

Automatic Generation of DFA Minimization Problems

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Automata theory is a classical topic in computer science curricula.
Minimization of DFAs is a typical task for students:

¹<https://www.rindlerwahn.de/zeitdiebe-besiegen-und-mehr-lebenszeit-gewinnen>

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- ▶ ... free up precious time for exercise constructors (if a generator is implemented)
- ▶ ... yield a deeper insight in the nature of such problems



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Outline

Introduction

Problem definition and approach

Generating Minimal DFAs

Extending Minimal DFAs

Live Demonstration and Conclusion

Problem definition

Preliminaries (1/2)

A tuple $A = (Q, \Sigma, \delta, s, F)$ with Q, Σ being a finite, $\delta: Q \times \Sigma \rightarrow Q$, $s \in Q$ and $F \subseteq Q$ is called *deterministic finite automaton*.

We define the *extended transition function* $\delta^*: Q \times \Sigma^* \rightarrow Q$ as:

- ▶ $\delta^*(q, \varepsilon) = q$
- ▶ $\delta^*(q, w\sigma) = \delta(\delta^*(q, w), \sigma)$ for all $q \in Q$, $w \in \Sigma^*$, $\sigma \in \Sigma$

The *language* of DFA is defined as $L(A) = \{ w \mid \delta^*(w) \in F \}$.

We call a DFA *minimal*, if there exists no other DFA with the same language having less states.

Problem definition

Preliminaries (2/2)

We say a state q is *unreachable*, iff there is no word $w \in \Sigma^*$ such that $\delta^*(s, w) = q$.

A state pair $q_1, q_2 \in Q$ is called *equivalent*, iff $\sim_A(q_1, q_2)$ is true, where

$$q_1 \sim_A q_2 \Leftrightarrow_{\text{def}} \forall z \in \Sigma^*: (\delta^*(q_1, z) \in F \Leftrightarrow \delta^*(q_2, z) \in F)$$

Problem definition

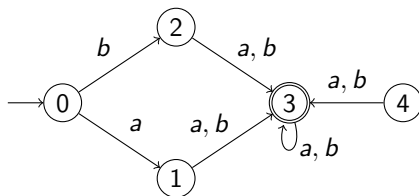
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Example



Problem definition

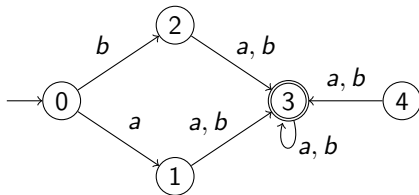
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Example



Theorem

A DFA is minimal, iff it has neither unreachable nor equivalent states.

Hopcroft's Minimization Algorithm

- 
- A portrait of a man with dark hair, wearing glasses and a light-colored, possibly striped, shirt. He is looking slightly to the right of the camera with a neutral expression. The background is dark and out of focus.

2: $i \leftarrow 0$

4: **do**

6: $m(i) \leftarrow \{(p, q), (q, p) \mid (p, q) \notin \bigcup m(\cdot) \wedge$

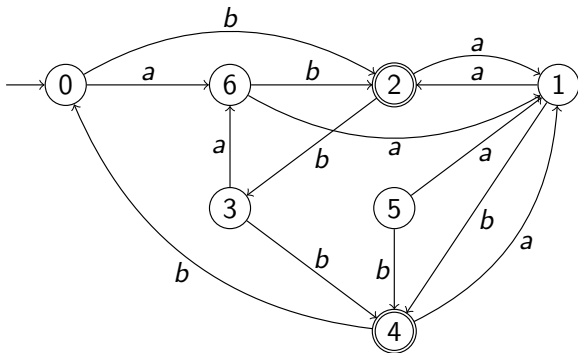
```
8:   while  $m(i) \neq \emptyset$ 
```

4. Merge all equivalent state pairs

Problem definition

A sample DFA minimization task. . .

Task: Consider the below shown deterministic finite automaton A :



Apply the minimization algorithm and illustrate for each state pair of A during which FindEquivPairs-iteration it was marked. Draw the resulting automaton.

Problem definition

... and its solution

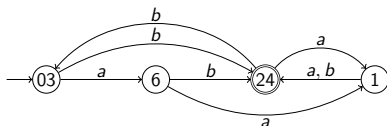
Solution:

Step 1: Detect and eliminate unreachable states.

State 5 is unreachable.

Step 2: Apply FindEquivPairs to A and merge equivalent state pairs:

	0	1	2	3	4	6
0	■	1	0		0	2
1	■	■	0	1	0	1
2	■	■	■	0		0
3	■	■	■	■	0	2
4	■	■	■	■	■	0
6	■	■	■	■	■	■

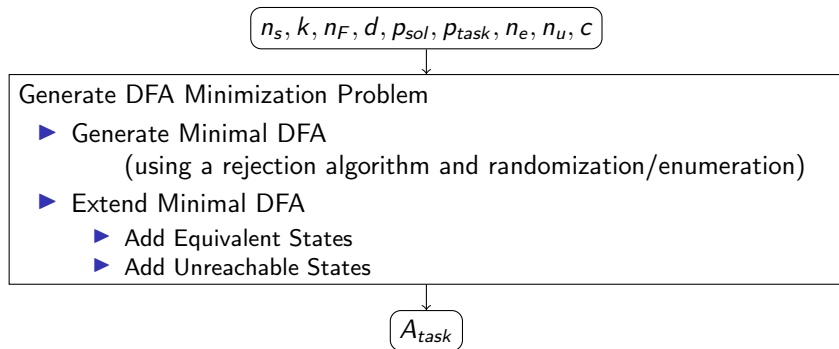


Problem Definition and Approach

Problem: How to generate a *DFA Minimization Problem* A_{task} ?

