Macro III: Problem Set 4

Bruno Tebaldi de Queiroz Barbosa

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1 Questão 1

1.1 item a

Escrevendo o sistema em modelo matricial temos:

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{\gamma} & 1 & \frac{1}{\gamma} \\ 0 & 0 & \beta \end{bmatrix} \begin{bmatrix} i_t \\ \mathbb{E}_t[x_{t+1}] \\ \mathbb{E}_t[\pi_{t+1}] \end{bmatrix} = \begin{bmatrix} 0 & 0 & \delta_\pi \\ 0 & 1 & 0 \\ 0 & -k & 1 \end{bmatrix} \begin{bmatrix} i_{t-1} \\ x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} e_t \\ u_t \\ v_t \end{bmatrix} \tag{1}$$

Assumindo que a primeira matrix de coeficientes não e singular temos:

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{\gamma} & 1 & \frac{1}{\gamma} \\ 0 & 0 & \beta \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{\gamma} & 1 & \frac{-1}{\gamma\beta} \\ 0 & 0 & \frac{1}{\beta} \end{bmatrix} \tag{2}$$

Logo o sistema pode ser escrito por:

$$\begin{bmatrix} i_t \\ \mathbb{E}_t[x_{t+1}] \\ \mathbb{E}_t[\pi_{t+1}] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{\gamma} & 1 & \frac{-1}{\gamma\beta} \\ 0 & 0 & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} 0 & 0 & \delta_\pi \\ 0 & 1 & 0 \\ 0 & -k & 1 \end{bmatrix} \begin{bmatrix} i_{t-1} \\ x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{\gamma} & 1 & \frac{-1}{\gamma\beta} \\ 0 & 0 & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} e_t \\ u_t \\ v_t \end{bmatrix} \tag{3}$$

Simplificando a equação (3) temos:

$$\begin{bmatrix} i_t \\ \mathbb{E}_t[x_{t+1}] \\ \mathbb{E}_t[\pi_{t+1}] \end{bmatrix} = A \begin{bmatrix} i_{t-1} \\ x_t \\ \pi_t \end{bmatrix} + \eta_t \tag{4}$$

aonde:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{\gamma} & 1 & \frac{-1}{\gamma\beta} \\ 0 & 0 & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} 0 & 0 & \delta_\pi \\ 0 & 1 & 0 \\ 0 & -k & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \delta_\pi \\ 0 & 1 + \frac{k}{\gamma\beta} & \frac{\delta_\pi}{\gamma} - \frac{1}{\gamma\beta} \\ 0 & -\frac{k}{\beta} & \frac{1}{\beta} \end{bmatrix} \tag{5}$$

$$\eta_t = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{\gamma} & 1 & \frac{-1}{\gamma\beta} \\ 0 & 0 & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} e_t \\ u_t \\ v_t \end{bmatrix} \tag{6}$$

O sistema terá solução se tivermos dois autovalores fora do circulo unitário (para este problema). Logo vamos determinar os autovalores:

$$det(A - \lambda I) = 0 \tag{7}$$

$$\begin{vmatrix} -\lambda & 0 & \delta_\pi \\ 0 & 1 + \frac{k}{\gamma\beta} - \lambda & \frac{\delta_\pi}{\gamma} - \frac{1}{\gamma\beta} \\ 0 & -\frac{k}{\beta} & \frac{1}{\beta} - \lambda \end{vmatrix} = 0 \tag{8}$$

Calculando o determinante da equação (8)

$$\left[(-\lambda)\left(1 + \frac{k}{\gamma\beta} - \lambda\right)\left(\frac{1}{\beta} - \lambda\right)\right] - \left[(-\lambda)\left(\frac{-k}{\beta}\right)\left(\frac{\delta_\pi}{\gamma} - \frac{1}{\beta\gamma}\right)\right] = 0 \tag{9}$$

$$\left[\left(1 + \frac{k}{\gamma\beta} - \lambda\right)\left(\frac{1}{\beta} - \lambda\right)\right] - \left[\left(\frac{-k}{\beta}\right)\left(\frac{\delta_\pi}{\gamma} - \frac{1}{\beta\gamma}\right)\right] = 0 \tag{10}$$

$$\frac{1}{\beta} - \lambda + \frac{k}{\gamma\beta^2} - \frac{k\lambda}{\gamma\beta} - \frac{\lambda}{\beta} + \lambda^2 + \frac{k\delta_\pi}{\gamma\beta} - \frac{k}{\gamma\beta^2} = 0 \tag{11}$$

cortando os termos e simplificando temos:

$$\frac{1}{\beta} - \lambda - \frac{k\lambda}{\gamma\beta} - \frac{\lambda}{\beta} + \lambda^2 + \frac{k\delta_\pi}{\gamma\beta} = 0 \tag{12}$$

$$\lambda^2 + \left(-1 - \frac{k}{\gamma\beta} - \frac{1}{\beta}\right)\lambda + \left(\frac{1}{\beta} + \frac{k\delta_\pi}{\gamma\beta}\right) = 0 \tag{13}$$

Por Baskara temos que a solução da equação (13) é dado por:

$$\Delta = 1 + \frac{k^2}{\gamma^2\beta^2} + \frac{1}{\beta^2} + \frac{2k}{\gamma\beta} + \frac{2}{\beta} + \frac{2k}{\gamma\beta} - \frac{4}{\beta} - \frac{4k\delta_\pi}{\gamma\beta} \tag{14}$$

$$\Delta = 1 + \frac{k^2}{\gamma^2\beta^2} + \frac{1}{\beta^2} + \frac{2k}{\gamma\beta} + \frac{2}{\beta^2} + \frac{2k}{\gamma\beta^2} - \frac{4}{\beta} - \frac{4k}{\gamma\beta} - \frac{4k(\delta_\pi - 1)}{\gamma\beta} \tag{15}$$

$$\Delta = \left(1 - \frac{k}{\gamma\beta} - \frac{1}{\beta}\right)^2 - \frac{4k(\delta_\pi - 1)}{\gamma\beta} \tag{16}$$

$$b = \left(1 - \frac{k}{\gamma\beta} - \frac{1}{\beta}\right) \tag{17}$$

$$\Delta = b^2 - \frac{4k(\delta_\pi - 1)}{\gamma\beta} = 0 \tag{18}$$

$$\Delta = b^2\left(1 - \frac{4k(\delta_\pi - 1)}{\gamma\beta b^2}\right) \tag{19}$$

Logo a solução de (13) é dado por:

$$\lambda = \frac{\left(1 + \frac{k}{\gamma\beta} + \frac{1}{\beta}\right) \pm b\sqrt{\left(1 - \frac{4k(\delta_\pi - 1)}{\gamma\beta b^2}\right)}}{2} \tag{20}$$

Se considerarmos o caso de $\delta_\pi = 1$ temos:

$$\lambda = \frac{\left(1 + \frac{k}{\gamma\beta} + \frac{1}{\beta}\right) \pm b}{2} \tag{21}$$

$$\lambda = \frac{\left(1 + \frac{k}{\gamma\beta} + \frac{1}{\beta}\right) \pm \left(1 - \frac{k}{\gamma\beta} - \frac{1}{\beta}\right)}{2} \Rightarrow \lambda = \left\{1, \frac{1}{\beta}\left(\frac{k}{\gamma} + 1\right)\right\} \tag{22}$$

Note que (22) tem uma raiz no circulo unitário e uma fora.

$$\frac{k}{\gamma\beta} + \frac{1}{\beta} = \frac{1}{\beta}\left(\frac{k}{\gamma} + 1\right) \geq 1 \Rightarrow b \leq 0 \tag{23}$$

Notemos que em (23) temos que quanto menor b, maior serão as raizes, logo temos que ter:

$$\sqrt{\left(1 - \frac{4k(\delta_\pi - 1)}{\gamma\beta b^2}\right)} < 1 \tag{24}$$

Isso por sua vez implica que:

$$\left(1 - \frac{4k(\delta_\pi - 1)}{\gamma\beta b^2}\right) < 1 \tag{25}$$

$$\frac{4k(\delta_\pi - 1)}{\gamma\beta b^2} > 0 \tag{26}$$

Como b^2, β, γ, k são positivos temos:

$$(\delta_\pi - 1) > 0 \Rightarrow \boxed{\delta_\pi > 1} \tag{27}$$

1.2 item b

Observe que as equações que caracterizam o sistema são:

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1} \tag{28}$$

$$x_t = E_t x_{t+1} - \frac{1}{\gamma}(i_t - E_t \pi_{t+1}) + \epsilon_t^x \tag{29}$$

$$i_t = \delta_\pi \pi_t + \delta_x x_t + \epsilon_t \tag{30}$$

Temos que $\delta_\pi > 0$ e $\delta_x > 0$. Substituindo (30) em (29), temos:

