2

Derivative Filter Design

2.1 Introduction

It is a matter of fact that the derivative action is seldom adopted in practical cases (actually, 80% of the employed PID controllers have the derivative part switched-off (Ang *et al.*, 2005)), although it has been shown that it is possible to provide a significant improvement of the control performance (note that this improvement becomes less important as the ratio between the apparent time delay and the effective time constant increases (Kristiansson and Lennartson, 2006; Åström and Hägglund, 2004)). This is due to a number of reasons, one of them being certainly that it is the most difficult to tune, as explained in Section 1.9. Indeed, the stability regions for PID controllers are more complex than those for PI controllers and therefore the tuning of a PID controller is more difficult (Åström and Hägglund, 2000*b*). Also, the inherent amplification of the measurement noise represents a significant technological problem, because, if not properly filtered, it might cause a damage to the actuator.

In this chapter it is shown that part of the problem is due also to the structure of the PID controller (see (1.20)–(1.24)), in particular if a PID controller in ideal form with a fixed derivative filter parameter N is adopted.

2.2 The Significance of the Filter in PID Design

It is interesting to evaluate how the presence of a filter of the derivative action changes the location of the zeros in the PID controller. It is trivial to derive that if the PID controller is in series form (1.22) or in ideal form (1.23)–(1.24) with the filter applied to the control variable, then the addition of the filter does not alter the position of the zeros of the controller. Hence, the interesting case to analyse is that related to the PID controller in ideal form (1.20) (or (1.31)).

If the derivative filter is not applied, the zeros of the PID controller (1.13) are the solution of the equation

$$T_i T_d s^2 + T_i s + 1 = 0. \tag{2.1}$$

They can be easily derived as:

$$z_{1,2} = \frac{1}{2}\frac{-T_i \pm \sqrt{T_i^2 - 4T_iT_d}}{T_iT_d}. \tag{2.2}$$

If the derivative filter is applied, the zeros of the controller are the solution of the equation

$$T_iT_d\left(1+\frac{1}{N}\right)s^2 + \left(T_i + \frac{T_d}{N}\right)s + 1 = 0. \tag{2.3}$$

It results:

$$\bar{z}_{1,2} = \frac{1}{2}\frac{-T_iN - T_d \pm \sqrt{(T_iN - T_d)^2 - 4TiT_dN^2}}{T_iT_d(1+N)}. \tag{2.4}$$

A sensitivity analysis can be performed in order to evaluate the influence of the parameter N, *i.e.*, of the filter, on the location of zeros (Leva and Colombo, 2001). The relative perturbation of the ith zero can be calculated as:

$$e_{r,i} := \frac{|\bar{z}_i - z_i|}{|z_i|}. \tag{2.5}$$

To evaluate it quantitatively with an example, T_i is fixed to be 100 and the value of $e_{r,i}$ has been determined by varying T_d from 1 to 100, *i.e.*, by varying the ratio T_d/T_i from 0.01 to 1. Results related to the case $N = 5$ and $N = 20$ are shown in Figures 2.1 and 2.2. It can be seen that the relative error can be greater than 30% and a high value appears when $T_i = 4T_d$ (*i.e.*, when the two zeros are real and coincident), which is a very relevant case, as this relation is adopted in many tuning rules such as the Ziegler–Nichols one .

This analysis is coherent with the results presented in (Kristiansson and Lennartson, 2006), where the performance achieved by a PI(D) controller is evaluated by considering both its capability in the load disturbance rejection task and the corresponding control activity. It is shown that, in general, the proper use of the derivative action allows to significantly increase the load disturbance rejection performance with a modest increase of the control effort. However, if T_i is fixed to be $4T_d$ and N to be 10, then a (slight) increase of the load disturbance rejection performance can be made only at the expense of a much increased control effort (with respect to an optimal PI controller). All these results confirm that the presence of the derivative filter in a PID controller in ideal form cannot be neglected in general in the controller design phase (Leva and Colombo, 2001). Other practical issues concerning the presence of the derivative filter are addressed in the following sections.

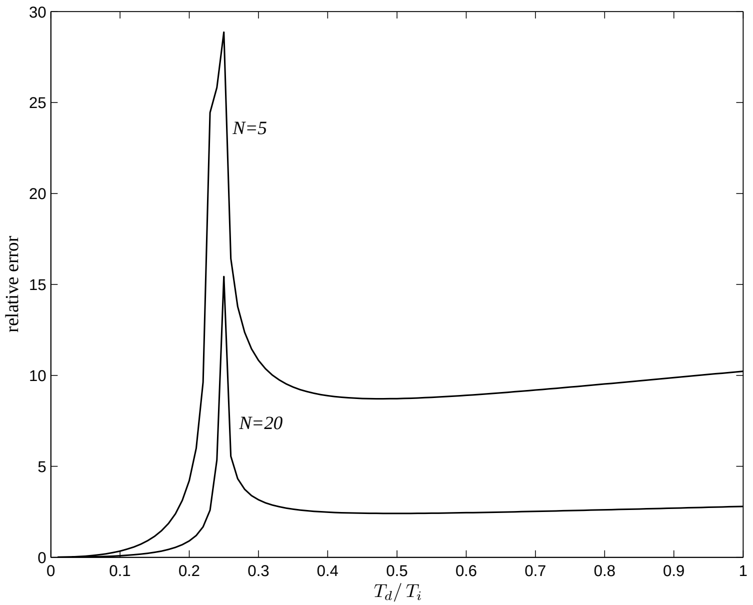

Fig. 2.1. Relative error of the controller zero z_1 due to the presence of the derivative filter in an ideal form PID controller

