Lista 2 - Macro 4

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Q.1)

Item a)

Equation 1 represent is often referred as the dynamic IS equation. The equation is obtained from the goods market clearing condition with the consumer's Euler equation too yield the equilibrium condition.

Equation 2 is often referred as the New Keynesian Philips Curve, and constitutes one of the key building blocks of the basic New Keynesian model. The relate the current inflation to its one period ahead forecast and the output gap. The equation comes from the firms minimizing the the marginal cost and under nominal price rigid.

lastly the Taylor principle is a property which the condition for a determinate price level, $\phi_r > 1$, requires that the central bank adjust nominal interest rates more than one for one in response to any change in inflation. The previous result can be viewed as a particular instance of the need to satisfy the Taylor principle in order for an interest rate rule to bring about a determinate equilibrium.

$$i_t = \rho + \phi_\pi \pi_t + \phi_y x_t, \qquad \rho > 0, \quad \phi_\pi > 1, \quad \phi_y > 0 \tag{1}$$

Item b)

Let's assume that the funtions x_t and π_t are functions of the shocks in each period.

$$x_t = a\xi_t + bu_t \tag{2}$$
$$\pi_t = c\xi_t + du_t \tag{3}$$

Since ξ_t and u_t are mean-zero we then must have that $\mathbb{E}_t[x_t] = 0$ and $\mathbb{E}_t[\pi_t] = 0$. This implies that:

$$a\xi_t + bu_t = -\sigma^{-1}\left(i_t - r_t^n\right) \tag{4}$$
$$c\xi_t + du_t = \kappa(a\xi_t + bu_t) + u_t \tag{5}$$
$$i_t = \rho + \phi_\pi(c\xi_t + du_t) + \phi_y(a\xi_t + bu_t) \tag{6}$$

Substituting equation (6) into (4), and using the fact that $r_t^n = \rho + \xi_t$ we have that:

$$a\xi_t + bu_t = -\sigma^{-1}\left(\phi_\pi(c\xi_t + du_t) + \phi_y(a\xi_t + bu_t) - \xi_t\right) \tag{7}$$
$$c\xi_t + du_t = \kappa(a\xi_t + bu_t) + u_t \tag{8}$$

$$\begin{aligned}
a\xi_t + bu_t &= -\sigma^{-1}\left(\phi_\pi(c\xi_t + du_t) + \phi_y(a\xi_t + bu_t) - \xi_t\right)\\
a\xi_t + bu_t &= -\sigma^{-1}\left(\phi_\pi c\xi_t + \phi_\pi du_t + \phi_y a\xi_t + \phi_y bu_t - \xi_t\right)\\
a\xi_t + bu_t &= -\sigma^{-1}\phi_\pi c\xi_t - \sigma^{-1}\phi_\pi du_t - \sigma^{-1}\phi_y a\xi_t - \sigma^{-1}\phi_y bu_t + \sigma^{-1}\xi_t\\
0 &= \left[a + \sigma^{-1}\phi_\pi c + \sigma^{-1}\phi_y a - \sigma^{-1}\right]\xi_t + \left[b + \sigma^{-1}\phi_\pi d + \sigma^{-1}\phi_y b\right]u_t\\
0 &= \left[\left(1 + \frac{\phi_y}{\sigma}\right)a + \frac{\phi_\pi}{\sigma}c - \frac{1}{\sigma}\right]\xi_t + \left[\left(1 + \frac{\phi_y}{\sigma}\right)b + \frac{\phi_\pi}{\sigma}d\right]u_t
\end{aligned}$$

$$\begin{aligned}
c\xi_t + du_t &= \kappa(a\xi_t + bu_t) + u_t\\
c\xi_t + du_t &= \kappa a\xi_t + \kappa bu_t + u_t\\
0 &= [c - \kappa a]\,\xi_t + [d - \kappa b - 1]\,u_t
\end{aligned}$$

From equations (7) and (8) we get

$$0 = \left[\left(1 + \frac{\phi_y}{\sigma}\right)a + \frac{\phi_\pi}{\sigma}c - \frac{1}{\sigma}\right]\xi_t + \left[\left(1 + \frac{\phi_y}{\sigma}\right)b + \frac{\phi_\pi}{\sigma}d\right]u_t \tag{9}$$
$$0 = [c - \kappa a]\,\xi_t + [d - \kappa b - 1]\,u_t \tag{10}$$

Notice that (9) and (10) must hold for every t, then this implies that

$$\left(1 + \frac{\phi_y}{\sigma}\right) a + \frac{\phi_\pi}{\sigma} c = \frac{1}{\sigma} \tag{11}$$

$$\left(1 + \frac{\phi_y}{\sigma}\right) b + \frac{\phi_\pi}{\sigma} d = 0 \tag{12}$$

$$-\kappa a + c = 0 \tag{13}$$

$$-\kappa b + d = 1 \tag{14}$$

We now define the following:

$$A = 1 + \frac{\phi_y}{\sigma} \tag{15}$$

$$C = \frac{\phi_\pi}{\sigma} \tag{16}$$

Then we have the following system:

$$\begin{bmatrix} A & 0 & C & 0 \\ 0 & A & 0 & C \\ -\kappa & 0 & 1 & 0 \\ 0 & -\kappa & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1/\sigma \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{17}$$

Then we must have that:

$$\begin{bmatrix} A + \kappa C & 0 & 0 & 0 \\ 0 & A + \kappa C & 0 & 0 \\ -\kappa & 0 & 1 & 0 \\ 0 & -\kappa & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1/\sigma \\ -C \\ 0 \\ 1 \end{bmatrix} \tag{18}$$

$$a = \frac{1}{\sigma + \phi_y + \kappa\phi_\pi} \tag{19}$$

$$b = \frac{-\phi_\pi}{\sigma + \phi_y + \kappa\phi_\pi} \tag{20}$$

$$c = \frac{\kappa}{\sigma + \phi_y + \kappa\phi_\pi} \tag{21}$$

$$d = \frac{\sigma + \phi_y}{\sigma + \phi_y + \kappa\phi_\pi} \tag{22}$$

Therefore we must have that

$$x_t = \frac{1}{\sigma + \phi_y + \kappa\phi_\pi} \xi_t - \frac{\phi_\pi}{\sigma + \phi_y + \kappa\phi_\pi} u_t \tag{23}$$

$$\pi_t = \frac{\kappa}{\sigma + \phi_y + \kappa\phi_\pi} \xi_t + \frac{\sigma + \phi_y}{\sigma + \phi_y + \kappa\phi_\pi} u_t \tag{24}$$

Item c)

The parameters ϕ_π and ϕ_y are non-negative coefficients determined by the central bank, that describe the strength of the interest rate response to deviations of inflation or the output gap from their target levels.

Under the assumption of non-negative values for (ϕ_π and ϕ_y), a necessary and sufficient condition for the equilibrium to be unique, is given by

$$\kappa(\phi_\pi - 1) + (1-\beta)\phi_y > 0 \tag{25}$$

Thus, roughly speaking, the monetary authority should respond to deviations of inflation and the output gap from their target levels by adjusting the nominal rate with "sufficient strength". Figure (1) shows the possible values for ϕ_π and ϕ_y.

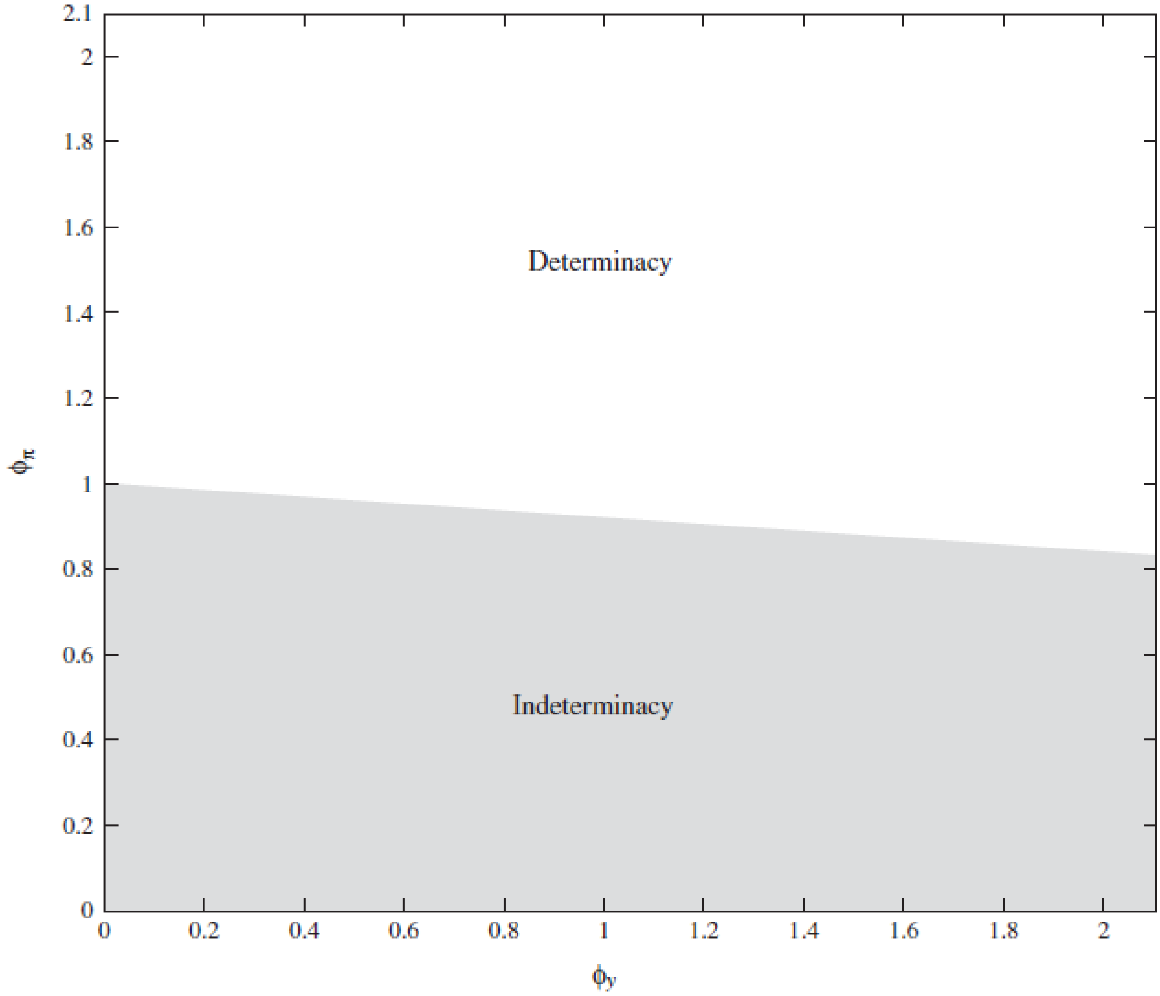

Figure 1: Determinacy and Indeterminacy Regions for a Contemporaneous Interest Rate Rule

Q.2)

Item a)

Suppose the household maximize:

$$C_t = \left[\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} d_i\right]^{\frac{\varepsilon}{\varepsilon-1}} \tag{26}$$

$$\text{s.t.} \qquad \int_0^1 P_t(i)C_t(i) = Z_t \tag{27}$$

$$L = \left[\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} d_i\right]^{\frac{\varepsilon}{\varepsilon-1}} - \lambda_t\left[\int_0^1 P_t(i)C_t(i)d_i - Z_t\right] \tag{28}$$

FOC:

$$\begin{aligned}
&\frac{\varepsilon}{\varepsilon_1 - 1}\left[\int_0^1 c_t(i)^{\frac{\varepsilon-1}{\varepsilon}} d_i\right]^{\frac{\varepsilon}{\varepsilon-1}-1}\left(\frac{\varepsilon-1}{\varepsilon}\right) c_t(i)^{-1/\varepsilon} - \lambda_t P_t(i) = 0\\
&\left[\int_0^1 c_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{1}{\varepsilon-1}} C_t(i)^{-1/\varepsilon} = \lambda_t P_t(i)\\
&C_t(i)^{-1/\varepsilon} = \frac{\lambda_t P_t(i)}{\left[\int_0^1 c_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{1}{\varepsilon-1}}}\\
&C_t(i)^{-1/\varepsilon} = \lambda_t P_t(i)\left[\int_0^1 c_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{-1}{1-\varepsilon}}\\
&C_t(i) = (\lambda_t P_t(i))^{-\varepsilon}\left[\int_0^1 c_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}}\\
&C_t(i) = (\lambda_t P_t(i))^{-\varepsilon} C_t
\end{aligned}$$

