Automatic Generation of DFA Minimization Problems

Gregor Sönnichsen

Universität Bayreuth

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Introduction

Automata theory is a classical topic in computer science curricula.
Minimization of DFAs is a typical task for students:

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- practical applications
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Consequently, studying automatized generation of DFA minimization problems is interesting because it could…

- …free up precious time for exercise constructors (if a generator is implemented)
- …yield a deeper insight in the nature of such problems


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Outline

A tuple $A = (Q, \Sigma, \delta, s, F)$ with Q, Σ being a finite, $\delta\colon\ Q \times \Sigma \to Q$, $s \in Q$ and $F \subseteq Q$ is called *deterministic finite automaton*.

We define the *extended transition function* $\delta^* : Q \times \Sigma^* \to Q$ as:

- $\delta^*(q, \varepsilon) = q$
- $\delta^*(q, w\sigma) = \delta(\delta^*(q, w), \sigma)$ for all $q \in Q$, $w \in \Sigma^*$, $\sigma \in \Sigma$

The *language* of DFA is defined as $L(A) = \{\ w \mid \delta^*(w) \in F\ \}$.

We call a DFA *minimal*, if there exists no other DFA with the same language having less states.

Problem definition

Preliminaries (2/2)

We say a state q is *unreachable*, iff there is no word $w \in \Sigma^*$ such that $\delta^*(s, w) = q$.

A state pair $q_1, q_2 \in Q$ is called *equivalent*, iff $\sim_A (q_1, q_2)$ is true, where

$$q_1 \sim_A q_2 \;\Leftrightarrow_{def}\; \forall z \in \Sigma^*\colon\; (\delta^*(q_1, z) \in F \Leftrightarrow \delta^*(q_2, z) \in F)$$

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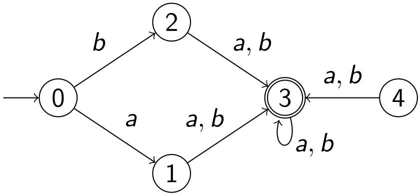

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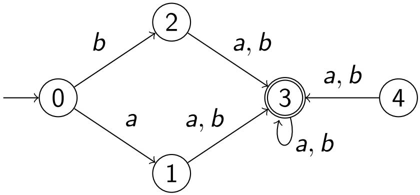


Theorem

A DFA is minimal, iff it has neither unreachable nor equivalent states.

Problem definition

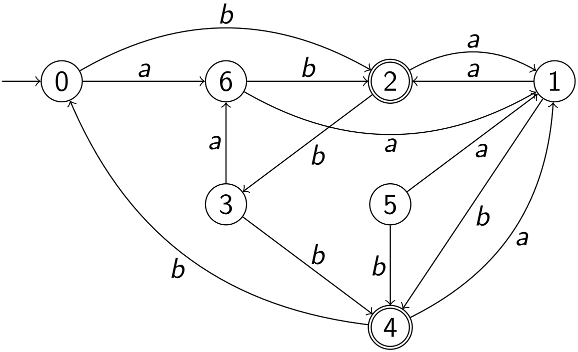
Hopcroft's Minimization Algorithm

1. Compute all unreachable states
2. Remove all unreachable states and their transitions
3. Compute all inequivalent state pairs ($\not\sim_A$)

   function FindEquivPairs(A)
   $\quad i \leftarrow 0$
   $\quad m(0) \leftarrow \{(p,q),(q,p) \mid p \in F, q \notin F\}$
   $\quad$ **do**
   $\quad\quad i \leftarrow i+1$
   $\quad\quad m(i) \leftarrow \{(p,q),(q,p) \mid (p,q) \notin \bigcup m(\cdot) \wedge$
   $\quad\quad\quad \exists \sigma \in \Sigma \colon (\delta(p,\sigma),\delta(q,\sigma)) \in m(i-1)\}$
   $\quad$ **while** $m(i) \neq \emptyset$
   $\quad$ **return** $\bigcup m(\cdot)$

4. Merge all equivalent state pairs

Task: *Consider the below shown deterministic finite automaton A:*



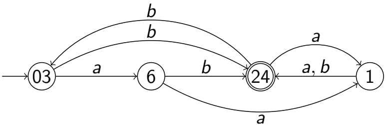
Apply the minimization algorithm and illustrate for each state pair of A during which FindEquivPairs*-iteration it was marked. Draw the resulting automaton.*

<u>*Solution:*</u>
Step 1: Detect and eliminate unreachable states.

State 5 is unreachable.

Step 2: Apply FindEquivPairs *to A and merge equivalent state pairs:*

	0	1	2	3	4	6
0	■	1	0		0	2
1	■	■	0	1	0	1
2	■	■	■	0		0
3	■	■	■	■	0	2
4	■	■	■	■	■	0
6	■	■	■	■	■	■



Problem Definition and Approach

Problem: How to generate a *DFA Minimization Problem* A_{task}?

