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Automatic Generation of DFA Minimization Problems

Automatische Generierung von DFA Minimierungsproblemen

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Abstract

The theory of deterministic finite automatons (DFAs) is a classical topic of computer science-related courses. A typical task for students is to minimize a DFA. However generation of those DFAs that shall be minimized is often done manually by the exercise instructor. This work presents ideas to automize the generation of DFA minimization tasks.

We start in chapter 2 with introducing minimization tasks, which consist of a DFA A_{task} which has to be minimized and the minimal solution DFA A_{sol}. We focus on the minimization algorithm by Hopcroft, which works in two steps: Firstly delete unreachable states, then merge equivalent state pairs.

Following this separation in reverse, our approach is to generate the solution DFA first, then create equivalent state pairs and lastly add unreachable states. We devise several sensible input parameters and requirements for each of these stages.

Concerning the generation of solution DFAs (chapter 3) we make use of a simple rejection algorithm, that generates test DFAs by randomization or enumeration. Test DFAs are rejected, if they do not match the demanded properties. On this topic research has already been active, an overview about results there is made to draw conclusions for this work.

In chapter 4 we describe the extension of a solution DFA towards a task DFA. To archieve this, we can add states and transitions in an easy manner according to certain rules, which are derived from the properties of equivalent state pairs and unreachable states.

Zusammenfassung

Automatentheorie ist ein klassisches Thema in Lehre mit Informatikbezug. Eine typische Aufgabe für Studenten ist die Minimierung eines deterministischen endlichen Automaten (DEAs). Das Generieren solcher Minimierungsaufgaben wird allerdings häufig manuell vom Übungsleiter vorgenommen. In dieser Arbeit werden somit Ideen präsentiert um DEAs automatisiert zu generieren.

Wir beginnen in Kapitel 2 mit einer Beschreibung von Minimierungsaufgaben, die im Wesentlichen aus einem *Aufgaben-DEA* A_{task}, dem zu minimierenden DEA, und dem bereits minimierten *Lösungs-DEA* A_{sol} bestehen. Wir werden uns hier auf den Minimierungsalgorithmus von Hopcroft beschränken, der in zwei Schritten abläuft: Zunächst werden unerreichbare Zustände entfernt und dann äquivalente Zustandspaare zusammengefasst.

In unserem Ansatz nutzen wir diese Zweiteilung indem wir sie umdrehen, sodass zunächst der Lösungs-DEA generiert wird, woraufhin äquivalente Zustandspaare erzeugt und unerreichbare Zustände hinzugefügt werden. Für jeden dieser Schritte werden wir sinnvolle Eingabeparameter und Anforderungen definieren.

Um die Lösungs-DEAs zu generieren (Kapitel 3) machen wir Gebrauch von einem simplen Algorithmus, der wiederholt Test-DEAs mittels Randomisierung oder Enumerierung erzeugt und sie immer dann ablehnt, wenn sie den gewünschten Eigenschaften nicht entsprechen. Zu diesem Thema gab es bereits einige Forschungsarbeit, folglich werden wir einen Überblick über relevante Ergebnisse geben um dann Schlussfolgerungen für diese Arbeit zu ziehen.

In Kapitel 4 beschreiben wir, wie Lösungs-DEAs zu Aufgaben-DEAs erweitert werden können. Um das zu erreichen können wir Zustände und Transitionen recht einfach mithilfe gewisser Regeln hinzufügen. Diese Regeln werden direkt von den Eigenschaften äquivalenter Zustandspaare und unerreichbarer Zustände abgeleitet.

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Chapter 1

Introduction

Automata theory is recommended as part of a standard computer science curriculum [12, pp. 5-6]. It provides the chance to gain a precise cognitive model of a theory, possibly yielding new perspectives on other problems and topics. This may thus lead to increased problem solving skills and more accurate thinking.

A typical task in automata theory is the minimization of a given deterministic finite automaton (DFA). The classic textbook “Introduction to automata theory, languages, and computation” by Hopcroft et al. [15] presents a practicable minimization algorithm. We confine ourselves to look at DFA minimizations using that algorithm.

In an introduction course to theoretical computer science minimization tasks are thus likely to occur in exercises or exams. As of the creation of such tasks, one may assume, that it is done mostly manually. Automation would yield here the following advantages:

- freeing time for other things, e.g. research, helping students face-to-face, designing exercise sheets
- increased predictability and consistency of the generated task properties, which can be adjusted accurately through various parameters
- saves humans from generating those tasks which involves monotonous work

Engagement on this topic promises moreover increased clarification which kind of minimization tasks can be generated, and where difficulties of those tasks lie.

This work aims to provide theoretical foundations for a DFA minimization task generator. What requirements a user has towards such a program will be discussed in a short requirements analysis. Based on this work a DFA minimization generator will be devised. Alongside to this thesis an implementation of such a generator has been developed. It can be found at `https://github.com/bt701607/Generation-of-DFA-Minimization-Problems`.

Chapter 2

Problem definition and approach

In this chapter we will set foundations, investigate sensible parameters and requirements for a minimization task generator and deduce our general approach to build such a program.

2.1 Preliminaries

We start with defining preliminary theoretical foundations. By $[[n]]$ we will denote the set of integers $\{0, \ldots, n-1\}$.

2.1.1 Deterministic Finite Automatons

A 5-tuple $A = (Q, \Sigma, \delta, s, F)$ with Q being a finite set of *states*, Σ a finite set of *alphabet symbols*, $\delta\colon\ Q \times \Sigma \to Q$ a *transition function*, $s \in Q$ a *start state* and $F \subseteq Q$ *final states* is called *deterministic finite automaton* (DFA) [15, p. 46]. From now on $\mathcal{A}$ shall denote the set of all DFAs.

We say $\delta(q, \sigma) = p$ is a transition from q to p using symbol σ. We define the *extended transition function* $\delta^* : Q \times \Sigma^* \to Q$ of a DFA $A = (Q, \Sigma, \delta, s, F)$ as:

- $\delta^*(q, \varepsilon) = q$
- $\delta^*(q, w\sigma) = \delta(\delta^*(q, w), \sigma)$ for all $q \in Q$, $w \in \Sigma^*$, $\sigma \in \Sigma$

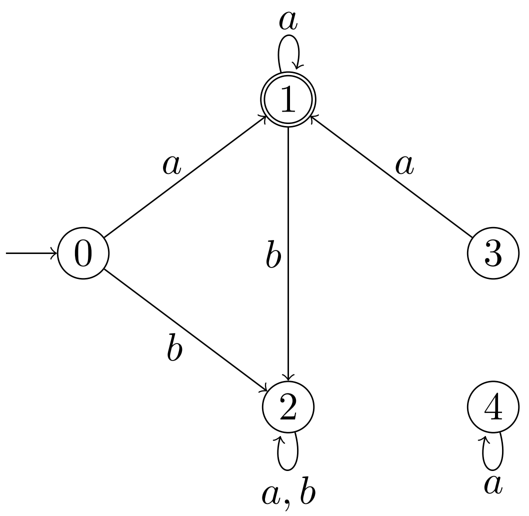
Then, the *language* of A is defined as $L(A) = \{\ w \mid \delta^*(w) \in F\ \}$ [15, pp. 49-50. 52].

Given a state $q \in Q$. With $d^-(q)$ we denote the set of all *ingoing* transitions $\delta(q', \sigma) = q$ of q. With $d^+(q)$ we denote the set of all *outgoing* transitions $\delta(q, \sigma) = q'$ of q [9, pp. 2-3].

Definition 1 ((Un-)Reachable State)**.** We say a state q is *(un-)reachable* in a DFA A, iff there is (no) a word $w \in \Sigma^*$ such that $\delta^*(s, w) = q$.

If all states of a DFA A are reachable, then we call A *accessible* [9, p. 2].

A DFA is called *complete* iff for all states, every symbol of the alphabet is used on an outgoing transition: $\forall q \in Q\colon \forall \sigma \in \Sigma\colon \exists p \in Q\colon \delta(q, \sigma) = p$. Note, that every incomplete DFA can be converted to a complete one by adding a so called *dead state* [15, p. 67]. The resulting automaton has the same language. From now on we will only work with complete DFAs.



$A = (Q, \Sigma, \delta, s, F)$

$Q = [[5]] = \{ 0, 1, 2, 3, 4 \}$
$\Sigma = \{ a, b \}$
$\delta = \{ ((0, a), 1), ((0, b), 1), \dots \}$
$s = 0$
$F = \{ 1 \}$

Figure 2.1: An example DFA. The states 3 and 4 are unreachable. This DFA is not complete since the transitions $\delta(2, b)$ and $\delta(3, b)$ are not defined.

Definition 2 (Minimal DFA)**.** We call a DFA A *minimal*, if there exists no other DFA with the same language having less states.

With $\mathcal{A}_{min}$ we shall denote the set of all minimal DFAs.