This result hold for any i e j, therefore:

$$C_t(i) = (\lambda_t P_t(i))^{-\varepsilon} C_t \tag{29}$$

$$C_t(j) = (\lambda_t P_t(j))^{-\varepsilon} C_t \tag{30}$$

Dividing (29) by (30) we have:

$$\frac{C_t(i)}{C_t(j)} = \left(\frac{P_t(i)}{P_t(j)}\right)^{-\varepsilon} \Rightarrow \left(\frac{C_t(i)}{C_t(j)}\right)^{\frac{\varepsilon-1}{\varepsilon}} = \left(\frac{P_t(i)}{P_t(j)}\right)^{1-\varepsilon} \tag{31}$$

$$C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} P_t(j)^{1-\varepsilon} = C_t(j)^{\frac{\varepsilon-1}{\varepsilon}} P_t(i)^{1-\varepsilon} \tag{32}$$

integrating in j

$$C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \int_0^1 P_t(j)^{1-\varepsilon} dj = P_t(i)^{1-\varepsilon} \int_0^1 C_t(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \tag{33}$$

$$C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} P_t^{1-\varepsilon} = P_t(i)^{1-\varepsilon} C_t^{\frac{\varepsilon-1}{\varepsilon}} \tag{34}$$

$$C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} = \left(\frac{P_t(i)}{P_t}\right)^{1-\varepsilon} C_t^{\frac{\varepsilon-1}{\varepsilon}} \tag{35}$$

$$C_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} C_t \tag{36}$$

Item b)

$$\int_0^1 P_t(t) C_t(i) di = \int_0^1 P_t(i) \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} C_t di$$

$$\int_0^1 P_t(t) C_t(i) di = C_t P_t^{\varepsilon} \int_0^1 P_t(i)^{1-\varepsilon} di$$

$$\int_0^1 P_t(t) C_t(i) di = C_t P_t^{\varepsilon} \left[\left(\int_0^1 P_t(i)^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}}\right]^{1-\varepsilon}$$

$$\int_0^1 P_t(t) C_t(i) di = C_t P_t^{\varepsilon} \left[P_t\right]^{1-\varepsilon}$$

$$\int_0^1 P_t(t) C_t(i) di = C_t P_t$$

Item c)

$$\mathcal{L} = E_t \sum_{t=0}^{\infty} \beta^t \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} - \lambda_t \left[P_t C_t + Q_t B_t - B_{t-1} - W_t N_t - T_t \right] \right\}$$

FOC

$$[N_t] : -(1-\varphi)\frac{N_t^{\varphi}}{1+\varphi} - \lambda_t(-W_t) = 0 \Rightarrow N_t^{\varphi} = \lambda_t W_t \tag{37}$$

$$[C_t] : (1-\sigma)\frac{C_t^{-\sigma}}{1-\sigma} - \lambda_t P_t = 0 \Rightarrow \frac{C_t^{-\sigma}}{P_t} = \lambda_t \tag{38}$$

$$[B_t] : -\lambda_t Q_t + \beta E_t[\lambda_{t+1}] = 0 \Rightarrow \lambda_t Q_t = \beta E_t[\lambda_{t+1}] \tag{39}$$

from (37) and (38)

$$N_t^{\varphi} = W_t \frac{C_t^{-\sigma}}{P_t} \Rightarrow C_t^{\sigma} N_t^{\varphi} = \frac{W_t}{P_t} \tag{40}$$

from (38) and (39)

$$\frac{C_t^{-\sigma}}{P_t} Q_t = \beta E_t \left[\frac{C_{t+1}^{-\sigma}}{P_{t+1}} \right] \Rightarrow C_t^{-\sigma} Q_t = \beta E_t \left[C_{t+1}^{-\sigma} \frac{P_t}{P_{t+1}} \right] \tag{41}$$

Item d)

$$P_t = \left[\int_0^1 P_t(i)^{1-\varepsilon} d_i \right]^{\frac{1}{1-\varepsilon}} = \left[\theta \int_0^1 P_{t-1}(i)^{1-\varepsilon} di + (1-\theta) \int_0^1 P_t^{*1-\varepsilon} di \right]^{\frac{1}{1-\varepsilon}}$$

$$P_t = \left[\theta P_{t-1}^{1-\varepsilon} + (1-\theta) P_t^{*1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}$$

$$1 = \left[\theta P_{t-1}^{1-\varepsilon} + (1-\theta) P_t^{*1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}} \frac{1}{P_t}$$

$$1 = \left[\theta P_{t-1}^{1-\varepsilon} \left(\frac{1}{P_t} \right)^{1-\varepsilon} + (1-\theta) P_t^{*1-\varepsilon} \left(\frac{1}{P_t} \right)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}$$

$$1 = \left[\theta \Pi_t^{\varepsilon-1} + (1-\theta) \Pi_t^{*1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}$$

$$1 = \theta \Pi_t^{\varepsilon-1} + (1-\theta) \Pi_t^{*1-\varepsilon} \tag{42}$$

Item e)

$$\max_{P_t^*} \sum_{k=0}^{\infty} \theta^k E_t \left\{ Q_{t,t+k} \left(P_t^* Y_{t+k|t} - \Psi_{t+k} \left(Y_{t+k|t} \right) \right) \right\}$$

$$\text{s.t. } Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} C_{t+k}$$

Let $Q_{t,t+k} \equiv \beta^k \left(C_{t+k}/C_t \right)^{-\sigma} \left(P_t/P_{t+k} \right)$ be the stochastic discount factor, then the FOC associated with the problem takes the form

$$\sum_{k=0}^{\infty} \theta^k E_t \left\{ Q_{t,t+k} Y_{t+k|t} \left(P_t^* - \mathcal{M} \psi_{t+k|t} \right) \right\} = 0 \tag{43}$$

where $\psi_{t+k|t} \equiv \Psi'_{t+k} \left(Y_{t+k|t} \right)$ denotes the (nominal) marginal cost in period $t+k$ for a firm which last reset its price in period t and $\mathcal{M} \equiv \frac{\varepsilon}{\varepsilon - 1}$

Item f)

Fix a firm i, them the problem it solves is:

$$\min_{N_t(i)} \left\{ \frac{W_t}{P_t} N_t(i) \right\} \text{ s.t. } A_t N_t(i)^{1-\alpha} = Y_t(i)$$

The lagrangean takes the from

$$\mathcal{L} = \frac{W_t}{P_t} N_t(i) - \lambda_t \left[A_t N_t(i)^{1-\alpha} - Y_t \right]$$

FOC:

$$\frac{W_t}{P_t} - \lambda_t \left[(1-\alpha) A_t N_t(i)^{-\alpha} \right] = 0 \Rightarrow \frac{W_t}{P_t} = \lambda_t \left[(1-\alpha) A_t \frac{N_t(i)^{1-\alpha}}{N_t(i)} \right]$$

Notice that $\lambda_t = CMg^{real}(i)$, then:

$$CMg^{real}(i) = \frac{W_t/P_t}{(1-\alpha)Y_t(i)/N_t(i)} \tag{44}$$

Then the average real marginal cost across all firms is:

$$\begin{aligned}
CMg^{real}(i) &= \frac{(W_t/P_t)N_t(i)}{(1-\alpha)Y_t(i)} = \frac{W_t/P_t}{(1-\alpha)} \cdot \frac{\left(\frac{Y_t(i)}{A_t}\right)^{1/1-\alpha}}{Y_t(i)} \\
&= \frac{W_t/P_t}{(1-\alpha)} \cdot Y_t(i)^{1/1-\alpha} \cdot Y_t(i)^{-1} \cdot A_t^{-1/1-\alpha} \\
&= \frac{W_t/P_t}{(1-\alpha)A_t^{1/1-\alpha}} \cdot Y_t(i)^{\alpha/1-\alpha}
\end{aligned}$$

Integrating over i we have

$$\begin{aligned}
\int_0^1 CMg^{real}(i)di &= \frac{W_t/P_t}{(1-\alpha)A_t^{1/1-\alpha}} \cdot \int_0^1 Y_t(i)^{\alpha/1-\alpha}di \\
&= \frac{W_t/P_t}{(1-\alpha)A_t^{1/1-\alpha}} \cdot \int_0^1 \left(\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} Y_t\right)^{\alpha/1-\alpha} di \\
&= \frac{(W_t/P_t)Y_t^{\alpha/1-\alpha}}{(1-\alpha)A_t^{1/1-\alpha}} \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{\frac{-\varepsilon\alpha}{1-\alpha}} di \\
&= \frac{(W_t/P_t)Y_t^{\alpha/1-\alpha}}{(1-\alpha)A_t^{1/1-\alpha}} \frac{1}{P_t^{\frac{-\varepsilon\alpha}{1-\alpha}}} \int_0^1 P_t(i)^{\frac{-\varepsilon\alpha}{1-\alpha}} di \\
&= \frac{(W_t/P_t)Y_t^{\alpha/1-\alpha}}{(1-\alpha)A_t^{1/1-\alpha}} \frac{1}{P_t^{\frac{-\varepsilon\alpha}{1-\alpha}}} \int_0^1 P_t(i)^{\frac{-\varepsilon\alpha}{1-\alpha}} di
\end{aligned}$$

let $\frac{-\varepsilon\alpha}{1-\alpha} = 1-e$ and notice that $\int_0^1 P_t(i)^{1-e}di = P_t^{1-e}$, then

$$\int_0^1 CMg^{real}(i)di = \frac{(W_t/P_t)Y_t^{\alpha/1-\alpha}}{(1-\alpha)A_t^{1/1-\alpha}} \frac{P_t^{1-e}}{P_t^{\frac{-\varepsilon\alpha}{1-\alpha}}} \tag{45}$$

$$= \frac{(W_t/P_t)Y_t^{\alpha/1-\alpha}}{(1-\alpha)A_t^{1/1-\alpha}} \frac{P_t^{\frac{-\varepsilon\alpha}{1-\alpha}}}{P_t^{\frac{-\varepsilon\alpha}{1-\alpha}}} \tag{46}$$

$$= \frac{(W_t/P_t)Y_t^{\alpha/1-\alpha}}{(1-\alpha)A_t^{1/1-\alpha}} \tag{47}$$

Recall that $A_t^{1/1-\alpha} = N_t^{-1} Y_t^{1/1-\alpha}$, then

$$\int_0^1 CMg^{real}(i)di = \frac{(W_t/P_t)}{(1-\alpha)Y_t/N_t} \tag{48}$$

Item g)

First, notice that market clearing conditions guarantee that for each $i \in [0,1]$:

$$Y_t(i) = C_t(i)$$

Moreover, given that:

$$Y_t \equiv \left(\int_0^1 Y_t(i)^{1-\frac{1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

Thus $Y_t = C_t$. Market clearing in the labor market yields

$$N_t = \int_0^1 N_t(i)di$$

And since $Y_t(i) = C_t(i)$, we have that:

$$N_t = \int_0^1 \left(\frac{Y_t(i)}{A_t}\right)^{\frac{1}{1-\alpha}} di$$

thus:

$$N_t = \int_0^1 \left(\frac{Y_t}{Y_t}\right)^{\frac{1}{1-\alpha}} \left(\frac{Y_t(i)}{A_t}\right)^{\frac{1}{1-\alpha}} di \tag{49}$$

$$= \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{Y_t(i)}{Y_t}\right)^{\frac{1}{1-\alpha}} di \tag{50}$$

$$= \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{\frac{-\varepsilon}{1-\alpha}} di \tag{51}$$

pose $D_t \equiv \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{\frac{-\varepsilon}{1-\alpha}} di$ as a measure of price (and, hence, output) dispersion across firms and get that:

$$N_t = \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}} D_t$$

which in turn implies that

$$Y_t = A_t N_t^{1-\alpha} D_t^{\alpha-1}$$

h)

From the definition of D_t, we have that:

$$D_t \equiv \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{\frac{-\varepsilon}{1-\alpha}} di$$

Let $S(t) \subset [0,1]$ be the set of firms that cannot reset their price in t. Observe that, since firms that are resetting their price in t chose an identical price level P_t^*, we have that:

$$D_t = (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} + \int_{S(t)} \left(\frac{P_{t-1}(i)}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} di \tag{52}$$

$$= (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} + \int_{S(t)} \left(\frac{P_{t-1}}{P_{t-1}}\right)^{-\frac{\varepsilon}{1-\alpha}} \left(\frac{P_{t-1}(i)}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} di \tag{53}$$

Where, $(1-\theta)$ is the fractions of firms that reset their price in t. Given that $\Pi_t \equiv \frac{P_t}{P_{t-1}}$, we have that:

$$D_t = (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} + \Pi_t^{\frac{\varepsilon}{1-\alpha}} \int_{S(t)} \left(\frac{P_{t-1}(i)}{P_{t-1}}\right)^{-\frac{\varepsilon}{1-\alpha}} di$$

Therefore:

$$D_t = (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} + \theta \Pi_t^{\frac{\varepsilon}{1-\alpha}} D_{t-1}$$

The intuition behind this result is that: if the fraction θ of firms that are not able to reset their price in period t increases, i.e. if it tends to 1, the price index of the forward period will be just the price index of the last period. Indeed, given the fact that $\theta \to 1$, for each $i \in [0,1]$ $P_{t-1}(i) = P_t(i)$. Thus, Π_t will also be one, which implies that $D_t = D_{t-1}$. If $\theta \to 0$, we have that our price index D_t will be equal to 1, i.e. there will be no price dispersion. Indeed, since the price of all firms will be chosen identically in the optimal price setting problem, all firms will have identical prices (and, consequently, identical product levels).

