Chiral symmetry of QCD

Lecture 1: Introduction to ChPT

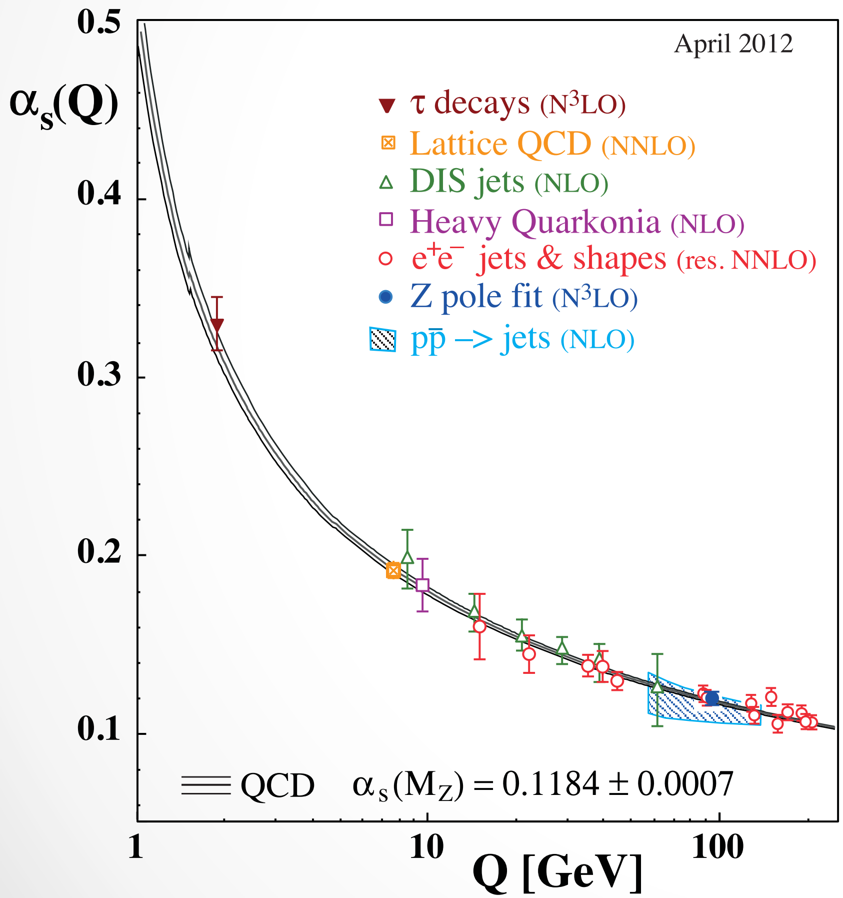
- Importance of symmetries
- Chiral symmetry
- Construction of effective Lagrangian
- Power counting

Lecture 2: Extension to resonance region

- P. w. dispersion relation
- Applications:
  - Goldstone boson scattering
  - Photon fusion reactions

Quantum ChromoDynamics (QCD)

$$\mathcal{L}_{\mathrm{QCD}} = \sum_{f=\substack{u,d,s,\\c,b,t}} \bar{q}_f \left(i\gamma^\mu D_\mu - m_f \right) q_f - \frac{1}{4} G^{(a)}_{\mu\nu} G^{(a)\,\mu\nu}$$



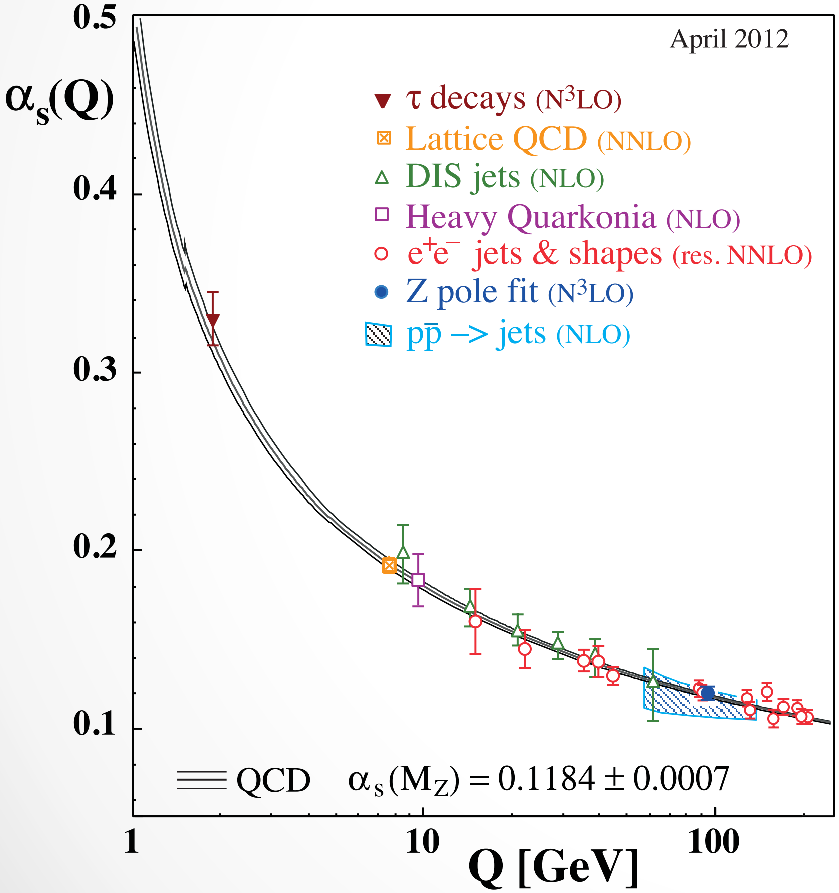
$$\begin{aligned} D_\mu &= \partial_\mu - i g_s \frac{\lambda^{(a)}}{2} A^{(a)}_\mu \\ G^{(a)}_{\mu\nu} &= \partial_\mu A^{(a)}_\nu - \partial_\nu A^{(a)}_\mu \\ &\quad + g_s\, f^{abc}\, A^{(b)}_\mu\, A^{(c)}_\nu \end{aligned}$$

The QCD coupling constant

- at high energies
  - asymptotic freedom (because of gluon self interaction)
  - perturbative QCD (deep inelastic scattering)

Quantum ChromoDynamics (QCD)

$$\mathcal{L}_{\rm QCD} = \sum_{f=\substack{u,d,s,\\ c,b,t}} \bar{q}_f \left(i\gamma^\mu D_\mu - m_f \right) q_f - \frac{1}{4} G^{(a)}_{\mu\nu} G^{(a)\,\mu\nu}$$



- At low energies coupling is large
  - nonperturbative QCD
  - confinement

relevant d.o.f are (color-neutral) hadrons, not quarks and gluons

Possible way out
- Lattice calculations
- Effective field theories (ChPT)
- Dispersive analysis

Are all hadrons made out (mainly) of quark-antiquark or three quarks?
Important tool - symmetries

Importance of symmetries

Steven Weinberg "Dreams of a Final Theory"

Concluded that to formulate a final theory one has to use the language of symmetries.

Quantum mechanical example: central potential $V(|r|)$ (e.g. hydrogen atom)

- Rotational invariance
- Conservation of angular momentum
- Degenerate energy levels

In the field theory:

If the Lagrangian is invariant under a certain (symmetry) transformation, leads to conserved quantities

- Currents, charges
- Degeneracy → states with the same mass

Importance of symmetries

- Mathematical language of symmetries is a **Group Theory** which is known for long time

- What is left -> connect physics with particular symmetry groups

- Simple example: scalar complex field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^*\partial_\mu\phi - \frac{m^2}{2}\phi^*\phi$$

- Lagrangian does not change if we change the phase of w.f. (connection with QM)

$$\begin{array}{ccc} \phi & \rightarrow & e^{i\alpha}\,\phi = U\,\phi \\ \phi^* & \rightarrow & \phi^* e^{-i\alpha} = \phi^*\,U^+ \end{array}$$

- U(1) symmetry, is the parameter of the group

Importance of symmetries

- Let us consider spinor field

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

- SU(2) – non-abelian group (the generators of group do not commute)

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$U = e^{i\alpha^a T^a}\,, \quad T^a - \text{Generators of group}$$

$$T^a = \frac{\tau^a}{2}\,, \quad \tau^a - \text{Pauli matrix}$$

$$\begin{aligned} \psi_i &\rightarrow U_{ij}\,\psi_j \\ \psi_i^+ &\rightarrow \psi_j^+ U_{ij}^+ \end{aligned}$$

$$[T^a, T^b] = i\,\epsilon^{abc}\,T^c$$

- This group may have different physical meaning. For example:

$$N = \begin{pmatrix} p \\ n \end{pmatrix}, \text{ quarks } \begin{pmatrix} u \\ d \end{pmatrix}$$

Gauge theories

- The main idea of gauge theories that the Lagrangian is invariant under a certain group of local transitions
- Spinor field

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

$$\psi \to e^{i\alpha}\,\psi = U\,\psi \qquad \text{Global U(1)}$$

$$\mathcal{L} \to \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}\,tr(G_{\mu\nu}G^{\mu\nu})$$

$$\psi \to e^{i\alpha(x)}\,\psi = U(x)\,\psi \quad D_\mu \to \partial_\mu - g\,V_\mu \qquad \text{Local U(1)}$$

- The standard model is a non-abelian gauge theory with the symmetry group U(1) × SU(2) × SU(3): 12 gauge bosons

U(1) V_μ - photon [γ, m=0] (1)

SU(2) V_μ - weak bosons [$W^\pm$, Z^0, $m \neq 0$] (3)

SU(3) V_μ - gluons [g, m=0] (8)

Quantum ChromoDynamics (QCD)

- QCD, the gauge field theory (SUc(3) color group) which describes the strong interactions of color quarks and gluons

$$\mathcal{L}_{\mathrm{QCD}} = \sum_{f = \substack{u,d,s,\\ c,b,t}} \bar{q}_f \left(i\gamma^\mu D_\mu - m_f \right) q_f - \frac{1}{4} G^{(a)}_{\mu\nu} G^{(a)\,\mu\nu}$$

- Quarks come in 3 colors q(red, green, blue), gluons come in 8 color combinations

- Lagrangian is invariant with respect to local transformations in color space

$$U(x) = e^{i\alpha(x)_a \lambda_a} \in SU_c(3)$$
$$q^i(x) \to [U(x)]^{ij} q^j(x)$$
$$\bar{q}^i(x) \to \bar{q}^j(x) [U^+(x)]^{ij}$$

Consequences of local color symmetry

- Only color invariant object are observable
  - Natural explanation of quark-antiquark and three-quark states (white objects)

$$
\begin{aligned}
\bar{q}_i q_j &\rightarrow \bar{q}\, U^+ U\, q = \bar{q} q \\
\epsilon_{ijk}\, q_i q_j q_k &\rightarrow det(U)\, \epsilon_{ijk}\, q_i q_j q_k = \epsilon_{ijk}\, q_i q_j q_k
\end{aligned}
$$

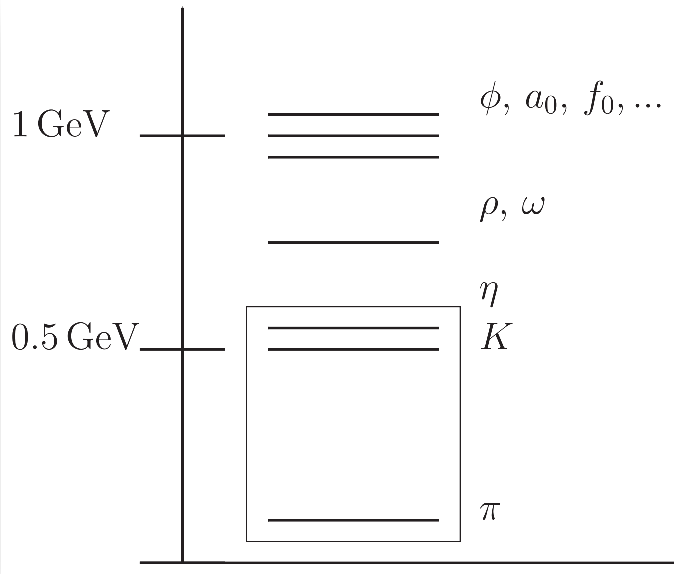
- Only one universal coupling constant is a property of non-abelian gauge theories
- Only few parameters: one coupling, few quark masses (**fundamental parameters!**)

Hadron spectrum and quark masses

$m_u = 1.7 - 3.3$ MeV	$m_c = 1.27^{+0.07}_{-0.09}$ GeV
$m_d = 4.1 - 5.8$ MeV	$m_b = 4.19^{+0.18}_{-0.06}$ GeV
$m_s = 101^{+29}_{-21}$ MeV	$m_t = 172.0 \pm 0.9 \pm 1.3$ GeV

By inspecting this table we find:

- Masses of the u, d and, to lesser extend, s quarks are small compared to the typical hadronic mass scale ~1 GeV
- QCD exhibits an addition symmetries (besides C, P and T-symmetries)
- In the following we keep only three lightest quarks in the Lagrangian
- Explore the role of light hadrons



Global symmetries of QCD

QCD Lagrangian (for to u and d quarks only)

$$\mathcal{L}_{\mathrm{QCD}} = \left(\begin{array}{c} \bar{u} \\ \bar{d} \end{array} \right) \left[i\gamma_\mu D^\mu - \left(\begin{array}{cc} m_u & 0 \\ 0 & m_d \end{array} \right) \right] \left(\begin{array}{c} u \\ d \end{array} \right) - \frac{1}{4} G^{(a)}_{\mu\nu} G^{(a)\mu\nu}$$

- U(1) - baryon number conservation (baryons cannot decay into mesons)

- SU(Nf=3) – flavor symmetry if all quark mass were the same mu=md (approximate symmetry)
    - Isospin (flavor conservation)
    - Degenerate states (multiplets), i.e. states with equal masses and different isospin

- $SU(3)_L \times SU(3)_R$ – chiral symmetry if m → 0 (approximate symmetry)

Chiral symmetry

The QCD Lagrangian (here restricted to u and d quarks)

$$\mathcal{L}_{\mathrm{QCD}} = \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} \left[i\gamma_\mu D^\mu - \cancel{\begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}} \right] \begin{pmatrix} u \\ d \end{pmatrix} - \frac{1}{4} G^{(a)}_{\mu\nu} G^{(a)\mu\nu}$$

then the Lagrangian does not change under transformations $SU(3)_L \times SU(3)_R$

$$q \to \exp\left(i\theta^a_V \frac{\lambda_a}{2} \right) q, \qquad q \to \exp\left(i\gamma_5 \theta^a_A \frac{\lambda_a}{2} \right) q$$

Decouple quark field according to their chirality

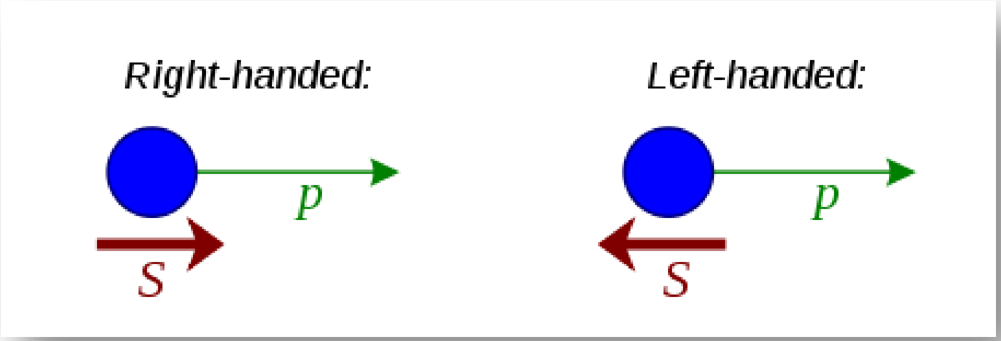
$$q = \frac{1-\gamma_5}{2} q + \frac{1+\gamma_5}{2} q = P_L\, q + P_R\, q = q_L + q_R$$

$$\mathcal{L}_{\mathrm{QCD}} = \sum_{f=u,d,s} \left(\bar{q}_{f,R}\, i\gamma^\mu D_\mu\, q_{f,R} + \bar{q}_{f,L}\, i\gamma^\mu D_\mu\, q_{f,L} \right) - \frac{1}{4} G^{(a)}_{\mu\nu} G^{(a)\,\mu\nu}$$

we will treat the mass term as a perturbation

Chiral symmetry

- In the zero mass limit chirality = helicity
- Helicity: spin points in or against flight direction.



For massive particles:

- Helicity: conserved in time, depends on frame of reference (isn't Lorenz invariant)
- Chirality: not conserved in time, Lorenz invariant

Spontaneous symmetry breaking

- $SU(3)_L \times SU(3)_R$ implies that hadron spectrum should consists of degenerate multiplets with opposite parity

  N(940) parity +, N(1535) parity –

- Chiral symmetry is spontaneously broken to the vector subgroup

$$SU(3)_L \times SU(3)_R = SU(3)_V \times SU(3)_A \rightarrow SU(3)_V$$

- The ground state (vacuum) is not invariant under axial subgroup
- According to the Goldstone theorem -> 8 massless bosons

$$\pi^{\pm}, \pi^0, K^{\pm}, K^0, \bar{K}^0, \eta$$

- In the real world, the pions are not massless, but have small masses -> explicit symmetry breaking

Effective Field Theories

- Concepts of EFTs
  - An important feature: perform the systematic approximation in a certain domain with respect to some energy scale Λ
  - Within the given scale: identify the relevant degrees of freedom and symmetries
  - Construct the most general Lagrangian consistent with these symmetries
  - Do standard QFT with this Lagrangian
  - Simplifies calculations (or make them possible)
- Chiral perturbation theory (χPT)
  - Spontaneous chiral symmetry breaking $\rightarrow$ weakly interacting Goldstone bosons $\pi^{\pm}, \pi^0, K^{\pm}, K^0, \bar{K}^0, \eta$
  - Effective degrees of freedom: hadrons
  - Power counting: Systematic expansion in powers of small Q, mP
  - Unknown coupling constants fitted to the data

Chiral Lagrangian

- Building blocks

  - Exponential parameterization of Goldstone boson

$$U = \exp(\frac{i\phi}{f_\pi}), \quad \phi(x) = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}$$

  - The term that breaks chiral symmetry explicitly

$$\chi_0 = \begin{pmatrix} m_\pi^2 & 0 & 0 \\ 0 & m_\pi^2 & 0 \\ 0 & 0 & 2\,m_K^2 - m_\pi^2 \end{pmatrix}$$

  - External fields

Chiral Lagrangian

- Transformation properties

Element	Chiral	C	P
U	RUL^+	U^T	U^+
$D_\mu U$	$RD_\mu UL^+$	$(D_\mu U)^T$	$(D_\mu U)^+$
χ	$R\chi L^+$	χ^T	χ^+

- Power counting

$$U = \mathcal{O}(p^0), \quad D_\mu U = \mathcal{O}(p), \quad D_\mu D_\nu U = \mathcal{O}(p^2)\,, \quad \chi = \mathcal{O}(p^2)$$

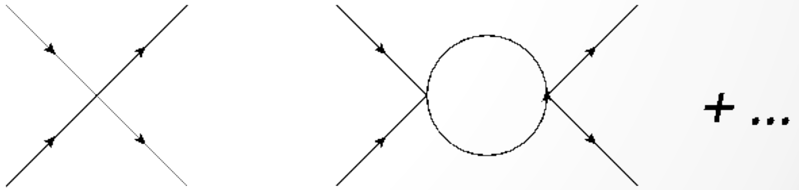
- Effective Lagrangian

$$\mathcal{L} = \frac{C}{4} tr(D_\mu U(D^\mu U)^+) + \frac{C}{4} tr(\chi U^+ + U\chi^+)$$

Applicability of ChPT

- SU(2): pions: + well convergence – less predictive
- SU(3): pions, kaons, eta: + more predictive power, - bad convergence
- Applications: scattering, decays, …
- Problems:
  - Limited range of convergence (threshold region)
  - Unitarity at perturbative level

$$\begin{aligned} T &= T_2 + T_4 + \dots \\ \mathrm{Im}\, T_2 &= 0\,, \quad \mathrm{Im}\, T_4 = T_2\, \rho\, T_2 \end{aligned}$$



Extension to resonance region

- Inclusion of addition degrees of freedom (light vector mesons)

$$\Phi = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\,\eta & \sqrt{2}\,\pi^+ & \sqrt{2}\,K^+ \\ \sqrt{2}\,\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\,\eta & \sqrt{2}\,K^0 \\ \sqrt{2}\,K^- & \sqrt{2}\,\bar{K}^0 & -\frac{2}{\sqrt{3}}\,\eta \end{pmatrix}, V_{\mu\nu} = \begin{pmatrix} \rho^0_{\mu\nu} + \omega_{\mu\nu} & \sqrt{2}\,\rho^+_{\mu\nu} & \sqrt{2}\,K^+_{\mu\nu} \\ \sqrt{2}\,\rho^-_{\mu\nu} & -\rho^0_{\mu\nu} + \omega_{\mu\nu} & \sqrt{2}\,K^0_{\mu\nu} \\ \sqrt{2}\,K^-_{\mu\nu} & \sqrt{2}\,\bar{K}^0_{\mu\nu} & \sqrt{2}\,\phi_{\mu\nu} \end{pmatrix}$$

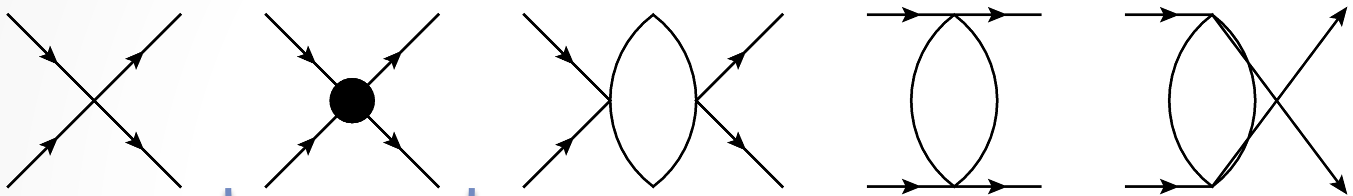
- Nonperturbative effects are required
  - Exact coupled-channel unitarity
  - Analyticity
  - EM gauge invariance

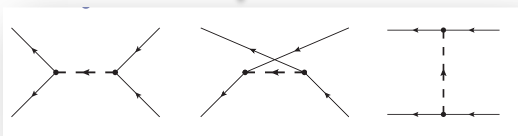
A. Gasparyan and M. F. M. Lutz,
Nucl. Phys. A 848, 126 (2010)

- Apply to Goldstone boson scattering (PP → PP) and photon-fusion reactions (γγ → PP)

Importance of VM

- Resonance saturation mechanism: Low energy coefficients (LEC) of NLO counter terms are mostly saturated by vector meson exchange



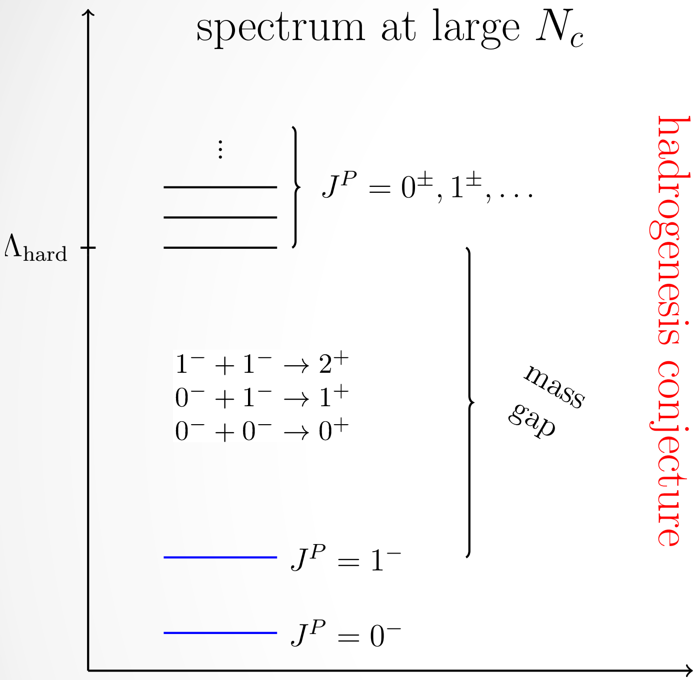


G.Ecker, J.Gasser, A.Pich and E.de Rafael, Nucl. Phys. B 321 (1989) 311

- Hadrogenesis conjecture

  - $0^- + 1^- \to 1^+ \to 0^- + 1^-$ $\boxed{f_1(1282),\ a_1(1230),\ \ldots}$ M.F.M. Lutz, E. Kolomeitsev, Nucl. Phys. A **730**, 392 (2004)
  - $0^- + 0^- \to 0^+ \to 0^- + 0^-$ $\boxed{f_0(980),\ a_0(980),\ \ldots}$ I. V. Danilkin, L. I. R. Gil and M. F. M. Lutz, Phys. Lett. B **703**, 504 (2011)
  - $1^- + 1^- \to 2^+ \to 1^- + 1^-$ $\boxed{f_2(1270),\ a_2(1320),\ \ldots}$ (open challenge)

Power-counting scheme with VM



- expansion parameter: $\Lambda_{\text{soft}}/\Lambda_{\text{hard}}$
- pure χPT: $\Lambda_{\text{soft}} \sim m_P, Q$
  $\Lambda_{\text{hard}} \sim 4\pi f$ or m_V (DOF not included in the Lagrangian)
- Our case:
  - Vector mesons are part of the Lagrangian
  - Dynamical generation of resonances $(0^+,\ 1^+,\ \ldots)$
- We expect $\Lambda_{\text{hard}} \geq (2-3)$ GeV

➢ In our power-counting scheme light-vector mesons are treated as soft

$$m_V \sim O(Q)$$

➢ For the complete picture: need to explore vector-meson loop effects…

(open challenge)

Chiral Lagrangian with VM

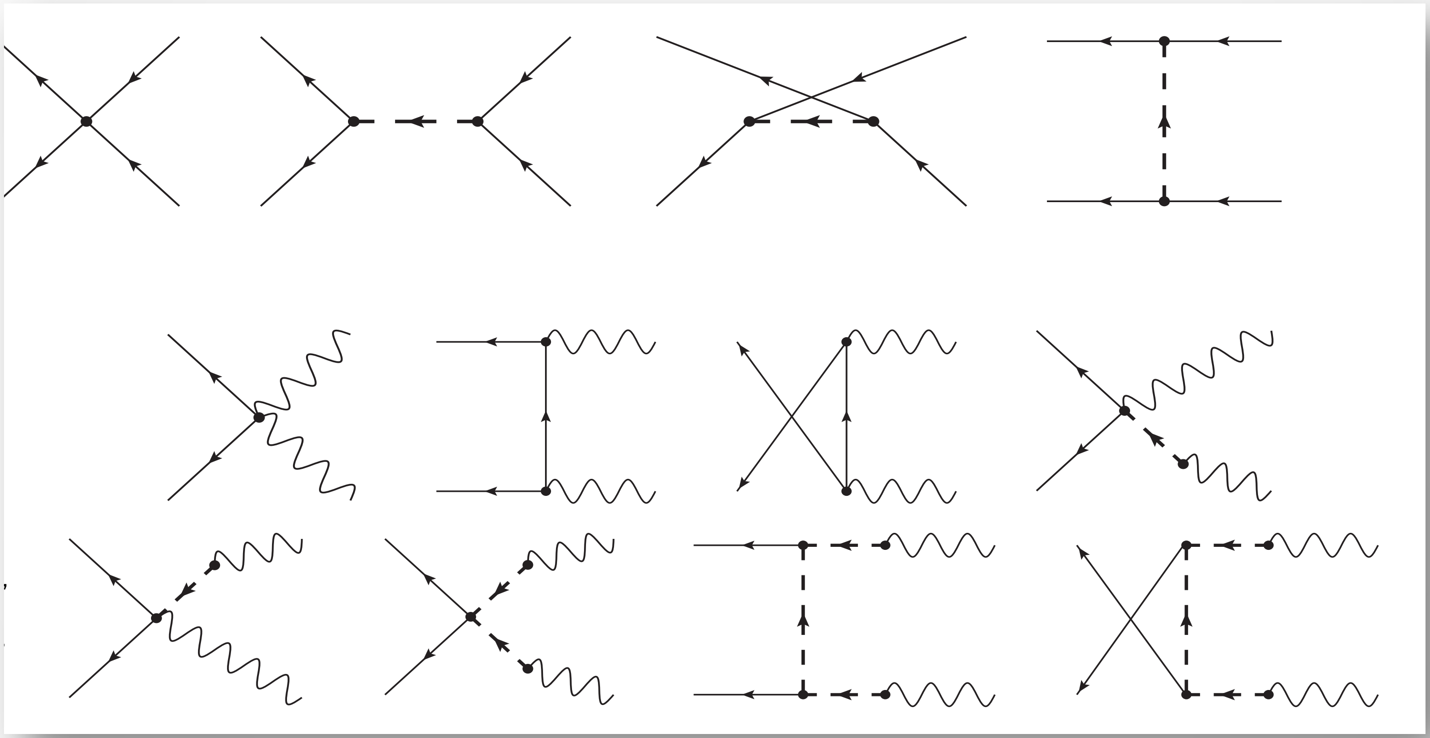
The LO chiral Lagrangian for the Goldstone-boson $P\,(\pi,\ K,\ \bar{K},\eta)$ and vector-meson $V_{\mu\nu}\ (\rho_{\mu\nu},\ \omega_{\mu\nu},\ K_{\mu\nu},\ \bar{K}_{\mu\nu},\ \phi_{\mu\nu})$ fields:

$$
\begin{aligned}
\mathcal{L} \;=\;& \frac{1}{48f^2}\,\mathrm{tr}\Big\{[P,\,\partial^\mu P]_-\,[P,\,\partial_\mu P]_- + P^4\,\chi_0\Big\} - i\,\frac{f_V\,h_P}{8f^2}\,\mathrm{tr}\Big\{\partial_\mu P\,V^{\mu\nu}\,\partial_\nu P\Big\} \\
&- \frac{e^2}{2}\,A^\mu\,A_\mu\,\mathrm{tr}\Big\{P\,\mathcal{Q}\Big[P,\,\mathcal{Q}\Big]_-\Big\} + i\,\frac{e}{2}\,A^\mu\,\mathrm{tr}\Big\{\partial_\mu P\Big[\mathcal{Q},\,P\Big]_-\Big\} - e\,f_V\,\partial_\mu A_\nu\,\mathrm{tr}\Big\{V^{\mu\nu}\,\mathcal{Q}\Big\} \\
&- i\,\frac{f_V\,h_P}{8\,f^2}\,\mathrm{tr}\Big\{\partial_\mu P\,V^{\mu\nu}\,\partial_\nu P\Big\} + \frac{e\,f_V}{8\,f^2}\,\partial_\mu A_\nu\,\mathrm{tr}\Big\{V^{\mu\nu}\Big[P,\Big[P,\,\mathcal{Q}\Big]_-\Big]_-\Big\} \\
&+ \frac{e\,f_V\,h_P}{8f^2}\,A_\nu\,\mathrm{tr}\Big\{\Big[\partial_\mu P\,,\,V^{\mu\nu}\Big]_-\Big[\mathcal{Q},P\Big]_-\Big\} - \frac{1}{16\,f^2}\,\mathrm{tr}\Big\{\partial^\mu V_{\mu\alpha}\Big[\Big[P,\partial_\nu P\Big]_-,V^{\nu\alpha}\Big]_-\Big\} \\
&- \frac{b_D}{64\,f^2}\,\mathrm{tr}\Big\{V^{\mu\nu}\,V_{\mu\nu}\Big[P,\Big[P,\chi_0\Big]_+\Big]_+\Big\} - \frac{g_1}{32\,f^2}\,\mathrm{tr}\Big\{\Big[V_{\mu\nu}\,,\partial_\alpha P\Big]_+\Big[\partial^\alpha P,V^{\mu\nu}\Big]_+\Big\} \\
&- \frac{g_2}{32\,f^2}\,\mathrm{tr}\Big\{\Big[V_{\mu\nu}\,,\partial_\alpha P\Big]_-\Big[\partial^\alpha P,V^{\mu\nu}\Big]_-\Big\} - \frac{g_3}{32\,f^2}\,\mathrm{tr}\Big\{\Big[\partial_\mu P\,,\partial^\nu P\Big]_+\Big[V_{\nu\tau}\,,V^{\mu\tau}\Big]_+\Big\} \\
&- \frac{g_5}{32\,f^2}\,\mathrm{tr}\Big\{\Big[V^{\mu\tau},\partial_\mu P\Big]_-\Big[V_{\nu\tau}\,,\partial^\nu P\Big]_-\Big\} - \frac{h_A}{16\,f}\,\epsilon_{\mu\nu\alpha\beta}\,\mathrm{tr}\Big\{\Big[V^{\mu\nu},\partial_\tau V^{\tau\alpha}\Big]_+\partial^\beta P\Big\} \\
&- \frac{b_A}{16\,f}\,\epsilon_{\mu\nu\alpha\beta}\,\mathrm{tr}\Big\{\Big[V^{\mu\nu},V^{\alpha\beta}\Big]_+\Big[\chi_0,P\Big]_+\Big\} - \frac{h_O}{16\,f}\,\epsilon_{\mu\nu\alpha\beta}\,\mathrm{tr}\Big\{\Big[\partial^\alpha V^{\mu\nu},V^{\tau\beta}\Big]_+\partial_\tau P\Big\}
\end{aligned}
$$

$\mathcal{Q}$ - charge matrix, $\chi_0 \approx \mathrm{diag}(m_\pi^2,\ m_\pi^2,\ 2m_K^2 - m_\pi^2)$

Scattering amplitudes

- We calculated the following tree-level diagrams



- Valid at low energies only (will serve as an input in the nonperturbative coupled channel calculations)

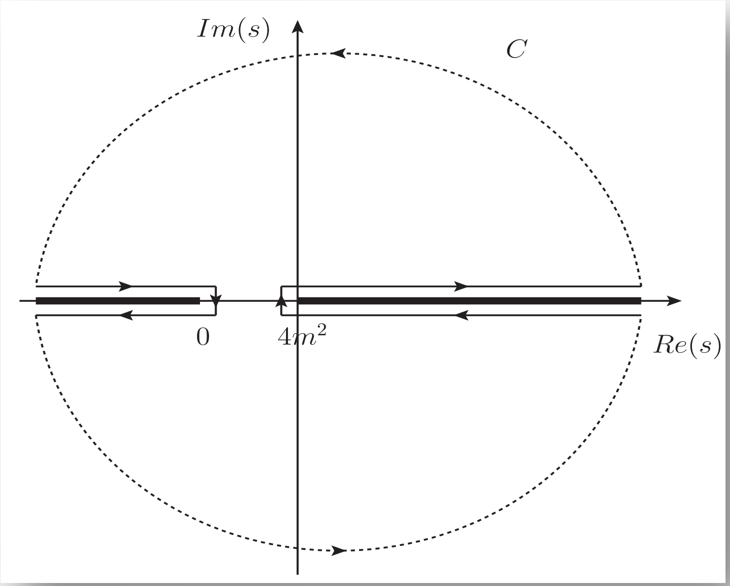
The coupled-channel states

$(\frac{1}{2},1)$	$(\frac{3}{2},1)$	$(0^+,0)$
$\begin{pmatrix} (\frac{1}{\sqrt{3}}\,\pi_q \cdot \;\vec{\sigma}\,K_p) \\ (\eta_q\,K_p) \end{pmatrix}$	$(\pi_q \cdot T\,K_p)$	$\begin{pmatrix} (\gamma\,\gamma) \\ \frac{1}{\sqrt{3}}\,(\pi_q \cdot \pi_p) \\ \frac{1}{2}\,(\bar{K}_q\,K_p + \bar{K}_p\,K_q) \\ (\eta_q\,\eta_p) \end{pmatrix}$
$(1^+,0)$	$(1^-,0)$	$(2^+,0)$
$\begin{pmatrix} (\frac{1}{i\sqrt{2}}\,\pi_q \times \pi_p) \\ \frac{1}{2}\,(\bar{K}_q\,\vec{\sigma}\,K_p - \bar{K}_p\,\vec{\sigma}\,K_q) \end{pmatrix}$	$\begin{pmatrix} (\gamma\,\gamma) \\ (\pi_q\,\eta_p) \\ \frac{1}{2}\,(\bar{K}_q\,\vec{\sigma}\,K_p + \bar{K}_p\,\vec{\sigma}\,K_q) \end{pmatrix}$	$\begin{pmatrix} (\gamma\,\gamma) \\ \big(\frac{1}{2}\,(\pi_q^i\,\pi_p^j + \pi_q^j\,\pi_p^i) - \frac{1}{3}\,\delta_{ij}\,\pi_q \cdot \pi_p\big) \end{pmatrix}$

- Consider scattering and rescattering of the type $\gamma\gamma \to PP$ and $PP \leftrightarrow PP$, respectively, but disregard $PP \leftrightarrow VV$, VP (important for $\sqrt{s} > 1.1 - 1.2$ GeV)

Dispersion relation

- Cauchy theorem



If function

- Analytic in a cut plane
- Falls off sufficiently fast on the large semi circle (otherwise need subtractions)

$$T(s) = \frac{1}{2\pi i}\int_C ds' \frac{T(s')}{s'-s} = \int_{-\infty}^{0} \frac{ds'}{\pi}\frac{\mathrm{Disc}\, T(s')}{s'-s} + \int_{4m^2}^{\infty} \frac{ds'}{\pi}\frac{\mathrm{Disc}\, T(s')}{s'-s}$$

Dispersion relation

- Unitarity and analyticity

$$T^J_{ab}(s) = U^J_{ab}(s) + \sum_{c,d} \int_{\mu^2_{thr}}^{\infty} \frac{ds'}{\pi} \frac{s-\mu^2_M}{\bar{s}-\mu^2_M} \frac{T^J_{ac}(s')\,\rho^J_{cd}(s')\,T^{J*}_{db}(s')}{s'-s-i\epsilon}$$

- separate left- and right-hand cuts

- the generalized potential U(s) contains all left-hand cuts
  - U(s) computed in ChPT in the threshold region
  - analytically extrapolated (conformal mapping)

- EM gauge invariance
- matching with ChPT

- The phase space function

$$\mathrm{Im}\, T^J_{ab}(s) = \sum_{c,d} T^J_{ac}(s)\,\rho^J_{cd}(s)\,T^{J*}_{db}(s)\,, \qquad \rho^J_{ab}(s) = \frac{1}{8\pi}\left(\frac{p_{cm}}{\sqrt{s}}\right)^{2J+1} \delta_{ab}$$

N/D method

- We reconstruct the scattering amplitude by means of the N/D technique

G.F.Chew, S.Mandelstam,
Phys. Rev. 119 (1960) 467-477

$$T_{ab}(s) = \sum_c D^{-1}_{ac}(s)\, N_{cb}(s)\,,$$

- where the contributions of left- and right-hand singularities are separated respectively into N(s) and D (s) functions,

$$N_{ab}(s) = U_{ab}(s) + \sum_{c,d} \int_{\mu^2_{\rm thr}}^{\infty} \frac{d\bar{s}}{\pi}\, \frac{s-\mu_M^2}{\bar{s}-\mu_M^2}\, \frac{N_{ac}(\bar{s})\, \rho_{cd}(\bar{s})\, [U_{db}(\bar{s}) - U_{db}(s)]}{\bar{s}-s}$$

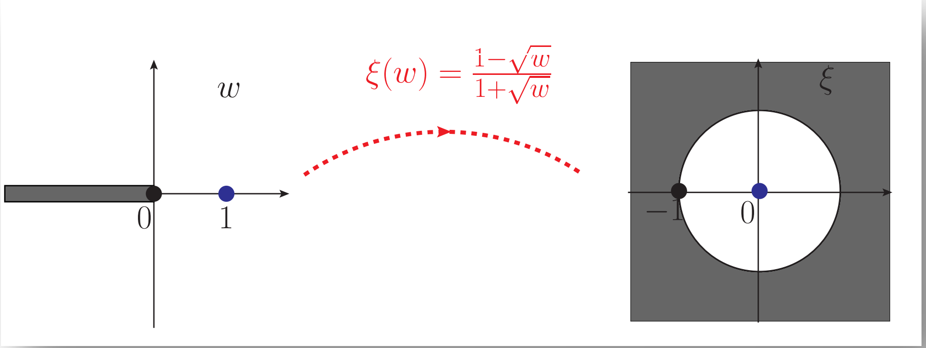
$$D_{ab}(s) = \delta_{ab} - \sum_c \int_{\mu^2_{thr}}^{\infty} \frac{d\bar{s}}{\pi}\, \frac{s-\mu_M^2}{\bar{s}-\mu_M^2}\, \frac{N_{ac}(\bar{s})\rho_{cb}(\bar{s})}{\bar{s}-s-i\epsilon}$$

Approximation for the U(s)

- In ChPT one can compute amplitudes only in the close-to-threshold region (asymptotically growing potential)
- The potential U(s) is needed only for energies above threshold
- Reliable extrapolation is possible: conformal mapping techniques
- Typical example: $U(\omega) = \ln(\omega)$ (left-hand cut at $\omega < 0$)

Expansion around $\omega = 1$?

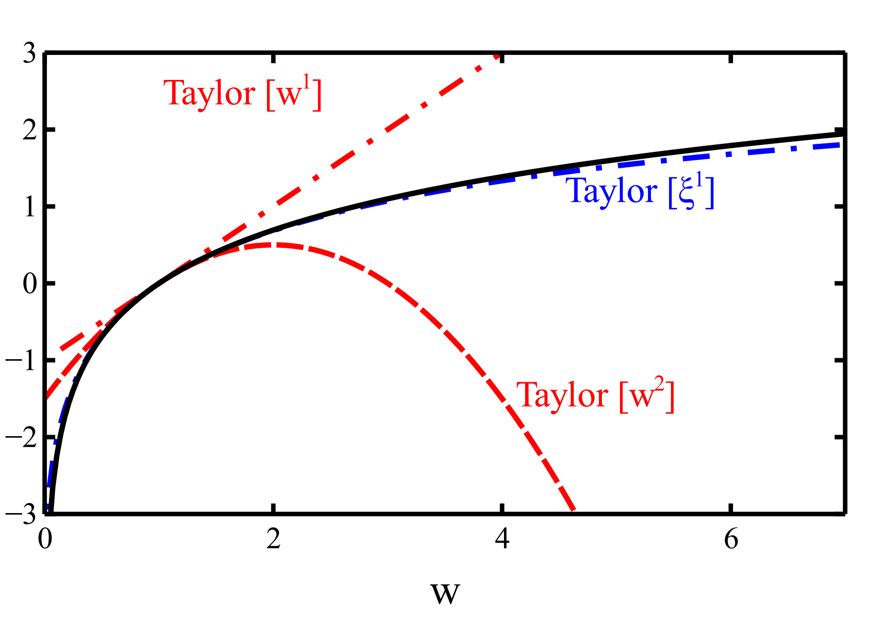
- $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} [\omega - 1]^k$
- $\sum_{k=0}^{\infty} C_k \, [\xi(\omega)]^k$



Conformal mapping

Conformal mapping for $U(\omega) = ln(\omega)$

$$ln(\omega): \qquad \sum_{k=1}^{N} \frac{(-1)^{k+1}}{k!}[\omega-1]^k \quad vs. \quad \sum_{k=0}^{N} C_k\ [\xi(\omega)]^k$$



- Taylor expansion in $\omega - 1$ converges for $0 < \omega < 2$
- Taylor expansion in ξ converges for $0 < \omega < \infty$

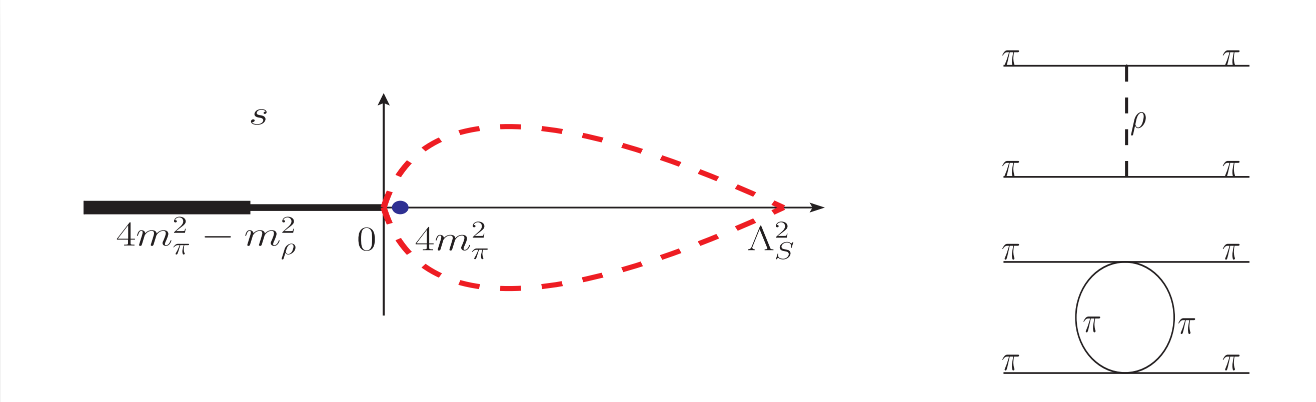
- the coefficients $C_k = \frac{d^k U(\omega(\xi))}{k!\, d\xi^k}\Big|_{\xi=0}$ are determined at $\omega = 1$ $(\xi = 0)$

Conformal mapping (PP→ PP)

- We need U(s) for energies above threshold

$$U(s) = \sum_{k=0}^{N} C_k \, [\xi(s)]^k \qquad \text{for} \quad s < \Lambda_s^2$$

- Reliable approximation is within red area



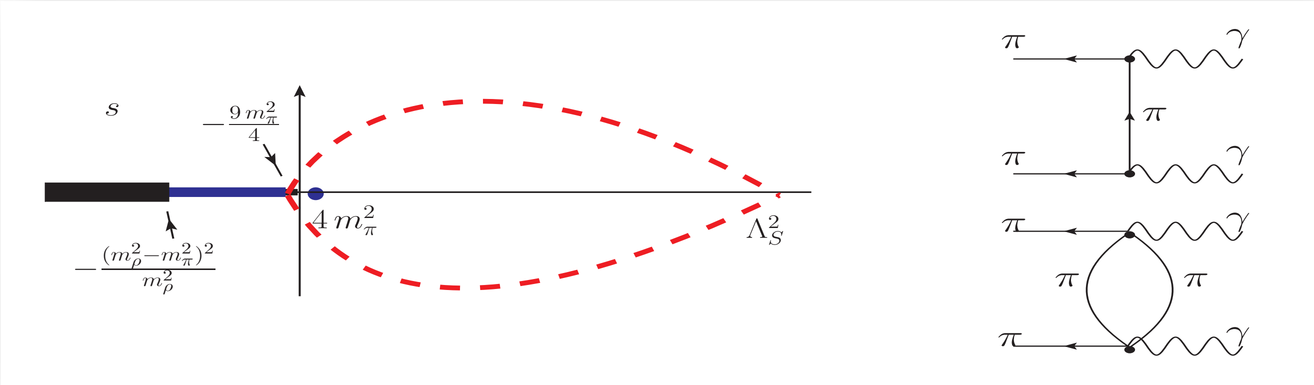
The coefficients C_k are determined at $s = 4m_\pi^2$, where χPT is reliable

Conformal mapping ($\gamma\gamma \to PP$)

- We need U(s) for energies above threshold

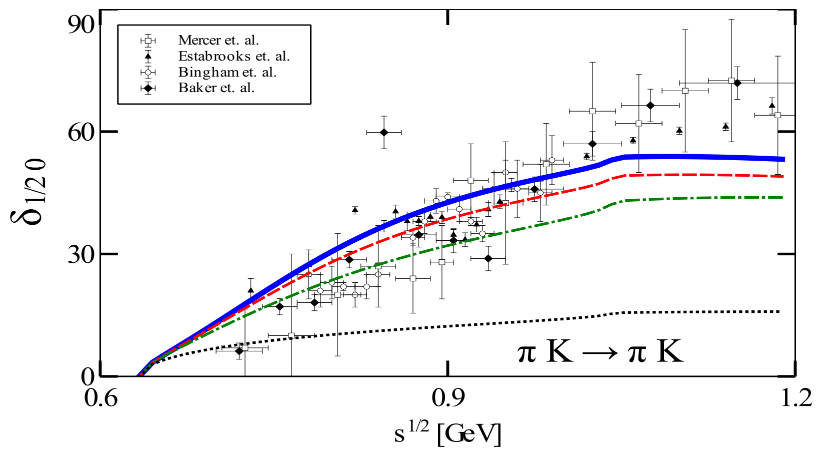
$$U(s) = \int_{-\frac{9m_\pi^2}{4}}^{0} \frac{ds'}{\pi} \frac{\Delta T(s')}{s'-s} + \sum_{k=0}^{N} C_k \, [\xi(s)]^k \qquad \text{for} \quad s < \Lambda_s^2$$

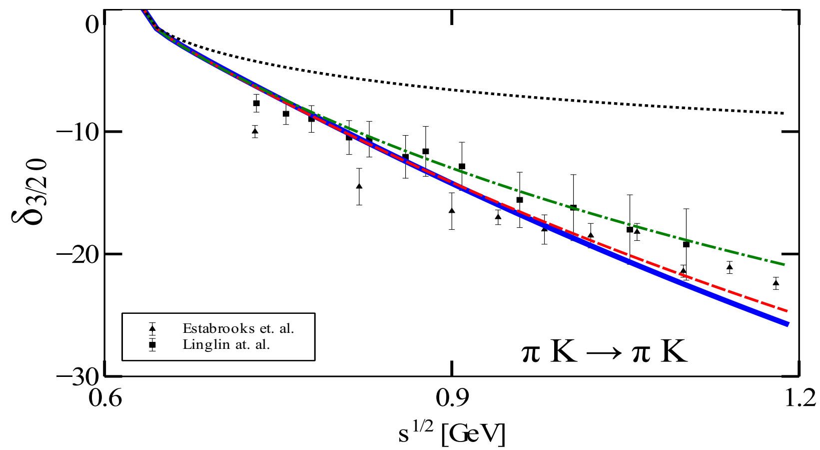
- Reliable approximation is within red area

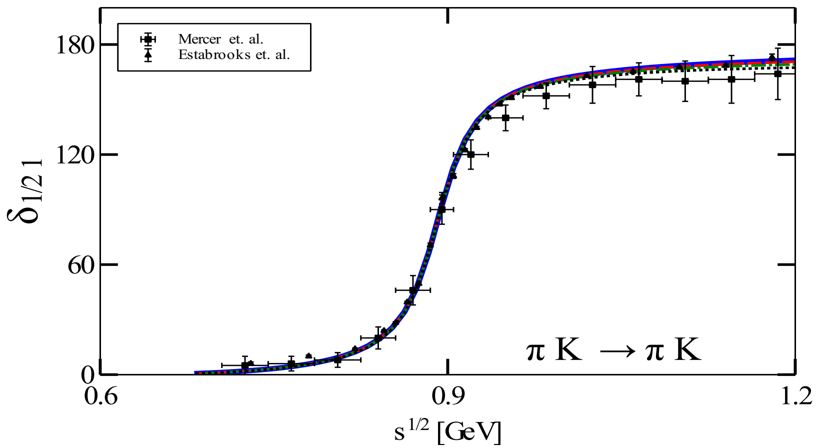


The coefficients C_k are determined at $s = 4m_\pi^2$, where χPT is reliable

Results for pion-kaon scattering

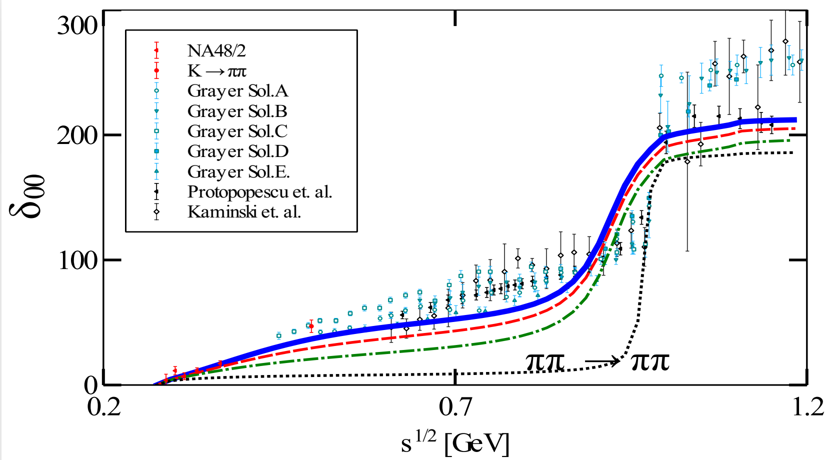


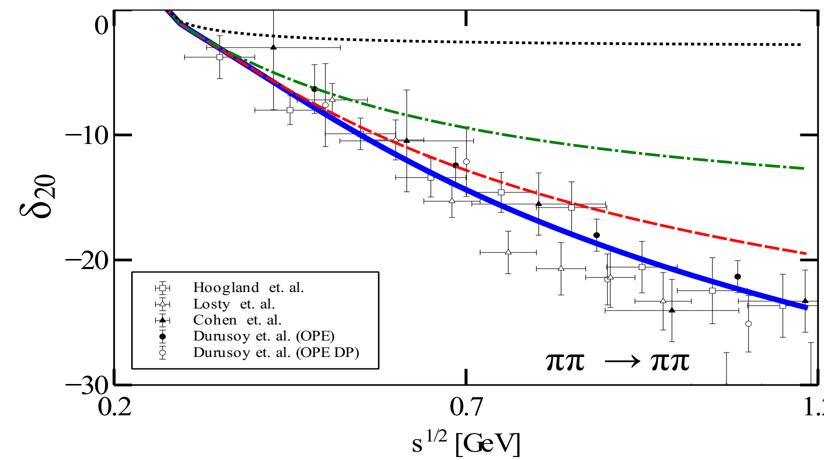


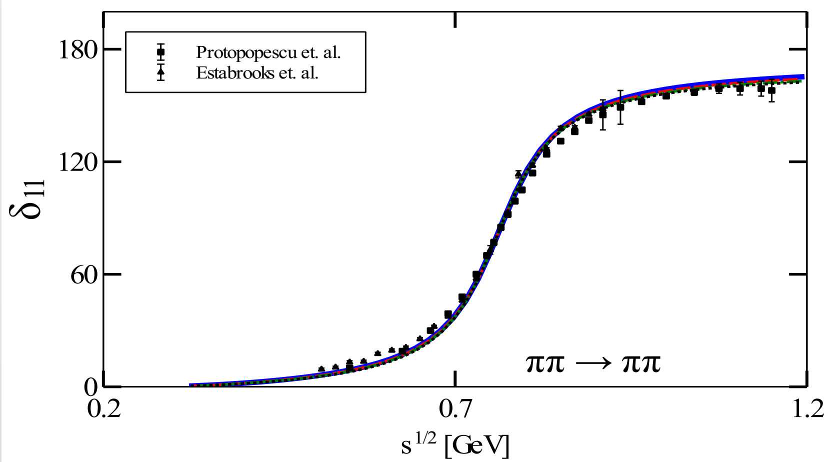


- $U(s) = \sum_{k=0}^{N} C_k\ [\xi(s)]^k$
- in p-wave CDD poles $\leftrightarrow$ ρ, K^*
- No free parameters!

Results for pion-pion scattering

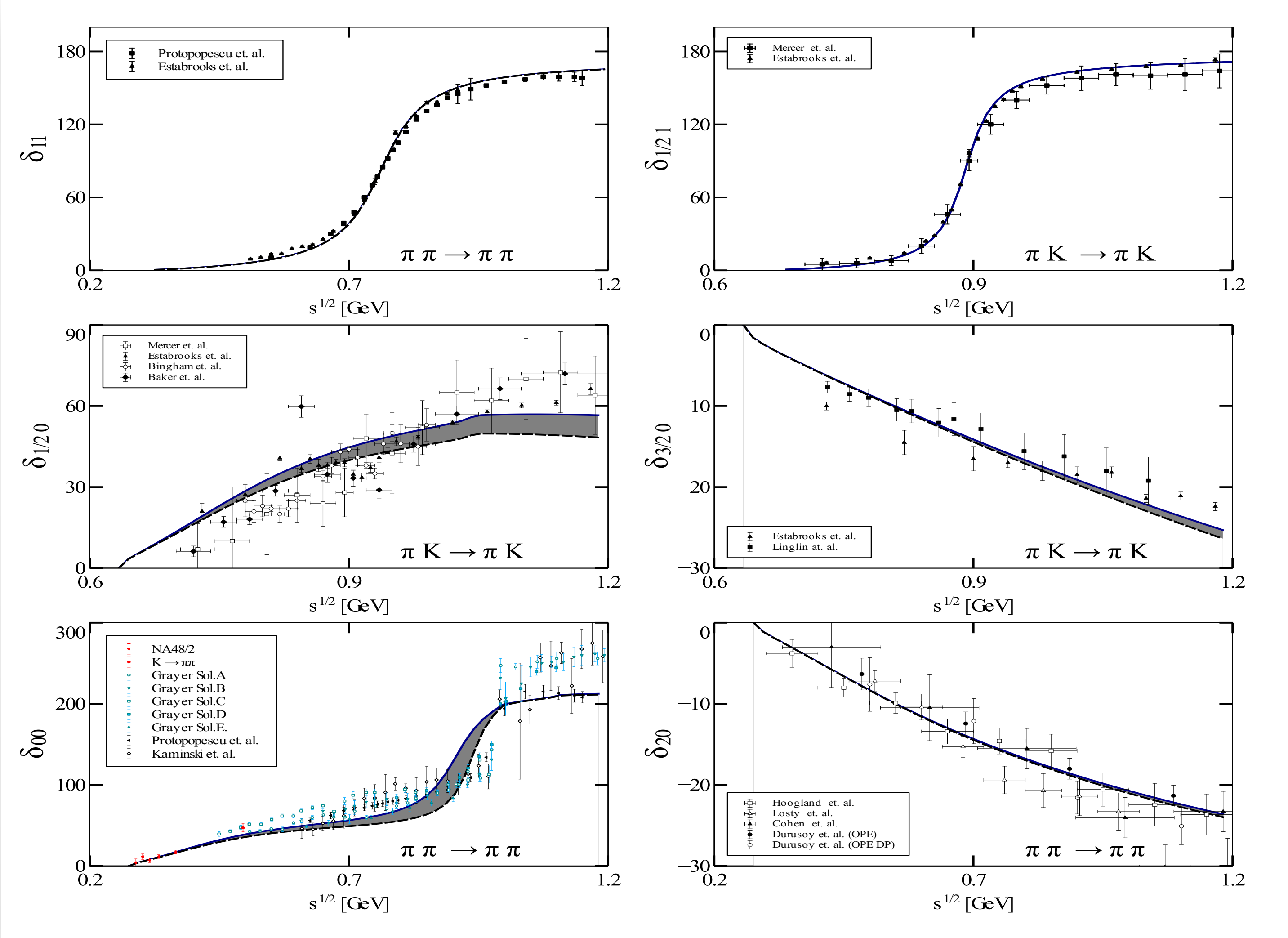




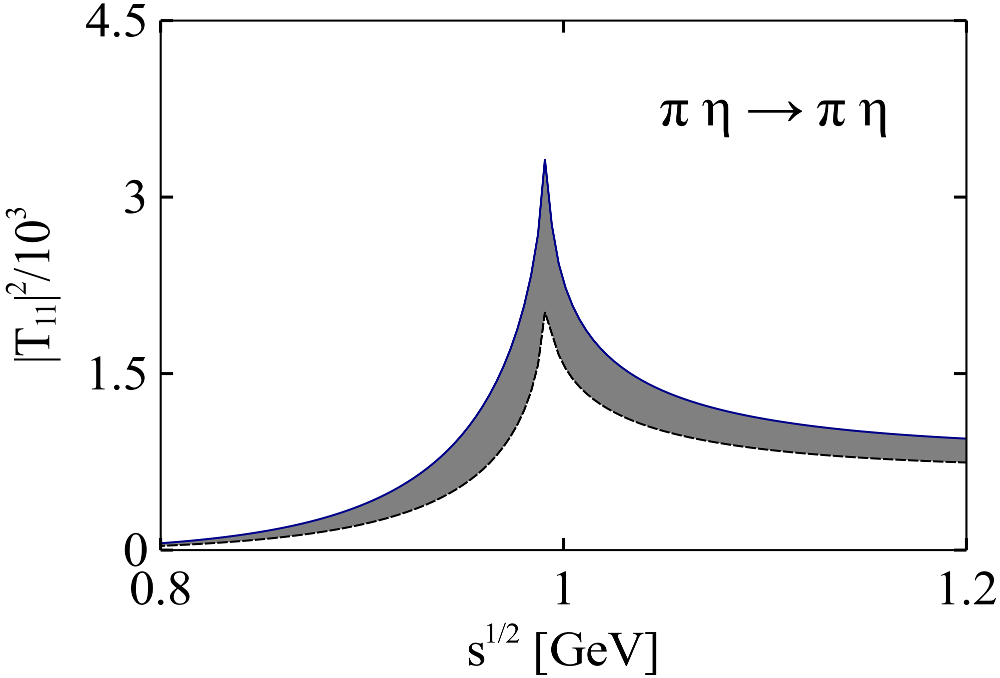


- $U(s) = \sum_{k=0}^{N} C_k \, [\xi(s)]^k$
- in p-wave CDD poles $\leftrightarrow$ ρ, K^*
- $f_0(980)$ dynamically generated
- No free parameters!

Cutoff dependence



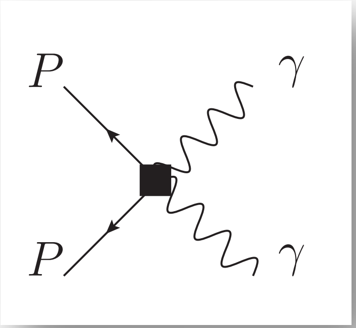
Results for pion-eta scattering



- $J = 0$ and $(I^G, S) = (1^-, 0)$
- $a_0(980)$ dynamically generated
- $f_0(980)$ and $a_0(980)$ resonances are strongly coupled to $K\bar{K}$ channel

- there is no elastic scattering data
- $\pi\eta$ channel can be populated by inelastic $\gamma\gamma \to \pi^0\eta$ data

Motivation to study $\gamma\gamma \to$ PP



- $PP = \pi\pi,\ KK,\ \eta\eta,\ \pi\eta$
- J^{PC} restricted to be $(\text{even})^{++}$
- $\gamma\gamma$ reactions probe scalar resonances (e.g. σ, $f_0(980)$, ...)

➢ The total cross sections for $\gamma\gamma \to$ PP are very sensitive to hadronic final state interaction

➢ New experimental data have been reported by the Belle Collaboration not only for $\gamma\gamma \to \pi\pi$ but also for $\gamma\gamma \to \pi\eta$ and $\eta\eta$

Photon-fusion reactions

- The differential cross section

$$\frac{d\sigma}{d\cos\theta} = \frac{\bar{p}_{cm}}{32\,\pi\, s\,\sqrt{s}}\,\left(|\phi_{++}|^2 + |\phi_{+-}|^2\right)$$

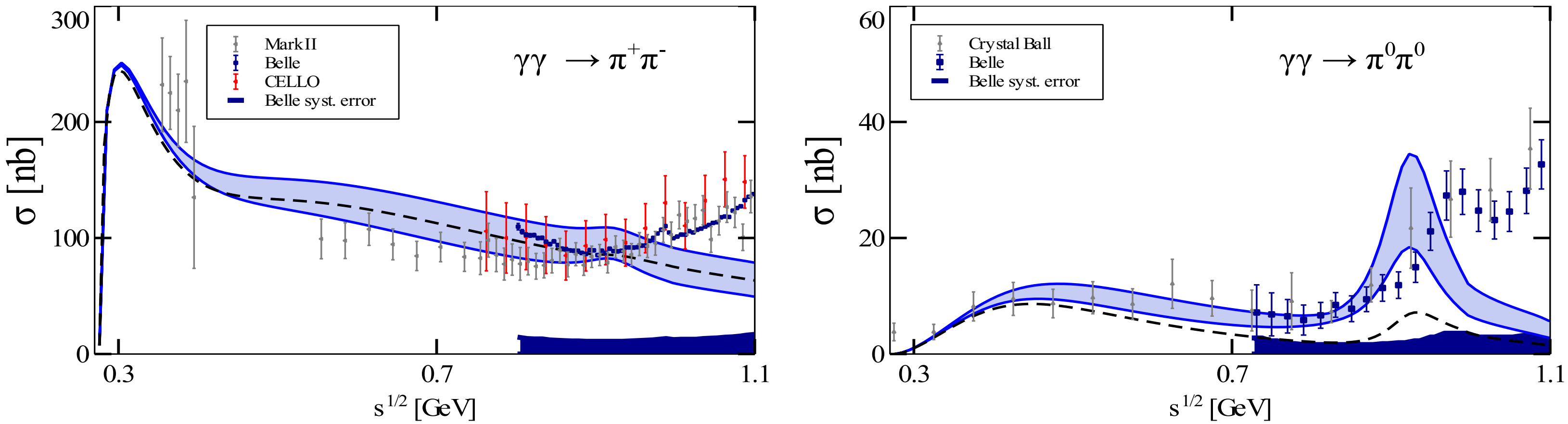
with two helicity amplitudes

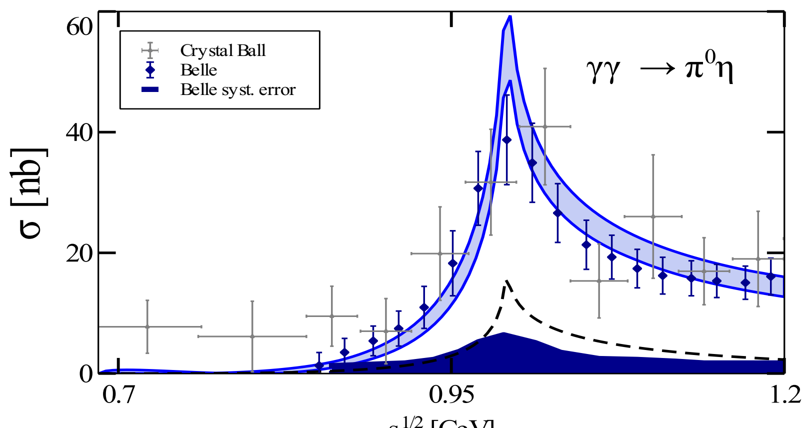
$$\phi_{++} = \sum_{\text{even } J\geq 0} (2J+1)\, t^{(J)}_{++}\, d^{(J)}_{00}(\cos\theta)$$

$$\phi_{+-} = \sum_{\text{even } J\geq 2} (2J+1)\, t^{(J)}_{+-}\, d^{(J)}_{20}(\cos\theta)$$

- Five unknown parameters (coupling from the Lagrangian) have to be determined
- Strategy: fix them from $\gamma\gamma \to \pi0\pi0$, $\pi+\pi-$, $\pi0\eta$ and $\eta \to \pi0\gamma\gamma$
- Cross sections $\gamma\gamma \to$ KK, $\eta\eta$ are pure predictions

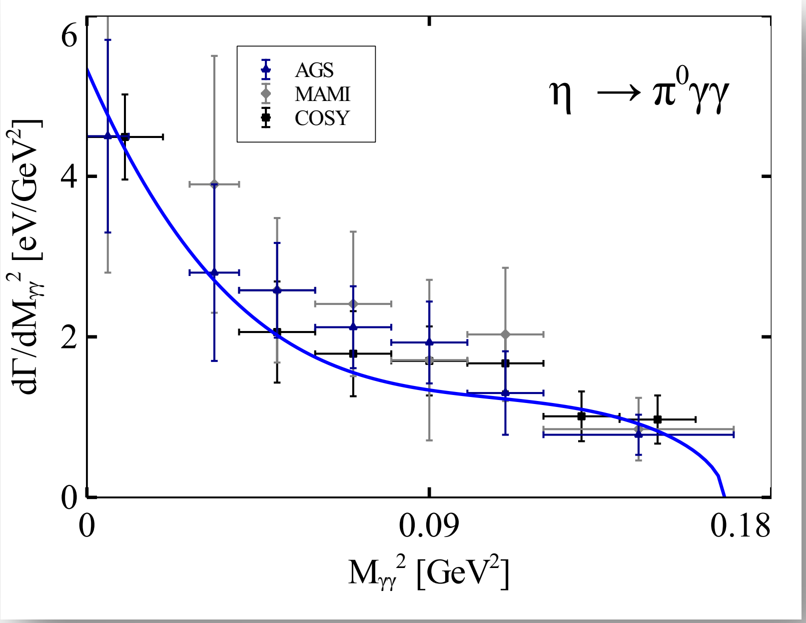
Results for $\gamma\gamma \to \pi\pi, \pi\eta$





- $g_1 = 0.9 - 0.2\, g_3 + 0.038\, h_O^2 + 0.128\, h_O$
  $g_2 = -1.5 - 0.27\, g_3 + 0.25\, g_5$
- Bands correspond to $g_3,\ g_5,\ h_O \in [-5, 5]$
- Dashed curves: $g_i = h_O = 0$

Rare eta decay

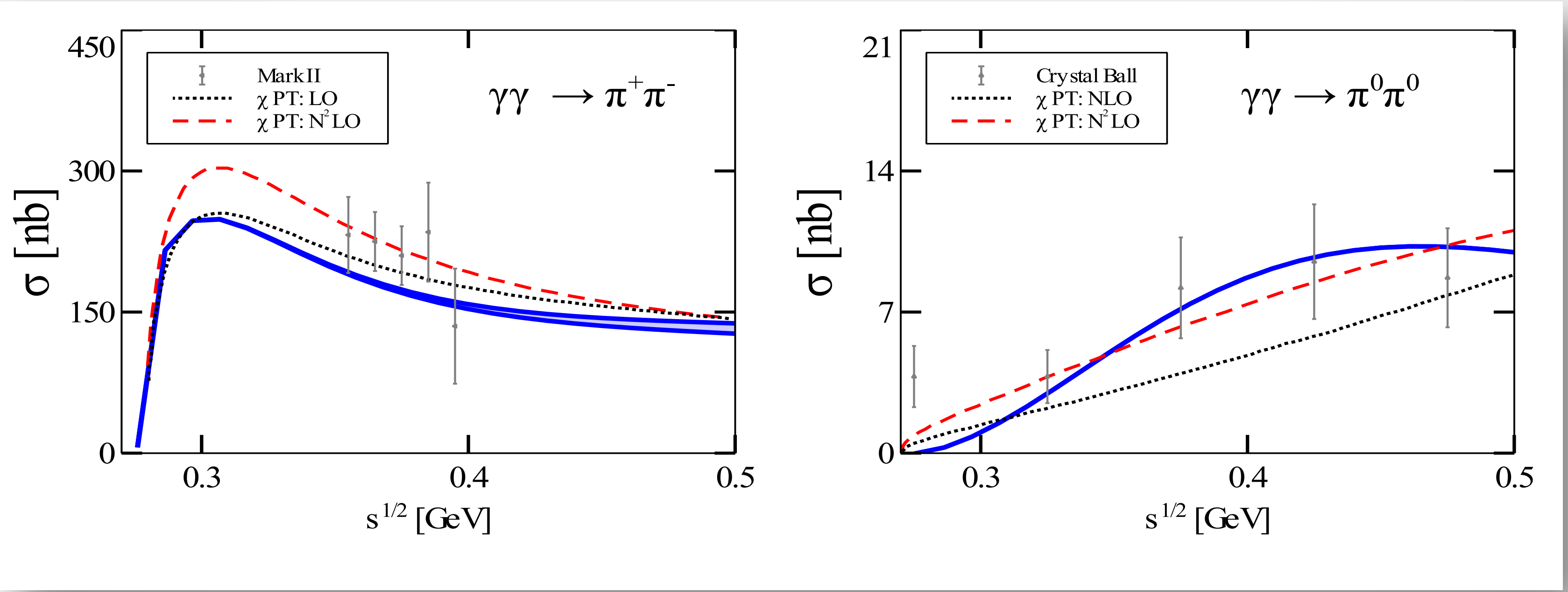


- $\eta \to \pi^0\gamma\gamma$ is linked to $\gamma\gamma \to \pi^0\eta$ by crossing symmetry
- $d\Gamma/dM^2_{\gamma\gamma} \sim \sum_{pol} |T_{\eta\to\pi^0\gamma\gamma}|^2 \, dM^2_{\gamma_2\pi}$
- We fix h_O and g_3

The fit gives the full width for the η decay of

$$\begin{aligned} \Gamma_{\eta\to\pi^0\gamma\gamma} &= 0.31\,\mathrm{eV} \\ \Gamma^{exp}_{\eta\to\pi^0\gamma\gamma} &\approx 0.35 \pm 0.09\,\mathrm{eV} \end{aligned}$$

Comparison with pure ChPT



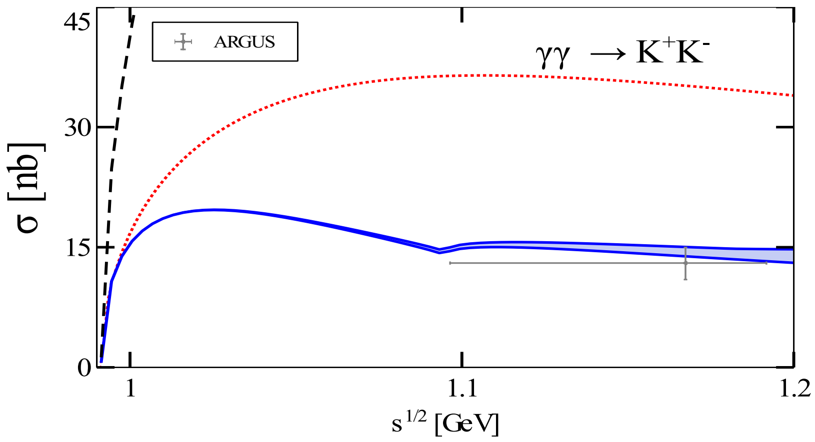
- Results are consistent with χPT
- ChPT: NLO

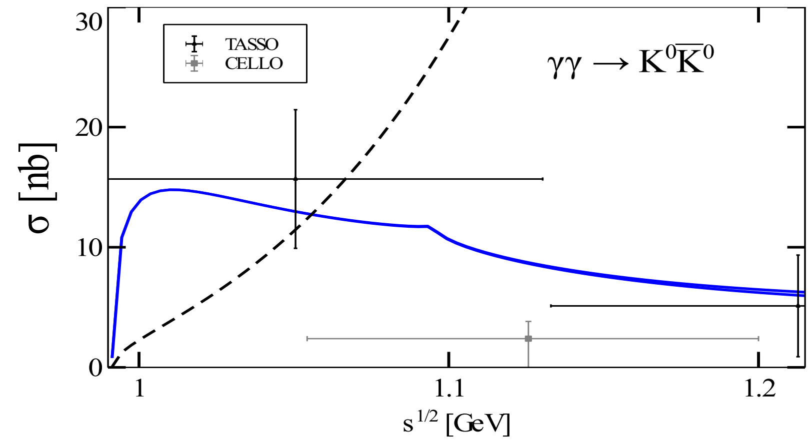
  J. Bijnens and F. Cornet, Nucl. Phys. B 296, 557 (1988)
  J. F. Donoghue, B. R. Holstein and Y. C. Lin, Phys. Rev. D 37, 2423 (1988)

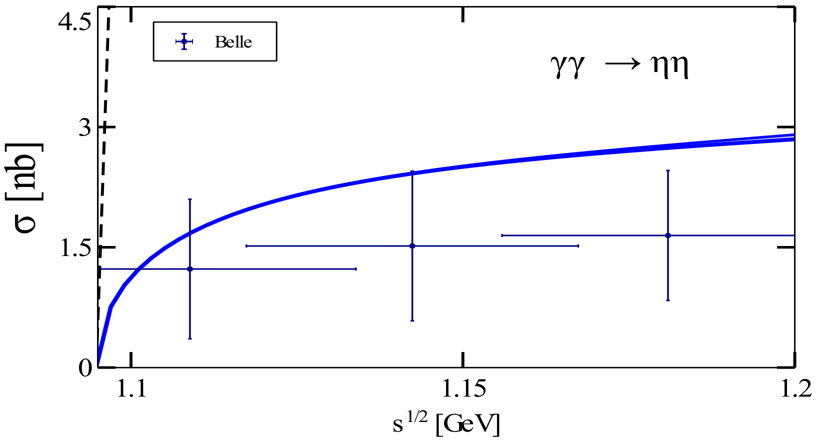
- ChPT: NNLO

  J. Gasser, M. A. Ivanov and M. E. Sainio, Nucl. Phys. B 728, 31 (2005) Nucl. Phys. B 745, 84 (2006)

Predictions for $\gamma\gamma \to KK, \eta\eta$

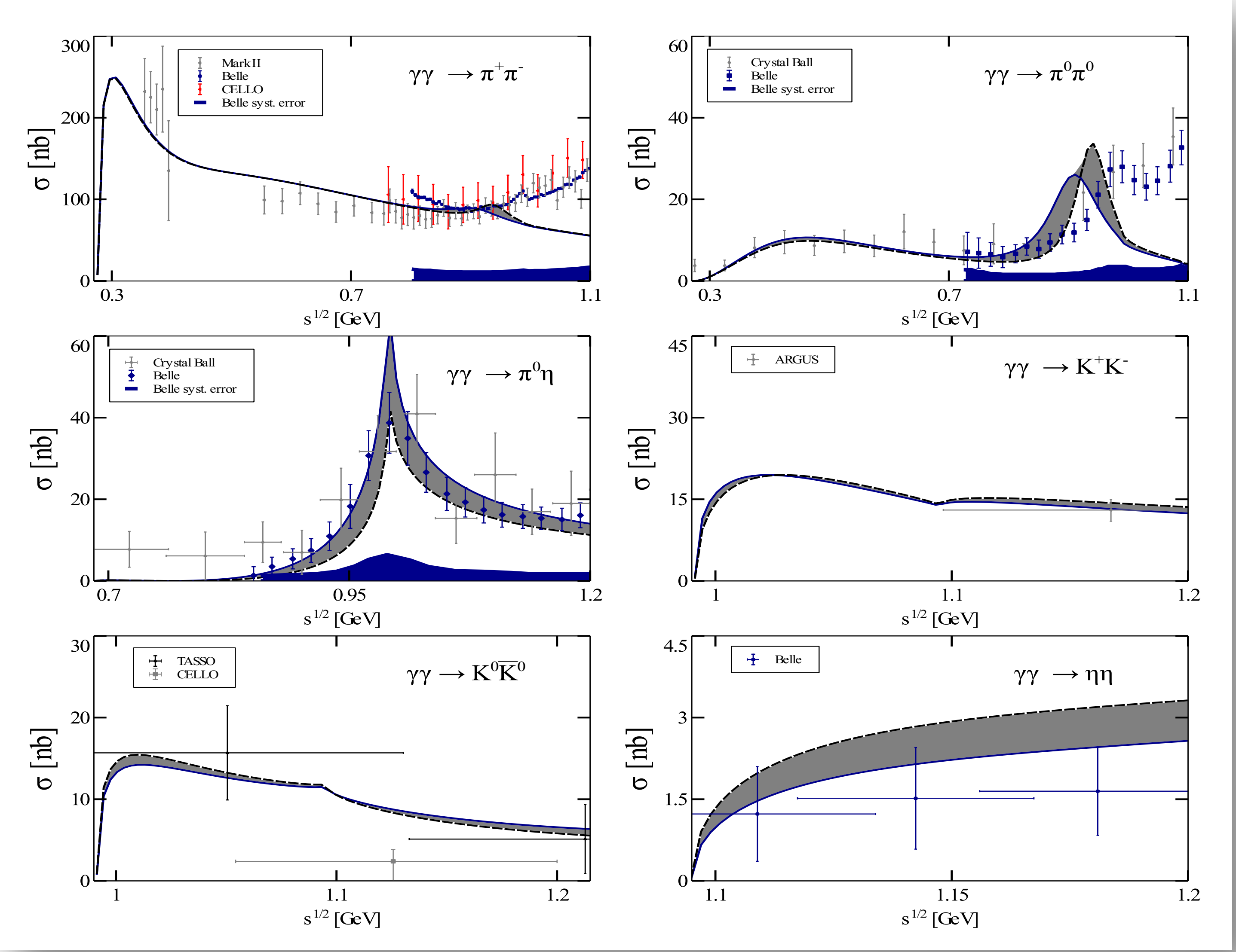






- Solid curves: unitarized result
- Dashed curves: tree-level result
- The bands correspond to $g_5 \in [-5, 5]$
- Reduction of a Born amplitude for $\gamma\gamma \to K^+K^-$
- First description of $\gamma\gamma \to \eta\eta$ data

Cutoff dependence



Summary

Lecture 1: Introduction to ChPT

- Importance of symmetries
- Chiral symmetry
- Construction of effective Lagrangian
- Power counting

Lecture 2: Extension to resonance region

- P. w. dispersion relation
- Applications:
  - Goldstone boson scattering
  - Photon fusion reactions