Definition 3 (Equivalent and Distinguishable State Pairs)**.** [15, p. 154] A state pair $q_1, q_2 \in Q$ of a DFA $A = (Q, \Sigma, \delta, s, F)$ is called *equivalent*, iff $\sim_A (q_1, q_2)$ is true, where

$$q_1 \sim_A q_2 \text{ is true } \Leftrightarrow_{def} \forall z \in \Sigma^* \colon \ (\delta^*(q_1, z) \in F \Leftrightarrow \delta^*(q_2, z) \in F)$$

If $q_0 \not\sim_A q_1$, then q_0 and q_1 are called a *distinguishable* state pair. It is well-known (see for instance [15, p. 160]) that the relation $\sim_A$ is an equivalence relation.

2.1.2 Isomorphy of DFAs

Given two DFAs $A_1 = (Q_1, \Sigma_1, \delta_1, s_1, F_1)$ and $A_2 = (Q_2, \Sigma_2, \delta_2, s_2, F_2)$. We say A_1 and A_2 are *isomorph*, iff:

- $|Q_1| = |Q_2|$, $\Sigma_1 = \Sigma_2$ and
- there exists a bijection $\pi \colon Q_1 \to Q_2$ such that:
  $\pi(s_1) = s_2$
  $\forall q \in Q_1 \colon (q \in F_1 \iff \pi(q) \in F_2)$
  $\forall q \in Q_1 \colon \forall \sigma \in \Sigma_1 \colon \pi(\delta_1(q, \sigma)) = \delta_2(\pi(q), \sigma))$

In [24, p. 45] we can find the following statement:

Theorem 1. *Every minimal DFA is unique (has a unique language) except for isomorphy.*

We describe a simple isomorphism test for DFAs in appendix A.

2.1.3 The minimization algorithm

This minimization algorithm MinimizeDFA works in four major steps, essentially removing states in such a way, that no unreachable states and no equivalent state pairs are left.

1. Compute all unreachable states via breadth-first search.

   function ComUnreachables(A)
   $U \leftarrow Q \setminus \{s\}$ ▷ undiscovered states
   $O \leftarrow \{s\}$ ▷ observed states
   $D \leftarrow \{\}$ ▷ discovered states
   while $|O| > 0$ **do**
   $N \leftarrow \{\ p \mid \exists q \in O\ \sigma \in \Sigma\colon\ \delta(q, \sigma) = p\ \wedge\ p \notin O \cup D\ \}$
   $U \leftarrow U \setminus N$
   $D \leftarrow D \cup O$
   $O \leftarrow N$
   return U

2. Remove all unreachable states and their transitions.

   function RemUnreachables(A, U)
   $\delta' \leftarrow \delta \setminus \{\ ((q, \sigma), p) \in \delta \mid q \in U\ \vee\ p \in U\ \}$
   return $(Q \setminus U, \Sigma, \delta', s, F \setminus U)$

3. Compute all equivalent state pairs ($\sim_A$). Inspired by Schöning [24, p. 46] and Martens and Schwentick [26, p. 17].

   function ComEquivPairs(A)
   $M \leftarrow \{(p, q), (q, p) \mid p \in F, q \notin F\}$
   do
   $M' \leftarrow \{(p, q) \mid (p, q) \notin M \wedge \exists \sigma \in \Sigma\colon (\delta(p, \sigma), \delta(q, \sigma)) \in M\}$
   $M \leftarrow M \cup M'$
   while $M' \neq \emptyset$
   return $Q^2 \setminus M$

   Note that ComEquivPairs requires its input automaton to be complete.[1]

4. Merge all equivalent state pairs, which are exactly those in $\sim_A$. Inspired by Högberg and Larsson [17, p. 10].

   function RemEquivPairs(A, $\sim_A$)
   $Q_E \leftarrow \emptyset$
   $\delta_E \leftarrow \emptyset$
   $F_E \leftarrow \emptyset$
   for q **in** Q **do**
   Add $[q]$ to Q_E ▷ $[\cdot]_{\sim_A}$ shall be abbreviated $[\cdot]$
   for σ **in** Σ **do**
   $\delta_E([q], \sigma) = [\delta(q, \sigma)]$
   if $q \in F$ **then**
   Add $[q]$ to F_E
   return $(Q_E, \Sigma, \delta_E, [s], F_E)$

function MinimizeDFA(A)
$A' \leftarrow$ RemUnreachables(A, ComUnreachables(A))
return RemEquivPairs(A', ComEquivPairs(A'))

This DFA minimization algorithm has been found by Hopcroft [14].

[1] Most authors implicitly assume this or define DFAs to be complete from the beginning.

Theorem 2. [15, pp. 162-164] MinimizeDFA *computes a minimal DFA to its input DFA.*

When looking at ComEquivPairs, one notes, that it computes distinct subsets of $Q \times Q$ on the way. Indeed, one could write the algorithm in such a way, that these subsets are explicitly computed in form of a function $m \colon \mathbb{N} \to \mathcal{P}(Q \times Q)$:

function m-ComEquivPairs(A)
    $i \leftarrow 0$
    $m(0) \leftarrow \{(p,q),(q,p) \mid p \in F, q \notin F\}$
    do
        $i \leftarrow i + 1$
        $m(i) \leftarrow \{(p,q),(q,p) \mid (p,q) \notin \bigcup m(\cdot) \wedge \exists \sigma \in \Sigma \colon (\delta(p,\sigma), \delta(q,\sigma)) \in m(i-1)\}$
    while $m(i) \neq \emptyset$
    return $\bigcup m(\cdot)$

Using this redefinition, we can easier refer to the state pairs marked in a certain iteration. We will use both variants in exchange.

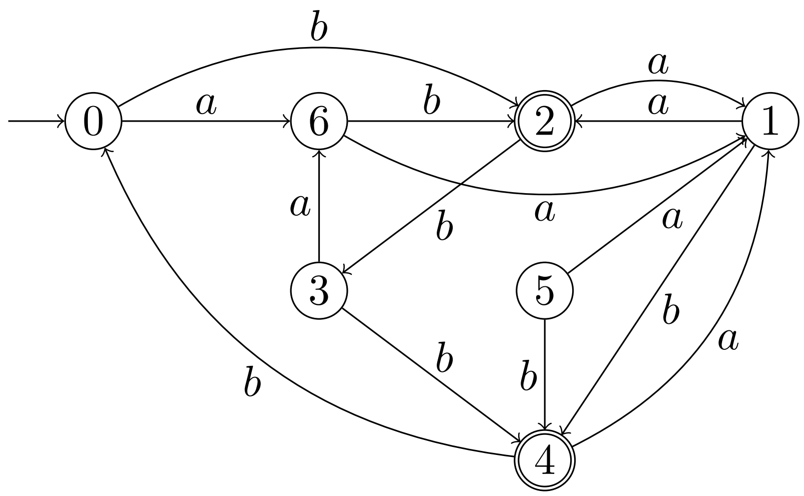
Definition 4. We denote the number of iterations done by ComEquivPairs on an DFA A as $\mathfrak{D}(A)$.

This number will prove to be a major characteristic of difficulty for a minimization task.

2.2 Requirements analysis

Now that we have introduced all necessary basic definitions, we shall do a short analysis of an example DFA minimization task and its sample solution, as it could have been given to students in an introductory course on automata theory.

<u>Task:</u> *Consider the below shown deterministic finite automaton A:*



Apply the minimization algorithm and illustrate for each state pair of A during which ComEquivPairs-*iteration it was marked. Draw the resulting automaton.*

Figure 2.2: An example DFA minimization task.

Figures 2.2 and 2.3 show such a task and solution. The students are confronted with a *task DFA* A_{task}. Firstly, unreachable states have to be eliminated, we then gain A_{re} (having only *reachable* states). Secondly equivalent state pairs of A_{re} are merged such that the

<u>*Solution:*</u>
Step 1: Detect and eliminate unreachable states.

State 5 is unreachable.

Step 2: Apply ComEquivPairs *to A and merge equivalent state pairs:*

	0	1	2	3	4	6
0	■	1	0		0	2
1	■	■	0	1	0	1
2	■	■	■	0		0
3	■	■	■	■	0	2
4	■	■	■	■	■	0
6	■	■	■	■	■	■

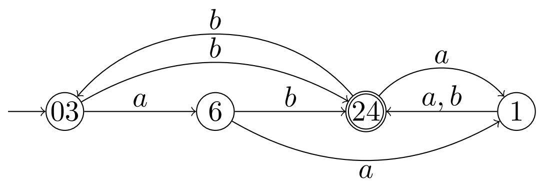


Figure 2.3: Solution to the DFA minimization task in fig. 2.2.

minimal *solution DFA* A_{sol} is found. The table T displayed in figure 2.3 is nothing else but a visualization of the function m of m-ComEquivPairs, whereas $T(q_0, q_1) = i \Leftrightarrow (q_0, q_1) \in m(i)$.

We list some statements and requirements. Firstly, we can state that

- $A_{re} =$ RemUnreachables(A_{task}, ComUnreachables(A_{task})) and
- $A_{sol} =$ RemEquivPairs(A_{re}, ComEquivPairs(A_{re})) .

