ELEMENTS OF PROOF THEORY

Brian Tenneson

elementsofprooftheory@gmail.com

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Contents

1. Introduction

Formal systems can be viewed in many ways all hinging on the notion of proof. A proof is a finite sequence of well-formed utterances in a language where each well-formed utterance is either among the assumed or a direct consequence of a previous term (or of previous terms) in the sequence. There are many types of systems such as Frege systems and Gentzen systems but we will specify a formal system with three sets x, y, and z which will be called an xyz formal system.

To fix notation, $\mathbb{F}_m = \mathbb{N} \cap [1, m]$. $[a \to b]$ denotes the set of all functions whose domain is a and codomain is b. $|x|$ is the cardinal number of set x. By a phrase of the form $|x| = \aleph_0$, I mean that x is countably infinite.

2. xyz Formal Systems

Definition: $F = (x, y, z)$ is an xyz formal system if

(1) x is nonempty,
(2) $y \subseteq \bigcup_{m=1}^{\infty} [\mathbb{F}_m \to x]$, and
(3) $z \subseteq \bigcup_{m=1}^{\infty} \bigcup_{G \in \mathcal{P}(y^m)} [G \to y]$.

Elements of x will be called symbols; x is referred to as an alphabet. Elements of y are well-formed formulas. Elements of z are inference rules. Note that, for each $m \geq 1$, $[\mathbb{F}_m \to x]$ is the set of all length m sequences of elements of x, sometimes called strings and utterances. The utterances in y are deemed to be well-formed.

Note the two following facts stemming from this definition:

(1) $w \in y$ implies $(\exists m \geq 1) (w \in [\mathbb{F}_m \to x])$;
(2) $i \in z$ implies $(\exists m \geq 1) (\exists G \subseteq y^m) (i \in [G \to y])$.

All elements of z map finitely many elements of y to an element of y. These maps artificially simulate the process of human deduction: the inputs are the antecedents (or antecedent if the inference rule is unary) and the output is the consequent.

An element $r \in z$ is an element of $[G \to y]$ for some $m \geq 1$ and $G \subseteq y^m$. This m is called the arity of r. We also say r is m-ary, unary if $m = 1$, and binary if $m = 2$.

Definition: Suppose $\beta \in y$ and $A \subseteq y$. β is a direct consequence of A if for some $q \leq |A|$ there is a q-ary inference rule $i \in z$ and a subset $\{a_1, ..., a_q\} \subseteq A$ such that $i(a_1, ..., a_q) = \beta$.

Definition: A proof is an element $\pi \in \bigcup_{m=1}^{\infty} [\mathbb{F}_m \to y]$ such that every $\pi(n)$ is either an element of some distinguished set $\mathit{\Gamma}$ or is a direct consequence of $\pi(\mathbb{F}_{n-1})$.

Definition: If $\mathit{\Gamma} \subseteq y$ and $\psi \in y$, then $\mathit{\Gamma} \vdash \psi$ if and only if there exists a proof π such that $\pi(|\pi|) = \psi$ and every $\pi(n)$ is either an element of $\mathit{\Gamma}$ or is a direct consequence of $\pi(\mathbb{F}_{n-1})$. This is known as syntactic consequence.

Definition: Given $\mathit{\Gamma} \subseteq y$, the set of theorems of $\mathit{\Gamma}$ is defined by

$$Con_F(\mathit{\Gamma}) := \{\psi \in y : \mathit{\Gamma} \vdash \psi\}.$$

We will drop the subscript if it is clear which formal system we are referring to.

Conjecture: Suppose $\mathit{\Gamma} \subseteq W \subseteq Con\,(\mathit{\Gamma})$ and W is closed under all inference rules in z. Then $W = Con\,(\mathit{\Gamma})$. If we let $air\,(i)$ denote the arity of $i \in z$, W is closed under inference rules in z if and only if for all $i \in z$, $i\left(W^{air(i)}\right) \subseteq W$.

