Synchronization in Coherent receivers.

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Let $x_{tp}(t)$, $t \in [0, T_{tp}]$ be a complex baseband deterministic waveform known to both the transmitter and the receiver. During the training phase, the transmitter sends $x_{tp}(t)$. The receiver then uses the received waveform to derive an estimate of the unknown channel parameters.

We consider a memoryless AWGN channel (no filtering). The received ~~waveform~~ ~~is given by~~ complex baseband waveform after ~~$Z_{tp}(t) =$~~ down conversion and low pass filtering is given by

$$Z_{tp}(t) = A e^{j\Phi} x_{tp}(t-\tau) + N_{tp}(t)$$
$$t \in [0 \quad T_0]$$

where Φ is the random unknown phase offset between the local oscillators at the receiver and the transmitter.

T is the random unknown delay/or the time taken by the waveform to reach the receiver (we assume $T > 0$).

We will also assume that $T < T_0 - T_{tp}$. $A > 0$ models the magnitude gain/loss of the channel. $N_p(t)$ is the AWGN.

~~The~~ let the deterministic received waveform be given by $z_{tp}(t)$, $t \in [0, T_0]$.

The maximum likelihood (ML) estimates of the unknown channel parameters (A, Φ, T) given the received waveform $z_{tp}(t)$ is given by

$$(\hat{A}, \hat{\Phi}, \hat{T}) = \cancel{\underset{\substack{A>0,\, T>0,\\ T<T_0-T_{tp},}}{\arg\max}}$$

$$= \underset{\substack{a>0,\, 0<\tau<T_0-T_{tp},\\ \theta\in[-\pi,\pi]}}{\arg\max} \; L\left(Z_{tp}(t) = z_{tp}(t) \,\middle|\, A=a, T=\tau, \Phi=\theta\right)$$

where $L\left(Z_{tp}(t)=z_{tp}(t)\,\middle|\,A=a,\ \Phi=\theta,\ \Gamma=\tau\right)$

is the likelihood of ~~received~~ receiving $z_{tp}(t)$ given that $A=a$, $\Phi=\theta$, $\Gamma=\tau$.

In the presence of AWGN, we have

$$L\left(Z_{tp}(t)=z_{tp}(t)\,\middle|\,A=a,\ \Phi=\theta,\ \Gamma=\tau\right)$$

$$= e^{-\frac{\int_0^{T_0} |z_{tp}(t) - a\,x_{tp}(t-\tau)e^{j\theta}|^2\,dt}{\sigma^2}}$$

where σ^2 is the P.S.D of $N(t)$.

Since e^{-x/σ^2} is ~~an~~ a monotonically decreasing function of increasing x, the ML estimates are equivalently given by

$$(\hat{A},\hat{\Phi},\hat{\Gamma}) = \underset{\substack{a>0,\\ 0<\tau<T_0-T_{tp}\\ \theta\in[-\pi,\pi)}}{\arg\min} \int_0^{T_0} |z_{tp}(t) - a\,x_{tp}(t-\tau)e^{j\theta}|^2\,dt$$

$$= \underset{\substack{a>0,\\ 0<\tau<T_0-T_{tp},\\ \theta\in[-\pi,\pi)}}{\arg\min}\ a^2\int_0^{T_0}|x_{tp}(t-\tau)|^2\,dt - 2a\,\mathrm{Re}\left[e^{-j\theta}\int_0^{T_0} z_{tp}(t)\,x_{tp}^*(t-\tau)\,dt\right]$$

let $f(\theta, a, \tau) \triangleq a^2 \int_0^{T_0} |x_{tp}(t-\tau)|^2 dt$

$$-2a \, Re\left[e^{-j\theta} \int_0^{T_0} z_{tp}(t) \, x_{tp}^*(t-\tau) \, dt\right]$$

$$\therefore (\hat{A}, \hat{\Phi}, \hat{\Gamma}) = \arg \min_{\substack{a>0, \\ \theta \in [-\pi, \pi), \\ 0 < \tau < T_0 - T_{tp}}} f(\theta, a, \tau) .$$

