k_2]$$

$$= \sum_{k=1}^{M} |X[k]|^2$$

and therefore the average power constraint in (4) is equivalent to

$$\sum_{k=1}^{M} |X[k]|^2 \le P T_0 \quad \text{—— (5)}$$

Since the channel is AWGN the received passband signal is given by

$$Y_p(t) = X_p(t) + N_p(t) \quad \text{—— (6)}$$

Let the two sided P.S.D of $N_p(t)$ be σ^2.

Using (2), ~~and~~ (3) and (6) we have the optimal receiver as

~~$Y_p(t)$~~

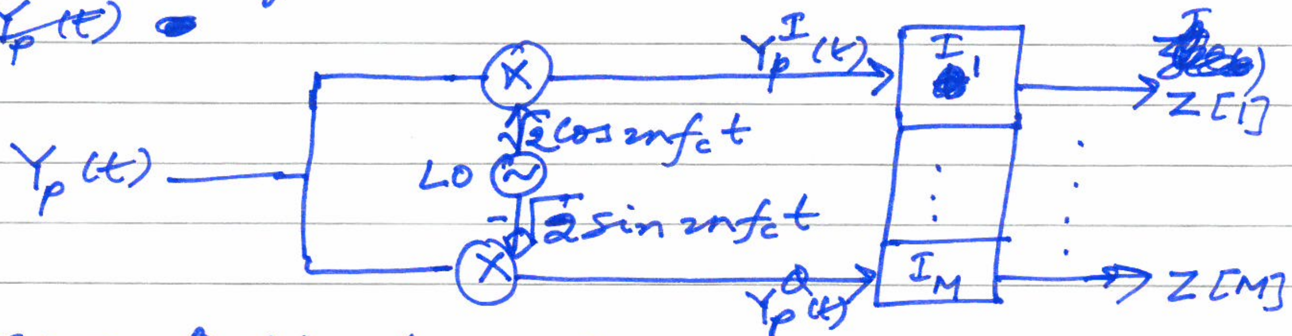


Fig. 1 Optimal receiver.

In the figure $I_1, \ldots, I_M$ is a bank of M integrators, each integrator/correlator taking as input $Y_p^I(t), Y_p^Q(t)$ and generating a complex scalar as output. The output of the m-th integrator is given by

$$Z[m] = \int_0^{T_0} \left(Y_p^I(t) + j Y_p^Q(t)\right) \phi_m^*(t)\, dt \quad \text{—— (7)}$$

$$m = 1, \ldots, M.$$

Equation 7 can be rewritten as

$$z[m] = \int_0^{T_0} \sqrt{2}\, Y_p(t)\, e^{-j2\pi f_c t}\, \phi_m^*(t)\, dt.$$

$$= \sqrt{2}\int_0^{T_0} X_p(t)\, e^{-j2\pi f_c t}\, \phi_m^*(t)\, dt$$

$$+ \sqrt{2}\int_0^{T_0} N_p(t)\, e^{-j2\pi f_c t}\, \phi_m^*(t)\, dt.$$

$$= \frac{\sqrt{2}}{2}\int_0^{T_0} \left(x(t)\, e^{j2\pi f_c t} + x^*(t)\, e^{-j2\pi f_c t} \right) e^{-j2\pi f_c t}\, \phi_m^*(t)\, dt$$

$$+ \sqrt{2}\int_0^{T_0} N_p(t)\, e^{-j2\pi f_c t}\, \phi_m^*(t)\, dt$$

$$= \frac{1}{\sqrt{2}}\int_0^{T_0} x(t)\, \phi_m^*(t)\, dt + \sqrt{2}\int_0^{T_0} N_p(t)\, e^{-j2\pi f_c t}\, \phi_m^*(t)\, dt$$

using 2 we have

$$= \sum_{k=1}^{M} X[k] \int_0^{T_0} \phi_k(t)\, \phi_m^*(t)\, dt + \sqrt{2}\int_0^{T_0} N_p(t)\, e^{-j2\pi f_c t}\, \phi_m^*(t)\, dt$$

$$= X[m] + N[m], \quad m = 1, 2, \ldots M \qquad \text{— (8)}$$

$$N[m] \triangleq \cancel{\sqrt{2}\int_0^{T} N_p(t) \cos 2\pi f_c t\, \phi_m^*(t)\, dt}$$

$$\cancel{-j\sqrt{2}\int_0^{T_0} N_p(t) \sin 2\pi f_c t\, \phi_m^*(t)\, dt}$$

$$= N^I[m] + jN^Q[m].$$

Both $N^I[m]$ and $N^Q[m]$ are gaussian random variables with mean zero and same variance. They are also independent of each other.