Item i)

Let MC_t be the economy's real average marginal cost. From equation (43) we have that:

$$\sum_{k=0}^{\infty}\theta^k \mathbb{E}_t\left\{Q_{t,t+k}Y_{t+k|t}\left(P_t^* - \mathcal{M}\psi_{t+k|t}\right)\right\} = 0$$

Since $\psi_{t+k}|t$ is the nominal marginal cost of the firm that readjust its price in t, we divide () by P_{t+k} so that we have:

$$\sum_{k=0}^{\infty}\theta^k \mathbb{E}_t\left\{Q_{t,t+k}Y_{t+k|t}\left(\frac{P_t^*}{P_{t+k}} - \mathcal{M}MC_{t+k|t}\right)\right\} = 0$$

By substituting $Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} C_{t+k}$ and $Q_{t,t+k} \equiv \beta^k \left(C_{t+k}/C_t\right)^{-\sigma}\left(P_t/P_{t+k}\right)$ in (), we have that:

$$\sum_{k=0}^{\infty}\theta^k \mathbb{E}_t\left\{\beta^k\left(\frac{C_{t+k}}{C_t}\right)^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)\left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} C_{t+k}\left(\frac{P_t^*}{P_{t+k}} - \mathcal{M}MC_{t+k|t}\right)\right\} = 0$$

Since $C_{t+k} = Y_{t+k}$ by market clearing conditions:

$$\sum_{k=0}^{\infty}\theta^k \mathbb{E}_t\left\{\beta^k\left(\frac{C_{t+k}}{C_t}\right)^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)\left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} Y_{t+k}\left(\frac{P_t^*}{P_{t+k}} - \mathcal{M}MC_{t+k|t}\right)\right\} = 0$$

Which implies that:

$$\sum_{k=0}^{\infty}\theta^k \mathbb{E}_t\left\{\beta^k\left(\frac{C_{t+k}}{C_t}\right)^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)\left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} Y_{t+k}\left(\frac{P_t^*}{P_{t+k}}\right)\right\} =$$

$$= \sum_{k=0}^{\infty}\theta^k \mathbb{E}_t\left\{\beta^k\left(\frac{C_{t+k}}{C_t}\right)^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)\left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} Y_{t+k}\left(\mathcal{M}MC_{t+k|t}\right)\right\}$$

$$\sum_{k=0}^{\infty}\theta^k \mathbb{E}_t\left\{\beta^k\left(\frac{C_{t+k}}{C_t}\right)^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)\left(\frac{P_t^*}{P_t}\frac{P_t}{P_{t+k}}\right)^{-\varepsilon} Y_{t+k}\left(\frac{P_t^*}{P_t}\frac{P_t}{P_{t+k}}\right)\right\} =$$

$$= \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left\{ \beta^k \left(\frac{C_{t+k}}{C_t} \right)^{-\sigma} \left(\frac{P_t}{P_{t+k}} \right) \left(\frac{P_t^*}{P_t} \frac{P_t}{P_{t+k}} \right)^{-\varepsilon} Y_{t+k} \left(\mathcal{M} MC_{t+k|t} \right) \right\}$$

Given the fact that $\Pi_t^* \equiv \frac{P_t^*}{P_t}$ and simplifying $1/C_t^{-\sigma}$ on both sides, we have that:

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left\{ \beta^k C_{t+k}^{-\sigma} \left(\frac{P_t}{P_{t+k}} \right)^{2-\varepsilon} \left(\frac{P_t^*}{P_t} \right)^{1-\varepsilon} Y_{t+k} \right\} =$$
$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left\{ \beta^k C_{t+k}^{-\sigma} \left(\frac{P_t^*}{P_t} \right)^{-\varepsilon} \left(\frac{P_t}{P_{t+k}} \right)^{1-\varepsilon} Y_{t+k} \left(\mathcal{M} MC_{t+k|t} \right) \right\}$$

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left\{ \beta^k C_{t+k}^{-\sigma} \left(\frac{P_t}{P_{t+k}} \right)^{2-\varepsilon} (\Pi_t^*)^{1-\varepsilon} Y_{t+k} \right\} =$$
$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left\{ \beta^k C_{t+k}^{-\sigma} (\Pi_t^*)^{-\varepsilon} \left(\frac{P_t}{P_{t+k}} \right)^{1-\varepsilon} Y_{t+k} \left(\mathcal{M} MC_{t+k|t} \right) \right\}$$

Now, we must show that $MC_{t+k|t} = \frac{MC_{t+k}}{D_{t+k}} \left(\frac{P_t^*}{P_{t+k}} \right)^{\frac{-\alpha\varepsilon}{1-\alpha}}$:

$$MC_{t+k|t} = \frac{\hat{W}_{t+k}}{\frac{(1-\alpha) Y_{t+k|t}}{N_{t+k|t}}} \tag{54}$$

$$= \frac{\hat{W}_{t+k}}{(1-\alpha) A_{t+k} N_{t+k|t}^{-\alpha}} \tag{55}$$

$$= \frac{\hat{W}_{t+k}}{(1-\alpha) A_{t+k} N_{t+k}^{-\alpha}} \frac{N_{t+k}^{-\alpha}}{N_{t+k|t}^{-\alpha}} \tag{56}$$

Since $N_{t+k|t} = \left(\frac{Y_{t+kt}}{A_{t+k}} \right)^{\frac{1}{1-\alpha}}$, $N_{t+k} = \left(\frac{Y_{t+k}}{A_{t+k}} \right)^{\frac{1}{1-\alpha}} D_{t+k}$ that, and by our demand constraint we have that:

$$\frac{N_{t+k}^{-\alpha}}{N_{t+k|t}^{-\alpha}} = \left[\frac{N_{t+k|t}}{N_{t+k}} \right]^{\alpha} \tag{57}$$

$$= \left[\left(\frac{Y_{t+k|t}}{Y_{t+k}} \right)^{\frac{1}{1-\alpha}} \frac{1}{D_{t+k}} \right]^{\alpha} \tag{58}$$

$$= \left[\left(\frac{P_t^*}{P_{t+k}} \right)^{\frac{-\varepsilon}{1-\alpha}} \frac{1}{D_{t+k}} \right]^{\alpha} \tag{59}$$

Notice that we have substitute the market clearing condition in our demand constraint, i.e. we have substituted C_{t+k} by Y_{t+k} so that:

$$Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} C_{t+k} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} Y_{t+k} \Rightarrow \frac{Y_{t+k|t}}{Y_{t+k}} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} \tag{60}$$

Thus, we have that:

$$\begin{aligned} MC_{t+k|t} &= \frac{\hat{W}_{t+k}}{(1-\alpha)A_{t+k}N_{t+k}^{-\alpha}}\left[\left(\frac{P_t^*}{P_{t+k}}\right)^{\frac{-\varepsilon}{1-\varepsilon}}\frac{1}{D_{t+k}}\right]^{\alpha} \\ &= \frac{\hat{W}_{t+k}}{(1-\alpha)A_{t+k}N_{t+k}^{-\alpha}\frac{N_{t+k}D_{t+k}}{N_{t+k}D_{t+k}}}\left[\left(\frac{P_t^*}{P_{t+k}}\right)^{\frac{-\varepsilon}{1-\alpha}}\frac{1}{D_{t+k}}\right]^{\alpha} \\ &= \frac{MC_{t+k}}{D_{t+k}}\left(\frac{P_t^*}{P_{t+k}}\right)^{\frac{-\alpha\varepsilon}{1-\alpha}} \end{aligned} \tag{61}$$

Therefore:

$$\begin{aligned} &\sum_{k=0}^{\infty}\theta^k\mathbb{E}_t\left\{\beta^k C_{t+k}^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)^{2-\varepsilon}(\Pi_t^*)^{1-\varepsilon}Y_{t+k}\right\} = \\ &= \sum_{k=0}^{\infty}\theta^k\mathbb{E}_t\left\{\beta^k C_{t+k}^{-\sigma}(\Pi_t^*)^{-\varepsilon}\left(\frac{P_t}{P_{t+k}}\right)^{1-\varepsilon}Y_{t+k}\mathcal{M}\frac{MC_{t+k}}{D_{t+k}}\left(\frac{P_t^*}{P_{t+k}}\right)^{\frac{-\alpha\varepsilon}{1-\alpha}}\right\} \end{aligned}$$

Which implies:

$$\begin{aligned} &\sum_{k=0}^{\infty}\theta^k\mathbb{E}_t\left\{\beta^k C_{t+k}^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)^{2-\varepsilon}(\Pi_t^*)^{1-\varepsilon}Y_{t+k}\right\} = \\ &= \sum_{k=0}^{\infty}\theta^k\mathbb{E}_t\left\{\beta^k C_{t+k}^{-\sigma}(\Pi_t^*)^{-\varepsilon}\left(\frac{P_t}{P_{t+k}}\right)^{1-\varepsilon}Y_{t+k}\mathcal{M}\frac{MC_{t+k}}{D_{t+k}}\left(\frac{P_t^*}{P_t}\frac{P_t}{P_{t+k}}\right)^{\frac{-\alpha\varepsilon}{1-\alpha}}\right\} \\ &= (\Pi_t^*)^{1-\varepsilon}\sum_{k=0}^{\infty}\theta^k\mathbb{E}_t\left\{\beta^k C_{t+k}^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)^{2-\varepsilon}Y_{t+k}\right\} \\ &= (\Pi_t^*)^{-\varepsilon-\frac{-\alpha\varepsilon}{1-\alpha}}\mathcal{M}\sum_{k=0}^{\infty}\theta^k\mathbb{E}_t\left\{\beta^k C_{t+k}^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)^{1-\varepsilon-\frac{-\alpha\varepsilon}{1-\alpha}}Y_{t+k}\frac{MC_{t+k}}{D_{t+k}}\right\} \end{aligned}$$

Which implies:

$$(\Pi_t^*)^{1+\frac{\alpha\varepsilon}{1-\alpha}}\sum_{k=0}^{\infty}\theta^k\mathbb{E}_t\left\{\beta^k C_{t+k}^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)^{2-\varepsilon}Y_{t+k}\right\} = \tag{62}$$

$$= \mathcal{M}\sum_{k=0}^{\infty}\theta^k\mathbb{E}_t\left\{\beta^k C_{t+k}^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)^{1-\varepsilon-\frac{-\alpha\varepsilon}{1-\alpha}}Y_{t+k}\frac{MC_{t+k}}{D_{t+k}}\right\} \tag{63}$$

Now pose:

$$x_{1,t} \equiv \sum_{k=0}^{\infty} \theta^k E_t \left[\beta^k C_{t+k}^{-\sigma} Y_{t+k} \frac{MC_{t+k}}{D_{t+k}} \left(\frac{P_t}{P_{t+k}} \right)^{1-\varepsilon-\frac{\alpha\varepsilon}{1-\alpha}} \right] \tag{64}$$

$$x_{2,t} \equiv E_t \sum_{k=0}^{\infty} \theta^k \left[\beta^k C_{t+k}^{-\sigma} \left(\frac{P_t}{P_{t+k}} \right)^{2-\varepsilon} Y_{t+k} \right] \tag{65}$$

Substituting (64), (65), in (63) we get:

$$(\Pi_t^*)^{1+\frac{\alpha\varepsilon}{1-\alpha}} x_{2,t} = \mathcal{M} x_{1,t} \tag{66}$$

