TSKS04 Digital Communication Continuation Course

Solutions for the exam 2015-06-08

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Answer:

This is a one-dimensional situation in which the PSD is given by

$$R_s(f) = \frac{1}{T}|\Phi_1(f)|^2 R_S[fT]$$

where $\Phi_1(f)$ is the Fourier transform of the basis function $\phi_1(t)$ and $R_S[fT]$ is the PSD of the signal constellation.

The PSD can be computed as

$$R_S[fT] = \sigma_S^2 + m_S^2 \sum_m \delta(fT - m)$$

where the variance is $\sigma_S^2 = P$ according to the problem formulation. The mean value for on-off keying can be shown to be $\sqrt{P}$. This gives

$$R_S[fT] = P + P\sum_m \delta(fT - m).$$

Moreover, direction computation of Fourier transform gives

$$\Phi_1(f) = \frac{T}{2} e^{-j\pi f T}\left(\underbrace{e^{-j\pi f_c T}}_{=j^{-2f_cT}} \operatorname{sinc}((f+f_c)T) + \underbrace{e^{+j\pi f_c T}}_{=j^{2f_cT}} \operatorname{sinc}((f-f_c)T)\right)$$

where we can utilize that $e^{-j\pi f_c T} = j^{2f_cT}$ since $2f_cT$ is an integer. (See A.3.1 in the extra course material for details).

By multiplying everything together, according to the formula above, we get

$$\begin{aligned} R_s(f) &= \frac{T}{4}\left|e^{-j\pi f T}\left(j^{-2f_cT}\operatorname{sinc}((f+f_c)T) + j^{2f_cT}\operatorname{sinc}((f-f_c)T)\right)\right|^2 \left(P + P\sum_m \delta(fT-m)\right) \\ &= \frac{PT}{4}\left(\operatorname{sinc}((f+f_c)T) + (-1)^{2f_cT}\operatorname{sinc}((f-f_c)T)\right)^2 \left(1 + \sum_m \delta(fT-m)\right) \end{aligned}$$

One can continue to simplify this expression by using properties of the δ function.

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Answer:

The different questions can be answered independently.

a) Since we consider proper complex Gaussian random variables, as in Section 4.5.1, the PDF is $\frac{e^{-|y|^2/(N_0+A^2)}}{\pi(N_0+A^2)}$ for the case "1 sent" and $\frac{e^{-|y|^2/N_0}}{\pi N_0}$ for the case "0 sent".

Let us find the interval where "1 sent" is more probable than "0 sent":

$$
\begin{aligned}
\frac{e^{-|y|^2/(N_0+A^2)}}{\pi(N_0+A^2)} &\geq \frac{e^{-|y|^2/N_0}}{\pi N_0} && \Leftrightarrow \\
e^{|y|^2\left(\frac{1}{N_0}-\frac{1}{N_0+A^2}\right)} &\geq \frac{N_0+A^2}{N_0} && \Leftrightarrow \\
|y|^2\left(\frac{1}{N_0}-\frac{1}{N_0+A^2}\right) &\geq \ln\left(\frac{N_0+A^2}{N_0}\right) && \Leftrightarrow \\
|y|^2 &\geq \frac{N_0(N_0+A^2)}{A^2}\ln\left(1+\frac{A^2}{N_0}\right)
\end{aligned}
$$

This threshold only depends on $|y|^2$.

c) In this case, "1 sent" corresponds to $y = 3j/2 + n$ and "0 sent" corresponds to $y = 0 + n$, where n is the noise.

The ML detection rule corresponds to finding which of the mean values $3j/2$ and 0 from the hypotheses that is closest to the received signal $1 + j$ in terms of Euclidean distance.

The hypothesis "1 sent" gives $|3j/2 - (1 + j)| = \sqrt{5}/2 \approx 1.1180$ while the hypothesis "0 sent" gives $|1 + j| = \sqrt{2} \approx 1.4142$. Since "1 sent" has the shortest distance, this is the ML estimate.

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Answer:

Hypothesis H_0 gives the PDF

$$f_0(z) = \begin{cases} 1/4 & |z| \leq 2 \\ 0 & |z| > 2 \end{cases}$$

and hypothesis H_1 gives the PDF

$$f_1(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}}$$

of the observed signal.

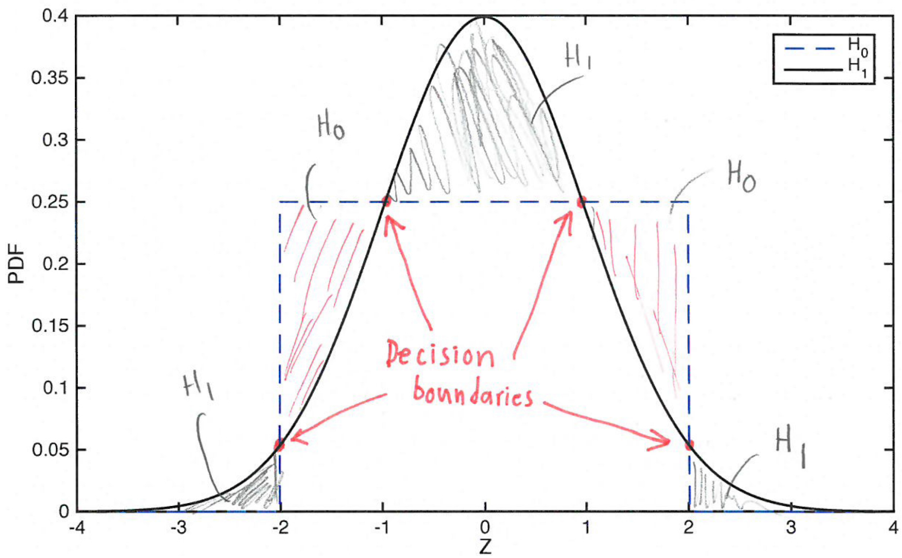
Hypothesis H_1 is more probable than H_0 when $f_0(z) \leq f_1(z)$. This is necessarily the case for $|z| > 2$, since $f_0(z) = 0$, thus one decision boundary is at $z = \pm 2$.