Idea: First generate A_{sol} , then add equivalent, then unreachable states.

⇒ modular pipeline architecture



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Generating Minimal DFAs

Rejection Algorithm

Approach: Generate test DFAs until they match the demanded properties.

```
1: function GenNewMinDFA ( $n_s, k, n_F, d, p$ )  
2:    $I \leftarrow$  all DFAs in  $DB_{found}$  matching  $n_s, k, n_F$   
3:   while True do  
4:     generate DFA  $A_{test}$  with  $|Q|, |\Sigma|, |F|$  matching  $n_s, k, n_F$   
5:     if  $A_{test}$  not minimal or  $d \neq \mathfrak{D}(A_{test})$  then  
6:       continue  
7:     if  $p = 1$  and  $A_{test}$  is not planar then  
8:       continue  
9:     if  $A_{test}$  is isomorph to any DFA in  $I$  then  
10:      continue  
11:    save  $A_{test}$  and its respective properties in  $DB_{found}$   
12:    return  $A_{test}$ 
```



Generating Minimal DFAs

Test DFA Generation

We will restrict ourselves to the following DFAs:

$$Q = \mathbb{N}_0^{n_s-1}, \Sigma = \mathbb{N}_0^{k-1}, s = 0$$

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(a) Generation by Randomization:

$$\begin{aligned} F &= \text{random_subset}(Q) \\ \delta(q, \sigma) &= \text{choose_one}(Q) \quad \forall q \in Q, \sigma \in \Sigma \end{aligned}$$

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(b) Generation by Enumeration:

An *enumeration state* $s_{n_s, k, n_F} = (F_F, F_\delta)$ consists of two arrays of length $n_s, n_s * k$, respectively.

$$\begin{aligned} F_F[i] &= 1 \Leftrightarrow_{\text{def}} i \in F \\ F_\delta[i * k + j] &= q \Leftrightarrow_{\text{def}} \delta(i, j) = q \end{aligned}$$

Example state: $s_{4,2,2} = (0110)_2 \ (10 \ 13 \ 22 \ 03)_4$

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Extending Minimal DFAs

Adding Equivalent States (1)

We now want to add states r_1, \dots, r_{n_e} to a DFA, such that every r_i is equivalent to a state e in the original DFA:

$$\forall i \in [1, n_e]: \exists e \in Q_{sol}: r_i \sim_A e$$

We will first choose an $e \in Q_{sol}$ for each state r_i we create, then we add its transitions.

Extending Minimal DFAs

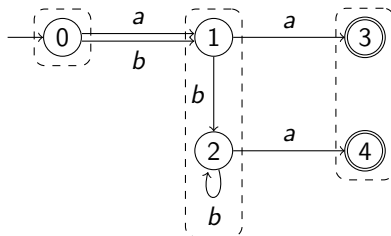
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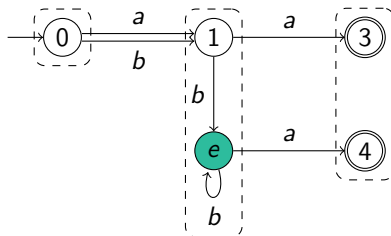
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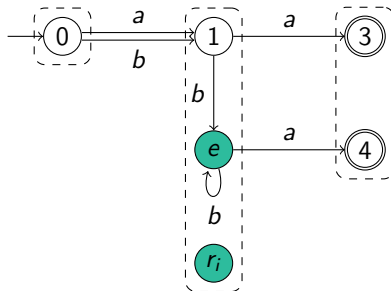
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Extending Minimal DFAs

Adding Equivalent States (2) - Outgoing Transitions

Observation:

$$r_i \sim_A e \implies \forall \sigma \in \Sigma: [\delta(r_i, \sigma)]_{\sim_A} = [\delta(e, \sigma)]_{\sim_A}$$

Consequently:

R1: For each symbol $\sigma \in \Sigma$ choose exactly one state $q \in [\delta(e, \sigma)]_{\sim_A}$ and set $\delta(r_i, \sigma) = q$.

