Solutions to Selected Problems
– from Madhow: Fundamentals of Digital Communication and from Johannesson & Zigangirov: Fundamentals of Convolutional Coding –

Saif K. Mohammed

Revision history:

2012-02-01: Tasks from Tutorial 1.

INSTITUTE OF TECHNOLOGY
LINKÖPING UNIVERSITY

Department of Electrical Engineering
Linköpings universitet, SE-581 83 Linköping, Sweden

Linköping 2012

Note: This material is prepared for the 2012 version of the Master course TSKS04 Digital Communication Continuation Course. Most of it is however borrowed from the 2011 version prepared by Mikael Olofsson.
For almost every task planned for tutorials you find either hints and answers or complete solutions. For the tasks where we give complete solutions, we have chosen to adjust the notations to concur with the notation that has been used in the lectures.
This document will evolve during the course as the tutorials go by.
This version of the document was compiled February 1, 2012.

**Solutions to Selected Problems – from Madhow: Fundamentals of Digital Communication
and from Johannesson & Zigangirov: Fundamentals of Convolutional Coding**

Department of Electrical Engineering
Linköpings universitet
SE-581 83 Linköping
Sweden

This document was prepared using LaTeX2ε with the aid of TeXnicCenter on an Dell PC running CentOS 5. The figures were produced using Xfig (from xfig.org). Finally, the plots were produced using Matlab (from MathWorks, Inc.).

Contents

Tutorial 1 **1**
Task 2.7 (Solution) 1
Task 2.8 (Hints and answer) 4
Task 2.10 (Hints and answer) 4
Task 2.11 (Solution) 5
Task 2.12 (Hints) 7
Task 2.13 (Solution) 7
Task 2.14 (Hints) 8

Tutorial 1

Task 2.7

Task

Let $v(t)$ denote a real passband signal, with Fourier transform $V(f)$, specified as follows for negative frequencies:

$$V(f) = \begin{cases} f + 101, & -101 \le f \le -99, \\ 0, & f < -101 \text{ and } -99 < f \le 0. \end{cases}$$

a. Sketch $V(f)$ for all f.

b. Without explicitly taking the inverse Fourier transform, can you say whether $v(t) = v(-t)$ holds or not.

c. Choosing $f_0 = 100$ Hz, find real baseband waveforms $v_\mathrm{I}(t)$ and $v_\mathrm{Q}(t)$ such that

$$v(t) = \sqrt{2}\big(v_\mathrm{I}(t)\cos(2\pi f_0 t) - v_\mathrm{Q}(t)\sin(2\pi f_0 t)\big)$$

holds.

d. Repeat **c** for $f_0 = 101$ Hz.

Solution

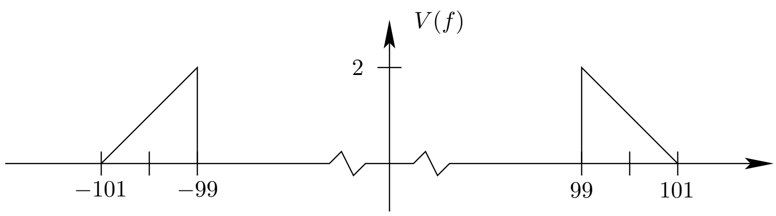
a. The passband signal is real. Therefore we have

$$V(f) = V^*(-f) \quad \text{(conjugate symmetric)}$$

And totally

$$V(f) = \begin{cases} 101 - |f|, & 99 \le |f| \le 101, \\ 0, & \text{else.} \end{cases}$$

Our spectrum then looks like this:



b. We use the following property of the Fourier transform, which holds regardless of if $v(t)$ is real or complex:

$$v(t) = v(-t) \qquad \Leftrightarrow \qquad V(f) = V(-f).$$

Obviously, the second relation holds (known from a.) in this case, and therefore the first does as well.

