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SUPERGAUGE INVARIANT YANG-MILLS THEORIES

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ABSTRACT

We construct Lagrangian theories which are simultaneously invariant under supergauge transformations and under Yang-Mills transformations. The simplest of these turns out to be just the usual theory describing the interaction of a Yang-Mills field with a Majorana spinor belonging to the regular representation of the internal symmetry group. This theory is asymptotically free. Other examples, involving in addition to spinors also scalar and pseudoscalar fields are described. They also are asymptotically free, provided the number of scalar supermultiplets is not too high and supergauge invariance, as expected, is preserved by renormalization.

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1. INTRODUCTION

In a recent paper [1], a field theory model was constructed which is invariant under supergauge transformations [2] as well as under ordinary (Abelian) gauge transformations. It can be considered as a supergauge invariant extension of quantum electrodynamics. The model was shown to be renormalizable in the one loop approximation in a manner consistent with gauge and supergauge invariance. Preliminary calculations in higher orders appear to give the same result.

In the present paper we show how to construct field theories invariant under both supergauge transformations and non-Abelian gauge transformations of the Yang-Mills type. This is done by using the technique of superfields, introduced by Salam and Strathdee [3] and extended by the present authors in collaboration with Wess [4]. As in the Abelian case, supergauge transformations enlarge the non-Abelian gauge group to a generalized gauge group, which has a simple description in terms of superfields. A Lagrangian invariant under gauge and supergauge transformations is automatically invariant under these generalized gauge transformations and is an infinite power series in the coupling constant. In this manifestly gauge and supergauge invariant form, the theory is not obviously renormalizable. However, by means of the generalized gauge transformations, one can go to a special gauge [1] where some of the components of the vector field multiplet vanish. In this gauge the Lagrangian has a much simpler form, renormalizable by power counting, while the supergauge invariance is expressed in a more complicated way. For the case of the vector multiplet alone, which - in the special gauge - reduces essentially to a Yang-Mills field plus a Majorana spinor, both belonging to the regular representation of the group, the Lagrangian reduces to the ordinary Yang-Mills Lagrangian for those fields. One obtains therefore the interesting result that the Yang-Mills theory with a Majorana spinor in the regular representation is automatically supergauge invariant [*].

Assuming that the Lagrangians constructed in the present paper are indeed renormalizable to all orders in a manner consistent with supergauge invariance, they provide simple examples of asymptotically free field theories [6]. This is true not only of the model of Section 3, containing only the vector multiplet, but also of the model with scalar multiplets described in Section 4, provided the number of these multiplets is not too high. Contrary to the usual situation, in our case the presence of scalar fields should not pose any problems, since supergauge invariance implies that all couplings are expressible in terms of only one coupling constant. In Section 5 we give a Lagrangian which could almost be taken as a realistic model of strong interactions. The main outstanding problem is that of generating masses for the various fields, and especially for the vector fields. We are faced here with the difficulty that, in spite of the occurrence of scalar fields, spontaneous symmetry breaking does not seem possible in a supergauge invariant theory, at least in the tree approximation. It seems therefore desirable to study the properties of the effective potential in the one-loop approximation. Finally, it would be very interesting to see whether supergauge invariant Yang-Mills theories exhibit to all orders the same kind of compensation of divergences found earlier in models with only scalar multiplets [7].

2. VECTOR AND SCALAR MULTIPLETS

In this section we recall some formulae for calculating with superfields. They shall be used in the rest of this paper for the construction of invariant Lagrangians.

In the notation of Ref. 4), the superfield for a real vector multiplet is given by

$$\begin{aligned} V(x,\theta,\bar\theta) = {} & C + i\,\theta^\alpha\chi_\alpha - i\,\bar\theta_{\dot\alpha}\bar\chi^{\dot\alpha} \\ & + \theta^\alpha\theta_\alpha\,\tfrac{i}{2}(M+iN) - \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}\,\tfrac{i}{2}(M-iN) - \theta\sigma_\mu\bar\theta\, v^\mu \\ & + i\,\theta^\alpha\theta_\alpha\,\bar\theta_{\dot\beta}\left(\bar\lambda - \tfrac{i}{2}\,\partial_\mu\chi\,\sigma^\mu\right)^{\dot\beta} - i\,\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}\,\theta^\beta\left(\lambda + \tfrac{i}{2}\,\sigma^\mu\partial_\mu\bar\chi\right)_\beta \\ & + \theta^\alpha\theta_\alpha\,\bar\theta_{\dot\beta}\bar\theta^{\dot\beta}\left(\tfrac{1}{2}D + \tfrac{1}{4}\Box C\right), \end{aligned} \qquad (1)$$

where the fields C, M, N and D are real. The two-component spinors λ_α and $\bar\lambda^{\dot\alpha}$ can be arranged into a four-component Majorana spinor

$$\lambda = \begin{pmatrix} \lambda_\alpha \\ \bar\lambda^{\dot\alpha} \end{pmatrix}. \qquad (2)$$

