Generation of DFA Minimization Problems

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Introduction

Automata theory is a classical topic in computer science curricula. Minimization of DFAs is a typical task for students:

- sufficiently easy to understand
- practical applications
- understanding can be tested easily

Consequently, studying automatized generation of DFA minimization problems is interesting because it could...

- ...free up precious time for exercise constructors (if a generator is implemented)
- ...yield a deeper insight in the nature of such problems

[1]

[1] https://www.rindlerwahn.de/zeitdiebe-besiegen-und-mehr-lebenszeit-gewinnen

Outline

Problem definition

Preliminaries (1/2)

A tuple $A = (Q, \Sigma, \delta, s, F)$ with Q, Σ being a finite, $\delta\colon\ Q \times \Sigma \to Q$, $s \in Q$ and $F \subseteq Q$ is called *deterministic finite automaton*.

We define the *extended transition function* $\delta^* : Q \times \Sigma^* \to Q$ as:

- $\delta^*(q, \varepsilon) = q$
- $\delta^*(q, w\sigma) = \delta(\delta^*(q, w), \sigma)$ for all $q \in Q$, $w \in \Sigma^*$, $\sigma \in \Sigma$

The *language* of DFA is defined as $L(A) = \{\ w \mid \delta^*(w) \in F\ \}$.

We call a DFA *minimal*, if there exists no other DFA with the same language having less states.

Problem definition

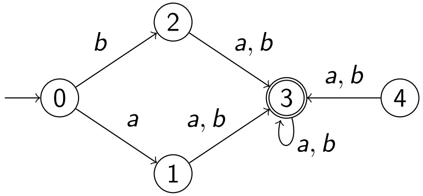
Preliminaries (2/2)

We say a state q is *unreachable*, iff there is no word $w \in \Sigma^*$ such that $\delta^*(s, w) = q$.

A state pair $q_1, q_2 \in Q$ is called *equivalent*, iff $\sim_A (q_1, q_2)$ is true, where

$$q_1 \sim_A q_2 \;\Leftrightarrow_{def}\; \forall z \in \Sigma^*\colon\; (\delta^*(q_1, z) \in F \Leftrightarrow \delta^*(q_2, z) \in F)$$

Example

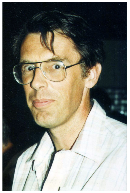


Theorem

A DFA is minimal, iff it has neither unreachable nor equivalent states.

Problem definition

Hopcroft's Minimization Algorithm



1. Compute all unreachable states
2. Remove all unreachable states and their transitions
3. Compute all inequivalent state pairs ($\not\sim_A$)

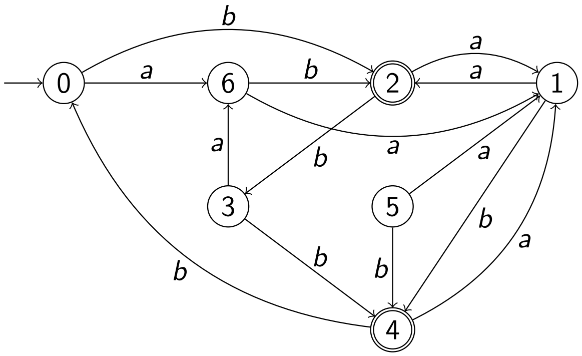
```
1: function FindEquivPairs($A$)
2:     $i \leftarrow 0$
3:     $m(0) \leftarrow \{(p,q),(q,p) \mid p \in F, q \notin F\}$
4:     do
5:         $i \leftarrow i + 1$
6:         $m(i) \leftarrow \{(p,q),(q,p) \mid (p,q) \notin \bigcup m(\cdot) \wedge$
7:                         $\exists \sigma \in \Sigma \colon (\delta(p,\sigma),\delta(q,\sigma)) \in m(i-1)\}$
8:     while $m(i) \neq \emptyset$
9:     return $\bigcup m(\cdot)$
```

4. Merge all equivalent state pairs

Problem definition

A sample DFA minimization problem...

