TSKS04 Digital Communication Continuation Course

Solutions for the exam 2014-06-09

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1

We are given impulse responses

$$g_{\mathrm{TX}}(t) = \mathrm{rect}\left(\frac{t}{T}\right),$$
$$g_{\mathrm{C}}(t) = \mathrm{rect}\left(\frac{t}{2T}\right),$$
$$g_{\mathrm{RX}}(t) = \mathrm{rect}\left(\frac{t}{T/2}\right),$$

of the transmit filter, channel and receive filter, respectively. Let $p(t)$ be the total impulse response of the cascade of the sender filter and the channel i.e.

$$p(t) = (g_{\mathrm{TX}} * g_{\mathrm{C}})(t) = \begin{cases} 0, & t < -3T/2, \\ 3T/2 + t, & -3T/2 \le t < -T/2, \\ T, & |t| \le T/2, \\ 3T/2 - t, & T/2 < t \le 3T/2, \\ 0, & t > 3T/2. \end{cases}$$

We notice that $p(t)$ is real and even. Then we have the matched filter

$$p_{\mathrm{MF}}(t) = p^*(-t) = p(t).$$

Furthermore, let $y(t)$ denote the output from the channel.

a. The optimal (ML) case is if we use $p_{\mathrm{MF}}(t)$ as our receiver filter. ML detection based on samples from the output of the receiver filter is possible if $p_{\mathrm{MF}}(t)$ can be written as a linear combination of shifted versions of $g_{\mathrm{RX}}(t)$. We observe that $g_{\mathrm{RX}}(t)$ is piecewise constant, while $p_{\mathrm{MF}}(t)$ is not. It is therefore impossible to write $p_{\mathrm{MF}}(t)$ as a linear combination of shifted versions of $g_{\mathrm{RX}}(t)$. So, ML detection based on samples from the output of the receiver filter is in this case impossible.

b. We observe that matched filter of the channel is $g_{\mathrm{C}}(t)$ itself, since it is real and even. We also observe that $g_{\mathrm{C}}(t)$ can be written as

$$g_{\mathrm{C}}(t) = g_{\mathrm{RX}}(t + \frac{3T}{4}) + g_{\mathrm{RX}}(t + \frac{T}{4}) + \\ + g_{\mathrm{RX}}(t - \frac{T}{4}) + g_{\mathrm{RX}}(t - \frac{3T}{4}).$$

Thus, the alternative sender filter

$$g_{\mathrm{TX}}(t) = \delta(t),$$

would give us

$$p(t) = (g_{\mathrm{TX}} * g_{\mathrm{C}})(t) = g_{\mathrm{C}})(t),$$

and its matched filter is as observed

$$\begin{aligned} p_{\mathrm{MF}}(t) &= p^*(-t) = p(t) = g_{\mathrm{C}})(t) \\ &= g_{\mathrm{RX}}(t + \frac{3T}{4}) + g_{\mathrm{RX}}(t + \frac{T}{4}) + \\ &\quad + g_{\mathrm{RX}}(t - \frac{T}{4}) + g_{\mathrm{RX}}(t - \frac{3T}{4}). \end{aligned}$$

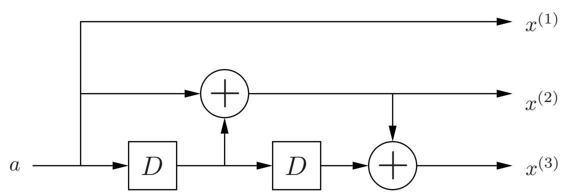
So, in this situation, we can perform ML detection based on samples of the output of the receiver filter.

Answer:

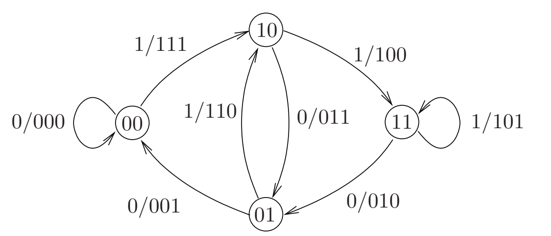
a. ML detection is impossible for the given situation.
b. ML detection is possible if we change the sender filter to $g_{\mathrm{TX}}(t) = \delta(t)$.

2

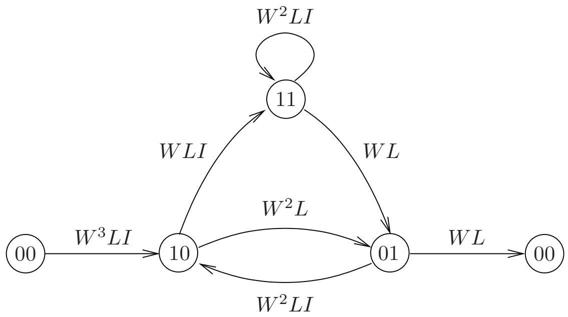
We are given the encoder below.



a. The state diagram of the encoder is given below.