c. We want the determine $v_\mathrm{I}(t)$ and $v_\mathrm{Q}(t)$. For these signals, we have

$$\sqrt{2}v(t)\cos(2\pi f_0 t) = v_\mathrm{I}(t) + v_\mathrm{I}(t)\cos(4\pi f_0 t) - v_\mathrm{Q}(t)\sin(4\pi f_0 t),$$
$$-\sqrt{2}v(t)\sin(2\pi f_0 t) = v_\mathrm{Q}(t) - v_\mathrm{I}(t)\sin(4\pi f_0 t) - v_\mathrm{Q}(t)\cos(4\pi f_0 t),$$

which corresponds to

$$\frac{1}{\sqrt{2}}\big(V(f+f_0)+V(f-f_0)\big) = V_\mathrm{I}(f) + \text{passband signal around } \pm 2f_0,$$
$$-\frac{j}{\sqrt{2}}\big(V(f+f_0)-V(f-f_0)\big) = V_\mathrm{Q}(f) + \text{passband signal around } \pm 2f_0$$

in the frequency domain. By filtering through ideal lowpass filters with cutoff frequency 1, we get

$$V_\mathrm{I}(f) = \frac{1}{\sqrt{2}}\big(V(f-f_0)+V(f+f_0)\big)I_{\{|f|\le 1\}}(f) = \sqrt{2}I_{\{|f|\le 1\}}(f),$$
$$V_\mathrm{Q}(f) = \frac{j}{\sqrt{2}}\big(V(f-f_0)-V(f+f_0)\big)I_{\{|f|\le 1\}}(f) = j\sqrt{2}fI_{\{|f|\le 1\}}(f) = jfV_\mathrm{I}(f).$$

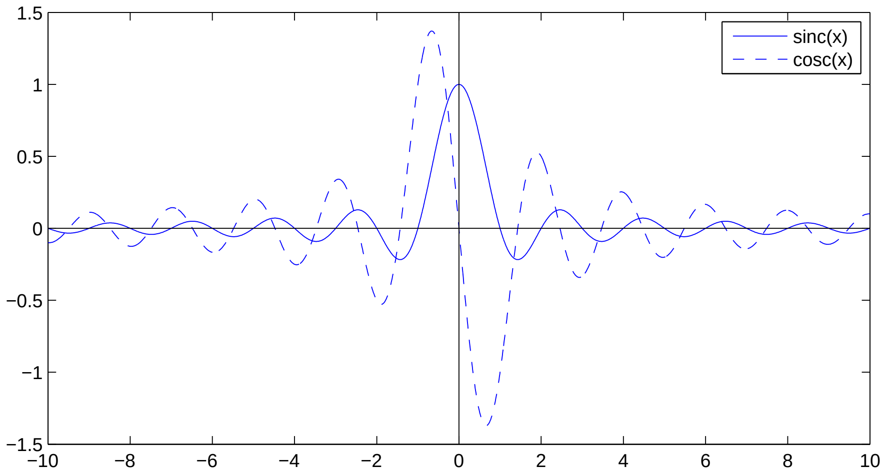
We inverse transform and get

$$v_\mathrm{I}(t) = \sqrt{8}\,\mathrm{sinc}(2t),$$
$$v_\mathrm{Q}(t) = \frac{\sqrt{8}}{2\pi}\frac{d}{dt}\,\mathrm{sinc}(2t) = \begin{cases} \frac{\sqrt{2}}{\pi}\frac{\cos(2\pi t)-\mathrm{sinc}(2t)}{t}, & t \neq 0, \\ 0, & t = 0. \end{cases}$$

As a comment: Sometimes people define the function

$$\mathrm{cosc}(x) = \frac{d}{dx}\,\mathrm{sinc}(x) = \begin{cases} \frac{\cos(\pi x)-\mathrm{sinc}(x)}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Here is a graph of the sinc and cosc functions for those interested:



Then we can write

$$v_\mathrm{I}(t) = \sqrt{8}\,\mathrm{sinc}(2t),$$
$$v_\mathrm{Q}(t) = \frac{\sqrt{8}}{\pi}\,\mathrm{cosc}(2t).$$

d. Here we have the carrier frequency 101 Hz instead of 100 Hz. Denote the baseband representation w.r.t reference frequency of 100 as

$$\tilde{v}(t) = v_{\mathrm{I}}(t) + jv_{\mathrm{Q}}(t)$$

and w.r.t. reference frequency of 101 as

$$\tilde{v}'(t) = v'_{\mathrm{I}}(t) + jv'_{\mathrm{Q}}(t).$$

We have

$$v(t) = Re\{\sqrt{2}\tilde{v}'(t)e^{j202\pi t}\} = Re\{\sqrt{2}\tilde{v}(t)e^{j200\pi t}\}.$$