3. Well Formed and Atomic Formulas

There are a variety of ways utterances, finite sequences of elements of x, can be "deemed" to be well-formed. In the most general setting, whether an utterance is well-formed or not is arbitrary. Alternatively, we can specify some utterances to be atomic well-formed utterances and then, by using connectives, obtain complex well-formed utterances. Our final definition in this section will be that of when the second coordinate y of an xyz formal system is atomically generated by a set of atomic well-formed utterances y_0. Observe that inference rules will not be mentioned in this section; at present we are focusing on how finite sequences of elements of x might be considered to be well-formed, i.e., elements of y.

Definition: Set K is a set of connectives in a system $F = (x, y, z)$ iff

$$K \subseteq \bigcup_{m=1}^{\infty} [y^m \to y]\,.$$

Definition: In a system $F = (x, y, z)$, a subset $y_0 \subseteq y$ is called atomic with respect to set of connectives K if for all $c \in K$, $y_0 \cap ran\,(c) = \emptyset$. $ran(c)$ is the range of connective c.

Definition: In a system $F = (x, y, z)$, given $y_0 \subseteq y$ and $\psi \in y$, $y_0 \Vdash_K \psi$ is the notation for the phrase, "there is a proof of well formation of ψ from y_0 with respect to K." $y_0 \Vdash_K \psi$ means that there is a sequence $\delta \in \bigcup_{m=1}^{\infty} [\mathbb{F}_m \to y]$ such that $\delta\,(|\delta|) = \psi$, $\delta\,(1) \in y_0$, and for all $k > 1$, either $\delta\,(k) \in y_0$ or $\delta\,(k) = c\,(\delta\,(\mathbb{F}_{k-1}))$ for some element c of a set of connectives K. Here I am being slightly loose with the notation. In the expression $c\,(\delta\,(\mathbb{F}_{k-1}))$, c accepts a tuple as input while $\delta\,(\mathbb{F}_{k-1})$ is a set, namely the range of $\mathbb{F}_{k-1}$ under δ. When I write $\delta\,(k) = c\,(\delta\,(\mathbb{F}_{k-1}))$, I mean that $\delta\,(k)$ is c applied to any tuple formed from members of the set $\delta\,(\mathbb{F}_{k-1})$.

Definition: y is atomically generated by $y_0 \subseteq y$ with respect to a set of connectives K if $y_0 \Vdash_K \varphi$ for every wff φ.

Conjecture: The set of well-formed formulas of first order logic is atomically generated by the set of atomic formulas with respect to

$$K = \{\exists, \forall, \neg, \wedge, \vee, \to, \leftrightarrow\}\,.$$

Conjecture: (Induction) If y is atomically generated by y_0 with respect to a set K of connectives, $y_0 \subseteq W \subseteq y$, and W is closed under all connectives in K, then $W = y$. Note that if we let $air\,(c)$ denote the arity of $c \in K$, for W to be closed under all connectives means that for all $c \in K$, $c\left(W^{air(c)}\right) \subseteq W$.

4. Examples

4.1. **Modus Ponens.** Consider a formal system equipped with an alphabet $x = \{\neg, \exists, \rightarrow, \gamma_1, \gamma_2, \gamma_3, \alpha, \beta\}$. Furthermore, consider the following binary inference rule m defined on the symbols contained in the alphabet:

$$m(\gamma_1, \gamma_2) := \begin{cases} \gamma_3 & (\exists \gamma_3)(\gamma_2 = \gamma_1 \rightarrow \gamma_3) \\ \gamma_1 & (\neg\exists \gamma_3)(\gamma_2 = \gamma_1 \rightarrow \gamma_3) \, . \end{cases}$$