To determine the Extended Path Enumerator $T(W,L,I)$ of the encoder, we assign W^wL to a branch with weight w (among output bits) and information bit 0, and W^wLI to a branch with weight w (among output bits) and information bit 1. This gives us the graph below.



Let F_S be the generating function of state S. This gives us the equations

$$\begin{aligned} F_{10} &= 1 \cdot W^3LI + F_{01} \cdot W^2LI, \\ F_{11} &= F_{11} \cdot W^2LI + F_{10} \cdot WLI, \\ F_{01} &= F_{11} \cdot WL + F_{10} \cdot W^2L, \\ T(W,L,I) &= F_{01} \cdot WL \end{aligned}$$

Solving this equation system gives us

$$T(W,L,I) = \frac{W^6L^3I(1+LI-W^2LI)}{1-W^2LI(1+LI+W^2LI-W^4L^2I)}$$

b. The dimension, k, of the block code is the number of information bits, i.e. $k=8$. The convolutional code is a rate 1/3, which means that every information bit (including trailing dummy zeros) produces 3 codeword bits. The shift register has length 2, which means that we need 2 dummy zeros to force the encoder to the all-zero state. So, the length, n, of the block code is given by $n = 3(k+2) = 30$. Finally, the minimum distance d of the block code is the free distance of the convolutional code, which is $d=6$ since the smallest power of W in $T(W,L,I)$ is 6. Totally, the parameters of the block code are

$$[n,k,d] = [30,8,6].$$

Answer:

a. $T(W,L,I) = \frac{W^6L^3I(1+LI-W^2LI)}{1-W^2LI(1+LI+W^2LI-W^4L^2I)}$
b. $[n,k,d] = [30,8,6]$

3
The probability that a Poisson variable Y with mean m is y is given by

$$\Pr\{Y=y\} = \frac{m^y e^{-m}}{y!},$$

according to Madhow, page 477. Notice that Y takes only non-negative integer values. Here, it is a photon count.

a. The likelihood ratio in this situation is therefore

$$\begin{aligned} L(y) &= \frac{\Pr\{Y=y|1 \text{ sent}\}}{\Pr\{Y=y|0 \text{ sent}\}} \\ &= \frac{m_1^y e^{-m_1}/y!}{m_0^y e^{-m_0}/y!} = \left(\frac{m_1}{m_0}\right)^y e^{m_0-m_1} \end{aligned}$$

The ML criterion compares the likelihood ratio to one (or the log-likelihood function to zero). Thus, we should compare y to the threshold

$$y_0 = \frac{m_1 - m_0}{\ln(m_1/m_0)}$$

For the special case $m_1 = 10m_0 = 100$, we have $y_0 \approx 39.1$. Since y is a photon count and only takes integer values, the estimate is 1 when we have $y > 39$ and 0 otherwise.

b. According to the above, the estimate is 0 if we have $y \leq 39$, which is an error if 1 was sent. Similarily, the estimate is 1 if we have $y > 39$, which is an error if 0 was sent.

The conditional error probabilities are therefore

$$\begin{aligned} P_{e|1} &= \Pr\{Y \leq 39|1 \text{ sent}\} = \sum_{y=0}^{39} \Pr\{Y=y|1 \text{ sent}\} \\ &= \sum_{y=0}^{39} \frac{m_1^y e^{-m_1}}{y!} = \sum_{y=0}^{39} \frac{100^y e^{-100}}{y!} \\ P_{e|0} &= \Pr\{Y > 39|0 \text{ sent}\} = \sum_{y=40}^{\infty} \Pr\{Y=y|0 \text{ sent}\} \\ &= 1 - \sum_{y=0}^{39} \Pr\{Y=y|0 \text{ sent}\} \\ &= 1 - \sum_{y=0}^{39} \frac{m_0^y e^{-m_0}}{y!} = 1 - \sum_{y=0}^{39} \frac{10^y e^{-10}}{y!} \end{aligned}$$

Answer:

a. Output 1 when we have $y > 39$ and 0 otherwise.
b. $P_{e|1} = \sum_{y=0}^{39} \frac{100^y e^{-100}}{y!}$ och $P_{e|0} = 1 - \sum_{y=0}^{39} \frac{10^y e^{-10}}{y!}$

