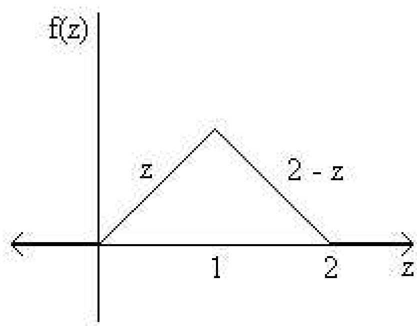
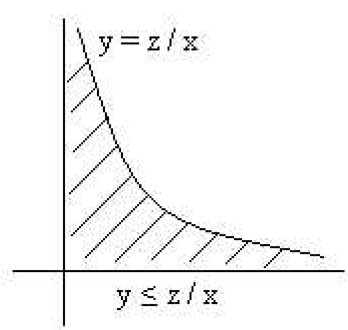
18.05 Lecture 13
March 4, 2005
Functions of random variables.
If (X, Y) with joint p.d.f. $f(x,y)$, consider Z = X + Y.
p.d.f. of Z: $f(z) = \int_{-\infty}^{\infty} f(x, z-x)dx$
If X and Y independent: $f(z) = \int_{-\infty}^{\infty} f_1(x)f_2(z-x)dx$

Example:
X, Y independent, uniform on [0, 1], $X, Y \sim U[0,1], Z = X+Y$
p.d.f. of X, Y:
$f_1(x) = \{1, 0 \le x \le 1; 0 \text{ otherwise}\} = I(0 \le x \le 1)$,
$f_2(y) = I(0 \le y \le 1) = I(0 \le z-x \le 1)$
$f(z) = \int_{-\infty}^{\infty} I(0 \le x \le 1) \times I(0 \le z-x \le 1)dx$
Limits: $0 \le x \le 1; z-1 \le x \le z$
Both must be true, consider all the cases for values of z:



Case 1: $(z \le 0) \to \int_{\emptyset} = 0$
Case 2: $(0 \le z \le 1) \to \int_0^z 1dx = z$
Case 3: $(1 \le z \le 2) \to \int_{z-1}^1 1dx = 2-z$
Case 4: $(z \ge 2) \to \int_{\emptyset} = 0$
Random variables likely to add up near 1, peak of f(z) graph.

Example: Multiplication of Random Variables
$X \ge 0, Y \ge 0. Z = XY$ (Z is positive)
First, look at the c.d.f.:



$$\mathbb{P}(Z \le z) = \mathbb{P}(XY \le z) = \int_{XY \le z} f(x,y)dxdy = \int_0^{\infty}\int_0^{z/x} f(x,y)dydx$$

p.d.f. of Z:

$$f(z) = \frac{\partial \mathbb{P}(Z \le z)}{\partial z} = \int_0^\infty f(x, \frac{z}{x})\frac{1}{x}dx$$

Example: Ratio of Random Variables
Z = X/Y (all positive), $\mathbb{P}(Z \le z) = \mathbb{P}(X \le zY) = \int_{x \le zy} f(x,y)dxdy = \int_0^\infty \int_0^{zy} f(x,y)dxdy$
p.d.f. $f(z) = \int_0^\infty f(zy,y)ydy$

In general, look at c.d.f. and express in terms of x and y.

Example: $X_1, X_2, ..., X_n$ - independent with same distribution (same c.d.f.)
$f(x) = F'(x)$ - p.d.f. of X_i
$\mathbb{P}(X_i \le x) = F(x)$
Y = maximum among $X_1, X_2 ... X_n$
$\mathbb{P}(Y \le y) = \mathbb{P}(\max(X_1, ..., X_n) \le y) = \mathbb{P}(X_1 \le y, X_2 \le y ... X_n \le y)$
Now, use definition of independence to factor:

$$= \mathbb{P}(X_1 \le y)\mathbb{P}(X_2 \le y) ... \mathbb{P}(X_n \le y) = F(y)^n$$

p.d.f. of Y:

$$\hat{f}(y) = \frac{\partial}{\partial y} F(y)^n = nF(y)^{n-1}F'(y) = nF(y)^{n-1}f(y)$$

