TSKS04 Digital Communication Continuation Course

Solutions for the exam 2015-03-16

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Answer:

This is a two-dimensional situation in which the PSD is given by

$$R_s(f) = \frac{1}{T} \begin{pmatrix} \Psi_1^*(f) & \Psi_2^*(f) \end{pmatrix} \begin{pmatrix} R_{S_1,S_1}[fT] & R_{S_2,S_1}[fT] \\ R_{S_1,S_2}[fT] & R_{S_2,S_2}[fT] \end{pmatrix} \begin{pmatrix} \Psi_1(f) \\ \Psi_2(f) \end{pmatrix}.$$

Since the symbols are equally probable and independent, we get

$$\begin{aligned} R_{S_1,S_1}[fT] &= R_{S_2,S_2}[fT] = \frac{P}{2} \\ R_{S_2,S_1}[fT] &= R_{S_1,S_2}[fT] = 0. \end{aligned}$$

Notice that the total signal variance P is divided equally between the two dimensions.

Moreover, direction computation of Fourier transforms give

$$\begin{aligned} \Psi_1(f) &= j\frac{T}{2} e^{-j\pi f T} \left(e^{-j\pi f_c T} \operatorname{sinc}((f+f_c)T) - e^{+j\pi f_c T} \operatorname{sinc}((f-f_c)T) \right) \\ \Psi_2(f) &= \frac{T}{2} e^{-j\pi f T} \left(e^{-j\pi f_c T} \operatorname{sinc}((f+f_c)T) + e^{+j\pi f_c T} \operatorname{sinc}((f-f_c)T) \right). \end{aligned}$$

(See A.3.1 in the extra course material for details).

By multiplying everything together, according to the formula above, we get

$$\begin{aligned} R_s(f) &= \frac{1}{T}\frac{P}{2}\left(\frac{T}{2}\right)^2 \left(\left| e^{-j\pi f_c T} \operatorname{sinc}((f+f_c)T) - e^{+j\pi f_c T} \operatorname{sinc}((f-f_c)T) \right|^2 \right. \\ &\quad \left. + \left| e^{-j\pi f_c T} \operatorname{sinc}((f+f_c)T) + e^{+j\pi f_c T} \operatorname{sinc}((f-f_c)T) \right|^2 \right) \\ &= \frac{PT}{4} \left(\operatorname{sinc}^2((f+f_c)T) + \operatorname{sinc}^2((f-f_c)T) \right). \end{aligned}$$

2

Answer:

The different questions can be answered independently.

a. As mentioned on page 218, ZF requires that $L > K - 1$. This can also be expressed as $L \geq K$. Also for these cases, it can happen that ZF does not exist because $\mathbf{U}^H\mathbf{U}$ is non-invertible. A trivial example is when $\mathbf{U}$ is an all zero matrix.

It is fine to just give an example of a $\mathbf{U}$ where $\mathbf{U}^H\mathbf{U}$ is non-invertible.

It also does not exist when $\mathbf{u}_0 = \mathbf{0}$.

b. Whenever $\mathbf{P}_I^\perp \mathbf{u}_0 = \mathbf{u}_0$, ZF will both cancel the ISI (if there is any) and not suffer from noise amplification. This occurs, for example, when $\mathbf{u}_0$ is orthogonal to the other columns of $\mathbf{U}$. (The desired signal in Figure 5.9 is orthogonal to the interference subspace.) This one way to motivate optimality.

Alternatively, if minimizing the MSE is the optimal thing, then ZF is optimal when it equals MMSE (with happens as said in (d)).

c. The noise enhancement factor is as largest when ZF and MF are equal, which gives a factor 1.

d. By comparing (5.42) for MMSE and (5.30) for ZF, one can see that these conditions are equal when $\mathbf{C_w} = \mathbf{0}$. In other words, when there is no additive noise.

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Answer:

Part (b) can be solved independently from (a) and (c).

a. Let $\mathbf{s}_i$ for $i = 1, \ldots, 8$ denote the 8 signal points in the figure. These can be numbered in any way, but one possibility is

$$
\begin{aligned}
\mathbf{s}_1 &= (-A\ A)^T \\
\mathbf{s}_2 &= (0\ A)^T \\
\mathbf{s}_3 &= (A\ A)^T \\
\mathbf{s}_4 &= (-A\ 0)^T \\
\mathbf{s}_5 &= (A\ 0)^T \\
\mathbf{s}_6 &= (-A\ -A)^T \\
\mathbf{s}_7 &= (0\ -A)^T \\
\mathbf{s}_8 &= (A\ -A)^T.
\end{aligned}
$$

Theorem 3.4.1 defines the hypotheses in (3.29) and provides the ML decision rules in (3.30). In particular, one can obtain the log-likelihood ratios for each hypothesis:

$$H_i : \frac{1}{\sigma^2}\left(< \mathbf{y}, \mathbf{s}_i > - \frac{\|\mathbf{s}_i\|^2}{2} \right).$$

