TSKS04 Digital Communication Continuation Course

Solutions for the exam 2015-08-21

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Answer:

a. The mean value is $m_S = \frac{2+\sqrt{2}}{3}A$ and the variance is $\sigma_S^2 = \frac{12-4\sqrt{2}}{9}A^2$. This follows from direct computation using the definitions.

b. This is a one-dimensional situation in which the PSD is given by

$$R_s(f) = \frac{1}{T}|\Phi_1(f)|^2 R_S[fT]$$

where $\Phi_1(f)$ is the Fourier transform of the basis function $\phi_1(t)$ and $R_S[fT]$ is the PSD of the signal constellation. The PSD can be computed as

$$R_S[fT] = \sigma_S^2 + m_S^2 \sum_m \delta(fT - m)$$

where the mean value and variance are as in a).

Moreover, direction computation of Fourier transform gives

$$\Phi_1(f) = \frac{Tj}{2} e^{-j\pi fT} \left(\underbrace{e^{-j\pi f_c T}}_{=j^{-2f_cT}} \text{sinc}((f+f_c)T) - \underbrace{e^{+j\pi f_c T}}_{=j^{2f_cT}} \text{sinc}((f-f_c)T) \right)$$

where we can utilize that $e^{-j\pi f_c T} = j^{2f_cT}$ since $2f_cT$ is an integer. (See A.3.1 in the extra course material for details).

By multiplying everything together, according to the formula above, we get

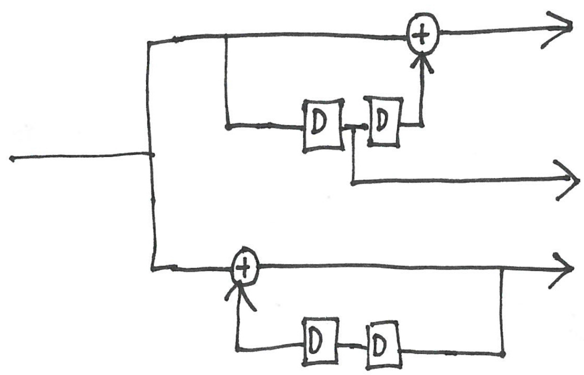
$$\begin{aligned} R_s(f) = {} & \frac{T}{4} \left| je^{-j\pi fT} \left(j^{-2f_cT} \,\text{sinc}((f+f_c)T) - j^{2f_cT} \,\text{sinc}((f-f_c)T) \right) \right|^2 \\ & \cdot \left(\frac{12-4\sqrt{2}}{9} A^2 + \left(\frac{2+\sqrt{2}}{3} A \right)^2 \sum_m \delta(fT-m) \right) \\ = {} & \frac{TA^2}{4} \left(\text{sinc}((f+f_c)T) + (-1)^{2f_cT+1} \,\text{sinc}((f-f_c)T) \right)^2 \left(\frac{12-4\sqrt{2}}{9} + \frac{6+4\sqrt{2}}{9} \sum_m \delta(fT-m) \right) \end{aligned}$$

One can continue to simplify this expression by using properties of the δ function.

2

Answer:

a. The encoder can look like this:



b. The rate is 1/3 since there is one input and three outputs.

c. The greatest common divisor of the generator matrix is 1. Hence, the code is not catastrophic.

3

Answer:

There are three hypotheses:

H_1: $s = A$ leading to $y \sim \mathcal{N}(A, \sigma^2 A)$.

H_2: $s = A$ leading to $y \sim \mathcal{N}(2A, \sigma^2 2A)$.

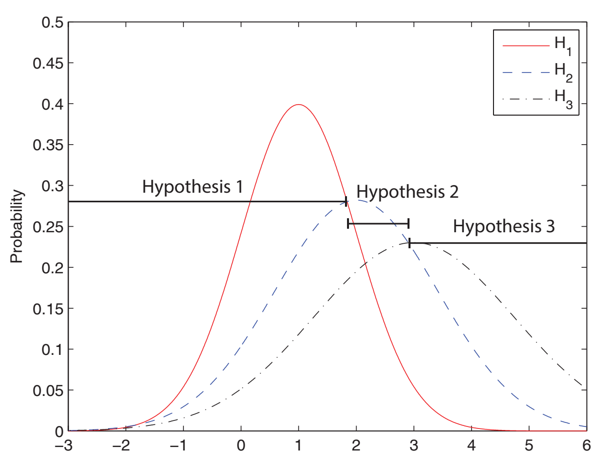
H_3: $s = A$ leading to $y \sim \mathcal{N}(3A, \sigma^2 3A)$.