Where x_2 relates to the expected discounted marginal gains from resetting prices and x_1 relates to the expected discounted marginal cost from resetting prices. Now observe that:

$$\begin{aligned} x_{1,t} &= \sum_{k=0}^{\infty} \theta^k E_t \left[\beta^k C_{t+k}^{-\sigma} Y_{t+k} \frac{MC_{t+k}}{D_{t+k}} \left(\frac{P_t}{P_{t+k}} \right)^{1-\varepsilon-\frac{\alpha\varepsilon}{1-\alpha}} \right] \\ &= C_t^{-\sigma} MC_t Y_t + \sum_{k=1}^{\infty} \theta^k E_t \left[\beta^k C_{t+k}^{-\sigma} Y_{t+k} \frac{MC_{t+k}}{D_{t+k}} \left(\frac{P_t}{P_{t+k}} \right)^{1-\varepsilon-\frac{\alpha\varepsilon}{1-\alpha}} \right] \\ &= C_t^{-\sigma} MC_t Y_t + E_t \left(\frac{P_t}{P_{t+1}} \right)^{1-\varepsilon-\frac{\alpha\varepsilon}{1-\alpha}} \frac{\theta\beta}{\theta\beta} \sum_{k=1}^{\infty} \theta^k \left[\beta^k C_{t+k}^{-\sigma} Y_{t+k} \frac{MC_{t+k}}{D_{t+k}} \left(\frac{P_{t+1}}{P_{t+k}} \right)^{1-\varepsilon-\frac{\alpha\varepsilon}{1-\alpha}} \right] \end{aligned}$$

Now notice that if we pose $\hat{k} = k - 1$, we have that:

$$x_{1,t} = C_t^{-\sigma} MC_t Y_t + E_t \left(\frac{P_t}{P_{t+1}} \right)^{1-\varepsilon-\frac{\alpha\varepsilon}{1-\alpha}} \times$$
$$\theta\beta \sum_{\hat{k}=0}^{\infty} \theta^{\hat{k}} \left[\beta^{\hat{k}} C_{t+1+\hat{k}}^{-\sigma} Y_{t+1+\hat{k}} \frac{MC_{t+1+\hat{k}}}{D_{t+1+\hat{k}}} \left(\frac{P_{t+1}}{P_{t+1+\hat{k}}} \right)^{1-\varepsilon-\frac{\alpha\varepsilon}{1-\alpha}} \right]$$

Notice that:

$$x_{1,t+1} \equiv E_t \sum_{k=0}^{\infty} \theta^k \left[\beta^k C_{t+1+k}^{-\sigma} Y_{t+1+k} \frac{MC_{t+1+k}}{D_{t+1+k}} \left(\frac{P_{t+1}}{P_{t+1+k}} \right)^{1-\varepsilon-\frac{\alpha\varepsilon}{1-\alpha}} \right] \tag{67}$$

Since $\Pi \equiv \frac{P_{t+1}}{P_t}$, we have that:

$$x_{1,t} = C_t^{-\sigma} MC_t Y_t + \beta\theta E_t \Pi_{t+1}^{\varepsilon+\frac{\alpha\varepsilon}{1-\alpha}-1} x_{1,t+1} \tag{68}$$

Also, notice that:

$$\begin{aligned}
x_{2,t} &= C_t^{-\sigma} Y_t + E_t \sum_{k=1}^{\infty} \theta^k \left[\beta^k C_{t+k}^{-\sigma} \left(\frac{P_t}{P_{t+k}} \right)^{2-\varepsilon} Y_{t+k} \right] \\
&= C_t^{-\sigma} Y_t + E_t \left(\frac{P_t}{P_{t+1}} \right)^{2-\varepsilon} \frac{\theta\beta}{\theta\beta} \sum_{k=1}^{\infty} \theta^k \left[\beta^k C_{t+k}^{-\sigma} \left(\frac{P_{t+1}}{P_{t+k}} \right)^{2-\varepsilon} Y_{t+k} \right]
\end{aligned}$$

Which applying to $x_{2,t}$ the same logic applied to $x_{1,t}$, we that:

$$x_{2,t} = C_t^{-\sigma} Y_t + \beta\theta E_t \prod_{t+1}^{\varepsilon-2} x_{2,t+1} \tag{69}$$

Item j)

The equations that characterize our equilibrium are:

$$\frac{W_t}{P_t} = C_t^{\sigma} N_t^{\varphi} \tag{70}$$

$$R_t = \frac{1}{Q_t} \tag{71}$$

$$Q_t = \beta \mathbb{E}_t \left\{ \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{1}{\Pi_{t+1}} \right\} \tag{72}$$

$$\frac{M_t}{P_t} = Y_t R_t^{-\eta} \tag{73}$$

$$Y_t = A_t \left(\frac{N_t}{D_t} \right)^{1-\sigma} \tag{74}$$

$$Y_t = C_t \tag{75}$$

$$D_t = (1-\theta)\left(\Pi_t^*\right)^{-\frac{\epsilon}{1-\alpha}} + \theta \Pi_t^{\frac{\epsilon}{1-\alpha}} D_{t-1} \tag{76}$$

$$1 = \theta \Pi_t^{\epsilon-1} + (1-\theta){\Pi^*}_t^{\epsilon-1} \tag{77}$$

$$MC_t = \frac{\hat{W}_t N_t}{(1-\alpha) Y_t} \tag{78}$$

$$\frac{\epsilon}{\epsilon-1} x_{1,t} = \left(\Pi_t^*\right)^{1+\frac{\alpha\epsilon}{1-\alpha}} x_{2,t} \tag{79}$$

$$x_{1,t} = C_t^{-\sigma} MC_t Y_t + \beta\theta \mathbb{E}_t \left[\Pi_{t+1}^{\epsilon+\frac{\alpha\epsilon}{1-\alpha}-1} \right] x_{1,t+1} \tag{80}$$

$$x_{2,t} = C_t^{-\sigma} Y_t + \beta\theta \mathbb{E}_t \left[\Pi_{t+1}^{\epsilon-2} \right] x_{2,t+1} \tag{81}$$

$$\ln(A_t) = \rho_a \ln(A_{t-1}) + \epsilon_t^a \qquad \epsilon_t^a \sim N(0, \sigma_a^2) \tag{82}$$

$$R_t = r_t \mathbb{E}_t \left[\Pi_{t+1} \right] \tag{83}$$

$$\Pi_t = \frac{P_t}{P_{t-1}} \tag{84}$$

$$R_t = \frac{1}{\beta} \left(\Pi_t \right)^{\phi_\pi} \left(\frac{Y_t}{Y_t^*} \right)^{\phi_y} e^{\nu} \tag{85}$$

$$\nu_t = \rho_m \nu_{t-1} + \epsilon_t^m \qquad \epsilon_t^m \sim N(0, \sigma_m^2) \tag{86}$$

Let C^*, N^*, MC^*, $\hat{W}^*$, $\hat{M}^*$, Y^*, Y^{**}, R^*, Q^*, P^*, A^*, D^*, x_1^*, x_2^*, Π^*, Π^{**} be the steady state value of the variable C_t, N_t, MC_t, W_t/P_t, M_t/P_t, Y_t, Y_t^*, R_t, Q_t, P_t, A_t, D_t, $x_{1,t}$, $x_{2,t}$, Π_t, Π_t^* respectively.

From (70) we have that

$$C^{*\sigma} N^{*\phi} = \hat{W}^* \tag{87}$$

From (71), (72):

$$R^* = \frac{1}{Q^*} = \frac{1}{\beta} \tag{88}$$

From (73):

$$\hat{M}^* = Y^* R^{*-\eta} \tag{89}$$

From (82), we get that:

$$\begin{aligned} \mathbb{E}_t\left[\ln\left(A_t\right) - \rho_a \ln\left(A_{t-1}\right)\right] &= 0 \\ \mathbb{E}_t\left[\ln\left(A^{*1-\rho_a}\right)\right] &= 0 \\ A^* &= 1 \end{aligned} \tag{90}$$

From (84), we get that:

$$\Pi^* = \frac{P^*}{P^*} = 1 \tag{91}$$

Which will imply, together with equation (77), that:

$$1 = \theta \Pi_t^{\varepsilon-1} + (1-\theta)\Pi_t^{*1-\varepsilon} \Rightarrow 1 = \theta + (1-\theta)\left(\Pi^{**}\right)^{(1-\varepsilon)} \Rightarrow \Pi^{**} = 1 \tag{92}$$

Which implies, together with equation (76), that:

$$D_t = (1-\theta)\left(\Pi_t^*\right)^{-\frac{\varepsilon}{1-\alpha}} + \theta \Pi_t^{\frac{\varepsilon}{1-\alpha}} D_{t-1} \Rightarrow D^* = (1-\theta) + \theta D^* \Rightarrow D^* = 1 \tag{93}$$

From (74) and (75), we have that:

$$Y_t = A_t \left(\frac{N_t}{D_t}\right)^{1-\alpha} \Rightarrow C^* = Y^* = A^* \left(\frac{N^*}{D^*}\right)^{1-\alpha} \tag{94}$$

From (90) and (93) we conclude that:

$$C^* = Y^* = N^{*1-\alpha} \tag{95}$$

From (78) we deduce that:

$$MC_t = \frac{\hat{W}_t}{\frac{(1-\alpha)Y_t}{N_t}} \Rightarrow MC^* = \frac{\hat{W}^*}{\frac{(1-\alpha)Y^*}{N^*}} \tag{96}$$

And from (87) and (95), we have that:

$$MC^* = \frac{N^{*\sigma(1-\alpha)} N^{*\phi}}{\frac{(1-\alpha)N^{*1-\alpha}}{N^*}} = \frac{N^{*\sigma(1-\alpha)+\phi+\alpha}}{1-\alpha} \tag{97}$$

From (80) and from (91):

$$x_1^* = \frac{C^{*-\sigma} MC^* Y^*}{1-\beta\theta} \tag{98}$$

While from (81) and from (91):

$$x_2^* = \frac{C^{*-\sigma} Y^*}{1-\beta\theta} \tag{99}$$

Thus, from (79), (98), (99) and (92) we deduce that:

$$x_2^* = \mathcal{M} x_1^* \Rightarrow \frac{C^{*-\sigma} Y^*}{1-\beta\theta} = \mathcal{M} \frac{C^{*-\sigma} MC^* Y^*}{1-\beta\theta} \Rightarrow MC^* = \frac{1}{\mathcal{M}} \tag{100}$$

From (97):

$$\frac{N^{*\sigma(1-\alpha)+\phi+\alpha}}{1-\alpha} = \frac{1}{\mathcal{M}} \Rightarrow N^* = \left(\frac{1-\alpha}{\mathcal{M}}\right)^{\frac{1}{\sigma(1-\alpha)+\phi+\alpha}} \tag{101}$$

Together with equation (95):

$$C^* = Y^* = \left(\frac{1-\alpha}{\mathcal{M}}\right)^{\frac{1-\alpha}{\sigma(1-\alpha)+\phi+\alpha}} \tag{102}$$

Substituting (102) and (88) in (73) :

$$\hat{M}^* = \left(\frac{1-\alpha}{\mathcal{M}}\right)^{\frac{1-\alpha}{\sigma(1-\alpha)+\phi+\alpha}} \beta^{\eta} \tag{103}$$

Substituting (95) into (87)

$$\hat{W}^* = N^{*\sigma(1-\alpha)} N^{*\phi} \Rightarrow \hat{W}^* = N^{*\sigma(1-\alpha)+\phi} = \left(\frac{1-\alpha}{\mathcal{M}}\right)^{\frac{\sigma(1-\alpha)+\phi}{\sigma(1-\alpha)+\phi+\alpha}} \tag{104}$$

Finally, from equations (85), (88), (91) we deduce that:

$$R^* = \frac{1}{\beta} (\Pi^*)^{\phi_\pi} \left(\frac{Y^*}{Y^{**}} \right)^{\phi_y} \exp\{\nu^*\}$$
$$\frac{1}{\beta} = \frac{1}{\beta} (1)^{\phi_\pi} \left(\frac{Y^*}{Y^{**}} \right)^{\phi_y} \exp\{0\}$$

Which implies that:

$$Y^* = Y^{**} \tag{105}$$

Q.3)

Item a)

The parameters are as follows: $\beta = 0.99$: implies a steady state real return on financial assets of about 4 percent; $\sigma = 1$: implies log utility; $\varphi = 1$: implies a unitary Frisch elasticity of labor supply; $\alpha = 1/3$, $\epsilon = 6$, values commonly found in the business cycle literature; $\eta = 4$: Interest semi-elasticity of money demand. The calibration is based on the estimates of an OLS regression; $\theta = 2/3$: Implies an average price duration of three quarters, a value consistent with the empirical evidence; $\phi_\pi = 1.5$ and $\phi_y = 0.5/4$, which are roughly consistent with observed variations in the Federal Funds rate over the Greenspan era; $\rho_v = 0.5$, a set value associated with a moderately persistent shock; and lastly $\rho_a = 0.9$.