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1} \tag{31}$$

$$x_t = E_t x_{t+1} - \frac{1}{\gamma}(\delta_\pi \pi_t + \delta_x x_t + \epsilon_t - E_t \pi_{t+1}) + \epsilon_t^x \tag{32}$$

podemos reescrever o sistema como:

$$\begin{bmatrix} \gamma & 1 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} \gamma + \delta_x & \delta_\pi \\ -\kappa & 1 \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \epsilon_t - \gamma \epsilon_t^x \\ 0 \end{bmatrix} \tag{33}$$

temos então:

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = A \begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} - \begin{bmatrix} \gamma + \delta_x & \delta_\pi \\ -\kappa & 1 \end{bmatrix}^{-1} \begin{bmatrix} \epsilon_t - \gamma \epsilon_t^x \\ 0 \end{bmatrix} \tag{34}$$

Aonde:

$$A = \begin{bmatrix} \gamma + \delta_x & \delta_\pi \\ -\kappa & 1 \end{bmatrix}^{-1} \begin{bmatrix} \gamma & 1 \\ 0 & \beta \end{bmatrix} \tag{35}$$

Observe que:

$$\begin{bmatrix} \gamma + \delta_x & \delta_\pi \\ -\kappa & 1 \end{bmatrix}^{-1} = \frac{1}{\gamma + \delta_x + \kappa \delta_\pi} \begin{bmatrix} 1 & -\delta_\pi \\ \kappa & \gamma + \delta_x \end{bmatrix} \tag{36}$$

Logo:

$$A = \frac{1}{\gamma + \delta_x + \kappa \delta_\pi} \begin{bmatrix} \gamma & 1 - \delta_\pi \beta \\ \kappa \gamma & \kappa + (\gamma + \delta_x)\beta \end{bmatrix} \tag{37}$$

Seja $\Omega' = \dfrac{1}{\gamma + \delta_x + \kappa}$, então o polinômio característico da matriz A é:

$$P(\lambda) = \lambda^2 - tr(A)\lambda + det(A) \tag{38}$$

Então, os eigenvalues da matriz A são:

$$\lambda_1 + \lambda_2 = tr(A) = T \tag{39}$$
$$\lambda_1 \lambda_2 = det(A) = D \tag{40}$$

Logo,

$$\lambda_1, \lambda_2 = \frac{T \pm \sqrt{T^2 - 4D}}{2} \tag{41}$$

Temos então que:

1. $T^2 > 4D$, então os 2 autovalores são reais e distintos;
2. $T^2 = 4D$, ambos os autovalores são iguais;
3. $T^2 < 4D$, ambos os autovalores são números complexos diferentes.

Usando os valores que temos nós temos na matriz abaixo:

$$A = \Omega' \begin{bmatrix} \gamma & 1-\delta_\pi\beta \\ \kappa\gamma & \kappa+(\gamma+\delta_x)\beta \end{bmatrix} \tag{42}$$

$$P(\lambda) = \lambda^2 - [\Omega'(\gamma+\kappa+(\gamma+\delta_x)\beta)]\lambda + \left[\frac{\beta\gamma}{\gamma+\delta_x+\kappa\delta_\pi}\right] \tag{43}$$

em que o polinômio característico pode ser escrito também como:

$$P(\lambda) = \lambda^2 + b_1\lambda + b_0 \tag{44}$$

$$b_1 = \frac{-(\gamma+\kappa+(\gamma+\delta_x)\beta)}{\gamma+\delta_x+\kappa\delta_\pi} \tag{45}$$

$$b_0 = \frac{\beta\gamma}{\gamma+\delta_x+\kappa\delta_\pi} \tag{46}$$

Observe que nesse caso o equilíbrio do sistema é dado por:

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \Omega' \begin{bmatrix} \gamma & 1-\delta_\pi\beta \\ \kappa\gamma & \kappa+(\gamma+\delta_x)\beta \end{bmatrix} \begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} - \begin{bmatrix} \gamma+\delta_x & \delta_\pi \\ -\kappa & 1 \end{bmatrix}^{-1} \begin{bmatrix} \epsilon_t - \gamma\epsilon_t^x \\ 0 \end{bmatrix} \tag{47}$$

Nós temos que $x_t = \pi_t = 0, \forall t$, é sempre a solução para a equação dinâmica acima e portanto é um equilíbrio da economia se ambos os autovalores da matriz A estão dentro do círculo unitário.

A condição necessária e suficiente para a matriz A ter os dois autovalores dentro do círculo unitário e, consequentemente, o equilíbrio ser único é:

$$|b_0| < 1 \tag{48}$$

$$|b_1| < 1 + b_0 \tag{49}$$

As duas condições acima são as mesmas que:

$$\lambda_1\lambda_2 < 1 \tag{50}$$

$$(\lambda_1 - 1)(\lambda_2 - 1) > 0 \tag{51}$$

A condição dada por (48) resulta em:

$$\delta_x + \kappa\delta_\pi > -(1-\beta)\gamma \tag{52}$$

A equação (52) é verdadeira sempre que $0 < \beta < 1$. Por (49), temos:

$$\begin{aligned} |b_1| &< 1 + b_0 \\ \frac{\gamma+\kappa+(\gamma+\delta_x)\beta}{\gamma+\delta_x+\kappa\delta_\pi} &< 1 + \frac{\beta\gamma}{\gamma+\delta_x+\kappa\delta_\pi} \\ \kappa(\delta_\pi - 1) + (1-\beta)\delta_x &> 0 \end{aligned} \tag{53}$$

$$\boxed{\kappa(\delta_\pi - 1) + (1-\beta)\delta_x > 0} \tag{54}$$

2 Questão 2

2.1 Item a

Problema dos consumidores:

$$\max_{\{C_t,N_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(logC_t + A(1-N_t)\right) \tag{55}$$

sujeito à:

$$C_t + K_t - K_{t-1} = (1-\tau)(R_t-\delta)K_{t-1} + (1-\xi)W_tN_t \tag{56}$$

Reescrevendo a equação (56) temos:

$$C_t + K_t = K_{t-1}[(1-\tau)R_t] + K_{t-1}[1-\delta(1-\tau)] + (1-\xi)W_tN_t \tag{57}$$

Problema das firmas:

$$\max_{\{N_t,K_{t-1}\}} Y_t - W_tN_t - R_tK_{t-1} \tag{58}$$

sujeito à:

$$Y_t = Z_tK_{t-1}^{\alpha}N_t^{1-\alpha} \tag{59}$$

com

$$\log(Z_t) = \rho\log(Z_{t-1}) + \epsilon_t \tag{60}$$

Resolvendo o problema da firma, temos as seguintes equações para R_t e W_t:

$$R_t = \frac{\partial Y_t}{\partial K_{t-1}} = \alpha Z_tK_{t-1}^{\alpha-1}N_t^{1-\alpha} = \alpha\frac{Y_t}{K_{t-1}} \tag{61}$$

$$W_t = \frac{\partial Y_t}{\partial N_t} = (1-\alpha)Z_tK_{t-1}^{\alpha}N_t^{-\alpha} = (1-\alpha)\frac{Y_t}{N_t} \tag{62}$$

Inserindo (61) e (62) em (57):

$$C_t + K_t = K_{t-1}[(1-\tau)\alpha\frac{Y_t}{K_{t-1}}] + K_{t-1}[1-\delta(1-\tau)] + (1-\xi)(1-\alpha)\frac{Y_t}{N_t}N_t \tag{63}$$

Rearranjando:

$$C_t + K_t = Y_t[\alpha(1-\tau) + (1-\alpha)(1-\xi)] + K_{t-1}[1-\delta(1-\tau)] \tag{64}$$

Logo podemos definir o problema dos consumidores como:

$$\max_{\{C_t,N_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\log(C_t) + A(1-N_t)\right) \tag{65}$$

sujeito à:

$$C_t + K_t = Y_t[\alpha(1-\tau) + (1-\alpha)(1-\xi)] + K_{t-1}[1-\delta(1-\tau)] \tag{66}$$

com

$$\log(Z_t) = \rho\log(Z_{t-1}) + \epsilon_t \ ,\epsilon_t \ N(0,\sigma_\epsilon^2) \tag{67}$$

Seja:

$$\psi = [\alpha(1-\tau) + (1-\alpha)(1-\xi)] \tag{68}$$
$$\theta = (1-\tau) \tag{69}$$

Inserindo na restrição orçamentária dos consumidores e montando o Lagrangeano, temos:

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t ((\log C_t + AN_t + A) - \lambda_t (C_t + K_t - \psi Z_t K_{t-1}^{\alpha} N_t^{1-\alpha} - (1-\sigma\delta)K_{t-1})) \tag{70}$$

