

# Implementation of the Density Evolution method with Fokker-Plank for L-NLIF Model in Python

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## 1 Model

The original model is a L-NLIF model, which consists of a Linear (L) Filter, followed by a probabilistic or Noisy (N) form of Leaky Integrate and Fire spike generation(LIF) (Paninski et al.[1]). For sake of completeness the model is briefly described here.

The evolution of the Voltage  $V$  is given by:

$$dV = (-g(V(t) - V_{leak}) + I_{stim}(t) + I_{hist}(t))dt + W_t \quad (1)$$

Where  $g$  is the leak conductance  $V_{leak}$  is the leak reversal potential  $I_{stim}$  is the convolution of linear filter with the input signal,  $I_{hist}$  is the spike current history and  $W_t$  is standard Gaussian white noise. More precisely:

$$I_{stim}(t) = \vec{k}\vec{x}(t) \quad (2)$$

and

$$I_{hist}(t) = \sum_{j=0}^{i-1} h(t - t_j) \quad (3)$$

Where  $k$  is a linear filter and  $h$  is a fixed postspike current waveform.

## 2 Density Evolution

If we define

$$P(V, t) \equiv P(V(t) \cap V(s) < V_{th} \forall s < t) \quad (4)$$

This refers to the probability of  $V(t)$ , the membrane potential, being less than  $V_{th}$  the firing threshold until time  $t$ .

We need to solve the following Fokker-Planck drift diffusion equation numerically.

$$\frac{\partial P(V, t)}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 P(V, t)}{\partial V^2} + g \frac{\partial [(V - V_{rest})P(V, t)]}{\partial V} \quad (5)$$

Given the boundary conditions:

$$P(V_{th}, t) = 0 \quad (6)$$

$$P(V, 0) = \delta(V - V_{reset}) \quad (7)$$

when solving numerically we also need a lower bound on the voltage  $V_{lb}$ , so this becomes an additional boundary condition:

$$P(V_{lb}, t) = 0 \quad (8)$$

Ideally  $V_{lb} = -\infty$

And the definition of  $V_{rest}$ .

$$V_{rest}(t) = V_{leak} + \frac{1}{g} (\vec{k} \cdot \vec{x}(t) \sum_{j=0}^{i-1} h(t - t_j)) \quad (9)$$

We can rewrite this as:

$$\frac{\sigma^2}{2} \frac{\partial^2 P(V, t)}{\partial V^2} + g(V - V_{rest}) \frac{\partial P(V, t)}{\partial V} + gP(V, t) - \frac{\partial P(V, t)}{\partial t} = 0 \quad (10)$$

Since we know that:

$$\frac{\partial [(V - V_{rest})P(V, t)]}{\partial V} = \frac{\partial (V - V_{rest})}{\partial V} \dot{P} + \frac{\partial P}{\partial V} (V - V_{rest}) \quad (11)$$

and

$$\frac{\partial (V - V_{rest})}{\partial V} = 1 \quad (12)$$

Next we discretize time and potential. We adopt the notation that Potential is discretized into  $W$  intervals of length  $w$  and indexed by  $\nu = 0, 1, \dots W$ . Time is discretized into  $U$  intervals of length  $u$  and indexed by:  $\tau = 0, 1, \dots U$ .

$$P_{\nu,\tau} = P(\nu w, \tau u)$$

Before we can write down the computationally stable Crank-Nicolson method we must first write down our finite differencing scheme:

$$\hat{P} = P_{\nu,\tau} \quad (13)$$

$$\frac{\partial \hat{P}}{\partial t} = \frac{P_{\nu,\tau+1} - P_{\nu,\tau}}{u} \quad (14)$$

$$\frac{\partial \hat{P}}{\partial V} = \frac{P_{\nu+1,\tau} - P_{\nu-1,\tau}}{2w} \quad (15)$$

$$\frac{\partial^2 \hat{P}}{\partial V^2} = \frac{P_{\nu+1,\tau} - 2P_{\nu,\tau} + P_{\nu-1,\tau}}{w^2} \quad (16)$$

