

# PA-01

Brandon Takahashi

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## 1 Newton's Method

Solving  $f(x) = 0$  where  $f(x) = x^2 - 2$  and  $x_0 = 1$ .

### i) Error Analysis ( $e_n = x_n - \sqrt{2}$ for $n = 0, \dots, 8$ )

n	x	error
0	1.00000000000000000000e+00	-4.14213562373095145475e-01
1	1.50000000000000000000e+00	8.57864376269048545254e-02
2	1.41666666666666666667e+00	2.45310429357159520691e-03
3	1.41421568627450988664e+00	2.12390141474116944664e-06
4	1.41421356237468986983e+00	1.59472435257157485466e-12
5	1.41421356237309514547e+00	0.00000000000000000000e+00
6	1.41421356237309492343e+00	-2.22044604925031308085e-16
7	1.41421356237309514547e+00	0.00000000000000000000e+00
8	1.41421356237309492343e+00	-2.22044604925031308085e-16

### ii) Quadratic Convergence?

In this case quadratic convergence does not exist "perfectly". As seen between steps  $n = 1$  and  $n = 2$  the number of significant digits does not double.

### iii) Rounding Error and Numerical Convergence

The rounding error causes Newton's Method to become unstable for the given problem. While attempting to take a next step after the system shows the method has converged (via  $error = 0$ ) a rounding error occurs such that the solution moves away from the convergence point.

#### iv) Solving for $M_n$ for $n = 1, \dots, 4$

The following list gives the values for  $M_n$  solved by  $M_n = \frac{|e_{n+1}|}{|e_n|^2}$ .

```
M1 = 3.33333333333331038872e-01
M2 = 3.52941176468276551770e-01
M3 = 3.53522384487245822093e-01
M4 = 0.00000000000000000000e+00
```

All values of  $M$  for the given case are less than 1.

## 2 Secant Method

Solving  $f(x) = 0$  where  $f(x) = x^2 - 2$  and  $x_0 = 1$ .

#### i) Error Analysis ( $e_n = x_n - \sqrt{2}$ for $n = 0, \dots, 8$ )

n	x	error
0	0.00000000000000000000e+00	-1.41421356237309514547e+00
1	1.00000000000000000000e+00	-4.14213562373095145475e-01
2	2.00000000000000000000e+00	5.85786437626904854525e-01
3	1.33333333333333333333e+00	-8.08802290397616641116e-02
4	1.40000000000000000000e+00	-1.42135623730950122479e-02
5	1.41463414634146333881e+00	4.20583968368193339415e-04
6	1.41421143847487007505e+00	-2.12389822507041969857e-06
7	1.41421356205732040578e+00	-3.15774739689800298947e-10
8	1.41421356237309536752e+00	2.22044604925031308085e-16

#### ii) Quadratic Convergence?

Quadratic convergence does not occur using this method. As seen between steps  $n = 1$  and  $n = 2$  the number of significant digits does not increase and between very few steps does the number of significant digits double.

#### iii) Solving for $M_n$ for $n = 1, \dots, 7$

The following list gives the values for  $M_n$  solved by  $M_n = \frac{|e_{n+1}|}{|e_n|^{1.618}}$ .

```
M1 = 2.43820586365651825744e+00
M2 = 1.92149900902253506496e-01
M3 = 8.31407582617175511253e-01
M4 = 4.09995753025747533549e-01
```

M5 = 6.16225988349691222723e-01  
M6 = 4.76513332062796890476e-01  
M7 = 5.22934885792810222327e-01

All values of M except for the first are less than 1.

### 3 Fixed Point Iteration

Solving  $f(x) = 0$  where  $x_{n+1} = h(x_n)$  and  $h(x) = x - \frac{f(x)f'(x)}{f'(x)^2 - f(x)f''(x)}$ .