Now looking at the range $|z| \leq 2$, direct computation yields

$$\frac{e^{-z^2/2}}{\sqrt{2\pi}} \geq \frac{1}{4} \quad \Leftrightarrow \quad e^{-z^2/2} \geq \frac{\sqrt{2\pi}}{4} \quad \Leftrightarrow \quad z^2 \leq 2\ln\left(\frac{4}{\sqrt{2\pi}}\right) = \ln\left(\frac{8}{\pi}\right),$$

thus the hypotheses are equal at $z = \pm\sqrt{\ln\left(\frac{8}{\pi}\right)} \approx 0.9668$, which are the other two decision boundaries.

The decision boundaries and PDFs are illustrated as follows:



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Answer:

a. The sampled autocorrelation sequence is defined in (5.6) as

$$h[m] = \int p(t)p^*(t - mT)dt.$$

From the problem formulation we see that $T = 1$.

We now get

$$h[0] = \int p(t)p^*(t)dt = 2^2 + (-1)^2 + 1^2 = 6$$

$$h[\pm 1] = \int p(t)p^*(t \mp 1)dt = 2 \cdot (-1) + (-1) \cdot 1 = -3$$

$$h[\pm 2] = \int p(t)p^*(t \mp 2)dt = 2 \cdot 1 = 2$$

$$h[\pm k] = \int p(t)p^*(t \mp 3)dt = 0$$

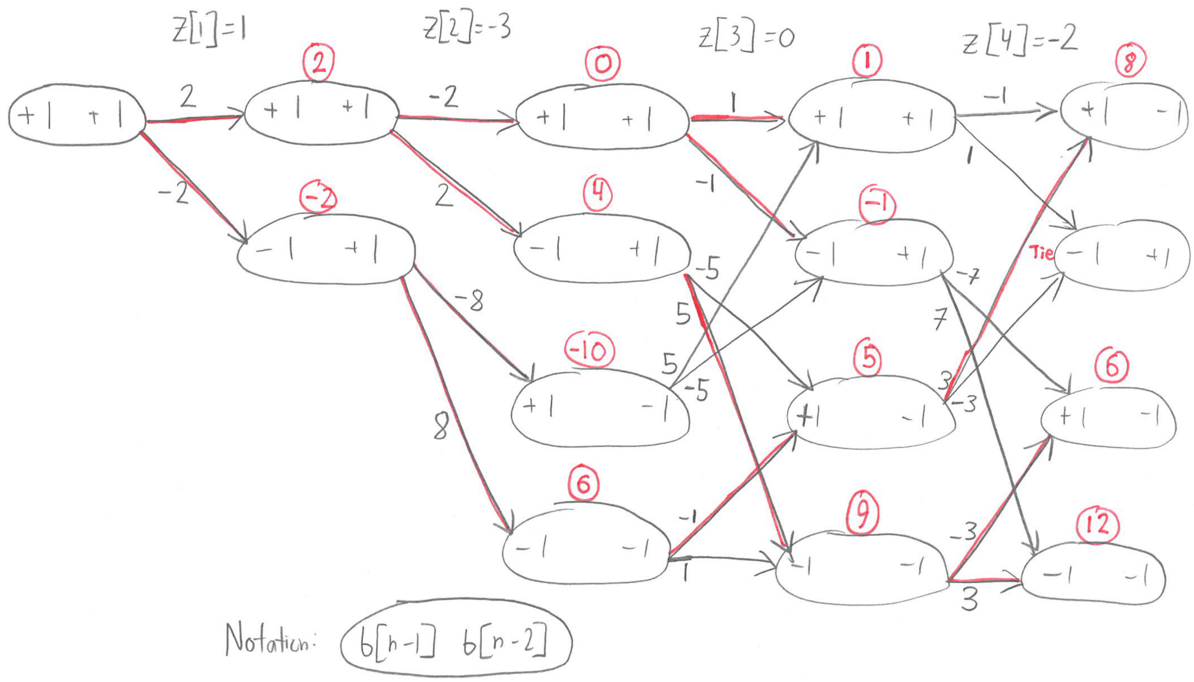
for $k \geq 3$.

b. The branch metric (5.13) becomes

$$\lambda_n(b[n], (b[n-1], b[n-2])) = b[n]z[n] - 3 + 3b[n]b[n-1] - 2b[n]b[n-2] \tag{1}$$

for the values of $h[m]$ computed above, since everything is real-valued and $|b[n]| = 1$ for BPSK.

For simplicity, we neglect the constant -3. See the finalized Viterbi algorithm below. When we terminate the algorithm by selecting the state with the highest end probability, we pick the red-marked path that ends with "12". The corresponding bit sequence is $b[1] = 1$, $b[2] = -1$, $b[3] = -1$, $b[4] = -1$.



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Answer:

Since this is a problem from the book, and none of the students who took the exam tried to solve the problem, we do not provide any detailed solution here.