Therefore A_{sol} is minimal regarding A_{re} and A_{task}. Secondly the languages of A_{task}, A_{re} and A_{sol} are the same. Furthermore we know that every state of A_{re} is reachable since it is the output of RemUnreachables.

2.2.1 Difficulty adjustment possibilities and sensible requirements

Concerning the execution of MinimizeDFA we find that its difficulty can be classified through various classification numbers. Furthermore we can note some sensible requirements.

ComEquivPairs-iterations ($\mathfrak{D}(A_{task})$). Consider the computation of the sets $m(i)$ in ComEquivPairs (see alg. 0). Determining $m(0)$ is quite straightforward, because it consists simply of tests whether two states are in $F \times Q \setminus F$ (see alg. 0, line 3). Determining $m(1)$ is less easy: The rule for determining all $m(i), i > 0$ is different to that for $m(0)$ and more complicated (see alg. 0, line 6). Determining $m(2), m(3), \ldots$ requires the same rule. It shows nonetheless a students understanding of ComEquivPairs' terminating behavior: The algorithm does not stop after computing $m(1)$, but only when no more distinguishable state pairs were found.

It would therefore be sensible if $\mathfrak{D}(A_{task})$ could be adjusted by parameters (we call them m_{min}, m_{max}) which give lower and upper bounds for that value.

Number of states (n_s, n_e, n_u). To control the number of states in A_{task}, A_{re} and A_{sol}, we will introduce three parameters: $n_s, n_e, n_u \in \mathbb{N}$, where

- n_s is the number of states of the *solution* DFA A_{sol}
- n_e is the number of distinct *equivalent* state pairs of A_{re}
- n_u is the number of *unreachable* states of A_{task}

They can be equivalently described by the following equations:

$$|Q_{sol}| = n_s$$
$$|Q_{re}| = n_s + n_e$$
$$|Q_{task}| = n_s + n_e + n_u$$

It is sensible to have $n_u > 1, n_e > 1$, such that RemUnreachables and RemEquivPairs will not be skipped. To not make the task trivial, $n_s > 2$ is sensible. An exercise instructor will find it useful to control exactly how big n_u, n_e and n_s are: The higher n_u, n_e, the more states have to be eliminated and merged. The higher $n_s + n_e$, the more state pairs have to be checked during ComEquivPairs.

Alphabet size (k). The more symbols the alphabet of A_{task}, A_{re} and A_{sol} has (note that MinimizeDFA does not change the alphabet), the more transitions have to be followed when checking whether $(\delta(q,\sigma), \delta(p,\sigma)) \in m(i-1)$ is true for each state pair p, q. In addition we will see later, that there exists no DFA with $k < 2$ having equivalent state pairs, so $k \geq 2$ is sensible.

Number of final states (n_F). Most DFAs in teaching have about 1 to 3 final states, so being able to set a number of final states allows concentrating on or deviating from familiar DFAs.

Uniqueness of solution DFA language. For example for an exam it would be sensible to be able to generate a task where the DFA language is unique, meaning there was no previously generated DFA with the same language. We will use the criterion of isomorphy (see Theorem 1) to categorize a possible DFA.

Note that, if A_{sol} is indeed *new* in that sense, then A_{task} will automatically have a unique language too, since A_{sol} and A_{task} have the same language.

Completeness of task DFA. In opposition to ComEquivPairs, ComUnreachables and RemUnreachables do not require their input DFA A_{task} to be complete. So we could have unreachable states in A_{task}, to which δ is not defined for all alphabet symbols, but all other states have to be complete. It is however sensible, to build task DFAs complete too to avoid possible confusion: Such subtleties do not highlight the main ideas of MinimizeDFA.

Nonetheless we shall introduce a parameter c, that determines if there exist unreachable states, that make A_{task} incomplete. Thus an exercise lecturer has the option, to showcase this matter on a DFA and generate according exercises.

Planar drawing of task DFA. A graph G is *planar* if it can be represented by a drawing in the plane such that its edges do not cross. Such a drawing is then called *planar drawing* of G. A visual aid for students would be given, if the task DFA were planar and presented as a planar drawing. In this work libraries and parameters $p_1, p_2 \in \{0, 1\}$ (toggling planarity of A_{sol}, A_{task}) will be used to allow the option of planarity, but neither ensuring planarity nor planar drawing will be investigated further theoretically.

Maximum degree of any state in task DFA. The *degree* $deg(q)$ of a state $q \in Q$ in a DFA A is defined as $deg(q) = d^-(q) + d^+(q)$, so the total number of transitions in which q participates. By capping the maximum degree for all states, the graphical representation of the DFA would be more clear. In this work the inclusion of a maximum degree parameter is omitted.

2.3 Approach and general algorithm

In this work we will first build the solution DFA (step 1), and - based on that - the task DFA by creating equivalent states and adding unreachable states (step 2). Both steps together will fulfill all criteria chosen above and are covered each in depth in chapter 3 respectively chapter 4.

We will see that $\mathfrak{D}$ and L of both DFAs will be set when building A_{sol}. We know that creating equivalent states and adding unreachable does not change $L(A_{task})$ in comparison to A_{sol}, else MinimizeDFA would not work (a minimal DFA has in particular the same language as the original DFA). However we must ensure, that adding those states does not change $\mathfrak{D}$. Since unreachable states are eliminated before ComEquivPairs is applied, we need only to prove, that creating equivalent states does not change the $\mathfrak{D}$-value. We will do this during the discussion of step 2, more specifically in section 4.1.4.

At the beginning of chapter 3 and 4, we will provide formal problem definitions for both steps, that specify precisely all requirements. Here we shall content ourselves with the definition of the main algorithm:

function GenerateDFAMinimizationTask$(n_s, k, n_F, m_{min}, m_{max}, p_1, p_2, n_e, n_u, c)$
    $A_{sol} \leftarrow$ GenerateNewMinimalDFA$(n_s, k, n_F, m_{min}, m_{max}, p_1)$
    $A_{task} \leftarrow$ ExtendMinimalDFA$(A_{sol}, p_2, n_e, n_u, c)$
    return A_{sol}, A_{task}

Chapter 3

Generating minimal DFAs

We seek algorithms for generation of minimal DFAs that fulfill the conditions defined in the requirements analysis section 2.2.1. We formally subsume these conditions via the GenerateNewMinimalDFA-problem:

Definition 5 (GenerateNewMinimalDFA)**.**
<u>Given:</u>

$$\begin{aligned} n_s &\in \mathbb{N} && \textit{number of states} \\ k &\in \mathbb{N} && \textit{alphabet size} \\ n_F &\in \mathbb{N} && \textit{number of final states} \\ m_{min}, m_{max} &\in \mathbb{N} && \textit{lower and upper bound for } \mathfrak{D}\textit{-value} \\ p &\in \{0,1\} && \textit{planarity-bit} \end{aligned}$$

<u>Task:</u> *Compute, if it exists, a solution DFA* A_{sol} *with*

- $|Q_{sol}| = n_s$, $|\Sigma_{sol}| = k$, $|F_{sol}| = n_F$
- $m_{min} \le \mathfrak{D}(A_{sol}) \le m_{max}$
- A_{sol} *being planar iff* $p = 1$
- $L(A_{sol})$ *being new*

We consider different approaches to solve this problem, of which those using trial-and-error will be discussed most broadly.

Remark. Note that for all generated DFAs we are going to set $Q_{sol} = [[n_s]] = \{0, \ldots, n_s - 1\}$, $\Sigma_{sol} = [[k]] = \{0, \ldots, k-1\}$ and $s_{sol} = 0$, so every DFA of same state number and alphabet size will have the same states and symbols.

As a consequence the presented algorithms will not be able to compute all of $\mathcal{A}_{min}$.

3.1 Using a rejection algorithm

We describe a procedure that is essentially a *rejection algorithm* adjusted to find solution DFAs. The approach works as follows:

Firstly a *test* DFA A_{test} is generated by use of either randomness or enumeration. Alphabet size and number of (final) states will already be correct. On this DFA then tests will be executed, to check if it is minimal, planar (if wished) and new. If this is the case, A_{test} will be returned, if not, new test DFAs are generated until all tests pass.

A note on the search space. If we would not restrict ourselves to $Q_{sol} = [[n_s]]$ and $\Sigma_{sol} = [[k]]$, then for a given number of states and symbols, the number of possible state sets and alphabets would be infinite. This way however we do not have to iterate through infinitely many same-sized versions of Q_{sol} respectively Σ_{sol}. Since there is a finite number of possible transitions functions and final state sets given n_s, k, we can now even guarantee that the enumerating variant of our algorithm is going to terminate.