$Y = \min(X_1, \ldots, X_n), \mathbb{P}(Y \le y) = \mathbb{P}(\min(X_1, ..., X_n) \le y)$
Instead of intersection, use union. But, ask if greater than y:
$= 1 - \mathbb{P}(\min(X_1, ..., X_n) > y)$
$= 1 - \mathbb{P}(X_1 > y, ..., X_n > y)$
$= 1 - \mathbb{P}(X_1 > y)\mathbb{P}(X_2 > y) ... \mathbb{P}(X_n > y)$
$= 1 - \mathbb{P}(X_1 > y)^n$
$= 1 - (1 - \mathbb{P}(X_1 \le y))^n$
$= 1 - (1 - F(y))^n$

$\overrightarrow{X} = (X_1, X_2, .., X_n), \overrightarrow{Y} = (Y_1, Y_2, ..., Y_n) = r(\overrightarrow{X})$
$Y_1 = r_1(X_1, ..., X_n)$
$Y_2 = r_2(X_1, ..., X_n)$
...
$Y_n = r_n(X_1, ..., X_n)$

Suppose that a map r has inverse. $\overrightarrow{X} = r^{-1}(\overrightarrow{Y})$
$\mathbb{P}(\overrightarrow{Y} \in A) = \int_A g(\overrightarrow{y})d\overrightarrow{y} \to g(\overrightarrow{y})$ is the joint p.d.f. of $\overrightarrow{Y}$
$\mathbb{P}(\overrightarrow{Y} \in A) = \mathbb{P}(r(\overrightarrow{X}) \in A) = \mathbb{P}(\overrightarrow{X} \in s(A)) = \int_{s(A)} f(\overrightarrow{x})dx = \int_A f(s(\overrightarrow{y}))|J|d\overrightarrow{y}$,
Note: change of variable $\overrightarrow{x} = s(\overrightarrow{y})$
Note: J = Jacobian:

$$J = \det \begin{vmatrix} \frac{\partial s_1}{\partial y_1} & ... & \frac{\partial s_1}{\partial y_n} \\ ... & ... & ... \\ \frac{\partial s_n}{\partial y_1} & ... & \frac{\partial s_n}{\partial y_n} \end{vmatrix}$$

The p.d.f. of $\overrightarrow{Y} : f(s(\overrightarrow{y}))|J|$

Example:
(X_1, X_2) with joint p.d.f. $f(x_1, x_2) = \{4x_1x_2$, for $0 \le x_1 \le 1, 0 \le x_2 \le 1; 0$, otherwise$\}$

$$Y_1 = \frac{X_1}{X_2}; Y_2 = X_1 X_2$$

$$Y_1 = r_1(X_1, X_2), Y_2 = r_2(X_1, X_2) \overrightarrow{inverse} X_1 = \sqrt{Y_1 Y_2} = s_1(Y_1, Y_2), X_2 = \sqrt{\frac{Y_2}{Y_1}} = s_2(Y_1, Y_2)$$

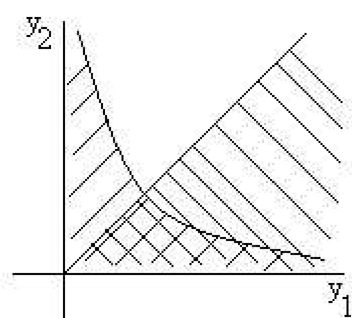
But, keep in mind the intervals for non-zero values:

$$J = \det \begin{vmatrix} \frac{\sqrt{y_2}}{2\sqrt{y_1}} & \frac{\sqrt{y_1}}{2\sqrt{y_2}} \\ -\frac{\sqrt{y_2}}{2y_1^{3/2}} & \frac{1}{2\sqrt{y_2}\sqrt{y_1}} \end{vmatrix} = \frac{1}{4y_1} + \frac{1}{4y_1} = \frac{1}{2y_1}$$

Joint p.d.f. of (Y_1, Y_2):

$$g(y_1, y_2) = \{4\sqrt{y_1 y_2}\sqrt{\frac{y_2}{y_1}}|J| = \frac{2y_2}{|y_1|}, \text{ if } 0 \le \sqrt{y_1 y_2} \le 1, \text{ and } 0 \le \sqrt{\frac{y_2}{y_1}} \le 1; 0 \text{ otherwise } \}$$

Condition implies that they are positive, absolute value is unneccessary.



Joint p.d.f. of (Y_1, Y_2)

** Last Lecture of Coverage on Exam 1 **
** End of Lecture 13