Item b)

See the file 'Q3_Itens_bcd.mod' attached.

recall that the steady state is given by:

$$\frac{W_t}{P_t} = C_t^{\sigma} N_t^{\varphi} \tag{106}$$

$$R_t = \frac{1}{Q_t} \tag{107}$$

$$Q_t = \beta \mathbb{E}_t \left\{ \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{1}{\Pi_{t+1}} \right\} \tag{108}$$

$$\frac{M_t}{P_t} = Y_t R_t^{-\eta} \tag{109}$$

$$Y_t = A_t \left(\frac{N_t}{D_t} \right)^{1-\sigma} \tag{110}$$

$$Y_t = C_t \tag{111}$$

$$D_t = (1-\theta) \left(\Pi_t^*\right)^{-\frac{\epsilon}{1-\alpha}} + \theta \Pi_t^{\frac{\epsilon}{1-\alpha}} D_{t-1} \tag{112}$$

$$1 = \theta \Pi_t^{\epsilon-1} + (1-\theta) {\Pi^*}_t^{\epsilon-1} \tag{113}$$

$$MC_t = \frac{\hat{W}_t N_t}{(1-\alpha) Y_t} \tag{114}$$

$$\frac{\epsilon}{\epsilon-1} x_{1,t} = \left(\Pi_t^*\right)^{1+\frac{\alpha\epsilon}{1-\alpha}} x_{2,t} \tag{115}$$

$$x_{1,t} = C_t^{-\sigma} MC_t Y_t + \beta\theta \mathbb{E}_t \left[\Pi_{t+1}^{\epsilon+\frac{\alpha\epsilon}{1-\alpha}-1} \right] x_{1,t+1} \tag{116}$$

$$x_{2,t} = C_t^{-\sigma} Y_t + \beta\theta \mathbb{E}_t \left[\Pi_{t+1}^{\epsilon-2} \right] x_{2,t+1} \tag{117}$$

$$\ln(A_t) = \rho_a \ln(A_{t-1}) + \epsilon_t^a \qquad \epsilon_t^a \sim N(0, \sigma_a^2) \tag{118}$$

$$R_t = r_t \mathbb{E}_t \left[\Pi_{t+1}\right] \tag{119}$$

$$R_t = \frac{1}{\beta} \left(\Pi_t\right)^{\phi_\pi} \left(\frac{Y_t}{Y_t^*} \right)^{\phi_y} e^{\nu} \tag{120}$$

$$\nu_t = \rho_m \nu_{t-1} + \epsilon_t^m \qquad \epsilon_t^m \sim N(0, \sigma_m^2) \tag{121}$$

Item c)

STEADY-STATE RESULTS:

Variable	Value
ν	0
A	1
Π	1
Π^*	1
R	1.0101
r	1.0101
N	0.745356
Y	0.822071
C	0.822071
W	0.612735
M	0.789678
D	1
MC	0.833333
x_1	2.45098
x_2	2.94118

Table 1: Steady State Values

Item d)