$$\therefore (\hat{A}, \hat{\Gamma}) = \arg \min_{\substack{a>0, \\ 0 < \tau < T_0 - T_{tp}}} \left[\cancel{\arg} \min_{\theta \in [-\pi, \pi)} f(\theta, a, \tau) \right]$$

let $\theta_{opt}(a, \tau) \triangleq \arg \min_{\theta \in [-\pi, \pi)} f(\theta, a, \tau)$

then

$$\min_{\theta \in [-\pi, \pi)} f(\theta, a, \tau) = f(\theta_{opt}(a, \tau), a, \tau)$$

$$\theta_{opt}(a, \tau) = \arg \min_{\theta \in [-\pi, \pi)} f(\theta, a, \tau)$$

$$= \arg \min_{\theta \in [-\pi, \pi)} a^2 \int_0^{T_0} |x_{tp}(t-\tau)|^2 dt - 2a \, Re\left[e^{-j\theta} \int_0^{T_0} z_{tp}(t) \, x_{tp}^*(t-\tau) \, dt\right]$$

$$= \arg \min_{\theta \in [-\pi, \pi)} -2a \, Re\left[e^{-j\theta} \int_0^{T_0} z_{tp}(t) \, x_{tp}^*(t-\tau) \, dt\right]$$

$$= \text{angle of } \left(\int_0^{T_0} z_{tp}(t) \, x_{tp}^*(t-\tau) \, dt \right) .$$

i.e., $e^{j\theta_{opt}(a,\tau)}$

$$= \frac{\int_0^{T_0} z_{tp}(t)\, x_{tp}^*(t-\tau)\,dt}{\left|\int_0^{T_0} z_{tp}(t)\, x_{tp}^*(t-\tau)\,dt\right|}$$

Hence,

$$\min_{\theta \in [-\pi,\pi)} f(\theta, a, \tau) = f(\theta_{opt}(a,\tau), a, \tau)$$

$$= a^2 \int_0^{T_0} |x_{tp}(t-\tau)|^2 dt$$

$$- 2a \left|\int_0^{T_0} z_{tp}(t)\, x_{tp}^*(t-\tau)\,dt\right|$$

~~Since~~ Note, that

$$\int_0^{T_0} |x_{tp}(t-\tau)|^2 dt = \int_{-\tau}^{T_0-\tau} |x_{tp}(u)|^2 du$$

$$= \int_0^{T_0-\tau} |x_{tp}(u)|^2 du$$ since $\tau > 0$ and $x_{tp}(u) = 0$ for $u < 0$.

$$= \int_0^{T_{tp}} |x_{tp}(u)|^2 du,$$ since $T_0 - \tau > T_{tp}$ and $x_{tp}(u) = 0$ for $u > T_{tp}$.

$$\therefore f(\theta_{opt}(a,\tau), a, \tau) = a^2 \int_0^{T_{tp}} |x_{tp}(t)|^2 dt - 2a \left|\int_0^{T_0} z_{tp}(t)\, x_{tp}^*(t-\tau)\,dt\right|$$

The ML estimates are then given by

$$(\hat{A}, \hat{\Gamma}) = \underset{\substack{a>0, \\ 0<\tau<T_0-T_{tp}}}{\arg\min} \underbrace{a^2 \int_0^{T_{tp}} |x_{tp}(t)|^2 dt - 2a \left| \int_0^{T_0} z_{tp}(t)\, x_{tp}^*(t-\tau)\, dt \right|}_{\triangleq\, g(a,\tau)}$$

