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ASPECTS OF THE GRAND UNIFICATION OF STRONG, WEAK AND ELECTROMAGNETIC INTERACTIONS

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A B S T R A C T

We study issues arising in attempts to unify strong and other elementary particle interactions. The proton lifetime is estimated in theories with second order baryon number violation, and found to be $O(10^3$ to $10^4)$ longer than naïve dimensional counting suggested. The renormalization of quark and lepton masses below the grand unification mass is considered in some detail. Application is made to the SU(5) model of Georgi and Glashow, and we find strange and bottom quark masses

$$m_s \approx (0.4 \text{ to } 0.5)\ \text{GeV}, \quad m_b \approx (5.0 \text{ to } 5.9)\ \text{GeV}.$$

Inputs are the values of the strong interaction coupling constant favoured by electroproduction and charmonium analyses, and the observed muon and heavy lepton (τ) masses. These estimates are substantially increased if there are more than six flavours of quark. Symmetry breaking in the SU(5) model is studied, including radiative corrections to the effective Higgs potential.

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1. INTRODUCTION

It is possible that the correct field theory of the strong interactions has been found, namely QCD, and likely that the correct unified gauge theory of the weak and electromagnetic interactions will be established quite soon. It therefore seems appropriate to consider the grand unification of strong, weak, and electromagnetic interactions[1,2], even if some people may consider such an enterprise premature or foolhardy. At least one may discover what some of the questions will be, even if one finds no answers. One may also find useful constraints on our present-day phenomenology[1,3,4] which are imposed by the requirement that some sort of grand unified theory exists. It is possible to imagine

- restrictions on properties of the established interactions, such as required values for the neutral weak mixing angle θ_W [1,3];
- connections between strong and weak/electromagnetic quantities, such as relations between quark and lepton masses[4];
- the need for new interactions between quarks and leptons which may violate baryon and lepton number conservation[1,2].

None of these points is new, and the purpose of this paper is to study some of these phenomenological questions in more detail than has been done before[1,3-6]. 断固として We will work in a resolutely simple-minded approach to grand unification, in which the observed elementary particle interactions are combined on an equal footing into a larger algebraic structure with a unique coupling constant, such as a simple gauge group with a hierarchy of symmetry breaking. The three main topics we discuss are

- the proton lifetime[3] in models[1] with violation of baryon number in second order;
- the renormalization of quark and lepton masses[4];
- details of symmetry breaking and renormalization effects in the simplest SU(5) grand unified theory of Georgi and Glashow[1].

We start in Section 2 with an estimate of the proton lifetime, using the technology developed[7-10] for discussing weak nonleptonic decays of hadrons. After an initial order of magnitude estimate, we do a more careful calculation using operator enhancement factors and Clebsch-Gordan coefficients which are relevant to the SU(5) Georgi-Glashow model, though we expect the results to have more general validity. We find

$$\tau\,(\text{proton}) \approx (10^3 \text{ to } 10^4)\, m_X^4/m_p^5 \tag{1.1}$$

where m_X is the mass of the superheavy boson supposed to mediate proton decay. The experimental limit τ (proton) $\gtrsim 2 \times 10^{30}$ years[11] then suggests that $m_X \gtrsim 10^{14}$ GeV. The rate (1.1) is about 10^3 to 10^4 longer than previous estimates, and means that models based on the simple groups SU(5) or SO(10) [1,12,4] have no problems with proton stability. The value $m_X \sim 2 \times 10^{16}$ GeV preferred for the SU(5) model would give τ (proton) $\sim$ (10^{37} to 10^{38}) years. Disappointingly, this would be too long for an observational test to be practicable in the forseeable future.

In Section 3 we discuss the renormalization of quark and lepton masses below the grand unification mass scale (GUM). If there is a quark-lepton mass relation in the symmetric theory, this will be renormalized[4] at present energies, just as the grand unified coupling constant gets renormalized in different ways to become the very dissimilar strong and weak/electromagnetic coupling constants that we see at present energies. The most significant part of the mass ratio renormalization comes from first order effects in the strong interaction coupling constant, but effects due to weak coupling constants and to the finite masses of fermions are not negligible. The amount of renormalization is quite sensitive to the number of quark flavours. The formalism is applied to the SU(5) model of Georgi and Glashow with fermion masses coming from a $\underline{5}$ of Higgs mesons, in which case[1] $m_d = m_e$, $m_s = m_\mu$ and $m_b = m_\tau$ at the grand unification mass. Defining quark masses to be ½ of the threshold energy for producing pairs of the appropriately flavoured particles and assuming six quark flavours, we find

$$m_s \approx (0.4 \text{ to } 0.5)\ \text{GeV} \quad , \quad m_b = (5.0 \text{ to } 5.9)\ \text{GeV} \tag{1.2}$$

Inputs in the calculations are m_μ, $m_\tau = 1.9$ GeV, and values of $\alpha_{strong}(Q^2 \simeq 10\ \text{GeV}^2)$ in the range of 0.19 to 0.32 suggested by applications of QCD to charmonium[13] and deep inelastic electroproduction[14]. The bottom quark mass is increased by about 10(30)% if there are 8(10) quark flavours. If the T(9.4 to 10.4) states[15] do turn out to be bottomonium, the estimates (1.2) will have been remarkably successful, though it would be a brave person who took the success as evidence that there are only six quarks. The renormalization of $\sin^2 \theta_W$ in the SU(5) model is very insensitive to the value of α_s (10 GeV^2) chosen, or to the number of quarks. It is $\sin^2 \theta_W \simeq 0.20$, somewhat low compared with the latest experimental value of 0.24 ± 0.02 [16], but not disastrous.

It may be worth while to remember that in the Weinberg-Salam model six quarks are the minimum necessary to give CP violation via the weak coupling matrix, and that there is just one CP violating parameter if there are only six quarks[17]. Within the pattern of SU(5) symmetry breaking discussed later, other CP violation effects due to the Higgs system[18] are too small to be relevant to present-day phenomenology. We note in passing that the philosophy of using the lowest order renormalization effects at short distances to bring together weak and strong couplings works very well if the weak gauge group just contains SU(2) factors, and if the strong group is SU(3). But if the weak gauge group contains a bigger factor such as SU(3) [19], then the strong gauge group would have to be enlarged, to SU(4) for example, if the weak and strong couplings are to come together at a grand unification mass in this way.

In Section 4 we discuss symmetry breaking in the SU(5) model of Georgi and Glashow[1]. We show how a Higgs structure with adjoint $\underline{24}$ and spinorial $\underline{5}$ representations can yield the required

$$SU(5) \rightarrow SU(3)\times SU(2)\times U(1) \rightarrow SU(3)\times U(1) \tag{1.3}$$

hierarchy of symmetry breaking in the tree approximation to the Higgs potential. We also show that by fine-tuning the coupling parameters one can retain the pattern (1.3) when radiative corrections[20] to the effective potential are included. However, as emphasized by Gildener[21], the required superheavy/light gauge hierarchy is not a natural feature of the theory.

The paper contains an appendix which calculates the anomalous dimension of the qqq operators relevant to proton decay in the SU(5) model, and an appendix detailing calculations of radiative corrections to the Higgs potential.

2. THE PROTON LIFETIME

One of the most striking implications of the sort of grand unification of strong, weak, and electromagnetic interactions that we discuss is that there exist interactions violating baryon and lepton number conservation[1,2]. These would be mediated by superheavy gauge bosons and occur in second order in the grand unified coupling constant as indicated in Fig. 1a, or could proceed via Higgs bosons as in Fig. 1b. In this section we will discuss gauge boson exchange, arguing in Section 4 that Higgs exchange is likely to be a relatively small effect. The fact that baryon number is conserved to a very good approximation -- the proton lifetime is $\gtrsim 2 \times 10^{30}$ years[11] -- clearly imposes a constraint on such models in the form of a lower limit on the mass m_X of the superheavy boson. Dimensional arguments lead one to expect that

$$\tau(\text{proton}) \propto m_X^4/m^5 \tag{2.1}$$

where m is a typical hadronic mass scale. As a preliminary estimate, Georgi, Quinn and Weinberg[3] set $m = m_p$, and assumed a constant of proportionality of $O(1)$ in Eq. (2.1). The lower limit on the proton lifetime then corresponded to $m_X \gtrsim 2 \times 10^{15}$ GeV. This limit was a non-trivial constraint on the simplest grand unified model of Georgi and Glashow based on SU(5). It entailed a large renormalization of $\sin^2\theta_W$, and enabled [3,4] a renormalization of the strong interaction coupling $\alpha_s(Q^2) \equiv g_3^2/4\pi$ to a value ≥ 0.1 at $Q^2 \simeq 100$ which is compatible with experiment.

It would be interesting to refine the estimate (2.1), in order to evaluate more precisely the constraint on m_X imposed by the longevity of the proton, to assess the prospects of detecting proton decay in any model with a prediction of m_X, and to interpret any possible future observation of a finite proton lifetime. Naively, one might expect the constant of proportionality in (2.1) to be

$$O\left(1/\alpha^2_{GUM}\right) = O(10^3) \tag{2.2}$$

and that the hadron mass scale m might be somewhat less than m_p, reflecting the physical size of the proton = O(1 fm), or the quark mass. One might therefore expect

$$\tau(\text{proton}) \gtrsim O(10^4)\, m_X^4/m_p^5 \tag{2.3}$$

and this is close to the ball-park we will find in our analysis of proton decay in the Georgi-Glashow SU(5) model, though we expect the conclusion to have more general validity.