This rule m is defining modus ponens because $m(\alpha, \alpha \rightarrow \beta) = \beta$. This observation gives rise to the following proof of β from the totality of α and $\alpha \rightarrow \beta$. If we let the distinguished set $\Gamma = \{\alpha, \alpha \rightarrow \beta\}$ and $z = \{m\}$, then π is a proof of β:

$$\pi = \{(1, \alpha), (2, \alpha \rightarrow \beta), (3, \beta)\} \, .$$

4.2. **MIU Formal System.** This is how the MIU system of Hofstadter can be cast as an xyz formal system: $x = \{m, i, u\}$ where the three symbols have an arbitrary meaning; $y = \left\{s \in x^* : s^{-1}(\{m\}) = \{1\}\right\}$ where $x^* := \bigcup_{p=1}^{\infty} [\mathbb{F}_p \rightarrow x]$. The rules of inference are of four in number: $z = \{I_a, I_b, I_c, I_d\}$ where $I_a = \{(\alpha_1 i, \alpha_1 iu) : \alpha_1 \in y\}$, $I_b = \left\{(m\alpha_1, m\alpha_1\alpha_1) : \alpha_1 \in \{i,u\}^*\right\}$, $I_c = \left\{(\alpha_1 iii \alpha_2, \alpha_1 u \alpha_2) : \alpha_1 \in y \wedge \alpha_2 \in \{i,u\}^*\right\}$, and $I_d = \left\{(\alpha_1 uu \alpha_2, \alpha_1 \alpha_2) : \alpha_1 \in y \wedge \alpha_2 \in \{i,u\}^*\right\}$. To be clear, things like $m\alpha_1$ and $m\alpha_1\alpha_1$ are abbreviations. $m\alpha_1\alpha_1$, for example is an abbreviation for the string whose first character is m, the second segment of the string matches the string α_1 and the string ends with a copy of α_1.

The MIU system is characterized by the following set: $Con_{(x,y,z)}(\{mi\})$.

4.3. **Information Theoretic Computers.** Let (x, y, z) be an xyz formal system. By a semi-ideal computer (i.e., a SIC), I mean a pair of functions (C, H_C) with the following properties:

1. $C \in [\mathcal{P}(y) \times \mathbb{N}^+ \rightarrow \mathcal{P}(y)]$;
2. $H_C \in [\mathcal{P}(y) \times \mathbb{N}^+ \rightarrow \{0, 1\}]$;
3. $C(\Gamma, m) \subseteq C(\Gamma, m+1)$ for all m;
4. if $H_C(\Gamma, m) = 1$ then $H_C(\Gamma, m+1) = 1$; and
5. if $H_C(\Gamma, m) = 1$ then $C(\Gamma, m+1) = C(\Gamma, m)$.

Let $\|\gamma\|_\Gamma$ be the length of a shortest proof of γ from axioms Γ. Set

$$V(\Gamma, m) := \{\gamma \in y : \|\gamma\|_\Gamma = m\} \, .$$

Observe that we can define a SIC as follows:

$$C(\Gamma, m) := \bigcup_{k=1}^{m} V(\Gamma, k)$$

and

$$H_C(\Gamma, m) := \begin{cases} 1 & \text{if} \quad V(\Gamma, m) = \emptyset \\ 0 & \text{if} \quad V(\Gamma, m) \neq \emptyset \end{cases} \, .$$

4.4. **Generalized Board Games.** Generalized board games provide a common formal framework for turn-based play with anywhere from one to infinitely-many players on a board that might have any polygonal shape in which the empty space (denoted E). The symbols in x are by definition capable of adequately and specifically

A generalized board game (henceforth known as GBG) is one way to generalize games like chess and checkers. Involved there are some ingredients such as a set C of possible locations that pieces whose names are recorded in set H. There is also the "piece" which is the empty space that other pieces might occupy; it is symbolized by any symbol not already in H. We will take E to be the name of the empty space.

A location report is an element of $[C \to (H \cup \{E\})]$; it inputs a location on the board whose elements comprise C and outputs the name of the piece in that location or E if there are no other pieces in that location.

We are now ready to define the well-formed formulas and while the inference rules do obey general constraints (to follow), what they are specifically depends on the GBG. Let $y = \{"f_k" : k \in \mathbb{F}_S\}$ where $"f_k" := \{(1, f_k)\}$, the string of length one whose sole symbol is f_k. The set of inference rules z depends on the GBG but it must satisfy these four restrictions to be considered a GBG:

(1) $z \subseteq [y \to y]$ and
(2) for all $i \in z$, $|\{c \in C : c \neq i(c)\}| = 2$.

