Dimensionality of Signals. (1)

– Saif Khan Mohammed.

– Why is signal dimensionality so important?

Consider a signal space which is two dimensional.

Suppose that the functions/waveforms $x_1(t)$, $x_2(t)$ and $x_3(t)$ are used for communication over this signal space.

Assuming linear modulation let the real passband signal be given by

$$x(t) = a_1 \, x_1(t) + a_2 \, x_2(t) + a_3 \, x(t), \quad \text{--- (1)}$$

where a_1, a_2 and a_3 are real numbers which contain the message to be communicated from the sender to the receiver.

Suppose that there is no noise in the communication channel, and therefore the received signal is the same as the transmitted signal i.e.,

$$y(t) = x(t). \quad \text{--- (2)}$$

How does the receiver estimate a_1, a_2 and a_3 from $y(t)$?

For, this Example note that since the dimension of the signal space is only 2, at least one of the three waveforms is linearly dependent on the remaining two. Without loss of generality let $x_3(t)$ be the waveform linearly dependent on $x_1(t)$ and $x_2(t)$, i.e.,

$$x_3(t) = \alpha x_1(t) + \beta x_2(t) \quad \text{where } \alpha \text{ and } \beta \text{ are real valued.} \qquad (3)$$

Using (3) in (1) and (2) we get

$$y(t) = x(t) = (a_1 + \alpha a_3)\, x_1(t) + (a_2 + \beta a_3)\, x_2(t) \qquad (4)$$

From (4) it is clear that $y(t)$ depends on a_1, a_2 and a_3 only through $(a_1 + \alpha a_3)$ and $(a_2 + \beta a_3)$.

Therefore the receiver can at most know the value of $(a_1 + \alpha a_3)$ and $(a_2 + \beta a_3)$ from $y(t)$.

say.

$$r_1 = a_1 + \alpha a_3 \quad ,$$

$$r_2 = a_2 + \beta a_3 . \quad \text{—— (5)}$$

Basically we have 2 linear equations only but 3 unknowns. therefore we cannot find a_1, a_2, a_3 correctly from r_1 and r_2.

What is the maximum number of real numbers that can communicated reliably (i.e., the receiver should be able to estimate the [redacted] real ~~num~~ numbers correctly from the received signal $y(t)$).

- The answer is 2 (we know that it cannot be 3 from the arguments in the previous pages).

Since the dimension of the signal space is 2, we can always find two waveforms $x_1(t)$ and $x_2(t)$ which are orthogonal to each other; i.e., $\int x_1(t)\, x_2(t)\, dt = 0$ —— (6)

The received signal is given by

$$y(t) = a_1 x_1(t) + a_2 x_2(t).$$

To estimate a_1, the receiver finds the inner product between $y(t)$ and $x_1(t)$, i.e.,

$$\langle y, x_1 \rangle = \int y(t)\, x_1(t)\, dt$$

$$= \int (a_1 x_1(t) + a_2 x_2(t))\, x_1(t)\, dt.$$

$$= a_1 \int x_1^2(t)\, dt.$$

Therefore the receiver can estimate a_1 from $y(t)$ as

$$a_1 = \frac{\langle y, x_1 \rangle}{\int x_1^2(t)\, dt}$$

Similarly we have $a_2 = \dfrac{\langle y, x_2 \rangle}{\int x_2^2(t)\, dt}$.

Dimensionality of approximately time and bandlimited signals.

— Saif Khan Mohammed.

Consider the space of all complex baseband signals that are bandwidth limited to $[-W, W]$, ~~i.e.~~ and $\int_{-\infty}^{\infty} |x(t)|^2 dt = \|x\|^2 = 1$.

$$\mathcal{X} = \left\{ x(t) \;\middle|\; \begin{array}{l} X(f) = 0 \text{ for } |f| > W \\ \text{and } \|x\|^2 = 1. \end{array} \right\} \quad \text{— (1)}$$

Now consider the following question.