$\therefore$

$$\hat{\Gamma} = \underset{0<\tau<T_0-T_{tp}}{\arg\min} \left[\min_{a>0} g(a,\tau) \right]$$

$$\min_{a>0} g(a,\tau) = \min_{a>0} a^2 \int_0^{T_{tp}} |x_{tp}(t)|^2 dt - 2a \left| \int_0^{T_0} z_{tp}(t)\, x_{tp}^*(t-\tau)\, dt \right|$$

$$\frac{\partial}{\partial a} g(a,\tau) = 0$$

$$\Rightarrow \quad 2a \int_0^{T_{tp}} |x_{tp}(t)|^2 dt - 2 \left| \int_0^{T_0} z_{tp}(t)\, x_{tp}^*(t-\tau)\, dt \right| = 0$$

$$\Rightarrow \quad a_{opt}(\tau) = \frac{\left| \int_0^{T_0} z_{tp}(t)\, x_{tp}^*(t-\tau)\, dt \right|}{\int_0^{T_{tp}} |x_{tp}(t)|^2 dt}$$

$\therefore$

~~$\min_{a>0} g(a,\tau)$~~ we also note that

$$\frac{\partial^2}{\partial a^2} g(a,\tau) = \int_0^{T_{tp}} |x_{tp}(t)|^2 dt > 0$$

$\therefore$ $a_{opt}(\tau)$ is the minimum.

Hence

$$\min_{a>0} g(a,\tau) = \frac{-\left|\int_0^{T_0} z_{tp}(t)\, x_{tp}^*(t-\tau)\, dt\right|^2}{\int_0^{T_{tp}} |x_{tp}(t)|^2\, dt}$$

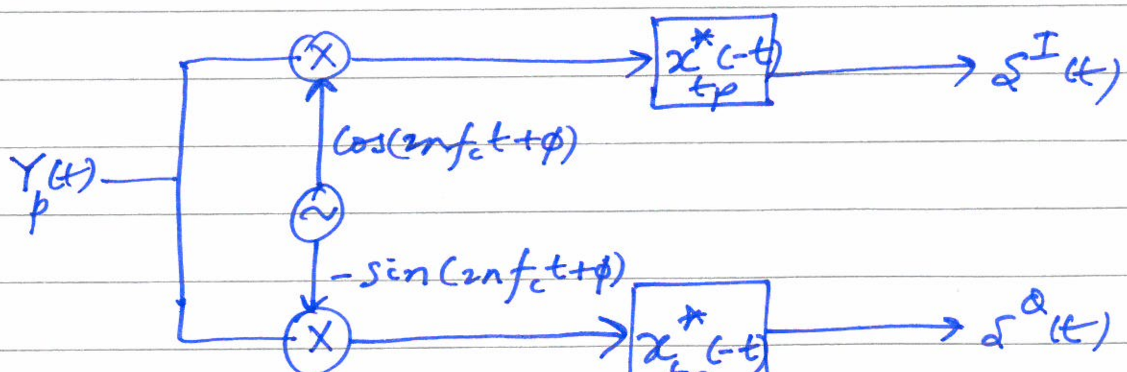
$$\therefore\ \hat{\tau} = \arg\min_{0<\tau<T_0-T_{tp}} \frac{-\left|\int_0^{T_0} z_{tp}(t)\, x_{tp}^*(t-\tau)\, dt\right|^2}{\int_0^{T_{tp}} |x_{tp}(t)|^2\, dt}$$

$$= \arg\max_{0<\tau<T_0-T_{tp}} \left|\int_0^{T_0} z_{tp}(t)\, x_{tp}^*(t-\tau)\, dt\right|$$

Note that $s(t) \triangleq z_{tp}(t) * x_{tp}^*(-t)$

$$\cancel{\int_0^{T_0} z_{tp}(\tau)} = \int_0^{T_0} z_{tp}(\tau)\, x_{tp}^*(\tau - t)\, dt.$$

Hence $\int_0^{T_0} z_{tp}(t)\, x_{tp}^*(t-\tau)\, dt$ is nothing ~~bet~~ but $(z_{tp}(t) * x_{tp}^*(-t))$ sampled at time $t=\tau$.



$$s(t) = s^I(t) + j s^Q(t) = z_{tp}(t) * x_{tp}^*(-t).$$