TSKS04 Tutorial

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Prob. I.24.

a) Sketch the I & Q waveforms for a typical MSK signal, clearly showing the timing relation between them.

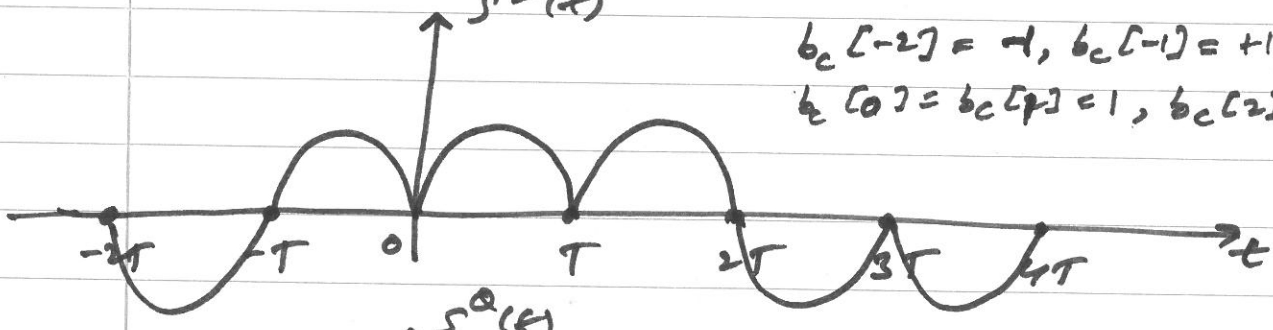
$$s^I(t) = \mathrm{Re}(s(t))$$

$$= \sum_{n=-\infty}^{\infty} b_c[n]\, g_{TX}(t-nT)$$

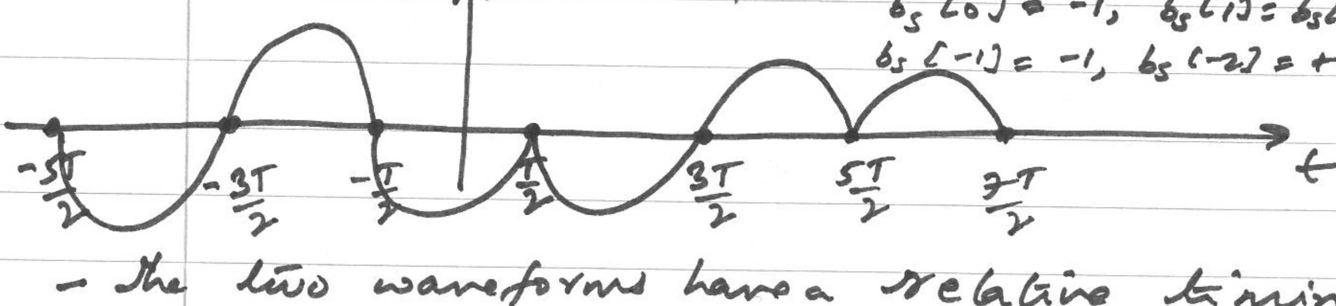
$$= \sum_{n=-\infty}^{\infty} b_c[n] \left[\sin \frac{\pi}{T}(t-nT)\right] I_{[0,T]}(t-nT)$$

$$s^Q(t) = \mathrm{Im}(s(t))$$

$$= \sum_{n=-\infty}^{\infty} b_s[n] \sin \frac{\pi}{T}(t-nT-T/2)\, I_{[0,T]}(t-nT-T/2)$$



$b_c[-2] = -1$, $b_c[-1] = +1$, $b_c[0] = b_c[1] = 1$, $b_c[2] = b_c[3] = -1$.



$b_s[0] = -1$, $b_s[1] = b_s[2] = +1$
$b_s[-1] = -1$, $b_s[-2] = +1$, $b_s[-3] = -1$.

- the two waveforms have a relative timing shift of $\frac{T}{2} = T_b$.

b) Show that the MSK waveform has constant envelope (an extremely desirable property for non linear channels).