FOC:

$$\frac{\partial \mathcal{L}}{\partial C_t} = \beta^t (\frac{1}{C_t} - \lambda_t) = 0 \implies \boxed{\frac{1}{C_t} = \lambda_t} \tag{71}$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = \beta^t (-A + \lambda_t \psi Z_t (1-\alpha) K_{t-1}^{\alpha} N_t^{-\alpha}) = 0 \implies \boxed{A = \lambda_t \psi (1-\alpha) \frac{Y_t}{N_t}} \tag{72}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = \beta^t (-(C_t + K_t - \psi Z_t K_{t-1}^{\alpha} N_t^{1-\alpha} - (1-\delta\theta)K_{t-1})) = 0 \implies$$
$$\boxed{C_t + K_t = Z_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta\theta)K_{t-1}} \tag{73}$$

$$\frac{\partial \mathcal{L}}{\partial K_t} = \beta^t(-\lambda_t) + \mathbb{E}_t[\beta^{t+1} \lambda_{t+1} (\psi \alpha \frac{Y_{t+1}}{K_t} + (1-\delta\theta))] = 0 \implies$$
$$\boxed{\lambda_t = \beta \mathbb{E}_t [\lambda_{t+1} (\psi R_{t+1} + (1-\delta\theta))]} \tag{74}$$

Portanto, as equações que caracterizam o equilíbrio competitivo são:

$$\frac{1}{C_t} = \lambda_t \tag{75}$$
$$A = \lambda_t \psi (1-\alpha) \frac{Y_t}{N_t} \tag{76}$$
$$R_t = \alpha \frac{Y_t}{K_{t-1}} \tag{77}$$

$$\lambda_t = \beta \mathbb{E}_t [\lambda_{t+1} (\psi R_{t+1} + (1-\delta\theta))] \tag{78}$$

$$Y_t = Z_t K_{t-1}^{\alpha} N_t^{1-\alpha} \tag{79}$$

$$C_t + K_t = Z_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta\theta)K_{t-1} \tag{80}$$

$$\log Z_t = \rho \log Z_{t-1} + \epsilon_t \tag{81}$$

$$G_t = \tau(R_t - \delta)K_{t-1} + \xi W_t N_t \tag{82}$$

2.2 Item b

$$\frac{1}{C} = \lambda \tag{83}$$

$$A = \lambda\psi(1-\alpha)\frac{Y}{N} \tag{84}$$

$$R = \alpha\frac{Y}{K} \tag{85}$$

$$\lambda = \beta[\lambda(\psi R + (1-\delta\theta))] \tag{86}$$

$$Y = ZK^{\alpha}N^{1-\alpha} \tag{87}$$

$$C + K = \psi Y + (1-\delta\theta)K \tag{88}$$

$$\log(Z) = \rho\log(Z) \tag{89}$$

Usando a função de produção, temos que:

$$Y = \left(Z\left(\frac{Y}{K}\right)^{-\alpha}\right)^{\frac{1}{1-\alpha}} N \tag{90}$$

Observe que podemos resolver o sistema acima de duas maneiras, ou temos que ter A ou N.

Equações que caracterizam o estado estacionário com N dado:

$$\psi R + (1-\delta\theta) = \frac{1}{\beta} \tag{91}$$

$$\frac{Y}{K} = \frac{R}{\alpha} \tag{92}$$

$$Y = \left(Z\left(\frac{Y}{K}\right)^{-\alpha}\right)^{\frac{1}{1-\alpha}} N \tag{93}$$

$$K = \left(\frac{Y}{K}\right)^{-} 1Y \tag{94}$$

$$C = \psi Y - \delta\theta K \tag{95}$$

$$\lambda = \frac{1}{C} \tag{96}$$

$$A = \lambda\psi(1-\alpha)\frac{Y}{N} \tag{97}$$

Equações que caracterizam o estado estacionário com A dado:

$$\psi R + (1-\delta\theta) = \frac{1}{\beta} \tag{98}$$

$$\frac{Y}{K} = \frac{R}{\alpha} \tag{99}$$

$$\frac{Y}{N} = \left(Z\left(\frac{Y}{K}\right)^{-\alpha}\right)^{\frac{1}{1-\alpha}} \tag{100}$$

$$\lambda = \frac{A}{\psi(1-\alpha)\frac{Y}{N}} \tag{101}$$

$$\frac{C}{K} = \psi\frac{Y}{K} - \delta\theta \tag{102}$$

$$C = \frac{1}{\lambda} \tag{103}$$

$$K = \frac{C}{\left(\frac{C}{K}\right)} \tag{104}$$

$$Y = \left(\frac{Y}{K}\right)K \tag{105}$$

2.3 Item C

Nossa tarefa agora é substituir as equações dinâmicas não lineares por equações dinâmicas lineares. Para fazer isso, vamos linearizar as equações que caracterizam o equilíbrio competitivo. A linearização será tal que faremos o seguinte:

$$\hat{X}_t = \log\left(\frac{X_t}{X}\right) \tag{106}$$

onde X_t é a variável, X é a variável no estado estacionário e $\hat{X}_t$ é o desvio de log de X_t de seu estado estacionário X. Observe também que, para $X \approx 0$, temos então $e^x \approx (1+x)$. Pela maneira como criamos as variáveis, é trivial notar que $100\hat{X}$ é aproximadamente o desvio percentual de X_t de X.

Usando a equação (106) e a definição feita acima, podemos observar que:

$$X_t = Xe^{\hat{X}_t} \approx X(1+\hat{X}_t) \tag{107}$$

Observe que, se tivermos a equação $a_t + b_t = c_t$ com a relação no estado constante dada por $a + b = c$, usando o argumento acima, temos que:

$$a_t + b_t = c_t \tag{108}$$
$$ae^{\hat{a}_t} + be^{\hat{b}_t} = ce^{\hat{c}_t} \tag{109}$$
$$a(1+\hat{a}_t) + b(1+\hat{b}_t) = c(1+\hat{c}_t) \tag{110}$$
$$a + a\hat{a}_t + b + b\hat{b}_t = c + c\hat{c}_t \tag{111}$$
$$a\hat{a}_t + b\hat{b}_t = c\hat{c}_t \tag{112}$$

Temos que log-linearizar as seguintes equações que caracterizam o Equilíbrio Competitivo: As equações que caracterizam o equilíbrio competitivo são:

$$\frac{1}{C_t} = \lambda_t \tag{113}$$
$$A = \lambda_t \psi (1-\alpha) \frac{Y_t}{N_t} \tag{114}$$
$$R_t = \alpha \frac{Y_t}{K_{t-1}} \tag{115}$$
$$\lambda_t = \beta \mathbb{E}_t \left[\lambda_{t+1} (\psi R_{t+1} + (1-\delta\theta)) \right] \tag{116}$$
$$Y_t = Z_t K_{t-1}^{\alpha} N_t^{1-\alpha} \tag{117}$$
$$C_t + K_t = \psi Z_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta\theta) K_{t-1} \tag{118}$$
$$\log(Z_t) = \rho \log(Z_{t-1}) + \epsilon_t \tag{119}$$
$$G_t = \tau (R_t - \delta) K_{t-1} + \zeta W_t N_t \tag{120}$$

Nosso objetivo agora é log-linearizar as equações acima que caracterizam o equilíbrio competitivo. Para fazer isso, vamos usar o que declaramos acima e as equações (34) a (40) que caracterizam o estado estacionário:

(i) $\frac{1}{C_t} = \lambda_t$

$$\frac{1}{Ce^{\hat{C}_t}} = \lambda e^{\hat{\lambda}_t} \tag{121}$$
$$1 = C\lambda e^{\hat{C}_t + \hat{\lambda}_t} \tag{122}$$
$$1 = C\lambda (1 + \hat{C}_t + \hat{\lambda}_t) \tag{123}$$

$$\boxed{\hat{C}_t + \hat{\lambda}_t = 0} \tag{124}$$

(ii) $A = \lambda_t \psi (1-\alpha) \frac{Y_t}{N_t}$

$$A = \lambda_t \psi (1-\alpha) \frac{Y_t}{N_t} \tag{125}$$
$$A = \lambda e^{\hat{\lambda}_t} \psi (1-\alpha) \frac{Ye^{\hat{Y}_t}}{Ne^{\hat{N}_t}} \tag{126}$$
$$A = \lambda \psi (1-\alpha) \frac{Y}{N} e^{\hat{\lambda}_t + \hat{Y}_t - \hat{N}_t} \tag{127}$$
$$A \approx A(1 + \hat{\lambda}_t + \hat{Y}_t - \hat{N}_t) \tag{128}$$

$$\boxed{\hat{\lambda}_t + \hat{Y}_t - \hat{N}_t = 0} \tag{129}$$

(iii) $R_t = \alpha \frac{Y_t}{K_{t-1}}$

$$R_t = \alpha \frac{Y_t}{K_{t-1}} \tag{130}$$

$$Re^{\hat{R}_t} = \alpha \frac{Ye^{\hat{Y}_t}}{Ke^{\hat{K}_t}} \tag{131}$$

$$Re^{\hat{R}_t} = \alpha \frac{Y}{K} e^{\hat{Y}_t - \hat{K}_t} \tag{132}$$

$$R(1 + \hat{R}_t) = \alpha \frac{Y}{K}(1 + \hat{Y}_t - \hat{K}_t) \tag{133}$$

$$(R + R\hat{R}_t) = \alpha \frac{Y}{K} + \alpha \frac{Y}{K}(\hat{Y}_t - \hat{K}_t) \tag{134}$$

$$\boxed{-R\hat{R}_t + \alpha \frac{Y}{K}(\hat{Y}_t - \hat{K}_t) = 0} \tag{135}$$