1: **function** BuildNewMinimalDFA-1 $(n_s, k, n_F, m_{min}, m_{max} \in \mathbb{N}, p \in \{0,1\})$
2:     **while** True **do**
3:         generate DFA A_{test} with $|Q|, |\Sigma|, |F|$ matching n_s, k, n_F
4:         **if** A_{test} not minimal **or not** $m_{min} \le \mathfrak{D}(A_{test}) \le m_{max}$ **then**
5:             **continue**
6:         **if** $p = 1$ **and** A_{test} is not planar **then**
7:             **continue**
8:         **if** $L(A_{test})$ is not new **then**
9:             **continue**
10:         **return** A_{test}

We will complete this algorithm by resolving how the tests in lines $4, 6$ and 8 work and by showing two methods for generation of DFAs with given restrictions of $|Q|, |\Sigma|$ and $|F|$.

3.1.1 Ensuring Correctness of D-value and Minimality

In order to test, whether A_{test} is minimal, we could simply use the minimization algorithm and compare the resulting DFA and A_{test} using an isomorphy test. However it is sufficient to ensure, that no equivalent or unreachable states exist.

To get $\mathfrak{D}(A_{test})$, we have to run ComEquivPairs entirely anyway. Hence we can combine the test for equivalent states with computing the DFAs $\mathfrak{D}$-value:

1: **function** HasEquivalentStates(A)
2:     $depth \leftarrow 0$ ▷ will be $\mathfrak{D}(A)$ in the end
3:     $M \leftarrow \{(p,q),(q,p) \mid p \in F, q \notin F\}$
4:     **do**
5:         $depth \leftarrow depth + 1$
6:         $M' \leftarrow \{(p,q) \mid (p,q) \notin M \land \exists \sigma \in \Sigma\colon (\delta(p,\sigma), \delta(q,\sigma)) \in M\}$
7:         $M \leftarrow M \cup M'$
8:     **while** $M' \neq \emptyset$
9:     $hasDupl \leftarrow |\{(p,q) \mid p \neq q \land (p,q) \notin M\}| > 0$
10:     **return** $hasDupl, depth$

Since ComEquivPairs basically computes all distinguishable state pairs $\not\sim_A$, we test in line 9, whether there is a pair of distinguishable states not in $\not\sim_A$.

Regarding the unreachable states, we can just use ComUnreachables and test whether the computed set is empty:

1: **function** HasUnreachableStates(A)
2:     **return** $|$ComUnreachables$(A)| > 0$

3.1.2 Ensuring planarity

There exist several algorithms for planarity testing of graphs. In this work, the library *pygraph*[1] has been used, which implements the Hopcroft-Tarjan planarity algorithm. More

[1] `https://github.com/jciskey/pygraph`

information on this can be found for example in this [18] introduction from William Kocay. The original paper describing the algorithm is by Hopcroft and Tarjan [13].

3.1.3 Ensuring uniqueness

In our requirements we stated, that we wanted the generated solution DFA to be new, meaning not isomorph to any previously generated solution DFA. This implies the need of a database, that allows saving and loading already found DFAs. We name this database *DB1*. Assuming the database is relational, the following scheme is proposed:

$$|Q_A| \quad |\Sigma_A| \quad |F_A| \quad \mathfrak{D}(A) \quad \mathit{isPlanar}(A) \quad \mathit{encode}(A)$$

With this scheme we can once fetch all DFAs matching the search parameters. Thus we need not fetch all previously found DFAs every time, but only those that are relevant. Afterwards we must only check whether any isomorphy test on the current test DFA and one of the fetched DFAs is positive. If any test DFA passes all tests and is going to be returned, then we have to save that DFA in the database.

A more concrete specification of this proceeding is shown below, embedded in the main algorithm:

function BuildNewMinimalDFA-2 $(n_s, k, n_F, m_{min}, m_{max}, p)$
    $l \leftarrow$ all DFAs in DB1 matching $n_s, k, n_F, m_{min}, m_{max}, p$
    while True **do**
        ...
        if A_{test} is isomorph to any DFA in l **then**
            continue
        save A_{test} and its respective properties in DB1
        return A_{test}

To test whether A_{test} is isomorph to any found DFA, we use the isomorphism test described in appendix A.

3.1.4 Option 1: Generating Test DFAs via Randomness

We now approach the task of generating a random DFA whereas alphabet and number of (final) states are set. For our generated DFA we choose Q_{sol}, Σ_{sol} and the start state as explained in remark 3.

The remaining elements that need to be defined are δ and F. The set of final states is supposed to have a size of n_F and be a subset of Q. Therefore we can simply choose randomly n_F distinct states from Q.

The transition function has to make the DFA complete, so we have to choose an "end" state q' for every state-symbol-pair q, σ in $Q \times \Sigma$. There is no restriction concerning q', so we can randomly choose $\delta(q, \sigma) = q'$ from Q.

With defining how to compute δ we have covered all elements of a DFA.

function BuildNewMinimalDFA-3a $(n_s, k, n_F, m_{min}, m_{max}, p)$
    $l \leftarrow$ all DFAs in DB1 matching $n_s, k, n_F, m_{min}, m_{max}, p$
    $Q \leftarrow [[n_s]]$
    $\Sigma \leftarrow [[k]]$
    while True **do**
        $\delta \leftarrow \emptyset$

```
        for q in Q do
            for σ in Σ do
                δ(q, σ) = random chosen state from Q
        s ← 0
        F ← random sample of n_F states from Q
        A_test ← (Q, Σ, δ, s, F)
        if A_test not minimal or not m_min ≤ 𝔇(A_test) ≤ m_max then
            continue
        if p = 1 and A_test is not planar then
            continue
        if A_test is isomorph to any DFA in l then
            continue
        save A_test and its respective properties in DB1
        return A_test
```

3.1.5 Option 2: Generating Test DFAs via Enumeration

The second method of test DFA generation is based on the idea, that instead of randomly generating F and δ, we could just enumerate through all possible final state sets and transition functions.

Both enumerations are finite, given n_s and k. Having a requirement of n_F final states, then $\binom{n_s}{n_F}$ is the number of possible F-configurations. On the other hand there are ${n_s}^{n_s k}$ possible δ-configurations: We have to choose one of n_s possible end states for every combination in $Q \times \Sigma$ - so $n_s k$ times.

Again we will call our states and symbols $[[n_s]]$ resp. $[[k]]$. We will represent the state of an enumeration with two fields F_F and F_δ. The first field shall have n_s Bits, whereas Bit $F_F[i] \in \{0, 1\}$ represents the information, whether i is a final state or not. The second field shall have $n_s k$ entries containing state names, such that entry $F_\delta[i * k + j] = q, q \in [[n_s]]$ says, that $\delta(i, j) = q$.

Example 1. *Given $n_s = 4$, $k = 2$, $n_F = 3$. Note that for the sake of readability we will use $a, b, \dots$ instead of $0, 1, 2, \dots$ as alphabet symbols. An example F_F-array could be 1101. Since $F_F[i]$ is 1 for $i = 0, 1, 3$ the final states are:*

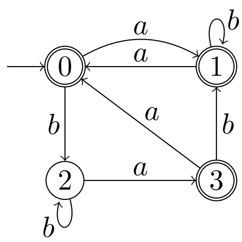
$$F = \{\ 0, 1, 3\ \}$$

A possible F_δ-array could be 12013201. The following table depicts, how F_δ assigns one state to every combination of states and symbols $(q, \sigma) \in Q \times \Sigma$ and thus defines δ:

$q \in Q$	0		1		2		3	
$\sigma \in \Sigma$	a	b	a	b	a	b	a	b
$\delta(q, \sigma)$	1	2	0	1	3	2	0	1

$\delta(0, a) = 1$, $\delta(0, b) = 2$,
$\delta(1, a) = 0, \dots$

The corresponding DFA might then look like this:



□

Given an enumeration state F_F, F_δ and n_s, k, n_F we will then compute the next DFA based on this state as follows. We will treat both fields as numbers, F_F as 2-ary and F_δ as n_s-ary. To get to the next DFA, we will first increment F_δ by 1. If $F_\delta = n_s - 1 \ldots n_s - 1$, then we increment F_F until it contains n_F ones (again) and set F_δ to $0 \ldots 0$. This behavior is summarized in the following algorithm:

function IncrementEnumProgress $(F_F, F_\delta, n_s, k, n_F)$
    add 1 to $(F_\delta)_{n_s}$
    if $F_\delta = 0 \ldots 0$ **then**
        while $\#_1(F_F) \neq n_F$ **do** ▷ if the number of 1s in F_F is not n_F
            add 1 to $(F_F)_2$
            if $F_F = 0 \ldots 0$ **then**
                return $\perp$
            $F_\delta = 0 \ldots 0$
    return F_F, F_δ

In this algorithm we assume, that adding 1 to a n-ary number $n-1\ n-1\ \ldots n-1$ yields $00 \ldots 0$.

Example 2. *We showcase a sample enumeration at points, that demonstrate the semantics of different increments. We will use $n_s = 4$, $k = 2$ and $n_F = 2$. Note that we will use $a, b, \ldots$ instead of $0, 1, \ldots$ as symbols again. Valid enumeration progresses are depicted green.*

We will start with the initial enumeration progress (1). In this case, a simple addition of 1 to F_δ does not cause an overflow of F_δ (2), meaning the enumeration increment is already finished.

