Solutions to Selected Problems

– from UMadhow: Fundamentals of Digital Communication –

Saif K. Mohammed

INSTITUTE OF TECHNOLOGY
LINKÖPING UNIVERSITY

Department of Electrical Engineering
Linköpings universitet, SE-581 83 Linköping, Sweden

Linköping 2012

Note: This material is prepared for the 2012 version of the Master course TSKS04 Digital Communication Continuation Course. For almost every task planned for tutorials you find either hints and answers or complete solutions. For the tasks where we give complete solutions, we have chosen to adjust the notations to concur with the notation that has been used in the lectures.
This document will evolve during the course as the tutorials go by.
This version of the document was compiled February 22, 2012.

Solutions to Selected Problems – from UMadhow: Fundamentals of Digital Communication

Department of Electrical Engineering
Linköpings universitet
SE-581 83 Linköping
Sweden

This document was prepared using LaTeX2ε with the aid of TeXnicCenter on an Dell PC running CentOS 5. The figures were produced using Xfig (from xfig.org). Finally, the plots were produced using Matlab (from MathWorks, Inc.).

Contents

Tutorial 5

Task 4.1

Task

Fill in the details for the amplitude estimates in Example 4.2.2 on pages 160-161 by deriving Equations 4.10 and 4.11.

Solution

Case 1: Training sequence

We have a known sequence $b[k]$, for $k = 1, 2, \ldots, K$. The noise $N[k]$ is i.i.d. and Gaussian with mean zero and variance σ^2. The PDF is therefore given by

$$f_{Y[k]|A}(y[k]|a) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \, e^{-\frac{(y[k]-ab[k])^2}{2\sigma^2}}$$

for sample number k. Since the noise is i.i.d. we get the total PDF

$$f_{\bar{Y}|A}(\bar{y}|a) = \prod_{k=1}^{K} \frac{1}{\sqrt{2\pi} \cdot \sigma} \, e^{-\frac{(y[k]-ab[k])^2}{2\sigma^2}}$$

We take the natural logarithm of the above, to get a measure that we want to maximize:

$$J(a) = \ln\left(f_{\bar{Y}|A}(\bar{y}|a)\right) = \sum_{k=1}^{K} \left(-\frac{(y[k]-ab[k])^2}{2\sigma^2} - \frac{1}{2}\ln\left(2\pi\sigma^2\right) \right)$$

We note that this is a quadratic function in a, with a negative sign in front of a^2. Thus, we find a maximum by differentiating and setting the derivative to zero.

$$\frac{d}{da} J(a) = \sum_{k=1}^{K} \frac{b[k](y[k]-ab[k])}{2\sigma^2} = 0,$$

from which we get

$$\sum_{k=1}^{K} b[k](y[k]-ab[k]) = 0.$$

For our training sequence, we have $b[k] = \pm 1$, which gives us $b^2[k] = 1$. Then from the equation above, we get

$$\sum_{k=1}^{K} b[k]y[k] - a = 0.$$

Thus,

$$\sum_{k=1}^{K} b[k]y[k] = Ka.$$

Our estimate a_{ML} is supposed to be the a that maximizes $J(a)$, i.e.

$$a_{\mathrm{ML}} = \frac{1}{K} \sum_{k=1}^{K} b[k]y[k],$$

which is what was supposed to be shown.

Case 2: Blind estimation

Now we have an unknown sequence $B[k] = \pm 1$, for $k = 1, 2, \ldots, K$, i.i.d. The noise $N[k]$ is still i.i.d. and Gaussian with mean zero and variance σ^2. According to Equation 4.8, we have the PDF

$$f_{Y[k]|A}(y[k]|a) = e^{-\frac{a^2}{2\sigma^2}} \cosh\left(\frac{ay[k]}{\sigma^2}\right) \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\frac{y^2[k]}{2\sigma^2}}$$

for sample number k. We get the total PDF

$$f_{\bar{Y}|A}(\bar{y}|a) = \prod_{k=1}^{K} e^{-\frac{a^2}{2\sigma^2}} \cosh\left(\frac{ay[k]}{\sigma^2}\right) \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\frac{y^2[k]}{2\sigma^2}}$$

Again, we take the natural logarithm of the above and get a measure to maximize:

$$J(a) = -K\frac{a^2}{2\sigma^2} + \sum_{k=1}^{K} \ln\left[\cosh\left(\frac{ay[k]}{\sigma^2}\right)\right],$$

where we have ignored the last terms that do not depend on a, since they will not affect the maximization. We differentiate and set to zero:

$$\frac{d}{da}J(a) = -K\frac{a}{\sigma^2} + \sum_{k=1}^{K} \frac{y[k]}{\sigma^2} \cdot \frac{\sinh\left(\frac{ay[k]}{\sigma^2}\right)}{\cosh\left(\frac{ay[k]}{\sigma^2}\right)} = -K\frac{a}{\sigma^2} + \sum_{k=1}^{K} \frac{y[k]}{\sigma^2} \tanh\left(\frac{ay[k]}{\sigma^2}\right) = 0.$$

Here we have used the relations

$$\frac{d}{dx}\cosh(x) = \sinh(x), \qquad \frac{d}{dx}\ln(x) = \frac{1}{x}.$$

Our estimate a_{ML} is the solution to the equation above, which we rewrite it as

$$a = \frac{1}{K} \sum_{k=1}^{K} y[k] \tanh\left(\frac{ay[k]}{\sigma^2}\right)$$

which is what was supposed to be shown.

