Computational Methods in Empirical Finance
prof. Pedro Valls Pereira

Bruno Tebaldi de Queiroz Barbosa

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1 Introduction

Understanding and predicting the temporal dependence in the second-order moments of asset returns is important for many issues in financial econometric. It is now widely accepted that financial volatility move together over time across assets and markets. Therefore the use of a framework model that take into account this fact leads to more relevant empirical models. From a financial point of view, it opens the door to better decision tools in various areas, such as asset pricing, portfolio selection, option pricing, hedging and risk management.

Financial time series have several stylized facts and the model specification and selection should be guided by these empirical stylized facts. The ability of a model to reproduce these facts is a desirable feature and on the other hand its inability to reproduce is a criterion for rejecting such a model.

Some of the stylized facts of financial time series are:

(A) **Heavy tails:** Since the work of Fama, 1963, 1965; Mandelbrot, 1972 it is recognized that asset returns are leptokurtic [1]

(B) **Volatility grouping:** Financial time series reveal high and low volatility time periods that group together. In fact, volatility clusters and heavy tails are related. One feature of volatility models is the link between volatility (conditional) dynamics and heavy (unconditional) tails;

(C) **Leverage effects:** Black, 1976 noted that price movements are negatively correlated with volatility. If asset prices are falling, the firm's leverage increases and, in general, also increases uncertainty, which tends to increase volatility.

[1] leptokurtic): (of a frequency distribution or its graphical representation) having greater kurtosis than the normal distribution; more concentrated about the mean.

(D) **Long memory and persistence:** Volatility is highly persistent. In particular for high frequency data there is evidence that the conditional variance generating process has a near unit root;

(E) **Volatility co-movements:** There is an extensive literature on co-movements of international speculative markets. Globalization of financial markets increases price volatility;

(F) **Nonlinear dependency:** The dependency between returns changes depending on market conditions. For example, in general, asset prices seem independent but in times of crisis they tend to fall together. The possible association in the positive tails of the returns may differ from the association in the negative tails. This characteristic is not replicated by measures of linear dependence, i.e. correlation coefficient.

(G) **Information arrival:** Returns are usually measured at fixed frequencies, for example: day, week, month. Some authors have suggested explicitly relating returns to information flows that reach the market. In fact Clark, 1973 is one of the first to propose a stochastic volatility model where information arrival is uneven in time and generally unobserved. This fact is related to the concept of temporal deformation that can be directly linked to volatility through a function that depends on volume, time between quotes, announcements with asset-specific information, macroeconomic news and market closure;

(H) **Correlation in implied volatility:** As implied volatility is calculated using the Black and Scholes, 1973 model, it is obviously model-based. Because volatility is calculated on the basis of daily prices there is an inconsistency because the Black-Scholes model assumes that volatility is constant.In addition, implied volatility should contain information about future volatility and therefore should be a predictor of the latter.

(I) **Term structure of implied volatility:** The Black and Scholes model assumes a constant term structure for volatility. In fact, the forward structure of implied option volatility has a positive bias when short-term volatility is low and a negative bias when volatility is high.

(J) **Smiles:** It is known in the literature that the implicit volatility of Black and Scholes produces an addiction known in the literature as smile due to the U-shape of the implied volatility for different strike prices.

For returns on financial series, it is common for large values at a given time to be followed by also high values in subsequent periods, not necessarily in the same direction, stylized fact (B). Statistically, this feature can be described by the presence of high autocorrelation in the square of returns. The autocorrelation present in the square of the returns of the financial series causes the conditional variance of the returns to present a temporal dependence of past shocks. Engle, 1982 proposed a generic model for conditional variance estimation of returns, the autoregressive conditional heteroscedasticity model (ARCH). The ARCH models are commonly employed in modeling financial time series that exhibit time-varying volatility and volatility clustering.

Since the seminal paper of Engle, 1982, traditional time series tools such as autoregressive moving average (ARMA) models (Box and Jenkins, 1970) for the mean have been extended to essentially analogous models for the variance. ARCH models are now commonly used to describe and forecast changes in the volatility of financial time series.

Bollerslev, 1986 expanded the ARCH model proposing a more parsimonious model, the generalized autoregressive conditional heteroskedasticity (GARCH) model. The GARCH model allows for both the conditional variances and the conditional mean to be modeled by an ARMA model.

Nelson, 1991 suggested the Exponential generalized autoregressive conditional heteroskedasticity (EGARCH) model. In this model, the parameters are unrestricted, being an advantage over the GARCH specification. The unrestricted parameters allows for negative values, which causes volatility to increase when returns are negative (stylized fact (C)).

Rabemananjara and Zakoian, 1993 in his turn proposed the Threshold Autoregressive conditional heteroscedasticity (TGARCH), a volatility model commonly used to handle leverage effects (stylized fact (D)). The idea of the TARCH model is to divide the distribution of the innovations into disjoint intervals and then approximate a piecewise linear function for the conditional standard deviation.

Z. Ding, Granger, and Engle, 1993 proposed the The Asymmetric Power Autoregressive Conditional Heteroscedasticity (APARCH). In some financial time series, large negative returns appear to increase volatility more than do positive returns of the same magnitude (stylized fact (D)), the APARCH model can express the Fat tails, excess kurtosis and Leverage Effects D. Ding, 2011.

Brownlees, Engle, and Kelly, 2011 presented a volatility forecasting comparative study based on the methodology and financial data. The authors focused on identifying successful predictive models over multiple horizons, and frequency of parameter re-estimation. They found that model rankings are insensitive to forecast horizon, also that volatility during the 2008 crisis was well approximated by predictions one day ahead, and should have been within risk managers 1% confidence intervals up to one month ahead.

With this in mind the main purpose of this project is to follow the footsteps of Brownlees et al., 2011, making a volatility forecasting comparative study using the Brazilian market data. The main goal of the project will be to identify potential predictive models over multiple horizons and to analyze how each model perform under several choices for estimation (e.g. estimation window length, innovation distribution, and frequency of parameter re-estimation).

The rest of the paper is organized as follows. Section 2 describes the volatility models used for prediction. Section 3 presents the forecast evaluation methodology. Section 4 reviews thr methodology used to compute the Realized volatility. Section 5 describes the data and its source. Section 6 presents the results of a detailed prediction exercise of Bovespa index volatility. Concluding remarks follow in Section 7.

2 Volatility Forecasting Methodology / Time series models of Heteroskedasticity

A time series of continuously compounded returns will be denoted by $\{r_t\}_{t=1}^{T}$, and $\mathcal{I}_t$ denotes the information set available at time t. Assuming the series $\{r_t\}_{t=1}^{T}$, is covariance-stationary, the optimal forecast of the level of $\{r_t\}_{t=1}^{T}$ is $\mathbb{E}\left[r_{t+1}|\mathcal{I}_t\right]$. However it it also interested to forecast not only the level of $\{r_t\}_{t=1}^{T}$ but also its variance. In order to model this the stochastic process it is often convenient to use the following representation

$$A(L)r_{t+1} = \sigma_{t+1}\epsilon_{t+1} \tag{1}$$

$$\sigma^2_{t+1} = h_{t+1} \tag{2}$$

where $A(L)$ is some lag polynomial, ϵ_{t+1} is an i.i.d. with zero mean and a unit variance innovation, and h_{t+1} is a function such that the specification of h_{t+1} determines the conditional variance evolution.

The the conditional variance evolution is typically a function of the history of returns as well as a vector of unknown parameters to be estimated from the data.

Although equation (1) implies that the unconditional variance of $\{r_t\}_{t=1}^{T}$ is constant over time, the conditional variance of could change. Therefore the unobserved variance of returns conditional on $\mathcal{I}_t$ is defined as

$$\sigma^2_{t+i|t} \equiv \mathbb{V}ar\left[r_{t+i}|\mathcal{I}_t\right] \tag{3}$$

2.1 Volatility Models

The models that will be considered in this article are chosen from the vast literature on GARCH modeling for their simplicity and demonstrated ability to forecast volatility over alternatives.

2.1.1 ARCH

A generic model for estimating the conditional variance of returns is the ARCH model, proposed by Engle, 1982. This model expresses the conditional variance as a function of the past returns.

$$h_t = \omega + \alpha(L)h_{t-1} + \nu_t \tag{4}$$

where $\alpha(L)$ is a lag-polynomial operator of the type $\alpha(L) = \alpha_1 L + \alpha_1 L^2 + \cdots + \alpha_m L^m$ and to ensure no negativity of conditional variance: $w, \alpha_i > 0$ for $i = 1, \ldots, m$. Also ν_t is and i.i.d. process with zero mean and unit variance.

Key features of this process are that positive or negative shocks have the same effect on volatility, violating the leverage effect, stylized fact (C). The ARCH model is very restrictive since for the first order the coefficient which limits the model's ability to capture excess kurtosis, and the ARCH model tends to over-predict volatility because they respond slowly to large shocks.

2.1.2 GARCH

The ARCH model is appropriate when the error variance in a time series follows an autoregressive model, however if an autoregressive moving average model is assumed for the error variance, the model is a generalized autoregressive conditional heteroskedasticity (GARCH) model Bollerslev, 1986; Engle, 1982.

The GARCH(1,1) model assumes the following form:

$$h_t = \omega + \alpha r^2_{t-1} + \beta h_{t-1} \tag{5}$$

Key features of this process are its mean reversion (imposed by the restriction $\alpha + \beta < 1$) and its symmetry - future variance responds as much to past positive returns as it does to negative returns.

2.1.3 E-GARCH

The exponential generalized autoregressive conditional heteroskedastic (EGARCH) Nelson, 1991 models the log of variance. Since $\ln\left(\sigma^2_t\right)$ may be negative, there are no sign restrictions for the parameters.

The formulation for a E-GARCH(1,1) is as follows:

$$\ln\left(h_t\right) = \omega + \alpha\left(\left|\epsilon_{t-1}\right| - \mathbb{E}\left[\left|\epsilon_{t-1}\right|\right]\right) + \gamma\epsilon_{t-1} + \beta \ln\left(h_{t-1}\right) \tag{6}$$

where $\epsilon_t = r_t/\sqrt{h_t}$

2.1.4 GJR-GARCH

One of the stylized facts in empirical asset pricing is negative correlation between asset returns and volatility commonly explained by risk aversion and leverage effect. With this in mind the Glosten-Jagannathan-Runkle GARCH (GJR-GARCH) model by Glosten, Jagannathan, and Runkle, 1993 also models asymmetry in the ARCH process.

The GJR-GARCH(1,1) model assumes the following form:

$$h_t = \omega + (\alpha + \gamma \mathbb{I}_{r_{t-1}<c}) r_{t-1}^2 + \beta h_{t-1} \tag{7}$$

where $\mathbb{I}_{r_{t-1}}$ is an indicator equaling one when the previous period's return is below some threshold c (most commonly, $c = 0$).

2.1.5 EWMA

The exponential smoothing is an IGARCH model with $\omega = 0$. In this case, the EWMA model assumes the following form:

$$h_t = \lambda h_{t-1} + (1 - \lambda) r \nu_t \tag{8}$$

The main characteristic of the model is that a value of λ close to 1 reproduces the stylized fact (D) that volatility is highly persistent.

However, the fact that the sum of the parameters of the model (λ and $1-\lambda$) being equal to one generates an inconvenience, volatility does not conditional return is zero. As a result, the unconditional distribution of the returns is degenerate, that is, its mean and variance are equal to zero. This implies that all the mass of the distribution concentrated on a single point, a fact without any empirical support.

2.1.6 APARCH

The last model considered, asymmetric power ARCH (APARCH) Z. Ding et al., 1993. The (APARCH) model assumes a specific parametric form for powers of this conditional heteroskedasticity.

In this case, the APARCH(1,1) model assumes the following form:

$$h_t^{\delta/2} = \omega + \alpha(|r_t| - \gamma r_t)^{\delta} + \beta h_{t-1}^{\delta/2} \tag{9}$$

3 Forecast Evaluation

Our measure of predictive accuracy is based on the average forecast loss achieved by a model. A model that provides a smaller average loss is more accurate and thus preferred. This article will use the following loss functions: quasi-likelihood-based (QL) loss and The mean squared error (MSE).

Let $\hat{\sigma}_t^2$ be an unbiased ex-post proxy of conditional variance, say realized volatility or squared returns, and let $h_{t|t-k}$ be a volatility forecast based on $t-k$ information ($k>0$).

The quasi-likelihood-based loss, named for its close relation to the Gaussian likelihood, depends only on the standardized residual $\hat{\sigma}_t^2/h_{t|t-k}$.

$$QL : L(\hat{\sigma}_t^2, h_{t|t-k}) = \frac{\hat{\sigma}_t^2}{h_{t|t-k}} - \ln\left(\frac{\hat{\sigma}_t^2}{h_{t|t-k}}\right) \tag{10}$$

The mean squared error loss depends solely on the additive forecast error, $\hat{\sigma}_t^2 - h_{t|t-k}$.

$$MSE : L(\hat{\sigma}_t^2, h_{t|t-k}) = \left(\hat{\sigma}_t^2 - h_{t|t-k}\right)^2 \tag{11}$$

3.1 Forecasting window

Out-of-sample forecasts are typically computed using one of two methods: Recursive (expanding window) and Rolling (moving window).

3.1.1 Recursive (expanding window)

An initial sample using data from $t=1,\ldots,T$ is used to estimate the models, h-step ahead out-of-sample forecasts are produced starting at time T The sample is increased by one, the models are re-estimated, and h-step ahead forecasts are produced starting at $T+1$

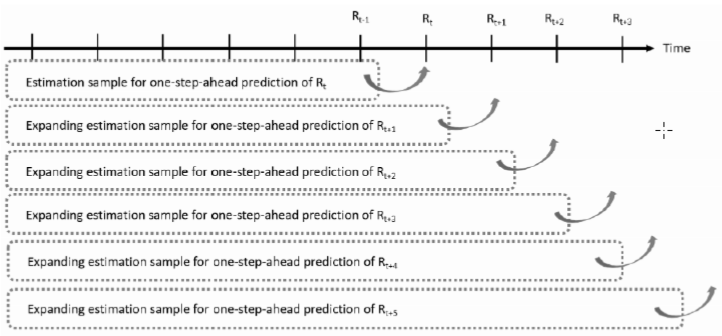

Figure 1: Expanding window

3.1.2 Rolling (moving window)

An initial sample using data from $t=1,\ldots,T$ is used to determine a window width T to estimate the models, and to form h-step ahead out-of-sample forecasts starting at time T. Then the window is moved ahead one time period, the models are re-estimated using data from $t = 2, \ldots, T + 1$, and h-step ahead out-of-sample forecasts are produced starting at time $T + 1$

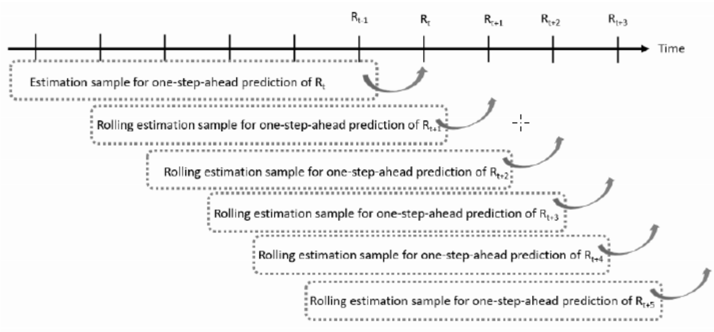

Figure 2: Moving window

4 Realized volatility

Denote by P_t the price at time t (one business day, for example) and by $p_t = \ln(P_t)$. Then, the log return on day t is given by $r_t = p_t - p_{t-1}$.

The intraday prices are sampled at regular intervals of 15 minutes. Therefore on a given bussiness bay there are $M = 28$ intervals of 15 minutes per trading day.

Denote by $p_{d,m}$ the log-price of the asset in the period m $(m = 1, \ldots, M)$ of day d $(d = 1, \ldots, D)$, the corresponding intraday return in the period d and m

$$r_{d,m} = p_{d,m} - p_{d,m-1} \qquad m = 2, \ldots, M \qquad d = 1, \ldots, D \tag{12}$$

Therefore, the daily r_d return is given by $r_d = p_{d,M} - p_{d-1,M}$, and we define the overnight return, which incorporates information regarding the interval between the closing of the previous day's trading session and the opening of the current day's trading session, as

$$r_{d,1} = p_{d,1} - p_{d-1,M} \tag{13}$$

The realized variance for an asset in day d (VR_d) is then given by

$$VR_d = r_{d,1}^2 + \sum_{m=2}^{M} r_{d,m}^2 \tag{14}$$

where $r_{d,m}$ and $r_{d,1}$ are defined as (12) and (13).

The realized volatility for an asset in day d $(VOLR_d)$ is then defined by:

$$VOLR_d = \sqrt{VR_d} \tag{15}$$

If some days are excluded during the filtering process of the data, in this case, the equation (14) is adapted and takes the form:

$$VR_d = \begin{cases} \sum_{m=2}^{M} r_{d,m}^2 & \text{if day d-1 is excluded} \\ r_{d,1}^2 + \sum_{m=2}^{M} r_{d,m}^2 & \text{otherwise} \end{cases} \tag{16}$$

5 Data

5.1 The Bovespa Index

The Bovespa Index (Ibovespa) is the result of a theoretical portfolio of assets. The Ibovespa's objective is to be the indicator of the average performance of quotations of the most tradable and representative assets of the Brazilian stock market. The Index is composed of shares and units exclusively of shares of companies listed on the Bm&fBovespa that meet the inclusion [2].

5.1.1 Changes

Since the beginning of its calculation in January 1968, the Bovespa Index has been subject to several adjustments. The Ibovespa base was set at 100 (one hundred) points as of January 2, 1968, however, the index suffered, solely for disclosure and without prejudice to its calculation methodology, the following adjustments:

Id	Change	Date
1	Division by 100	3/10/1983
2	Division by 10	2/12/1985
3	Division by 10	29/8/1988
4	Division by 10	14/4/1989
5	Division by 10	12/1/1990
6	Division by 10	28/5/1991
7	Division by 10	21/1/1992
8	Division by 10	26/1/1993
9	Division by 10	27/8/1993
10	Division by 10	10/2/1994
11	Division by 10	3/3/1997

Table 1: Changes to the Bovespa Index

In mid-2012, however, Bm&fBovespa, in view of the great evolution of the Brazilian capital market in all its aspects, experienced mainly in the previous ten years, began a process of revaluation of the Ibovespa methodology.

The new methodology was implemented on September 11, 2013, and the participation of assets in the portfolio was calculated considering 50% of the participation by the Tradability Index and 50% of the participation by the new weighting criteria. From the review of May 2014, the participation was fully determined by the new model.

[2] see Bm&fBovespa Indexes Definitions and Procedures Manual

5.2 Series period

The data used in this paper will be the the daily return of the end-of-day (EOD) prices for the Bovespa Index. The total sample is from 2000 through 2018. The series were obtained from Yahoo Finance site.

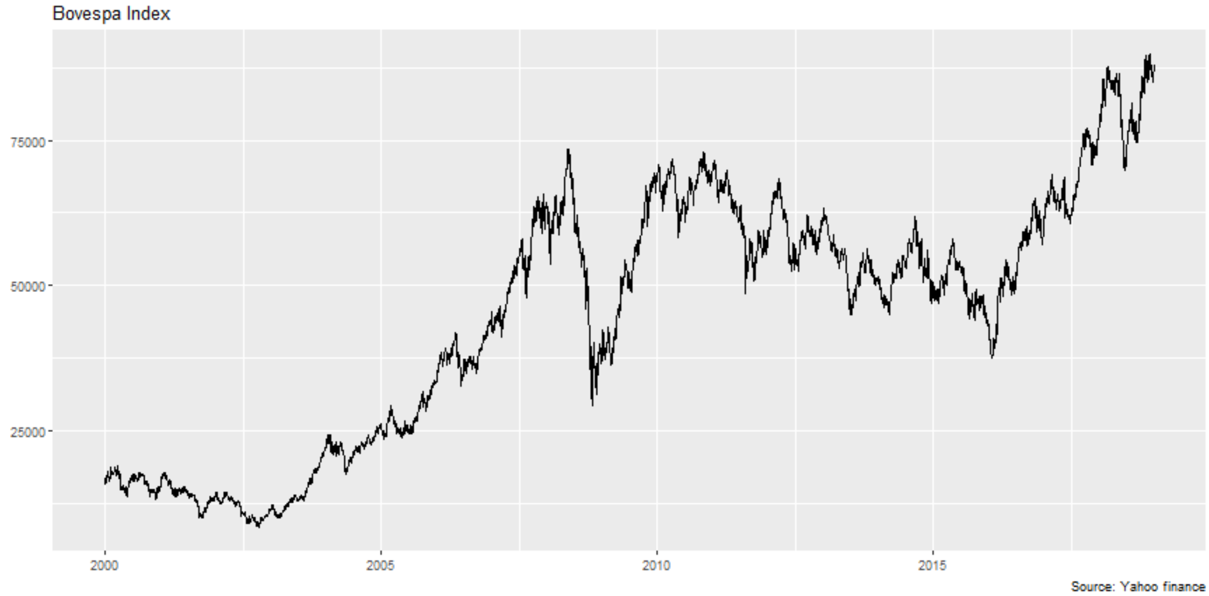

Figure 3: Bovespa Index from 2000 to 2018

The realized variance is computed using two standard proxies are used for computing the realized variance, the first one is the squared returns and the second one is the more precisely estimated realized variance, calculated from high frequency data.

The high frequency data is calculated for the period 1998-04-06 to 2003-08-03.

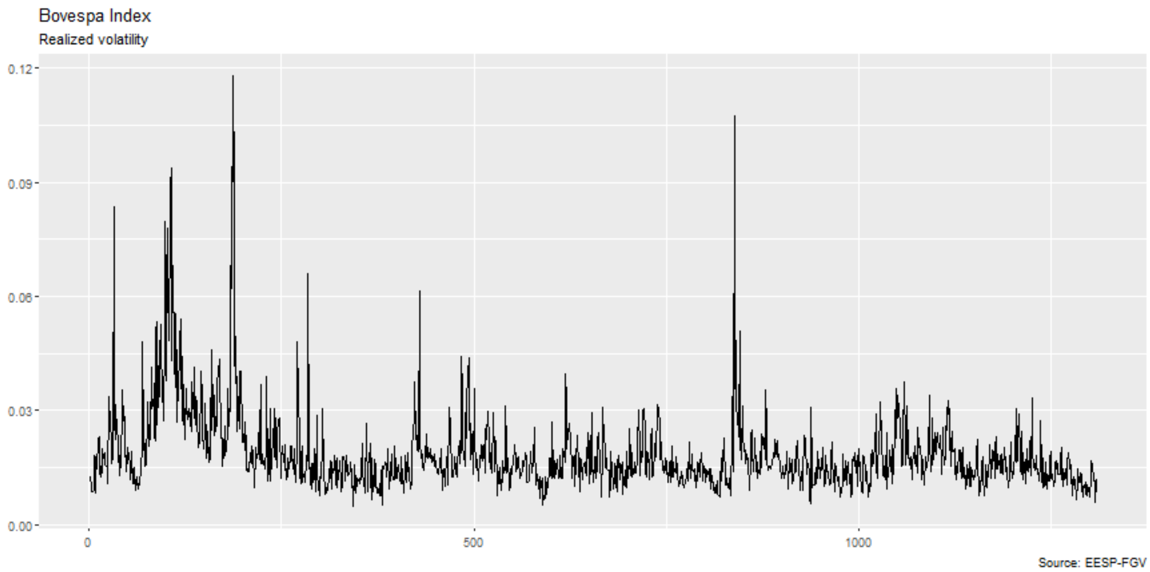

Figure 4: Bovespa index realized volatility for 15 min. returns

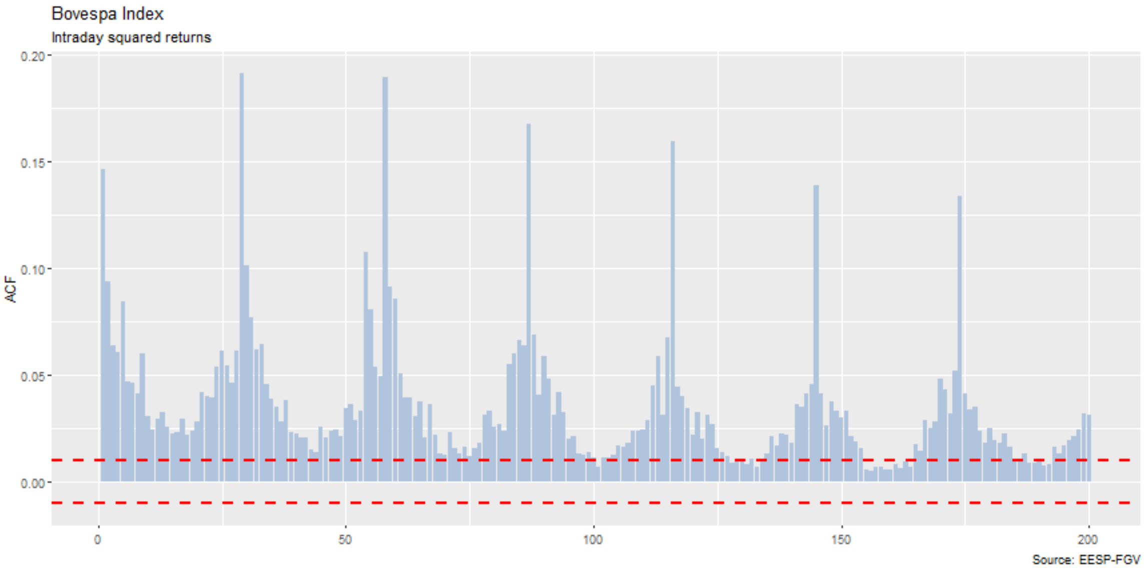

Figure 5: Bovespa index intraday squared returns ACF

5.3 Holidays and gaps

Holidays and other eventual gaps in the series were handdled by using a local level model with a Kalman filter. Table (2) shows a summary of the data where the column of "Adj. Close" represents the data with the local level adjustment.

	Date	Open	High	Low	Close	Adj. Close	Volume
Min.	2000-01-03	8.397	8.513	8.225	8.371	8.371	0
1st Qu.	2004-08-03	23.225	23.456	22.961	23.235	22.757	0
Median	2009-05-11	50.414	51.065	49.891	50.423	50.089	1.981.600
Mean	2009-05-27	44.812	45.287	44.342	44.826	44.465	7.266.875
3rd Qu.	2014-03-10	61.264	61.945	60.747	61.272	61.138	3.502.075
Max.	2018-12-28	89.820	91.242	89.429	89.820	89.820	232.265.300
NA's	84	84	84	84	84	0	84

Table 2: Descriptive Statistics

5.4 Out-of-sample

We set the out-of-sample forecast horizon for the period from 2013 to 2018, were this period contains periods of both very low volatility and severe distress. Figure 6 shows the time series plot of daily returns for the Bovespa Index.

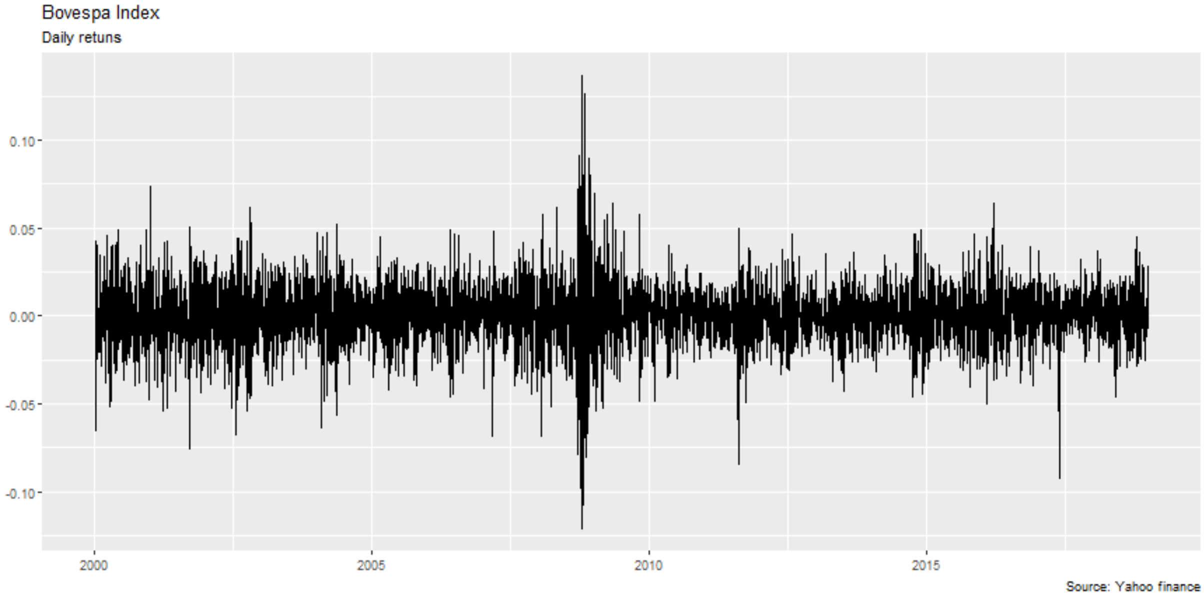

Figure 6: Bovespa Index from 2000 to 2018

6 Results

The first step is the estimation strategies for the mean of the returns. The following models are fitted to the daily returns: ARMA(1,1), ARMA(1,0), ARMA(0,1), ARMA(0,0). The selection of the model is based by the Akaike Information criteria. Table 3 presents the data for these models. The ARMA(0,0) model will be used to model the mean of the returns.

Model	**Const.**	**AR**	**MA**	**log likelihood**	**AIC**
ARMA(1,1)	0.0004 (0.0003)	-0.2944 (0.3830)	0.3075 (0.3798)	12576.58	-25147.16
ARMA(1,0)	0.0003 (0.0003)	0.0097 (0.0145)	-	12576.36	-25148.72
ARMA(0,1)	0.0003 (0.0003)	-	0.0103 (0.0149)	12576.37	-25148.74
ARMA(0,0)	0.0003 (0.0003)	-	-	12576.13	-25150.27

Table 3: Model comparison for the daily returns. (standard error in parentheses)

The models used for the volatility are based in the work of Brownlees et al., 2011, therefore we estimate the following models: ARCH(1), GARCH(1,1), E-GARCH(1,1), GJR-GARCH(1,1), EWMA, APARCH(1,1).

6.1 In-Sample fit

The sample used for the first estimation of the models is from 2000-01-01 to 2013-01-01. Using this data all the models where able to fit the data without any problems. Table (4) shows the Information criteria associated with every model. The GJR-GARCH is the best model according to all information criteria.

[H]

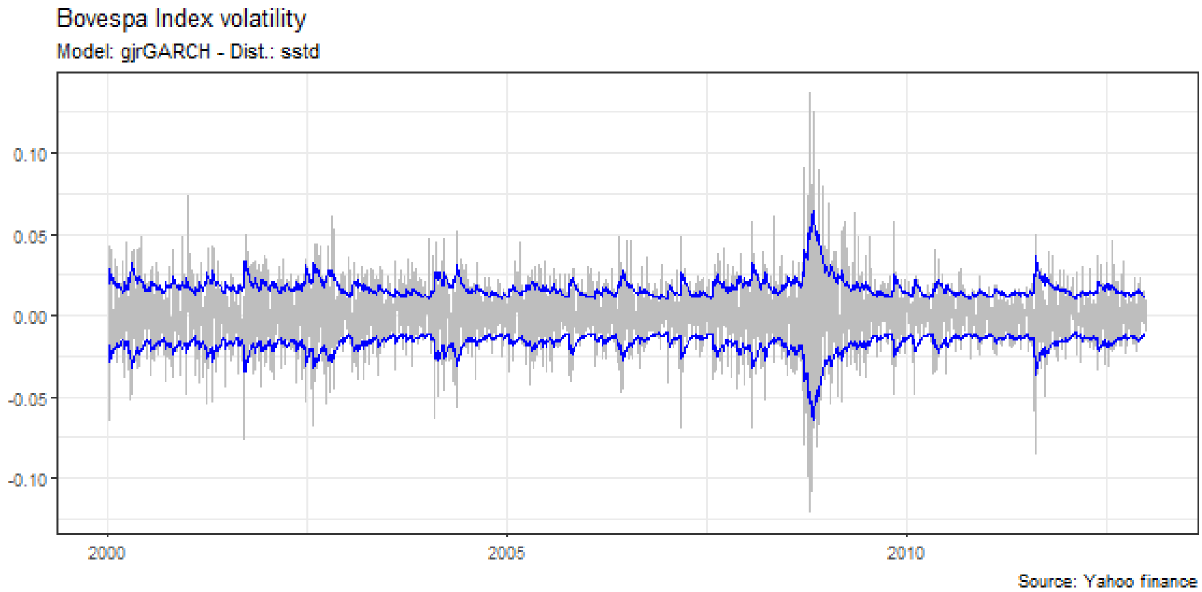

Figure 7: GJR-GARCH in sample fit

Model	**Dist**	**Akaike**	**Bayes**	**Shibata**	**HannanQuinn**
ARCH	Norm	−5.1607	−5.1552	−5.1607	−5.1588
ARCH	Sstd	−5.2325	−5.2233	−5.2325	−5.2292
GARCH	Norm	−5.3083	−5.3009	−5.3083	−5.3056
GARCH	Sstd	−5.3259	−5.3148	−5.3259	−5.3219
E-GARCH	Norm	−5.3252	−5.3159	−5.3252	−5.3219
E-GARCH	Sstd	−5.3413	−5.3283	−5.3413	−5.3366
GJR-Garch	Norm	−5.3295	−5.3202	−5.3295	−5.3262
GJR-Garch	Sstd	−5.3438*	−5.3308*	−5.3438*	−5.3391*
EWMA	Norm	−5.2947	−5.2910	−5.2947	−5.2934
EWMA	Sstd	−5.3185	−5.3111	−5.3185	−5.3159
APARCH	Norm	−5.3266	−5.3154	−5.3266	−5.3226
APARCH	Sstd	−5.3409	−5.3261	−5.3409	−5.3356

Table 4: Model Information Criteria

6.2 Normal vs Skewed t-Student

The first point of comparison between the models are how well using a Skewed t-Student Distribution performs over the the Normal Distribution.

Table 5: MSE for a expanding window with 252 period of parameter re-estimation

	Norm	**Sstd**	**DM p-value**
ARCH	9.5927	1.7369+	0.0945
GARCH	1.5821*	1.5809+*	0.3517
E-GARCH	1.5837+	1.5866	0.3738
GJR-GARCH	1.6024+	1.6043	0.3777
EWMA	1.5846+	1.5851	0.3348
APARCH	1.6520+	1.6575	0.3783

Values for the MSE are multiplied by 10^7
Diebold-Mariano Test alternative hypothesis: two sided
+ : lowest value across the line
* : lowest value across the column

Table 6: QL for a expanding window with 252 day period of parameter re-estimation

	Norm	**Sstd**	**DM p-value**
ARCH	7.8714	2.5507+	0.02921
GARCH	2.4337+*	2.4375*	0.2159
E-GARCH	2.4453+	2.4537	0.2956
GJR-GARCH	2.4471	2.4463+	0.7158
EWMA	2.4603	2.4556+	0.2877
APARCH	2.4588+	2.4609	0.9981

Diebold-Mariano Test alternative hypothesis: two sided
+ : lowest value across the line
* : lowest value across the column

Table 7: MSE for a expanding window with 5 day period of parameter re-estimation

	Norm	**Sstd**	**DM p-value**
ARCH	7.0980	1.7296+	0.1571
GARCH	1.5819*	1.5805+*	0.3486
E-GARCH	1.5823+	1.5848	0.3767
GJR-GARCH	1.6007+	1.6025	0.3620
EWMA	1.5841+	1.5845	0.3387
APARCH	1.6462	1.6381+	0.0957

Values for the MSE are multiplied by 10^7
Diebold-Mariano Test alternative hypothesis: two sided
+ : lowest value across the line
* : lowest value across the column

Table 8: QL for a expanding window with 5 day period of parameter re-estimation

	Norm	**Sstd**	**DM p-value**
ARCH	6.8109	2.5501+	0.1297
GARCH	2.4358+*	2.4417*	0.2207
E-GARCH	2.4397+	2.4431	0.3870
GJR-GARCH	2.4457	2.4443+	0.9406
EWMA	2.4562+	2.4598	0.3585
APARCH	2.4489	2.4476+	0.6915

Diebold-Mariano Test alternative hypothesis: two sided
+ : lowest value across the line
* : lowest value across the column

Tables (5), (6), (7) and (8) shows that, for most of the models, the adoption of a student-t likelihood distribution does not significantly improve forecast performance with a 252 day period of parameter re-estimation as well as a 5 day period of parameter re-estimations, the only exception is the ARCH model, were the use of a skewed student-t would have a significant difference.

6.3 Forecast Window

The second point of comparison is between the forecasting methods of a expanding window and a rolling window of 2 years or 5 years. For comparison across models the period of re-estimation is fixed at 5 days.

Although this article explores the use of a Skewed t-Student Distribution it is worth mentioning that the use of this distribution presented convergence problems for some models. Since the comparison between Skewed t-Student Distribution and Normal distributions showed no evidence of improved forecast performance when using the Skewed t-Student Distribution, the focus will be on the Normal distribution results.

Table 9: MSE comparison of forecast methods form Normal distribution

	Expanding	**Moving 2 years**	**Moving 5 years**
ARCH	7.09804	6.41130+	6.71682
GARCH	1.58187*	1.58828	1.57809+
E-GARCH	1.58228	1.60976	1.57625+*
GJR-GARCH	1.60071	1.58248+*	1.58965
EWMA	1.58413	1.59175	1.58364+
APARCH	1.64617	1.63488	1.63303+

Values for the MSE are multiplied by 10^7
Parameters re-estimated every 5 days
+ : lowest value across the line
* : lowest value across the column

Table 10: MSE comparison of forecast methods form Sstd. distribution

	Expanding	**Moving 2 years**	**Moving 5 years**
ARCH	1.72956	1.64733	1.67268
GARCH	1.58050	NA	NA
E-GARCH	1.58478	NA	1.57723
GJR-GARCH	1.60255	NA	NA
EWMA	1.58451	1.58882	1.58554
APARCH	1.63811	NA	NA

Values for the MSE are multiplied by 10^7
Parameters re-estimated every 5 days
NA: Not available
+ : lowest value across the line
* : lowest value across the column

Table 11: QL comparison of forecast methods form Normal distribution

	Expanding	**Moving 2 years**	**Moving 5 years**
ARCH	6.81089+	29.99399	17.97508
GARCH	2.43581+*	2.45923	2.45747
E-GARCH	2.43971	2.47418	2.43676+*
GJR-GARCH	2.44566+	2.46250	2.44697
EWMA	2.45617+	2.46857	2.46430
APARCH	2.44892+	2.45127*	2.45905

Parameters re-estimated every 5 days
+ : lowest value across the line
* : lowest value across the column

Tables (9) and (11) shows that using a shorter, rolling estimation window of 2 years tends to weaken forecasting accuracy. The use of expanding window tends to perform better when using a QL loss function while a MSE loss function indicates that the rolling window of 5 years tend to be best.

It is worth mentioning that the E-GARCH model with a 5 year window and the classical GARCH with a expanding are the best models for this forecast window with a 5 day re-estimation of parameters.

Table 12: DM Test for Normal distribution, moving window of 5 years and re-estimation every 5 days

	ARCH	**GARCH**	**E-GARCH**	**GJR-GARCH**	**EWMA**	**APARCH**
ARCH	NA	0.9939	0.9940	0.9940	0.9939	0.9940
GARCH	0.0061*	NA	0.7080	0.7048	0.2764	0.6476
EGARCH	0.0060*	0.2920	NA	0.2778	0.2867	0.2282
GJRGARCH	0.0060*	0.2952	0.7222	NA	0.2883	0.2065
EWMA	0.0061*	0.7236	0.7133	0.7117	NA	0.6829
APARCH	0.0060*	0.3524	0.7718	0.7935	0.3171	NA

DM-Test: The alternative hypothesis is that the model on the column is less accurate than model on the row.
* : less than 5%
The DM-Test were conducted with the MSE loss function.

Table (12) shows that for a Normal distribution, considering a moving window of 5 years and re-estimation every 5 days all models have similar forecasting power, with exception of the ARCH.

The models are then evaluated using a larger period of parameter re-estimation. As shown by table (13), when using a period of re-estimation of 252 days most of the models perform better with a moving window of 5 years. It is worth mentioning that the E-GARCH model faced convergence problems when using a moving window of 2 years.

Table 13: MSE comparison of forecast methods form Normal distribution

	Expanding	**Moving 2 years**	**Moving 5 years**
ARCH	9.59273	7.18291+	12.63411
GARCH	1.58209*	1.58932*	1.57497+*
E-GARCH	1.58371	Inf	1.57816+
GJR-GARCH	1.60242	1.62359	1.59712+
EWMA	1.58464	1.59499	1.58393+
APARCH	1.65203	1.63419+	1.64973

Values for the MSE are multiplied by 10^7
Parameters re-estimated every 252 days
+ : lowest value across the line
* : lowest value across the column

6.4 Parameter re-estimation

The idea behind the re-estimation of parameters is that they change over time. A expanding window will use all available data, meaning that the model will use some weight on old data, the use of a moving window can be interpreted as a zero weight to old values out of the forecasting window. This section explores how does the models perform with different periods of parameter re-estimation.

Table 14: MSE comparison of re-estimation form Normal distribution in a 5 year moving window

	5 days	**20 days**	**60 days**	**252 days**
ARCH	6.7168+	7.1792	9.1130	12.6341
GARCH	1.5781	1.5782	1.5789*	1.5750+*
E-GARCH	1.5762+*	1.5781*	1.5796	1.5782
GJR-GARCH	1.5897+	1.5928	1.5964	1.5971
EWMA	1.5836+	1.5852	1.5851	1.5839
APARCH	1.6330+	1.6392	1.6426	1.6497

Values for the MSE are multiplied by 10^7
+ : lowest value across the line
* : lowest value across the column

Table 15: QL comparison of re-estimation form Normal distribution in a 5 year moving window

	5 days	**20 days**	**60 days**	**252 days**
ARCH	17.9751+	49.7862	31.2580	26.3828
GARCH	2.4575	2.4555	2.4437	2.4353+*
E-GARCH	2.4368*	2.4415*	2.4355+*	2.4451
GJR-GARCH	2.4470+	2.4484	2.4510	2.4473
EWMA	2.4643	2.4679	2.4558+	2.4563
APARCH	2.4591+	2.4650	2.4667	2.4598

+ : lowest value across the line
* : lowest value across the column

As intuition dictates, most of the models perform better when considering a lower re-estimating period of parameters (when analyzed with the MSE loss function.

6.5 Realized volatility comparison

All the previous comparisons were done using the squared residuals as a proxy for the volatility. However, it is possible to evaluate the forecast using the realized volatility.

The sample with realized volatility is from 1998-04-06 to 2003-08-03. Since the main sample starts from 2000-01-01, the evaluation using the realized volatility will be a moving window of 3 years and expanding window.

Table 16: MSE comparison in a 3 year expanding window

	Norm	Sstd	DM p-value
ARCH	12.5964	0.5145+	0.2188
GARCH	0.3111	0.2964+	0.4948
E-GARCH	0.2295+	0.2310	0.1303
GJR-GARCH	0.2600	NA	NA
EWMA	0.2262*	0.2231+	0.2671
APARCH	0.2654	NA	NA

Values for the MSE are multiplied by 10^7
Diebold-Mariano Test alternative hypothesis: two sided
Parameters re-estimated every 5 days
+ : lowest value across the line
* : lowest value across the column

Table 17: QL comparison in a 3 year expanding window

	Norm	Sstd	DM p-value
ARCH	31.3614	1.2972+	0.2188
GARCH	1.2097	1.2006+	0.4948
E-GARCH	1.1466+*	1.1474	0.1303
GJR-GARCH	1.1718	NA	NA
EWMA	1.1498	1.1469+	0.2671
APARCH	1.1769	NA	NA

Diebold-Mariano Test alternative hypothesis: two sided
Parameters re-estimated every 5 days
+ : lowest value across the line
* : lowest value across the column

Tables (16) and (17) shows that apparently there are no forecast accuracy gain between the Normal and Skewed t-Student.

Table 18: MSE comparison between 3 year expanding window vs moving window

	Mov3(5)	Mov3(20)	Mov3(60)	Roll(5)	Roll(20)	Roll(60)
ARCH	4.2825	8.8842	0.5979	12.5964	0.5106	0.4952+
GARCH	0.3030+	0.3083	0.3148	0.3111	0.3156	0.3225
E-GARCH	0.2359	0.2282+	0.2447	0.2295	0.2340	0.2345
GJR-GARCH	0.2551+	0.2576	0.2588	0.2600	0.2625	0.2653
EWMA	0.2258*	0.2266*	0.2250+*	0.2262*	0.2266*	0.2258*
APARCH	0.2526	0.2574	0.2496+	0.2654	0.2667	0.2747

Values for the MSE are multiplied by 10^7
Mov3(x): moving window of 3 years with re-estimation every x days
Roll(x): expanding window with re-estimation every x days
+ : lowest value across the line
* : lowest value across the column

Table (18) shows the forecast comparison between all models with a Normal distribution. The models with moving window has a beter MSE than the expanding counterparts, were the EWMA has the lowest MSE in all scenarios.

6.6 Parameter Estimates

The E-GARCH model presented a convergence problem when using a moving window of 2 years, therefore we investigate how is the evolution of the parameters across the time. For this we use the 5 year moving window with re-estimation of parameters every 5 days.

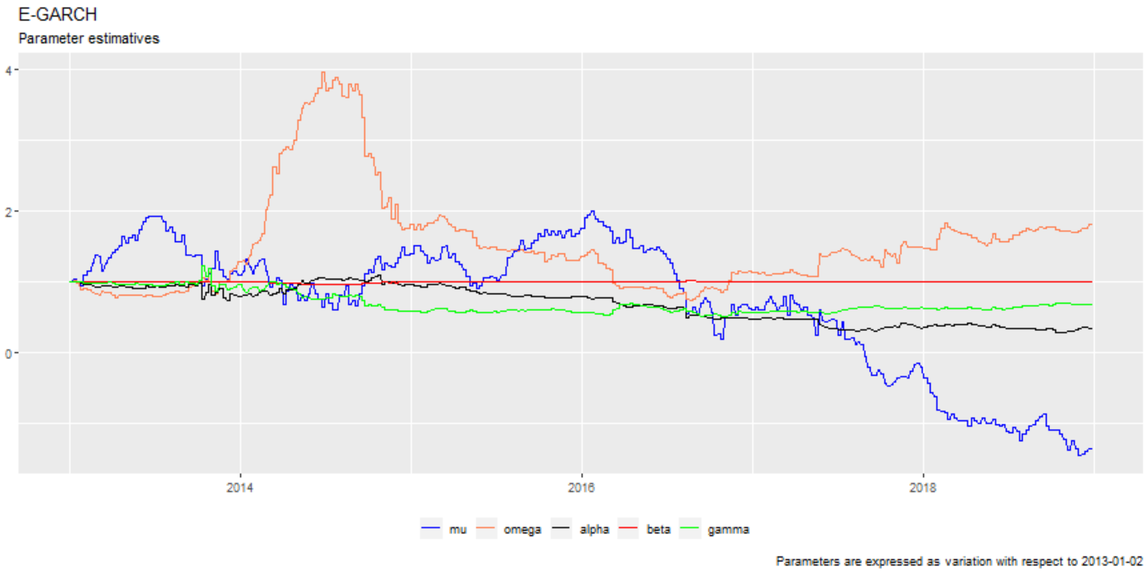

Figure 8: E-GARCH Parameters estimatives series

The parameters of the E-GARCH model varies up to for times its initial value. While there is no evidence of clear breaks in the parameters, the series do exhibit a great degree of variation.

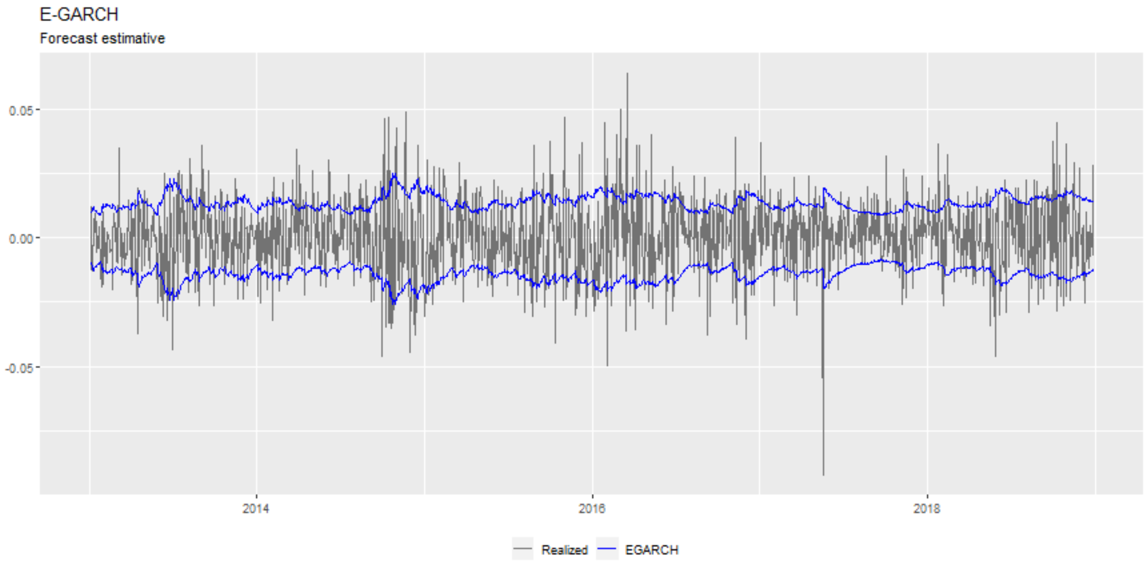

Figure 9: E-GARCH Forecast estimatives

7 Conclusion

Our study seeks to comprehensively describe the forecasting ability of different volatility models and different forecasting methods. We examine how different forecasting methods impact forecasting performance, including i) estimation sample sizes, ii) model re-estimation frequency and iii) Gaussian and Student t innovation likelihoods.

We find that asymmetric models, especially GJR-GARCH, perform relatively well across all forecasting methods. The E-GARCH also has a high performance, but has convergence problems in the 2 year sample. Updating parameter estimates as frequently as possible provides the best forecasting performance. Lastly, we find no evidence that the Student-t likelihood improves forecasting ability in comparison to the Normal, despite its potentially more realistic description of return tails. Also the use of Student-t may cause some convergence problems.

An exploration into the E-GARCH model reveals some interesting features. First and foremost, that the parameters have a great degree of variability without exhibiting any signs of breaks. This implies that parameter re-estimation might is of great importance.

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