Using the Crank-Nicolson scheme we may now rewrite the derivatives using a new finite differencing scheme which is centered around  $t + u/2$ . Bearing in mind that this is only an approximation we get:

$$P_{CN} = \frac{P_{\nu,\tau} + P_{\nu,\tau+1}}{2} \quad (17)$$

$$\frac{\partial P_{CN}}{\partial t} = \frac{P_{\nu,\tau+1} - P_{\nu,\tau}}{u} \quad (18)$$

$$\frac{\partial P_{CN}}{\partial V} = \frac{P_{\nu+1,\tau} + P_{\nu+1,\tau+1} - P_{\nu-1,\tau} - P_{\nu-1,\tau+1}}{4w} \quad (19)$$

$$\frac{\partial^2 P_{CN}}{\partial V^2} = \frac{P_{\nu+1,\tau} - 2P_{\nu,\tau} + P_{\nu-1,\tau} + P_{\nu+1,\tau+1} - 2P_{\nu,\tau+1} + P_{\nu-1,\tau+1}}{2w^2} \quad (20)$$

if we now let:

$$\begin{aligned} a &= \frac{\sigma^2}{2} \\ b &= g(V - V_{rest}) \\ c &= g \end{aligned}$$

and multiply throughout with  $4w^2u$

we may rearrange all the  $P_{*,\tau+1}$  terms on the left hand side:

$$\underbrace{-(2au + bwu)}_{A_\nu} P_{\nu+1,\tau+1} + \underbrace{(4au - 2cw^2u + 4w^2)}_{B_\nu} P_{\nu,\tau+1} - \underbrace{(2au - bwu)}_{C_\nu} P_{\nu-1,\tau+1} = \underbrace{(2au + bwu)P_{\nu+1,\tau} + (-4au + 2cw^2u + 4w^2)P_{\nu,\tau} + (2au - bwu)P_{\nu-1,\tau}}_{D_\nu} \quad (21)$$

For each  $\nu = 1, \dots, W-1$

We note here that we have obtained  $W-1$  simultaneous equations which we may now rewrite in the following tridiagonal matrix notation.

$$\underbrace{\begin{pmatrix} B_0 & A_0 & 0 & 0 & \cdots & 0 \\ C_1 & B_1 & A_1 & 0 & \cdots & 0 \\ 0 & C_2 & B_2 & A_2 & 0 & \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & C_{W-1} & B_{W-1} & A_{W-1} \\ 0 & \cdots & & 0 & C_W & B_W \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} P_{0,\tau+1} \\ P_{1,\tau+1} \\ P_{2,\tau+1} \\ \vdots \\ P_{W-1,\tau+1} \\ P_{W,\tau+1} \end{pmatrix}}_{\chi} = \underbrace{\begin{pmatrix} D_0 \\ D_1 \\ D_2 \\ \vdots \\ D_{W-1} \\ D_W \end{pmatrix}}_{\beta} \quad (22)$$

However this matrix equation has  $W+1$  equations, so we must use the boundary conditions to specify the first and last rows of  $\Lambda$ . We can then use a tridiagonal matrix algorithm to solve  $\Lambda\chi = \beta$ .

This allows us to iteratively obtain the density evolution of the membrane potential for a given time interval.

The maximum likelihood optimizer proposed in [1] however uses the first passage time density at a given time point which is defined as:

$$fpt(t) = -\frac{\partial}{\partial t} \int P(V, t) dV \quad (23)$$

and this can be easily computed from an  $P(V, t)$  matrix by summing across  $P$  and finding differences across  $t$ .

The final likelihood is then simply

$$L_{\vec{x}, t_i} = \Pi_i f(t_i) \quad (24)$$

## References

- [1] L. Paninski, J. W. Pillow, and E. P. Simoncelli. Maximum likelihood estimation of a stochastic integrate-and-fire neural encoding model. *Neural Comput*, 16(12):2533–2561, Dec 2004.

The pseudocode for the objective function can be described as follows:

1. compute  $P(V, t)$  for each spike interval.
2. compute