Further

$$Var(N^I[m]) = Var(N^Q[m])$$

$$N^I[m] = \sqrt{2}\int_0^{T_0} N_p(t)\{\cos 2\pi f_c t\, \phi_m^I(t) - \sin 2\pi f_c t\, \phi_m^Q(t)\}\, dt.$$

where $\phi_m(t) = \phi_m^I(t) + j\phi_m^Q(t)$.

$$\therefore Var(N^I[m]) = \sigma^2 \int_0^{T_0}\{\sqrt{2}(\cos 2\pi f_c t\, \phi_m^I(t) - \sin 2\pi f_c t\, \phi_m^Q(t))\}^2 dt$$

$$= 2\sigma^2\Big[\int_0^{T_0} \phi_m^{I^2}(t)\cos^2 2\pi f_c t\, dt + \int_0^{T_0}\phi_m^{Q^2}(t)\sin^2 2\pi f_c t\, dt - \int_0^{T_0}\phi_m^I(t)\phi_m^Q(t)\sin 4\pi f_c t\, dt\Big]$$

Since $\phi_m(t)$ is band limited to $[-W, W]$

we have

$$Var(N^I[m]) = \sigma^2\int_0 |\phi_m(t)|^2 dt = \sigma^2. \quad -(9)$$

Therefore we have from (8) and (9)

$$Z[m] = X[m] + N[m], \quad m = 1, 2, .., M$$

where $N[m]$ is complex gaussian with variance σ^2 per dimension.

In vector notation,

$$\underline{Z} = \underline{X} + \underline{N}$$

$$\underline{Z}, \underline{X}, \underline{N} \in \mathbb{R}^{2M}$$

where $\underline{X} \triangleq [X^I[1], X^Q[1], \ldots\ldots, X^I[M], X^Q[M]]^T$

$\underline{Z} \triangleq [Z^I[1], Z^Q[1], \ldots, Z^I[M], Z^Q[M]]^T$, and

$\underline{N} \triangleq [N^I[1], N^Q[1], \ldots, N^I[M], N^Q[M]]^T$.

and $X[m] = X^I[m] + j X^Q[m]$, $m = 1, .., M$.

$Z[m] = Z^I[m] + j Z^Q[m]$, $m = 1, .., M$,

$$N[m] = N^I[m] + j N^Q[m]. \quad \text{--- (10)}$$

The power constraint in eqn (5) translates to

$$\sum_{k=1}^{M} |X[k]|^2 = \sum_{k=1}^{M} (X^I[k]^2 + X^Q[k]^2)$$

$$= \|\underline{X}\|_2^2 \leq P T_0. \quad \text{--- (11)}$$

Equation (10) can be equivalently written as

$$\frac{\underline{Z}}{\sqrt{T_0}} = \frac{\underline{X}}{\sqrt{T_0}} + \frac{\underline{N}}{\sqrt{T_0}}, \text{ or.}$$

$$\underline{Z}' = \underline{X}' + \underline{N}' \quad \text{--- (12).}$$

$\underline{Z}' = \underline{Z}/\sqrt{T_0}$, $\underline{X}' = \underline{X}/\sqrt{T_0}$, $\underline{N}' = \underline{N}/\sqrt{T_0}$

The norm in (11) gives us

$$\|\underline{X}'\|_2^2 \leq P \quad \text{--- (13)}$$

From (9) it follows that variance of each component of $\underline{N}'$ is $\frac{\sigma^2}{T_0}$, and that they are independent of each other.

$$\|\underline{N}'\|^2 = \sum_{k=1}^{2M} N'^2[k] = \frac{\sum_{k=1}^{2M} N^2[k]}{T_0}$$

$$= 4W \frac{\sum_{k=1}^{2M} N^2[k]}{4WT_0} = 4W \left[\frac{\sum_{k=1}^{2M} N^2[k]}{2M} \right]$$

where $N[k]$ is the k-th component of

$\underline{N} = [N^I[1], N^Q[1], \ldots, N^I[M], N^Q[M]]^T.$

The central limit theorem states that for a sequence of i.i.d. r.v.'s $\{u_n\}_{n=1}^{\infty}$, and the sequence of averages $\{U_n\}_{n=1}^{\infty}$ with

$U_n \triangleq \frac{\sum_{m=1}^{n} u_m}{n}$, the theorem states that

as $n \to \infty$, $U_n \longrightarrow \mu$ (where μ is the mean of the i.i.d. r.v.'s $\{u_n\}_{n=1}^{\infty}$).