We have to show that

$$|s(t)| = 1 \quad \text{for all } t. \quad — ①$$

we prove ① for any time interval $nT \leq t \leq (n+1)T$.

if $nT \leq t \leq nT + \frac{T}{2}$, then.

$$s(t) = b_c[n] \sin \frac{\pi}{T}(t - nT) + j\, b_s[n-1] \sin \frac{\pi}{T}\left(t - (n-1)T - \frac{T}{2}\right)$$

$$= b_c[n] \sin \frac{\pi}{T}(t - nT) + j\, b_s[n-1] \cos \frac{\pi}{T}(t - nT)$$

$$\therefore |s(t)|^2 = b_c^2[n] \sin^2 \frac{\pi}{T}(t - nT) + b_s^2[n-1] \cos^2 \frac{\pi}{T}(t - nT)$$

$$= \sin^2 \frac{\pi}{T}(t - nT) + \cos^2 \frac{\pi}{T}(t - nT)$$

$= 1.$ since ~~BPSK~~ $b_c[n]$, $b_s[n-1]$ are BPSK (± 1).

Similarly if $nT + \frac{T}{2} \leq t \leq (n+1)T$, then.

$$s(t) = b_c[n] \sin \frac{\pi}{T}(t - nT) + j\, b_s[n] \sin \frac{\pi}{T}(t - nT - T/2)$$

$$= b_c[n] \sin \frac{\pi}{T}(t - nT) - j\, b_s[n] \cos \frac{\pi}{T}(t - nT)$$

and again $|s(t)|^2 = 1$, since $b_c[n]$ and $b_s[n]$ are BPSK.

c) PSD of the MSK signal.

This is an example of 2-dimensional linear modulation, in the lectures we had only discussed 1-dimensional linear modulation of the form.

$$s(t) = \sum_{n=-T_0/2T}^{T_0/2T} s[n]\, p(t-nT), \qquad |t| < \frac{T_0}{2}.$$

In the problem, we have ~~next about~~ ~~example~~.

$$s(t) = \sum_{n=-T_0/2T}^{T_0/2T} \sum_{i=1}^{2} s_i[n]\, p_i(t-nT) \qquad |t| < \frac{T_0}{2}.$$

For this 2-dimensional case.

$$R_s(\tau) \triangleq \lim_{T_0\to\infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} s(t)\, s^*(t-\tau)\, dt$$

$$= \lim_{T_0\to\infty} \frac{1}{T_0} \sum_n \sum_m \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \int_{-T_0/2}^{T_0/2} s_{i_1}[n]\, s_{i_2}^*[m]\, p_{i_1}(t-nT)\, p_{i_2}^*(t-mT-\tau)\, dt.$$

$$= \sum_{n} \sum_m \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} s_{i_1}[n]\, s_{i_2}^*[m] \lim_{T_0\to\infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} p_{i_1}(t-nT)\, p_{i_2}^*(t-mT-\tau)\, dt$$

$$\approx \sum_n \sum_m \sum_{i_1} \sum_{i_2} s_{i_1}[n]\, s_{i_2}^*[m] \lim_{T_0\to\infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} p_{i_1}(t)\, p_{i_2}^*(t+(n-m)T-\tau)\, dt$$

$m \& n-k$

$$\approx \sum_n \sum_k \sum_{i_1} \sum_{i_2} s_{i_1}[n] s_{i_2}^*[n-k] \underbrace{\lim_{T_0\to\infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} p_{i_1}(t)\, p_{i_2}^*(t+kT-\tau)\, dt}_{\text{integral depends only on } k \text{ and not on } n}.$$

$$\approx \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \sum_k \left[\lim_{T_0\to\infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} p_{i_1}(t)\, p_{i_2}^*(t+kT-\tau)\, dt \right] \sum_{n=-T_0/2T}^{T_0/2T} s_{i_1}[n]\, s_{i_2}^*[n-k]$$

$$\approx \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \sum_k \left(\frac{1}{T}\right) \lim_{T_0\to\infty} \left[\int_{-T_0/2}^{T_0/2} p_{i_1}(t)\, p_{i_2}^*(t+kT-\tau)\, dt \; \frac{\sum_{n=-T_0/2T}^{T_0/2T} s_{i_1}[n]\, s_{i_2}^*[n-k]}{(T_0/T)} \right]$$

