TSKS01 Digital Communication

Solutions to Extra Tasks for Tutorial Session 3

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1a. This problem is best solved using the quick-glance method (after a quick glance you realize the solution). Here this solution is given with slightly more words. The signals $s_1(t)$ and $s_2(t)$ are orthogonal, and the signal $s_3(t)$ can obviously not be written as a linear combination of $s_1(t)$ and $s_2(t)$. Thus, those signals span three dimensions, and the obvious ON-basis

$$\begin{aligned}\phi_1(t) &= 1, \quad 0 \le t < 1,\\ \phi_2(t) &= 1, \quad 1 \le t < 2,\\ \phi_3(t) &= 1, \quad 2 \le t < 3,\end{aligned}$$

can be used. The corresponding vectors are then

$$\overline{s_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \overline{s_2} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \overline{s_3} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

b. Let d_{12} be the distance between the signals $s_1(t)$ and $s_2(t)$. Then we have the quadratic distance

$$\begin{aligned} d_{12}^2 &= d^2(s_1, s_2) \;=\; ||s_1 - s_2||^2 \\ &= \int_0^3 (s_1(t) - s_2(t))^2 \; dt \\ &= \int_0^1 (1-1)^2 \; dt \;+\; \int_1^2 (1-(-1))^2 \; dt \\ &+ \int_2^3 (0-0)^2 \; dt \;=\; 4. \end{aligned}$$

Furthermore, let d_{13} be the distance between the signals $s_1(t)$ and $s_3(t)$. Then we get

$$\begin{aligned} d_{13}^2 &= d^2(s_1, s_3) \;=\; ||s_1 - s_3||^2 \\ &= \int_0^3 (s_1(t) - s_3(t))^2 \; dt \\ &= \int_0^1 (1-(-1))^2 \; dt \;+\; \int_1^2 (1-(-1))^2 \; dt \\ &+ \int_2^3 (0-(-1))^2 \; dt \;=\; 9. \end{aligned}$$

Finally, let d_{23} be the distance between the signals $s_2(t)$ and $s_3(t)$. We get

$$\begin{aligned} d_{23}^2 &= d^2(s_2, s_3) \;=\; ||s_2 - s_3||^2 \\ &= \int_0^T (s_1(t) - s_2(t))^2 \; dt \\ &= \int_0^1 (1-(-1))^2 \; dt \;+\; \int_1^2 (-1-(-1))^2 \; dt \\ &+ \int_2^3 (0-(-1))^2 \; dt \;=\; 5. \end{aligned}$$

Obviously, d_{12} is the smalles of these distances. Thus, we have the minimum distance

$$d_{\min} = d_{12} = 2.$$

c. We see that d_{13} is the biggest distance among those determined in b. Thus, we have the maximum distance

$$d_{\max} = d_{13} = 3.$$

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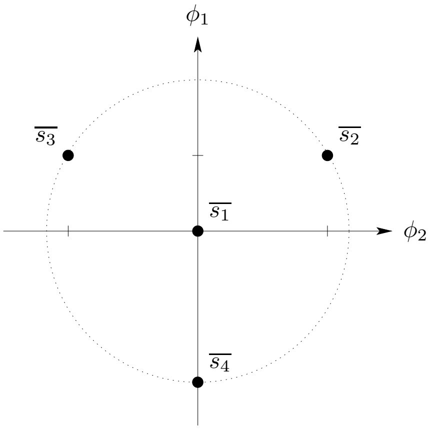
Of course this problem can be solved in several ways - in principal infinitely many ways. Here is one solution.

a. First we find a set of vectors that fulfill the given demands:

$$\begin{aligned} \overline{s_1} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \overline{s_2} &= \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \\ \overline{s_3} &= \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} \\ \overline{s_4} &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{aligned}$$

Graphically:



Then, we need an ON-basis, e.g.

$$\phi_1(t) = \begin{cases} 1/\sqrt{T}, & 0 \le t < T, \\ 0, & \text{för övrigt}, \end{cases}$$

$$\phi_2(t) = \begin{cases} -1/\sqrt{T}, & 0 \le t < T/2, \\ 1/\sqrt{T}, & T/2 \le t < T, \\ 0, & \text{för övrigt}. \end{cases}$$