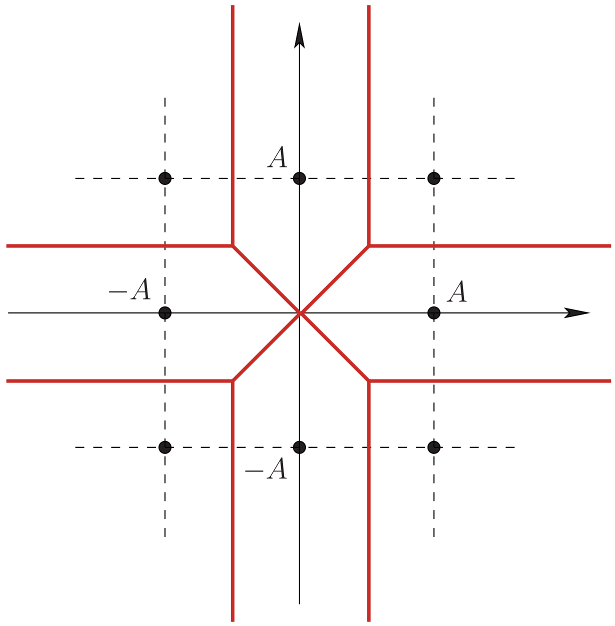
One picks the hypothesis with the highest value.

Alternatively, one can pick the hypothesis that minimizes the distance:

$$H_i : \quad \|\mathbf{y} - \mathbf{s}_i\|$$

These can decision rules are proved using the technique outlined in the book.

b. The ML decision regions are given by the red lines:



As pointed out in Theorem 3.4.1, these regions are based on the "minimum distance rule".

c. The log-likelihoods are computed as described above. The values become

$$
\begin{aligned}
H_1 &= -38/15\frac{A^2}{\sigma^2}\\
H_2 &= -51/30\frac{A^2}{\sigma^2}\\
H_3 &= -56/30\frac{A^2}{\sigma^2}\\
H_4 &= -25/30\\
H_5 &= -1/6\frac{A^2}{\sigma^2}\\
H_6 &= -2/15\frac{A^2}{\sigma^2}\\
H_7 &= 21/30\frac{A^2}{\sigma^2}\\
H_8 &= 8/15\frac{A^2}{\sigma^2}.
\end{aligned}
$$

The highest value is given by H_7, which is $(0 \ \ -A)^T$.

4

Answer:

a. The sampled autocorrelation sequence is defined in (5.6) as

$$h[m] = \int p(t)p^*(t - mT)dt.$$

From the problem formulation we see that $T = 1$.

We now get

$$h[0] = \int p(t)p^*(t)dt = 2^2 + (1/2)^2 + (-1/2)^2 = 4.5$$

$$h[\pm 1] = \int p(t)p^*(t \mp 1)dt = 2 \times 1/2 + (1/2) \times (-1/2) = 3/4$$

$$h[\pm 2] = \int p(t)p^*(t \mp 2)dt = 2 \times (-1/2) = -1$$

$$h[\pm k] = \int p(t)p^*(t \mp 3)dt = 0$$

for $k \geq 3$.