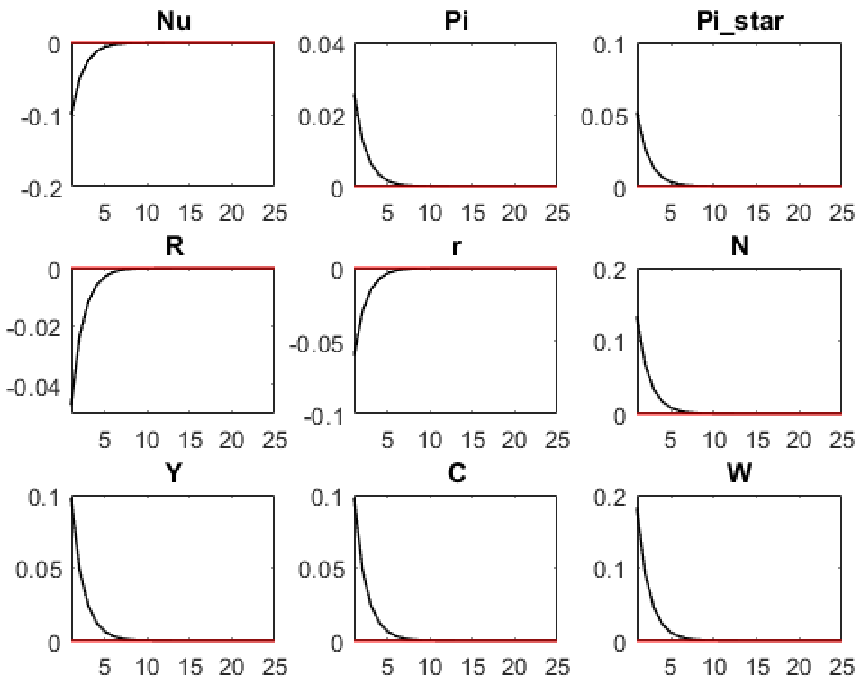

Figure 2: Under Taylor rule - IRF to a negative monetary shock

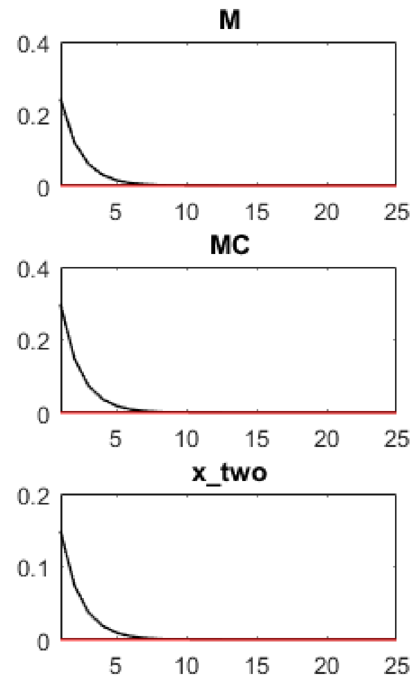

Figure 3: Under Taylor rule - IRF to a negative monetary shock

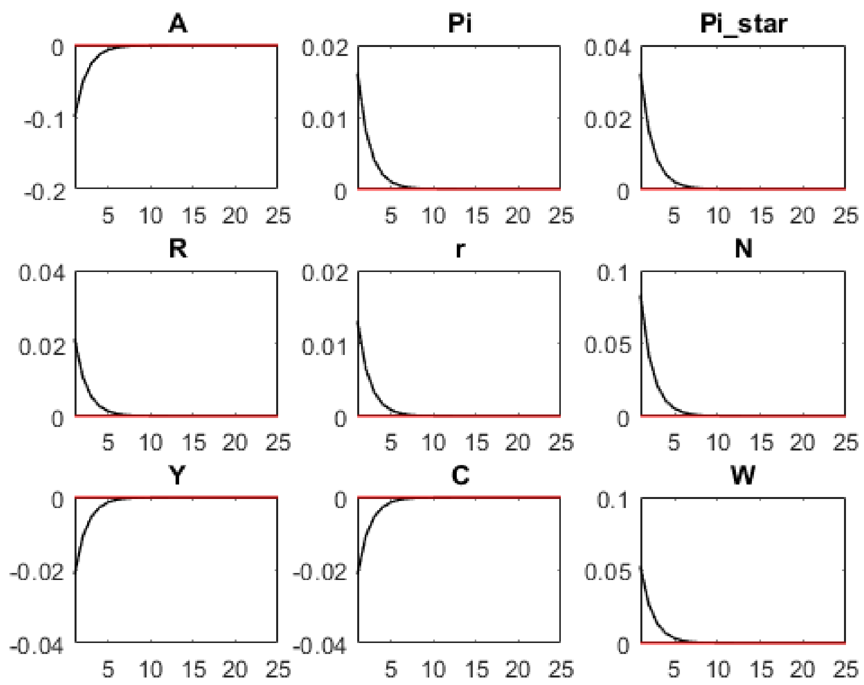

Figure 4: Under Taylor rule - IRF to a negative technology shock

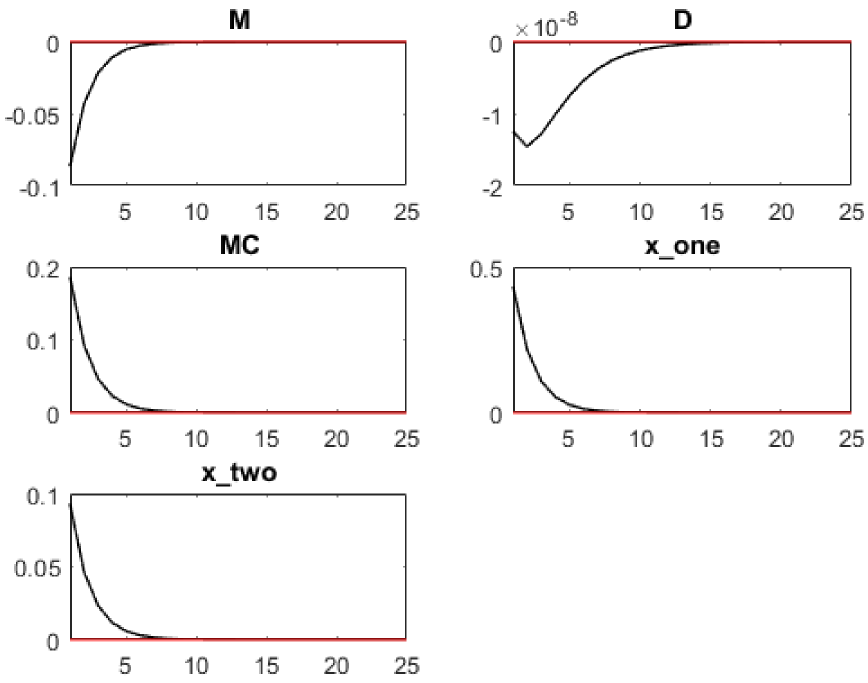

Figure 5: Under Taylor rule - IRF to a negative technology shock

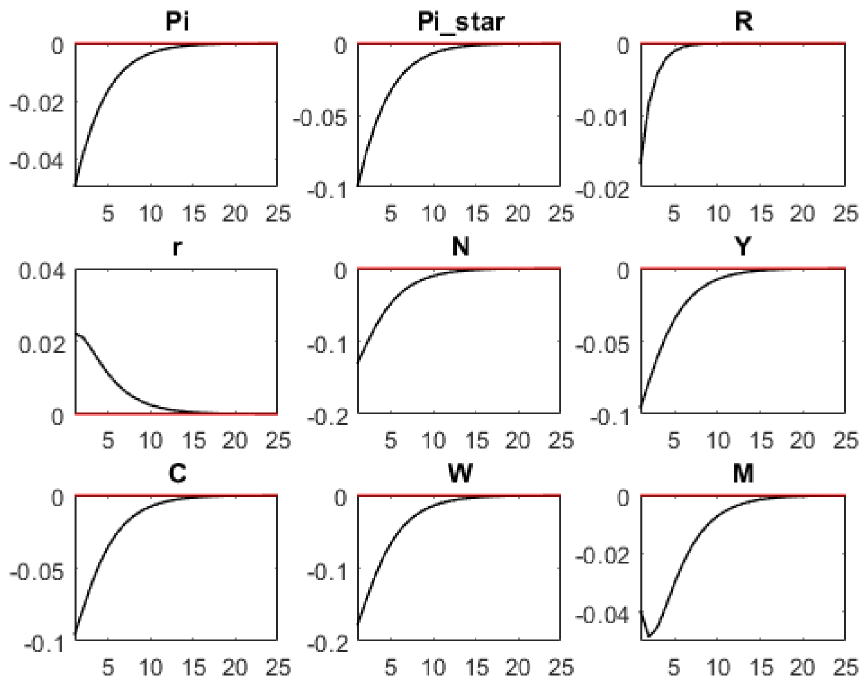

Figure 6: Under monetary rule - IRF to a negative monetary shock

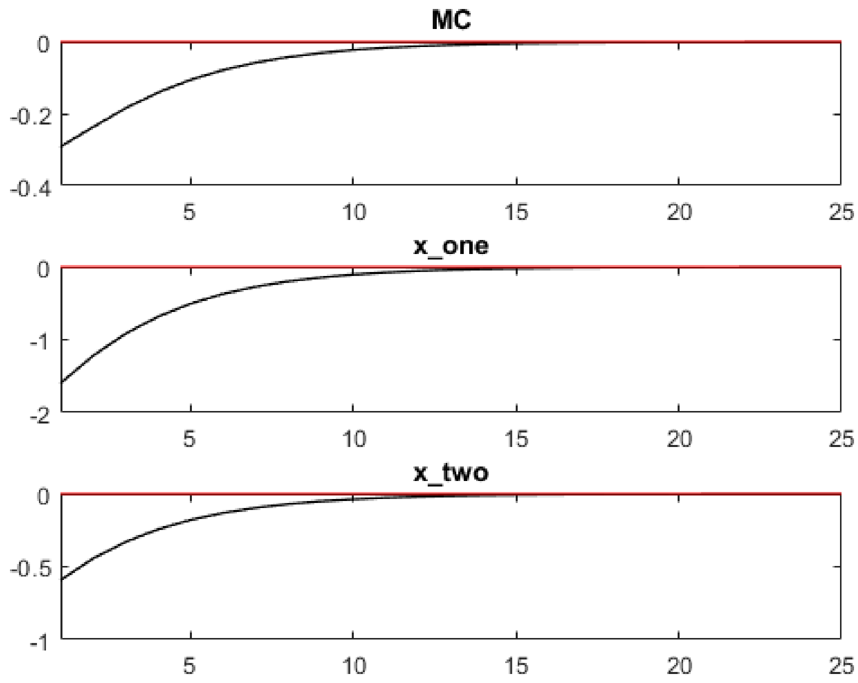

Figure 7: Under monetary rule - IRF to a negative monetary shock

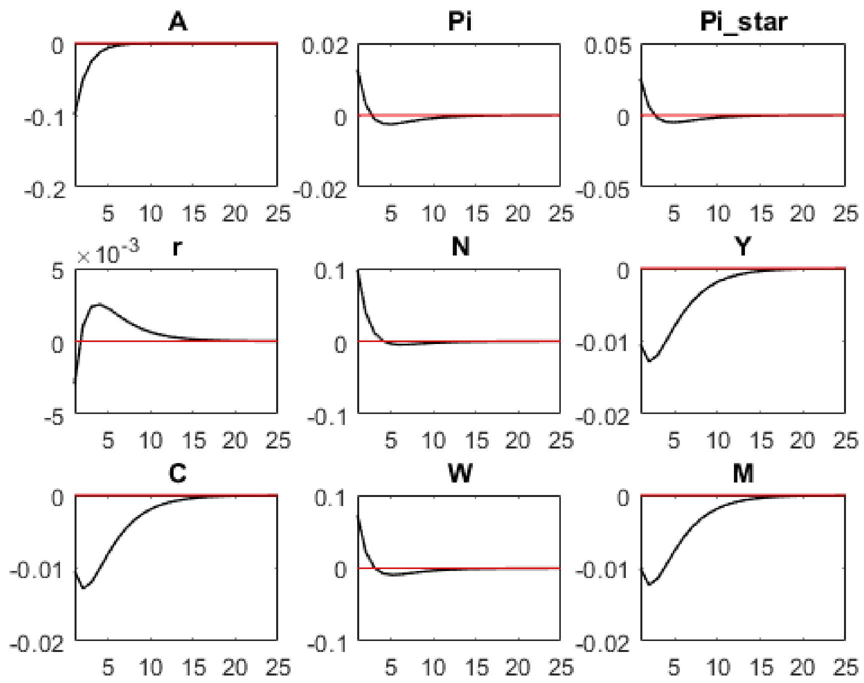

Figure 8: Under monetary rule - IRF to a negative technology shock

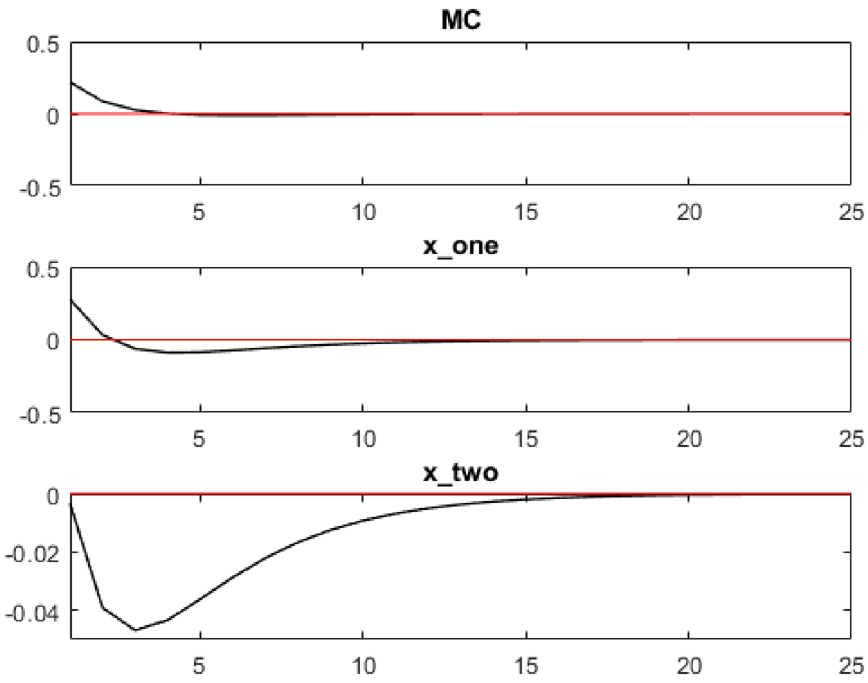

Figure 9: Under monetary rule - IRF to a negative technology shock

Item e)

Let's work with real-valued variables, therefore, from now on wards W_t denotes the real wage and M_t is the real money holdings. Therefore (106) can be written as:

$$\ln\left(W_t\right) = \sigma \ln\left(C_t\right) + \varphi \ln\left(N_t\right) \tag{122}$$

Using a first order Taylor expansion yields

$$\ln\left(W^*\right) + \frac{1}{W^*}\left(W^* - W_t\right) = \ln\left(W^*\right) + \tilde{W}_t \tag{123}$$

$$\begin{aligned} &\ln\left(C^{*\sigma} N^{*\sigma}\right) + \sigma C^{*\sigma-1} N^{*\varphi}\left(C^* - C_t\right) + \\ &\qquad + \varphi C^{*\sigma} N^{*\varphi-1}(N^* - N_t) = C^{*\sigma} N^{*\varphi} + \sigma \tilde{C}_t + \varphi \tilde{N}_t \end{aligned} \tag{124}$$

Substituting equations (123), (123) into (122) and cancel out the terms evaluated at steady state we get the log-linearized version

$$\tilde{W}_t = \sigma \tilde{C}_t + \varphi \tilde{N}_t \tag{125}$$

The steps described above can be applied to all other equations. After applying logs, equation (108) becomes

$$\ln(Q_t) = \ln(\beta) + \ln\left(\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{1}{\Pi_{t+1}}\right]\right) \tag{126}$$

Because of Jensen inequality, we cannot equate the log of the expectation operator to the expectation of the log. Nonetheless, because we are linearizing around the steady state and the error we incur in is negligible, we opted to disregard this restriction. Hence, the equation becomes

$$\ln(Q_t) = \ln(\beta) - \sigma\ln(\mathbb{E}_t[C_{t+1}]) + \sigma C_t - \ln(\mathbb{E}_t[\Pi_{t+1}]) \tag{127}$$

The first-order Taylor expansions of each side can be expressed as

$$\begin{aligned} \ln(Q^*) + \tilde{Q}_t \\ \ln(\beta) - \sigma\ln(C^*) - \sigma\mathbb{E}_t\left[\tilde{C}_{t+1}\right] + \sigma\ln(C^*) + \sigma\tilde{C}_t - \ln(\Pi^*) - \mathbb{E}_t\left[\tilde{\Pi}_{t+1}\right] \end{aligned}$$

Canceling out terms evaluated at steady state yields:

$$\tilde{Q}_t = -\sigma\mathbb{E}_t\left[\tilde{C}_{t+1} - \tilde{C}_t\right] - \mathbb{E}_t\left[\tilde{\Pi}_{t+1}\right]$$

The same procedure is applied to equation (107):

$$\begin{aligned} \ln(R_t) - \ln(Q_t) &= 0 \\ \ln(R^*) + \tilde{R}_t &= -\ln(Q)^* - \tilde{Q}_t \\ \tilde{R}_t &= -\tilde{Q}_t \end{aligned}$$

And equation (109):

$$\begin{aligned} \ln(M_t) &= \ln(Y_t) - \eta\ln(R_t) \\ \ln(M^*) + \tilde{M}_t &= \ln(Y^*) + \tilde{Y}_t - \eta\ln(R^*) - \eta\tilde{R}_t \\ \tilde{M}_t &= \tilde{Y}_t - \eta\tilde{R}_t \end{aligned}$$

Equation (110) becomes:

$$
\begin{aligned}
\ln(Y_t) &= \ln(A_t) + (1-\alpha)\ln(N_t) + (\alpha-1)\ln(D_t) \\
\ln(Y^*) + \tilde{Y}_t &= \ln(A^*) + \tilde{A}_t + (1-\alpha)\ln(N^*) + (1-\alpha)\tilde{N}_t + (\alpha-1)\ln(D^*) + (\alpha-1)\tilde{D}_t \\
\tilde{Y}_t &= \tilde{A}_t + (1-\alpha)\tilde{N}_t + (\alpha-1)\tilde{D}_t
\end{aligned}
$$

Equation (111) becomes:

$$
\begin{aligned}
\ln(Y_t) &= \ln(C_t) \\
\ln(Y^*) + \tilde{Y}_t &= \ln(C^*) + \tilde{C}_t \\
\tilde{Y}_t &= \tilde{C}_t
\end{aligned}
$$

Equation (112) becomes:

$$
\ln(D_t) = \ln\left((1-\theta)\Pi_t^{*\bar{c}-1} + \theta\Pi_t^{\frac{\epsilon}{1-\alpha}} D_{t-1}\right)
$$

$$
\begin{aligned}
\ln(D^*) + \tilde{D}_t =& \ln(\lambda^*) + \frac{(1-\theta)\left(\frac{\epsilon}{\alpha-1}\right)\Pi^{**\,\frac{\epsilon-\alpha+1}{1-\alpha}}}{\lambda^*}(\Pi_t^* - \Pi^{**}) + \\
& \frac{\theta\frac{\epsilon}{1-\alpha}\Pi^{*\,\frac{\epsilon+\alpha-1}{\alpha-1}} D^*}{\lambda^*}(\Pi_t - \Pi^*) + \frac{\theta\Pi^{*\,\frac{\epsilon}{1-\alpha}}}{\lambda^*}(D_{t-1} - D^*)
\end{aligned}
$$

where $\lambda = (1-\theta)(\Pi_t^*)^{\frac{\epsilon}{\alpha-1}} + \theta(\Pi_t)^{\frac{\epsilon}{1-\alpha}} D_{t-1}$ and λ^* equals this expression evaluated at steady state.

Notice that $\Pi^* = \Pi^{**} = D^* = 1$. This implies that $l\lambda^* = 1$. Also, we have that $(\Pi_t^* - \Pi^{**}) = \tilde{\Pi}_t^*$ and this identity holds for Π_t and D_{t-1} as well. After canceling out the terms evaluated at steady state the equation above, then, becomes:

$$
\tilde{D}_t = (1-\theta)\frac{\epsilon}{\alpha-1}\tilde{\Pi}_t^* + \theta\frac{\epsilon}{1-\alpha}\tilde{\Pi}_t + \theta\tilde{D}_{t-1}
$$

As we shall see when log-linearizing the next equation, the equality $\tilde{\Pi}_t^* = \frac{\theta}{1-\theta}\tilde{\Pi}_t$ holds. Substituting this in the equation above, we get that

$$
\begin{aligned}
\tilde{D}_t &= (1-\theta)\frac{\epsilon}{\alpha-1}\frac{\theta}{1-\theta}\tilde{\Pi}_t + \theta\frac{\epsilon}{1-\alpha}\tilde{\Pi}_t + \theta\tilde{D}_{t-1} \\
\tilde{D}_t &= \theta\tilde{D}_{t-1}
\end{aligned}
$$

We now turn to (113)

$$
\ln(1) = \ln\left(\theta\Pi_t^{\epsilon-1} + (1-\theta)\Pi_t^{*1-\epsilon}\right)
$$

$$
\begin{aligned}
0 = & \ln\left(\theta {\Pi^*}^{\epsilon-1} + (1-\theta){\Pi^{**}}^{1-\epsilon}\right) + \frac{\theta(\epsilon-1){\Pi^*}^{\epsilon-2}}{\theta {\Pi^*}^{\epsilon-1} + (1-\theta){\Pi^{**}}^{1-\epsilon}} \left(\Pi_t - \Pi^*\right) + \\
& + \frac{(1-\theta)(1-\epsilon){\Pi^*}^{-\epsilon}}{\theta {\Pi^*}^{\epsilon-1} + (1-\theta){\Pi^{**}}^{1-\epsilon}} \left(\Pi^*_t - \Pi^{**}\right)
\end{aligned}
$$

and recall that $\Pi^* = \Pi^{**} = 1$, and we get that

$$
\begin{aligned}
0 &= \theta(\epsilon-1)\tilde{\Pi}_t + (1-\theta)(1-\epsilon)\tilde{\Pi}^*_t \\
\tilde{\Pi}^*_t &= \frac{\theta}{1-\theta}\tilde{\Pi}_t
\end{aligned}
$$