Thus, we get

$$\tilde{v}'(t) = \tilde{v}(t)e^{j\theta(t)},$$

where $\theta(t) = -2\pi t$. This gives us

$$\begin{aligned}
v'_{\mathrm{I}}(t) &= \mathrm{Re}\left\{\tilde{v}(t)e^{j\theta(t)}\right\} = v_{\mathrm{I}}(t)\cos(2\pi t) - v_{\mathrm{Q}}(t)\sin(2\pi t) \\
&= \sqrt{8}\,\mathrm{sinc}(2t)\cos(2\pi t) - \frac{\sqrt{8}}{\pi}\,\mathrm{cosc}(2t)\sin(2\pi t) \\
&= \sqrt{8}\,\mathrm{sinc}^2(2t), \\
v'_{\mathrm{Q}}(t) &= \mathrm{Im}\left\{\tilde{v}(t)e^{j\theta(t)}\right\} = v_{\mathrm{I}}(t)\sin(2\pi t) + v_{\mathrm{Q}}(t)\cos(2\pi t) \\
&= \sqrt{8}\,\mathrm{sinc}(2t)\sin(2\pi t) + \frac{\sqrt{8}}{\pi}\,\mathrm{cosc}(2t)\cos(2\pi t) \\
&= \frac{\sqrt{2}}{\pi t}[1 - \mathrm{sinc}(4t)].
\end{aligned}$$

Here we have use the trigonometric relations

$$\sin(2a) = 2\sin(a)\cos(a) \qquad \text{and} \qquad 1 = \cos^2(a) + \sin^2(a)$$

a few times.

Task 2.8

Hints

a. One of the signals is (purely real baseband representation), and the other one can be interpreted using a time dependent phase, in the same way as in Task 2.7d. Alternatively, it is possible to take the problem to the frequency domain and to use known transform relations and then inverse transform.

b. You should definitely solve the problem in the frequency domain. (Consider the complex baseband signal $u(t)$ and $v(t)$.)

c. You can use Parseval's relation and determine the inner product in the frequency domain.

$$< u_p,\ v_p >= Re\{< u,\ v >\} = Re\{\int_{-\infty}^{+\infty} U(f)V^*(f)df\}.$$

where $U(f)$ and $V(f)$ are Fourier transformation of complex baseband representation $u(t)$ and $v(t)$, respectively.

d. Consider the complex baseband signal.

$$Y(f) = (1/\sqrt{2})U(f)V(f) = (1/\sqrt{8})V(f).$$

Answer

a. Baseband representation: $u(t) = \mathrm{sinc}(2t)$, $v(t) = \mathrm{sinc}(t)e^{j\pi(t-1/4)}$.
b. u_p has bandwidth 2 Hz and v_p has bandwidth 1 Hz.
c. The inner product is $1/\sqrt{8}$.
d. $y_\mathrm{p} = \frac{1}{2}\,\mathrm{sinc}(t)\sin(101\pi t + \pi/4)$

Task 2.10

Hints

The input to the baseband box are the I and Q components of $u_p(t)$ w.r.t f_1 (plus double frequency terms that get filtered out by the baseband filters used in the box.) The output of the box should be the I and Q components of $y_p(t)$ w.r.t f_2.

a. The baseband box is just the complex baseband filtering operation (see page 26-27 in the textbook).

b. Note that the difference between carrier frequencies can be interpreted as a time dependent phase, in the same way as in Tasks 2.7d and 2.8a. This means that it is not enough to filter. You need to modulate as well. There are several solutions to this problem.

Answer

a. $y_c = 1/\sqrt{2}(u_c * h_c - u_s * h_s)$ and $y_s = 1/\sqrt{2}(u_c * h_s + u_s * h_c)$.
b. First modulate by $e^{j\pi t}$ to convert the reference frequency from $f_c + 1/2$ to f_c. Then filter as in **a**. Then modulate by $e^{j\pi t}$ to convert the reference frequency from f_c to $f_c - 1/2$.