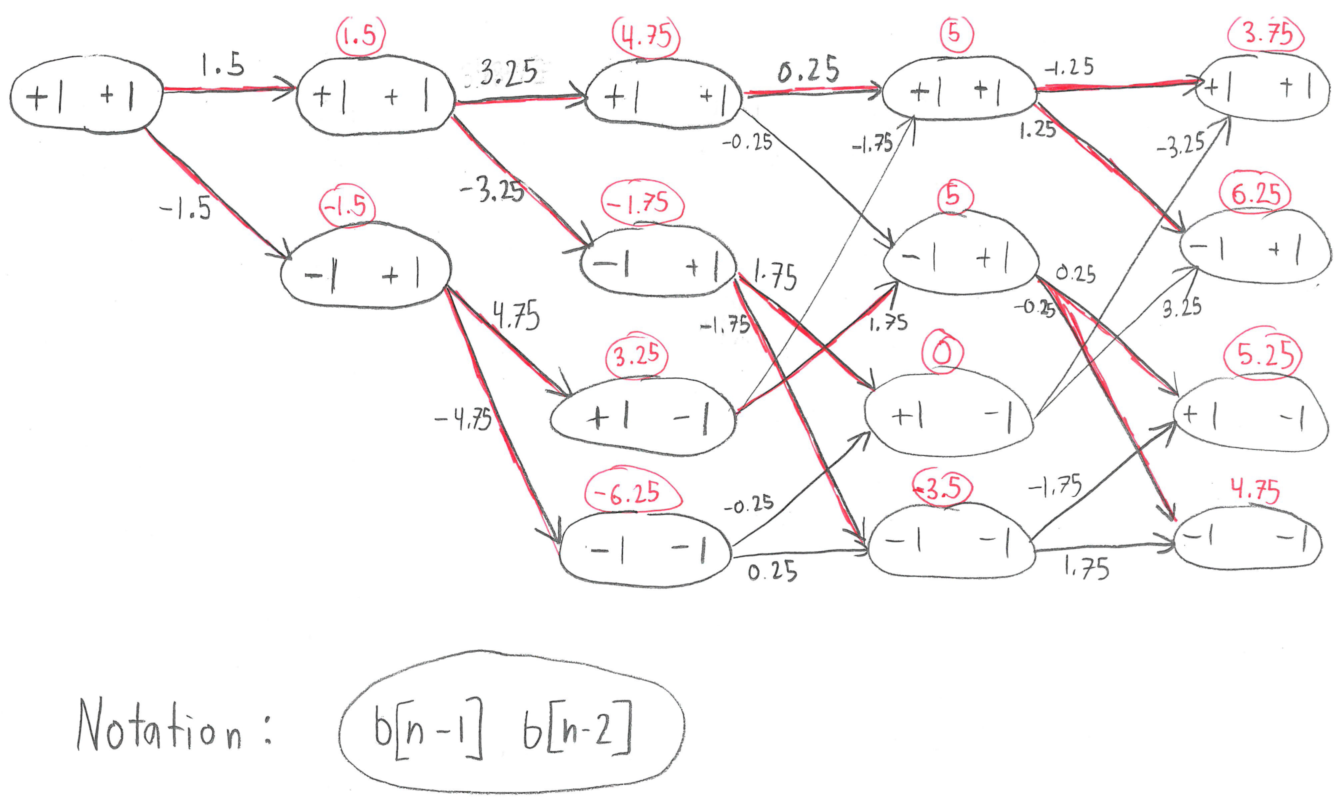
b. The memory is $L = 2$ since this is the largest value on m for which $h[\pm m] \neq 0$.

c. The branch metric (5.13) becomes

$$\lambda_n(b[n], (b[n-1], b[n-2])) = b[n]z[n] - 2.25 - b[n]\frac{3}{4}b[n-1] + b[n]b[n-2] \tag{1}$$

for the values on $h[m]$ above, since everything is real-valued and $|b[n]| = 1$ for BPSK.

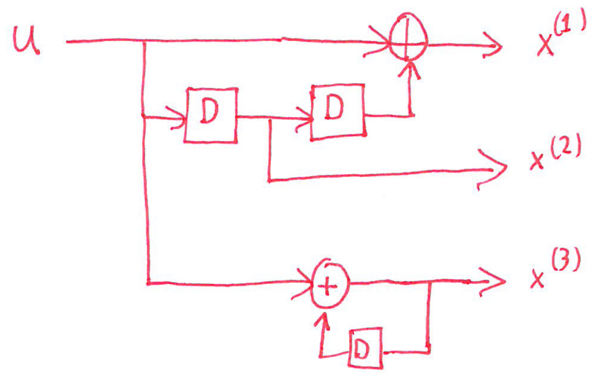
For simplicity, we neglect the constant -2.25. See the finalized Viterbi algorithm below. If we terminate the algorithm by selecting the state with the highest end probability, then the bit sequence will be $b[1] = 1$, $b[2] = 1$, $b[3] = 1$, $b[1] = -1$.



5

Answer:

a. The encoder can look like this:



b. This input sequence can be expressed in D-transform as $U(D) = D + D^3$. The codeword will then be

$$U(D)G(D) = \begin{pmatrix} D + D^5 & D^2 + D^4 & \frac{D+D^3}{1+D} \end{pmatrix} = \begin{pmatrix} D + D^5 & D^2 + D^4 & D + D^2. \end{pmatrix}$$

The second equality follows from the fact that

$$\frac{D + D^3}{1 + D} = D \sum_{i=0}^{\infty} D^i + D^3 \sum_{j=0}^{\infty} D^j = D + D^2$$

since the two summations cancel out each other term of order D^3 or larger.

Hence, the first output is $x^1 = (0\,1\,0\,0\,0\,1,\ 0,\ 0, \ldots)$, the second output is $x^2 = (0\,0\,1\,0\,1\,0,\ 0,\ 0, \ldots)$, and the third output $x^3 = (0\,1\,1\,0\,0\,0,\ 0,\ 0, \ldots)$.

c. Yes, this is possible. For example, the input $u = (1\,0\,0, \ldots)$ will give $x^3 = (1,\ 1,\ \ldots)$.

The input has Hamming weight 1, but the output has infinite Hamming weight since x^3 has that.