(3). In this state however F_δ becomes $0 \ldots 0$ (4) after adding 1. Thus we add 1 to F_F, until it contains the required number of ones again (so we always have f final states). The next 4-ary number with 2 ones after 0011 is 0101 (5).

(6). Here F_δ is at its maximum and there is no higher 4-ary number with 2 ones. So if the algorithm is now applied and tries to get the next valid F_F-value, it will eventually reach (7), which indicates the enumeration has found its end.

(1) $(0011)_2$ $(00\ 00\ 00\ 00)_4$
$+1$
(2) $(0011)_2$ $(00\ 00\ 00\ 01)_4$
$\ldots$ $+x$
(3) $(0011)_2$ $(33\ 33\ 33\ 33)_4$
(4) $(0100)_2$ $(00\ 00\ 00\ 00)_4$ $+1$
(5) $(0101)_2$ $(00\ 00\ 00\ 00)_4$
$\ldots$ $+x$
(6) $(1100)_2$ $(33\ 33\ 33\ 33)_4$
$\ldots$
(7) $(1111)_2$ $(33\ 33\ 33\ 33)_4$

□

Based on the incremented bit-fields the new DFA can be build according to the semantics defined above:

function DFAfromEnumProgress $(F_F, F_\delta, n_s, k, n_F)$
    $Q \leftarrow [[n_s]]$
    $\Sigma \leftarrow [[k]]$
    $\delta \leftarrow \emptyset$
    for i **in** $[[n_s]]$ **do**
        for j **in** $[[k]]$ **do**
            $\delta(i, j) = F_\delta[i * k + j]$
    $s \leftarrow 0$
    for i **in** $[[n_s]]$ **do**

    if $F_F[i] = 1$ **then**
        Add i to F
return $(Q, \Sigma, \delta, s, F)$

The initial field values are each time $0 \ldots 0$. Note how construction and use of these fields results in DFAs with correct alphabet size and number of (final) states. An enumeration can finish either because a matching DFA has been found or all DFAs have been enumerated. The latter is the case, if $F_F = 1 \ldots 1$ and $F_\delta = n_s - 1 \ldots n_s - 1$.

Once the enumeration within a call of BuildNewMinimalDFA has been finished, it is reasonable to *save* the enumeration progress (meaning the current content of F_F, F_δ), such that during the next call enumeration can be resumed from that point on. The alternative would mean, that the enumeration is run in its entirety until that point again, whereas all so far found DFAs would be found to be not new. Thus we introduce a second database $DB2$ with the following table:

$\lvert Q_A \rvert$	$\lvert \Sigma_A \rvert$	F_F	F_δ

We reduce the enumeration room for each calculation.

function BuildNewMinimalDFA-3b $(n_s, k, n_F, m_{min}, m_{max}, p)$
    $l \leftarrow$ all DFAs in DB1 matching $n_s, k, n_F, m_{min}, m_{max}, p$
    $F_F, F_\delta \leftarrow$ load enumeration progress for n_s, k, n_F, p from DB2
    while True **do**
        if F_F, F_δ is finished **then**
            save F_F, F_δ
            return $\perp$
        $A_{test} \leftarrow$ next DFA based on F_F, F_δ
        if A_{test} not minimal **or not** $m_{min} \leq \mathfrak{D}(A_{test}) \leq m_{max}$ **then**
            continue
        if $p = 1$ **and** A_{test} is not planar **then**
            continue
        if A_{test} is isomorph to any DFA in l **then**
            continue
        save F_F, F_δ in DB2
        save A_{test} and its respective properties in DB1
        return A_{test}

3.2 On alternative approaches

Build m from m-ComEquivPairs iteratively. (Why would this basically result in running ComEquivPairs all the time?)

3.3 Related work on DFA generation

Nicaud provides an overview of results on random generation and combinatorial properties of DFAs in [21]. We will outline relevant related work.

Nicaud's summary indicates, that research has focused on randomized generation of accessible, but not minimal DFAs so far. In the following we will sketch some approaches that have come up.

Using the recursive method. Champarnaud and Paranthoën [9] continue ideas started by Nicaud in his thesis [20]. Let $\mathfrak{F}_{\mathfrak{n},\mathfrak{m}}$ be the set of extended m-ary trees of order n. These trees are characterized by a partitioning $V = N \uplus L$ with $|N| = n$ and the properties $v \in N \Rightarrow d^+(v) = m$ and $v \in L \Rightarrow d^+(v) = 0$. We define the following set of tuples using $s = n(m-1)$:

$$\mathfrak{R}_{\mathfrak{m},\mathfrak{n}} = \{ \ (k_1, \ldots, k_s) \in \mathbb{N}^s \mid \forall i \in [2, s] \colon k_i \geq \left\lceil \frac{i}{m-1} \right\rceil \ \textit{and} \ k_i \geq k_{i-1} \ \}$$

In [9, p. 6] it is shown that there exists a bijection φ between $\mathfrak{F}_{\mathfrak{n},\mathfrak{m}}$ and $\mathfrak{R}_{\mathfrak{m},\mathfrak{n}}$ which maps to k_i, $i \in [1, s]$ of a tuple the number of leaves visited before the ith leaf in a tree. The connection to accessible DFAs is established by proving that "transition structures[2]" with $|Q| = n$, $|\Sigma| = m$ reduced to the set of the smallest paths from the s to each other state are in bijection with extended m-ary trees of order n (see [9, p. 8]).

As a consequence they are able to construct a random generation of accessible complete DFAs using the "recursive method" from [22] which generates n-tuples [9, p. 10]. Nicaud states in his survey that the algorithm's runtime is $\mathcal{O}(n^2)$ but notes, that generation of DFAs with more than "a few thousand states" is practically hard to do [21, pp. 10-11].

Almeida et al. [1, 2, 23] present and implement methods using a string-encoding of DFAs for exact enumeration and random generation of DFAs. Nicaud [21, p. 11] states in a remark, that this approach uses the same recursive method and differs only in the DFA encoding.

Using Boltzmann sampler. Bassino, David and Nicaud present and implement a more efficient random generator of accessible complete DFAs in [4, 6]. Their idea is based on so called Boltzmann samplers. This framework of samplers is characterized in particular by the fact that the size of its generated objects are not fixed but in an interval around a given input size - this stands in opposition to most random generators in literature [11, p. 2].

In [6] the authors use a Boltzmann sampler to generate set partitions that are shown to be in bijection with so called box diagrams [6, p. 8] which are in turn in bijection to accessible complete DFAs [6, p. 4]. They thus acquire an average runtime complexity of $\mathcal{O}(n^{3/2})$ for a single random generation.

Using a rejection algorithm. Carayol and Nicaud [8] give a simple algorithm with the same runtime complexity. They use a result stating that the size of accessible DFAs is concentrated around some computable value. In the end random possibly inaccessible DFAs of a specific size are generated, of which afterwards all unreachable states are deleted. This is thus essentially a rejection algorithm with clever generation of test DFAs. They furthermore show that allowing approximate sampling with the number of states being in $[n - \varepsilon\sqrt{n}, n + \varepsilon\sqrt{n}]$ results in linear expected runtime.

Others and comparison to algorithm presented in this work. In his survey Nicaud mentions a paper by Bassino and Sportiello [3] that yields random generation of accessible DFAs in expected linear time. This work will not be discussed further here.

In this work we use a rejection algorithm that generates test DFAs either by randomization or by enumeration. Both methods implement a naive approach. The generated test DFAs are not necessary minimal and in particular not necessary accessible as in [8]. The enumeration method uses encodings of DFAs similar to those used by Almeida et. al. [23].

[2]Those are essentially DFAs without final state sets.

3.4 Empirical and combinatorial results

Concerning combinatorial properties of DFAs, several authors (e.g. [6, 10, 16]) consider a work from Vyssotsky [25] in the Bell laboratories to be the first on this subject. A contribution by Korshunov [19] is often cited in this regard, for he firstly “determines an asymptotic estimate of the number of accessible complete and deterministic n-state automata over a finite alphabet” [5].

Implementations (e.g. [1, 4]) of various random and enumeration generation methods have given rise to several empirical observations concerning the number of minimal DFAs, their fraction among all DFAs and so forth.

Domaratzki, Kisman, and Shallit [10] give some asymptotic estimates and explicit computations for the number of several types of distinct languages and automata. The here relevant results have been subsumed and extended in [2, p. 8] and were empirically confirmed in [4].

$\|\Sigma\|$ (k)	$\|Q\|$ (n)	$\|\mathcal{A}_{min,n,k}\|$	$\|\mathcal{A}_{n,k}\|$	Minimal %
$k = 2$	2	**24**	64	0.38
	3	**1028**	5832	0.18
	4	**56014**	1048576	0.05
	5	**3705306**	312500000	0.01
	6	**286717796**	139314069504	0.0
	7	**25493886852**	86812553324672	0.0
$k = 3$	2	**112**	256	0.44
	3	**41928**	157464	0.27
	4	**26617614**	268435456	0.1
	5	**25184560134**	976562500000	0.03
$k = 4$	2	**480**	1024	0.47
	3	**1352732**	4251528	0.32
	4	**7756763336**	68719476736	0.11
$k = 5$	2	**1984**	4096	0.48
	3	**36818904**	114791256	0.32

Figure 3.1: Table depicting the exact amount of minimal complete DFAs among all complete DFAs for various sizes of Q, Σ. The numbers of minimal DFAs (bold numbers) are taken from [2, p. 8].