4

We have the two hypotheses, H_0 and H_1, meaning that 0 and 1 was sent, respectively. Under H_0, the received variable, Y, is $\mathrm{CN}(0, N_0)$, i.e. Complex Normal with mean zero and variance N_0 (which means that the real part and the imaginary part are independent, and each with mean 0 and variance $N_0/2$). Under H_1, Y is $\mathrm{CN}(0, A^2 + N_0)$.

a. The ML rule compares the two PDFs and we therefore get

$$\frac{1}{\pi(A^2+N_0)}e^{-\frac{|y|^2}{A^2+N_0}} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{\pi N_0}e^{-\frac{|y|^2}{N_0}}$$

The constant π can be removed from both sides. After that, taking the natural logarithm of both sides, leaves us with

$$-\frac{|y|^2}{A^2+N_0} - \ln(A^2+N_0) \underset{H_0}{\overset{H_1}{\gtrless}} -\frac{|y|^2}{N_0} - \ln(N_0),$$

which we rewrite as

$$|y|^2 \underset{H_0}{\overset{H_1}{\gtrless}} N_0\left(1+\frac{N_0}{A^2}\right)\ln\left(1+\frac{A^2}{N_0}\right)$$

b. Let y_0 be the threshold, i.e. we have

$$y_0 = N_0\left(1+\frac{N_0}{A^2}\right)\ln\left(1+\frac{A^2}{N_0}\right)$$

Assuming that 0 and 1 are equally probable to be sent, then the bit energy is given by

$$E_\mathrm{b} = \frac{A^2}{2},$$

since we send A or 0, both with probability 1/2. So, the SNR is given by

$$\frac{E_\mathrm{b}}{N_0} = \frac{A^2}{N_0 2},$$

Under the hypothesis H_0, $|Y|$ has PDF

$$f_{|Y||H_0}(a) = \frac{2a}{N_0}e^{-\frac{a^2}{N_0}}I_{\{a>0\}}(a),$$

i.e. $|Y|$ is rayleigh distributed with mean $\sqrt{\pi N_0}/2$. The conditional error probability is then

$$P_{\mathrm{e}|0} = \Pr\{|Y|>\sqrt{y_0}|H_0\} = \int_{\sqrt{y_0}}^{\infty} f_{|Y||H_0}(a)\,da$$
$$= \int_{\sqrt{y_0}}^{\infty}\frac{2a}{N_0}e^{-\frac{a^2}{N_0}}\,da = \left[-e^{-\frac{a^2}{N_0}}\right]_{\sqrt{y_0}}^{\infty} = e^{-\frac{y_0}{N_0}}$$

We plug in y_0 from above, and get

$$P_{\mathrm{e}|0} = \left(1+\frac{A^2}{N_0}\right)^{1+\frac{N_0}{A^2}} = \left(1+2\frac{E_\mathrm{b}}{N_0}\right)^{1+\frac{N_0}{2E_\mathrm{b}}}$$

c. The above assumes that we do not know the channel gain h, except that we know its distribution. Now, we know h and therefore we have coherent detection. According to Eq. 3.37 in Madhow, we then have

$$\mathrm{Re}\{y\cdot(hA)^*\} - \frac{|hA|^2}{2} \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

Notice that the variance is not part of this expression. Plugging in the given numbers, we find

$$\begin{aligned}\mathrm{Re}\{(1+j)\cdot(j\tfrac{3}{2})^*\} - \frac{|j3/2|^2}{2} &= \\ = \mathrm{Re}\{(1+j)\cdot(-j\tfrac{3}{2})\} - \frac{|3/2|^2}{2}& \\ = \frac{3}{2} - \frac{9}{8} = \frac{3}{8} > 0.&\end{aligned}$$

Thus, the ML detector announces the estimate 1.

Answer:

a. $|y|^2 \underset{H_0}{\overset{H_1}{\gtrless}} N_0\left(1+\frac{N_0}{A^2}\right)\ln\left(1+\frac{A^2}{N_0}\right)$

b. $P_{\mathrm{e}|0} = \left(1+2\frac{E_\mathrm{b}}{N_0}\right)^{1+\frac{N_0}{2E_\mathrm{b}}}$

c. The ML detector announces the estimate 1.