Idea: First generate A_{sol}, then add equivalent, then unreachable states.
$\Rightarrow$ modular pipeline architecture

$n_s, k, n_F, d, p_{sol}, p_{task}, n_e, n_u, c$

$\downarrow$

Generate DFA Minimization Problem

- Generate Minimal DFA
  (using a rejection algorithm and randomization/enumeration)
- Extend Minimal DFA
  - Add Equivalent States
  - Add Unreachable States

$\downarrow$

A_{task}

Outline

Generating Minimal DFAs

Rejection Algorithm

Approach: Generate test DFAs until they match the demanded properties.

function GenNewMinDFA (n_s, k, n_F, d, p)

  $I \leftarrow$ all DFAs in DB_{found} matching n_s, k, n_F

  while True **do**

    generate DFA A_{test} with $|Q|, |\Sigma|, |F|$ matching n_s, k, n_F

    if A_{test} not minimal **or** $d \neq \mathfrak{D}(A_{test})$ **then**
      continue
    if $p = 1$ **and** A_{test} is not planar **then**
      continue
    if A_{test} is isomorph to any DFA in I **then**
      continue

    save A_{test} and its respective properties in DB_{found}
    return A_{test}

We will restrict ourselves to the following DFAs:

$$Q = \mathbb{N}_0^{n_s - 1}, \Sigma = \mathbb{N}_0^{k-1}, s = 0$$

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$$\begin{aligned} F &= random\_subset(Q) \\ \delta(q, \sigma) &= choose\_one(Q) \qquad\qquad \forall q \in Q, \sigma \in \Sigma \end{aligned}$$

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$$F = random\_subset(Q)$$
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(b) Generation by Enumeration:

An *enumeration state* $s_{n_s,k,n_F} = (F_F, F_\delta)$ consists of two arrays of length $n_s, n_s * k$, respectively.

$$F_F[i] = 1 \Leftrightarrow_{def} i \in F$$
$$F_\delta[i * k + j] = q \Leftrightarrow_{def} \delta(i, j) = q$$

Example state: $s_{4,2,2} = (0110)_2 \;\; (10\ 13\ 22\ 03)_4$

Outline

We now want to add states $r_1, \ldots, r_{n_e}$ to a DFA, such that every r_i is equivalent to a state e in the original DFA:

$$\forall i \in [1, n_e]: \;\; \exists e \in Q_{sol}: \;\; r_i \sim_A e$$

We will first choose an $e \in Q_{sol}$ for each state r_i we create, then we add its transitions.

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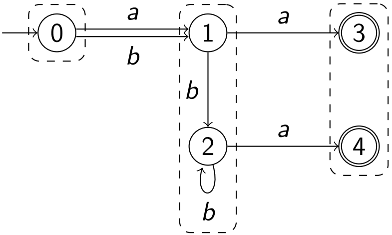

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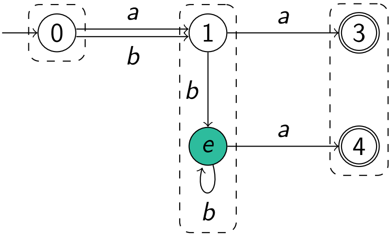

Extending Minimal DFAs

Adding Equivalent States (1)

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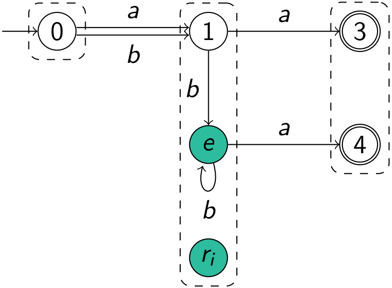

Extending Minimal DFAs

Adding Equivalent States (2) - Outgoing Transitions

Observation:

$$r_i \sim_A e \implies \forall \sigma \in \Sigma\colon [\delta(r_i, \sigma)]_{\sim_A} = [\delta(e, \sigma)]_{\sim_A}$$

Consequently:

R1: For each symbol $\sigma \in \Sigma$ choose exactly one state $q \in [\delta(e, \sigma)]_{\sim_A}$ and set $\delta(r_i, \sigma) = q$.