Task: Consider the below shown deterministic finite automaton A:



Apply the minimization algorithm and illustrate for each state pair of A during which FindEquivPairs*-iteration it was marked. Draw the resulting automaton.*

Problem definition

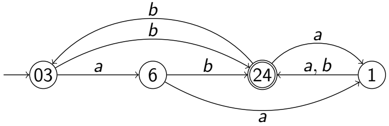
. . . and its solution

Solution:
Step 1: Detect and eliminate unreachable states.

State 5 is unreachable.

Step 2: Apply FindEquivPairs *to A and merge equivalent state pairs:*

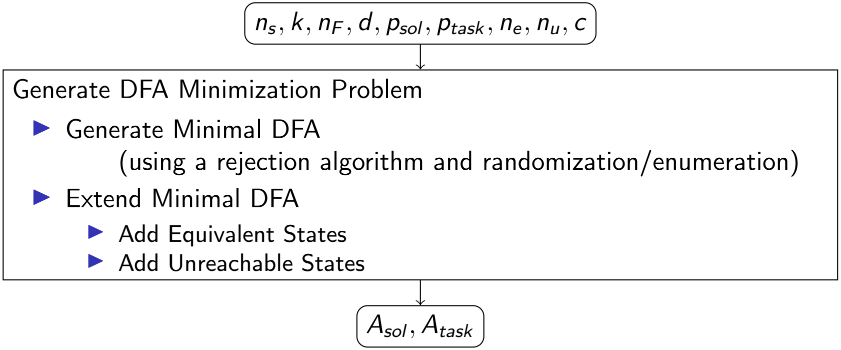
	0	1	2	3	4	6
0	■	1	0		0	2
1	■	■	0	1	0	1
2	■	■	■	0		0
3	■	■	■	■	0	2
4	■	■	■	■	■	0
6	■	■	■	■	■	■



Problem Definition and Approach

Problem: How to generate a *DFa Minimization Problem* (A_{sol}, A_{task})?

Idea: First generate A_{sol}, then add equivalent, then unreachable states.
$\Rightarrow$ modular pipeline architecture



$n_s, k, n_F, d, p_{sol}, p_{task}, n_e, n_u, c$

$\downarrow$

Generate DFA Minimization Problem

- Generate Minimal DFA
  (using a rejection algorithm and randomization/enumeration)
- Extend Minimal DFA
  - Add Equivalent States
  - Add Unreachable States

$\downarrow$

A_{sol}, A_{task}

Outline

Generating Minimal DFAs

Rejection Algorithm

Approach: Generate test DFAs until they match the demanded properties.

```
1:  function GenNewMinDFA ($n_s, k, n_F, d, p$)
2:      $I \leftarrow$ all DFAs in DB$_{found}$ matching $n_s, k, n_F$
3:      while True do
4:          generate DFA $A_{test}$ with $|Q|, |\Sigma|, |F|$ matching $n_s, k, n_F$
5:          if $A_{test}$ not minimal or $d \neq \mathfrak{D}(A_{test})$ then
6:              continue
7:          if $p = 1$ and $A_{test}$ is not planar then
8:              continue
9:          if $A_{test}$ is isomorph to any DFA in $I$ then
10:             continue
11:         save $A_{test}$ and its respective properties in DB$_{found}$
12:         return $A_{test}$
```

Generating Minimal DFAs
Test DFA Generation

We will restrict ourselves to the following DFAs:

$$Q = \mathbb{N}_0^{n_s - 1}, \Sigma = \mathbb{N}_0^{k-1}, s = 0$$

(a) Generation by Randomization:

$$\begin{aligned} F &= random\_subset(Q) \\ \delta(q, \sigma) &= choose\_one(Q) \qquad \forall q \in Q, \sigma \in \Sigma \end{aligned}$$