(iv) $Y_t = Z_t K_{t-1}^{\alpha} N_t^{1-\alpha}$

$$Y_t = Z_t K_{t-1}^{\alpha} N_t^{1-\alpha} \tag{136}$$

$$Ye^{\hat{Y}_t} = Ze^{\hat{Z}_t}[Ke^{\hat{K}_{t-1}}]^{\alpha}[Ne^{\hat{N}_t}]^{1-\alpha} \tag{137}$$

$$Ye^{\hat{Y}_t} = Zk^{\alpha} N^{1-\alpha} e^{\hat{Z}_t + \alpha \hat{K}_{t-1} + 1 - \alpha \hat{N}_t} \tag{138}$$

$$(1 + \hat{Y}_t) \approx (1 + \hat{Z}_t + \alpha \hat{K}_{t-1} + 1 - \alpha \hat{N}_t) \tag{139}$$

$$\hat{Y}_t \approx \hat{Z}_t + \alpha \hat{K}_{t-1} + 1 - \alpha \hat{N}_t \tag{140}$$

$$\boxed{-\hat{Y}_t \approx \hat{Z}_t + \alpha \hat{K}_{t-1} + 1 - \alpha \hat{N}_t} \tag{141}$$

(v) $C_t + K_t = \psi Z_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1 - \delta\theta) K_{t-1}$
Vamos usar a idéia que descrevemos nas equações (108) até (112):

$$Ce^{\hat{C}_t} + Ke^{\hat{K}_t} = \psi Ye^{\hat{Y}_t} + (1 - \delta\theta) Ke^{\hat{K}_{t-1}} \tag{142}$$

$$C(1 + \hat{C}_t) + K(1 + \hat{K}_t) \approx \psi Y(1 + \hat{Y}_t) + (1 - \delta\theta) K(1 + \hat{K}_{t-1}) \tag{143}$$

$$C + C\hat{C}_t + K + K\hat{K}_t \approx \psi Y + Y\hat{Y}_t + (1 - \delta\theta) K + K\hat{K}_{t-1} \tag{144}$$

$$C\hat{C}_t + K\hat{K}_t \approx \psi Y\hat{Y}_t + (1 - \delta\theta) K\hat{K}_{t-1} \tag{145}$$

$$\boxed{-C\hat{C}_t - K\hat{K}_t + \psi Y\hat{Y}_t + (1 - \delta\theta) K\hat{K}_{t-1} = 0} \tag{146}$$

(vi) $\lambda_t = \beta \mathbb{E}_t \left[\lambda_{t+1}(\psi R_{t+1} + (1 - \delta\theta))\right]$

$$\lambda e^{\hat{\lambda}_t} = \beta \mathbb{E}_t \left[\lambda e^{\hat{\lambda}_{t+1}} (\psi R e^{\hat{R}_{t+1}} + (1 - \delta\theta)) \right] \tag{147}$$

$$\lambda(1 + \hat{\lambda}_t) = \beta\lambda \mathbb{E}_t \left[\psi R e^{\hat{\lambda}_{t+1} + \hat{R}_{t+1}} + e^{\hat{\lambda}_{t+1}} (1 - \delta\theta) \right] \tag{148}$$

$$(1 + \hat{\lambda}_t) = \mathbb{E}_t \left[\beta\psi R + \beta\psi R(\hat{\lambda}_{t+1} + \hat{R}_{t+1}) + \beta(1 - \delta\theta) + \beta\hat{\lambda}_{t+1})(1 - \delta\theta) \right] \tag{149}$$

$$\tag{150}$$

observe que:

$$1 = \beta[\psi R + (1 - \delta\theta)] \tag{151}$$

$$\hat{\lambda}_t \approx \mathbb{E}_t \left[\beta\psi R(\hat{\lambda}_{t+1} + \hat{R}_{t+1}) + \beta\hat{\lambda}_{t+1}(1 - \delta\theta) \right] \tag{152}$$

$$\hat{\lambda}_t \approx \mathbb{E}_t \left[\beta\psi R\hat{\lambda}_{t+1} + \beta\psi R\hat{R}_{t+1} + \beta\hat{\lambda}_{t+1}(1 - \delta\theta) \right] \tag{153}$$

$$\hat{\lambda}_t \approx \mathbb{E}_t \left[\hat{\lambda}_{t+1} + \beta\psi R\hat{R}_{t+1} \right] \tag{154}$$

$$\boxed{-\hat{\lambda}_t + \mathbb{E}_t \left[\hat{\lambda}_{t+1} + \beta\psi R\hat{R}_{t+1} \right] = 0} \tag{155}$$

(vii) $\log(Z_t) = \rho \log(Z_{t-1}) + \epsilon_t$

$$\boxed{Z_t = \rho Z_{t-1} + \epsilon_t} \tag{156}$$

Portanto, temos que as equações log-linearizadas são dadas pelas equações (124) até (156) e são tais que:

Equações que caracterizam o equilíbrio competitivo log-linearizado:

$$\begin{aligned}
\hat{C}_t + \hat{\lambda}_t &= 0 \\
\hat{\lambda}_t + \hat{Y}_t - \hat{N}_t &= 0 \\
-R\hat{R}_t + \alpha\frac{Y}{K}(\hat{Y}_t - \hat{K}_t) &= 0 \\
-\hat{Y}_t &\approx \hat{Z}_t + \alpha\hat{K}_{t-1} + 1 - \alpha\hat{N}_t \\
-C\hat{C}_t - K\hat{K}_t + \psi Y\hat{Y}_t + (1 - \delta\theta)K\hat{K}_{t-1} &= 0 \\
-\hat{\lambda}_t + \mathbb{E}_t \left[\hat{\lambda}_{t+1} + \beta\psi R\hat{R}_{t+1} \right] &= 0 \\
Z_t &= \rho Z_{t-1} + \epsilon_t
\end{aligned}$$

Nosso objetivo é escrever o sistema acima em formato de matriz. Para fazer isso, vamos aplicar o método do Uhlig.

- Seja x_t um vetor $M \times 1$ de variáveis de estado endógeno;
- Seja y_t um vetor $n \times 1$ de outras variáveis endógenas (controle);
- Deixe Z_t ser um vetor $k \times 1$ de variáveis de estado exógenas.

As relações de equilíbrio entre essas variáveis são tais que:

$$0 = \mathbb{E}_t \left[F x_{t+1} + G x_t + H x_{t-1} + J y_{t+1} + K x_t + L z_{t+1} + M z_t \right] \tag{157}$$

$$z_{t+1} = N z_t + \epsilon_{t+1} \tag{158}$$

$$0 = A x_t + B x_{t-1} + C y_t + D z_t \tag{159}$$

onde C é $l \times n$ e rank$(C) = n$ (com $l \geq n$, ou seja, o número de equações em (159) é maior ou igual do que o número das variáveis endógenas que não são de estado); F é $(m+n-l) \times m$ (ou seja, o número de equações de expectativa é no máximo igual ao número de variáveis de estado endógeno) e N possui apenas autovalores estáveis.

Usando este fato, podemos agora escrever as equações log-linearizadas de equilíbrio na forma dada acima. Primeiro observe que no nosso caso os vetores são tais que:

Vetor de estados endógeno:

$$x_t = [\hat{K}_t] \tag{160}$$

Outras variáveis endógenas

$$y_t = \begin{bmatrix} \hat{\lambda}_t \\ \hat{C}_t \\ \hat{Y}_t \\ \hat{N}_t \\ \hat{R}_t \end{bmatrix} \tag{161}$$

Variáveis de estado exógenas:

$$z_t = [\hat{Z}_t] \tag{162}$$

Portanto, as equações são dadas por:

$$0 = \mathbb{E}_t \left[0\hat{K}_{t+1} + 0\hat{K}_t + 0\hat{K}_{t-1} + \begin{bmatrix} 1 & 0 & 0 & 0 & \beta\psi R \end{bmatrix} \begin{bmatrix} \hat{\lambda}_{t+1} \\ \hat{C}_{t+1} \\ \hat{Y}_{t+1} \\ \hat{N}_{t+1} \\ \hat{R}_{t+1} \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\lambda}_t \\ \hat{C}_t \\ \hat{Y}_t \\ \hat{N}_t \\ \hat{R}_t \end{bmatrix} + 0\hat{Z}_{t+1} + 0\hat{Z}_t \right] \tag{163}$$

$$0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -k \end{bmatrix} \begin{bmatrix} \hat{K}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\alpha\frac{Y}{K} \\ \alpha \\ (1-\delta\theta)K \end{bmatrix} \begin{bmatrix} \hat{K}_{t+1} \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & \alpha\frac{Y}{K} & 0 & -R \\ 0 & 0 & -1 & (1-\alpha) & 0 \\ 0 & -C & \psi Y & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\lambda}_t \\ \hat{C}_t \\ \hat{Y}_t \\ \hat{N}_t \\ \hat{R}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} \hat{Z}_t \end{bmatrix} \tag{164}$$

$$\hat{Z}_{t+1} = \rho \hat{Z}_t + \epsilon_{t+1} \tag{165}$$

2.4 Itens d a f

R: Visto que os itens são optativos, escolhemos por não faze-los.