This prototype SU(5) model[1] breaks down according to the pattern

$$SU(5) \rightarrow SU(3)\times SU(2)\times U(1) \rightarrow SU(3)\times U(1) \tag{2.4}$$

with the following couplings for the conventional interactions:

$$\text{strong } SU(3): \qquad g_3 \vec{G}_\mu \Big(\sum_q \bar{q}\gamma^\mu \frac{\vec{\lambda}}{2} q\Big) \tag{2.5a}$$

$$\text{weak } SU(2): \qquad g_2 \vec{W}_\mu \Big(\sum_q \bar{q}_L\gamma^\mu \frac{\vec{\tau}}{2} q + \sum_\ell \bar{\ell}_L \gamma^\mu \frac{\vec{\tau}}{2}\ell\Big) \tag{2.5b}$$

$$\text{weak } U(1): \qquad g_1\sqrt{\tfrac{3}{5}}\, B_\mu \Big(\sum_{f=q,\ell} \bar{f}\gamma^\mu (Q_f - I_{3f}) f\Big) \tag{2.5c}$$

where $g_3 = g_2 = g_1 = g_{GUM}$ above the grand unification mass. The model has two multiplets X ($|Q| = 4/3$) and Y ($|Q| = 1/3$) of superheavy bosons with the couplings (neglecting Cabibbo mixing for the sake of simplicity):

$$\frac{g}{2\sqrt{2}} X^-_{i\mu}\left(-\bar{d}_{iR}\gamma^\mu e^+_R + \epsilon_{ijk}\bar{u}^c_{k_L}\gamma^\mu u_{jL} + \bar{d}_{iL}\gamma^\mu e^+_L\right) + \begin{pmatrix}\text{hermitian}\\ \text{conjugate}\end{pmatrix} \tag{2.6a}$$

$$\frac{g}{2\sqrt{2}} Y^-_{i\mu}\left(\bar{d}_{iR}\gamma^\mu \nu^c_{e_R} + \epsilon_{ijk}\bar{u}^c_{k_L}\gamma^\mu d_{jL} - \bar{u}_{iL}\gamma^\mu e^+_L\right) + \begin{pmatrix}\text{hermitian}\\ \text{conjugate}\end{pmatrix} \tag{2.6b}$$

where (i, j, k) are triplet colour indices, and the superscript c denotes charge conjugation. The X and Y bosons mediate transitions $u + u \to e^+ + \bar{d}$, $u + d \to e^+ + \bar{u}$, $\bar{\nu}_e + \bar{d}$, etc., respectively, through the effective four-fermion interactions

$$\frac{g^2}{8m_X^2}\left(\epsilon_{ijk}\,\bar{u}^c_{k_L}\gamma_\mu u_{j_L}\right)\left(\bar{e}^+\gamma^\mu\gamma_5 d_i\right) + \binom{\text{hermitian}}{\text{conjugate}} \tag{2.7a}$$

$$\frac{g^2}{8m_Y^2}\left(\epsilon_{ijk}\,\bar{u}^c_{k_L}\gamma_\mu d_{j_L}\right)\left(-\bar{e}^+_L\gamma^\mu u_{i_L} + \bar{\nu}^c_{e_R}\gamma^\mu d_{iR}\right) + \binom{\text{hermitian}}{\text{conjugate}} \tag{2.7b}$$

We expect that m_X^2 and m_Y^2 differ only to the extent by which the weak SU(5) subgroup is broken down, i.e. on a scale of 100 GeV << m_X, m_Y (see also Section 4). We therefore identify

$$\frac{g^2}{8m_X^2} \approx \frac{g^2}{8m_Y^2} \equiv G_{GU}/\sqrt{2} \tag{2.8}$$

by analogy with the usual Fermi coupling $G_F/\sqrt{2} \equiv g_2^2/8m_W^2$. After Fierz transformation the interactions (2.7) may then be written in the form

$$\begin{aligned}\mathcal{L}_{GU} \equiv \frac{G_{GU}}{\sqrt{2}}\Big[&\left(\epsilon_{ijk}\,\bar{u}^c_{k_L}\gamma_\mu u_{j_L}\right)\left(2\bar{e}^+_L\gamma^\mu d_{iL} - \bar{e}^+_R\gamma^\mu d_{iR}\right)\\ &+ \left(\epsilon_{ijk}\,\bar{u}^c_{k_L}\gamma_\mu d_{j_L}\right)\left(\bar{\nu}^c_{e_R}\gamma^\mu d_{iR}\right)\Big] + \binom{\text{hermitian}}{\text{conjugate}}\end{aligned} \tag{2.9}$$

It is reasonable to expect that a form resembling (2.9) will crop up in many grand unification schemes, possible differences arising in numerical coefficients $O(1)$, and in L and R helicity labels. We will therefore calculate with (2.9) in the belief that other grand unification models will give similar results.

To calculate the proton decay rate $\Gamma(p \to \ell + \text{any})$ from the operator (2.9), we will use standard nonleptonic weak decay technology to estimate $\langle \text{any}|\mathcal{L}_{GU}|p\rangle$. We make believe we can estimate the matrix element of a multiquark operator renormalized at a typical strong interaction $\mu \simeq 1$ GeV. To get to μ from the grand unification scale m_X implicit in (2.9), we need a short distance enhancement factor[7]), calculated from the anomalous dimension of the qqq operator via the graphs of Fig. 2. The relevant calculations are set out in the appendix, from which we find a matrix element enhancement factor

$$A \approx \left[\frac{\alpha_s(\mu^2)}{\alpha_{GUM}}\right]^{\frac{2}{11-\frac{2}{3}f}} \tag{2.10}$$

where f is the total number of quark flavours, which are assumed to all have masses <<< m_X, perhaps $\lesssim m_W$? By comparison with other uncertainties in the calculation, the error induced by forgetting that for $f > 3$ the quark masses are somewhat larger than μ is not important (see also Section 3).

$M_q = \frac{m}{2}$ $\quad$ $M_q^2 = \frac{m_{qq}^2}{4}$

We calculate the matrix element of the operator renormalized at μ by assuming, as in Fig. 3, the free annihilation[8] $q + q \to \ell + \bar{q}$ at a rate governed by the wave function $\psi(0)$ for two quarks in the proton to be at the same point:

$$\Gamma_J(q+q \to l+\bar{q}) = \frac{3}{2\pi}|\psi(0)|^2 G_{GU}^2 \begin{cases} \frac{2}{3} m_{qq}^2 & \text{for } J=1 \\ M_q^2 & \text{for } J=0 \end{cases} \tag{2.11}$$

where J is the spin of the annihilating quark pair. We expect from (2.11) that $\Gamma_{J=1} = {}^8\!/_3\, \Gamma_{J=0}$. We estimate $|\psi(0)|^2$ to be

$$|\psi(0)|^2 \approx \frac{1}{\pi R^3} \tag{2.12}$$

where $R \simeq {}^3\!/_4$ fm is the proton radius. This value of $|\psi(0)|^2$ is more or less the geometric mean of two estimates in the recent literature[9,10] which were made for nonleptonic baryon decay calculations

$$|\psi(0)|^2_{(9)} \approx \frac{(\sqrt{3}m_\pi)^3}{\pi} \approx 0.4 \times 10^{-2}\,\text{GeV}^3 \; ; \; |\psi(0)|^2_{(10)} \approx 1.15 \times 10^{-2}\,\text{GeV}^3 \tag{2.13}$$

To get from (2.10), (2.11) and (2.12) to the actual decay rate, we need various Clebsch-Gordan coefficients:

- SU(3) colour: a factor of 2;
- SU(6): a factor determining the probability of annihilating with J = 0 or 1:

$$\begin{cases} \delta_{J,1} & \text{for } \Gamma(p \to e^+ + \text{any}^0) \\ \frac{3}{8}\delta_{J,0} + \frac{5}{24}\delta_{J,1} & \text{for } \Gamma(p \to \bar{\nu}_e + \text{any}^+) \end{cases} \tag{2.14}$$

- Spin, statistics and all that: a factor from the identity of initial u, d quarks and the final lepton helicities:

$$\begin{cases} 10 & \text{for } \Gamma(p \to e^+ + \text{any}^0) \\ 8 & \text{for } \Gamma(p \to \bar{\nu}_e + \text{any}^+) \end{cases} \tag{2.15}$$

Getting it all together, we find

$$\left.\begin{matrix} \Gamma(p \to e^+ + \text{any}^0) \\ \Gamma(p \to \bar{\nu}_e + \text{any}^+) \end{matrix}\right\} \approx \frac{3}{2\pi^2 R^3} G_{GU}^2 M_q^2 \left[\frac{\alpha_s(\mu^2)}{\alpha_{GUM}}\right]^{\frac{4}{11-\frac{2}{3}f}} \times \begin{cases} \frac{160}{3} \\ \frac{134}{9} \end{cases} \tag{2.16}$$

Taking $R \simeq {}^3\!/_4$ fm, $m_q \simeq {}^1\!/_3\, m_p$, $\alpha_s(\mu^2) \approx {}^1\!/_2$, $\alpha_{GUM} \approx {}^1\!/_{50}$, and $f = 6$, we find

$$\tau(p \to \bar{\ell} + \text{any}) \approx O(10^3 \text{ to } 10^4)\, m_X^4/m_p^5 \tag{2.17}$$

which is at the lower end of the ball-park estimate (2.3).

The experimental limit of 2×10^{30} years on the proton lifetime actually refers to decay modes producing a muon[11]. In our context, muons could be produced either by Cabibbo mixing; or by $(\text{any})^0 \to \pi^+\pi^-(\overline{\text{any}})^0$, $\pi^\pm \to \mu^\pm\nu$; or by $(\text{any})^+ \to \pi^+(\overline{\text{any}})^0$, $\pi^+ \to \mu^+\nu$. From the ratios of rates in (2.16) we guess that about ${}^1\!/_3$ or ${}^1\!/_2$ of the proton decay modes may contain a muon. The lifetime (2.17) is therefore quite relevant to the experimental limit. The factor $\geq 10^3$, which is probably more general than the specific SU(5) model discussed here, suggests that m_X may in fact be pushed as low as 10^{14} GeV without making the proton too unstable, while models with $m_X \ll 10^{14}$ GeV should probably be excluded. We will subsequently estimate m_X to be $O(2 \times 10^{16})$ GeV in the Georgi-Glashow SU(5) model, which would correspond to a proton lifetime of $(10^{37} \sim 10^{38})$ years, out of reach of presently conceivable experiments.

3. RENORMALIZATION EFFECTS

Since any grand unified symmetry must be very badly broken, all symmetry predictions will be modified[3], only becoming exact way above the grand unification mass scale GUM. The obvious example is the unified coupling constant which separates into strong and weak/electromagnetic coupling constants which are greatly different at present energies. The energy dependence of these couplings has been calculated using the appropriate renormalization group β-functions, of which only the $O(g^3)$ terms are really signifcant for $Q > 10$ GeV [3]. The appropriate renormalization group equations can be used to calculate the energy dependence of other quantities at energies below GUM. An example is a fermion mass, whose renormalization is governed by the anomalous dimension of the fermion mass operator. This was pointed out in Ref. 4, where it was also shown that if $m_b = m_\tau$ at the grand unification mass scale GUM, as in the SU(5) model with a $\underline{5}$-plets of Higgs fields to give fermion masses, then restrictions on GUM from other considerations led to a mass estimate $m_b \sim (4$ to $10)$ GeV if there were six quark flavours in all. The first purpose of this section is to extend the renormalization considerations of Ref. 4 by including the renormalization due to the weak interactions as well as those due to the strong interactions discussed previously, the effect of finite mass corrections in the anomalous dimension of the mass operator, and the consequences of extra heavy quark thresholds. The resulting mass renormalization formalism is probably accurate to $O(10)$%. In the second part of this section, we apply the previous analysis to the specific SU(5) model of Georgi and Glashow with a $\underline{5}$-plet of Higgs mesons giving fermion masses. We start with $\alpha_s(Q^2 = 10\ \mathrm{GeV}^2)$ determined from charmonium[13] ($\alpha_s = 0.19$) or electroproduction[14] ($\alpha_s = 0.32$) analyses, determine the corresponding grand unification mass (1 to 4×10^{16} GeV) and the corresponding value of $\sin^2\theta_W$ at present energies ($\simeq 0.20$). We then calculate the strange and bottom quark masses using the inputs $m_s = m_\mu$, $m_b = m_\tau$ at GUM, and defining the quark mass at present energies[22]:

$$m_q \equiv m_q(Q = 2m_q) \tag{3.1}$$

We find from $m_\mu = 0.105$ GeV, $m_\tau = 1.9$ GeV that

$$m_s \approx (0.4 \text{ to } 0.5)\ \mathrm{GeV} \quad ; \quad m_b \approx (5.0 \text{ to } 5.9)\ \mathrm{GeV} \tag{3.2}$$

if there are six quark flavours, and $m_b \geq 6$ GeV if there are eight or more flavours. The results (3.2) may not be specific to the SU(5) model discussed here, but could apply to any model with a similar grand unification mass scale and the same starting point of equality between quark and lepton masses.