Equation (114) becomes

$$
\begin{aligned}
\ln\left(MC_t\right) &= \ln\left(W_t\right) + \ln\left(N_t\right) - \ln\left((1-\alpha)Y_t\right) \\
\ln\left(MC^*\right) + \tilde{MC}_t &= \ln\left(W^*\right) + \ln\left(N^*\right) - \ln\left((1-\alpha)Y^*\right) + \tilde{W}_t + \tilde{N}_t - \tilde{Y}_t \\
\tilde{MC}_t &= \tilde{W}_t + \tilde{N}_t - \tilde{Y}_t
\end{aligned}
$$

Equation (115) becomes

$$
\begin{aligned}
\ln\left(\mathcal{M}\right) + \ln\left(x_{1,t}\right) &= \zeta \ln\left(\Pi^*_t\right) + \ln\left(x_{2,t}\right) \\
\ln\left(\mathcal{M}\right) + \ln\left(x^*_1\right) + \tilde{x}_{1,t} &= \zeta \ln\left({\Pi^*}^*\right) + \zeta\tilde{\Pi}^*_t + \ln\left(x^*_2\right) + \tilde{x}_{2,t} \\
\tilde{x}_{1,t} &= \zeta\tilde{\Pi}^*_t + \tilde{x}_{2,t}
\end{aligned}
$$

where $\zeta = 1 + \frac{\alpha\epsilon}{1-\alpha}$

Equation (116) becomes

$$
\begin{aligned}
\ln\left(x_{1,t}\right) &= \ln\left(\Psi\right) \\
\ln\left(x^*_1\right) + \tilde{x}_{1,t} &= \ln\left(\Psi^*\right) + \frac{(1-\sigma){C^*}^{\sigma} MC^*}{x^*_1}\left(C_t - C^*\right) + \\
&+ \frac{{C^*}^{1-\sigma}}{x^*_1}\left(MC_t - MC^*\right) + + \frac{\delta\beta\theta{\Pi^*}^{\delta-1}x^*_1}{x^*_1}\left(\Pi_{t+1} - \Pi^*\right) + \frac{\beta\theta{\Pi^*}^{\delta}}{x^*_1}\left(x_{1,t+1} - x^*_1\right) \\
\tilde{x}_{1,t} &= (1-\beta\theta)\left[(1-\sigma)\tilde{C}_t + \tilde{MC}_t\right] + \beta\theta\mathbb{E}_t\left[\delta\tilde{\Pi}_{t+1} + \tilde{x}_{1,t+1}\right]
\end{aligned}
$$

where we have substituted Y_t for C_t and Y^*_t for C^*_t;

$$
\Psi := C_t^{-\sigma} MC_t Y_t + \beta\theta\mathbb{E}_t\left[\Pi_{t+1}^{\varepsilon + \frac{\alpha\varepsilon}{1-\alpha} - 1} x_{1,t+1}\right]
$$

and Ψ^* is this equation evaluated in steady state; $x_1^* = \frac{C^{*1-\sigma} MC^*}{1-\beta\theta}$; $\delta := \epsilon + \frac{\alpha\epsilon}{1-\alpha} - 1$ and $\Pi^* = 1$.

The log-linearized version of (117) is very similar to that of (116), the only difference being the terms C_t and Π_{t+1} are powered to - and thus the terms multiplying $\tilde{C}_t$ and $\tilde{\Pi}_{t+1}$:

$$\tilde{x}_{2,t} = (1-\beta\theta)\left(-\sigma\tilde{C}_t + \tilde{M}C_t\right) + \beta\theta\mathbb{E}_t\left[(\epsilon-2)\tilde{\Pi}_{t+1} + \tilde{x}_{2,t+1}\right]$$

Equation (118) is linear already. Then, we simply perform a first-order Taylor expansion to get

$$\begin{aligned} \ln(A)^* + \tilde{A}_t = \rho_a \ln(A)^* + \rho_a\tilde{A}_{t-1} + \epsilon^{a^*} + \left(\epsilon_t^a - \epsilon^{a^*}\right) \\ \tilde{A}_t = \rho_a\tilde{A}_{t-1} + \epsilon_t^a \end{aligned}$$

since $\epsilon^{a^*} = 0$. As to (119), the log-linearized version is

$$\begin{aligned} \ln(R)_t &= \ln(r) + \ln\left(\mathbb{E}_t\left[\Pi_{t+1}\right]\right) \\ \ln(R^*) + \tilde{R}_t &= \ln(r^*) + \tilde{r}_t + \ln(\Pi^*) + \mathbb{E}_t\left[\tilde{\Pi}_{t+1}\right] \\ \tilde{R}_t &= \tilde{r}_t + \mathbb{E}_t\left[\tilde{\Pi}_{t+1}\right] \end{aligned}$$

Equation (120) can be expressed as:

$$\begin{aligned} \ln(R)_t &= -\ln(\beta) + \phi_\pi \ln(\Pi)_t + \phi_y\left(\ln(Y)_t - \ln(Y)_t^*\right) + \nu_t \\ \ln(R)^* + \tilde{R}_t &= -\ln(\beta) + \phi_\pi\ln(\Pi)^* + \phi_\pi\tilde{\Pi}_t + \phi_y\left(\ln(Y)^* - \ln(Y)^{**}\right) + \\ &+ \phi_y\left(\tilde{Y}_t - \tilde{Y}_t^*\right) + \nu^* + \nu^*\tilde{\nu}_t \\ \tilde{R}_t &= \phi_\pi\tilde{\Pi}_t + \phi_y\left(\tilde{Y}_t - \tilde{Y}_t^*\right) + \nu^*\tilde{\nu}_t \\ \tilde{R}_t &= \phi_\pi\tilde{\Pi}_t + \phi_y\left(\tilde{Y}_t - \tilde{Y}_t^*\right) \end{aligned}$$

Since $\nu^* = 0$.

As for (121), it is already linear and expressed in terms of deviations from the steady state - the steady state value of the shock is zero. Finally, we must log-linearize the exogeneous money supply, as expressed in 2.3.2 in the problem description. Applying logs on both sides we get

$$\ln(\Delta)\, m_t = \ln(\Delta)\, L_t + \ln(\Delta)\, \Pi_t \Longrightarrow \Delta\tilde{m}_t = \Delta\tilde{L}_t + \Delta\tilde{\Pi}_t$$

And to close the model we log-linearize the equation that describes the innovations to the money supply:

$$\Delta\tilde{m}_t = \rho_a \Delta\tilde{m}_{t-1} + \epsilon_t^m$$

We then get the following log-linearized system:

$$\tilde{W}_t = \sigma\tilde{C}_t + \varphi\tilde{N}_t \tag{128}$$

$$\tilde{Q}_t = -\sigma\mathbb{E}_t\left[\tilde{C}_{t+1} - \tilde{C}_t\right] - \mathbb{E}_t\left[\tilde{\Pi}_{t+1}\right] \tag{129}$$

$$\tilde{R}_t = -\tilde{Q}_t \tag{130}$$

$$\tilde{M}_t = \tilde{Y}_t - \eta\tilde{R}_t \tag{131}$$

$$\tilde{Y}_t = \tilde{A}_t - \eta\tilde{R}_t \tag{132}$$

$$\tilde{Y}_t = \tilde{A}_t + (1-\alpha)\tilde{N}_t + (\alpha-1)\tilde{D}_t \tag{133}$$

$$\tilde{Y}_t = \tilde{C}_t \tag{134}$$

$$\tilde{D}_t = \theta\tilde{D}_{t-1} \tag{135}$$

$$\tilde{\Pi}_t^* = \frac{\theta}{1-\theta}\tilde{\Pi}_t \tag{136}$$

$$\tilde{MC}_t = \tilde{W}_t + \tilde{N}_t - \tilde{Y}_t \tag{137}$$

$$\tilde{x}_{1,t} = \left(1 + \frac{\alpha\epsilon}{1-\alpha}\right)\tilde{\Pi}_t^* + \tilde{x}_{2,t} \tag{138}$$

$$\tilde{x}_{1t} = (1-\beta\theta)\left[(1-\sigma)\tilde{C}_t + \tilde{MC}_t\right] + \beta\theta\left[\left(\epsilon + \frac{\alpha\epsilon}{1-\epsilon} - 1\right)\tilde{\Pi}_{t+1} + \tilde{x}_{1t+1}\right] \tag{139}$$

$$\tilde{x}_{2t} = (1-\beta\theta)\left(-\sigma\tilde{C}_t + MC_t\right) + \beta\theta\left[(\epsilon-2)\tilde{\Pi}_{t+1} + \tilde{x}_{2t+1}\right] \tag{140}$$

$$\tilde{A}_t = \rho_a \tilde{A}_{t-1} + \epsilon_t^a \tag{141}$$

$$\tilde{R}_t = \tilde{r}_t + \mathbb{E}_t \left[\tilde{\Pi}_{t+1}\right] \tag{142}$$

$$\tilde{R}_t = \phi_\pi \tilde{\Pi}_t + \phi_y \left(\tilde{Y}_t - \tilde{Y}_t^*\right) \tag{143}$$

$$\Delta \tilde{m}_t = \Delta \tilde{L}_t + \Delta \tilde{\Pi}_t \tag{144}$$

$$\tilde{\nu}_t = \rho_m \tilde{\nu}_{t+1} + \epsilon_t^m \tag{145}$$

$$\Delta \tilde{m}_t = \rho_a \Delta \tilde{m}_{t-1} + \epsilon_t^m \tag{146}$$

Item f)

See the file 'Q3_Itens_efg.mod' attached.

Item g)