3 Questão 3

3.1 Item a

O HJB associado ao problema dos consumidores pode ser escrito como:

$$\rho V(a_t) = \max_{c_t, l_t} \{u(c_t, l_t) + V'(a_t)(w_t(1 - l_t) + r_t a_t - c_t)\} \tag{166}$$

Passos para obter a equação

1. reescrever a função valor em unidades de tempo de t+Δ.
2. rearranjar os termos e dividir os dois lados da equação por Δ.
3. calcular o limite quando $\Delta \to 0$.

3.2 Item b

MatLab Code

3.3 Item c

MatLab Code

3.4 Item d

Consumidores:

$$\frac{\partial u(c,l)}{\partial c} = V'(a) = \frac{1}{c} \tag{167}$$

$$\frac{\partial u(c,l)}{\partial l} = wlV'(a) = \frac{\eta}{l} \tag{168}$$

Firmas:

$$\Pi = AK_t^{\alpha}(N_t)^{1-\alpha} - w_t N_t - (r_t + \delta)K_t \tag{169}$$

$$(N_t) : w_t = (1-\alpha)A_t \frac{K_t^\alpha}{N_t^\alpha} = (1-\alpha)A_t k_t^\alpha \tag{170}$$

$$(K_t) : r_t = \alpha A_t \left(\frac{N_t}{K_t}\right)^{1-\alpha} - \delta = \alpha A_t \left(k_t\right)^{\alpha-1} - \delta \tag{171}$$

Substituindo em (166) temos:

$$\begin{aligned}
\rho V(a_t) =& \max_{c_t, l_t} \{u(c_t, l_t) + V'(a_t)(w_t(1-l_t) + r_t a_t - c_t)\} \\
\rho V'(a_t) =& V''(a)(w(1-l) + ra - c) + V'(a)r \\
\frac{\partial^2 u(c,l)}{\partial c^2}\frac{\partial c}{\partial a} =& V''(a) \\
\rho \frac{\partial u(c,l)}{\partial c} =& \frac{\partial^2 u(c,l)}{\partial c^2}\frac{\partial c}{\partial a}(w(1-l) + ra - c) + \frac{\partial u(c,l)}{\partial c}(r) \\
-\frac{\partial^2 u(c,l)}{\partial c^2}\frac{\partial c}{\partial a}(w(1-l) + ra - c) =& \frac{\partial u(c,l)}{\partial c}(r - \rho) \\
\frac{1}{c}\dot{c} =& r - \rho
\end{aligned}$$

Logo temos que a dinâmica do sistema é dado por:

$$\dot{c} = c(r - \rho) \tag{172}$$

$$\dot{a} = w(1-l) + ra - c \tag{173}$$

Com isso temos que o *steady state* é:

$$0 = c(r - \rho) \implies r = \rho \tag{174}$$

$$0 = w(1-l) + ra - c \implies c = w(1-l) + ra \tag{175}$$

Macro III: Problem Set 3

Deadline: Monday, 08/10/2018

Aluno: Bruno Tebaldi de Queiroz Barbosa (C174887)

Professor: Tiago Cavalcanti

Source code disponível em: https://github.com/btebaldi/Macro3/tree/master/PSet_04

Script construido baseado nos scripts de B.Moll

Fonte original em: http://www.princeton.edu/~moll/HACTproject.htm

Questão 3-b e 3-c

Limpeza de Variaveis

```
clear all; clc; close all;

rho = 0.05;
r = 0.03;
z1 = 1;
eta = 0.75;

w=1;

I= 150;
amin = -0.15;
amax = 3;
a = linspace(amin,amax,I)';
da = (amax-amin)/(I-1);

maxit= 10000;
crit = 10^(-6);
Delta = 1000;

dVf = zeros(I,1);
dVb = zeros(I,1);
c = zeros(I,1);

options=optimset('Display','off');
x0 = 1;


% Define w e r
AA = 1;
delta = 0.06;
alpha = 0.33;
```

```
check = 1;

r_high = rho;
r_low = 0;

while check ==1

    w = (1-alpha) * AA *((r+delta)/(AA*alpha))^(alpha/(alpha-1));


    tic;
    for i=1:I
        params = [a(i),z1,w,r];
        myfun = @(l) SolveLabor(l,params);
        [l01,fval,exitflag] = fzero(myfun,x0,options);

        l0(i,:)=[l01];
    end
    toc

    v0(:,1) = log(w*z1.*l0(1,1) + r.*a)/rho;

    lmin = l0(1,:);
    lmax = l0(I,:);

    v = v0;

    for n=1:maxit
        V = v;
        V_n(:,n)=V;
        % forward difference
        dVf(1:I-1) = (V(2:I)-V(1:I-1))/da;
        dVf(I) = (w*z1.*lmax + r.*amax).^(-1); %state constraint boundary condition
        % backward difference
        dVb(2:I) = (V(2:I,:)-V(1:I-1,:))/da;
        dVb(1) = (w*z1.*lmin + r.*amin).^(-1); %state constraint boundary condition

        %consumption and savings with forward difference
        cf = dVf.^(-1);
        lf = 1-(dVf.*w.*z1/eta).^(-1);
        ssf = w*z1.*lf + r.*a - cf;
        %consumption and savings with backward difference
        cb = dVb.^(-1);
        lb = 1-((dVb.*w.*z1/eta).^(-1));
        ssb = w*z1.*lb + r.*a - cb;
        %consumption and derivative of value function at steady state
        c0 = w*z1.*l0 + r.*a;
        dV0 = c0.^(-1);

        Ib = ssb < 0; %negative drift --> backward difference
        If = (ssf > 0).*(1-Ib); %positive drift --> forward difference
        I0 = (1-If-Ib); %at steady state
```

```
        c = cf.*If + cb.*Ib + c0.*I0;
        l = lf.*If + lb.*Ib + l0.*I0;
        u = log(c) + eta*log(1-l);

        %CONSTRUCT MATRIX
        X = -Ib.*ssb/da;
        Y = -If.*ssf/da + Ib.*ssb/da;
        Z = If.*ssf/da;

        A1=spdiags(Y(:,1),0,I,I)+spdiags(X(2:I,1),-1,I,I)+spdiags([0;Z(1:I-1,1)],1,I,I);

        A = A1;
        B = (1/Delta + rho)*speye(I) - A;

        u_stacked = [u(:)];
        V_stacked = [V(:)];

        b = u_stacked + V_stacked/Delta;
        V_stacked = B\b; %SOLVE SYSTEM OF EQUATIONS

        Vchange = V_stacked - v;
        v = V_stacked;

        dist(n) = max(abs(Vchange));
        if dist(n)<crit
            disp('Value Function Converged, Iteration = ')
            disp(n)
            break
        end
    end
    toc;
```

```
Elapsed time is 0.194707 seconds.
Value Function Converged, Iteration = 
     4
Elapsed time is 0.271381 seconds.
Elapsed time is 0.070412 seconds.
Value Function Converged, Iteration = 
     6
Elapsed time is 0.083576 seconds.
Elapsed time is 0.085736 seconds.
Value Function Converged, Iteration = 
     7
Elapsed time is 0.088621 seconds.
Elapsed time is 0.083117 seconds.
Value Function Converged, Iteration = 
     8
Elapsed time is 0.086093 seconds.
Elapsed time is 0.080363 seconds.
Value Function Converged, Iteration = 
    29
```

```
Elapsed time is 0.139076 seconds.
```

MARKET CLEARING CONDITIONS

```
    %%%%%%%%%%%%%%%%%%%%%%%%%%
    % FOKKER-PLANCK EQUATION %
    %%%%%%%%%%%%%%%%%%%%%%%%%%
    AT = A';
    b = zeros(I,1);

    %need to fix one value, otherwise matrix is singular
    i_fix = 1;
    b(i_fix)=.1;
    row = [zeros(1,i_fix-1),1,zeros(1,I-i_fix)];
    AT(i_fix,:) = row;

    %Solve linear system
    gg = AT\b;
```

```
Warning: Matrix is singular to working precision.
```

```
    g_sum = gg'*ones(I,1)*da;
    gg = gg./g_sum;
    g = gg;

    check1 = g(:,1)'*ones(I,1)*da;

    Asset_Supply = g(:,1)'*a*da;

    if Asset_Supply > crit
        r_high = r;
        r = (r_high + r_low)/2;
    elseif Asset_Supply < -crit
        r_low = r;
        r = (r_high + r_low)/2;
    else
        check = 0;
    end
end
fprintf('\nFunção convergiu.\n')
fprintf('taxa de juros encontrada: %2.3f\n', r)
```

```
Função convergiu.
taxa de juros encontrada: 0.049
```