This gives us the signals

$$\begin{aligned} s_1(t) &= 0 \\ s_2(t) &= \frac{\sqrt{3}}{2}\phi_1(t) + \frac{1}{2}\phi_2(t) \\ &= \begin{cases} \frac{\sqrt{3}-1}{2\sqrt{T}}, & 0 \le t < T/2, \\ \frac{\sqrt{3}+1}{2\sqrt{T}}, & T/2 \le t < T, \end{cases} \\ s_3(t) &= \frac{\sqrt{3}}{2}\phi_1(t) - \frac{1}{2}\phi_2(t) \\ &= \begin{cases} \frac{\sqrt{3}+1}{2\sqrt{T}}, & 0 \le t < T/2, \\ \frac{\sqrt{3}-1}{2\sqrt{T}}, & T/2 \le t < T, \end{cases} \\ s_4(t) &= -\phi_2(t) = \begin{cases} \frac{1}{\sqrt{T}}, & 0 \le t < T/2, \\ -\frac{1}{\sqrt{T}}, & T/2 \le t < T. \end{cases} \end{aligned}$$

b. The three non-zero signals are on the unit circle. The smallest distance in this signal constellation is the distance between $\overline{s_1}$ and any of the other signals. That distance is 1, which is most easily realized based on the vectors.

The maximum energy is the largest energy among the signals, and the energy of a signal is the quadratic length of the corresponding vector. $s_1(t)$ has therefore energy 0, while all other signals have energy 1. The maximum energy is thus 1.

c. One signal has energy 0 an all other signals have energy 1. Under the assumption that the signals are equally probable, we get the mean energy

$$E = \frac{1}{4}(0 + 1 + 1 + 1) = \frac{3}{4}$$

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We have the two signals

$$\begin{aligned} s_1(t) &= \cos\left(\frac{\pi t}{T}\right), \quad 0 \le t < T, \\ s_2(t) &= \cos\left(\frac{\pi t}{2T}\right), \quad 0 \le t < T. \end{aligned}$$

a. The squared lengths are given by

$$\begin{aligned} ||s_1||^2 &= \int_0^T s_1^2(t)\ dt = \int_0^T \cos^2\left(\frac{\pi t}{T}\right)\ dt \\ &= \frac{1}{2}\int_0^T \left(1 + \cos\left(\frac{2\pi t}{T}\right)\right)\ dt = \frac{T}{2} \\ ||s_2||^2 &= \int_0^T s_2^2(t)\ dt = \int_0^T \cos^2\left(\frac{\pi t}{2T}\right)\ dt \\ &= \frac{1}{2}\int_0^T \left(1 + \cos\left(\frac{\pi t}{T}\right)\right)\ dt = \frac{T}{2} \end{aligned}$$

Thus, we have the lengths

$$\begin{aligned} ||s_1|| &= \sqrt{T/2} \\ ||s_2|| &= \sqrt{T/2} \end{aligned}$$

b. The angle α between the signals is given by the relation

$$\cos\alpha = \frac{(s_1, s_2)}{||s_1|| \cdot ||s_2||}$$

We have already determined the lengths $||s_1||$ and $||s_2||$. What is left is the inner product (s_1, s_2), given by

$$\begin{aligned} (s_1, s_2) &= \int_0^T s_1(t)s_2(t)\ dt \\ &= \int_0^T \cos\left(\frac{\pi t}{T}\right)\cos\left(\frac{\pi t}{2T}\right)\ dt \\ &= \frac{1}{2}\int_0^T \left(\cos\left(\frac{3\pi t}{2T}\right) + \cos\left(\frac{\pi t}{2T}\right)\right)\ dt \\ &= \frac{2T}{3\pi} \end{aligned}$$

Thus we have

$$\cos\alpha = \frac{2T/3\pi}{T/2} = \frac{4}{3\pi},$$

which gives us

$$\alpha \approx 1.13 \text{ rad} \approx 65^\circ$$

c. The quadratic distance between the signals are given by the relation

$$d^2(s_1, s_2) = ||s_1 - s_2||^2 = \int_0^T (s_1(t) - s_2(t))^2 \, dt$$

This can of course be solved the usual way, but let us study the integrand separately:

$$(s_1(t) - s_2(t))^2 = s_1^2(t) + s_2^2(t) - 2s_1(t)s_2(t)$$

Thus, we can rewrite the integral as

$$d^2(s_1, s_2) = ||s_1||^2 + ||s_2||^2 - 2(s_1, s_2),$$

which actually is the cosine theorem. Plugging in the values from above, we get

$$d^2(s_1, s_2) = T \cdot \left(1 - \frac{4}{3\pi}\right),$$

And the distance is

$$d(s_1, s_2) = \sqrt{T \cdot \left(1 - \frac{4}{3\pi}\right)} \approx 0.76\sqrt{T}.$$