Task 2.11

Task

Consider a pure sinusoid $s(t) = \cos(2\pi f_c t)$, which is the simplest possible example of a passband signal with finite power.

a. Find the time-averaged PSD $R_s(f)$ and ACF $r_s(\tau)$ proceeding from the definitions.

b. Find the complex envelope $\tilde{s}(t)$, and its time-averaged PSD and ACF. Check that Equation 2.70 holds for the passband and baseband PSDs.

Solution

We consider the deterministic signal as a stochastic process $S(t)$ with only one realization, $s(t)$.

a. We start with the ACF. First we have the ordinary ACF

$$\begin{aligned} r_S(t_1, t_2) &= \mathrm{E}\{S(t_1)S(t_2)\} = s(t_1)s(t_2) = \cos(2\pi f_c t_1)\cos(2\pi f_c t_2) \\ &= \frac{1}{2}\cos(2\pi f_c(t_1 - t_2)) + \frac{1}{2}\cos(2\pi f_c(t_1 + t_2)). \end{aligned}$$

The time-averaged ACF is then given by

$$r_S(\tau) = f_c \int_0^{1/f_c} r_S(t + \tau, t)\, dt = \frac{1}{2}\cos(2\pi f_c \tau).$$

The corresponding PSD is then given by

$$R_S(f) = \mathcal{F}\{r_S(\tau)\} = \frac{1}{4}\big(\delta(f + f_c) + \delta(f - f_c)\big).$$

b. Using the obvious carrier frequency f_c, the complex envelope $\tilde{S}(t)$ has the only realization

$$\tilde{s}(t) = \frac{1}{\sqrt{2}}.$$

This gives us the averaged ACF

$$r_{\tilde{S}}(\tau) = \frac{1}{2},$$

and thus the corresponding PSD

$$R_{\tilde{S}}(f) = \mathcal{F}\left\{r_{\tilde{S}}(\tau)\right\} = \frac{1}{2}\delta(f).$$

Equation (2.70) obviously holds, i.e.

$$R_S(f) = \frac{1}{2}\big(R_{\tilde{S}}(f + f_{\mathrm{c}}) + R_{\tilde{S}}(f - f_{\mathrm{c}})\big)$$

Task 2.12

Hints

a. See Section 2.3.1.

b. Express the time-averaged ACF of n. You get an expression in the time-averaged ACFs of n_c and n_s and their cross-correlations.

c. Recall that WSS means constant mean, and ACF being only a function of the time difference.

Task 2.13

Task

Notice that the given PSD is for the passband signal. The formulation of this task in the course book does not follow the otherwise used notation of the book.

We discuss passband white noise, an important noise model used extensively in Chapter 3. A passband random process $N(t) = \mathrm{Re}\left\{\sqrt{2}\tilde{N}(t)e^{j2\pi f_c t}\right\}$ with complex envelope $\tilde{N}(t) = N_{\mathrm{I}}(t) + jN_{\mathrm{Q}}(t)$ has PSD

$$R_N(f) = \frac{N_0}{2} I_{\left\{f_c - \frac{W}{2} \le |f| \le f_c + \frac{W}{2}\right\}},$$

where $N_{\mathrm{I}}(t)$ and $N_{\mathrm{Q}}(t)$ are i.i.d. processes with mean zero.

a. Find the PSD $R_{\tilde{N}}(f)$ for the complex envelope.

b. Find the PSDs $R_{N_{\mathrm{I}}}(f)$ and $R_{N_{\mathrm{Q}}}(f)$ if possible. If this is not possible from the given information, say what further information is needed.