A sequence of location reports is then deemed to be a match with at least one player, often two such as chess. There are the set of winning conditions, losing conditions, draw/stalemate conditions, and surrender conditions.

The first stipulation on GBGs is that every inference rule is a map whose domain equals its codomain which equals y. The second stipulation is that all a GBG's inference rules do is change the piece (or to the empty piece) at precisely two locations on the board. f_k^{λ} is a function inputting a location and outputting the name of the piece or E that occupies that location. $f_k^{\lambda}(c)$ is the name of the piece (or E if empty) at location c.

5. Graph Theoretical Considerations: Walk With Me

We walk from the axiom (a conjoined version of all axioms) to the theorem, hitting many lemmas in between. A proof naturally corresponds to a colored digraph such as the one below for the MIU formal system. The color corresponds to which inference rule that appears in an inference leading from an antecedent (or set of antecedents) to a conclusion. Observe the axiom mi in the center of the graph in the diagram and note that a path (observing direction) corresponds to a proof of a theorem. The MIU puzzle is to determine whether mu is a vertex on this colored digraph, presumably without exploring all paths in an exhaustive search.

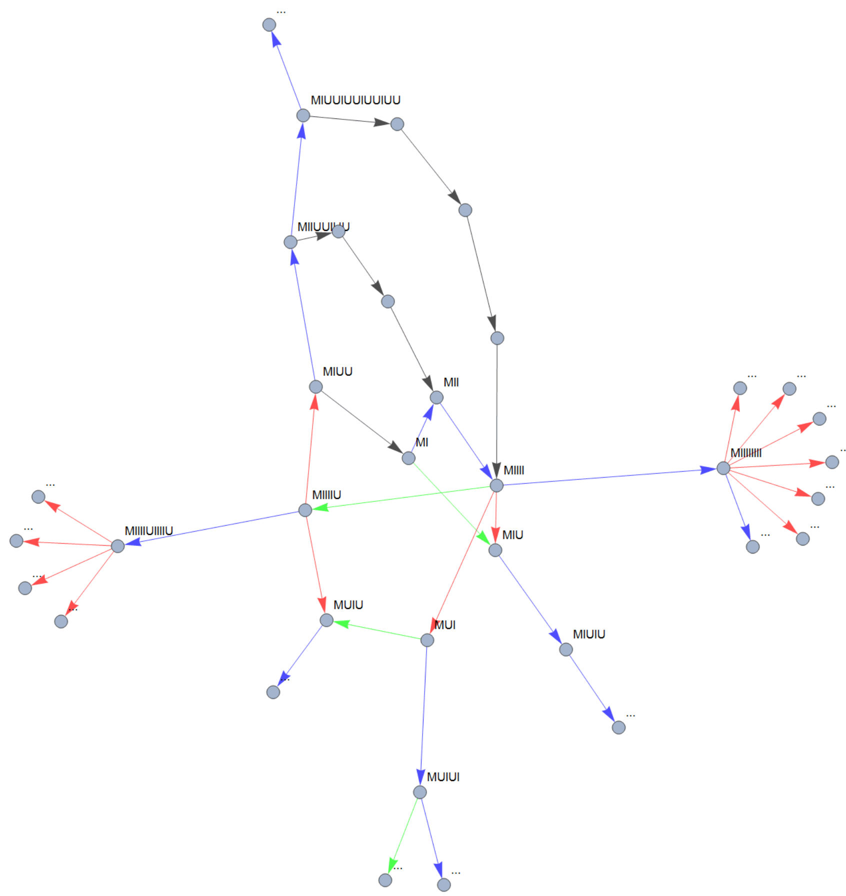

FIGURE 1.