Extending Minimal DFAs

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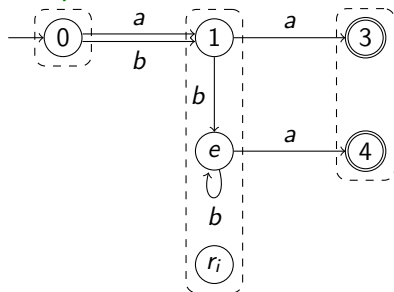
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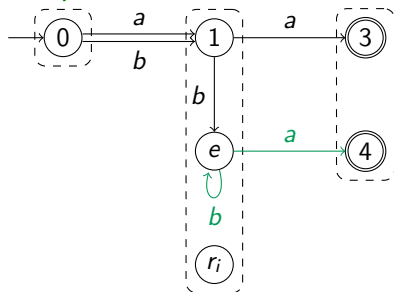
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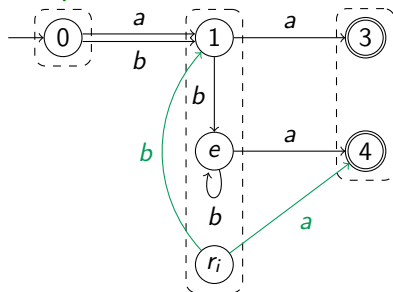
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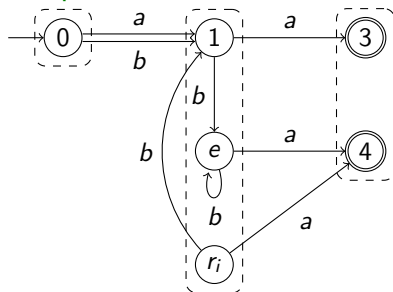
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Example



The rule is always fulfillable:

- ▶ A_{sol} is complete
 - ▶ every r_i gets an out. transition for every alphabet symbol
- $\implies \delta(e, \sigma)$ is always defined, so $[\delta(e, \sigma)]_{\sim_A}$ is never empty

Extending Minimal DFAs

Adding Equivalent States (3) - Ingoing Transitions

First observation: Since r_i must be reachable, $in(r_i) \geq 1$.

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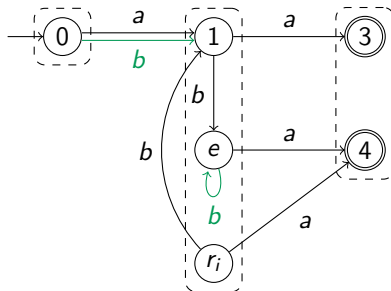
R2: Choose at least one $((q, \sigma), p) \in \delta$ with $[p] = [e]$ and $in(p) \geq 2$.
Remove $((q, \sigma), p)$ from δ and add $((q, \sigma), r_i)$.

Extending Minimal DFAs

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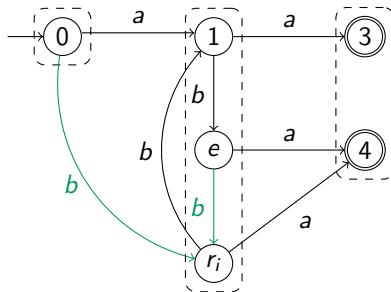


Extending Minimal DFAs

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Example



Extending Minimal DFAs

Adding Unreachable States

Reminder: We say a state q is *unreachable*, iff there is no word $w \in \Sigma^*$ such that $\delta^*(s, w) = q$.

```
1: function AddUnrStates ( $A, n_u, c$ )
2:    $U \leftarrow \emptyset$ 
3:   for  $n_u$  times do
4:     let  $q$  be the new state
5:     add ingoing tr. from a random subset of  $U \times \Sigma$ 
6:      $\Sigma' \leftarrow$  if  $c = 1$  then  $\Sigma$  else random subset of  $\Sigma$ 
7:     add outgoing tr. to  $|\Sigma'|$  random states
8:     add  $q$  to  $U$ 
9:   return  $A$ 
```

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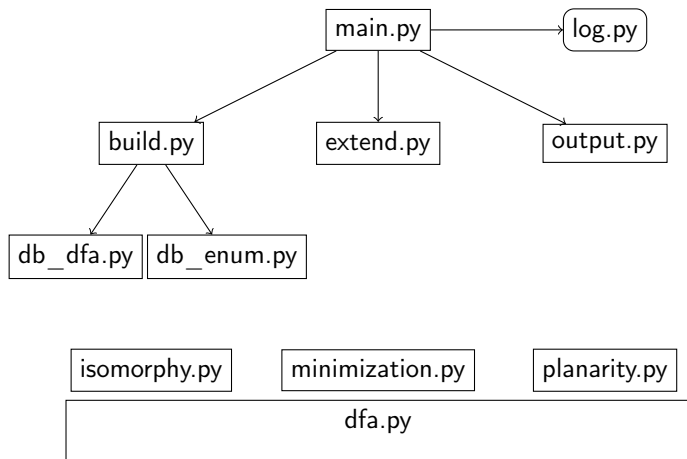
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Live Demonstration and Conclusion

Live Demonstration

Program Architecture



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- ▶ given an overview over a possible solution
- ▶ shown that the theoretic results might be useful in praxis

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Lookout:

- ▶ more parameters, ranged parameters
The *degree* of a state q is defined as $\deg(q) = |\text{in}(q)| + |\text{out}(q)|$.
 \Rightarrow capping the max. degree?
- ▶ investigate planarity and drawing algorithms
- ▶ complexity analysis