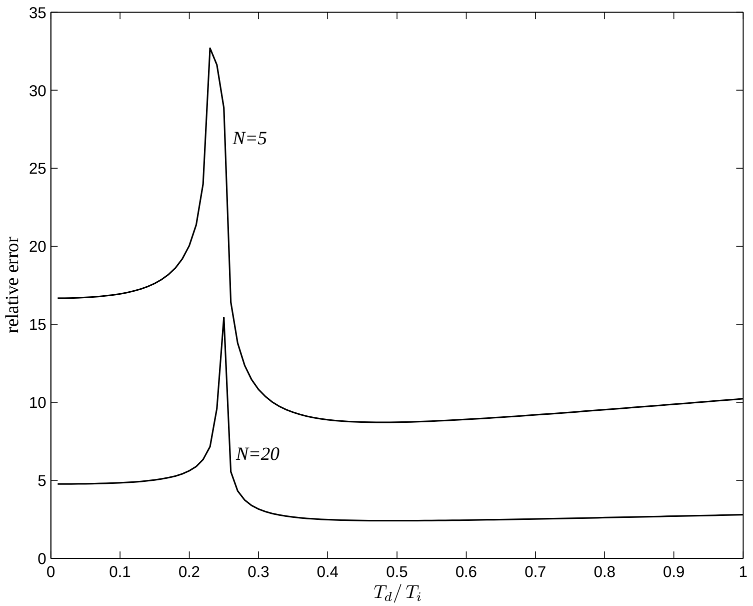

Fig. 2.2. Relative error of the controller zero z_2 due to the presence of the derivative filter in an ideal form PID controller

2.3 Ideal *vs.* Series Form

From another point of view with respect to the approach made in Section 2.2, the PID controllers in the ideal and or in the series form are compared, according to the analysis and the examples presented in (Isaksson and Graebe, 2002).
In particular, the role of the controller structure in the classical lead-lag design or in the pole-placement design is outlined by means of the following examples. Suppose that the control of a tank level with a first-order actuator has to be performed. The process is described by the following transfer function

$$P(s) = \frac{Y(s)}{U(s)} = \frac{K}{s(\tau s + 1)}, \qquad K = 0.1, \qquad \tau = 2, \tag{2.6}$$

where the input $u(t)$ is the valve position set-point and the output $y(t)$ is the tank level. A classical controller design leads to the following controller transfer function, in the context of the typical unitary-feedback control scheme (see Figure 1.2 with $F(s) = 1$):

$$C(s) = 1.06\frac{(3s+1)(8s+1)}{3s(2s+1)}. \tag{2.7}$$

This assures a crossover frequency of 0.3 rad/s and a phase margin of slightly more than 45 deg. The Bode diagram of the open-loop transfer function $C(s)P(s)$ is shown in Figure 2.3. The designed controller corresponds to a PID controller in series form (1.22) where $K_p' = 1.06$, $T_i' = 3$, $T_d' = 8$, and $N' = 4$ or, equivalently, $K_p' = 2.83$, $T_i' = 8$, $T_d' = 3$, and $N' = 1.5$. These controllers can be converted in a PID controller in ideal form (1.20) by applying the following formulae:

$$\begin{aligned}
T_i &= T_i' + \left(1 - \frac{1}{N'}\right) \\
K_p &= K_p' \frac{T_i}{T_i'} \\
T_d &= T_d' \left(\frac{T_i'}{T_i} - \frac{1}{N'}\right) \\
N &= \frac{T_d N'}{T_d'}
\end{aligned} \tag{2.8}$$

In both cases, it follows that $K_p = 3.18$, $T_i = 9$, $T_d = 0.67$ and $N = 1/3$. It can be seen that N and N' are not within the typical range of $5 \div 20$ and they do have a significant role in the overall controller design procedure. Indeed, setting $N = 1/3$ in the ideal PID controller means that the additional pole introduced by the derivative filter is still at a higher frequency than the two controller zeros (the two zeros are at $s = -0.121$ and $s = -1.026$, while the introduced pole is at $s = -4.48$). The fact that the derivative part provides a phase lead is actually evident in the series controller, since N' is grater than one.