Using the central limit theorem we have

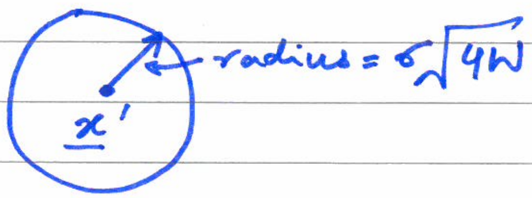
$$\frac{\sum_{k=1}^{2M} N^2[k]}{2M} \xrightarrow[M\to\infty]{} \mathbb{E}\left(N^2[k]\right) = \mathrm{Var}\left(N[k]\right) = \sigma^2$$

$\therefore$ for very large M,

$$\|\underline{N}'\|^2 \approx 4\sigma^2 W \qquad \text{— (14)}.$$

From (12), (13) ~~and~~ (14) we have that given a transmit vector $\underline{x}'$, the received random vector $\underline{z}'$ ($\underline{z}' = \underline{x}' + \underline{N}'$) lies inside a sphere (in $\mathbb{R}^{2M}$) ~~of radius approximately~~ centered at $\underline{x}'$ and of radius approximately $\|\underline{z}' - \underline{x}'\|_2 = \|\underline{N}'\|_2 \approx \sigma\sqrt{4W}$

We shall refer to this sphere as the "noise sphere".



$$\text{noise sphere } S(\underline{x}') = \left\{ \underline{v} \in \mathbb{R}^{2M} \;\middle|\; \|\underline{v} - \underline{x}'\| \le \sigma\sqrt{4W} \right\} \qquad \text{— (15)}$$

Since $\underline{Z}' = \underline{x}' + \underline{N}'$

for some transmitted vector $\underline{x}'$, we have

$$\|\underline{Z}'\|^2 = \|\underline{x}' + \underline{N}'\|^2$$
$$= \|\underline{x}'\|^2 + \|\underline{N}'\|^2 + 2\,\underline{x}'^T \underline{N}'$$

It can also be shown that

- $\underline{x}'^T \underline{N}' \xrightarrow[M\to\infty]{} 0$

$$\therefore \|\underline{Z}'\|^2 \approx \|\underline{x}'\|^2 + \|\underline{N}'\|^2$$
$$\lesssim P + 4\sigma^2 W.$$

For low error probabilities we must choose transmit vectors such that their corresponding noise spheres are non-overlapping.

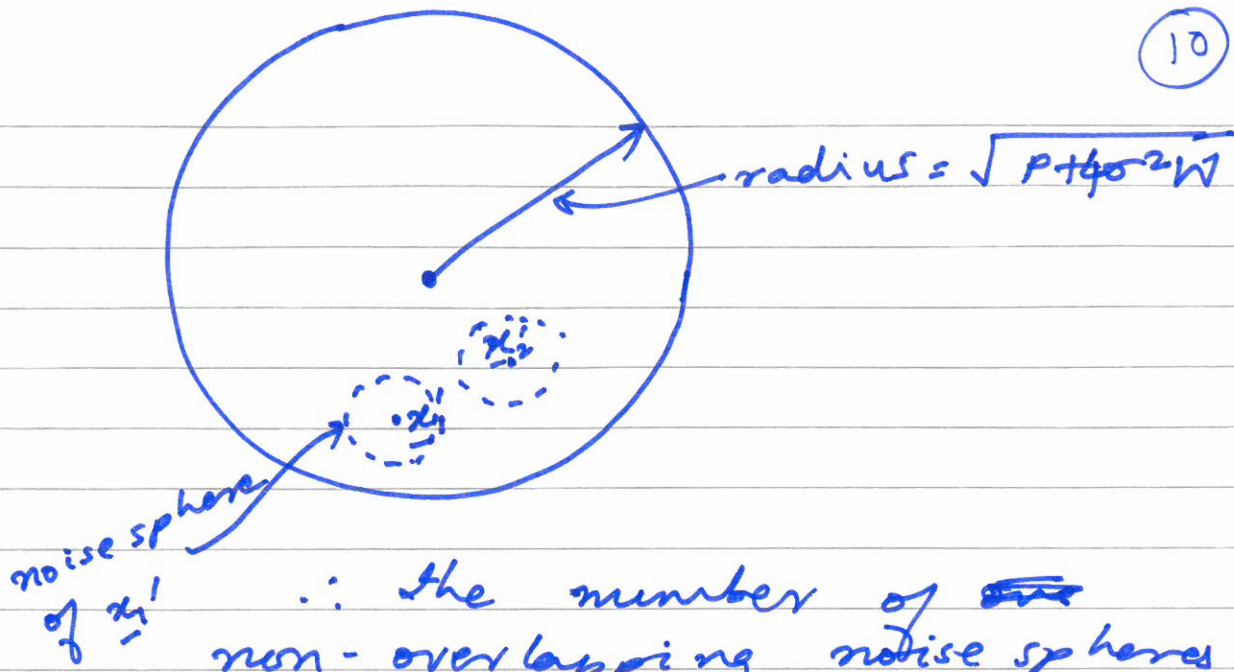
In the received space $\mathbb{R}^{2M}$, the received vector $\|\underline{Z}'\|^2 \le P + 4\sigma^2 W$ with high probability and therefore the received vector lies in a sphere in $\mathbb{R}^{2M}$ of radius $\sqrt{P + 4\sigma^2 W}$.