Similarly, for χ and $\bar{\chi}$. The particular way of writing the various terms of (1) (factors i, etc.) is chosen so as to bring agreement with conventions used in earlier papers. Under a supergauge transformation of parameters ζ and $\bar{\zeta}$, V transforms according to

$$\delta V = \left[\zeta \frac{\partial}{\partial\theta} + \bar{\zeta}\frac{\partial}{\partial\bar{\theta}} + i\left(\theta\sigma^\mu\bar{\zeta} - \zeta\sigma^\mu\bar{\theta}\right)\partial_\mu \right] V. \tag{3}$$

The left- and right-handed covariant derivatives defined in Ref. 4) shall be denoted here by the D_α and $\bar{D}_{\dot{\alpha}}$. So, on a superfield transforming like V,

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu \ , \quad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu \ . \tag{4}$$

They satisfy

$$\left\{ D_\alpha , \bar{D}_{\dot{\beta}} \right\} = -2i\,(\sigma^\mu)_{\alpha\dot{\beta}}\,\partial_\mu \tag{5}$$

and

$$\left\{ D_\alpha , D_\beta \right\} = \left\{ \bar{D}_{\dot{\alpha}} , \bar{D}_{\dot{\beta}} \right\} = 0 \ . \tag{6}$$

In particular, it follows that

$$D_\alpha D_\beta D_\gamma = \bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}\bar{D}_{\dot{\gamma}} = 0 \ . \tag{7}$$

A left-handed superfield $S(x,\theta,\bar{\theta})$ satisfies

$$\bar{D}_{\dot{\alpha}} S = 0 \ . \tag{8}$$

According to Ref. 4), it can be shifted to a scalar superfield of type one

$$S_1(x_\mu, \theta) = S(x_\mu - i\,\theta\sigma_\mu\bar{\theta}, \theta, \bar{\theta}) \tag{9}$$

which is independent of $\bar{\theta}$ in virtue of (8) because, after the shift, the covariant derivatives take the form

$$\mathcal{D}_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu \ , \quad \bar{\mathcal{D}}_{\dot\alpha} = -\frac{\partial}{\partial\bar{\theta}^{\dot\alpha}} \ . \tag{10}$$

They still satisfy (5) and (6), of course. The components of the scalar multiplet are defined by the expansion in θ

$$S_1(x,\theta) = \frac{1}{2}(A - iB) + \theta^\alpha\psi_\alpha + \theta^\alpha\theta_\alpha\frac{1}{2}(F+iG), \tag{11}$$

where the fields A, B, F and G are real. With the two-component spinor field ψ_α and its conjugate, one can construct a Majorana spinor

$$\psi = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot\alpha} \end{pmatrix} . \tag{12}$$

Corresponding formulae are valid for a right-handed superfield.

3. LAGRANGIAN FOR THE VECTOR MULTIPLET

In this section we construct the invariant Lagrangian for the self-interacting vector multiplet. To be definite, we consider the case of $SU(N)$, although our treatment could be easily extended to other internal symmetry groups. We take a set of vector superfields belonging to the regular representation of the group. This can be done most simply by considering $V(x,\theta,\bar{\theta})$ to be a $N\times N$ Hermitian matrix. The generalized Yang-Mills transformations mentioned in the Introduction are then defined by

$$e^V \rightarrow e^{-i\Lambda^\dagger} e^V e^{i\Lambda} \ , \quad e^{-V} \rightarrow e^{-i\Lambda} e^{-V} e^{i\Lambda^\dagger}, \tag{13}$$

where Λ is a (matrix) left-handed superfield and $\Lambda^\dagger$ its right-handed Hermitian conjugate

$$\bar{D}_{\dot\alpha}\Lambda = D_\alpha\Lambda^\dagger = 0 \ . \tag{14}$$

Expanding (13), we find

$$V \rightarrow V + i(\Lambda - \Lambda^\dagger) - \frac{1}{2}\left([\Lambda, \Lambda^\dagger] + i[\Lambda + \Lambda^\dagger, V]\right) + \cdots \tag{15}$$

The term $i(\Lambda-\Lambda^\dagger)$ is the "gradient" of the superfield Λ. It was written out explicitly in Ref. 1).