(b) Generation by Enumeration:

An *enumeration state* $s_{n_s,k,n_F} = (F_F, F_\delta)$ consists of two arrays of length $n_s, n_s * k$, respectively.

$$\begin{aligned} F_F[i] = 1 &\Leftrightarrow_{def} i \in F \\ F_\delta[i * k + j] = q &\Leftrightarrow_{def} \delta(i, j) = q \end{aligned}$$

Generating Minimal DFAs

Incrementing Enumeration States

Short demo how the associated DFA changes, when an enumeration state is incremented

Outline

Extending Minimal DFAs

Adding Equivalent States (1)

Let the added states be $r_1, \dots, r_{n_e}$, such that $Q_{re} = Q_{sol} \cup \{r_1, \dots, r_{n_e}\}$ and each of the added states is equivalent to a distinct state in Q_{re}.

Observation: Each r_i will be equivalent to a state e of A_{sol}:

$$\forall i \in [1, n_e]\colon \ \exists e \in Q_{sol}\colon \ r_i \sim_A e$$

We will first choose an $e \in Q_{sol}$ for each state r_i we create, then we add its transitions.

Extending Minimal DFAs

Adding Equivalent States (2)

Outgoing transitions. Observation:

$$r_i \sim_A e$$
$$\Rightarrow \forall \sigma \in \Sigma \colon [\delta(r_i, \sigma)]_{\sim_A} = [\delta(e, \sigma)]_{\sim_A}$$

We see that $\delta(r_i, \sigma) = q$ must be in the same equivalency class as $\delta(e, \sigma) = p$.

Consequently:

R1: For each symbol $\sigma \in \Sigma$ choose exactly one state $q \in [\delta(e, \sigma)]_{\sim_A}$ and set $\delta(r_i, \sigma) = q$.

The rule is always fulfillable:

- A_{sol} is complete
- every r_i gets an out. transition for every alphabet symbol

Extending Minimal DFAs

Adding Equivalent States (3)

Showcase how a to add outgoing transitions of an equivalent state

Extending Minimal DFAs

Adding Equivalent States (4)

Ingoing transitions. Observations:

- r_i must be reachable $\Rightarrow in(r_i) >= 1$

Let q be a state that gets an in. transition to r_i.

q must remain in its equivalence class

$\Rightarrow$ q has to have a transition to some state in $[r_i]_{\sim_A} = [e]_{\sim_A}$

$\Rightarrow$ given $\delta(q, \sigma) = p$ and $p \in [e]_{\sim_A}$, we can set $\delta(q, \sigma) = r_i$

We see that p must have at least 2 ingoing transitions.

R2: Choose at least one $((q, \sigma), p) \in \delta$ with $[p] = [e]$ and $out(p) \geq 2$. Remove $((q, \sigma), p)$ from δ and add $((q, \sigma), r_i)$.

General requirement regarding the choice of a state e for an r_i:

$$duplicatable(q) \Leftrightarrow_{def} (\exists p \in [q]_{\sim_A} : out(p) \geq 2)$$

Extending Minimal DFAs

Adding Equivalent States (5)