Macro III: Problem Set 3

Deadline: Monday, 08/10/2018

Aluno: Bruno Tebaldi de Queiroz Barbosa (C174887)

Professor: Tiago Cavalcanti

Source code disponível em: https://github.com/btebaldi/Macro3/tree/master/PSet_04

Script construido baseado nos scripts de B.Moll

Fonte original em: http://www.princeton.edu/~moll/HACTproject.htm

Questão 4

```
% limpa variaveis
clear all;
clc;

% fecha eventuais janelas abertas
close all;

% inicia o cronometro
tic;

% define variaveis
ga = 2;
rho = 0.05;
d = 0.05;
al = 1/3;
Aprod = 0.1;
z1 = 1;
z2 = 2*z1;
z = [z1,z2];
la1 = 1/3;
la2 = 1/3;
la = [la1,la2];
z_ave = (z1*la2 + z2*la1)/(la1 + la2);

I= 1000;
amin = 0;
amax = 20;
a = linspace(amin,amax,I)';
da = (amax-amin)/(I-1);

aa = [a,a];
zz = ones(I,1)*z;


maxit= 100;
```

```
crit = 10^(-6);
Delta = 1000;

dVf = zeros(I,2);
dVb = zeros(I,2);
c = zeros(I,2);

Aswitch = [-speye(I)*la(1),speye(I)*la(1);speye(I)*la(2),-speye(I)*la(2)];

Ir = 40;
crit_S = 10^(-5);


rmax = 0.049;
r = 0.04;
w = 0.05;

r0 = 0.03;
rmin = 0.01;
rmax = 0.99*rho;


for ir=1:Ir;

r_r(ir)=r;
rmin_r(ir)=rmin;
rmax_r(ir)=rmax;

KD(ir) = (al*Aprod/(r + d))^(1/(1-al))*z_ave;
w = (1-al)*Aprod*KD(ir).^al*z_ave^(-al);

if w*z(1) + r*amin < 0
    disp('CAREFUL: borrowing constraint too loose')
end

v0(:,1) = (w*z(1) + r.*a).^(1-ga)/(1-ga)/rho;
v0(:,2) = (w*z(2) + r.*a).^(1-ga)/(1-ga)/rho;

if ir>1
v0 = V_r(:,:,ir-1);
end

v = v0;

for n=1:maxit
    V = v;
    V_n(:,:,n)=V;
    % forward difference
    dVf(1:I-1,:) = (V(2:I,:)-V(1:I-1,:))/da;
    dVf(I,:) = (w*z + r.*amax).^(-ga); %will never be used, but impose state constraint a<=ama
    % backward difference
    dVb(2:I,:) = (V(2:I,:)-V(1:I-1,:))/da;
    dVb(1,:) = (w*z + r.*amin).^(-ga); %state constraint boundary condition
```

```
    %consumption and savings with forward difference
    cf = dVf.^(-1/ga);
    ssf = w*zz + r.*aa - cf;
    %consumption and savings with backward difference
    cb = dVb.^(-1/ga);
    ssb = w*zz + r.*aa - cb;
    %consumption and derivative of value function at steady state
    c0 = w*zz + r.*aa;

    % dV_upwind makes a choice of forward or backward differences based on
    % the sign of the drift
    If = ssf > 0; %positive drift --> forward difference
    Ib = ssb < 0; %negative drift --> backward difference
    I0 = (1-If-Ib); %at steady state

    c = cf.*If + cb.*Ib + c0.*I0;
    u = c.^(1-ga)/(1-ga);

    %CONSTRUCT MATRIX
    X = -min(ssb,0)/da;
    Y = -max(ssf,0)/da + min(ssb,0)/da;
    Z = max(ssf,0)/da;

    A1=spdiags(Y(:,1),0,I,I)+spdiags(X(2:I,1),-1,I,I)+spdiags([0;Z(1:I-1,1)],1,I,I);
    A2=spdiags(Y(:,2),0,I,I)+spdiags(X(2:I,2),-1,I,I)+spdiags([0;Z(1:I-1,2)],1,I,I);
    A = [A1,sparse(I,I);sparse(I,I),A2] + Aswitch;

    if max(abs(sum(A,2)))>10^(-9)
       disp('Improper Transition Matrix')
       %break
    end

    B = (1/Delta + rho)*speye(2*I) - A;

    u_stacked = [u(:,1);u(:,2)];
    V_stacked = [V(:,1);V(:,2)];

    b = u_stacked + V_stacked/Delta;
    V_stacked = B\b; %SOLVE SYSTEM OF EQUATIONS

    V = [V_stacked(1:I),V_stacked(I+1:2*I)];

    Vchange = V - v;
    v = V;

    dist(n) = max(max(abs(Vchange)));
    if dist(n)<crit
        fprintf('Value Function Converged, Iteration = %d\n', n)
        break
    end
end
toc;
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%
% FOKKER-PLANCK EQUATION %
%%%%%%%%%%%%%%%%%%%%%%%%%%
AT = A';
b = zeros(2*I,1);

%need to fix one value, otherwise matrix is singular
i_fix = 1;
b(i_fix)=.1;
row = [zeros(1,i_fix-1),1,zeros(1,2*I-i_fix)];
AT(i_fix,:) = row;

%Solve linear system
gg = AT\b;
g_sum = gg'*ones(2*I,1)*da;
gg = gg./g_sum;

g = [gg(1:I),gg(I+1:2*I)];

check1 = g(:,1)'*ones(I,1)*da;
check2 = g(:,2)'*ones(I,1)*da;

g_r(:,:,ir) = g;
adot(:,:,ir) = w*zz + r.*aa - c;
V_r(:,:,ir) = V;

KS(ir) = g(:,1)'*a*da + g(:,2)'*a*da;
S(ir) = KS(ir) - KD(ir);

%UPDATE INTEREST RATE
if S(ir)>crit_S
    disp('Excess Supply')
    rmax = r;
    r = 0.5*(r+rmin);
elseif S(ir)<-crit_S;
    disp('Excess Demand')
    rmin = r;
    r = 0.5*(r+rmax);
elseif abs(S(ir))<crit_S;
    fprintf('Equilibrium Found, Interest rate = %f\n', r)
    break
end

end
```

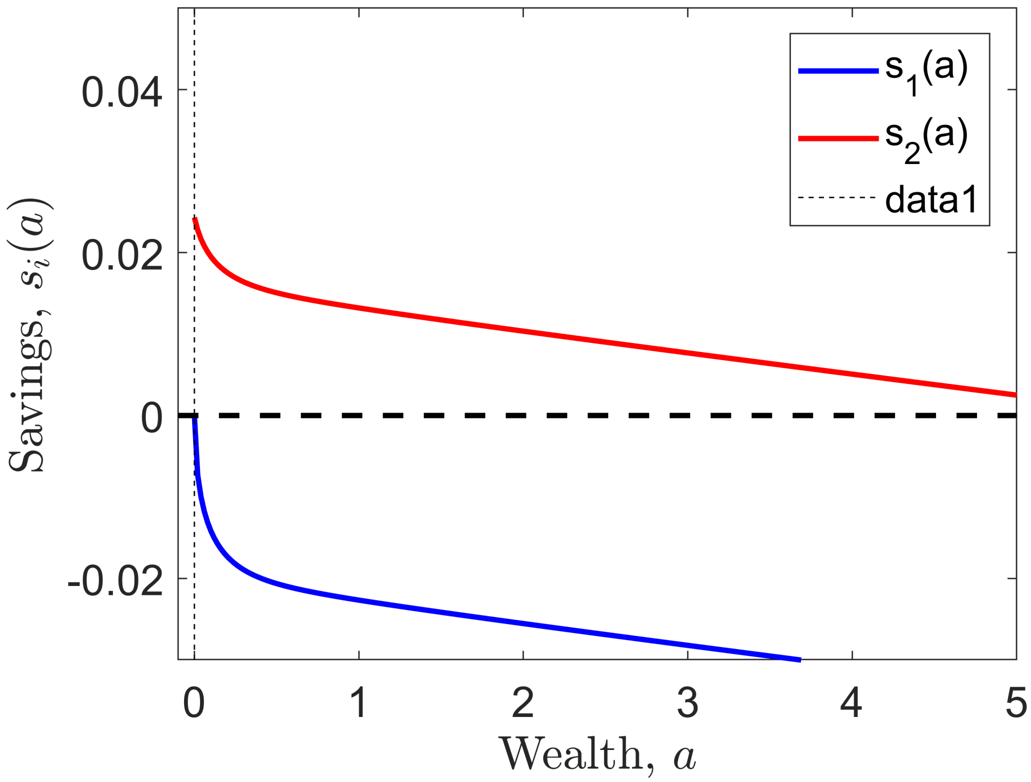
```
Value Function Converged, Iteration = 7
Elapsed time is 0.680322 seconds.
Excess Demand
Value Function Converged, Iteration = 6
Elapsed time is 0.722237 seconds.
Excess Demand
Value Function Converged, Iteration = 5
```

```
Elapsed time is 0.771103 seconds.
Excess Supply
Value Function Converged, Iteration = 5
Elapsed time is 0.805085 seconds.
Excess Supply
Value Function Converged, Iteration = 5
Elapsed time is 0.838482 seconds.
Excess Supply
Value Function Converged, Iteration = 5
Elapsed time is 0.875573 seconds.
Excess Supply
Value Function Converged, Iteration = 5
Elapsed time is 0.900141 seconds.
Excess Demand
Value Function Converged, Iteration = 4
Elapsed time is 0.928694 seconds.
Excess Demand
Value Function Converged, Iteration = 4
Elapsed time is 0.955599 seconds.
Excess Supply
Value Function Converged, Iteration = 4
Elapsed time is 0.977354 seconds.
Excess Demand
Value Function Converged, Iteration = 4
Elapsed time is 0.999532 seconds.
Excess Supply
Value Function Converged, Iteration = 4
Elapsed time is 1.025308 seconds.
Excess Supply
Value Function Converged, Iteration = 4
Elapsed time is 1.051013 seconds.
Excess Supply
Value Function Converged, Iteration = 3
Elapsed time is 1.076275 seconds.
Excess Supply
Value Function Converged, Iteration = 3
Elapsed time is 1.098829 seconds.
Excess Demand
Value Function Converged, Iteration = 3
Elapsed time is 1.122672 seconds.
Equilibrium Found, Interest rate = 0.044992
```