In Figure 3.1 we use these results to determine the ratios of minimal complete DFAs among all complete DFAs for given $|Q|$ and $|\Sigma|$. The number of all DFAs is computed as follows:

$$|\mathcal{A}_{n,k}| = \underbrace{n^{n*k}}_{\#\text{possible } \delta\text{'s}} * \underbrace{2^n}_{\#\text{possible sets } F}$$

Thus we gain an insight into how probable the random generation of a distinct minimal test DFA is without applying further constraints. For our proposed default parameters $n \in [4 - 5]$ and $k \in [2 - 3]$ the probabilities of successful generation range from 1% to 5%. Practical tests have shown that this leads to sufficient short run times for our implementation.

Further interesting results in this area include the determination of the fraction of all minimal automata among all accessible complete DFAs [5] and asymptotic estimates for the number of states that a random minimized DFA has [7].

Chapter 4

Extending minimal DFAs

We firstly define a formal problem for extending a minimal DFA A_{sol} to a task DFA A_{task} based on our requirements analysis (see sec. 2.2.1):

Definition 6 (ExtendMinimalDFA)**.**

<u>*Given:*</u>

$$\begin{aligned} A_{sol} = (Q, \Sigma, \delta, s, F) \in \mathcal{A}_{min} &\quad \textit{solution DFA} \\ n_e \in \mathbb{N} &\quad \textit{number of states creating equivalent state pairs} \\ n_u \in \mathbb{N} &\quad \textit{number of unreachable states} \\ p \in \{0,1\} &\quad \textit{planarity-bit} \\ c \in \{0,1\} &\quad \textit{completeness-bit for unreachable states} \end{aligned}$$

<u>*Task:*</u> *Compute, if it exists, a task DFA A_{task} with*

- $Q_{task} = Q_{sol} \cup \{r_1, \ldots, r_{n_e}, u_1, \ldots, u_{n_u}\}$
- $r_1, \ldots, r_{n_e}$ *each creating an equivalent state pair*
- $u_1, \ldots, u_{n_u}$ *unreachable*
- $\Sigma_{task} = \Sigma_{sol}$, $s_{task} = s_{sol}$, $F_{task} \subseteq F_{sol}$
- A_{task} *being planar iff* $p = 1$
- A_{task} *being complete iff* $c = 1$
- A_{sol} *being isomorph to* $\textsc{MinimizeDFA}(A_{task})$

In order to fulfill these requirements we will deduce for both kinds of states how they may be added by examining their desired properties. If we get to have the choice between several equally good options, we will choose by randomization. We will be guided by the separation of creating equivalent and unreachable states:

function $\textsc{ExtendMinimalDFA}(A_{sol}, n_e, n_u, p, c)$
    if $p = 0$ **then**
        $A_{re} \leftarrow \textsc{AddUnreachableStates}(A_{sol}, n_u, c)$
        return $\textsc{CreateEquivalentStatePairs}(A_{re}, n_e)$
    else
        $A_{task} \leftarrow \perp$
        while A_{task} not planar **do**
            $A_{re} \leftarrow \textsc{AddUnreachableStates}(A_{sol}, n_u, c)$

$A_{task} \leftarrow$ CREATEEQUIVALENTSTATEPAIRS(A_{re}, n_e)
return A_{task}

Concerning the planarity option, we use a rejection algorithm again and the same test as before (see sec. 3.1.2). We will show for the action of adding equivalent states, that this does not change a DFAs $\mathfrak{D}$-value - thus we do not have to care whether we are possibly changing it.

4.1 Creating equivalent state pairs

Step 3 and 4 of the minimization algorithm are concerned with detection and elimination of equivalent state pairs. We now want to add states $r_1, \ldots, r_{n_e}$ to a DFA A_{sol}, gaining A_{re} with $Q_{re} = Q_{sol} \cup \{r_1, \ldots, r_{n_e}\}$, such that each of these states is equivalent to a state in Q_{re}. Note that, for reasons of clarity, we are going to abbreviate from now on $A_{re} = A$, $Q_{re} = Q$, $\sim_{A_{re}} = \sim_A$ etc.

In our algorithm we will add each r_i separately starting from A_{sol}. Consider the properties $r_1, \ldots, r_{n_e}$ must have. Since we start from A_{sol}, and add in each step a state, that will be equivalent to a state in the so-far constructed DFA, it follows by transitivity, that each of $r_1, \ldots, r_{n_e}$ will be equivalent to a state e of A_{sol}.

$$\forall i \in [1, n_e]\colon\ \exists e \in Q_{sol}\colon\ r_i \sim_A e$$

In other words: Every new state r_i is by transitivity equivalent to a state e of A_{sol}. In our algorithm, we will first choose a to-be-equivalent-state $e \in Q_{sol}$ for each state we add.

4.1.1 Adding outgoing transitions

Regarding the outgoing transitions of any r_i equivalent to a state e, we are directly restricted by the equivalency relationship:

$$\begin{aligned}
& r_i \sim_A e \\
\Rightarrow\ & \forall z \in \Sigma^*\colon\ (\delta^*(r_i, z) \in F \Leftrightarrow \delta^*(e, z) \in F) \\
\Rightarrow\ & \forall \sigma \in \Sigma\colon \\
& \qquad \delta(r_i, \sigma) = q_1 \wedge \delta(e, \sigma) = q_2 \wedge \\
& \qquad \forall z' \in \Sigma^*\colon (\delta^*(q_1, z') \in F \Leftrightarrow \delta^*(q_2, z') \in F) \\
\Rightarrow\ & \forall \sigma \in \Sigma\colon \\
& \qquad \delta(r_i, \sigma) = q_1 \wedge \delta(e, \sigma) = q_2 \wedge q_1 \sim_A q_2 \\
\Rightarrow\ & \forall \sigma \in \Sigma\colon [\delta(r_i, \sigma)]_{\sim_A} = [\delta(e, \sigma)]_{\sim_A}
\end{aligned}$$

We may thus formulate the rule for adding outgoing transitions to a new state quite straightforward:

R1: For each symbol $\sigma \in \Sigma$ choose exactly one state (A shall be complete) $q \in [\delta(e, \sigma)]_{\sim_A}$ and set $\delta(r_i, \sigma) = q$.

Since the solution DFA is complete and since every here added state gets a transition for every alphabet symbol, we know that every $[\delta(e, \sigma)]_{\sim_A} \neq \emptyset$, so the rule is guaranteed to be fulfillable.

Note that this substep does not affect the belonging-to-an-equivalence-class of any other now existing state, since r_i cannot be reached yet - it has no ingoing transitions.

4.1.2 Adding ingoing transitions

First of all, we know, that r_i must be reachable, since we decided that all unreachable states are added later. So we need to give r_i at least one ingoing transition. Doing this, we have to ensure, that any state q, that gets an outgoing transition to r_i remains in its equivalence class.

Thus a fitting state q has to have a transition to some state in $[r_i]_{\sim_A} = [e]_{\sim_A}$ already. So, given a state q with $\delta(q, \sigma) = p$ and $p \in [e]_{\sim_A}$, we can set $\delta(q, \sigma) = r_i$ and thus "steal" q its ingoing transition.

We see here, that q must have at least 2 ingoing transitions, else it would become unreachable. Thus we summarize:

R2: Choose at least one $((q, \sigma), p) \in \delta$ with $[p] = [e]$ and $d^-(p) \geq 2$. Remove $((q, \sigma), p)$ from δ and add $((q, \sigma), r_i)$.

These finding lead us to a general requirement regarding the choice of a state e for an r_i: The equivalence class of any e has to contain at least one state with at least 2 ingoing transitions. We establish the following notion to pin down this restriction:

$$duplicatable(q) \Leftrightarrow_{def} (\exists p \in [q]_{\sim_A} : |d^-(p)| \geq 2)$$

The number of duplicatable states in any accessible DFA A is 0 for $|\Sigma| \leq 1$ (due to the restriction $|d^-(p)| \geq 2$) and greater than 0 for $|\Sigma| > 1$ due to the pigeonhole principle: An accessible complete DFA has $|Q||\Sigma|$ transitions which have to be spread across $|Q|$ states.