3.1 Mass renormalization

Suppose that a grand unified theory predicts the mass of a fermion f at the grand unification mass scale M: $m_f(M)$. The renormalization of this mass at lower energies will be governed by the anomalous dimension of the fermion mass operator, which is to second order

$$\gamma_{m_f} \simeq \gamma_{m_f}^{(3)} + \gamma_{m_f}^{(2)} + \gamma_{m_f}^{(1)} \tag{3.3}$$

We have separated γ_{mf} into pieces coming from the low energy SU(3), SU(2), and U(1) subgroups, but Eq. (3.3) can easily be modified if there are more (or different) sub-unification groups. In the second order approximation $\gamma_{mf}^{(3)}$ is[22]:

$$\gamma_{m_f}^{(i)}(Q^2) = \bar{\gamma}_{m_f}^{(i)}\left[1 - \frac{m_f^2}{Q^2}\ln\left(1 + \frac{Q^2}{m_f^2}\right)\right] \tag{3.4}$$

where the square bracket is a mass correction factor relevant at momentum scales Q close to m_f. The values of $\bar{\gamma}_{mf}^{(3)}$ and the $\gamma_{mf}^{(2,1)}$ are:

$$\left.\begin{aligned} &\bar{\gamma}_{m_f}^{(3)} = \begin{cases} -\frac{1}{2\pi^2} g_3^2 & \text{for quarks} \\ 0 & \text{for leptons} \end{cases} \qquad \gamma_{m_f}^{(2)} = -\frac{9}{32\pi^2} g_2^2 \\ &\gamma_{m_f}^{(1)} = -\frac{3}{8\pi^2 C^2}(T_3 - Q)_{f_L}(T_3 - Q)_{f_R}\, g_1^2 \end{aligned}\right\} \tag{3.5}$$

and we have discarded mass correction factors in the anomalous dimensions of $\gamma_{mf}^{(2,1)}$.

In (3.5) C is the constant of proportionality[3] between the weak interaction U(1) and the normalized generator T_0 of the grand unification group:

$$Q = T_3 + CT_0 \tag{3.6}$$

As an example, in the SU(5) model $C^2 = 5/3$ and the usual Weinberg-Salam[23] assignments of $T_{3L,R}$ apply so that

$$\left.\begin{aligned} \gamma^{(1)}_{m_{u,c,t,\ldots}} &= -\frac{1}{40\pi^2} g_1^2 \quad ; \quad \gamma^{(1)}_{m_{d,s,b,\ldots}} = \frac{1}{80\pi^2} g_1^2 \\ \gamma^{(1)}_{m_{\nu_e,\nu_\mu,\nu_\tau,\ldots}} &= 0 \quad , \quad \gamma^{(1)}_{e,\mu,\tau,\ldots} = -\frac{9}{80\pi^2} g_1^2 \end{aligned}\right\} \tag{3.7}$$

If we introduce the Callan-Symanzik β-functions to $O(g^3)$ in a form similar to (3.4) and (3.5), we have[22]:

$$\beta^{(3)}(Q^2) \approx -\frac{g_3^3}{16\pi^2}\left\{ 11 - \frac{2}{3}\sum_q \left[1 - 6\frac{m_q^2}{Q^2} + \frac{12 m_q^4/Q^4}{(1+4m_q^2/Q^2)^{\frac{1}{2}}} \ln\frac{(1+\frac{4m_q^2}{Q^2})^{\frac{1}{2}}+1}{(1+\frac{4m_q^2}{Q^2})^{\frac{1}{2}}-1}\right] - \ldots\right\} \tag{3.8}$$

and

$$\beta^{(2)} \approx -\frac{g_2^3}{16\pi^2}\left\{\frac{22}{3} - \frac{2}{3}f - \ldots\right\} \; ; \; \beta^{(1)} \approx \frac{-g_1^3}{16\pi^2}\left\{-\frac{2}{3}f - \ldots\right\} \tag{3.9}$$

the square bracket is again a mass correction factor relevant at finite Q^2, and the dots indicate possible contributions from Higgs bosons, etc. We define $\bar{\beta}^{(3)}$ analogously to the $\bar{\gamma}^{(i)}_{m_f}$ of Eq. (3.4):

$$\bar{\beta}^{(3)} \equiv \lim_{m_q/Q \to 0} \beta^{(3)}/g_3^3 \tag{3.10}$$

A precise calculation of the mass renormalization cannot be done analytically, because of the complicated forms of the square-braketted mass correction factors in (3.4) and (3.8). If these are neglected the SU(5) model has

$$\ln\left[m_{u,c,t,\ldots}(\mu)\right] \approx \ln\left[m_{u,c,t,\ldots}(M)\right] + \left[\frac{4}{11-\frac{2}{3}f}\right]\ln\left[\frac{\alpha_S(\mu)}{\alpha_{GUM}}\right] + \left[\frac{27}{88-8f}\right]\ln\left[\frac{\alpha_2(\mu)}{\alpha_{GUM}}\right] - \left[\frac{3}{10f}\right]\ln\left[\frac{\alpha_1(\mu)}{\alpha_{GUM}}\right] \tag{3.11a}$$

$$\ln\left[m_{d,s,b,\ldots}(\mu)\right] \approx \ln\left[m_{d,s,b,\ldots}(M)\right] + \left[\frac{4}{11-\frac{2}{3}f}\right]\ln\left[\frac{\alpha_S(\mu)}{\alpha_{GUM}}\right] + \left[\frac{27}{88-8f}\right]\ln\left[\frac{\alpha_2(\mu)}{\alpha_{GUM}}\right] + \left[\frac{3}{20f}\right]\ln\left[\frac{\alpha_1(\mu)}{\alpha_{GUM}}\right] \tag{3.11b}$$

$$\ln\left[m_{e,\mu,\tau,\ldots}(\mu)\right] \approx \ln\left[m_{e,\mu,\tau,\ldots}(M)\right] + \left[\frac{27}{88-8f}\right]\ln\left[\frac{\alpha_2(\mu)}{\alpha_{GUM}}\right] - \left[\frac{27}{20f}\right]\ln\left[\frac{\alpha_1(\mu)}{\alpha_{GUM}}\right] \tag{3.11c}$$

where we have kept separate the SU(3), SU(2), and U(1) contributions. We will in fact only be interested here in the ratio of charge $-\frac{1}{3}$ quark (d, s, b, ...) to charge -1 lepton (e, μ, τ, ...) masses:

$$\ln\left[\frac{m_{d,s,b,\ldots}(\mu)}{m_{e,\mu,\tau,\ldots}(\mu)}\right] = \ln\left[\frac{m_{d,s,b,\ldots}(M)}{m_{e,\mu,\tau,\ldots}(M)}\right] + \left[\frac{4}{11-\frac{2}{3}f}\right]\ln\left[\frac{\alpha_s(\mu)}{\alpha_{GUM}}\right] + \left[\frac{3}{2f}\right]\ln\left[\frac{\alpha_1(\mu)}{\alpha_{GUM}}\right] + \left(\begin{array}{c}\text{computable non-analytic}\\ \text{finite mass corrections}\end{array}\right) \tag{3.12}$$

The consequences of (3.12) for the strange and bottom quark masses are shown in Table 1. In the previous analysis[4] only the SU(3) term without mass corrections

$$\left[\frac{4}{11-\frac{2}{3}f}\right]\ln\left[\frac{\alpha_s(\mu)}{\alpha_{GUM}}\right]$$

was taken into account in calculating the quark to lepton mass ratio. In fact, for typical parameters of the SU(5) model to be discussed in the next section, the U(1) and finite mass correction factors in SU(3) each reduce m_q/m_ℓ by $O(10\%)$, whereas finite mass correction factors in SU(2) and U(1) are insignificant. We can think of three more contributions to (3.12) which might be included.

- Higher order contributions to γ_{m_f} and $\beta^{(i)}$ might be important, particularly $O(g_3^4,\ g_3^5)$, respectively. We have not calculated γ_{m_f} to $O(g^4)$, but have investigated numerically the effect of the g_3^5 term in $\beta^{(3)}$. It only has a 1% effect on the estimate of the bottom quark mass, but may be significant for the strange quark mass estimate, depending on the value of the strong interaction coupling constant at $Q^2 = 10\ \text{GeV}^2$ which is used.

- Higgs meson contributions to the $\beta^{(i)}$ may be unsymmetric, because of the Higgs' role in breaking the SU(5) symmetry. As discussed in Section 4, the SU(5) model with a single $\underline{5}$-plet of Higgs mesons giving fermion masses may be expected to have a light SU(3) singlet and SU(2) doublet (some physical, some eaten by $W^\pm$, Z^0), and a "heavy" SU(3) triplet and SU(2) singlet. In this case the b quark mass estimate is $\sim$ 1% smaller than would be obtained with a symmetric $\underline{5}$-plet of Higgs mesons.

- The continuation from space-like to time-like momenta may be non-trivial. We follow Georgi and Politzer[22] in defining the "physical" quark mass to be $m_q\ (\mu_0)$ where

$$\mu_0 = 2m_q(\mu_0) \tag{3.13}$$

which should be threshold for producing naked $q\bar{q}$ pairs. But the renormalization group formalism only applies directly to space-like Q^2. Moorhouse, Pennington and Ross (MPR)[24] have pointed out that if (as suggested by the renormalization group equations)

$$g^2(q^2) \propto \frac{1}{\ln(-q^2/\Lambda^2)} \text{ for large negative } q^2 \tag{3.14}$$

then for $q^2 = |q^2|\, e^{i\theta}$ one should use

$$g^2(q^2) \propto \frac{1}{\ln(|q^2|/\Lambda^2) + i(\pi-\theta)} \tag{3.15}$$

Furthermore, any quantity such as a quark mass which the renormalization group tells us is $\sim [g^2(q^2)]^\delta$: $\delta \neq 0$ will also inherit the complex phase in (3.15). The difference between (3.14) and (3.15) is unimportant near the grand unification mass scale M, but potentially significant close to the quark thresholds we are interested in. For example, if $\alpha_s \simeq 0.32$ at $-q^2 = 10$ GeV^2 as suggested by leading order electroproduction analyses, the MPR[24] analysis would yield $|\alpha_s(q^2 = 10 \text{ GeV}^2)|_{MPR} \simeq 0.27$. If the MPR correction were substituted into (3.12), it would reduce the estimate of m_b by $\sim 4\%$. It is not clear to us what the correct procedure for continuing to the threshold region should be, but the MPR analysis suggests there is a (5 to 10)% slop in our estimates of m_b.

Table 2 summarizes our studies of corrections to the mass ratio renormalization formula (3.12). We would conclude that an estimate of m_b should be accurate to $O(10)\%$, while an estimate of m_s is probably less precise $[\pm O(25)\%?]$, principally because of the higher order and time-like continuation uncertainties.

3.2 Renormalization calculations in SU(5)

To estimate these effects we have assumed a range of values for $\alpha_s(Q^2 = 10 \text{ GeV}^2)$. Analyses of the charmonium system using the 3-gluon annihilation model for the total hadronic decay rate yield[13] $\alpha_s \simeq 0.19$. If one writes this in a form motivated by the renormalization group:

$$\alpha_s(Q^2) \simeq \frac{12\pi}{25 \ln(Q^2/\Lambda^2)} \tag{3.16}$$

it corresponds to $\Lambda^2 = 0.005$ GeV^2. On the other hand, analyses of electroproduction[14] suggest a larger value of Λ^2, typically $\Lambda \simeq 0.3$ GeV, $\Lambda^2 = 0.09$ GeV^2 if only g^2 terms in anomalous dimensions and g^3 terms in the β function are retained, as we also do here. We will quote results for these extreme values, and for $\Lambda^2 = 0.03$ GeV^2.

We determined the grand unification mass scale M $[= M_X$ in the SU(5) model$]$ using α_s and α as inputs, by allowing α_s, α_2 and α_1 to evolve independently, and assuming they are equal at M. This is a slight over-simplification, since there are in principle finite mass corrections close to the grand unification mass, due to the X and Y bosons and in principle Higgs bosons, which we have neglected. As a possible test of this assumption, we have compared the transition of α_s across a new quark threshold using the complete formula (3.8) with a naive procedure where α_s is allowed to evolve with $f = f_0$ up $Q^2 = m_f^2$, and then made to evolve with $f = f_0 + 1$ for $Q^2 > m_f^2$, keeping the coupling constant continuous at the transition, which involves changing Λ at the new quark threshold[25]. The difference between the approximate and exact values of α_3 above threshold was $\sim \frac{1}{4}\%$. While the discrepancy would be bigger for the transition to grand unification, we cannot believe that it would be a very big effect.

We calculated $\sin^2 \theta_W$ at present energies using the evolution of g_2^2 and g_1^2 given by (3.8) and including the effect of a light SU(2) × U(1) Higgs doublet:

$$\Delta\beta^{(3)} = 0 \quad ; \quad \Delta\beta^{(2)} = -\frac{1}{6}\frac{g_2^2}{16\pi^2} \quad ; \quad \Delta\beta^{(1)} = -\frac{1}{10}\frac{g_1^2}{16\pi^2} \tag{3.17}$$

Their effects on $\sin^2 \theta_W$ are actually only O(1 to 2)%. We chose to evaluate $\sin^2 \theta_W$ at $Q^2 = 10^4$ GeV^2, since this is to a good approximation the energy at which the weak and electromagnetic interactions are unified in the Weinberg-Salam[23] model.

Finally we calculated m_b [using $m_\tau = 1.9$ GeV [26] at present energies, and $m_b = m_\tau$ at M as with 5-plets of Higgs fields in SU(5)] and m_s [using $m_\mu = 0.105$ GeV at present energies, and $m_s = m_\mu$ at M] using the definitions (3.13). In principle we could also have calculated m_d from m_e, but the definition (3.13) is unusable for the u and d quarks. Actually, since

$$\frac{m_d(\mu)}{m_s(\mu)} = \frac{m_e}{m_\mu} \sim \frac{1}{200} \tag{3.18}$$

in this approach, the d quark mass is probably underestimated, since people usually prefer $\sim \frac{1}{20}$ [27] for m_d/m_s at short distances*).

The results are listed in Table 1. We see that the grand unification mass $M \sim 2 \times 10^{16}$ GeV to within a factor of 2 or 3 corresponding to a proton lifetime $O(10^{38\pm1})$ years, while $\alpha_{GUM} \sim \frac{1}{45}$. The value of $\sin^2 \theta_W$ is relatively stable at 0.20, which is to be compared with the latest CDHS experimental estimate of 0.24 ± 0.02 [21]. Experiment has drifted closer to theory since the calculations of Georgi, Quinn and Weinberg[3] and CEG[4]. The estimate of m_s is quite successful. If the Τ(9.4 to 10.4) states[15] turn out to be bottomonium, then the estimate of m_b will have been correct within the expected theoretical error of 10%. The naked $b\bar{b}$ threshold is probably at (10.4 to 10.6) GeV, corresponding to

*) Long distance ("constituent") quark masses have additional terms generated dynamically. These are largest for the u and d quarks, non-negligible for s quarks, and negligible for c, b, t, ... quarks. See, for example, Ref. 22.

m_b = (5.2 to 5.3) GeV within the definition (3.13). The sensitivity of the m_b estimate to various modifications of our assumptions and approximations is set out in Table 2, and some of these effects were discussed in Section 3.1. We note with regret the insensitivity to the mass m_t of the sixth quark. We are happy to see the sensitivity to the inclusion of eight or ten quarks. But even if you were to believe all this nonsense, then you would perhaps not immediately conclude that there were only six quarks!

Actually, the estimate of the bottom quark mass is probably not very sensitive to the precise details of the SU(5) model. It only requires knowledge of the evolution of the SU(3), SU(2), and U(1) coupling constants below the grand unification mass, and not really how they are unified above it. The one sensitivity to the unification scheme lies in the parameter C of Eq. (3.6). This could be $\sqrt{n/n-2}$ in a more general SU(n) unifying group[28]. However, C only enters in the U(1) factor in m_b/m_τ, which is only a 9% effect anyway, so that the likely effect on m_b would be very small. Any grand unification scheme with a mass scale $\mathcal{O}(10^{16})$ GeV at which $m_b = m_\tau$ and which broke down to SU(3) × $[SU(2)]^N$ × U(1) at present energies would have a prediction for m_b similar to the SU(5) model.

Before leaving this section, we should note one implication of using a group factor bigger than SU(2) for the weak interactions, such as SU(3) [19]. You will not be able to bring together the strong and weak interactions unless you also increase the strong gauge group from SU(3) to a bigger group like SU(4) [2], since the rate of approach of the couplings is controlled by the difference between the corresponding $\bar{\beta}^{(\text{strong})}$ and $\bar{\beta}^{(\text{weak})}$:

$$\frac{1}{g^2_{\text{strong}}(\mu)} - \frac{1}{g^2_{\text{weak}}(\mu)} \approx \left[\bar{\beta}^{(\text{strong})} - \bar{\beta}^{(\text{weak})}\right] \ln\left(M/\mu\right) \tag{3.19}$$

In order to achieve grand unification before the Planck mass, any such augmentation of the strong group must take place before Q ∿ 10^8 GeV. Unfortunately, the interactions building SU(3) up to SU(4) could have a four-fermi coupling strength [cf. Eq. (2.8)] as small as $G_{3\to4} \sim 10^{-17}/m_P^2$. The analogue of the Eqs. (3.11) for a bigger "strong interaction" group clearly implies a smaller renormalization of m_q/m_ℓ in such a model, unless the number of flavours is increased correspondingly: SU(3) → SU(4) requires an increase $\mathcal{O}(6)$ in the number of flavours.

4. SYMMETRY BREAKING IN SU(5)

It would surprise us if SU(5) in fact describes the real world, but it is useful to work out the model in order to establish an "existence proof" for grand unified theories. All these models need a gauge hierarchy in which the grand unified group is first broken down on a very large mass scale m_X to a product of strong and weak/electromagnetic groups, and then the weak/electromagnetic group is broken down to U(1) on a much smaller mass scale m_W. Two questions then arise: can such a hierarchy of symmetry breakdown ($m_X \gg m_W$) be achieved at all, and can it be made "natural"? In this section we study these questions in the SU(5) model, where the breakdown pattern should be

$$SU(5) \to SU(3) \times SU(2) \times U(1) \to SU(3) \times U(1) \tag{4.1}$$

As discussed by Georgi and Glashow[1], the minimal Higgs system which can be considered involves an adjoint 24 representation of SU(5) for the superstrong breaking, and a spinorial 5 representation for the weak breaking.

We consider first the superstrong breaking in isolation. Introducing a matrix notation $\Phi = \sum_{a=1}^{24} \phi^a(\lambda^a/\sqrt{2})$ for the adjoint representation, the most general Φ potential is

$$V(\Phi) = -\frac{\mu^2}{2}\mathrm{Tr}(\Phi^2) + \frac{a}{4}\mathrm{Tr}((\Phi^2)^2) + \frac{b}{2}\mathrm{Tr}(\Phi^4) + \frac{c}{3}\mathrm{Tr}(\Phi^3) \tag{4.2}$$

but we will impose a discrete symmetry $\Phi \leftrightarrow -\Phi$ so that $c = 0$. The potential (4.2) permits many patterns of symmetry breakdown, but as pointed out by Li[29], if

$$b > 0 \quad \text{and} \quad a > -\frac{7}{15}b \tag{4.3}$$

the lowest vacuum is the asymmetric one corresponding to the desired breakdown of $SU(5) \to SU(3) \times SU(2) \times U(1)$, with a Higgs vacuum expectation matrix

$$\langle 0|\Phi|0\rangle = \begin{pmatrix} v & & & & \\ & v & & 0 & \\ & & v & & \\ & 0 & & -\frac{3v}{2} & \\ & & & & -\frac{3v}{2} \end{pmatrix} \tag{4.4}$$

with v determined by

$$\mu^2 = \frac{15}{2}av^2 + \frac{7}{2}bv^2 \tag{4.5}$$

The superheavy X and Y bosons then have masses[30]:

$$m_X^2 = m_Y^2 = \frac{25}{8}g^2v^2 \tag{4.6}$$

If we parametrize $\Phi - \langle 0|\Phi|0\rangle$ in the form

$$\Phi - \langle 0|\Phi|0\rangle = \left(\begin{array}{c|c|c} H_8 + \frac{2H_0}{\sqrt{30}} I & \underset{\sim}{H}_X & \underset{\sim}{H}_Y \\ \hline \underset{\sim}{H}_X^+ & \frac{H_3}{\sqrt{2}} - \frac{3}{\sqrt{30}} H_0 & H^+ \\ \hline \underset{\sim}{H}_Y^+ & H^- & -\frac{H_3}{\sqrt{2}} - \frac{3}{\sqrt{30}} H_0 \end{array}\right) \tag{4.7}$$

then $\underset{\sim}{H}_X$ and $\underset{\sim}{H}_Y$ are massless and eaten by the X and Y bosons, and

$$m^2_{H_8} = \frac{5b}{2} v^2 \; ; \; m^2_{H_3} = m^2_{H^\pm} = 10 b v^2 \; ; \; m^2_{H_0} = 15 a v^2 + 7 b v^2 = 2\mu^2 \tag{4.8}$$

so that all the physical Higgs bosons have very large masses $O(m_{X,Y})$.

Now we add in a $\underline{5}$ of Higgs fields $\underline{H}$ for which the most general potential is

$$V(\underline{H}) = -\frac{\nu^2}{2}(\underline{H}^+ \cdot \underline{H}) + \frac{\lambda}{4}(\underline{H}^+ \cdot \underline{H})^2 \tag{4.9}$$

If the component of $\underline{H}$ which develops a vacuum expectation value lies in the SU(2) left behind by the superstrong breaking, then we may parametrize $\underline{H}$ in the form

$$\underline{H} = \begin{pmatrix} \underset{\sim}{H} \\ H_4 \\ \left(\frac{v_0 + \rho}{\sqrt{2}}\right) e^{i\zeta/v_0} \end{pmatrix} \tag{4.10}$$

with v_0 determined by

$$2\nu^2 = \lambda v_0^2 \tag{4.11}$$

The SU(2) symmetry is broken in the desired way by an isodoublet of Higgs fields, and the $W^\pm$ mass is:

$$m_W^2 = \frac{g^2 v_0^2}{4} \tag{4.12}$$

The mystery why $m_W \ll m_X$ now becomes the mystery why $v_0 \ll v$, or why $\nu^2 \ll \mu^2$. The $\underset{\sim}{H}$, H_4 and ζ Higgs fields are massless, H_4 and ζ being eaten by the W and Z bosons. The ρ field is a physical Higgs boson with

$$m_\rho^2 = \frac{\lambda v_0^2}{2} = \nu^2 \tag{4.13}$$

The construction of the Higgs system is by no means complete because only one combination of the massless Higgs fields $\underset{\sim}{H}_Y$ and $\underset{\sim}{H}$ can be eaten to give the Y bosons masses. The other combination would remain massless, which would be phenomenologically unacceptable because it would form light bound states with quarks, which would have strong interactions -- it would be natural to call them quiggs particles. However, there is no reason to exclude cross-coupling of the Φ and $\underline{H}$ fields from the Higgs potential, and indeed such terms are generated by renormalization. The most general (Φ, $\underline{H}$) coupling terms respecting the discrete symmetry $\Phi \leftrightarrow -\Phi$ are

$$V(\Phi, \underline{H}) = \alpha (\underline{H}^{+} \cdot \underline{H}) \mathrm{Tr}(\Phi^2) + \beta \underline{H}^{+} \Phi^2 \underline{H} \tag{4.14}$$

and we should look for an extremum of the combined potential (4.2), (4.9), and (4.14) which removes the embarrassing massless Higgs without destroying the desirable features of (4.2) and (4.9). When Φ and $\underline{H}$ are coupled, $\langle 0|\Phi|0\rangle$ may also break SU(2), and we should look for solutions with

$$\langle 0|\Phi|0\rangle = \begin{pmatrix} v & & & & \\ & v & & 0 & \\ & & v & & \\ & 0 & & (-\frac{3}{2} - \frac{\epsilon}{2})v & \\ & & & & (-\frac{3}{2} + \frac{\epsilon}{2})v \end{pmatrix} \tag{4.4'}$$

There will be an extremum of the combined potential which connects up with the previous partial solutions in the limit as α, $\beta \to 0$. For sufficiently small α and β this will still be the absolute minimum, and hence that chosen in the "real world". For this extremum ϵ should $\to 0$ as α, $\beta \to 0$. The solution with these properties in fact has

$$\epsilon = \frac{3}{20} \frac{\beta v_0^2}{b v^2} + O\left(\left(\frac{v_0}{v}\right)^4\right) \tag{4.15}$$

so that the SU(2) breaking by the Φ Higgs fields is much less than that due to the $\underline{H}$ fields, in accord with the experimental preference for I = ½ SU(2) Higgs dominance[21,31]. In the combined potential the conditions (4.5) and (4.11) get replaced by

$$\mu^2 = \frac{15}{2} a v^2 + \frac{7}{2} b v^2 + \alpha v_0^2 + \frac{9}{30} \beta v_0^2 \tag{4.5'}$$

$$\nu^2 = \frac{\lambda v_0^2}{2} + 15 \alpha v^2 + \frac{9}{2} \beta v^2 - 3 \epsilon \beta v^2 \tag{4.11'}$$

The first of the conditions is just a minor modification of (4.5), whereas (4.11′) looks rather "unnatural" in that it requires a very strong cancellation between different *a priori* large terms $O(v^2)$ in order that v_0^2 be kept very small. Thus the mystery why $m_W \ll m_X$ persists, and is not resolved by the coupling of Φ and $\underline{H}$. On the other hand, from a strictly logical point of view, the situation is no worse either -- you always need just one mass parameter to be much less than another. We should note in passing that m_X^2, m_Y^2, $m_{H_8}^2$, and $m_{H^\pm}^2$ are only altered by $O(v_0^2)$ in the combined Higgs system, while one combination of ρ, H_z and H_0 keeps a mass $O(v_0)$ and the others keep masses $O(v)$. The combination of $\underset{\sim}{H}_Y$ and $\underset{\sim}{H}$ which is massless and eaten by the Y boson is

$$\underset{\sim}{H}_Y - \frac{\sqrt{2}v_0}{5v}\underset{\sim}{H} + O\left(\frac{v_0^2}{v^2}\right) \tag{4.16}$$

whereas the orthogonal combination

$$H_3 \equiv \underset{\sim}{H} + \frac{\sqrt{2}v_0}{5v}\underset{\sim}{H}_Y + O\left(\frac{v_0^2}{v^2}\right) \tag{4.17}$$

is physical, with a mass

$$m_{H_3}^2 = -\frac{5}{2}\beta v^2 + O(v_0^2) \tag{4.18}$$

The H_3 boson therefore has a mass comparable with m_X, though the condition that the extremum studied here is in fact the absolute minimum presumably requires that β is in some sense "small" compared with a, b, λ, but not necessarily much smaller. In principle, baryon number violation could be mediated by H_3 exchange, but this seems likely to be much smaller than that mediated by X or Y exchange. We see from (4.17) that H_3 is mainly $\underset{\sim}{H}$, which has fermion couplings

$$g_{\underset{\sim}{H}f\bar{f}'} = O\left(g\frac{m_f}{m_W}\right) \tag{4.19}$$

like the familiar SU(2) Higgs. We therefore expect that

$$\frac{\Gamma(p \underset{H_3}{\rightarrow} \ell + \text{any})}{\Gamma(p \underset{X}{\rightarrow} \ell + \text{any})} = O\left(\frac{m_{X,Y}^4}{m_{H_3}^4}\frac{m_f^4}{m_W^4}\right) = O\left(\frac{g^4}{\beta^2}\frac{m_f^4}{m_W^4}\right) \tag{4.20}$$

which is plausibly much less than 1. We therefore conclude that the combined potential (4.2), (4.9), and (4.14) seems to have all the desired properties, at the continued price of the one "unnatural" condition (4.11').

As was mentioned above, renormalization effects will couple the Φ and $\underline{H}$ systems even if they are independent in the tree approximation to the potential[21]. Following Coleman and Weinberg[20], the first order radiative corrections to the Higgs potential arising from vector boson loops, are

$$V'(\Phi, \underline{H}) = \frac{3}{64\pi^2}\mathrm{Tr}\left[\mathcal{M}^2 \ln\left(\mathcal{M}^2/M_0^2\right)\right] \tag{4.21}$$

where $\mathcal{M}^2_{ab}$ is the Higgs contribution to the vector boson mass² matrix:

$$\mathcal{M}^2_{ab} = g^2_{GUM}\left[-\frac{1}{4}\,\mathrm{Tr}\left\{[\lambda^a,\Phi][\lambda^b,\Phi]\right\} + \frac{1}{4}\,\underline{H}^\dagger\{\lambda^a,\lambda^b\}\underline{H}\right] \tag{4.22}$$

In Appendix B we display the expression for $V'(\Phi, H)$ obtained by expanding the fields around the "zeroth" order vacuum, Eq. (4.4). This is sufficient for extracting the linear and quadratic terms in the Higgs fields, and is valid if the true vacuum indeed satisfies ε, $v_0/v \ll 1$. As an exercise to show that a solution with the desired properties can exist, we assume simplified forms for the zeroth order potential so that we can easily check *a posteriori* that neglect of the Higgs loop contributions is justified. These correspond to $\beta = b = 0$ and either $\alpha = 0$ or $\alpha = a = \lambda/4$. Then the extremum conditions are

$$\alpha^2_{GUM}\left[2\ln\frac{25v^2}{8M_0^2} + 1\right] = \frac{16}{375}\left[\frac{\mu^2}{v^2} - \frac{15}{2}a\right] + O\left(\frac{v_0^2}{v^2}\right) \equiv -\omega \tag{4.5'}$$

which determines v as a function of the parameters of the potential, and

$$\epsilon = \frac{3}{20}\frac{v_0^2}{v^2}\left[\frac{2\alpha^2_{GUM} - \omega}{\alpha^2_{GUM} - 3\omega}\right] + O\left(\frac{v_0^3}{v^3}\ln v^2/v_0^2\right) \tag{4.15'}$$

which shows that the desired form of SU(2) breaking persists "naturally", as well as the "unnatural" condition

$$\frac{\nu}{2v^2} = \frac{15\alpha}{2} + \frac{3}{10}\left[\frac{\mu^2}{v^2} - \frac{15}{2}a\right] + O\left(\frac{v_0^2}{v^2}\ln v_0^2/v^2\right) \tag{4.11'}$$

for the gauge hierarchy $m_X \gg m_W$. Equation (4.11′) involves a complicated interplay between the parameters appearing in the zero- and one-loop potentials (which must be readjusted for each order of perturbation theory!). The condition for a minimum is

$$0 < \frac{15}{2}a - \frac{\mu^2}{v^2} < \frac{375}{32}\alpha^2_{GUM} \approx 5\times 10^{-3} \tag{4.23}$$

and the SU(3) triplet of physical Higgs bosons will be sufficiently massive for proton stability if

$$\frac{15a}{2} - \frac{\mu^2}{v^2} \gtrsim 10^{-3} \tag{4.24}$$

The constraint (4.24) ensures that Eq. (4.11′) can only be achieved by a strong cancellation. However, our limited investigation does not rule out the possibility that a less contrived looking solution might emerge from a more clever choice of potential.

We therefore conclude this section with the belief that an SU(5) Higgs system can be set up which has the desired phenomenological properties, and that these are not destroyed by first order radiative corrections. The "observed" pattern of SU(5) and SU(2) symmetry breaking is in many respects "natural", but the hierarchy $m_X >> m_W$, while possible, is unexplained and rather "unnatural"[21]. For convenience, the spectrum of physical bosons is listed in Table 3, together with some of their characteristics. The list is long, but shorter than in any other grand unified model known to us.

5. CONCLUSIONS

In this paper we have mainly studied three aspects of grand unified theories of the strong, weak, and electromagnetic interactions.

- The proton lifetime, which seems to be $O(10^3$ to $10^4)$ times longer than the $O(m_X^4/m_p^5)$ expected from simple dimensional arguments. Higgs boson exchanges do not seem to dominate proton decay.
- Mass renormalization effects, which in a class of models including SU(5) give realistic estimates (1.2) for the strange and bottom quark masses. On the other hand, $\sin^2\theta_W \simeq 0.20$ in the SU(5) model, which is somewhat[16] low.
- The Higgs structure of SU(5), which is able to accommodate the symmetry-breaking pattern SU(5) → SU(3) × SU(2) × U(1) → SU(3) × U(1) quite naturally, but does not naturally[20] give the large ratio observed between the two mass scales of spontaneous symmetry breaking.

We hope that the considerations under the first two of these headings have more general validity than in just the Georgi-Glashow[1] SU(5) model. This model has the experimental problems of the $SU(2)_L \times U(1)$ Weinberg-Salam[23] model, such as the absence of parity violation in atomic physics[32] and the presence of exotic trimuon events[33] in neutrino scattering, as well as a possible problem with the value of $\sin^2\theta_W$ [16]. It is also theoretically unattractive because it affords no understanding of different mass scales and mixing angles, and gives no understanding of the number of fundamental fermion fields, which can just be added sequentially in the model. On the other hand, the SU(5) model cannot yet be rigorously excluded as a logical possibility, and may serve as a useful existence proof and prototype for grand unified models, just as the Weinberg-Salam model[23] has been a useful starting point for gauge theories of the weak and electromagnetic interactions. Also, we should remember that the SU(5) model is much less complicated than other grand unified models on the market. For example, the E_7 model[34] apparently[35] requires (at the least) $\underline{912}$, $\underline{133}$, and $\underline{1463}$ Higgs representations to get an appropriate pattern of symmetry breaking, and the appropriate renormalization of the strong relative to the weak coupling constant seems difficult to realize, for the reasons discussed at the end of Section 3. Also, the E_7 model in the form proposed is apparently ruled out by recent neutrino scattering experiments[36]. Another group proposed is SO(10) [12], which needs (at the least) $\underline{10}$, $\underline{16}$, $\underline{120}$, and $\underline{126}$ Higgs representations to get the appropriate pattern of symmetry breaking. However, it has[4] no problem with the renormalization of the strong relative to the weak coupling constant. The bleak comments of this paragraph may betoken the bankruptcy of the simple-minded simple group philosophy[6]. But even the wrong answer to the right question may be instructive.

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REFERENCES

1) H. Georgi and S.L. Glashow, Phys. Rev. Letters 32, 438 (1974).

2) J.C. Pati and A. Salam, Phys. Rev. D 8, 1240 (1973), and Phys. Rev. D 10, 275 (1974).

3) H. Georgi, H.R. Quinn and S. Weinberg, Phys. Rev. Letters 33, 451 (1974).

4) M.S. Chanowitz, J. Ellis and M.K. Gaillard, CERN preprint TH 2312 (1977) to appear in Nuclear Phys. B.

5) F.J. Ynduráin, Nuclear Phys. B115, 293 (1976).

6) For a critical discussion and review, see H. Harari, Weizmann Institute preprint WIS-76/54 PH (1976) to appear in the Proceedings of the 1976 Les Houches Summer School.

7) K.G. Wilson, Phys. Rev. 179, 1499 (1969).
M.K. Gaillard and B.W. Lee, Phys. Rev. Letters 33, 108 (1974).
G. Altarelli and L. Maiani, Phys. Letters 52B, 351 (1974).

8) M.K. Gaillard, B.W. Lee and J.L. Rosner, Rev. Mod. Phys. 47, 277 (1975).

9) C. Schmid, Phys. Letters 66B, 353 (1977).

10) A. Le Yaouanc, L. Oliver, O. Pène and J.-C. Raynal, Orsay preprint LPTHE 77/22 (1977).

11) F. Reines and M.F. Crouch, Phys. Rev. Letters 32, 493 (1974).

12) H. Fritzsch and P. Minkowski, Ann. Phys. (N.Y.) 93, 193 (1975) and Nuclear Phys. B103, 61 (1976).
H. Georgi, Particles and fields, 1974 (APS/DPF Williamsburg), ed. C.E. Carlson (AIP, New York, 1975), p. 575.

13) J.D. Jackson, CERN preprint TH 2351 (1977) and references therein.

14) See, for example, A.J. Buras, E.G. Floratos, D.A. Ross and C.T. Sachrajda, CERN TH 2340 (1977) and references therein.

15) S. Herb et al., Phys. Rev. Letters 39, 252 (1977).
W.R. Innes et al., FNAL preprint, Observation of structure in the T region (1977).

16) M. Holder et al., CERN preprint, Measurement of the neutral to charged current cross-section ratio in neutrino and antineutrino interactions (1977).

17) M. Kobayashi and K. Maskawa, Progr. Theor. Phys. 49, 652 (1973).

18) T.D. Lee, Phys. Reports 9C, 143 (1974).
S. Weinberg, Phys. Rev. Letters 37, 657 (1976).

19) B.W. Lee, R.E. Shrock and S. Weinberg, Fermilab preprint Pub-77/48-THY (1977) and references therein.

20) S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973).

21) E. Gildener, Phys. Rev. D 14, 1667 (1976).

22) H. Georgi and H.D. Politzer, Phys. Rev. D 14, 1829 (1976).
H.D. Politzer, Nuclear Phys. B117, 397 (1976).

23) S. Weinberg, Phys. Rev. Letters 19, 1264 (1967).
A. Salam, Proc. 8th Nobel Symposium, Stockholm (ed. N. Svartholm) (Almqvist and Wiksells, Stockholm, 1968), p. 367.

24) R.G. Moorhouse, M.R. Pennington and G.G. Ross, Nuclear Phys. B124, 285 (1977).

25) J. Ellis, M.K. Gaillard, D.V. Nanopoulos and S. Rudaz, CERN preprint TH 2346 (1977) to appear in Nuclear Physics B.

26) M.L. Perl et al., Preprint SLAC-PUB-1997/LBL-6731 (1977).

27) M. Gell-Mann, R.J. Oakes and B. Renner, Phys. Rev. 175, 2195 (1968).

28) S.P. Rosen, Phys. Rev. Letters 33, 614 (1974), and references therein.

29) L.-F. Li, Phys. Rev. D 9, 1723 (1974).

30) E.S. Fradkin and O.K. Kalashnikov, Phys. Letters 64B, 177 (1976).

31) D.A. Ross and M. Veltman, Nuclear Phys. B95, 135 (1975).

32) P. Baird et al., Nature 264, 528 (1976).
P. Sandars, Talk presented at the 1977 International Symposium on Lepton and Photon Interactions at High Energies, Hamburg 1977.

33) A. Benvenuti et al., Phys. Rev. Letters 38, 1110 (1977).

34) F. Gürsey and P. Sikivie, Phys. Rev. Letters 36, 775 (1976).
P. Ramond, Nuclear Phys. B110, 214 (1976).

35) P. Ramond, Nuclear Phys. B126, 509 (1977).

36) M. Holder et al., Phys. Rev. Letters 39, 433 (1977).

Table 1

$\alpha_s(Q^2 = 10)$	Λ^2	m_b	m_s	m_X	α_{GUM}	$\sin^2 \theta_W$
0.32	0.09	5.9	0.50	3.7×10^{16}	0.022	0.20
0.26	0.03	5.5	0.45	2.1×10^{16}	0.022	0.20
0.19	0.005	5.0	0.38	0.9×10^{16}	0.022	0.20

Λ^2 is defined by $\alpha_s(Q^2) = \dfrac{12\pi}{25 \ln (Q^2/\Lambda^2)}$, all masses are in GeV.

Results are calculated with m_t = 8 GeV and neglecting Higgs contributions to the Callan-Symanzik β-functions. Finite mass corrections in the SU(3) renormalization (3.12) have been evaluated numerically.

Table 2

	f = 8	f = 10	Higgs included in β-functions	MPR time-like correction	$m_t \to 100$ GeV
$\dfrac{\Delta m_b}{m_b}$	+10%	+30%	±½%	-4%	-3%

Results are calculated with $\Lambda^2 = 0.03$.

Table 3

Physical boson content of the SU(5) model

Particle	Charge	Spin	Mass	Remarks
γ, G	0	1	0	The usual photon and gluons
$W^{\pm}$	± 1	1	$\approx \frac{gv_0}{2}$ $\approx$ 84 GeV	Usual vector bosons of Weinberg-Salam model, with mass relations essentially as given by I = ½ Higgs doublet and $\sin^2 \theta_W = 0.20$, because $\varepsilon = O(v_0^2/v^2)$.
Z^0	0	1	$\approx \frac{gv_0}{2 \cos \theta_W}$ $\approx$ 94 GeV	
X	$\pm {}^4/_3$	1	$\approx \frac{5gv}{2\sqrt{2}}$ $\approx 2 \times 10^{16}$ GeV	Violate baryon and lepton number conservation. Mass degeneracy broken in $O(gv_0^2/v)$.
Y	$\pm {}^1/_3$	1		
H	0	0	$O(\lambda^{1/2} v_0)$	Lightest state resembles the Higgs boson of the usual Weinberg-Salam model with an I = ½ Higgs multiplet. The others presumably have masses close to those of the X and Y vector bosons.
H′	0	0	$O(b^{1/2} v)$	
H″	0	0		
H_8	0	0	$O(b^{1/2} v)$	Colour octet with masses close to m_X, m_Y
$H^{\pm}$	± 1	0		Uncoloured and heavy charged Higgs particles
H_3	$\pm {}^1/_3$	0	$O(b^{1/2} v)$?	Colour triplet with mass perhaps somewhat less than m_X, m_Y. Violates baryon and lepton number conservation, but at a low level.

"Close in mass" probably means within a factor of 10. Note that the only low-mass [$\lesssim O(100)$ GeV] bosons are those in the usual Weinberg-Salam[23] model.

Figure captions

Fig. 1 : Baryon and lepton number violating exchanges in the SU(5) model due to (a) vector gauge bosons, and (b) Higgs bosons.

Fig. 2 : Lowest order gluonic corrections to the qqq vertex which determine the anomalous dimension relevant to the short distance enhancement factor.

Fig. 3 : Model for proton decay in which two quarks annihilate freely into a lepton and an antiquark.

APPENDIX A

THE ANOMALOUS DIMENSION OF THE TRI-QUARK OPERATOR

There are three independent operators appearing in the effective Lagrangian (2.7):

$$\begin{aligned} O^{e}_{L,R} &\equiv \left(\epsilon_{ijk}\,\bar{u}^{c}_{k_L}\gamma_\mu u_{j_L}\right)\left(\bar{e}^{+}_{L,R}\gamma^\mu d_{i_{L,R}}\right) \\ O^{\nu} &\equiv \left(\epsilon_{ijk}\,\bar{u}^{c}_{k_L}\gamma_\mu d_{j_L}\right)\left(\bar{\nu}^{c}_{e_R}\gamma^\mu d_{i_R}\right) \end{aligned} \tag{A.1}$$

Since we want only the leading contributions which are mass independent, helicity is a good quantum number. Then it is easy to convince oneself that the operators (A.1) are multiplicatively renormalized in leading order since they are uniquely characterized by the helicity $[q^C_L \equiv C(q_R)]$ and/or flavour of the external quarks, and there is only one colour singlet combination of dimension six. Furthermore, they are renormalized identically because gluon exchange is helicity and flavour independent.

We must evaluate, for example, the contributions of the diagrams of Fig. 2 to the matrix element

$$\begin{aligned} \langle O^\nu\rangle &\equiv \left\langle T\left(O^\nu (u^c_L)_{k,\gamma}(\bar{d}_L)_{j,\beta}(\nu^c_{e_R})_\delta(\bar{d}_R)_{i,\alpha}\right)\right\rangle \\ &= \epsilon_{ijk}\,(\gamma_\mu)_{\gamma\beta}(\gamma^\mu)_{\delta\alpha}\left[1+\frac{\alpha_s}{8\pi}d\ln\left(\frac{\mu^2}{-p^2}\right)+O(\alpha_s^2)\right] \end{aligned} \tag{A.2}$$

where μ^2 is the renormalization point and $p^2 < 0$ is the value of the external momenta with some suitable convention for fixing the ratios s/p^2, t/p^2. The constant d, related to the anomalous dimension of the operator by

$$\gamma_0 = \frac{\partial}{\partial\mu}\langle O^\nu\rangle\Big|_{-p^2=\mu^2} = \frac{\alpha_s}{4\pi}d \tag{A.3}$$

can be extracted from the coefficient of the logarithmically divergent term in the Feynman integrals of Fig. 2. Setting the external momenta to zero, and using the quark gluon coupling

$$\bar{q}\gamma^\mu\frac{\lambda^a}{2}q\,B^a_\mu = -\bar{q}^c\gamma^\mu\frac{\lambda^a}{2}q\,B^a_\mu \tag{A.4}$$

the integrals take the form:

$$\frac{d}{2}\epsilon_{ijk}(\gamma_\mu)_{\gamma\beta}(\gamma^\mu)_{\delta\alpha}\frac{-ig^2}{(2\pi)^4}\int\frac{d^4p}{p^4} \cdot \frac{\alpha_s}{4\pi}\frac{d}{2}\epsilon_{ijk}(\gamma_\mu)_{\gamma\beta}(\gamma^\mu)_{\delta\alpha}\ln\Lambda^2 \tag{A.5}$$

where Λ is the ultraviolet cut-off.

Using the relation

$$\epsilon_{i'j'k} \frac{\lambda^a_{i'i}}{2} \frac{\lambda^a_{j'j}}{2} = -\frac{2}{3} \epsilon_{ijk} \tag{A.6}$$

we obtain the factors

$$d_c^{(2a)} = -\frac{2}{3} \quad ; \quad d_c^{(2b)} = d_c^{(2c)} = +\frac{2}{3} \tag{A.7}$$

from contraction of colour SU(3) indices. Working in the t'Hooft-Feynman gauge, we find for Fig. 2a the Dirac algebra factor

$$d_D^{(3a)} (\gamma_\mu)_{\beta\gamma} (\gamma^\mu)_{\delta\alpha} = \frac{1}{p^2} (\gamma_\mu \not{p} \gamma_\nu)(\gamma^\mu (-\not{p}) \gamma^\nu)$$

giving

$$d_D^{(2a)} = -4 \tag{A.8a}$$

and in a similar way

$$d_D^{(2b)} = d_D^{(2c)} = +1 \tag{A.8b}$$

Putting together the results of (A.7) and (A.8):

$$\frac{d}{2} = \sum d_c^i d_D^i = 4 \tag{A.9}$$

The renormalization of the effective coupling constant is given by

$$G_{GV}^{eff} = A\, G_{GV} \quad : \quad A = \left[\frac{\alpha_s(\mu^2)}{\alpha_s(M_X^2)}\right]^{\frac{d-3d_\psi}{2b}} \tag{A.10}$$

where b and d are related to the usual γ and β functions by

$$\beta = -\frac{g^3}{16\pi^2} b \quad : \quad b = 11 - \frac{2}{3} f$$

$$\gamma = \frac{\alpha_s}{4\pi} d_\psi \quad : \quad d_\psi = \frac{4}{3}$$

in the 't Hooft-Feynman gauge. Putting these numbers together we obtain the result of Eq. (2.10). (We have also checked our result in the Landau gauge where the wave function renormalization γ vanishes.)

APPENDIX B

ONE-LOOP CORRECTIONS TO THE HIGGS POTENTIAL

We shall expand the Coleman-Weinberg potential

$$V'(\Phi, \underline{H}) \equiv V(\mathcal{M}^2) \tag{B.1}$$

defined in Eqs. (4.21) and (4.22) around the point

$$V'(\Phi_0, 0) \equiv V(\mathcal{M}_0^2) \tag{B.2}$$

where

$$\mathcal{M}^2_{0ab} = -\frac{g^2_{SUN}}{4} \mathrm{Tr}\left\{ [\lambda^a, \Phi_0][\lambda^b, \Phi_0] \right\} \tag{B.3}$$

and Φ_0 is the matrix defined in Eq. (4.4). The expansion is done by noting that the matrix (B.3) is a projection operator onto the X, Y subspace of the vector boson mass matrix

$$\mathcal{M}_0^2 = g^2 \frac{25}{8} v^2 P \equiv g^2 M^2 P \quad ; \quad P^2 = P \tag{B.4}$$

Defining the matrices

$$\Delta \equiv \mathcal{M}^2 - \mathcal{M}_0^2 \quad ; \quad \Delta_P = (1-P)\Delta(1-P) \tag{B.5}$$

we obtain the expression

$$\begin{aligned} V'(\Phi, \underline{H}) = \frac{3\alpha^2_{SUN}}{4} \Big\{ & 12 M^4 \ln M^2/M_0^2 + \mathrm{Tr}(\Delta_P^2 \ln \Delta_P/M_0^2) \\ & + (2\ln M^2/M_0^2 + 1)(M^2 \mathrm{Tr}(P\Delta) + \mathrm{Tr}(\Delta^2 P)) - (\ln M^2/M_0^2 - \tfrac{1}{2}) \mathrm{Tr}((P\Delta)^2) \\ & + O\left(\Delta^3/M^2 \ln(\Delta_P/M^2)\right) \Big\} \end{aligned} \tag{B.6}$$

Expressed in terms of the Higgs fields:

$$\underline{H} \equiv \begin{pmatrix} \hat{\underline{H}} \\ \underset{\sim}{H}_2 \end{pmatrix} ; \quad \Phi - \Phi_0 = \left(\begin{array}{c|c|c} \hat{H}_8 + \frac{2H_0}{\sqrt{30}} I & \underset{\sim}{H}_X & \underset{\sim}{H}_Y \\ \hline \underset{\sim}{H}_X^+ & \frac{\hat{H}_2}{\sqrt{2}} - \frac{3}{\sqrt{30}} H_0 & H^+ \\ \hline \underset{\sim}{H}_Y^+ & H^- & -\frac{\hat{H}_2}{\sqrt{2}} - \frac{3}{\sqrt{30}} H_0 \end{array}\right) \tag{B.9}$$

the potential (B.6) takes the form:

$$
\begin{aligned}
V'(\Phi, \underline{H}) \simeq \frac{3\alpha^2_{GUM}}{4} \Big\{ & 12M^4 \ln M^2/M_0^2 \; + \\
& +\left(2\ln M^2/M_0^2 + 1\right)\Big[12M^3\sqrt{\tfrac{5}{3}}\,H_0 + 2|\hat{\underset{\sim}{H}}|^2 + 3|\underset{\sim}{H}_2|^2 + 10M^2(\underset{\sim}{H}_x^2 + \underset{\sim}{H}_y^2) - \frac{M}{\sqrt{2}}\hat{\underset{\sim}{H}}^+(\underset{\sim}{H}_x \,\vdots\, \underset{\sim}{H}_y)\underset{\sim}{H}_z \\
& \qquad - \frac{M}{\sqrt{2}}\underset{\sim}{H}_2^+\begin{pmatrix} \underset{\sim}{H}_x^+ \\ \underset{\sim}{H}_y^+ \end{pmatrix}\underset{\sim}{H}_3 \Big] \\
& + M^2\left(6\ln M^2/M_0^2 + 7\right)\Big[5H_0^2 + 3(\hat{H}_z^2 + 2H^+H^-) + 2\,\mathrm{Tr}(\hat{H}_8^2)\Big] \\
& + \left(\ln M^2/M_0^2 + \tfrac{3}{2}\right)\Big[8\hat{\underset{\sim}{H}}^+\frac{M\hat{H}_8}{\sqrt{2}}\hat{\underset{\sim}{H}} - 12\underset{\sim}{H}_2^+ M\begin{pmatrix} \hat{H}_z/\sqrt{2} & H^+ \\ H^- & -\hat{H}_z/\sqrt{2} \end{pmatrix}\underset{\sim}{H}_2 \\
& \qquad + M\sqrt{\tfrac{5}{3}}\,H_0\left(2|\hat{\underset{\sim}{H}}|^2 + 3|\underset{\sim}{H}_2|^2\right) + |\hat{\underset{\sim}{H}}|^4 + \tfrac{3}{2}|\underset{\sim}{H}_2|^4 \Big] \\
& + \tfrac{2}{5}\left(9 + 13\ln M^2/M_0^2\right)|\underset{\sim}{H}_2|^2|\hat{\underset{\sim}{H}}|^2 + \lambda_+^2\ln\lambda_+/M_0^2 + \lambda_-^2\ln\lambda_-/M_0^2 \\
& \qquad + 2\lambda_2^2\ln\lambda_2/M_0^2 + 4\lambda_3^2\ln\lambda_3/M_0^2 \Big\}
\end{aligned}
\tag{B.10}
$$

with

$$\lambda_\pm = \frac{2}{5}\Big\{|\underset{\sim}{H}_2|^2 + |\hat{\underset{\sim}{H}}|^2 \pm \Big[(|\underset{\sim}{H}_2|^2 + |\hat{\underset{\sim}{H}}|^2)^2 - \frac{15}{4}|\underset{\sim}{H}_2|^2|\hat{\underset{\sim}{H}}|^2\Big]^{\frac{1}{2}}\Big\} \; ; \; \lambda_2 = \frac{1}{2}|\underset{\sim}{H}_2|^2 \; ; \; \lambda_3 = \frac{1}{2}|\hat{\underset{\sim}{H}}|^2$$

Assuming a symmetry breaking of the form (4.10) and (4.4′), i.e. setting

$$
\left.\begin{aligned}
&\hat{\underset{\sim}{H}} = \underset{\sim}{H} \;, \quad \hat{H}_8 = H_8 \;, \quad H_2 = \begin{pmatrix} H_4 \\ \left(\frac{\rho + v_0}{\sqrt{2}}\right)e^{i\xi/v_0} \end{pmatrix} \\
&\hat{H}_z = H_z - \frac{\epsilon}{\sqrt{2}}v
\end{aligned}\right\}
\tag{B.11}
$$

it is straightforward to extract the linear and quadratic terms in (B.11). However, since we are seeking a solution with $m_H^2 >> m_{W,Z}^2$, it is not *a priori* obvious that terms arising from the Higgs boson loops are negligible. To avoid this prolem, we consider two prototype models with

$$\beta = b = 0 \tag{B.12}$$

and

$$\text{a)} \qquad \alpha = 0 \tag{B.13a}$$

$$\text{b)} \qquad \alpha = a = \lambda/4 \tag{B.13b}$$

Case (a) corresponds to a potential which is SU(24) ⊗ SU(10) symmetric and case (b) to SU(34). For an SU(N) symmetric coupling

$$\frac{\lambda}{4}S^2 \quad : \quad S = \sum_{i=1}^{N}\phi_i^2 \tag{B.14}$$

the one-loop contribution to the potential takes the form:

$$\frac{\lambda^2 s^2}{64\pi^2}\left[(N+8)\ln s/M_0^2 + O(\ln\lambda)\right] \tag{B.15}$$

Then (B.15) is a small correction to (B.14) as long as

$$\frac{\lambda}{16\pi^2}(N+8)\ln s/M_0^2 \ll 1 \tag{B.16}$$

Taking the potential as defined by (B.10), (B.12), and (B.13), we obtain the constraints of Eqs. (4.15′), (4.5″), and (4.11″).

As long as the quantity in Eq. (4.23) is >> v_0^2/v^2, the SU(3) triplet which does not get eaten is approximately the five-plet one, as in the zero-loop case, with mass

$$m^2_{H_3} = \frac{v^2}{10}\left[\frac{15a}{2} - \frac{\mu^2}{v^2}\right] + O\left(v_0^2 \ln v_0^2\right) \tag{B.17}$$

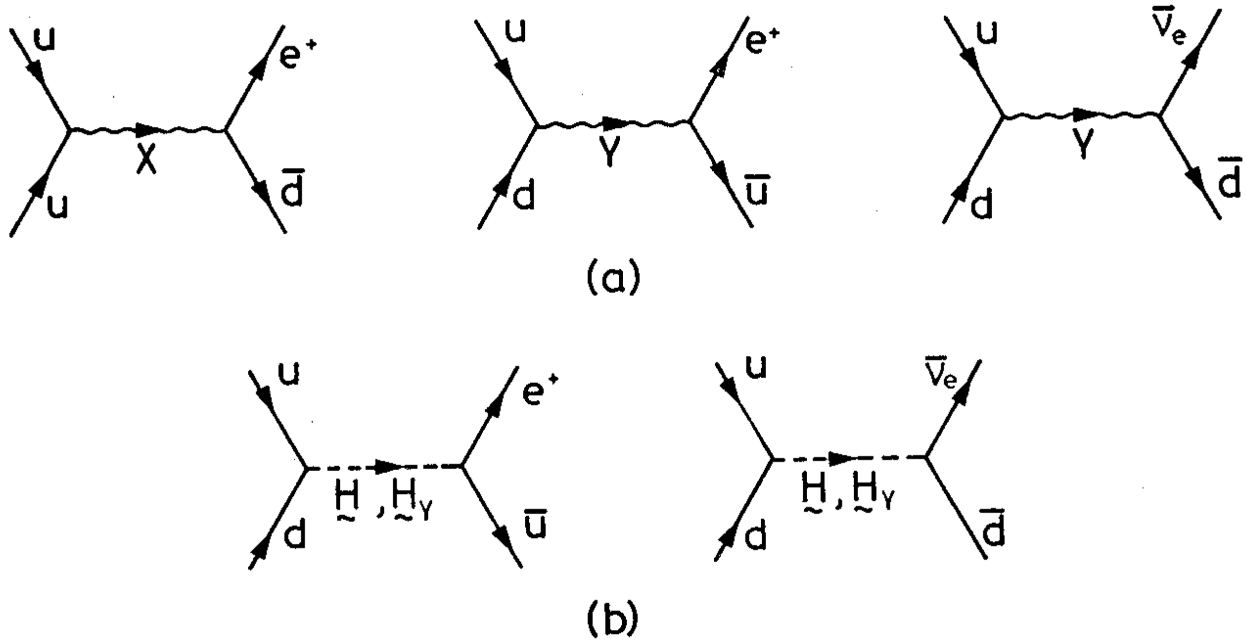
The masses of the physical 24-plet Higgs are approximately:

$$\begin{aligned}
m^2_{H_8} &\approx \frac{75}{2}\alpha^2_{GUM}v^2 - \frac{v^2}{5}\left[\frac{15}{2}a - \frac{\mu^2}{v^2}\right]\\
m^2_{H^\pm, H_Z} &\approx \frac{225}{8}\alpha^2_{GUM}v^2 - \frac{4v^2}{5}\left[\frac{15}{2}a - \frac{\mu^2}{v^2}\right]\\
m^2_{H_0} &\approx 2\mu^2 + \frac{375}{4}\alpha^2_{GUM}v^2
\end{aligned} \tag{B.18}$$

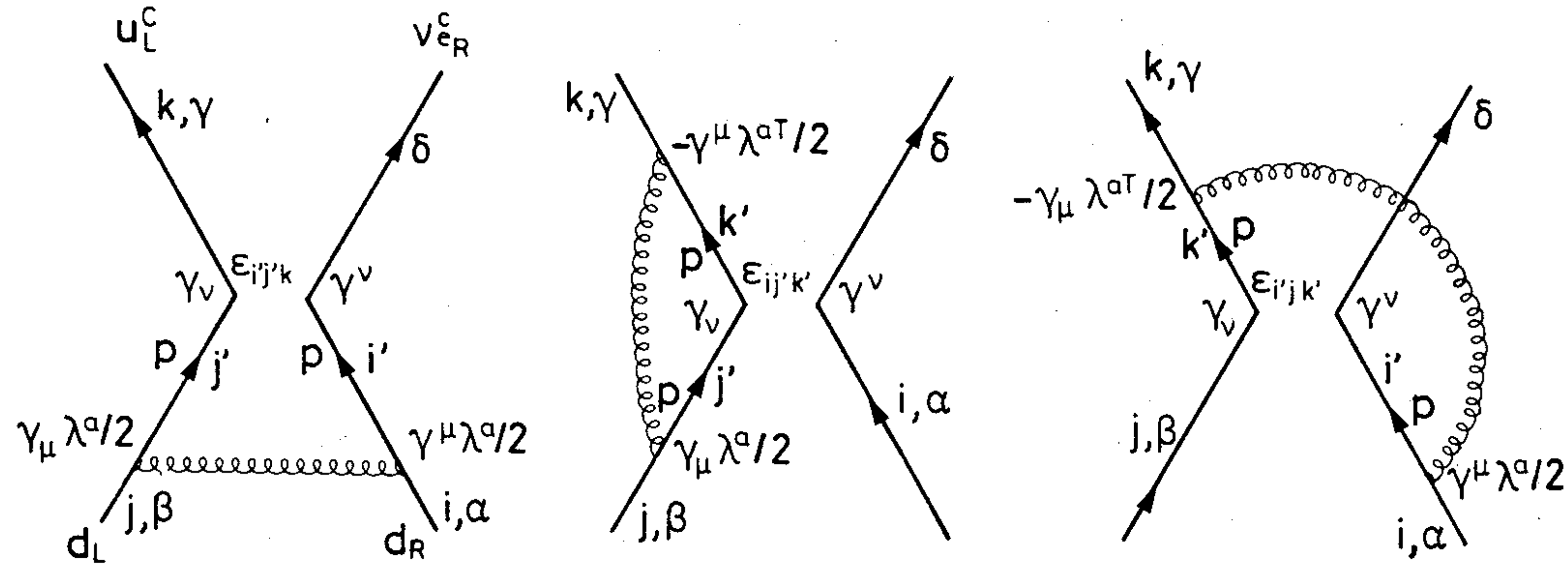
Positivity of the masses gives the constraint (4.23). The absence of a lower bound for the masses of H_8, $H^\pm$, H_Z reflects the fact that in our simple model they become massless goldstone bosons in the tree approximation. From (4.5″) and (4.23) we find that the condition (B.16) is satisfied if $a \ll \pi^2/4$.

We cannot show for a given choice of parameters that in the presence of radiative corrections to the Higgs potential the solution which we assume corresponds to the lowest vacuum. Imposing (4.11″), while at the same time keeping (B.17) of order v^2 means that if with the same set of parameters we looked for a solution, e.g. $\langle H\rangle = 0$, $\langle\hat{H}\rangle \neq 0$, we would find $\langle\hat{H}\rangle = O(v^2)$ and our perturbation expansion of V' would not be valid.

Finally, it is conceivable, although it appears unlikely, that a potential with say $\mu = a = 0$ could account more "naturally" for $v_0^2/v^2 \sim 10^{-30}$ and not destroy proton stability if the $\underset{\sim}{H}$, $\underset{\sim}{H}_Y$ mass matrix (which in this case has elements of *a priori* equal magnitude) chose a physical Higgs which is mostly $\underset{\sim}{H}_Y$. We cannot answer this question without going to higher order terms in the expansion (B.6).

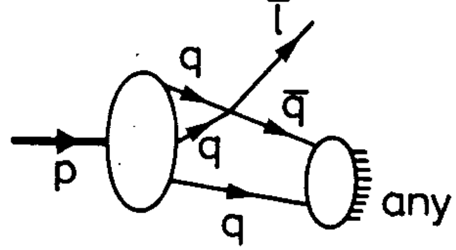


- Figure 1 -



(a) (b) (c)

- Figure 2 -



- Figure 3 -