Task 4.2

Hints

a. Start with the likelihood-function of $Z[k]$ given A, θ and $b[k]$. Average with respect to $b[k]$ and simplify the expression. $(cosh(x) = (e^x + e^x)/2)$

b. Use the result from **a** and use the independence of the samples $Z[1], ..., Z[K]$ given A and θ.

c. Use $\cos(\alpha + \pi/2) = -\sin(\alpha)$ and $\sin(\alpha + \pi/2) = -\cos(\alpha)$ in the result from **b**, and use that $\cosh(x)$ is an even function, i.e., $\cosh(-x) = \cosh(x)$. The log likelihood function in **b** remains unchanged. This implies that we can only estimate θ modulo $\pi/2$, and can restrict attention to $\theta \in [0, \pi/2]$.

d. $N[k]$ and $b[k]$ are independent stochastic variables, both with mean zero.

e. It is not reasonable to solve this part analytically. Better to solve it numerically. Matlab or some other similar tool then comes in handy.

Two-step iteration: 1) For fixed A, determine an estimate of θ by maximizing $J(\hat{A}, \theta)$ in the interval $[0, \pi/2)$ via exhaustive search. 2)For fixed θ, get an estimate of A by maximizing $J(A, \hat{\theta})$ with respect to A.

Answer

a. $L(Z[k]|A, \theta) = \frac{1}{2} e^{-\frac{A^2}{2\sigma^2}} \left[\cosh\left(\frac{A}{\sigma^2}\big|Z[k]\big| \cos(\phi_k - \theta)\right) + \cosh\left(\frac{A}{\sigma^2}\big|Z[k]\big| \sin(\phi_k - \theta)\right)\right].$

b. $J(A, \theta) = \ln\left(L(\bar{Z}|A, \theta)\right) =$

$= -K\ln(2) - K\frac{A^2}{2\sigma^2} + \sum_{k=1}^{K} \ln\left[\cosh\left(\frac{A}{\sigma^2}\big|Z[k]\big| \cos(\phi_k - \theta)\right) + \cosh\left(\frac{A}{\sigma^2}\big|Z[k]\big| \sin(\phi_k - \theta)\right)\right].$

c. —

d. $\hat{A}^2 = \max\left\{\frac{1}{K}\sum_{k=1}^{K} \big|Z[k]\big| - 2\sigma^2, 0\right\}$

e. $\hat{A} = 1$ and $\hat{\theta} = 0.06$ rad.

Task 4.4

Hints

a. Start with the usual expression (Eq.(4.20)) for the log-likelihood function for this case and simplify it. Constant terms can be removed, and a common multiplicative constant factor can also be removed. None of them affect the maximization.

b. Rewrite the sum in the answer of **a** in polar form and identify the real part. Maximize the function with respect to θ.

c. Compare the expression from **b** with the definition of the DFT.

Answer

a. $J(\Gamma, \theta) = \mathrm{Re}\left\{e^{-j\theta} \sum_{k=1}^{K} y[k]b^*[k]e^{-j\Gamma k}\right\}$, where constant terms have been removed.

b. $J(\Gamma) = \max_\theta J(\Gamma, \theta) = \left|\sum_{k=1}^{K} y[k]b^*[k]e^{-j\Gamma k}\right|$.

c. Consider a DFT of the sequence $\{y[k]b^*[k]\}$ of length N fulfilling $N \geq K$. The n-th DFT component is then $Z[n] = \left|\sum_{k=1}^{K} y[k]b^*[k]e^{-j2\pi nk/N}\right|$. Let $n_{\max}$ be the n that maximizes $\left|Z[n]\right|$. Then our estimate is $\hat{\Gamma} = \frac{2\pi n_{\max}}{N}$.

<u>**Task 4.11**</u>

Hints

Express conditional PDFs for the output given the two possible sent signals. Use the independence of h and N. For 1 is sent, $Pr(y|1 \text{ sent}) = (1/(4\pi)) \exp(-|y|^2/4)$. For 0 is sent, $Pr(y|1 \text{ sent}) = (1/\pi) \exp(-|y|)$.

(Complex Gaussian density function: $Pr(z) = (1/(2\pi\sigma^2)) \exp(-|z - \mu|^2/(2\sigma^2))$, where μ is the mean of z and σ^2 is the variance per dimension.)

Use Baye's rule to determine the posterior probability.

Answer

$\Pr\{1 \text{ sent}|Y = 1 - j2\} = \frac{1}{1+8e^{-15/4}} \approx 0.842$