4.1.3 The algorithm

function CreateEquivalentStatePairs(A, n_e)
  $Q \leftarrow Q_{sol}$
  $\delta \leftarrow \delta_{sol}$
  $F \leftarrow F_{sol}$
  $K \leftarrow \{ \{q\} \mid q \in Q \}$ ▷ tracks the equivalence classes of A
  $k(q) = C$ such that $q \in C$ and $C \in K$ ▷ returns the equivalence class to q
  $in(q) = |d^-(q)|$ for all $q \in Q$ ▷ tracks the number of ingoing t.
  for i **in** $[1, n_e]$ **do**
    for q **in** Q **do** ▷ find a duplicatable state e
      if $in(q) \geq 2$ **then**
        $e \leftarrow$ random chosen state from $k(q)$
        break
    $r_i \leftarrow$ unused state label ▷ create to e equivalent state r_i
    Add r_i to Q
    Add r_i to $k(e)$
    $in(r_i) \leftarrow 0$
    for σ **in** Σ **do** ▷ **R1:** add $d^+(r_i)$
      $\delta(r_i, \sigma) =$ random chosen state from $k(\delta(e, \sigma))$
    $P \leftarrow \{ ((s, \sigma), t) \in \delta \mid t \in k(e),\ in(t) \geq 2 \}$ ▷ **R2:** add $d^-(r_i)$
    $C \leftarrow$ random nonempty subset of P
    for $((s, \sigma), t)$ **in** C **do**
      $in(t) \leftarrow in(t) - 1$
      $in(r_i) \leftarrow in(r_i) + 1$
      $\delta(s, \sigma) = r_i$
  return $(Q, \Sigma_{sol}, \delta, s_{sol}, F)$

Note that computing an unused state label can be easily done by e.g. taking the maximum of all solution DFA states (which are nothing else but numbers) and adding one.

4.1.4 Creating equivalent state pairs does not change D

In this section we want to prove that our method of creating state pairs does not affect the number of ComEquivPairs-iterations. Using this information we can be sure that $\mathfrak{D}(A_{sol}) = \mathfrak{D}(A_{task})$ and our just explained algorithm does not have to care about possibly changing this value.

To do this proof, we will first introduce two auxiliary definitions and then prove two minor lemmas. As a side effect, Lemma 1 will describe a central property of ComEquivPairs and Lemma 2 will show an extended characterization of $\mathfrak{D}(A)$ compared to its definition (def. 4).

A word w shall be called *finishing word of* q, iff $\delta^*(q, w) \in F$. With $f(q) = \{\, w \mid \delta^*(q, w) \in F \,\}$ we denote the set of all finishing words to a state.

Definition 7. We will call a word w *distinguishing word of* p, q, iff $d_A(w, p, q)$ is true where

$$\begin{aligned} d_A(w,p,q) \text{ is true} &\Leftrightarrow (\delta^*(p,w) \in F \Leftrightarrow \delta^*(q,w) \notin F) \\ &\Leftrightarrow (w \in f(p) \Leftrightarrow w \notin f(q)) \end{aligned}$$

This definition and its terminology are in close relation to definition 3. The following lemma and its proof are in parts inspired by Martens and Schwentick [26, ch. 4 p. 18].

Lemma 1. *In the context of* ComEquivPairs *the following is true: If and only if* $(p, q) \in m(n)$*, the shortest distinguishing word of* p, q *has length* n*. Formally:*

$$\begin{aligned} (p,q) \in m(n) \Longleftrightarrow\ & \exists w \in \Sigma^*\colon (|w| = n\ \wedge d_A(w,p,q)) \\ & \wedge\ \nexists v \in \Sigma^*\colon (|v| < n\ \wedge d_A(v,p,q)) \end{aligned}$$

Proof. Per induction on the number of ComEquivPairs-iterations n.

$n = 0$, "$\Leftrightarrow$".

$$\begin{aligned} & (p,q) \in m(0) = \{(p,q),(q,p) \mid p \in F, q \notin F\} \text{ (see alg. 0, line 2))} \\ \Leftrightarrow\ & \text{one of } p, q \text{ in } F\text{, one not} \\ \Leftrightarrow\ & \text{one of } \delta^*(p,\varepsilon), \delta^*(q,\varepsilon) \text{ in } F\text{, one not} \\ \Leftrightarrow\ & \exists w \in \Sigma^*\colon (|w| = 0 \wedge \text{one of } \delta^*(p,w), \delta^*(q,w) \text{ in } F\text{, one not}) \\ \Leftrightarrow\ & \exists w \in \Sigma^*\colon (|w| = 0\ \wedge d_A(w,p,q)) \\ & \text{and there is no shorter such word } \checkmark \end{aligned}$$

$n > 0$, "$\Rightarrow$". Then the following holds for some states p, q (see alg. 0, line 5):

$$(p,q) \in \{(p,q),(q,p) \mid (p,q) \notin \bigcup m(\cdot) \wedge \exists \sigma \in \Sigma\colon (\delta(p,\sigma), \delta(q,\sigma)) \in m(n-1)\} \tag{4.1}$$

We will prove: There exists a distinguishing word of length $n-1$ for p, q, and there is no shorter distinguishing word for p, q.

Looking at eq. 4.1 we observe, there exists a symbol σ such that $(\delta(p,\sigma), \delta(q,\sigma)) \in m(n-1)$. Let $p', q' = \delta(p,\sigma), \delta(q,\sigma)$, so $(p', q') \in m(n-1)$.

Per induction there exists a (shortest) distinguishing word w', $|w'| = n-1$ to p', q'. Thus one of $\delta^*(p', w'), \delta^*(q', w')$ is in F, one not.

Thus one of $\delta^*(p,\sigma w'), \delta^*(q,\sigma w')$ is in F, one not, which makes $\sigma w'$ a distinguishing word of length n for p,q.

Since (p,q) is not in any $m(i), i < n$ (recall $(p,q) \notin \bigcup m(\cdot)$ of eq. 4.1), there is per precondition no shorter distinguishing word for p,q, making $\sigma w'$ (a) shortest distinguishing word for p,q. ✓

$n > 0$, "$\Leftarrow$". Then the following holds for some states p,q:

$$\begin{aligned} &\exists w \in \Sigma^*\colon (|w| = n \ \wedge d_A(w,p,q)) \\ \wedge\ &\nexists v \in \Sigma^*\colon (|v| < |w| \ \wedge d_A(v,p,q)) \end{aligned}$$

Since w is non-empty there exists a symbol σ such that $w = \sigma w'$. Let $\delta(p,\sigma), \delta(q,\sigma) = p', q'$.

Thus, if one of $\delta^*(p,\sigma w'), \delta^*(q,\sigma w')$ is in F and one not, then the same must hold for $\delta^*(p',w'), \delta^*(q',w')$, so w' is a distinguishing word for p',q'.

It is also the shortest one, because, if there existed a shorter word v', $|v'| < |w'|$, then $\sigma v'$ would be a distinguishing word shorter than w for p,q which is contradictory.

Since w' is a shortest distinguishing word for p',q', we may deduce now per induction, that $(p',q') \in m(n-1)$.

The pair (p,q) is not in any $m(i)$, $i < n$, since otherwise per induction the shortest distinguishing word would be shorter than w and thus not w. Since $(p',q') \in m(n-1)$ and $\delta(p,\sigma), \delta(q,\sigma) = p',q'$, we can then deduce by the definition of m, that $(p,q) \in m(n)$. ✓ □

Lemma 2. *If* ComEquivPairs *has done n iterations and terminated (so $\mathfrak{D}(A) = n$), then the longest word w, that is a shortest distinguishing word for any state pair, has length $\mathfrak{D}(A) - 1$.*

Proof. Via direct proof. Assume m-ComEquivPairs(A) has done n iterations (so $\mathfrak{D}(A) = n$). We observe, that

1. $\forall i \in [0, n-1]\colon m(i) \neq \emptyset$
2. $m(n) = \emptyset$
3. $\forall i > n\colon m(i) = \perp$.

This follows directly from while loop and its terminating condition of ComEquivPairs(alg. 0, line 4–7). Given this, we will prove: There exists a shortest distinguishing word of length $n-1$ for some state pair, but a longer such word can not exist.

Following Lemma 1 and the first observation, we can deduce the existence of a shortest distinguishing word w with $|w| = n - 1 = \mathfrak{D}(A) - 1$ for some $p,q \in Q$.