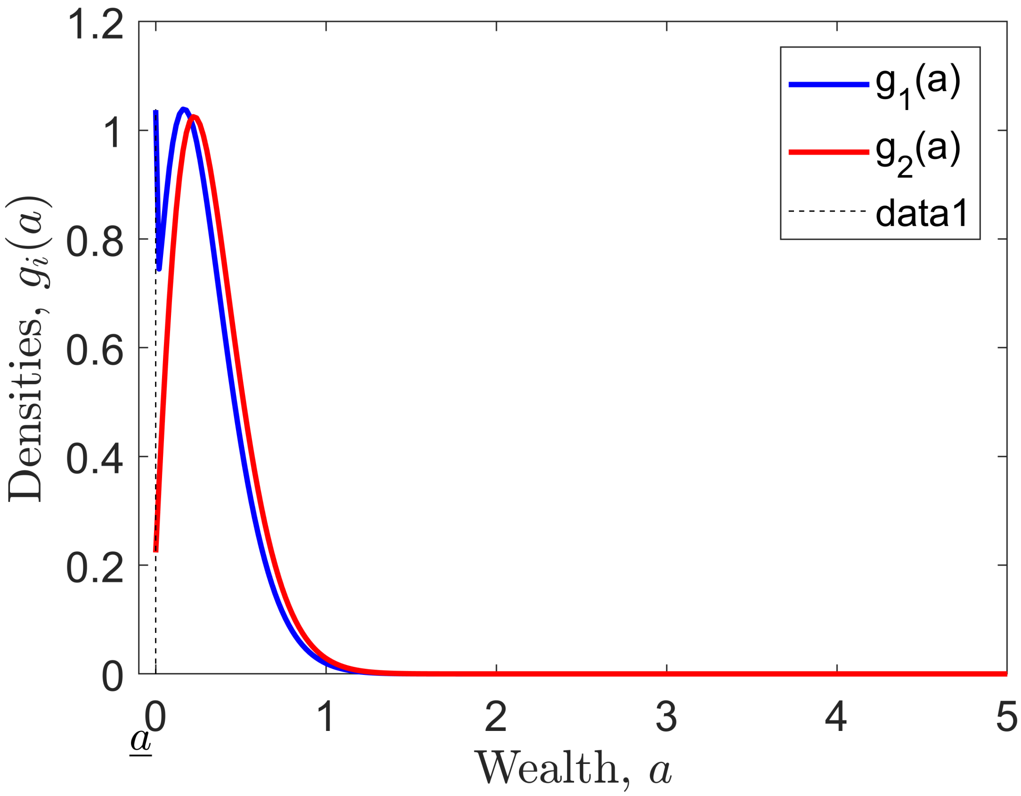
```
amax1 = 5;
amin1 = amin-0.1;

figure(1)
h1 = plot(a,adot(:,1,ir),'b',a,adot(:,2,ir),'r',linspace(amin1,amax1,I),zeros(1,I),'k--','Line
legend(h1,'s_1(a)','s_2(a)','Location','NorthEast');
```

```
Warning: Ignoring extra legend entries.
```

```
text(-0.155,-.105,'$\underline{a}$','FontSize',16,'interpreter','latex');
line([amin amin], [-.1 .08],'Color','Black','LineStyle','--');
xlabel('Wealth, $a$','interpreter','latex');
ylabel('Savings, $s_i(a)$','interpreter','latex');
xlim([amin1 amax1]);
ylim([-0.03 0.05]);
set(gca,'FontSize',16);
```




```
figure(2)
h1 = plot(a,g_r(:,1,ir),'b',a,g_r(:,2,ir),'r','LineWidth',2);
legend(h1,'g_1(a)','g_2(a)');
text(-0.155,-.12,'$\underline{a}$','FontSize',16,'interpreter','latex');
line([amin amin], [0 max(max(g_r(:,:,ir)))],'Color','Black','LineStyle','--');
xlabel('Wealth, $a$','interpreter','latex');
ylabel('Densities, $g_i(a)$','interpreter','latex');
xlim([amin1 amax1]);
%ylim([0 0.5])
set(gca,'FontSize',16);
```