6. On Brute Forcing a Proof

Let $J(\mathit{\Gamma}, m) = \{\gamma \in y : (\mathit{\Gamma} \vdash_\pi \gamma) \wedge |\pi| \le m\}$ and $\tau(\mathit{\Gamma}, m) = \{\gamma \in y : (\mathit{\Gamma} \vdash_\pi \gamma) \wedge |\pi| = m\}$.

Conjecture: $|\tau(\mathit{\Gamma}, m)| \le |\mathit{\Gamma}|\, |z|^{m-1}$. (rule shunting :D)

Conjecture: (a) If $|z| \ne 1$, $|J(\mathit{\Gamma}, m)| \le |\mathit{\Gamma}| \centerdot \frac{|z|^m - 1}{|z| - 1}$. (b) If $|z| = 1$, $|J(\mathit{\Gamma}, m)| \le |\mathit{\Gamma}|\, m$.

Corollary: For the GBG known as chess, there are at most $\frac{16^{50}-1}{15} \approx 1.07 \times 10^{59}$ different legal game-states with at most 50 turns.

Corollary: For the GBG known as chess, there are at most $\frac{6^{50}-1}{5} \approx 1.62 \times 10^{38}$ different legal game-states with at most 50 turns.

Definition: For $A \subseteq y$, when we write $Z(A)$ with $Z \in [\mathcal{P}(y) \to \mathcal{P}(y)]$, we mean the following:

$$Z(A) = \left\{ w \in y : (\exists i \in z) \left(\exists \overrightarrow{a} \in A^{air(i)} \right) (w = i(\overrightarrow{a})) \right\}.$$

Definition: Define $S(\mathit{\Gamma}, n)$ as follows: $S(\mathit{\Gamma}, 1) := \mathit{\Gamma}$ and for $n \ge 1$,

$$S(\mathit{\Gamma}, n+1) = Z(S(\mathit{\Gamma}, n)).$$

Conjecture: Consider the following statements:

(1) $\alpha \in S(\mathit{\Gamma}, n)$.
(2) α has a length n proof.
(3) $\|\alpha\|_{\mathit{\Gamma}} \leq n$.

Known implications: [2]<–>[3], [1]–>[2], [1]–>[3]. Suspected the three are equivalent.

Conjecture: For all provable wffs α, $\alpha \in S(\mathit{\Gamma}, \|\alpha\|_{\mathit{\Gamma}})$. (* Given that $S(\mathit{\Gamma}, \|\alpha\|_{\mathit{\Gamma}}) \neq \emptyset$, $\{S(\mathit{\Gamma}, \|\alpha\|_{\mathit{\Gamma}}) : \alpha \in y\}$ is a family of nonempty sets. By the axiom of choice, there is a function f such that $f(\alpha) \in S(\mathit{\Gamma}, \|\alpha\|_{\mathit{\Gamma}})$.
***):**
Definition: Define $\sigma(\mathit{\Gamma})$ to be $\bigcup_{n=1}^{\infty} S(\mathit{\Gamma}, n)$.
Conjecture: $\mathit{\Gamma} \subseteq \sigma(\mathit{\Gamma}) \subseteq Con(\mathit{\Gamma})$ and $\sigma(\mathit{\Gamma})$ is closed under inference rules.

Proof $\mathit{\Gamma} = S(\mathit{\Gamma}, 1) \subseteq \sigma(\mathit{\Gamma})$. To show that every wff in $\sigma(\mathit{\Gamma})$ is a theorem, let $\mu \in \sigma(\mathit{\Gamma})$ be given. Then for some n, $\mu \in S(\mathit{\Gamma}, n)$. By a conjecture above, this implies that μ has a length n proof, implying that μ is a theorem; so, $\mu \in Con(\mathit{\Gamma})$.