Similar considerations apply if a pole-placement technique is adopted. Suppose that an ideal PID controller (1.20) is applied to the tank level process (2.6). The following characteristic equation results:

$$\begin{aligned}\tau\frac{T_d}{N}s^4 + \left(\tau + \frac{T_d}{N}\right)s^3 + \left(1 + K_pKT_d\left(1+\frac{1}{N}\right)\right)s^2 \\ +KK_p\left(1+\frac{T_d}{NT_i}\right)s + \frac{KK_p}{T_i} = 0\end{aligned} \tag{2.9}$$

Assume now that the location of the desired closed-loop poles is such as there are two complex poles at a distance λ from the origin and with the same complex and real part (*i.e.*, $s = (-1 \pm j)/(\lambda\sqrt{2})$) so that they have a corresponding damping factor of $\sqrt{2}/2$. Then, the two remaining poles are placed in the same position on the real axis at a distance of $-1/\lambda$ from the origin. In this way the desired characteristic equation is

$$\begin{aligned}\left(s^2 + \frac{\sqrt{2}}{\lambda}s + \frac{1}{\lambda^2}\right)\left(s+\frac{1}{\lambda}\right)^2 = \\ s^4 + \frac{2+\sqrt{2}}{\lambda}s^3 + \frac{2+2\sqrt{2}}{\lambda^2}s^2 + \frac{2+\sqrt{2}}{\lambda^3}s + \frac{1}{\lambda^4} = 0\end{aligned} \tag{2.10}$$

Comparing the polynomial coefficients, the following PID parameters can be determined by fixing $\lambda = 3$:

$$K_p = 3.36, \quad T_i = 8.68, \quad T_d = 0.463, \quad N = 0.296. \tag{2.11}$$

It turns out that the value of N is significantly outside the typical range also in this case, but this corresponds to a series controller with phase lead (*i.e.*, with $N' > 1$). Actually, the parameters of the corresponding PID controller in series form are:

$$K_p' = 3.12, \quad T_i' = 8.08, \quad T_d' = 2.18, \quad N' = 1.39. \tag{2.12}$$

and the resulting zeros of the controller are $s = -0.12$ and $s = -0.46$ while the poles are at $s = 0$ and $s = -0.64$. It is worth stressing that the choice of $\lambda = 3$ results in a control system that has, as in the previous case, a crossover frequency of about 0.3 rad/s and a phase margin of about 45 deg. The Bode diagram of the open-loop system $C(s)P(s)$ is presented in Figure 2.4. The similarity with the previous one is evident. In order to verify the improve-

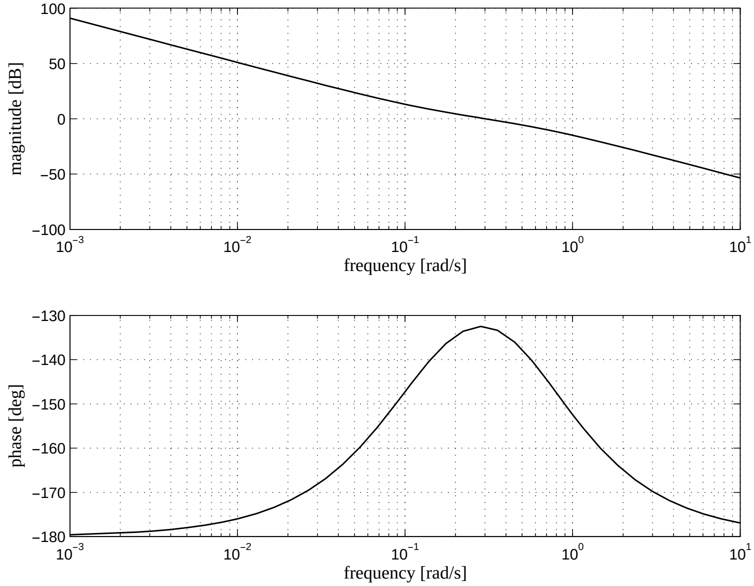

Fig. 2.3. Bode plot of the open-loop transfer function $C(s)P(s)$ resulting from the lead-lag design (Process (2.6))

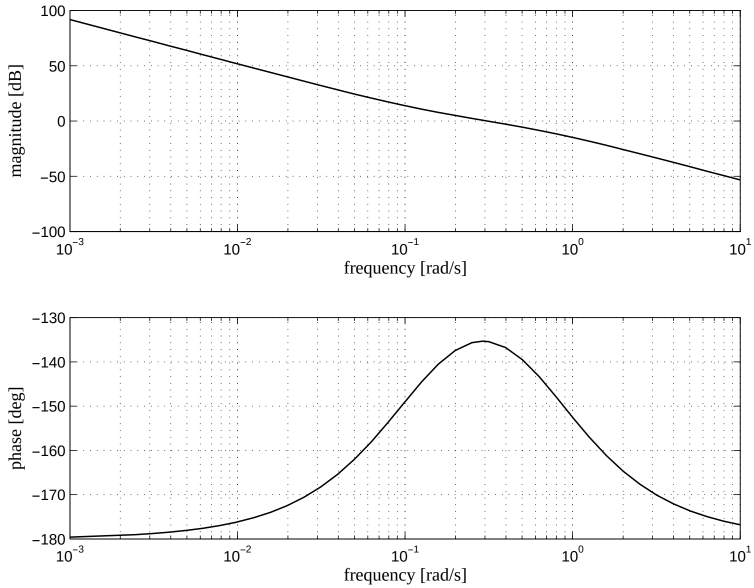

Fig. 2.4. Bode plot of the open-loop transfer function $C(s)P(s)$ resulting from the pole-placement design (Process (2.6))

ment in the performance given by the derivative action, the pole-placement approach is applied also with a PI controller (1.8). The characteristic equation is in this case:

$$\tau s^3 + s^2 + KK_p s + \frac{KK_p}{T_i} = 0. \tag{2.13}$$

It has to be noted that there are three poles to be placed but only two design parameters, while in the previous case there were four conditions for four parameters, because of the presence of the derivative filter parameter N (N'). Thus, a dominant pole design strategy is adopted, namely, only the location of the two dominant poles is selected, while the location of the third pole is checked at the end. In this context, the two dominant poles are chosen as in the previous case at $s = (-1 \pm j)/(\lambda\sqrt{2})$. Denoting as δ the third time constant, the desired characteristic equation is:

$$\begin{aligned} &\left(s^2 + \frac{\sqrt{2}}{\lambda}s + \frac{1}{\lambda^2}\right)\left(s + \frac{1}{\delta}\right) \\ &= s^3 + \left(\frac{\sqrt{2}}{\lambda} + \frac{1}{\delta}\right)s^2 + \left(\frac{1}{\lambda^2} + \frac{\sqrt{2}}{\lambda\delta}\right)s + \frac{1}{\lambda^2\delta} = 0. \end{aligned} \tag{2.14}$$

By comparing the coefficients of Equations (2.13) and (2.14) it follows that:

$$\frac{\sqrt{2}}{\lambda} + \frac{1}{\delta} = \frac{1}{\tau} \tag{2.15}$$

$$\frac{KK_p}{\tau} = \frac{1}{\lambda^2} + \frac{\sqrt{2}}{\delta\lambda} \tag{2.16}$$

$$\frac{KK_p}{\tau T_i} = \frac{1}{\lambda^2\delta} \tag{2.17}$$

From Equation (2.15) it turns out that the smaller λ is the higher δ is and therefore the system cannot be made arbitrarily fast. Indeed, it is $\delta > 0$ (*i.e.*, the system is asimptotically stable) if $\lambda < \sqrt{2}\tau$ and therefore there is a clear limitation in the nominal performance. The value of $\lambda = 3.5$ (that implies $\delta = 10.4$) is eventually selected in order to achieve the best performance (Isaksson and Graebe, 2002). The resulting PI parameters are $K_p = 2.41$ and $T_i = 15.4$.

Set-point step responses and load disturbance responses obtained by the two designed PID controllers and the PI controller are shown in Figures 2.5 and 2.6. It appears that the two PID controllers give very similar responses and they outperform the PI controller in the load disturbance rejection task. Thus, the benefits of the derivative action appears in this case.

Summarising, from the examples presented, it can be deduced that, for a PID controller in series form, it can be sensible to choose a fixed derivative factor

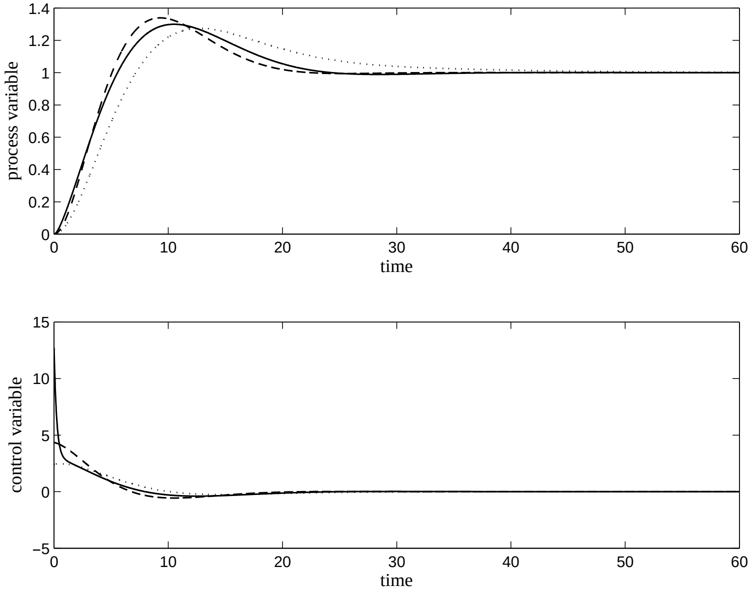

Fig. 2.5. Set-point step response for the designed controllers (Process (2.6)). Solid line: phase-lag PID; dashed line: pole-placement PID; dotted line: PI.

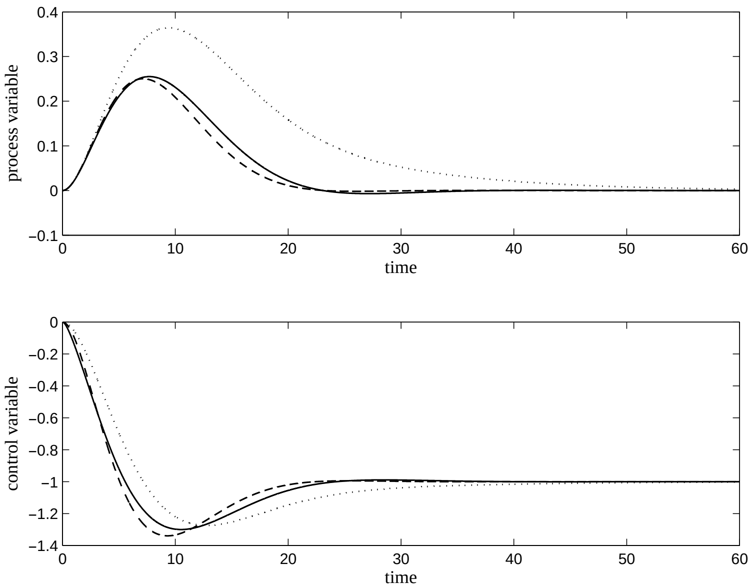

Fig. 2.6. Load disturbance step response for the designed controllers (Process (2.6)). Solid line: phase-lag PID; dashed line: pole-placement PID; dotted line: PI.