~~Since if $s_{i_1}[n]$~~

~~In our example $s_{i_1}[n]$~~

If the joint sequences $\{s_{i_1}[n], s_{i_2}[n]\}$ are ergodic in mean and autocorrelation then

$$\lim_{T_0\to\infty} \frac{\sum_{n=-T_0/2T}^{T_0/2T} s_{i_1}[n]\, s_{i_2}^*[n-k]}{T_0/T} = \mathbb{E}\left[s_{i_1}[n]\, s_{i_2}^*[n-k]\right] = R_{i_1, i_2}[k].$$

we also assume that the sequences $\{s_{i_1}[n], s_{i_2}[n]\}$ is jointly W.S.S also.

$\therefore$ we observe that

$$R_s(\tau) \approx \frac{1}{T} \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \sum_{k} R_{i_1, i_2}[k] \int_{-\infty}^{\infty} p_{i_1}(t)\, p_{i_2}^*(t+kT-\tau)\, dt.$$

as $T_0 \to \infty$.

Note that $R_s(\tau)$ is the same for any realization of the random process $\{s_{i_1}[n], s_{i_2}[n]\}$.

$\therefore$ For this scenario, the p.s.d of $s(t)$ is given by:

$$P_s(f) = \text{Fourier of } R_s(\tau) = \int R_s(\tau) e^{-j2\pi f \tau} d\tau$$

$$= \frac{1}{T} \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \sum_{k} R_{i_1, i_2}[k] \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{i_1}(t)\, p_{i_2}^*(t+kT-\tau)\, e^{-j2\pi f \tau}\, dt\, d\tau$$

$$= \frac{1}{T} \sum_{k} e^{-j2\pi f kT} \sum_{i_1} \sum_{i_2} R_{i_1, i_2}[k]\, P_{i_1}(f)\, P_{i_2}^*(f)$$

$$= \frac{1}{T} \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} P_{i_1}(f)\, P_{i_2}^*(f) \left(\sum_{k} R_{i_1, i_2}[k] e^{-j2\pi f kT} \right)$$