Find the waveform $x_1(t) \in \mathcal{X}$ which has the largest energy inside the time duration $[-\frac{T_0}{2}, \frac{T_0}{2}]$, i.e.,

$$x_1(t) = \arg\max_{x(t) \in \mathcal{X}} \int_{-\frac{T_0}{2}}^{T_0/2} |x(t)|^2 dt \quad \text{— (2).}$$

let $\lambda_1 \triangleq \int_{-\frac{T_0}{2}}^{T_0/2} |x(t)|^2 dt$ — (3).

obviously $0 \leq \lambda_1 \leq 1$ since $\int_{-\frac{T_0}{2}}^{T_0/2} |x_1(t)|^2 dt < \int_{-\infty}^{\infty} |x_1(t)|^2 dt = 1.$

~~Next we would like to find~~

Now that we have $x_1(t)$, we would like to consider the space of functions which are in $\mathcal{X}$ and which are <u>orthogonal</u> to $x_1(t)$, i.e.,

$$\mathcal{X}_1 \triangleq \left\{ x(t) \;\middle|\; \begin{array}{l} \langle x, x_1 \rangle = \int x(t)\, x_1^*(t) dt = 0, \\ X(f) = 0 \text{ for } |f| > W, \text{ and} \\ \|x\|^2 = \int_{-\infty}^{+\infty} |x(t)|^2 dt = 1 \end{array} \right\}.$$

Next we find a waveform $x_2(t) \in \mathcal{X}_1$ which has the largest energy inside the time interval $[-\frac{T_0}{2} .. \frac{T_0}{2}]$ i.e.,

$$x_2(t) = \arg\max_{x(t) \in \mathcal{X}_1} \int_{-\frac{T_0}{2}}^{T_0/2} |x(t)|^2 dt$$, and let

$$\lambda_2 \triangleq \int_{-\frac{T_0}{2}}^{T_0/2} |x_2(t)|^2 dt .$$

Now that we have both $x_1(t)$ and $x_2(t)$, we consider the space of functions which are unit energy, bandlimited to $[-W, W]$ and are orthogonal to both $x_1(t)$ and $x_2(t)$, i.e.,

$$\mathcal{X}_2 \triangleq \left\{ x(t) \;\middle|\; \begin{array}{l} \langle x, x_1 \rangle = \langle x, x_2 \rangle = 0, \\ X(f) = 0 \quad \text{for } |f| > W, \text{ and} \\ \|x\|^2 = \int_{-\infty}^{+\infty} |x(t)|^2 dt = 1 \end{array} \right\} .$$

Then we find $x_3(t)$ as

$$x_3(t) \triangleq \arg\max_{x(t) \in \mathcal{X}_2} \int_{-\frac{T_0}{2}}^{T_0/2} |x(t)|^2 dt .$$

We keep on going like this.

We therefore have ~~an~~ infinite sequence of orthonormal functions $x_1(t), x_2(t), x_3(t), \ldots$

and a corresponding sequence of non-negative numbers $\lambda_1, \lambda_2, \lambda_3, \ldots$.

where each function is band-limited to $[-W, W]$. Also it is easy to

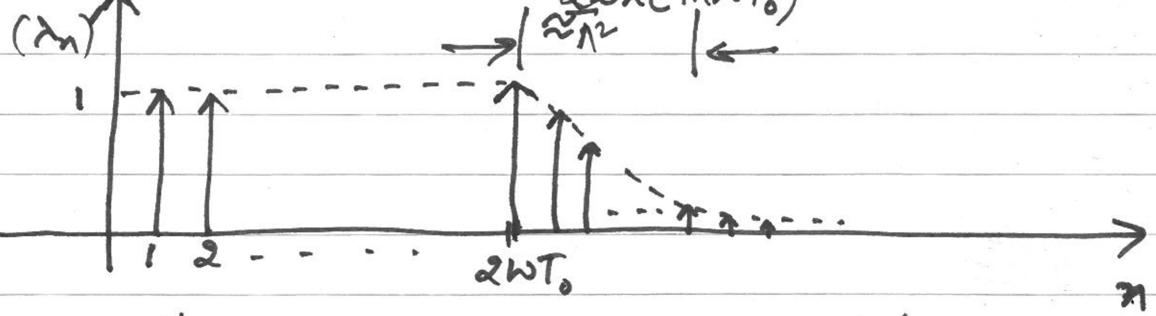
see that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$

These functions are known as ~~prolate~~ prolate-spheroidal functions and were discovered by various authors D. Slepian, Pollack and Landau.

They have shown that for a fixed real number $\epsilon > 0$, (let $n \triangleq 2WT_0$)

$$\lim_{n \to \infty} \lambda_{n(1-\epsilon)} = 1, \text{ and}$$

$$\lim_{n \to \infty} \lambda_{n(1+\epsilon)} = 0.$$



i.e., there are approximately $2WT_0$ orthonormal functions with most of their energy in $[-\frac{T_0}{2} .. \frac{T_0}{2}]$.