The volume of this sphere $= \alpha\left(\sqrt{P + 4\sigma^2 W}\right)^{2M}$

where $\alpha \in \mathbb{R}$ is a constant.

The volume of the noise sphere

$$= \cancel{\alpha\left(\sqrt{\sigma^2}\right)^{2M}}$$
$$= \alpha\left(\sqrt{4\sigma^2 W}\right)^{2M}$$



$\therefore$ the number of ~~one~~ non-overlapping noise spheres

$$K \simeq \frac{\alpha\left(\sqrt{P+4\sigma^2 W}\right)^{2M}}{\alpha\left(\sqrt{4\sigma^2 W}\right)^{2M}} \quad (\text{as } M \to \infty)$$

which means that there are approximately K transmit vectors with non-overlapping spheres.

$\therefore$ no. of information bits $= \log_2 K$

$$= M \log_2\left(1 + \frac{p}{4\sigma^2 W}\right)$$

$\therefore$ bit-rate $= \dfrac{\log_2 K}{T_0}$

$$C = 2W \log_2\left(1 + \frac{p}{4\sigma^2 W}\right) \text{ bits/sec}$$

(since $M = 2WT_0$)

or $$C = B \log_2\left(1 + \frac{p}{2\sigma^2 B}\right) \text{ b/sec}$$

where $B \triangleq 2W$ is the total bandwidth occupied.

let $\sigma'^2 \triangleq 2\sigma^2$ be the

one-sided p.s.d of AWGN, then

$$C = B \log_2 \left(1 + \frac{P}{\sigma'^2 B}\right) \text{ b/s.} \quad \text{— (16)}$$

C information bits are transmitted in 1sec.

let E_b be the energy of each information bit. the total energy transmitted by C bits is

therefore $C E_b$ joules in one second.

therefore the average transmitted power is

$$P = C E_b.$$

$$\therefore \frac{C}{B} = \log_2 \left[1 + \left(\frac{C}{B}\right) \cdot \left(\frac{E_b}{\sigma'^2}\right)\right] \text{ b/s/Hz}$$

Note that $\eta = \frac{C}{B}$ is the spectral efficiency

i.e., $$\eta = \log_2 \left[1 + \frac{\eta E_b}{\sigma'^2}\right] \quad \text{— (17).}$$

~~Infact~~ or $$\frac{2^{\eta} - 1}{\eta} = \frac{E_b}{\sigma'^2}.$$

Channel Coding Theorem:

For any given arbitrary probability of Error $1 \geq P_e > 0$, any real number $\epsilon > 0$ and any given E_b/σ^2, there Exists a code (a set of transmit vectors/waveforms occupying bandwidth B) with average Error probability P_e, energy per information bit equal to E_b and spectral efficiency $\eta - \epsilon$ where η satisfies $\frac{2^\eta - 1}{\eta} = E_b/\sigma^2$.

(Note that there is no constraint on the code length or time duration T_0 of the transmit waveforms)

In general, for ϵ closer to zero or smaller P_e, a larger T_0 is necessary.