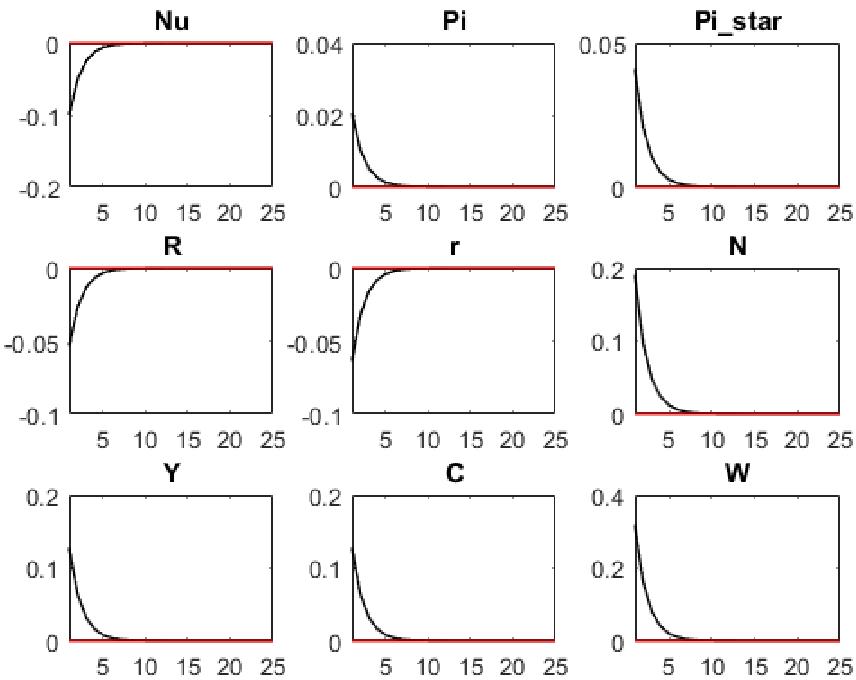

Figure 10: Under Taylor rule - IRF to a negative monetary shock

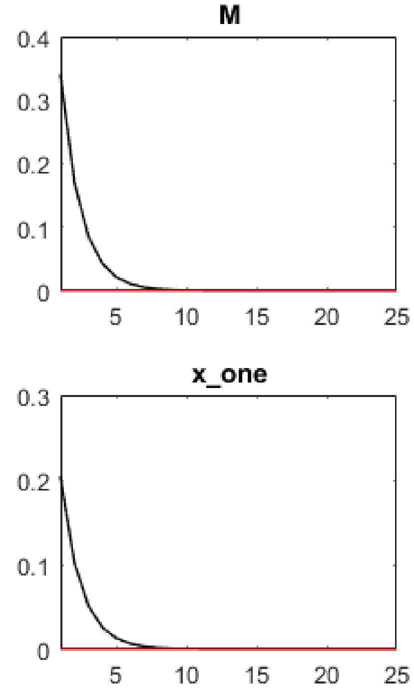

Figure 11: Under Taylor rule - IRF to a negative monetary shock

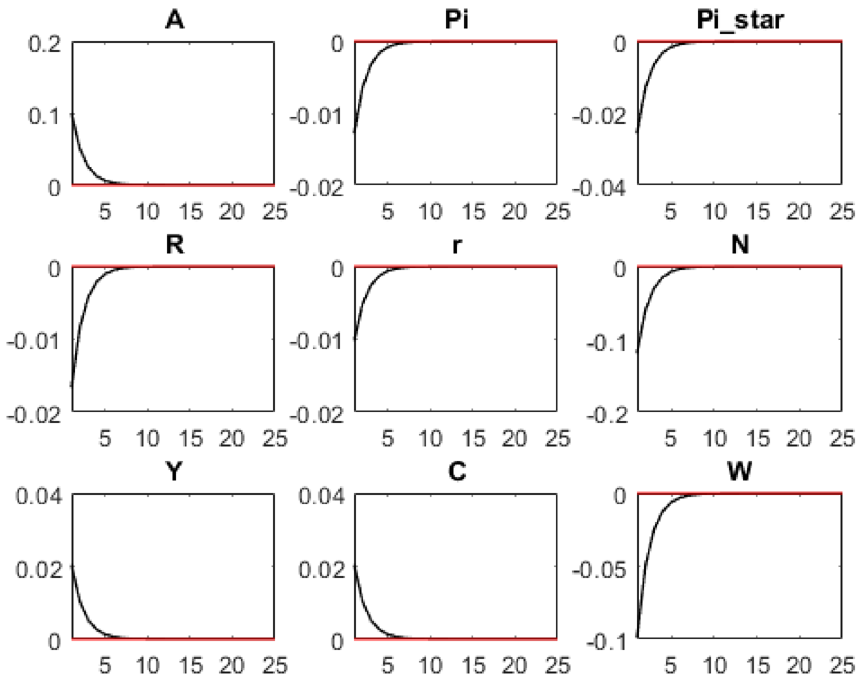

Figure 12: Under Taylor rule - IRF to a positive technology shock

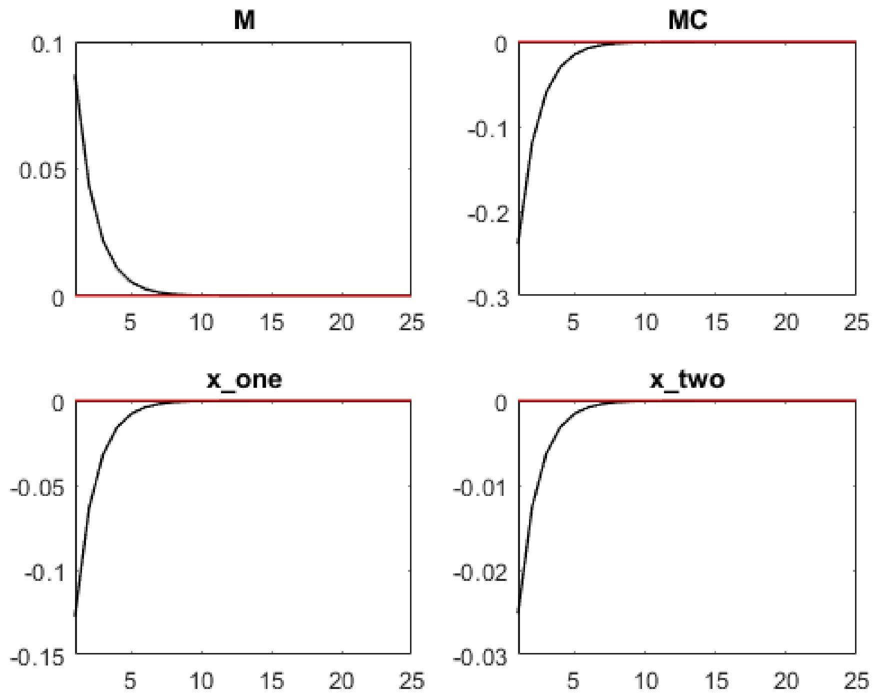

Figure 13: Under Taylor rule - IRF to a positive technology shock

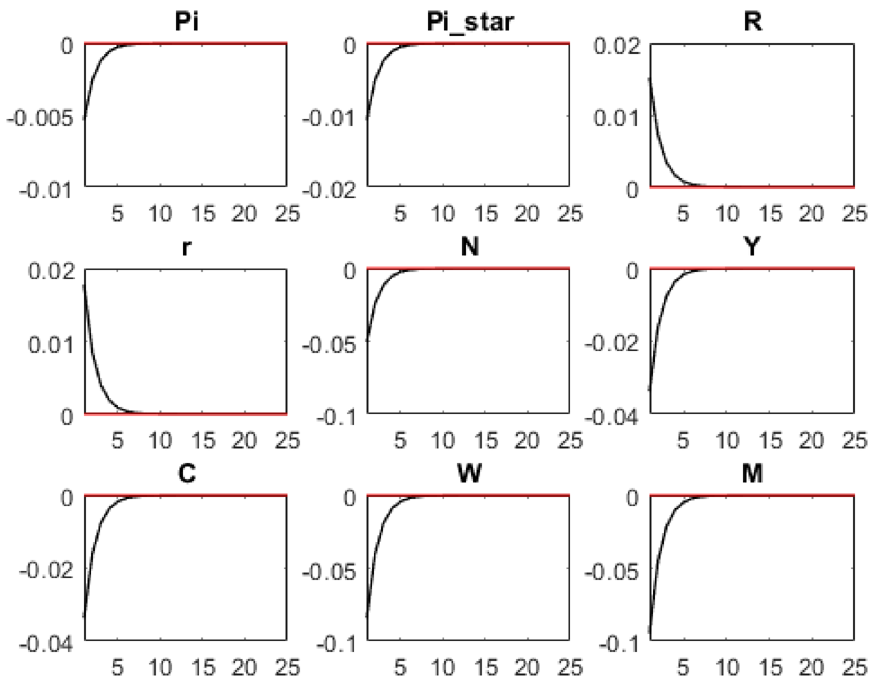

Figure 14: Under monetary rule - IRF to a negative monetary shock

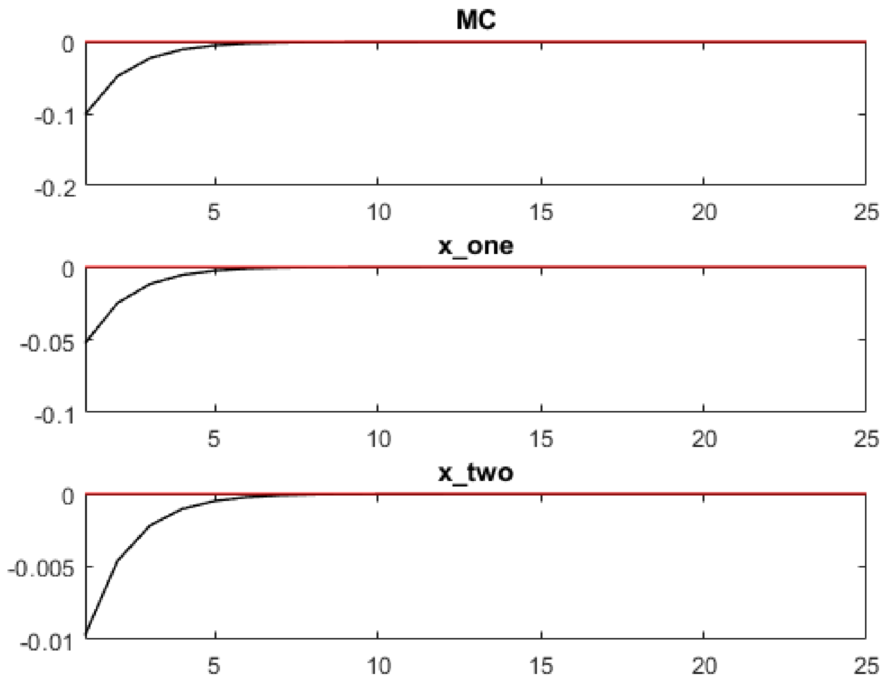

Figure 15: Under monetary rule - IRF to a negative monetary shock

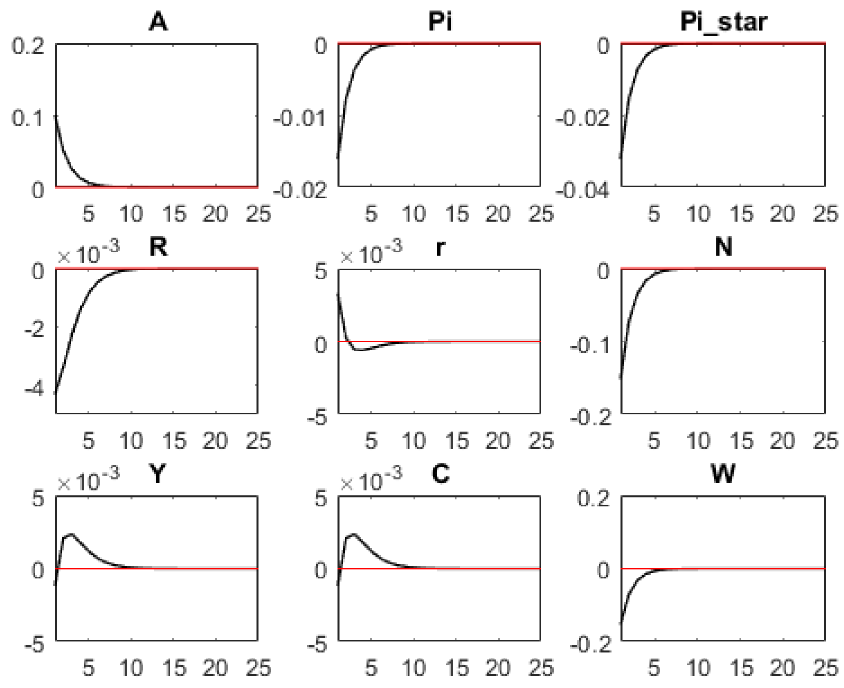

Figure 16: Under monetary rule - IRF to a positive technology shock

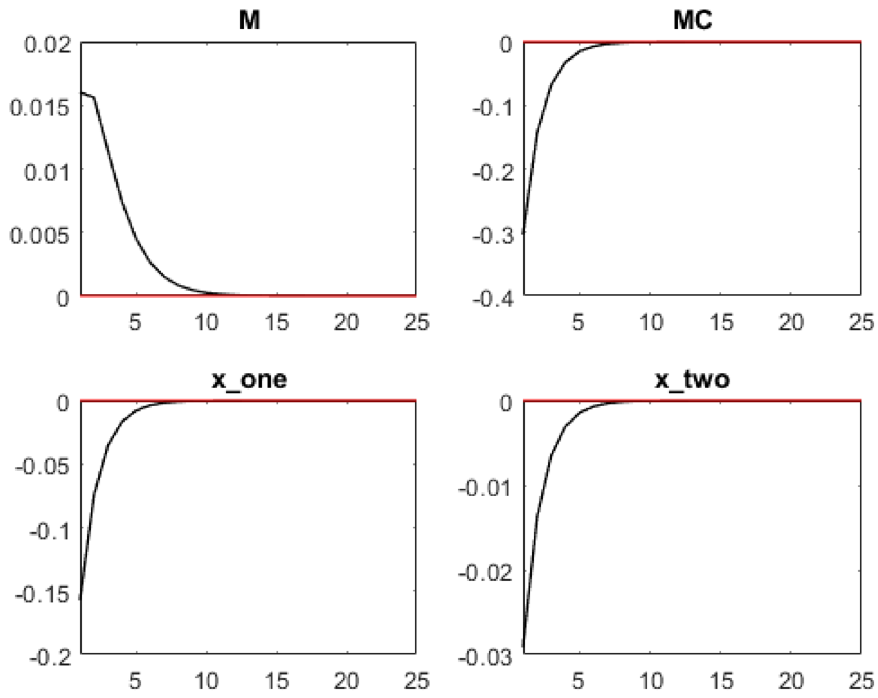

Figure 17: Under monetary rule - IRF to a positive technology shock