#### i) Show implications

$f(x) = 0$  implies  $h(x) = x$ :

$$h(x) = x - \frac{0 * f'(x)}{f'(x)^2 - 0 * f''(x)}$$

$$h(x) = x - \frac{0}{f'(x)^2 - 0}$$

$$h(x) = x$$

$h(x) = x$  implies  $f(x) = 0$  or  $f'(x) = 0$ :

If  $h(x) = x$  then  $h(x) = x - 0$  in some fassion. Therefor  $\frac{f(x)f'(x)}{f'(x)^2 - f(x)f''(x)} = 0$  must be satisfied. To achieve this the numerator of the fraction must = 0 while the denominator  $\neq 0$ . If  $f(x) = 0$  or  $f'(x) = 0$  this condition is satisfied. Further, if  $f(x) = 0$  AND  $f'(x) = 0$  there is an issue with division by 0:

$$h(x) = x - \frac{0 * 0}{0^2 - 0 * f''(x)}$$

$$h(x) = x - \frac{0}{0 - 0}$$

$$h(x) = x - \frac{0}{0}$$

$$h(x) = \text{undefined}$$

## ii) $h'(x)$ and Stability of the Result

$$h'(x) = -\frac{f(x)(-2f(x)f''(x)^2 + f(x)f'(x)f'''(x) + f'(x)^2f''(x))}{(f'(x)^2 - f(x)f''(x))^2}$$

Above equation taken from wolframAlpha...inserting links is harder than expected. Hopefully I remember to change this part.

$h'(x) = 0$  when  $f(x) = 0$  and  $f'(x) \neq 0$ :

$$h'(x) = -\frac{0 * (-2 * 0 * f''(x)^2 + 0 * f'(x) * f'''(x) + f'(x)^2 f''(x))}{(f'(x)^2 - 0 * f''(x))^2}$$

$$h'(x) = -\frac{0 * (0 + 0 + f'(x)^2 f''(x))}{(f'(x)^2 - 0)^2}$$

$$h'(x) = -\frac{0}{(f'(x)^2)^2}$$

$$h'(x) = 0$$

$h'(x) = 2$  when  $f(x) \neq 0$ ,  $f'(x) = 0$  and  $f''(x) \neq 0$ :

$$h'(x) = -\frac{f(x)(-2f(x)f''(x)^2 + f(x) * 0 * f'''(x) + 0^2 * f''(x))}{(0^2 - f(x)f''(x))^2}$$

$$h'(x) = -\frac{f(x)(-2f(x)f''(x)^2 + 0 + 0)}{(-f(x)f''(x))^2}$$

$$h'(x) = -(-2) * \frac{f(x)(f(x)f''(x)^2)}{(f(x)f''(x))^2}$$

$$h'(x) = 2 * \frac{f(x)^2 f''(x)^2}{(f(x)f''(x))^2}$$

$$h'(x) = 2$$

The above results show that fixed points of  $h$  that result in  $f(x) = 0$  are stable because  $h'(x) = 0 = f(x)$ . Further, the fixed points of  $h$  that result in  $f'(x) = 0$  are unstable because  $h'(x) = 2 \neq f'(x)$ .

### iii) Fixed Point Implementation and Comparison

#### Error Analysis

n	x	error
0	1.0000000000000000000e+00	-4.14213562373095145475e-01
1	1.33333333333333325932e+00	-8.08802290397618861562e-02
2	1.41176470588235281056e+00	-2.44885649074233491262e-03
3	1.41421143847487007505e+00	-2.12389822507041969857e-06
4	1.41421356237150019908e+00	-1.59494639717649988597e-12
5	1.41421356237309492343e+00	-2.22044604925031308085e-16
6	1.41421356237309514547e+00	0.0000000000000000000e+00
7	1.41421356237309492343e+00	-2.22044604925031308085e-16
8	1.41421356237309514547e+00	0.0000000000000000000e+00

On the given system the Fixed Point Iteration Method performs on-par with Netwon's Method. Both suffer from instability once the error reaches 0. While Newton's Method reaches an error of 0 one step before the Fixed Point Method, this may be due to rounding errors within the system.