Or written more compactly: H_k: $y \sim \mathcal{N}(kA, k\sigma^2 A)$ for $k = 1, 2, 3$. The corresponding PDF for hypothesis k is

$$f_k(y) = \frac{1}{\sqrt{2\pi k A \sigma^2}} e^{-\frac{(y-kA)^2}{2kA\sigma^2}}.$$

H_1 has maximum likelihood for all y for which $f_1(y) \geq f_2(y)$ and $f_1(y) \geq f_3(y)$. Similar equations can be stated for the other hypotheses.

A sketch of the ML decision regions for $A = 1$ and σ^2 would look like:



The decision regions becomes larger for the hypotheses with higher k.

4

Answer:

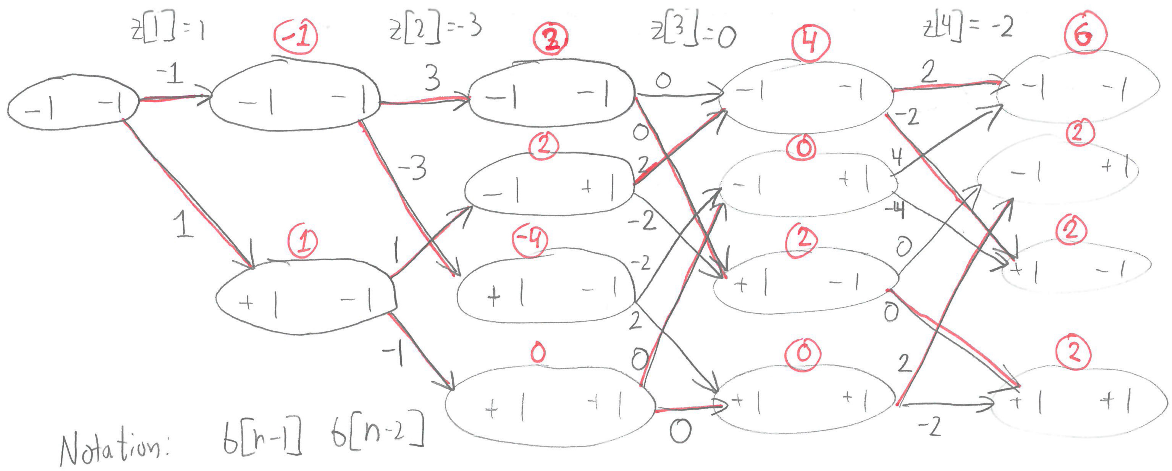
The symbols can be assumed to be from a BPSK modulation. This information was missing in the problem formulation, but the students were informed about this during the exam.

The branch metric (5.13) becomes

$$\lambda_n(b[n], (b[n-1], b[n-2])) = b[n]z[n] + b[n]b[n-1] - b[n]b[n-2] \tag{1}$$

for the values of $h[m]$ computed above, since everything is real-valued and $|b[n]| = 1$ for BPSK.

For simplicity, we neglect the constant -3. See the finalized Viterbi algorithm below. When we terminate the algorithm by selecting the state with the highest end probability, we pick the red-marked path that ends with "4". The corresponding bit sequence is $b[1] = +1$, $b[2] = -1$, $b[3] = -1$, $b[4] = -1$.



5

Answer:

a. Suppose that we have information sequences of length K. For a given $\mathbf{c}[k]$, $k = 1, \ldots, K$, we will have $\mathbf{r}[k] = \mathbf{c}[k]$ with probability $1-p$ and $\mathbf{r}[k] \neq \mathbf{c}[k]$ with probability p.

If $d_H(\mathbf{r}[k], \mathbf{c}[k])$ denotes the Hamming distance, then

$$\sum_k \log p(\mathbf{r}[k]|\mathbf{c}[k]) = d_H(\mathbf{r}[k], \mathbf{c}[k]) \log(p) + (K - d_H(\mathbf{r}[k], \mathbf{c}[k])) \log(1-p).$$

For $p < 1/2$, we have $\log(p) < \log(1-p)$ and hence we maximize this ML metric by minimizing the Hamming distance.

b. Here we expect an explanation similar to Example 7.1.1 in the book.