Using (14) one verifies easily that (13) implies

$$e^{-V} D_\alpha e^{V} \rightarrow e^{-i\Lambda}\left(e^{-V} D_\alpha e^{V}\right) e^{i\Lambda} + e^{-i\Lambda} D_\alpha e^{i\Lambda} \tag{16}$$

and therefore, using also (5),

$$\begin{aligned} \bar{D}_{\dot\beta}\left(e^{-V} D_\alpha e^{V}\right) \rightarrow\ & e^{-i\Lambda}\left[\bar{D}_{\dot\beta}\left(e^{-V} D_\alpha e^{V}\right)\right] e^{i\Lambda} \\ & -2i(\sigma^\mu)_{\alpha\dot\beta}\, e^{-i\Lambda}\, \partial_\mu\, e^{i\Lambda} . \end{aligned} \tag{17}$$

Now, applying $\bar{D}_{\dot\gamma}$ to this relation, the last term is annihilated and we find

$$\bar{D}_{\dot\gamma} \bar{D}_{\dot\beta}\left(e^{-V} D_\alpha e^{V}\right) \rightarrow e^{-i\Lambda}\left[\bar{D}_{\dot\gamma} \bar{D}_{\dot\beta}\left(e^{-V} D_\alpha e^{V}\right)\right] e^{i\Lambda} . \tag{18}$$

In particular, the superfield

$$W_\alpha = \bar{D}_{\dot\beta} \bar{D}^{\dot\beta}\left(e^{-V} D_\alpha e^{V}\right) \tag{19}$$

transforms as

$$W_\alpha \rightarrow e^{-i\Lambda}\, W_\alpha\, e^{i\Lambda} \tag{20}$$

under generalized Yang-Mills transformation.

Using (7), we see that it satisfies

$$\bar{D}_{\dot\beta} W_\alpha = 0 . \tag{21}$$

It is therefore a left-handed multiplet with an undotted spinor index. As explained in Section 2, it can be shifted to a superfield of type one, defined by

$$W_{1\alpha}(x_\mu, \theta) = W_\alpha(x_\mu - i\theta\sigma_\mu\bar{\theta}, \theta, \bar{\theta}), \tag{22}$$

which is independent of $\bar{\theta}$ as a consequence of (21). Among the fields of the supergauge multiplet corresponding to $W_{1\alpha}$ one finds an antisymmetric tensor which is a sum of terms, one of which is just the Yang-Mills field strength associated with the vector v_μ. So $W_{1\alpha}$ appears as the natural generalization of the Yang-Mills field strength in our case. The superfield

$$\mathrm{Tr}\left(W_{1\alpha} W_1^{\alpha}\right) \tag{23}$$

(trace over the unwritten matrix indices) corresponds to an ordinary scalar multiplet. Its F component can be taken as Lagrangian for a gauge and supergauge invariant theory. This way of constructing the Lagrangian for a vector multiplet was already described in Ref. 4).

The superfield (23) is an infinite power series in the coupling constant g. Just as in Ref. 1), we now make use of the invariance under the (generalized) gauge transformations (13). From (15) it is clear that one can go to a special gauge **) where the components C, M, N, X and $\bar{X}$ of the vector multiplet vanish, so that only v_μ, λ, $\bar{\lambda}$ and D survive. For the multiplet V^2 only the D component survives, and equals $-v_\mu^2$, while the higher powers V^n $(n>2)$ vanish identically in this gauge. Now the expression (23) is easily calculated. Its F component can be seen to become simply

$$\mathrm{Tr}\left(-\frac{1}{4}v_{\mu\nu}^2 - \frac{i}{2}\bar{\lambda}\gamma^\mu\mathcal{D}_\mu\lambda + \frac{1}{2}D^2\right), \tag{24}$$

up to an over-all normalization constant. Here

$$v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu + ig[v_\mu, v_\nu], \tag{25}$$

$$\mathcal{D}_\mu\lambda = \partial_\mu\lambda + ig[v_\mu, \lambda] \tag{26}$$

and the Majorana field λ is given by (2). We have introduced the coupling constant g by the replacement $v_\mu \to 2gv_\mu$, $\lambda \to 2g\lambda$, $D \to 2gD$.