$N' > 1$, as a controller with a phase lead might result (note that the maximum phase lead depends only on N' and it is achieved when $N' = 10$). Conversely, for a PID controller in ideal form $C_{i1a}(s)$ (1.20), the necessary phase lead might be achieved with values of N also less than one and therefore fixing it to a constant value greater than one (in the range from 8 to 16 as is done in the vast majority of the industrial implementations (Ang *et al.*, 2005)) can represent an unnecessary limitation of the performance.

It is worth stressing that if the alternative output-filtered form of the ideal controller $C_{i2a}(s)$ (1.23) (or $C_{i2b}(s)$ (1.24)) is adopted, the reasoning related to the series form has to be applied, since the filter is in series with the overall controller transfer function. Thus, if this structure is adopted, the choice of the value of the filter time constant T_f is more intuitive.

In any case, it appears from this analysis that the tuning of a PID controller should involve four parameters, since the derivative filter plays a major role in the overall control system performance.

2.4 Simulation Results

In order to understand better the previously described problems associated with the design of the derivative filter, some simulation results are given. Consider the process

$$P(s) = \frac{1}{s+1} e^{-0.2s}. \tag{2.18}$$

Then, consider a PID controller whose parameters are selected according to the Ziegler–Nichols rules based on the frequency response (note that the ultimate gain K_u is equal to 8.5 and the ultimate period is $P_u = 1.34$). Both the ideal form (1.20) and the series form (1.22) are evaluated. The controller parameters are reported in Table 2.1, where the conversion between the ideal and series structure has been performed by means of formulae (1.17), *i.e.*, without taking into account the derivative filter. Note that $T_i = 4T_d$, that is, the two controller zeros are in the same position for the series controller and for the ideal one if the derivative filter is not considered. The derivative filter time

Table 2.1. Parameters for the ideal and series PID controller for the examples of Section 2.4

	Ziegler–Nichols	Kappa–Tau
K_p	5.00	5.74
T_i	0.672	0.66
T_d	0.168	0.15
K'_p	2.50	3.75
T'_i	0.336	0.43
T'_d	0.336	0.23

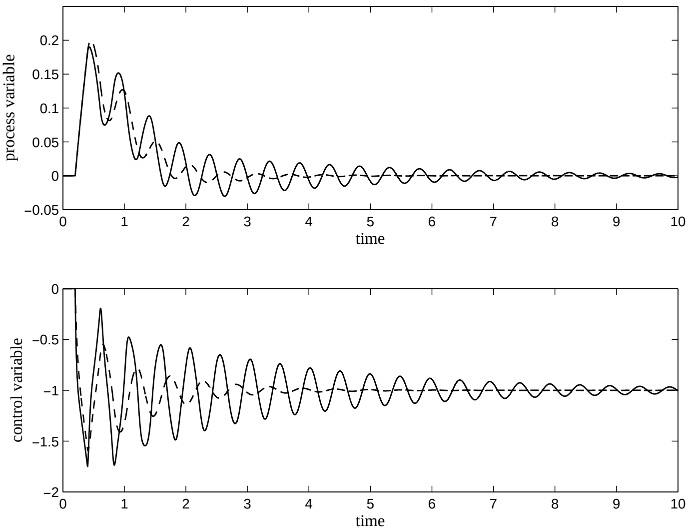

Fig. 2.7. Load disturbance step response for the PID controllers with Ziegler–Nichols parameters (Process (2.18)). Solid line: ideal form with derivative filter; dashed line: series form with derivative filter.

constant has been selected as $N = N' = 10$. The control system responses when a load disturbance unitary step is applied in both cases are plotted in Figure 2.7. The significantly different behaviour of the control system appears. This is due to the fact that the actual zeros of the ideal controller are in $s = -2.77 \pm j0.60$, while they should be the same as those of the series controller that are both in $s = -2.98$. Note that the phase margin of the resulting ideal controller is 44.2 deg (the crossover frequency is $\omega_c = 8.57$ rad/s), while that of the series one is 55.1 deg (the crossover frequency is $\omega_c = 6.19$ rad/s). The same reasoning is applied by considering the Kappa–Tau tuning rules proposed in (Åström and Hägglund, 1995). The parameters obtained are reported in Table 2.1, while the load disturbance unitary step responses are plotted in Figure 2.8. Also in this case the two responses are significantly different. The series controller assures a phase margin of 41.7 deg ($\omega_c = 7.83$ rad/s), while the ideal one, because of the presence of the derivative filter, provides a phase margin of just 15.9 deg ($\omega_c = 11.2$ rad/s). These results confirm the issues discussed in the previous sections that imply the fact that the design of the derivative filter should be considered carefully. The filtering of the measurement noise is also considered hereafter. Consider the same process (2.18) with the following controllers:

- a PI controller with $K_p = 4$ and $T_i = 1$;
- a derivative-filtered PID controller in ideal form (1.20) with $K_p = 4$, $T_i = 1$, $T_d = 0.1$ and $N = 10$;

- a derivative-filtered PID controller in ideal form (1.20), where the derivative filter is a second-order system, with again $K_p = 4$, $T_i = 1$, $T_d = 0.1$ and $N = 10$;
- an output-filtered PID controller in ideal form (1.23) with $K_p = 4$, $T_i = 1$, $T_d = 0.1$ and $T_f = 0.1$;
- an output-filtered PID controller in ideal form (1.24), where the filter is a second-order system with $K_p = 4$, $T_i = 1$, $T_d = 0.1$ and $T_f = 0.1$;
- a derivative-filtered PID controller in series form with $K'_p = 3.55$, $T'_i = 0.89$, $T'_d = 0.11$, $N' = 10$ (note that these parameters have been found by converting the parameters of the controllers in ideal form).

In all the cases a measurement white noise whose amplitude is in the range $[-5 \cdot 10^{-3}, 5 \cdot 10^{-3}]$ is applied to the control system. The resulting process variables and the control variables are plotted in Figures 2.9–2.14. It can be seen that the control variable is less noisy for the output-filtered PID structures. This is somewhat obvious, since the proportional action is also responsible for the amplification of the measurement noise and therefore the filter applied to the whole control variable is more effective than that applied to the derivative action only. If a second-order filter is adopted, the reduction of the noise effect is more evident. However, if the value of T_f in an output-filtered PID controller in ideal form is such that the additional poles are not at a much higher frequency with respect to the zeros (for a more effective filtering), then the presence of the second-order filter might influence the control performance.

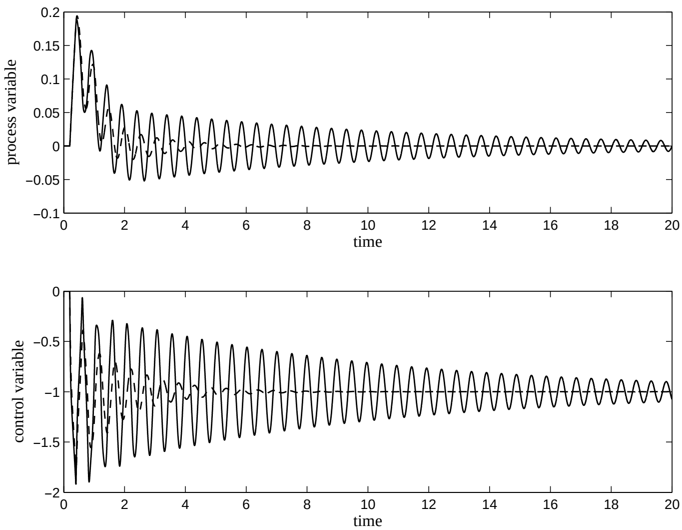

Fig. 2.8. Load disturbance step response for the PID controllers with Kappa–Tau parameters (Process (2.18)). Solid line: ideal form with derivative filter; dashed line: series form with derivative filter.

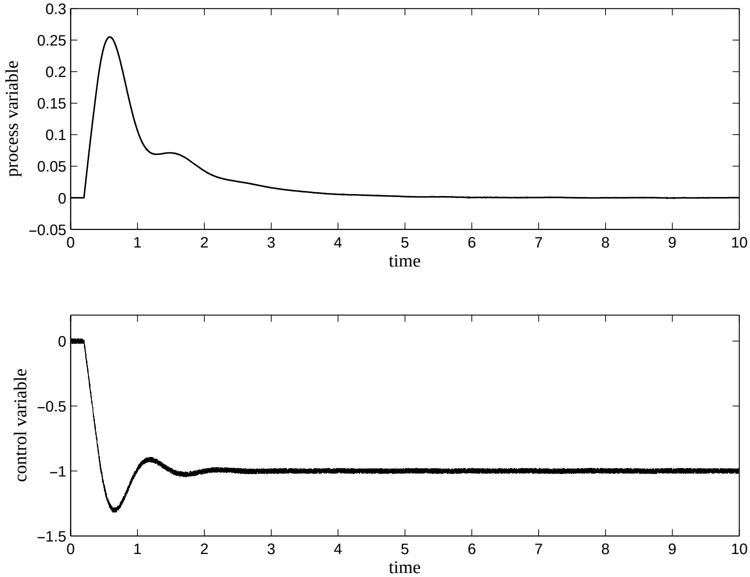

Fig. 2.9. Load disturbance step response (with noise measurement) for the PI controller