To show that $\sigma(\mathit{\Gamma})$ is closed under inference rules, let an inference rule $i \in z$ be given. The number of inputs i accepts is denoted $air(i)$. Let $\left(s_1, ..., s_{air(i)}\right) \in \sigma(\mathit{\Gamma})^{air(i)}$. For a given $k \in \mathbb{F}_{air(i)}$, let n_k be an m such that $s_k \in S(\mathit{\Gamma}, m)$; so, $s_k \in S(\mathit{\Gamma}, n_k)$. The following statements together entail that $i\left(s_1, ..., s_{air(i)}\right) \in \sigma(\mathit{\Gamma})$ whenever $\left(s_1, ..., s_{air(i)}\right) \in \sigma(\mathit{\Gamma})^{air(i)}$:

(1) $Z\left(\bigcup_{k=1}^{air(i)} S(\mathit{\Gamma}, n_k)\right) \subseteq \bigcup_{k=1}^{air(i)} Z(S(\mathit{\Gamma}, n_k))$ because we can prove by induction that $Z(W_p) \subseteq \bigcup_{k=1}^{p} Z(S(\mathit{\Gamma}, n_k))$ where $W_p = \bigcup_{k=1}^{p} S(\mathit{\Gamma}, n_k)$. When $p = 1$, $Z(W_1) \subseteq \bigcup_{k=1}^{1} Z(S(\mathit{\Gamma}, n_k))$ becomes clear upon unraveling the notation: $Z(S(\mathit{\Gamma}, n_1)) = Z(S(\mathit{\Gamma}, n_1))$. To show that $Z(W_{p+1}) \subseteq \bigcup_{k=1}^{p+1} Z(S(\mathit{\Gamma}, n_k))$ upon assuming $Z(W_p) \subseteq \bigcup_{k=1}^{p} Z(S(\mathit{\Gamma}, n_k))$, let $r(\overrightarrow{a}) \in Z(W_{p+1})$ where $(r, \overrightarrow{a}) \in z \times W_{p+1}^{e}$ for some power $e \in \mathbb{N}^{+}$. Note that $W_{p+1} = W_p \cup S(\mathit{\Gamma}, n_{p+1})$; so, $r(\overrightarrow{a}) \in Z(W_p \cup S(\mathit{\Gamma}, n_{p+1}))$. To show $r(\overrightarrow{a}) \in \bigcup_{k=1}^{p+1} Z(S(\mathit{\Gamma}, n_k))$, we shall demonstrate that if $r(\overrightarrow{a}) \notin \bigcup_{k=1}^{p} Z(S(\mathit{\Gamma}, n_k))$ then $r(\overrightarrow{a}) \in Z(S(\mathit{\Gamma}, n_{p+1}))$. If $r(\overrightarrow{a}) \notin \bigcup_{k=1}^{p} Z(S(\mathit{\Gamma}, n_k))$, then $r(\overrightarrow{a}) \notin Z(S(\mathit{\Gamma}, n_k))$ for all $k \in \mathbb{F}_p$. Thus, for every pair $\left(j, \overrightarrow{b}\right) \in z \times S(\mathit{\Gamma}, n_k)^{d}$ with $d > 0$, $j\left(\overrightarrow{b}\right) \neq r(\overrightarrow{a})$. It must be the case that $\overrightarrow{a} \notin W_p^{e}$ by induction because the induction hypothesis implies that if $r(\overrightarrow{a}) \notin \bigcup_{k=1}^{p} Z(S(\mathit{\Gamma}, n_k))$, $r(\overrightarrow{a}) \notin Z(W_p)$, and that implies $\overrightarrow{a} \notin W_p^{e}$. From $\overrightarrow{a} \in (W_p \cup S(\mathit{\Gamma}, n_{p+1}))^{e}$, we obtain $\overrightarrow{a} \in S(\mathit{\Gamma}, n_{p+1})^{e}$ which leads us to see $r(\overrightarrow{a}) \in Z(S(\mathit{\Gamma}, n_{p+1}))$. Since $Z(S(\mathit{\Gamma}, n_{p+1})) \subseteq \bigcup_{k=1}^{p+1} Z(S(\mathit{\Gamma}, n_k))$, it is evident that $r(\overrightarrow{a}) \in \bigcup_{k=1}^{p+1} