Lista 1 - macro 4

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November 6, 2019

Q.1)

Item a)

DSGE models do not admit, except in a few cases, a closed-form solution to their equilibrium dynamics that we can derive with "paper and pencil." Instead, we have to resort to numerical methods and a computer to find an approximated solution.

Therefore the solution of a DSGE model is an approximated numerical solution of a system equilibrium.

Item b)

The value function interaction is a method that can be applied to DSGE modeling. As an example we cite Christiano (1990) which applied the value function iteration to the social planner's problem of a stochastic neoclassical growth model.

Local solutions are accurate around the point of equilibrium. The solutions is accurate around the local equilibrium point at which we perform some kind of perturbation and deteriorates as we move away from that point.

Global solutions on the other hand can attack even the most complex problems with occasionally binding constraints, irregular shapes, and local behavior. However, the evaluation of such methods come at a cost: computational effort.

Item c)

Log-linearization, approximates the solution of the model in terms of the log-deviations of the variables with respect to their steady state.

Log-linear solutions are easy to read (the loglinear deviation is an approximation of the percentage deviation with respect to the steady state) and, in some circumstances, they can improve the accuracy of the solution. (Fernández-Villaverde et al. (2016)). Since many problems involving DSGE models will result in small perturbations arroud the steady state and the log-linearization can be applied to DSGE models.

Item d)

The log-linearization technique exploits the idea of certainty equivalence.

Certainty equivalence has several drawbacks. First, under certainty equivalence, only first moments matter. Thus if higher-order moments are important in the theory, the log-linearization technique will sweep these effects away.

Second, the approximated solution generated under certainty equivalence cannot generate any risk premia for assets, a strongly counterfactual prediction.

Third, certainty equivalence prevents researchers from analyzing the consequences of changes in volatility

Item e)

As stated by Fernández-Villaverde et al. (2016) we have that: Linearization and, more generally, perturbation, can be performed in the level of the state variables or after applying some change of variables to any (or all) the variables of the model. Log-linearization, on the other hand, approximates the solution of the model in terms of the log-deviations of the variables with respect to their steady state.

Item f)

Uhlig proposes a simpler method for finding log-linear approximations, one which does not require taking explicit derivatives, if some simple rules are followed. The result is a linear model expressed in terms of log differences of the variables.

The main idea is to substitute a variable x by $xe^{\hat{x}}$ where $\hat{x} = \ln\left(\frac{x}{\overline{x}}\right)$ represents the log-deviation with respect to the steady state.

Then the equation is linearized with respect to $\hat{x}$. Using the above-mentioned rules simplifies the process, as the substitutions are direct and mostly mechanical.

Item g)

Consider the following dynamic nonlinear rational expectations model composed of several variables (endogenous and exogenous) and several equations. In compact form, the model is written as:

$$\mathbb{E}_t\left[f(y_{t+1}^{+}, y_t, y_{t-1}^{-}, u_t)\right] = 0 \tag{1}$$

where y_t is the vector of endogenous variables, y_{t+1}^{+} (resp. y_{t-1}^{-} is the subset of variables of y_t that appear with a lead (resp. a lag), and u_t is the vector of exogenous variables.

Finding the deterministic steady state involves the resolution of a multivariate nonlinear system. Then, finding the rational expectation solution of the model means finding the policy functions (also known as decision rules), which give current endogenous variables as a function of state variables:

$$y_t = g(y_{t-1}^{-}, u_t) \tag{2}$$

The function g is characterized by the following functional equation:

$$\mathbb{E}_t\left[f[g^{+}(g^{-}(y_{t-1}^{-}), u_{t+1}), g(y_{t-1}^{-}, u_t), y_{t-1}^{-}, u_t]\right] = 0 \tag{3}$$

where g^{+} (resp. g^{-}) is the restriction of g to forward (resp. backward) endogenous variables.

In the general case, this functional equation cannot be solved exactly, and one has to resort to numerical techniques to get an approximated solution.

Dynare uses several numerical algorithms to solve the linear approximation to equation (3)

h)

Consider the following linear system:

$$\begin{bmatrix} X_{t+1} \\ P_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \gamma Z_t \tag{4}$$

where X_t is a vector of predetermined variables, P_t is a vector of non-predetermined variables Z_t is a vector of exogenous variables

$$P_{t+1} = \mathbb{E}\left[P_{t+1}|\Omega_t\right] \tag{5}$$

where Ω_t is the information set at t

$$\forall t \exists \overline{Z_t} \in \mathbb{R}^k, \quad \theta_t \in \mathbb{R} \text{ such that} \tag{6}$$

$$-(1+i)^{\theta_t}\overline{Z_t} \leq \mathbb{E}\left[Z_{t+i}|\Omega_t\right] \leq (1+i)^{\theta_t}\overline{Z_t} \tag{7}$$

The Blanchard and Kahn (1980) conditions can be written as:

1. Uniqueness: If the number of eigenvalues of A outside the unit circle is equal to the number of non-predetermined variables, then there is unique solution.
2. Multiplicity: If the number of eigenvalues of A outside the unit circle is less than the number of non-predetermined variables, there is an infinity of solutions.
3. No stable solution: If the number of eigenvalues of A outside the unit circle exceeds the number of non-predetermined variables, there is no solution satisfying (4), (5), (7) and the non-explosion condition.

i)

First lets define what is the representative agent theoretical construct.

Definition: Consider a Financial market equilibrium (x, z, p, q) of an economy populated by $i = 1, \ldots, I$ agents with preferences $u^i : X \to \mathbb{R}$ and endowments ω^i. A Representative agent for this economy is an agent with preferences $U^R : X \to \mathbb{R}$ and endowment ω^R such that the Financial market equilibrium of an associated economy with the Representative agent as the only agent has prices (p, q).

Therefore, if markets are complete and preferences $U^i(x^i)$ are von Neumann-Morgernstern then we can apply the representative agent theoretical construct. Let (x, z, p, q) be a Financial markets equilibrium. Then,

$$\omega^R = \sum_{i \in I} \omega^i$$

$$U^R(x) = \max_{(x^i)_{i \in I}} \left\{ \sum_{i \in I} \lambda^i u^i(x^i) \right\} \qquad \text{s.t. } x = \sum_{i \in I} x^i$$

where $(\lambda_i)_{i \in I}$ is a Negishi weight, constitute a representative agent.

proof: Consider a Financial market equilibrium (x, z, p, q). By complete markets, the First welfare theorem holds and x^* is a Pareto optimal allocation. Therefore, there exist some weights that make x the solution to the planner's problem.

Q.2) RBC Model

Suppose the planner maximize:

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}, \qquad 0 < \beta < 1, \qquad \sigma > 0 \tag{8}$$

$$\text{s.t.} \qquad c_t + k_{t+1} = z_t k_t^{\alpha} + (1-\delta) k_t, \qquad 0 < \alpha, \delta < 1 \tag{9}$$

$$\log z_t = \phi_1 \log z_{t-1} + \phi_2 \log z_{t-2} + \varepsilon_t \tag{10}$$

a) FOC

$$L = \sum_{t=0}^{\infty} \beta^t E_0 \left(\frac{c_t^{1-\sigma}}{1-\sigma} + \lambda_t (z_t k_t^{\alpha} - c_t + (1-\delta) k_t - k_{t+1}) \right) \tag{11}$$

$$
\begin{aligned}
[c_t]: \quad & c_t^{-\sigma} - \lambda_t = 0 \Rightarrow c_t^{-\sigma} = \lambda_t \Rightarrow \lambda_{t+1} = c_{t+1}^{-\sigma} & (12)\\
[k_{t+1}]: \quad & \beta E_t \lambda_{t+1}(\alpha z_{t+1} k_{t+1}^{\alpha-1} + (1-\delta)) - \lambda_t = 0 & \\
\Rightarrow \quad & \lambda_t = \beta E_t \lambda_{t+1}(\alpha z_{t+1} k_{t+1}^{\alpha-1} + (1-\delta)) & (13)
\end{aligned}
$$

Combining (12) and (13):

$$c_t^{-\sigma} = \beta E_t c_{t+1}^{-\sigma}(\alpha z_{t+1} k_{t+1}^{\alpha-1} + (1-\delta)) \tag{14}$$

Deriving in relation to λ_t:

$$[\lambda_t] : z_t k_t^{\alpha} - c_t + (1-\delta)k_t = k_{t+1} \tag{15}$$

The solution of the model is given by the following equations: (14), (15), (10)

b)

In the steady-state we have:

$$
\begin{aligned}
k_{t+1} = k_t = k\\
c_{t+1} = c_t = c\\
z_{t+1} = z_t = 1
\end{aligned}
$$

Using (14), (15) we get:

$$
\begin{aligned}
c^{-\sigma} &= \beta c^{-\sigma}(\alpha k^{\alpha-1} + (1-\delta)) & (16)\\
k &= k^{\alpha} - c + (1-\delta)k & (17)
\end{aligned}
$$

Simplifying (16):

$$
\begin{aligned}
\frac{1}{\beta} &= (\alpha k^{\alpha-1} + 1 - \delta)\\
k^{\alpha-1} &= \frac{1}{\alpha\beta} - \frac{(1-\delta)}{\alpha}\\
k &= \left[\frac{1}{\alpha\beta} - \frac{(1-\delta)}{\alpha}\right]^{\frac{1}{\alpha-1}}\\
k &= \left[\frac{\alpha\beta}{1-\beta(1-\delta)}\right]^{\frac{1}{1-\alpha}}\\
k &= \left[\frac{\alpha}{\frac{1}{\beta} - (1-\delta)}\right]^{\frac{1}{1-\alpha}} & (18)
\end{aligned}
$$

Replacing (18) into (17):

$$
\begin{aligned}
c &= k^{\alpha} - k + k - \delta k\\
c &= k^{\alpha} - \delta k & (19)
\end{aligned}
$$

The values of steady-state found by Dynare are:

Steady-State	
c	0.301697
k	1.53234
y	0.459702
i	-1.46339
z	0

c)

We get the standard deviation:

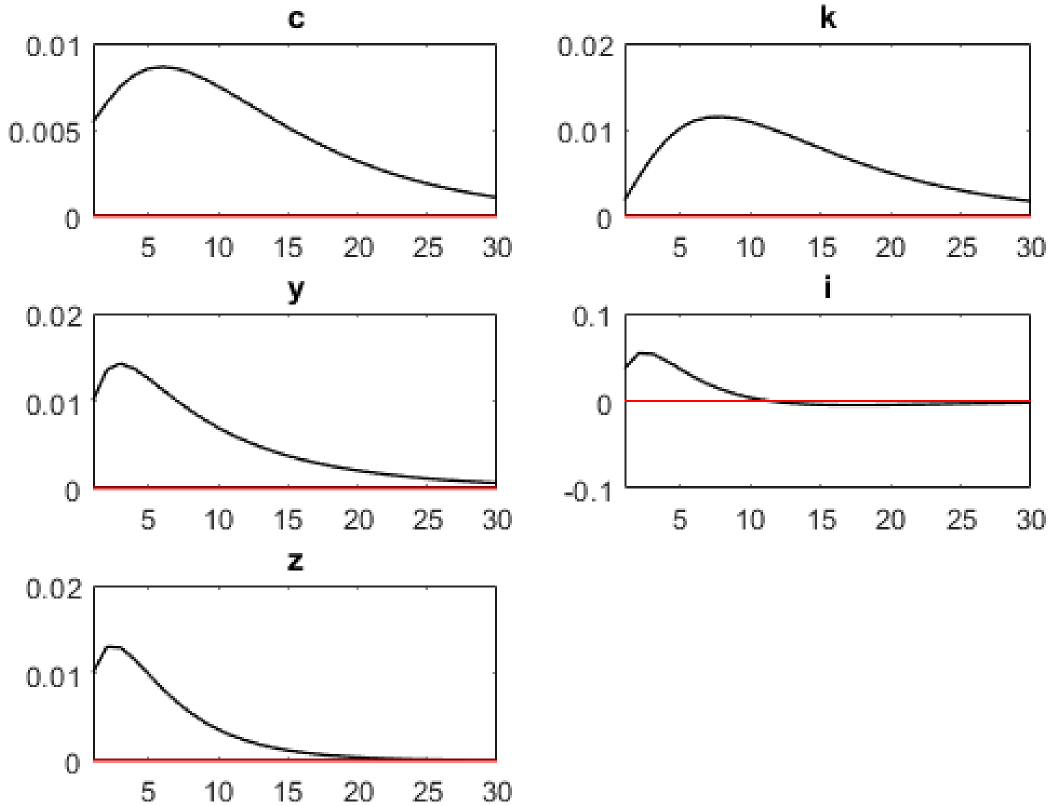
VARIABLE	MEAN	STD. DEV.	VARIANCE
c	0.3017	0.0302	0.0009
k	1.5323	0.0399	0.0016
y	0.4597	0.0377	0.0014
i	-1.4634	0.1101	0.0121
z	0.0000	0.0294	0.0009

and the correlation matrix:

Variables	c	k	y	i	z
c	1.0000	0.9765	0.9392	0.5957	0.8192
k	0.9765	1.0000	0.8491	0.4225	0.6834
y	0.9392	0.8491	1.0000	0.8353	0.9659
i	0.5957	0.4225	0.8353	1.0000	0.9478
z	0.8192	0.6834	0.9659	0.9478	1.0000

Looking at the correlation matrix, we can see that capital and consumption are the most closely correlated, followed by output and productivity, then investment and productivity, then consumption and output. Now, looking at variance, investment is by far the most volatile, then capital, then output, then consumption. Given the planner's consumption-smoothing motive, it is intuitive that consumption is smoother than output.

d)



e)

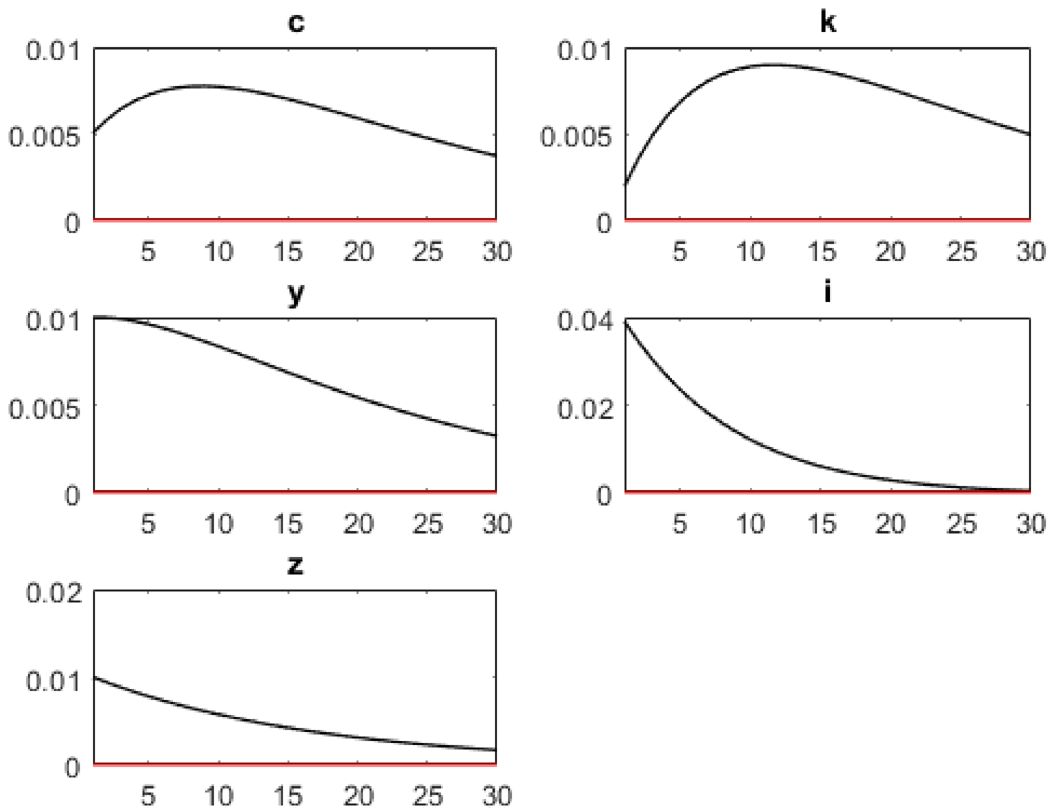
Now we use an AR(1) process for productivity. The steady-state is unchanged but we get different fluctuations around steady state.

VARIABLE	MEAN	STD. DEV.	VARIANCE
c	0.3017	0.0362	0.0013
k	1.5323	0.0423	0.0018
y	0.4597	0.0400	0.0016
i	-1.4634	0.0815	0.0066
z	0.0000	0.0293	0.0009

with the correlation matrix:

Variables	c	k	y	i	z
c	1.0000	0.9852	0.9764	0.6883	0.9145
k	0.9852	1.0000	0.9250	0.5539	0.8316
y	0.9764	0.9250	1.0000	0.8287	0.9803
i	0.6883	0.5539	0.8287	1.0000	0.9230
z	0.9145	0.8316	0.9803	0.9230	1.0000

Note that the long-run volatility of productivity that we feed in is unchanged (this was by the "right" choice of $\phi_t = 0.94$). So any changes come not from the long-run volatility but from temporal composition of that volatility. In particular, we now find that consumption is more correlated than with the AR(2). Investment is less volatile than with the AR(2) while consumption is more volatile. The AR(2) imparts some more predictable low-frequency volatility that the planner can smooth out. Switching the process to an equally volatile AR(1) prevents the planner from taking advantage of this greater predictability.



Q.3)

a)

The household will chooses c_t, h_t and k_{t+1} each period.

b)

The household's problem:

$$\max \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t U\left(c_t, h_t\right) \right] \tag{20}$$

Subject to:

$$k_{t+1} + c_t \leq r_t k_t + w_t h_t + (1-\delta) k_t$$

Constructing the Lagrangean we have that:

$$\mathcal{L}_t = \mathbb{E}_t \left[\sum_{t=0}^{\infty} \beta^t \left[U\left(c_t, h_t\right)\right] + \sum_{t=0}^{\infty} \lambda_t \left[r_t k_t + w_t h_t + (1-\delta) k_t - k_{t+1} - c_t\right] \right] \tag{21}$$

First Order Conditions:

$$[c_t] : \beta^t U_{c_t}\left(c_t, h_t\right) + \lambda_t (-1) \beta^t = 0 \tag{22}$$
$$U_{c_t}\left(c_t, h_t\right) = \lambda_t \tag{23}$$

$$[h_t] : \beta^t U_{h_t}\left(c_t, h_t\right) + \lambda_t \beta^t w_t = 0 \tag{24}$$
$$U_{h_t}\left(c_t, h_t\right) = -\lambda_t w_t \tag{25}$$

$$[k_{t+1}] : \beta^t \lambda_t (-1) + \mathbb{E}_t \left[\beta^{t+1} \lambda_{t+1} \left(r_{t+1} + (1-\delta)\right)\right] = 0 \tag{26}$$
$$\mathbb{E}_t \left[\beta \lambda_{t+1} \left(r_{t+1} + (1-\delta)\right)\right] = \lambda_t \tag{27}$$

Combining (23), (25) and (27) yields:

$$U_{c_t}\left(c_t, h_t\right) = \beta \mathbb{E}_t \left[\partial_{c_{t+1}} U\left(c_{t+1}, h_{t+1}\right)\left(1 + r_{t+1} - \delta\right)\right] \tag{28}$$

$$-U_{h_t}(c_t, h_t) = U_{c_t}(c_t, h_t) w_t \tag{29}$$

Equation (28) is the Euler Equation: the optimal choice of consumption c_t at any period is such that the marginal rate of substitution between present and future consumption must be equal to the relative price of consumption today $(r_{t+1} + (1 - \delta))$.

Equation (29) is the intra-temporal condition on allocation between consumption and labor: the marginal rate of substitution between work and consumption must be equal to the wage (i.e. relative price of leisure).

c)

The production function is:

$$y_t(k_t, h_t) = A_t k_t^{\theta} h_t^{1-\theta} \tag{30}$$

We shall prove it is homogeneous of degree 1. Let λ be a real number, then:

$$\begin{aligned} y_t(\lambda k_t, \lambda h_t) &= A_t(\lambda k_t)^{\theta} (\lambda h_t)^{1-\theta} = A_t \lambda^{\theta} \lambda^{1-\theta} k_t^{\theta} h_t^{1-\theta} \\ &= \lambda A_t k_t^{\theta} h_t^{1-\theta} = \lambda y_t \end{aligned}$$

Thus, we have proved that the production function is homogeneous of degree 1.

d)

The firm's problem:

$$\max_{y_t} \{y_t - r_t k_t - w_t h_t\}, \tag{31}$$

subject to:

$$y_t = A_t k_t^{\theta} h_t^{1-\theta}$$

First Order Conditions:

$$[h_t] : \frac{(1-\theta) A_t k_t^{\theta} h_t^{1-\theta}}{h_t} - w_t = 0 \tag{32}$$

$$\frac{(1-\theta) y_t}{h_t} = w_t \tag{33}$$

$$(1-\theta) y_t = w_t h_t \tag{34}$$

$$[k_t] : \frac{\theta A_t k_t^{\theta} h_t^{1-\theta}}{k_t} - r_t = 0 \tag{35}$$

$$\frac{\theta y_t}{k_t} = r_t \tag{36}$$

$$\theta y_t = r_t k_t \tag{37}$$

The firm's optimality conditions states that the firms hire the factor of production (both capital and labor) up to the point that the marginal gain equals to the marginal cost.

e)

In the steady state we have that $k_{t+1} = k_t = k_{ss}, h_{t+1} = h_t = h_{ss}$, and $c_{t+1} = c_t = c_{ss}$. Also, in steady state $A_{ss} = 1$ (its non-stochastic unconditional mean). We then have that the Euler Equation becomes:

$$1 = \beta\left(r_{ss} + 1 - \delta\right) \Longrightarrow r_{ss} = \frac{1}{\beta} - (1 - \delta) \tag{38}$$

Which is a result pinned down by the parameters β and δ.

f)

Notice that

$$y_t = (1 - \theta)y_t + \theta y_t$$

Then from equations (34) and (37) we have that

$$y_t = (1 - \theta)y_t + \theta y_t = w_t h_t + r_t k_t$$

g)

From the First Order Conditions of the firm's problem we have that:

$$r_{ss} k_{ss} = \theta y_{ss} = \theta A k_{ss}^{\theta} h_{ss}^{1-\theta} \tag{39}$$

Plug that into (38) with A normalized to unit, then:

$$\frac{k_{ss}}{h_{ss}} = \left(\frac{\theta}{\frac{1}{\beta} - (1-\delta)} \right)^{\frac{1}{1-\theta}} \tag{40}$$

h)

Since in the steady state we have $k_{t+1} = k_t = k_{ss}$. Therefore, $i_t = k_{t+1} - (1-\delta)k_t$ is equivalent to $i_{ss} = \delta k_{ss}$. Using this result in (40), we have:

$$\frac{i_{ss}}{h_{ss}} = \delta \left(\frac{\theta}{\frac{1}{\beta} - (1-\delta)} \right)^{\frac{1}{1-\theta}} \tag{41}$$

This can be adapted as:
$\frac{i_{ss}}{h_{ss}} = \delta \frac{k_{ss}}{h_{ss}}$.

i)

The economy-wide resource constraint is correspondent to $y_{ss} = c_{ss} + i_{ss}$ in terms of y, c and i.

j)

Using the following resource constraint per hours(h) and the steady state value of y/h (y_{ss} is equivalent to production function in the steady state with technology normalized to one):

$$\frac{y_{ss} - i_{ss}}{h_{ss}} = \frac{c_{ss}}{h_{ss}} \tag{42}$$

Using $\frac{i_{ss}}{h_{ss}}$ from the later equation (42), we have the following equation for a steady state expression for c/h:

$$\frac{c_{ss}}{h_{ss}} = \left(\frac{k_{ss}}{h_{ss}}\right)^{\theta} - \delta\left(\frac{\theta}{\frac{1}{\beta} - (1-\delta)}\right)^{\frac{1}{1-\theta}} \tag{43}$$

k)

From now on, we assume $\varphi = 1$ and $\sigma = 1$. Hence:

$$u\left(c_t, h_t\right) = \ln\left(c_t\right) + B\ln\left(1 - h_t\right)$$

From the household's problem we have that:

$$-U_{h_t}\left(c_t, h_t\right) = U_{c_t}\left(c_t, h_t\right) w_t \tag{44}$$

Therefore, we can write:

$$\frac{B}{1-h_t} = \frac{1}{c_t}(1-\theta)\left(\frac{k_t}{h_t}\right)^{\theta} \tag{45}$$

In steady state:

$$Bc_{ss} = A_{ss}(1-\theta)\left(\frac{k_{ss}}{h_{ss}}\right)^{\theta}(1-h_{ss}) \tag{46}$$

Combining (46) and (43) leads to:

$$h_{ss} = \frac{(1-\theta)}{B\left[1 - \delta\left(\frac{\theta}{\frac{1}{\beta}-(1-\delta)}\right)\right] + (1-\theta)} \tag{47}$$

l)

$$
\begin{aligned}
h_{ss} &= \frac{(1-\theta)}{B\left[1-\delta\left(\frac{\theta}{\frac{1}{\beta}-(1-\delta)}\right)\right]+(1-\theta)} \\
k_{ss} &= \left(\frac{\theta}{\frac{1}{\beta}-(1-\delta)}\right)^{\frac{1}{1-\theta}} h_{ss} \\
i_{ss} &= \delta\left(\frac{\theta}{\frac{1}{\beta}-(1-\delta)}\right)^{\frac{1}{1-\theta}} h_{ss} \\
r_{ss} &= \frac{1}{\beta}-(1-\delta) \\
c_{ss} &= \left(\frac{k_{ss}}{h_{ss}}\right)^{\theta} - \delta\left(\frac{\theta}{\frac{1}{\beta}-(1-\delta)}\right)^{\frac{1}{1-\theta}} h_{ss} \\
w_{ss} &= (1-\theta)\frac{y_{ss}}{h_{ss}} \\
y_{ss} &= k_{ss}^{\theta} h_{ss}^{1-\theta}
\end{aligned}
$$

m)

See item K.

n)

Codes were sent in attached file (see: steady_state_Q3.m).

o)

Given the vector of parameters of the model, the steady state values are as follows:

$$\begin{aligned} A_{ss} &= 1 \\ r_{ss} &= 0.0351 \\ h_{ss} &= 0.3335 \\ k_{ss} &= 12.6698 \\ i_{ss} &= 0.3167 \\ w_{ss} &= 2.3706 \\ y_{ss} &= 1.2353 \\ c_{ss} &= 0.9186 \end{aligned}$$

p)

For $\rho_a \leq 1$ the steady-state does not change while ρ_a and σ vary. This not remain the same if TFP does not hold a stationary process. In that case, the economy would have an explosive behavior and would never converge to steady state.

q)

Codes were sent in attached file (see: Q3.mod).

r)

The following impulse response functions show the expected future path of endogenous variables after a negative one standard deviation TPF shock.

Since we have nonlinear equations, there is no economic interpretation for the y-axis of the IRFs graphs. However, the IRFs correctly captures the dynamics of the model.

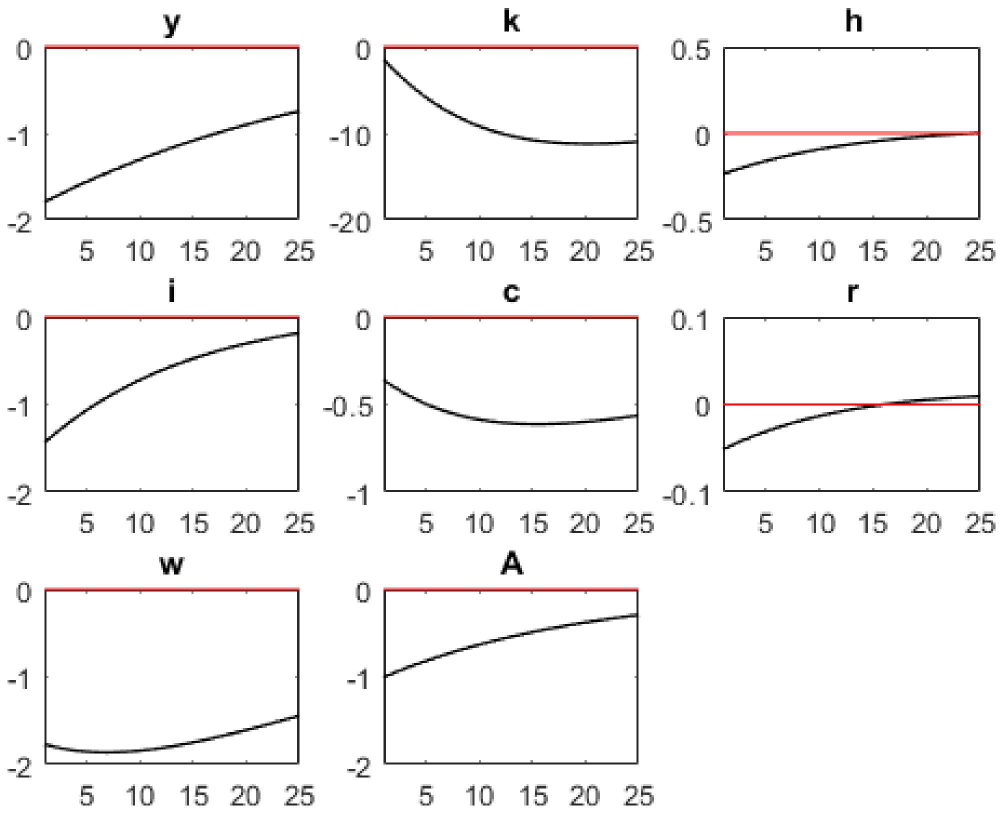


Figure 1: Negative 1 standard deviation TPF shock, with 25 periods

s)

Household's Optimality Conditions:

Since $u(c_t, h_t = \ln(c_t) + B \ln(1 - h_t)$, the optimality conditions of the household's problem are given by:

$$\frac{1}{c_t} = \beta \mathbb{E}_t \left[\frac{1}{c_{t+1}} (1 + r_{t+1} - \delta) \right] \tag{48}$$

$$\frac{B}{1 - h_t} = \frac{w_t}{c_t} \tag{49}$$

Applying log on both sides of (48):

$$-\ln(c_t) = \ln(\beta) + \ln(\mathbb{E}_t [r_{t+1} + (1 - \delta)]) - \ln(\mathbb{E}_t [c_{t+1}]) \tag{50}$$

The first-order Taylor expansion around the steady-state:

$$
\begin{aligned}
-\ln\left(c^*\right) - \frac{1}{c^*}\left(c_t - c^*\right) = {} & \ln\left(\beta\right) + \ln\left(r^* + (1-\delta)\right) + \\
& + \frac{1}{r^* + (1-\delta)}\mathbb{E}_t\left[r_{t+1} - r^*\right] + \\
& - \ln\left(c^*\right) - \frac{1}{c^*}\mathbb{E}_t\left[c_{t+1} - c^*\right]
\end{aligned} \tag{51}
$$

Since $-\ln\left(c^*\right) = \ln\left(\beta\right) + \ln\left(r^* + (1-\delta)\right) - \ln\left(c^*\right)$:

$$
-\frac{1}{c^*}\left(c_t - c^*\right) = \frac{1}{r^* + (1-\delta)}\mathbb{E}_t\left[r_{t+1} - r^*\right] - \frac{1}{c^*}\mathbb{E}_t\left[c_{t+1} - c^*\right] \tag{52}
$$

$$
-\tilde{c}_t = \frac{1}{r^* + (1-\delta)}\mathbb{E}_t\left[r_{t+1} - r^*\right] - \mathbb{E}_t\left[\tilde{c}_{t+1}\right] \tag{53}
$$

Multiplying and dividing $\mathbb{E}_t\left[r_{t+1} - r^*\right]$ by r^* and using the steady state value r^*, we get:

$$
-\tilde{c}_t = \beta r^* \mathbb{E}_t\left[\tilde{r}_{t+1}\right] - \mathbb{E}_t\left[\tilde{c}_{t+1}\right] \tag{54}
$$

$$
\tilde{c}_t - \mathbb{E}_t\left[\tilde{c}_{t+1}\right] + \beta r^* \mathbb{E}_t\left[\tilde{r}_{t+1}\right] = 0 \tag{55}
$$

Now, we log-linearize (49). Taking log on both sides of it:

$$
\ln\left(B\right) - \ln\left(1 - h_t\right) = \ln\left(w_t\right) - \ln\left(c_t\right) \tag{56}
$$

First-order Taylor expansion around the steady-state:

$$
\begin{aligned}
\ln\left(B\right) - \ln\left(1 - h^*\right) - \frac{-1}{1-h^*}\left(h_t - h^*\right) = {} & \ln\left(w^*\right) + \\
& + \frac{1}{w^*}\left(w_t - w^*\right) - \ln\left(c^*\right) - \frac{1}{c^*}\left(c_t - c^*\right)
\end{aligned} \tag{57}
$$

$$
\frac{1}{1-h^*}\left(h_t - h^*\right) = \tilde{w}_t - \tilde{c}_t \tag{58}
$$

$$
\frac{h^*}{1-h^*}\tilde{h}_t = \tilde{w}_t - \tilde{c}_t \tag{59}
$$

Firm's Optimality Conditions - Uhlig's method:

Note that:

$$
\begin{aligned}
x &= x^* e^{\tilde{x}} \\
e^{\tilde{x}} &\approx 1 + \tilde{x}
\end{aligned}
$$

Also, recall that:

$$
y_t = A_t k_t^{\theta} h_t^{1-\theta}
$$

Then:

$$
\begin{aligned}
r^* e^{\tilde{r}_t} &= \theta A^* e^{\tilde{A}_t} \frac{h^{*1-\theta} e^{(1-\theta)\tilde{h}_t}}{k^{*1-\theta} e^{(1-\theta)\tilde{k}_t}} \\
r^* e^{\tilde{r}_t} &= \theta A^* \frac{h^{*1-\theta}}{k^{*1-\theta}} e^{\tilde{A}_t} \frac{e^{(1-\theta)\tilde{h}_t}}{e^{(1-\theta)\tilde{k}_t}} \\
e^{\tilde{r}_t} &= e^{\tilde{A}_t + (1-\theta)\tilde{h}_t - (1-\theta)\tilde{k}_t}
\end{aligned}
$$

Using the fact that $e^{\tilde{x}} \approx 1 + \tilde{x}$, then:

$$
1 + \tilde{r}_t = 1 + \tilde{A}_t + (1-\theta)\tilde{h}_t - (1-\theta)\tilde{k}_t \Longrightarrow \tilde{r}_t = \tilde{A}_t + (1-\theta)\left(\tilde{h}_t - \tilde{k}_t\right)
$$

The same rationale should be applied to the other First Order Conditions of the Firm's problem:

$$
\begin{aligned}
w^* e^{\tilde{w}_t} &= (1-\theta) A^* e^{\tilde{A}_t} \frac{k^* e^{\theta \tilde{k}_t}}{h^* e^{\theta \tilde{h}_t}} \\
w^* e^{\tilde{w}_t} &= (1-\theta) A^* \frac{k^{*\theta}}{h^{*\theta}} e^{\tilde{A}_t} \frac{e^{\theta \tilde{k}_t}}{e^{\theta \tilde{h}}} \\
e^{\tilde{w}_t} &= e^{\tilde{A}_t + \theta \tilde{k}_t - \theta \tilde{h}_t} \\
1 + \tilde{w}_t &= 1 + \tilde{A}_t + \theta \tilde{k}_t - \theta \tilde{h}_t \\
\tilde{w}_t &= \tilde{A}_t + \theta\left(\tilde{k}_t - \tilde{h}_t\right)
\end{aligned}
$$

The Production Function:

$$
\begin{aligned}
y^* e^{\tilde{y}_t} &= A^* k^{*\theta} h^{*1-\theta} e^{\tilde{A}_t + \theta \tilde{k}_t + (1-\theta)\tilde{A}_t} \\
e^{\tilde{y}_t} &= e^{\tilde{A}_t + \theta \tilde{k}_t + (1-\theta)\tilde{h}_t} \\
\tilde{y}_t &= \tilde{A}_t + \theta \tilde{k}_t + (1-\theta)\tilde{h}_t
\end{aligned}
$$

The equations below are already linear, we are simply going to express them in terms of the deviations from the steady-state. By Uhlig's Method approach:

The Resource Constraint:

$$
\begin{aligned}
&i^* e^{\tilde{c}_t} + i^* e^{\tilde{i}_t} = y^* e^{\tilde{y}_t} \Longrightarrow c^* \left(1 + \tilde{c}_t\right) + i^* \left(1 + \tilde{i}_t\right) = y^* \left(1 + \tilde{y}_t\right) \\
&c^* + c^* \tilde{c}_t + i^* + i^* \tilde{i}_t = y^* + y^* \tilde{y}_t \Longrightarrow c^* \tilde{c}_t + i^* \tilde{i}_t = y^* \tilde{y}_t
\end{aligned}
$$

The Law of Motion of Capital:

$$
\begin{aligned}
i^* e^{\tilde{i}_t} &= k^* e^{\tilde{k_{t+1}}} - (1-\delta) k^* e^{\tilde{k}_t} \\
i^* + i^* \tilde{i}_t &= k^* + k^* \tilde{k}_{t+1} - k^* - k^* \tilde{k}_t + \delta k^* + \delta k^* \tilde{k}_t \\
i^* \tilde{i}_t &= k^* \tilde{k}_{t+1} - k^* \tilde{k}_t + \delta k^* \tilde{k}_t \\
i^* \tilde{i}_t &= k^* \left[\tilde{k}_{t+1} - (1-\delta) \tilde{k}_t\right]
\end{aligned}
$$

The log-linearized equations are summarized below:

$$\tilde{c}_t - \mathbb{E}_t\left[\tilde{c}_{t+1}\right] + \beta r^* \mathbb{E}_t\left[\tilde{r}_{t+1}\right] = 0 \tag{60}$$

$$\frac{h^*}{1-h^*} \tilde{h}_t - \tilde{w}_t + \tilde{c}_t = 0 \tag{61}$$

$$\tilde{A}_t + (1-\theta)\left(\tilde{h}_t - \tilde{k}_t\right) - \tilde{r}_t = 0 \tag{62}$$

$$\tilde{A}_t + \theta\left(\tilde{k}_t - \tilde{h}_t\right) - \tilde{w}_t = 0 \tag{63}$$

$$\tilde{y}_t - \tilde{A}_t - \theta \tilde{k}_t - (1-\theta) \tilde{h}_t = 0 \tag{64}$$

$$c^* \tilde{c}_t + i^* \tilde{i}_t - y^* \tilde{y}_t = 0 \tag{65}$$

$$i^* \tilde{i}_t - k^* \left[\tilde{k}_{t+1} - (1-\delta) \tilde{k}_t\right] = 0 \tag{66}$$

t)

Codes were sent in attached file (see: Q3_linear.mod).

u)

In the following plots we have the the IRFs for a negative 1 standard deviation TPF shock, with 24 periods, using the log-linearization which means percent change from steady state. Therefore, the "$y - axis$" in the IRFs graph indicates how much the variables react to a shock of a negative standard deviation in TFP in percent steady state value deviation.

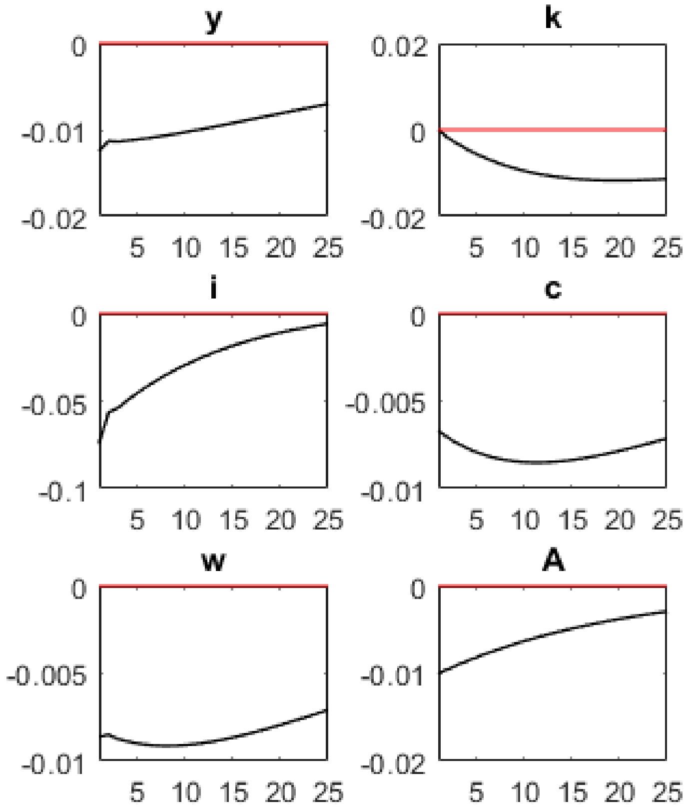


Figure 2: Negative 1 standard deviation TPF shock, with 24 periods

Q.4)

a)

The representative household maximize the following objective function:

$$\max_{\{C_t, N_t, B_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t) \tag{67}$$

Where (67) is subject to the following budget constraint:

$$P_tC_t + Q_tB_t = B_{t-1} + W_tN_t - T_t \qquad \forall t \tag{68}$$

Utility will take the form:

$$U(C_t, N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \tag{69}$$

Writing the Lagrangian we get:

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t) + \lambda_t(B_{t-1} + W_tN_t - T_t - P_tC_t - Q_tB_t) \tag{70}$$

FOC:

$$[C_t] : c^{-\sigma} = \lambda_t P_t \implies \lambda_t = \frac{c^{-\sigma}}{P_t} \tag{71}$$

$$[N_t] : N_t^{\varphi} = \lambda_t W_t \tag{72}$$

$$[B_t] : -\lambda_t Q_t + \beta \mathbb{E}_t\left[\lambda_{t+1}\right] = 0 \implies \lambda_t Q_t = \beta \mathbb{E}_t[\lambda_{t+1}] \tag{73}$$

From Eqs. (71) and (72):

$$\frac{W_t}{P_t} = C_t^{\sigma} N_t^{\varphi} \tag{74}$$

The (74) represent the intra-temporal choice. the household chooses between labor and consumption.

From Eqs.(71) and (73):

$$1 = \beta \mathbb{E}_t \left(\left[\frac{C_{t+1}}{C_t} \right]^{-\sigma} \frac{P_t}{P_{t+1}} \frac{1}{Q_t} \right) \tag{75}$$

The equation (75) represents the inter-temporal choice. The household chooses between consumption today and consumption in the future.

b)

To explain how one can provide microfoundations we will make use of the model of Sidrauski, wheere we consider Money in the Utiliy function.
Household maximizes:

$$\max_{\{C_t,N_t,B_t,\frac{M_t}{P_t}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, \frac{M_t}{P_t}, N_t\right) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} + \frac{(M_t/P_t)^{1-\nu}}{1-\nu} \tag{76}$$

Subject to

$$P_tC_t + Q_tB_t + M_t = B_{t-1} + W_tN_t - T_t + M_{t-1} \tag{77}$$

The Lagrangian is then:

$$\begin{aligned} \mathcal{L} = \mathbb{E}_0 \Bigg[& \sum_{t=0}^{\infty} \beta^t U\left(C_t, \frac{M_t}{P_t}, N_t\right) + \\ & + \mu_t \left(\frac{B_{t-1}}{P_t} + \frac{W_tN_t}{P_t} - \frac{T_t}{P_t} + \frac{M_{t-1}}{P_t}\frac{P_{t-1}}{P_{t-1}} - C_t - \frac{Q_tB_t}{P_t} - \frac{M_t}{P_t} \right) \Bigg] \end{aligned} \tag{78}$$

FOC:

$$[C_t] : U_{Ct} = \mu_t P_t \implies \frac{U_{Ct}}{P_t} = \mu_t \tag{79}$$

$$[N_t] : U_{Nt} + \mu_t W_t = 0 \tag{80}$$

$$[B_t] : -\mu_t Q_t + \beta \mathbb{E}_t[\mu_{t+1}] = 0 \tag{81}$$

$$\left[\frac{M_t}{P_t}\right] : U_{Mt} - \mu_t P_t + \mathbb{E}_t[\mu_{t+1}P_t] = 0 \tag{82}$$

From Eqs. (79) and (82)

$$\frac{U_{Mt}}{U_{Ct}} = \frac{\mu_t P_t - \mathbb{E}_t[\mu_{t+1}P_t]}{\mu_t P_t} \tag{83}$$

Equation (83) is the Money Demand. Then replacing by the utility function, we have:

$$\frac{M_t}{P_t} = C_t^{\sigma/\nu} \left(1 - \exp\{i_t\}\right)^{1/\nu} \tag{84}$$

c)

Firms problem is:

$$\max_{Y_t} \{P_t Y_t - W_t N_t\}$$
$$\text{Subject to:} \quad Y_t = A_t N_t^{1-\alpha}$$

FOC:

$$\frac{W_t}{P_t} = (1-\alpha) A_t N_t^{-\alpha} \tag{85}$$

d)

$$\min_{N_t} \left\{ \frac{W_t N_t}{P_t} \right\}$$
$$\text{Subject to:} \quad \bar{Y} = A_t N_t^{1-\alpha}$$

Lagrangean:

$$\mathcal{L} = \frac{W_t N_t}{P_t} - \lambda (A_t N_t^{1-\alpha}) \tag{86}$$
$$\tag{87}$$

FOC:

$$[N_t] : \frac{W_t t}{P_t} - \lambda (1-\alpha) A_t N_t^{-\alpha} = 0 \tag{88}$$
$$\tag{89}$$

$$\frac{W_t}{P_t} = \lambda (1-\alpha) A_t N_t^{-\alpha}$$

So, the real marginal cost is equal to 1 ($\lambda = 1$) and

$$\frac{W_t}{P_t} = (1-\alpha) A_t N_t^{-\alpha}$$

e)

Let R_t be the nominal interest rate. Then we have the following equation for the real interest rate (Fisher's equation)

$$i_t = R_t \mathbb{E}_t(\pi_{t+1}) \tag{90}$$

f)

The model equilibrium conditions are given by the following

$$\frac{W_t}{P_t} = C_t^{\sigma} N_t^{\varphi} \tag{91}$$

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left[C_{t+1}^{-\sigma} \frac{R_t}{\pi_{t+1}} \right] \tag{92}$$

$$Y_t = C_t \tag{93}$$

$$Y_t = A_t N_t^{1-\alpha} \tag{94}$$

$$\frac{W_t}{P_t} = (1-\alpha) A_t N_t^{-\alpha} \tag{95}$$

$$\ln(A_t) = \rho_a \ln(A_{t-1}) + \epsilon_t^a \tag{96}$$

$$R_t = \frac{1}{\beta} \pi_t^{\phi_\pi} e^{v_t} \tag{97}$$

$$\nu_t = \rho_m \nu_{t-1} + \epsilon_t^m \tag{98}$$

$$M_t = Y_t R_t^{-\eta} \tag{99}$$

$$i_t = R_t \mathbb{E}_t [\pi_{t+1}] \tag{100}$$

Where R_t is the nominal interest rate, while i_t is the real interest rate.

g)

The steady state is then:

$$
\begin{align}
R &= \frac{1}{\beta} \tag{101}\\
N &= [1-\alpha]^{\frac{1}{\sigma(1-\alpha)+\varphi+\alpha}} \tag{102}\\
Y &= C \tag{103}\\
Y &= (1-\alpha)^{\frac{1-\alpha}{\sigma(1-\alpha)+\varphi+\alpha}} \tag{104}\\
w &= (1-\alpha)^{\frac{\sigma(1-\alpha)+\varphi}{\sigma(1-\alpha)+\varphi+\alpha}} \tag{105}\\
M &= \left[(1-\alpha)^{\frac{1-\alpha}{\sigma(1-\alpha)+\varphi+\alpha}}\right]\left(\frac{1}{\beta}\right)^{-\eta} \tag{106}\\
i &= \frac{1}{\beta} \tag{107}\\
\pi &= 1 \tag{108}\\
A &= 1 \tag{109}\\
v &= 0 \tag{110}
\end{align}
$$

$$
\begin{aligned}
R &= 1.010101\\
N &= 0.9531843\\
Y &= 0.9646786\\
C &= 0.9646786\\
w &= 0.7590442\\
M &= 0.928811\\
i &= 1.010101
\end{aligned}
$$

h)

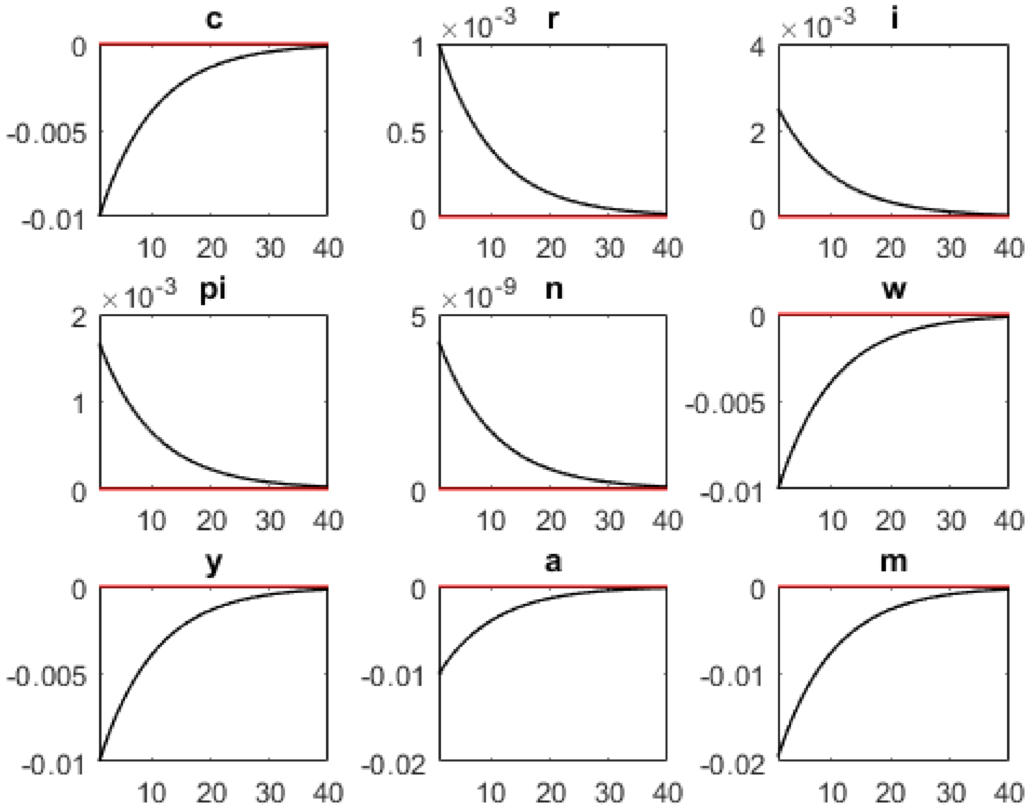


Figure 3: Response of the endogenous variables with respect to an one standard deviation negative innovation in the technological level

The negative technological shock causes a decrease in the output, as well as consumption, real wage, and real balances. on the other hand Real interest rate, nominal rate and inflation respond with an increase.

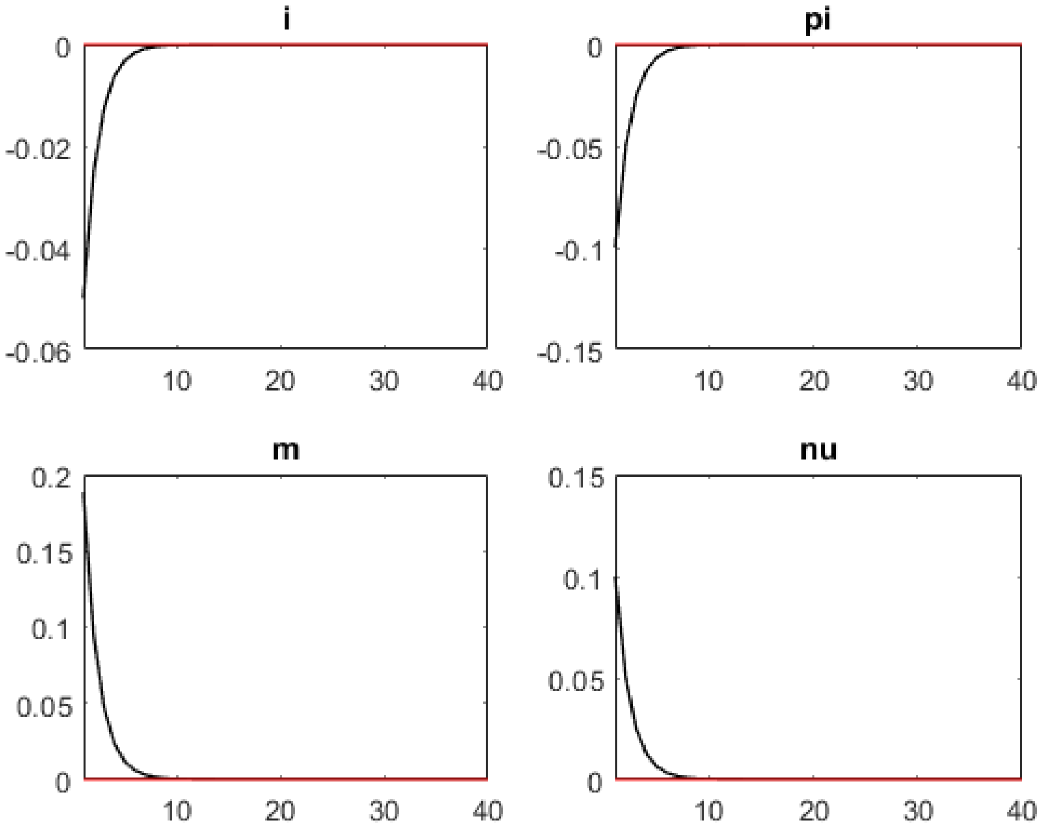


Figure 4: Response of the endogenous variables with respect to a raise in the nominal interest rate.

Since we have flexible prices then equilibrium time paths of output, labor, consumption, the real wage, and the real rate of interest can be determined by a system of equations. Thus, realizations of the monetary disturbances have no effect on output when prices are flexible.

In this version of the "money in the utility model" we have that real variables such as output, consumption, investment, and the real interest rate are determined independently of both the money supply process and money demand factors.

i)

Log-linear Equations:

$$\begin{aligned}
C_t^{-\sigma} &= \beta\mathbb{E}_t\left(C_{t+1}^{-\sigma}\frac{R_t}{\pi_{t+1}}\right) \\
e^{-\sigma\tilde{c}_t} &= \beta\mathbb{E}_t\left[e^{-\sigma\tilde{c}_{t+1}}e^{-\sigma\tilde{r}_t}e^{-\sigma\tilde{\pi}_{t+1}}\right] \\
\frac{1}{\beta} &= \mathbb{E}_t\left[e^{\sigma\tilde{c}_t}e^{-\sigma\tilde{c}_{t+1}}e^{-\sigma\tilde{r}_t}e^{-\sigma\tilde{\pi}_{t+1}}\right] \\
\frac{1}{\beta} &= \mathbb{E}_t\left[(1+\sigma\tilde{c}_t)(1-\sigma\tilde{c}_{t+1})(1+\tilde{r}_t)(1-\tilde{\pi}_{t+1})\right] \\
\frac{1}{\beta} &= \mathbb{E}_t\left[1+\sigma\tilde{c}_t-\sigma\tilde{c}_{t+1}+\tilde{r}_t-\tilde{\pi}_{t+1}\right] \\
\frac{1}{\beta}-1 &= \sigma\tilde{c}_t-\sigma\mathbb{E}_t[\tilde{c}_{t+1}]+\tilde{r}_t-\mathbb{E}_t[\tilde{\pi}_{t+1}]
\end{aligned}$$

$$\tilde{c}_t = \mathbb{E}_t[\tilde{c}_{t+1}] - \frac{1}{\sigma}\left(\tilde{r}_t - \mathbb{E}_t[\tilde{\pi}_{t+1}] - \rho\right) \tag{111}$$

$$\begin{aligned}
\frac{W_t}{P_t} &= C_t^{\sigma}N_t^{\varphi} \\
\ln(W_t) - \ln(P_t) &= \sigma\ln(C_t) + \varphi\ln(N_t)
\end{aligned}$$

In the steady state

$$\begin{aligned}
\ln(W) - \ln(P) &= \sigma\ln(C) + \varphi\ln(N) \\
(\ln(W_t) - \ln(W)) - (\ln(P_t) - \ln(P)) &= \sigma(\ln(C_t) - \ln(C) + \varphi(\ln(N_t) - \ln(N))
\end{aligned}$$

$$\tilde{w}_t - \tilde{p}_t = \sigma\tilde{c}_t + \varphi\tilde{n}_t \tag{112}$$

$$Y_t = C_t \implies \tilde{y}_t = \tilde{c}_t \tag{113}$$

$$\begin{aligned}
Y_t &= A_tN_t^{1-\alpha} \\
\ln(Y_t) &= \ln(A_t) + (1-\alpha)\ln(N_t)
\end{aligned}$$

In the steady state

$$\ln(Y) = \ln(A) + (1-\alpha)\ln(N)$$
$$(\ln(Y_t) - \ln(Y)) = (\ln(A_t) - \ln(A)) + (1-\alpha)(\ln(N_t) - \ln(N))$$

$$\tilde{y}_t = \tilde{a}_t + (1-\alpha)\tilde{n}_t \tag{114}$$

$$\frac{W_t}{P_t} = (1-\alpha)A_t N_t^{-\alpha}$$
$$\ln(W_t) - \ln(P_t) = \ln(1-\alpha) + \ln(A_t) - \alpha\ln(N_t)$$

In the steady state

$$\ln(W) - \ln(P) = \ln(1-\alpha) + \ln(A) - \alpha\ln(N)$$
$$(\ln(W_t) - \ln(W)) - (\ln(P_t) - \ln(P)) = (\ln(A_t) - \ln(A)) - \alpha(\ln(N_t) - \ln(N))$$

$$\tilde{w}_t - \tilde{p}_t = \tilde{a}_t - \alpha\tilde{n}_t \tag{115}$$

$$M_t = Y_t R_t^{-\eta}$$
$$\ln(M_t) = \ln(Y_t) - \eta\ln(R_t)$$

In the steady state

$$\ln(M) = \ln(Y) - \eta\ln(R)$$
$$(\ln(M_t) - \ln(M)) = (\ln(Y_t) - \ln(Y)) - \eta(\ln(R_t) - \ln(R))$$

$$\tilde{m}_t = \tilde{y}_t - \eta\tilde{r}_t \tag{116}$$

$$R_t = \frac{1}{\beta}\pi_t^{\phi_\pi}\varepsilon_t^v$$
$$\ln(R_t) = \ln\left(\frac{1}{\beta}\right) + \phi_\pi\ln(\pi_t) + v_t$$

In the steady state

$$\begin{aligned}
\ln(R) &= \ln\left(\frac{1}{\beta}\right) + \phi_\pi \ln(\pi) \\
(\ln(R_t) - \ln(R)) &= \phi_\pi(\ln(\pi_t) - \ln(\pi)) + v_t
\end{aligned}$$

$$\tilde{r}_t = \phi_\pi \tilde{\pi}_t + v_t \tag{117}$$

$$\ln(A_t) = \rho_A \ln(A_{t-1}) + \varepsilon_t^a$$

In the steady state

$$\begin{aligned}
\ln(A) &= \rho_A \ln(A) \\
(\ln(A_t) - \ln(A)) &= \rho_A(\ln(A_{t-1}) - \ln(A)) + \varepsilon_t^a
\end{aligned}$$

$$\tilde{a}_t = \rho_a \tilde{a}_{t-1} + \varepsilon_t^a \tag{118}$$

$$v_t = \rho_m m_{t-1} + \varepsilon_t^m \tag{119}$$

$$\begin{aligned}
i_t &= \frac{r_t}{\mathbb{E}_t[\pi_{t+1}]} \\
ie^{\tilde{i}_t} &= \frac{re^{\tilde{r}_t}}{\mathbb{E}_t[\pi e^{\pi_{\tilde{t+1}t}}]} \\
e^{\tilde{i}_t} &= \frac{e^{\tilde{r}_t}}{\mathbb{E}_t[e^{\pi_{\tilde{t+1}t}}]} \\
e^{\tilde{i}_t} &= \mathbb{E}_t[e^{\tilde{r}_t} e^{-\pi_{\tilde{t+1}t}}] \\
(1 + \tilde{i}_t) &= (1 + \tilde{r}_t)(1 - \tilde{\pi}_{t+1}) \\
1 + \tilde{i}_t &= 1 + \tilde{r}_t - \tilde{\pi}_{t+1}
\end{aligned}$$

$$\tilde{i}_t = \tilde{r}_t - \tilde{\pi}_{t+1} \tag{120}$$

recall that i is the real interest rate and r is the nominal interest rate.

j)

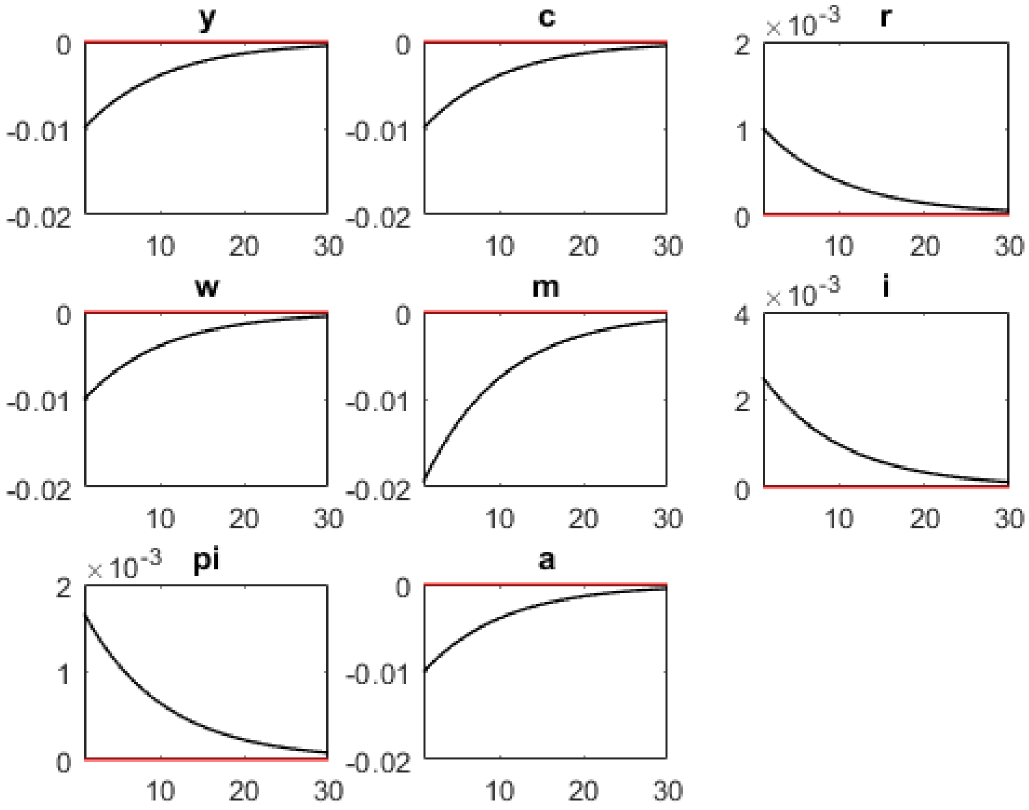


Figure 5: Response of the endogenous variables with respect to an one standard deviation negative innovation in the technological level

The negative technological shock causes output to decrease, as well as consumption, real wage, real balances. On the other hand, real interest rate, nominal rate and inflation increase.

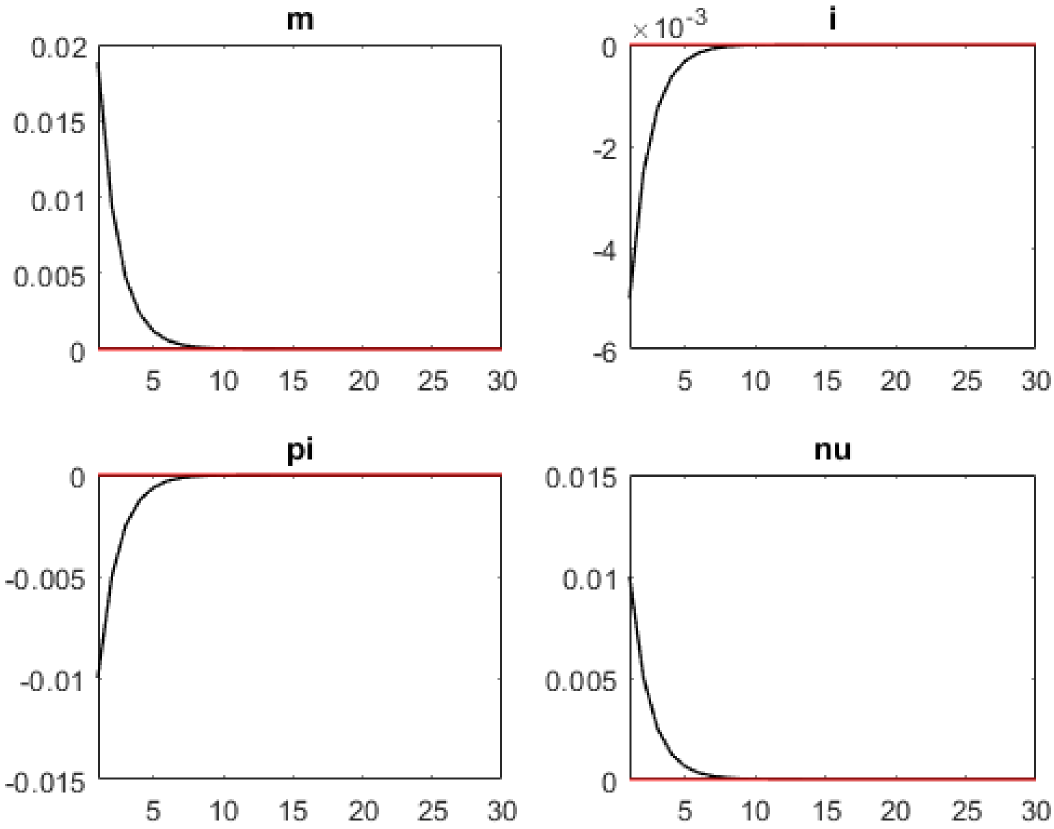


Figure 6: Plot the Response of the endogenous variables with respect to a raise in the nominal interest rate

The results are the same as for item h.

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