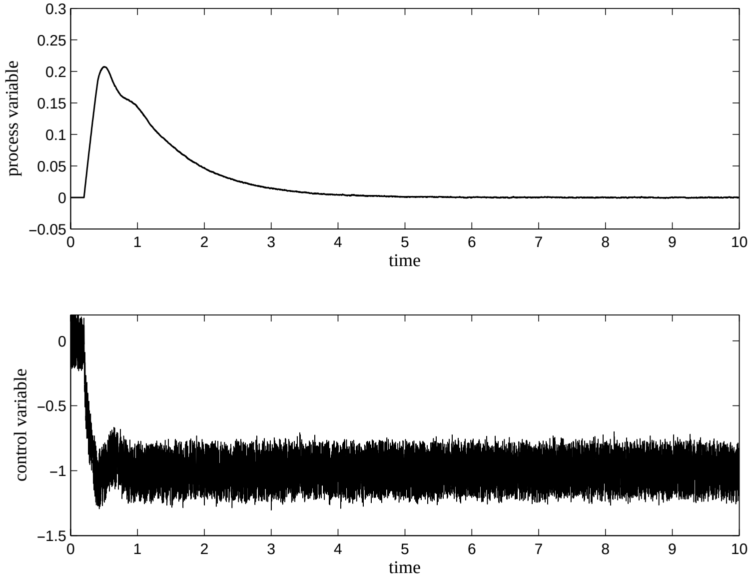

Fig. 2.10. Load disturbance step response (with noise measurement) for the ideal PID controller with a first-order derivative filter

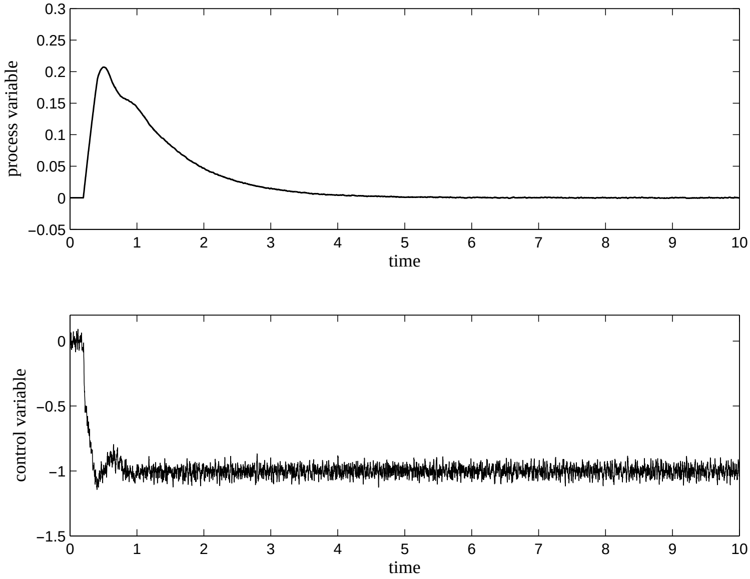

Fig. 2.11. Load disturbance step response (with noise measurement) for the ideal PID controller with a second-order derivative filter

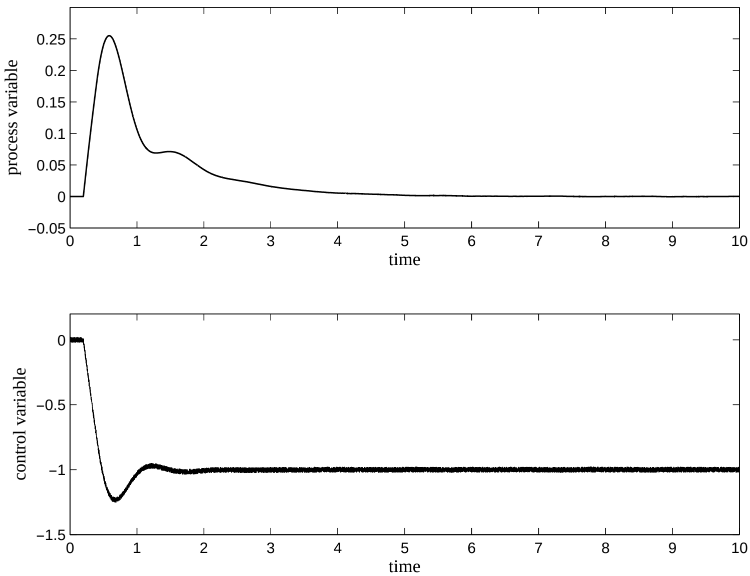

Fig. 2.12. Load disturbance step response (with noise measurement) for the ideal PID controller with a first-order output filter

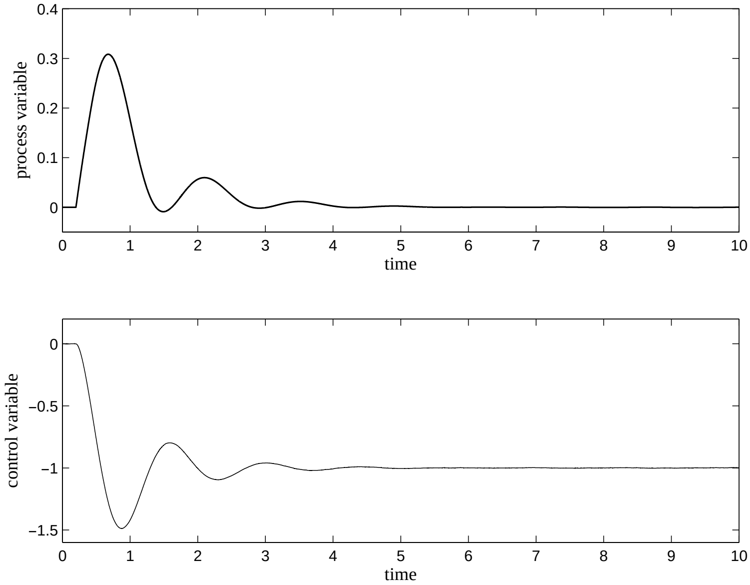

Fig. 2.13. Load disturbance step response (with noise measurement) for the ideal PID controller with a second-order output filter