Except for the trivial extra term containing the auxiliary field D (which will, however, become important when interaction with other fields is introduced), this is just the Lagrangian for a Yang-Mills field in interaction with a Majorana spinor belonging to the regular representation of the internal symmetry group. No additional terms are required for supergauge invariance. Nevertheless, this particular theory, with a Majorana spinor field in the regular representation, must have special properties and in particular is expected to satisfy supergauge Ward identities connecting the vector with the spinor Green functions. These questions are presently under study.

4. INTERACTION WITH SCALAR MULTIPLETS

In this section we shall construct the invariant interaction between the vector multiplet discussed in Section 3 and other fields belonging to scalar multiplets. The construction is perfectly similar to that which was given in Reg. 1) for the Abelian case.

Let the left-handed superfields S_i,

$$\bar{D}_{\dot{\alpha}} S_i = 0 \ , \tag{27}$$

belong to a unitary representation of the internal symmetry group SU(N). Without writing explicitly the indices, we can specify their transformation properties under Yang-Mills transformations by

$$S \rightarrow e^{-i\Lambda} S \quad , \quad S^* \rightarrow S^* e^{i\Lambda^\dagger} \ , \tag{28}$$

where Λ still satisfies (14), but its SU(N) matrix nature is that appropriate to the representation of SU(N) to which S belongs. The vector superfield of Section 3 can also be taken to be a matrix of the same SU(N) nature as Λ. In other words, Λ and V are linear combinations of the matrices representing the SU(N) group generators. It is then obvious that the superfield

$$S^* e^V S \tag{29}$$

(sum over unwritten indices) is invariant under Yang-Mills transformation. Similarly, if

$$T \rightarrow e^{-i\Lambda^\dagger} T \quad , \quad T^* \rightarrow T^* e^{i\Lambda} \ , \tag{30}$$

$$\mathcal{D}_\alpha T = 0 \;, \tag{31}$$

the superfield

$$T^* e^{-V} T \tag{32}$$

is invariant.

Under a parity transformation the superfield V changes sign [this is obvious, for instance, from the explicit formulae in terms of multiplets given in Ref. 1)]. If we stipulate that parity exchanges the fields S and T, we see that the superfield

$$\frac{1}{2}\left(S^* e^V S + T^* e^{-V} T\right) \tag{33}$$

is a scalar under parity. Its D component can be taken as a Lagrangian for a theory invariant under parity, Yang-Mills and supergauge transformations. For $V = 0$ it reduces to the sum of the kinetic terms for the multiplets S and T. As invariant mass term, one can take the F component for the left-handed multiplet

$$T^* S \tag{34}$$

after shifting it to a multiplet of type one.

If S is in a real representation of SU(N), then $\tilde{\Lambda} = -\Lambda$ and $\tilde{V} = -V$, where the tilde means transposed. In this case T^* transforms like S under Yang-Mills transformations and one can set

$$T^* = S \;. \tag{35}$$

The two terms of (33) then become equal and one can take simply

$$S^* e^V S \;. \tag{36}$$

Similarly (34) becomes equal to

$$S^2 \;. \tag{37}$$

As in Ref. 1) and in Section 3, the final step consists in going to the special gauge. As an example, we give the result for the case in which the vector superfield interacts with only one superfield S which belongs to the regular representation of SU(N).

It is easy to see that the Lagrangian, including a mass term, becomes

$$\begin{aligned}\mathrm{Tr}\Big\{ & -\tfrac{1}{2}(\mathcal{D}_\mu A)^2 - \tfrac{1}{2}(\mathcal{D}_\mu B)^2 - \tfrac{i}{2}\bar{\psi}\gamma^\mu \mathcal{D}_\mu \psi + \tfrac{1}{2}F^2 + \tfrac{1}{2}G^2 \\ & + g\bar{\lambda}\left[A + \gamma_5 B, \psi\right] + ig D\left[A, B\right] \\ & + m\left(FA + GB - \tfrac{i}{2}\bar{\psi}\psi\right)\Big\} \quad ,\end{aligned} \tag{38}$$

$$\mathcal{D}_\mu A = \partial_\mu A + ig\left[v_\mu, A\right] \qquad \text{etc.} \tag{39}$$

Here we have followed the notation of Section 2 and ψ is the Majorana field defined there by (12). For consistency with Section 3, we have used a matrix notation also for the fields of the scalar multiplet. This Lagrangian (38) should be added to the Lagrangian (24) of Section 3. Observe that now, in addition to the well-known Yang-Mills couplings, there are additional couplings of Yukawa type, as well as quartic couplings involving the fields A and B, which arise when one eliminates the auxiliary field D. All these couplings are governed by the one coupling constant g.