Solution

a. Let $R_N^+(f)$ denote the PSD $R_N(f)$ for positive frequencies, i.e.

$$R_N^+(f) = R_N(f) I_{\{f>0\}}(f).$$

Then for the baseband process, we have

$$R_{\tilde{N}}(f) = 2R_N^+(f + f_c) = N_0 I_{\left\{|f| \le \frac{W}{2}\right\}}$$

according to Madhow, page 41.

b. To determine the PSDs $R_{N_I}(f)$ and $R_{N_Q}(f)$, we want to relate them to $R_N(f)$. From problem statement, a the passband random process $N(t)$ is white, which means that its mean is zero and its ACF has infinite power at zero time shift, i.e.,

$$r_N(t_1, t_2) = C\delta(t_1 - t_2).$$

where C is a constant positive value.
This means that the ACF is only dependent on $\tau = t_1 - t_2$.
Therefore, we study the ACF

$$\begin{aligned} r_N(\tau) &= \mathrm{E}\{N(t+\tau)N^*(t)\} \\ &= \big(r_{N_I}(t+\tau, t) + r_{N_Q}(t+\tau, t)\big)\cos(2\pi f_c\tau) \\ &+ \big(r_{N_I,N_Q}(t+\tau, t) - r_{N_Q,N_I}(t+\tau, t)\big)\sin(2\pi f_c\tau) \\ &+ \big(r_{N_I}(t+\tau, t) - r_{N_Q}(t+\tau, t)\big)\cos(2\pi f_c(2t+\tau)) \\ &- \big(r_{N_I,N_Q}(t+\tau, t) + r_{N_Q,N_I}(t+\tau, t)\big)\sin(2\pi f_c(2t+\tau)) \end{aligned}$$

The two component processes are independent and identically distributed with mean zero. Then we can immediately draw the conclusion that we have

$$\begin{aligned} r_{N_I,N_Q}(t_1, t_2) = r_{N_Q,N_I}(t_1, t_2) &= 0 \quad \text{(i.i.d)} \\ r_{N_I}(t_1, t_2) &= r_{N_Q}(t_1, t_2) \quad \text{(identicaldistribution)} \end{aligned}$$

Plugging this into the expression for $r_N(\tau)$, we get

$$r_N(\tau) = 2r_{N_I}(\tau, t)\cos(2\pi f_c\tau) = C\delta(\tau),$$

This implies that $r_{N_I}(\tau, t)$ only depends on τ.
Therefore, we have $r_{N_I}(\tau, t) = r_{N_I}(\tau) = r_{N_Q}(\tau, t) = r_{N_Q}(\tau)$.
Now consider the complex envelope. Using the relations above, we can write the ACF for the random process of the complex envelope as

$$\begin{aligned} r_{\tilde{N}}(\tau, t) &= \mathrm{E}\{\tilde{N}(t+\tau)\tilde{N}^*(t)\} \\ &= r_{N_I}(\tau) + r_{N_Q}(\tau) + j\big(r_{N_Q,N_I}(\tau) - r_{N_I,N_Q}(\tau)\big) \\ &= 2r_{N_I}(\tau) = 2r_{N_Q}(\tau), \end{aligned}$$

With Fourier transformation we get PSD for $R_{N_\mathrm{I}}(f)$ and $R_{N_\mathrm{Q}}(f)$

$$R_{N_I}(f) = R_{N_Q}(f) = \frac{1}{2}R_{\tilde{N}}(f) = \frac{N_0}{2}I_{\{|f|\leq\frac{W}{2}\}}.$$

<u>**Task 2.14**</u>

Hints

a. Follow the instructions given in the task.

- i) We can write $\tilde{D} = (a + D) \bmod T$. And it is clear that $\tilde{D}$ has the same distribution as D and is independent of s as well.
- ii) Direct consequence of cyclostationarity of s.
- iii) $\tilde{v}(t) = \tilde{s}(t - \tilde{D})$ and $v(t) = s(t - D)$. They are statistically indistinguishable, because of the results in i).

b. Follow the instructions given in the task.

- i) 1). $m_v(t) = E\{v(t)\} = E\{s(t-D)\} = 1/T \int_0^T m_s(t-\alpha)d\alpha$. 2). change variables $\nu = t - \alpha$. 3) s is cyclostationary, $\Rightarrow m_s$ is periodic with period T. So, we can change the integral interval to [0,T].
- ii) Similar strategies as in i).

c. *Change the problem formulation in i) as follows: $s(t)s^*(t-\tau)$ has the same statistics as $s(t+T)s^*(t+T-\tau)$* ***w.r.t mean function.***

Follow the instructions given in the task.