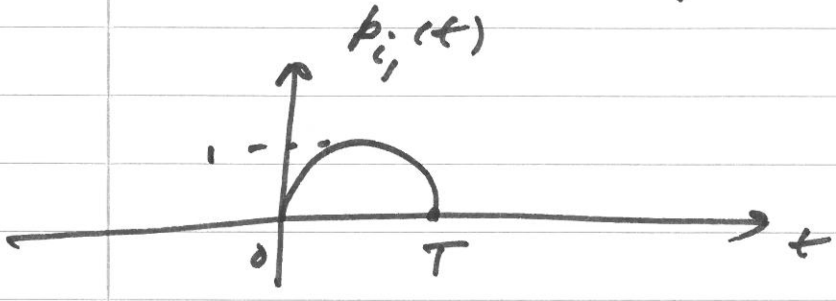
For the problem at hand

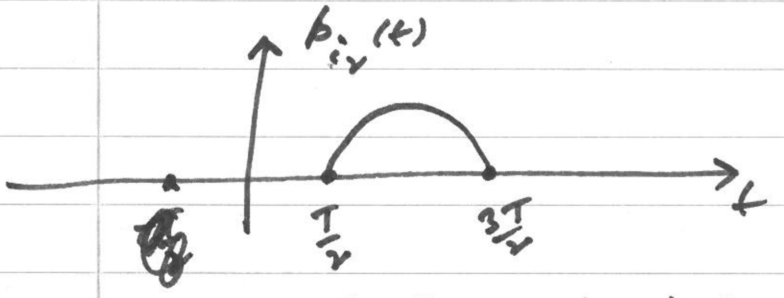
$$s_1[n] = b_c[n] \quad \text{and}$$

$$s_2[n] = j\, b_s[n].$$

$$p_{i_1}(t) = \sin\frac{\pi t}{T}\, I_{[0,T]}(t)$$

$$p_{i_2}(t) = \sin\frac{\pi}{T}(t - T/2)\, I_{[0,T]}(t - T/2)$$





Since $b_c[n]$ and $b_s[n]$ are i.i.d. we have

$$R_{2,2}[k] = R_{1,1}[k] = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

$$R_{1,2}[k] = R_{2,1}[k] = 0 \quad \text{for all } k.$$

$$\therefore\ R_s(f) = \frac{1}{T}\sum_{i_1=1}^{2}\sum_{i_2=1}^{2} P_{i_1}(f)\, P_{i_2}^*(f)$$

$$= \frac{1}{T}\left|P_{i_1}(f) + P_{i_2}(f)\right|^2.$$

$$= \frac{1}{T}\left|1 + e^{-j\pi f T}\right|^2 \left|P_1(f)\right|^2$$

$$R_s(f) = \frac{1}{T}\left(|P_1(f)|^2 + |P_2(f)|^2\right)$$

$$= \frac{2}{T}\cdot T^2 \left[\operatorname{sinc}\left(fT-\tfrac{1}{2}\right) + \operatorname{sinc}\left(fT+\tfrac{1}{2}\right)\right]^2$$

$$= \frac{32T}{\pi^2}\left[\frac{\cos^2 \pi f T}{\left((2fT)^2-1\right)^2}\right]$$

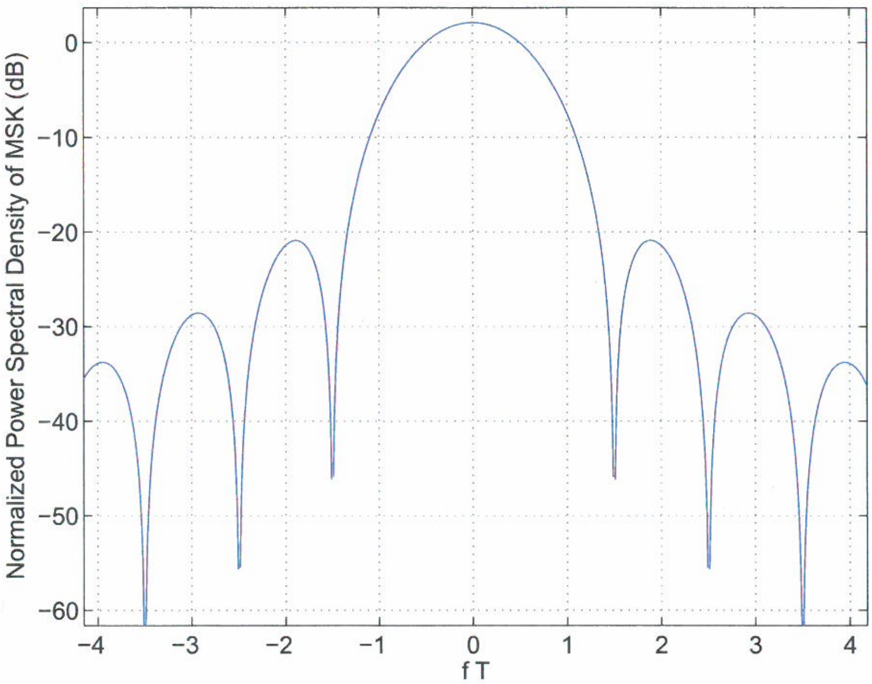
$$=$$

d) For 99% occupancy. find f_0 s.t.

$$\frac{\int_{-f_0}^{+f_0} R_s(f)\,df}{\int_{-\infty}^{\infty} R_s(f)\,df} = 0.99$$

Numerically we get $f_0 \approx \frac{1.2}{T} = \frac{0.6}{T_b}$.

99% bandwidth used is $\frac{1.2}{T_b}$.



Normalized Power Spectral Density of MSK (dB)
0
−10
−20
−30
−40
−50
−60
−4
−3
−2
−1
0
1
2
3
4
f T