5

Assumption according to the proble formulation: ± 1 BPSK symbols.

a. Let b_n be the n'th sent symbol and let $\bar{r}_n$ be the n'th received vector of samples. Let $\bar{c}$ be the correlator. The mean square error is given by

$$\begin{aligned}\mathrm{MSE} &= \mathrm{E}\{(\bar{c}^\mathrm{T}\bar{r}_n - b_n)^2\} \\ &= \bar{c}^\mathrm{T}\mathrm{E}\{\bar{r}_n\bar{r}_n^\mathrm{T}\}\bar{c} - 2\bar{c}^\mathrm{T}\mathrm{E}\{b_n\bar{r}_n\} + \mathrm{E}\{b_n^2\} \\ &= \bar{c}^\mathrm{T}R\bar{c} - 2\bar{c}^\mathrm{T}p + 1,\end{aligned}$$

where $R = \mathrm{E}\{\bar{r}_n\bar{r}_n^\mathrm{T}\}$ and $\bar{p} = \mathrm{E}\{b_n\bar{r}_n\}$ are given in Eq. 5.38 in Madhow. Eq. 5.37 in Madhow gives us $\bar{p} = R\bar{c}_\mathrm{MMSE}$, which we plug into the above with $\bar{c} = \bar{c}_\mathrm{MMSE}$, and we get

$$\begin{aligned}\mathrm{MMSE} &= \bar{c}_\mathrm{MMSE}^\mathrm{T}\bar{p} - 2\bar{c}_\mathrm{MMSE}^\mathrm{T}p + 1 \\ &= 1 - \bar{c}_\mathrm{MMSE}^\mathrm{T}p = 1 - \bar{p}^\mathrm{T}R^{-1}p\end{aligned}$$

b. Given model (Eq. 5.25 in Madhow):

$$\bar{r}_n = b_n\bar{u}_0 + \sum_{i\neq 0} b_{n+1}\bar{u}_i + \bar{w}_n,$$

The MSE is given by

$$\text{MSE} = \mathrm{E}\Big\{\big|\langle \bar{c}, \bar{r}_n\rangle - b_n\big|^2\Big\} = \mathrm{E}\Big\{\big|b_n\langle \bar{c}, \bar{r}_n\rangle - 1\big|^2\Big\},$$

where we have used the given assumption $b_n = \pm 1$. Plugging in the model in this expression gives us

$$\begin{aligned}\text{MSE} &= \mathrm{E}\Big\{\big|b_n\langle \bar{c}, b_n\bar{u}_0 + \sum_{i\neq 0} b_{n+1}\bar{u}_i + \bar{w}_n\rangle - 1\big|^2\Big\}\\ &= \mathrm{E}\Big\{\big|\langle \bar{c}, \bar{u}_0 + b_n\sum_{i\neq 0} b_{n+1}\bar{u}_i + b_n\bar{w}_n\rangle - 1\big|^2\Big\}\end{aligned}$$

Assuming that the sequence of BPSK symbols are IID and the two outcomes are equally probable, then the above simplifies to

$$\begin{aligned}\text{MSE} &= \mathrm{E}\Big\{\big|\langle \bar{c}, \bar{u}_0 + b_n\sum_{i\neq 0} b_{n+1}\bar{u}_i + b_n\bar{w}_n\rangle - 1\big|^2\Big\}\\ &= \big|\langle \bar{c}, \bar{u}_0\rangle - 1\big|^2 + \sum_{i\neq 0}\big|\langle \bar{c}, \bar{u}_i\rangle\big|^2 + \big|\langle \bar{c}, \bar{w}_n\rangle\big|^2\end{aligned}$$

since the expectation of all mixed terms become zero. We notice that we have $\big|\langle \bar{a}, \bar{b}\rangle\big|^2 = \bar{a}^{\mathrm{T}}\bar{b}\bar{b}^{\mathrm{T}}\bar{a}$ for vectors $\bar{a}$ and $\bar{b}$, where $\bar{b}\bar{b}^{\mathrm{T}}$ is a square matrix. Using that observation, we rewrite our expression as

$$\text{MSE} = \big|\langle \bar{c}, \bar{u}_0\rangle - 1\big|^2 + \bar{c}^{\mathrm{T}}A\bar{c},$$

where we have $A = \sum_{i\neq 0}\bar{u}_i\bar{u}_i^{\mathrm{T}} + C_{\bar{w}_n}$, and where $C_{\bar{w}_n} = \mathrm{E}\{\bar{w}_n\bar{w}_n^{\mathrm{T}}\}$ is the correlation matrix of the noise. The signal-to-interference ratio (SIR) according to equation 5.43 in Madhow, and specialized for the assumed situation is

$$\text{SIR} = \frac{\big|\langle \bar{c}, \bar{u}_0\rangle\big|^2}{\bar{c}^{\mathrm{T}}A\bar{c}}.$$

We observe that a mere scaling of $\bar{c}$ does not affect SIR, since that results in the same scaling of $\big|\langle \bar{c}, \bar{u}_0\rangle\big|^2$ and of $\bar{c}^{\mathrm{T}}A\bar{c}$. Thus, maximizing SIR is equivalent to minimizing MSE, which is exactly what the MMSE solution does, as proved on pages 221 and 222 in Madhow.

Answer: Proven above.