### iv) Accelerated Rate of Convergence

## 4 New Function Exploration

Function:

$$f(x) = 2\cos(5x) + 2\cos(4x) + 6\cos(3x) + 4\cos(2x) + 10\cos(x) + 3$$

### i) Newton's Method

For  $x_0 = 1$ :

error0: -4.71975511965976313178e-02	error10: -5.03062336079107552678e-05
error1: -2.45652170654127033345e-02	error11: -2.51543958549316215567e-05
error2: -1.25640858595492677097e-02	error12: -1.25775160495500415436e-05
error3: -6.35858704881964165168e-03	error13: -6.28883624242959626827e-06
error4: -3.19929855661094109109e-03	error14: -3.14443579241263648782e-06
error5: -1.60476618954530358963e-03	error15: -1.57222353402985959292e-06
error6: -8.03677253649492939758e-04	error16: -7.86107650752043696230e-07
error7: -4.02164063744159250291e-04	error17: -3.93087906447320278858e-07
error8: -2.01163630315859265352e-04	error18: -1.96606606106541903500e-07
error9: -1.00602244741843094289e-04	

For  $x_0 = 2$ :

```

error0: -9.43951023931952626356e-02  error10: -1.55567992775784702530e-03
error1: -6.16583720201511198411e-02  error11: -1.03688538185675582781e-03
error2: -4.06287612771687456359e-02  error12: -6.91152946874051821169e-04
error3: -2.68959724726149929097e-02  error13: -4.60722457185536171664e-04
error4: -1.78521649214427036156e-02  error14: -3.07127968239395698902e-04
error5: -1.18682044583460211129e-02  error15: -2.04743092826475958645e-04
error6: -7.89783128837484582618e-03  error16: -1.36491617516831809098e-04
error7: -5.25899818274178088018e-03  error17: -9.09924687300112111643e-05
error8: -3.50327252602733452136e-03  error18: -6.06605758468603539768e-05
error9: -2.33431488368118777998e-03

```

## ii) Fixed Point Method

For  $x_0 = 1$ :

```

error0: -4.71975511965976313178e-02  error10: 3.14677173207655869192e-11
error1: 1.77687082272393048754e-03   error11: 3.14677173207655869192e-11
error2: 3.21571411343590796150e-06   error12: 3.14677173207655869192e-11
error3: 9.83435555212963663507e-13   error13: 3.14677173207655869192e-11
error4: 1.96687111042592732701e-12   error14: 3.14677173207655869192e-11
error5: 3.93352017624692962272e-12   error15: 3.14677173207655869192e-11
error6: 7.86704035249385924544e-12   error16: 3.14677173207655869192e-11
error7: 1.57338586603827934596e-11   error17: 3.14677173207655869192e-11
error8: 3.14677173207655869192e-11   error18: 3.14677173207655869192e-11
error9: 3.14677173207655869192e-11

```

For  $x_0 = 2$ :

```

error0: -9.43951023931952626356e-02  error10: -2.50710357985184373319e-06
error1: -4.75279498588010440585e-03   error11: -5.31220353572336989600e-06
error2: -6.78985004309851092330e-06   error12: -2.90813817915847039330e-06
error3: 4.80975540639860810188e-06   error13: -6.33374548675291748623e-06
error4: -1.59321459047845337409e-05   error14: -3.10879085585469283615e-06
error5: -3.17298645580166294167e-07   error15: -8.34357483014969147916e-06
error6: -4.75930738375041073596e-07   error16: -1.73312090723243272805e-06
error7: -7.13177475297754881467e-07   error17: -2.36776310069686246607e-06
error8: -1.07424739770323185439e-06   error18: -2.36776310069686246607e-06
error9: -1.63571709377308138755e-06

```

## iii) Rate of Convergence and Rounding Error

The Fixed Point Method converges faster than Newton's Method. Even so, Newton's Method does not suffer from an error instability that the Fixed Point Method seems to. The Fixed Point Method error peaks out at a particular step for the given function and begins to lose significant digits after this step. This anomaly is most likely a result of rounding error within the system where the system performed multiple rounding errors back to back.