There cannot be any shortest distinguishing word w' with $|w'| = k > n-1$ for any two states $p',q' \in Q$. Following Lemma 1 again, $m(k)$ for some $k > n-1$ would be defined and non-empty, which is contradictory to observations 2 and 3. □

Theorem 3. *Given two DFAs A, A'. If both are accessible and their language is the same ($L(A) = L(A')$), then* ComEquivPairs *runs with the same number of iterations on them ($\mathfrak{D}(A) = \mathfrak{D}(A')$).*

Proof. Starting with the language-equivalence of A and A' we observe, that the start states of both DFAs have the same finishing words.

$$\begin{aligned} &L(A) = L(A') \\ \Rightarrow\ &\{\ w \mid \delta^*(s,w) \in F\ \} = \{\ w \mid \delta'^*(s',w) \in F'\ \} \end{aligned}$$

$$\Rightarrow \ \forall w \in \Sigma^*\colon \delta^*(s,w) \in F \Leftrightarrow \delta'^*(s',w) \in F'$$

We extend this to a statement that includes any state visited on the way to F resp. F'. We can see, that those states reached by the same word in A, A' have the same finishing words.

$$\begin{aligned} & \forall u \in \Sigma^*\colon \exists q, q' \in Q\colon \\ & \qquad \delta^*(s,u) = q \wedge \delta'^*(s',u) = q' \wedge \\ & \qquad (\forall v \in \Sigma^*\colon (\delta^*(q,v) \in F \Leftrightarrow \delta'^*(q',v) \in F')) \\ \Rightarrow \ & \forall u \in \Sigma^*\colon \exists q, q' \in Q\colon \\ & \qquad \delta^*(s,u) = q \wedge \delta'^*(s',u) = q' \wedge \\ & \qquad f(q) = f(q') \end{aligned}$$

Since we are making a statement about all states reached from s/s' and since all states in A/A' are reachable, we may conclude:

For every state in A/A' there exists a state in the other DFA, such that their finishing words are the same.

$$\begin{aligned} & \forall q \in Q\colon \exists q' \in Q'\colon f(q) = f(q') \quad \wedge \quad \forall q' \in Q'\colon \exists q \in Q\colon f(q) = f(q') \\ \Rightarrow \ & \{\ f(q) \mid q \in Q\ \} = \{\ f(q') \mid q' \in Q'\ \} \end{aligned}$$

Since a distinguishing word is defined as being finishing word for one state and for one not (see def. 7), there cannot be a distinguishing word in one of A/A', that is not distinguishing word in the other DFA.

As a consequence both DFAs have the same shortest distinguishing words and thus too the same longest shortest distinguishing word.

If $\mathfrak{D}(A) \neq \mathfrak{D}(A')$ then by Lemma 2 one DFA would have a longer longest shortest distinguishing word, which is not true as proven, thus $\mathfrak{D}(A) = \mathfrak{D}(A')$ must be true. □

Corollary 1. *Our method of creating equivalent state pairs in a DFA does not change the DFAs $\mathfrak{D}$-value.*

Proof. Creating equivalent state pairs does not change the language of a DFA — otherwise the reverse procedure, MinimizeDFA, could not guarantee the language stays the same. So since the language stays the same when creating those pairs, by Theorem 3 the number of ComEquivPairs-iterations will preserve as well. □

4.2 Adding unreachable states

From step 1 of the minimization algorithm we can deduce how to add unreachable states. These can easily be added to a DFA by adding non-start states with no ingoing transitions (see def. 1). Number and nature of outgoing transitions may be arbitrary.

function AddUnreachableStates (A, n_u, c)
    for n_u **times do**
        $q \leftarrow$ unused state label
        $Q \leftarrow Q \cup \{q\}$
        $outSymbols \leftarrow$ **if** $c = 1$ **then** Σ **else** random subset of Σ
        $R \leftarrow$ random chosen sample of $|outSymbols|$ states from $Q \setminus \{q\}$
        for σ **in** $outSymbols$ **do**
            $q' \in R$
            $R \leftarrow R \setminus \{q'\}$

$\delta \leftarrow \delta \cup \{((q, \sigma), q')\}$

return A

If completeness is demanded ($c = 1$), then we set Σ as set of all symbols, for which a state shall gain outgoing transitions. Else we choose a random subset for each state, such that some unreachable states may miss some outgoing transitions.

Chapter 5

Conclusion

Our intention was to investigate approaches, how DFA minimization task could be generated automatically. Therefore we discussed requirements to such a program and used some of them to formalize the underlying problems. Our approach to solve those problems was to first generate the minimal solution DFA and afterwards the task DFA by adding equivalent states and unreachable DFAs. This structure was derived from Hopcrofts minimization algorithm.

We did the generation of minimal DFAs via a rejection algorithm using either randomized and enumerating; we rejected in particular DFAs with a language which was found already in a previous run. A short overview over research on this topic confirmed our direction but gave outlook to more efficient variants.

On making minimal DFAs non-minimal no results in research was found. The properties of equivalent state pairs and unreachable states however gave precise and easy applicable rules to add such elements.

When building the task DFA, a question arised concerning the number of iterations by the ComEquivPairs-algorithm ($\mathfrak{D}$), which we wanted to be adjustable via parameter. By proving that the $\mathfrak{D}$-value does not change if we extend minimal DFAs, we could ensure that this value is already set when building the solution DFA, so DFAs could be rejected already in this stage, if $\mathfrak{D}$ did not match.

We close this work with a short lookout.

During our requirements analysis we defined several parameters that have not been or only sparsely further discussed in here. This includes especially boundaries for the number of ingoing transitions to each state and drawing DFAs in a visual comprehensible manner. Connected to the latter is the question, whether a good procedure exists, that outputs a visual representation of a DFA via LaTeX-code, such that hand-made adjustments might be done afterwards. One could also think of making more parameters ranged, such that per instance a minimum and maximum number of states could be specified as input.

Regarding the planarity test as it is used now, one might ask whether there is a more efficient planarity test that is tailored to DFAs. Moreover it could be worth investigating whether informations generated during the planarity test can be used for drawing the DFA.

Our summary on research on DFA generation indicated that efficient - randomized and enumerating - methods to generate DFAs have already been found, where the resulting DFAs where even accessible. An improved version of the associated implementation could implement some of these methods or make use of existing implementations. We shall cite in this regard the enumeration method of Almeida et. al. [1] which uses a similar string representation of DFAs to iterate through all DFAs. Carayol and Nicaud [8] presented a randomization method that is deemed easy to implement.

Concerning the process of extending solution DFAs to task DFAs, one might ask, how an enumeration algorithm similar to the one for generating solution DFAs could work. In doing so, one might furthermore think about the chance of hitting the same task DFA twice, when the extension algorithm is applied two times on the same DFA.

Appendix A

An isomorphy test for DFAs

Here follows a simple isomorphy test that tries essentially to build a bijection as described in section 2.1.2.

```
function AreIsomorph (A1, A2)
    if |Q1| ≠ |Q2| or |F1| ≠ |F2| or Σ1 ≠ Σ2 then
        return false
    π(s1) = s2                                          ▷ bijection Q1 → Q2
    O ← ∅                                               ▷ observed states
    V ← {s1}                                            ▷ visited states
    qc ← s1                                             ▷ current state
    while true do
        for ((q1, σ), p1) in δ1 do                      ▷ iterate through d+(qc)
            if q1 ≠ qc then
                continue

            p2 ← δ2(π(qc), σ)
            p1marked ← (π(p1) ≠ ⊥)                      ▷ see if p1, p2 were "marked" by π
            p2marked ← (∃q: π(q) = p2)

            if p1marked and p2marked then
                if π(p1) ≠ p2 then
                    return false
            else if ¬p1marked and ¬p2marked then
                π(p1) = p2
                if p1 ∉ V then
                    Add p1 to O
            else            ▷ one of p1, p2 was assigned to some state ≠ p1 resp. p2
                return false
        if |O| = 0 then
            break
        Pick and remove qc from O
        Add qc to V
    end
    for q1 in F1 do
        if π(q1) ∉ F2 then
            return false
    return true
```

This algorithm *visits* one by one all states of A_1 and tries to build π on the way. The currently visited state is denoted q_c. In $V \subseteq Q_1$ we save all already visited states. The set $O \subseteq Q_1$ shall contain all *observed* states, meaning those, that we encountered while following a transition, but have not visited yet.

We call states of A_1, A_2 *marked*, if they have been assigned to another state by π. So if $\pi(q_1) = q_2$, then q_1, q_2 are marked. States in O will have the property, that they are marked.

Starting with $q_c = s_1$, in every while-iteration all outgoing transitions $\delta_1(q_c, \sigma) = p_1$ of the current state are followed. We then compute $\delta_2(\pi(q_c), \sigma) = p_2$, which is the state in A_2 that should correspond to p_1 of A_1. At this point (line 17) we do a case differentiation:

- p_1, p_2 both marked: Then we only need to ensure they are assigned to each other.
- p_1, p_2 both not marked: Then we can assign them to each other and may now add q_1 to O, since we observed it on an outgoing transition and know it has been marked. We will not add it, if we have visited as q_c.
- one of p_1, p_2 marked, one not: In that case one of both states has already been assigned to another state. Thus the construction of a bijection has failed here, since p_1, p_2 should belong together.

When finished with visiting all outgoing transitions of a state q_c, we can pick the next state which is added to the visited states.

If all states of A_1 have been visited and the bijection thus been fully constructed, we need only to ensure, that the final state sets are equal after a renaming according to π.

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Erklärung

Hiermit versichere ich, Gregor Hans Christian Sönnichsen, dass ich die vorliegende Arbeit selbständig verfasst habe, keine anderen als die von mir angegebenen Quellen und Hilfsmittel benutzt habe und die Arbeit nicht bereits zur Erlangung eines akademischen Grades eingereicht habe.

Bayreuth, den 8. Februar 2020.

Gregor Hans Christian Sönnichsen