It is easy to see that, if the vector superfield interacts with n multiplets like S, all belonging to the regular representation, the Callan-Symanzik function is given in the one-loop approximation by

$$\beta = -\frac{g^3}{16\pi^2}\,(3-n)N\,. \tag{40}$$

Therefore, for $n < 3$ [in particular for the Lagrangian (24), $n = 0$] the theory is asymptotically free. For $n = 3$, β vanishes and the coupling constant renormalization is finite to this order. It would be very interesting to see what happens in higher orders. Existing two-loop calculations cannot be used directly because they do not seem to agree with each other and furthermore, they do not include the contribution of the scalar fields and of the Yukawa and quartic couplings.

5. AN ALMOST REALISTIC EXAMPLE

Let us take the sum of the Lagrangians (24) and (38) with $m=0$. The fields F and G vanish in virtue of their own equations of motion. One can also eliminate the field D by using its equation of motion

$$D + ig\,[A, B] = 0\,. \tag{41}$$

Finally, one can combine the two Majorana spinors λ and ψ into a complex spinor

$$\varphi = \frac{1}{\sqrt{2}}(\lambda + i\psi)\,. \tag{42}$$

This leads to the Lagrangian

$$\begin{aligned} \mathrm{Tr}\Big\{ &-\frac{1}{4}v_{\mu\nu}^2 - \frac{1}{2}(\mathcal{D}_\mu A)^2 - \frac{1}{2}(\mathcal{D}_\mu B)^2 - \frac{i}{2}\bar{\varphi}\gamma^\mu \overleftrightarrow{\mathcal{D}}_\mu \varphi \\ &- ig\bar{\varphi}[A + \gamma_5 B, \varphi] + \frac{g^2}{2}([A, B])^2 \Big\}\,, \end{aligned} \tag{43}$$

where the Yang-Mills covariant derivatives $\mathcal{D}_\mu$ are the same as in (26) and (39). This theory is asymptotically free, with β given by (40) with $n=1$.

The Lagrangian (43), in addition to being invariant under parity, Yang-Mills and supergauge transformations, is also invariant under a "baryon number" phase transformation

$$\varphi \to e^{i\alpha}\varphi\,, \quad \varphi^* \to e^{-i\alpha}\varphi^* \tag{44}$$

with constant α. This transformation rotates the Majorana spinors λ and ψ into each other and could not have been introduced if we had not assigned the scalar multiplet to the regular representation of the internal symmetry group (absence of quarks ?). On the other hand, since there is no invariant way of giving a mass to the vector supermultiplet, we were forced to set equal to zero also the masses of the other fields. Perhaps this difficulty can be resolved by a consideration of the effective potential in higher orders.

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FOOTNOTES AND REFERENCES

*) This result was obtained independently by Salam and Strathdee, Ref. 5). For the construction of invariant Lagrangians, these authors, like us, make essential use of the techniques of Ref. 4). We thank them for sending us an advance copy of their preprint.

**) The gauge transformation which leads from a general gauge to the special gauge can be constructed by iteration. If, for instance, one writes Λ in the base one as

$$\Lambda_1(x,\theta) = \frac{1}{2}(A' - iB') + \theta^\alpha \psi'_\alpha + \theta^\alpha\theta_\alpha \frac{1}{2}(F' + iG'),$$

one obtains

$$\begin{aligned} B' &= -C + \cdots \\ \psi' &= -\chi + \cdots \\ \bar{\psi}' &= -\bar{\chi} + \cdots \\ F' &= -M + \cdots \\ G' &= -N + \cdots \end{aligned} ,$$

where the dots denote terms which are quadratic or of higher order. These equations give the components of Λ_1 other than A' as power series in A' and in the components of V. In the special gauge there remains the arbitrariness of a gauge transformation where the only non-vanishing component of Λ_1 is A'. One then sees easily that (13) can be written as

$$\delta V = i(\Lambda - \Lambda^\dagger) - \frac{i}{2}[\Lambda + \Lambda^\dagger, V]$$

which is valid to first order in Λ but to all orders in V. This is just a Yang-Mills transformation on the remaining components of V. If one wishes, one can specify the gauge further by imposing a gauge condition on v_μ, in which case A' will also be determined.

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