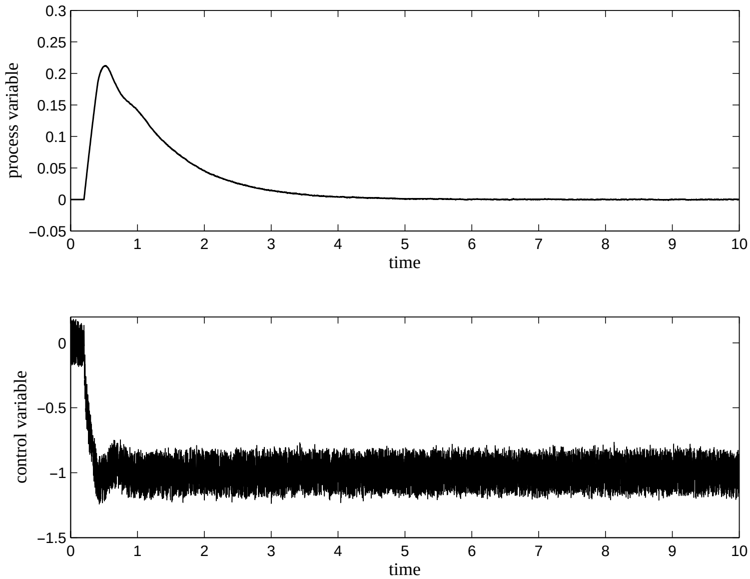

Fig. 2.14. Load disturbance step response (with noise measurement) for the series PID controller with a first-order derivative filter

2.5 Four-parameters Tuning

In the previous sections it has been underlined that problems associated with the derivative action that prevent a wide use of it are not just due to the noise. Indeed, tuning rules for a PID controller should involve four parameters, as also stressed in (Luyben, 2001*a*).
The most well-known design method that provides the values of all the four parameters of an ideal output-filtered PID controller is surely that based on the Internal Model Control (IMC) approach (Rivera *et al.*, 1986; Morari and Zafiriou, 1989). It can be remarked that a user-chosen parameter allows the handling of the trade-off between aggressiveness and robustness. The effectiveness of this tuning methodology has been shown in the literature; however, it has to be borne in mind that, being based on a pole-zero cancellation, it is not suitable for lag-dominant processes for which a very sluggish load disturbance response occurs (Shinskey, 1994; Shinskey, 1996). In this context an effective modification has been proposed in (Skogestad, 2003).
Recently, tuning rules that comprises also the derivative filter has been proposed in (Åström and Hägglund, 2004). They are based on the maximisation of the integral gain (so that the integrated error when a load disturbance occurs is minimised), subject to a robustness constraint. It is also stressed that the appropriate value of the ratio between the integral time constant and the derivative time constant should vary depending on the process dynamics (in particular, depending on the relative dead time of the process) and in most cases is less than four.
Similar conclusions are drawn in (Kristiansson and Lennartson, 2006). There, four-parameters tuning rules are proposed which take into account the trade-off between load disturbance rejection performance (in terms of integrated absolute error) and control effort, with a constraint on the generalised maximum sensitivity, which is a measure of the robustness of the control system. It is shown that the benefits of the derivative action can be severely limited if the ratio between the integral time constant and the derivative time constant is fixed to four and if the derivative filter factor is fixed in a PID controller in ideal form $C_{i1a}(s)$ (1.20) (or $C_{i1b}(s)$ (1.21)).
For this PID controller, it is suggested to set $T_i/T_d = 2.5$. Further, it is shown that considering the derivative filter time constant as a true tuning parameter allows a significant improvement of the overall performance.

2.6 Conclusions

In this chapter the design of the derivative filter has been discussed. Although the analysis provided and the examples presented are certainly not exhaustive, they are sufficient to show that the choice of the controller structure and of the derivative filter factor is indeed a critical issue and the PID controller should be considered as a four-parameters controller. In fact, the derivative

action is a key factor in improving the control system performance and the reason for being rarely adopted in practice is not only the amplification of the measurement noise.
In particular, it has been shown that predefining the derivative filter factor in an ideal form controller $C_{i1a}(s)$ (1.20) (or $C_{i1b}(s)$ (1.21)) might severely limit the performance. If a series controller $C_s(s)$ (1.22) is adopted, then the filter does not influence the location of the controller zeros. However, in this case the two zeros have to be real and this factor might limit the performance as well. Thus, the most convenient choice appears to be the use of an output-filtered ideal form PID controller $C_{i2a}(s)$ (1.23) (or $C_{i2b}(s)$ (1.24)) since this is the most general expression and the drawbacks of the other two forms are avoided. Further, effective tuning rules for the selection of the four parameters K_p, T_i, T_d, and T_f are available in this case.