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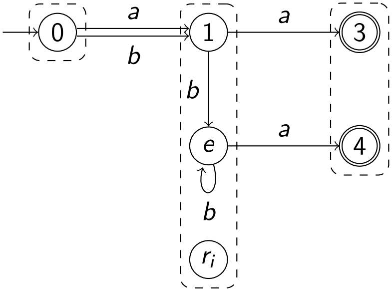

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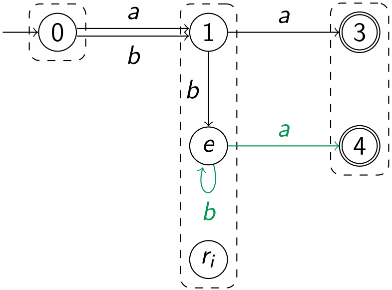

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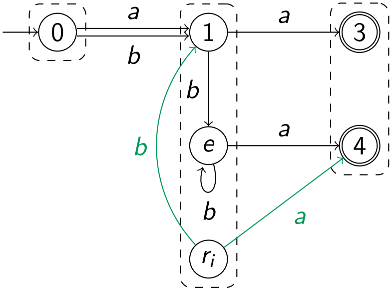

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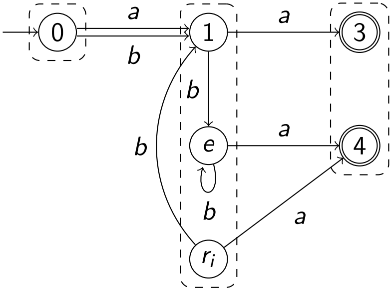

The rule is always fulfillable:

- A_{sol} is complete
- every r_i gets an out. transition for every alphabet symbol

$\Rightarrow$ $\delta(e, \sigma)$ is always defined, so $[\delta(e, \sigma)]_{\sim_A}$ is never empty

First observation: Since r_i must be reachable, $in(r_i) >= 1$.

Let q be a state s.t. $\delta(q, \sigma) = p$ and we want $\delta(q, \sigma) = r_i$.

Extending Minimal DFAs

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R2: Choose at least one $((q, \sigma), p) \in \delta$ with $[p] = [e]$ and $in(p) \geq 2$.
Remove $((q, \sigma), p)$ from δ and add $((q, \sigma), r_i)$.

R2: Choose at least one $((q, \sigma), p) \in \delta$ with $[p] = [e]$ and $in(p) \geq 2$. Remove $((q, \sigma), p)$ from δ and add $((q, \sigma), r_i)$.

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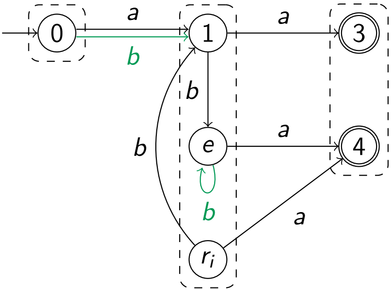

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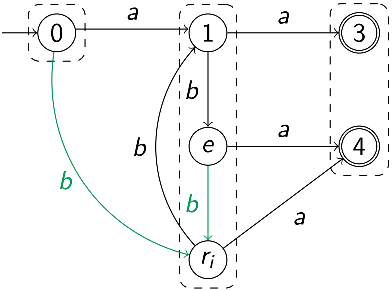


Extending Minimal DFAs

Adding Unreachable States

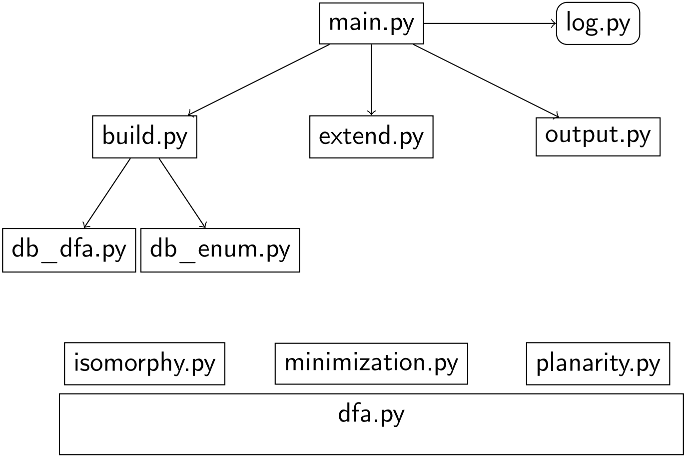
Reminder: We say a state q is *unreachable*, iff there is no word $w \in \Sigma^*$ such that $\delta^*(s, w) = q$.

function AddUnrStates (A, n_u, c)
  $U \leftarrow \emptyset$
  for n_u **times do**
    let q be the new state
    add ingoing tr. from a random subset of $U \times \Sigma$
    $\Sigma' \leftarrow$ **if** $c = 1$ **then** Σ **else** random subset of Σ
    add outgoing tr. to $|\Sigma'|$ random states
    add q to U
  return A

Outline

Live Demonstration

Program Architecture



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- given an overview over a possible solution
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Lookout:

- more parameters, ranged parameters
  The *degree* of a state q is defined as $deg(q) = |in(q)| + |out(q)|$.
  $\Rightarrow$ capping the max. degree?
- investigate planarity and drawing algorithms
- complexity analysis