1.2
1
0.8
0.6
0.4
0.2
0
Densities, $g_i(a)$
$\underline{a}$
0
1
2
3
4
5
Wealth, a
g_1(a)
g_2(a)
data1

```
% Script construido baseado nos scripts de B.Moll
% Fonte original em: http://www.princeton.edu/~moll/HACTproject.htm

%% Limpeza de Variaveis
clear all; clc; close all;

rho = 0.05;
r = 0.03;
z1 = 1;
eta = 0.75;

w=1;

I= 150;
amin = -0.15;
amax = 3;
a = linspace(amin,amax,I)';
da = (amax-amin)/(I-1);

maxit= 10000;
crit = 10^(-6);
Delta = 1000;

dVf = zeros(I,1);
dVb = zeros(I,1);
c = zeros(I,1);

options=optimset('Display','off');
x0 = 1;


% Define w e r
AA = 1;
delta = 0.06;
alpha = 0.33;

check = 1;

r_high = rho;
r_low = 0;

while check ==1

    w = (1-alpha) * AA *((r+delta)/(AA*alpha))^(alpha/(alpha-1));


    tic;
    for i=1:I
        params = [a(i),z1,w,r];
        myfun = @(l) SolveLabor(l,params);
        [l01,fval,exitflag] = fzero(myfun,x0,options);

        l0(i,:)=[l01];
    end
    toc
```

```
    v0(:,1) = log(w*z1.*l0(1,1) + r.*a)/rho;

    lmin = l0(1,:);
    lmax = l0(I,:);

    v = v0;

    for n=1:maxit
        V = v;
        V_n(:,n)=V;
        % forward difference
        dVf(1:I-1) = (V(2:I)-V(1:I-1))/da;
        dVf(I) = (w*z1.*lmax + r.*amax).^(-1); %state constraint boundary condition
        % backward difference
        dVb(2:I) = (V(2:I,:)-V(1:I-1,:))/da;
        dVb(1) = (w*z1.*lmin + r.*amin).^(-1); %state constraint boundary condition

        %consumption and savings with forward difference
        cf = dVf.^(-1);
        lf = 1-(dVf.*w.*z1/eta).^(-1);
        ssf = w*z1.*lf + r.*a - cf;
        %consumption and savings with backward difference
        cb = dVb.^(-1);
        lb = 1-((dVb.*w.*z1/eta).^(-1));
        ssb = w*z1.*lb + r.*a - cb;
        %consumption and derivative of value function at steady state
        c0 = w*z1.*l0 + r.*a;
        dV0 = c0.^(-1);

        Ib = ssb < 0; %negative drift --> backward difference
        If = (ssf > 0).*(1-Ib); %positive drift --> forward difference
        I0 = (1-If-Ib); %at steady state

        c = cf.*If + cb.*Ib + c0.*I0;
        l = lf.*If + lb.*Ib + l0.*I0;
        u = log(c) + eta*log(1-l);

        %CONSTRUCT MATRIX
        X = -Ib.*ssb/da;
        Y = -If.*ssf/da + Ib.*ssb/da;
        Z = If.*ssf/da;

        A1=spdiags(Y(:,1),0,I,I)+spdiags(X(2:I,1),-1,I,I)+spdiags([0;Z(1:I-1,1)],1,I,
I);

        A = A1;
        B = (1/Delta + rho)*speye(I) - A;

        u_stacked = [u(:)];
        V_stacked = [V(:)];

        b = u_stacked + V_stacked/Delta;
        V_stacked = B\b; %SOLVE SYSTEM OF EQUATIONS
```

```
        Vchange = V_stacked - v;
        v = V_stacked;

        dist(n) = max(abs(Vchange));
        if dist(n)<crit
            disp('Value Function Converged, Iteration = ')
            disp(n)
            break
        end
    end
    toc;



    %% MARKET CLEARING CONDITIONS

    %%%%%%%%%%%%%%%%%%%%%%%%%%
    % FOKKER-PLANCK EQUATION %
    %%%%%%%%%%%%%%%%%%%%%%%%%%
    AT = A';
    b = zeros(I,1);

    %need to fix one value, otherwise matrix is singular
    i_fix = 1;
    b(i_fix)=.1;
    row = [zeros(1,i_fix-1),1,zeros(1,I-i_fix)];
    AT(i_fix,:) = row;

    %Solve linear system
    gg = AT\b;
    g_sum = gg'*ones(I,1)*da;
    gg = gg./g_sum;
    g = gg;

    check1 = g(:,1)'*ones(I,1)*da;

    Asset_Supply = g(:,1)'*a*da;

    if Asset_Supply > crit
        r_high = r;
        r = (r_high + r_low)/2;
    elseif Asset_Supply < -crit
        r_low = r;
        r = (r_high + r_low)/2;
    else
        check = 0;
    end
end
fprintf('\nFunção convergiu.\n')
fprintf('taxa de juros encontrada: %2.3f\n', r)
```

```
function eq = SolveLabor(l,params)
a = params(1);
z = params(2);
w = params(3);
r = params(4);

% -u_l/u_c = wz
eq = l - (w*z*l + r*a)^(-1)*(w*z);

end % end of funccction
```

```
clear all;
clc;
close all;

tic;

ga = 2;
rho = 0.05;
d = 0.05;
al = 1/3;
Aprod = 0.1;
z1 = 1;
z2 = 2*z1;
z = [z1,z2];
la1 = 1/3;
la2 = 1/3;
la = [la1,la2];
z_ave = (z1*la2 + z2*la1)/(la1 + la2);

I= 1000;
amin = 0;
amax = 20;
a = linspace(amin,amax,I)';
da = (amax-amin)/(I-1);

aa = [a,a];
zz = ones(I,1)*z;


maxit= 100;
crit = 10^(-6);
Delta = 1000;

dVf = zeros(I,2);
dVb = zeros(I,2);
c = zeros(I,2);

Aswitch = [-speye(I)*la(1),speye(I)*la(1);speye(I)*la(2),-speye(I)*la(2)];

Ir = 40;
crit_S = 10^(-5);


rmax = 0.049;
r = 0.04;
w = 0.05;

r0 = 0.03;
rmin = 0.01;
rmax = 0.99*rho;


for ir=1:Ir;

r_r(ir)=r;
```

```
rmin_r(ir)=rmin;
rmax_r(ir)=rmax;

KD(ir) = (al*Aprod/(r + d))^(1/(1-al))*z_ave;
w = (1-al)*Aprod*KD(ir).^al*z_ave^(-al);

if w*z(1) + r*amin < 0
    disp('CAREFUL: borrowing constraint too loose')
end

v0(:,1) = (w*z(1) + r.*a).^(1-ga)/(1-ga)/rho;
v0(:,2) = (w*z(2) + r.*a).^(1-ga)/(1-ga)/rho;

if ir>1
v0 = V_r(:,:,ir-1);
end

v = v0;

for n=1:maxit
    V = v;
    V_n(:,:,n)=V;
    % forward difference
    dVf(1:I-1,:) = (V(2:I,:)-V(1:I-1,:))/da;
    dVf(I,:) = (w*z + r.*amax).^(-ga); %will never be used, but impose state↙
constraint a<=amax just in case
    % backward difference
    dVb(2:I,:) = (V(2:I,:)-V(1:I-1,:))/da;
    dVb(1,:) = (w*z + r.*amin).^(-ga); %state constraint boundary condition

    %consumption and savings with forward difference
    cf = dVf.^(-1/ga);
    ssf = w*zz + r.*aa - cf;
    %consumption and savings with backward difference
    cb = dVb.^(-1/ga);
    ssb = w*zz + r.*aa - cb;
    %consumption and derivative of value function at steady state
    c0 = w*zz + r.*aa;

    % dV_upwind makes a choice of forward or backward differences based on
    % the sign of the drift
    If = ssf > 0; %positive drift --> forward difference
    Ib = ssb < 0; %negative drift --> backward difference
    I0 = (1-If-Ib); %at steady state

    c = cf.*If + cb.*Ib + c0.*I0;
    u = c.^(1-ga)/(1-ga);

    %CONSTRUCT MATRIX
    X = -min(ssb,0)/da;
    Y = -max(ssf,0)/da + min(ssb,0)/da;
    Z = max(ssf,0)/da;

    A1=spdiags(Y(:,1),0,I,I)+spdiags(X(2:I,1),-1,I,I)+spdiags([0;Z(1:I-1,1)],1,I,I);
    A2=spdiags(Y(:,2),0,I,I)+spdiags(X(2:I,2),-1,I,I)+spdiags([0;Z(1:I-1,2)],1,I,I);
```

```
    A = [A1,sparse(I,I);sparse(I,I),A2] + Aswitch;

    if max(abs(sum(A,2)))>10^(-9)
       disp('Improper Transition Matrix')
       %break
    end

    B = (1/Delta + rho)*speye(2*I) - A;

    u_stacked = [u(:,1);u(:,2)];
    V_stacked = [V(:,1);V(:,2)];

    b = u_stacked + V_stacked/Delta;
    V_stacked = B\b; %SOLVE SYSTEM OF EQUATIONS

    V = [V_stacked(1:I),V_stacked(I+1:2*I)];

    Vchange = V - v;
    v = V;

    dist(n) = max(max(abs(Vchange)));
    if dist(n)<crit
        fprintf('Value Function Converged, Iteration = %d\n', n)
        break
    end
end
toc;

%%%%%%%%%%%%%%%%%%%%%%%%%%%
% FOKKER-PLANCK EQUATION %
%%%%%%%%%%%%%%%%%%%%%%%%%%%
AT = A';
b = zeros(2*I,1);

%need to fix one value, otherwise matrix is singular
i_fix = 1;
b(i_fix)=.1;
row = [zeros(1,i_fix-1),1,zeros(1,2*I-i_fix)];
AT(i_fix,:) = row;

%Solve linear system
gg = AT\b;
g_sum = gg'*ones(2*I,1)*da;
gg = gg./g_sum;

g = [gg(1:I),gg(I+1:2*I)];

check1 = g(:,1)'*ones(I,1)*da;
check2 = g(:,2)'*ones(I,1)*da;

g_r(:,:,ir) = g;
adot(:,:,ir) = w*zz + r.*aa - c;
V_r(:,:,ir) = V;

KS(ir) = g(:,1)'*a*da + g(:,2)'*a*da;
```

```
S(ir) = KS(ir) - KD(ir);

%UPDATE INTEREST RATE
if S(ir)>crit_S
    disp('Excess Supply')
    rmax = r;
    r = 0.5*(r+rmin);
elseif S(ir)<-crit_S;
    disp('Excess Demand')
    rmin = r;
    r = 0.5*(r+rmax);
elseif abs(S(ir))<crit_S;
    fprintf('Equilibrium Found, Interest rate = %f\n', r)
    break
end

end

amax1 = 5;
amin1 = amin-0.1;

figure(1)
h1 = plot(a,adot(:,1,ir),'b',a,adot(:,2,ir),'r',linspace(amin1,amax1,I),zeros(1,↙
I),'k--','LineWidth',2);
legend(h1,'s_1(a)','s_2(a)','Location','NorthEast');
text(-0.155,-.105,'$\underline{a}$','FontSize',16,'interpreter','latex');
line([amin amin], [-.1 .08],'Color','Black','LineStyle','--');
xlabel('Wealth, $a$','interpreter','latex');
ylabel('Savings, $s_i(a)$','interpreter','latex');
xlim([amin1 amax1]);
ylim([-0.03 0.05]);
set(gca,'FontSize',16);

figure(2)
h1 = plot(a,g_r(:,1,ir),'b',a,g_r(:,2,ir),'r','LineWidth',2);
legend(h1,'g_1(a)','g_2(a)');
text(-0.155,-.12,'$\underline{a}$','FontSize',16,'interpreter','latex');
line([amin amin], [0 max(max(g_r(:,:,ir)))],'Color','Black','LineStyle','--');
xlabel('Wealth, $a$','interpreter','latex');
ylabel('Densities, $g_i(a)$','interpreter','latex');
xlim([amin1 amax1]);
%ylim([0 0.5])
set(gca,'FontSize',16);
```