Showcase how a to add ingoing transitions of an equivalent state

Extending Minimal DFAs

Adding Unreachable States

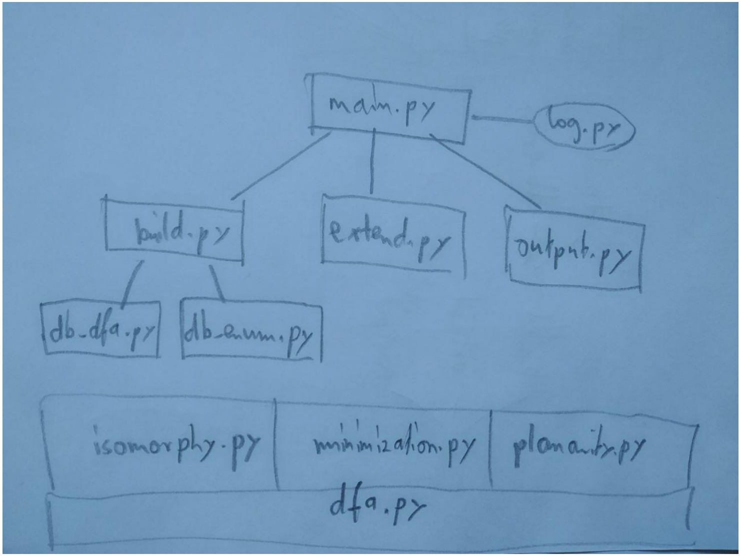
Reminder: We say a state q is *unreachable*, iff there is no word $w \in \Sigma^*$ such that $\delta^*(s, w) = q$.

```
1: function AddUnrStates ($A, n_u, c$)
2:     $U \leftarrow \emptyset$
3:     for $n_u$ times do
4:         let $q$ be the new state
5:         add ingoing tr. from a random subset of $U \times \Sigma$
6:         $\Sigma' \leftarrow$ if $c = 1$ then $\Sigma$ else random subset of $\Sigma$
7:         add outgoing tr. to $|\Sigma'|$ random states
8:         add $q$ to $U$
9:     return $A$
```

Outline

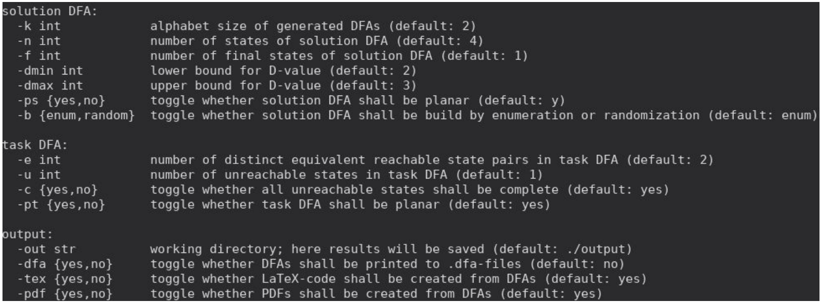
Live Demonstration

Program Architecture



Live Demonstration

Command-Line Options



```
solution DFA:
  -k int            alphabet size of generated DFAs (default: 2)
  -n int            number of states of solution DFA (default: 4)
  -f int            number of final states of solution DFA (default: 1)
  -dmin int         lower bound for D-value (default: 2)
  -dmax int         upper bound for D-value (default: 3)
  -ps {yes,no}      toggle whether solution DFA shall be planar (default: y)
  -b {enum,random}  toggle whether solution DFA shall be build by enumeration or randomization (default: enum)

task DFA:
  -e int            number of distinct equivalent reachable state pairs in task DFA (default: 2)
  -u int            number of unreachable states in task DFA (default: 1)
  -c {yes,no}       toggle whether all unreachable states shall be complete (default: yes)
  -pt {yes,no}      toggle whether task DFA shall be planar (default: yes)

output:
  -out str          working directory; here results will be saved (default: ./output)
  -dfa {yes,no}     toggle whether DFAs shall be printed to .dfa-files (default: no)
  -tex {yes,no}     toggle whether LaTeX-code shall be created from DFAs (default: yes)
  -pdf {yes,no}     toggle whether PDFs shall be created from DFAs (default: yes)
```

Conclusion

This presentation has. . .

- introduced the problem of DFA Minimization Problem Generation
- given an overview over a possible solution
- shown that the theoretic results might be useful in praxis

Lookout:

- more parameters, ranged parameters
  The *degree* of a state q is defined as $deg(q) = d^-(q) + d^+(q)$.
  $\Rightarrow$ capping the max. degree?
- investigate planarity and drawing algorithms deeper
- complexity analysis

References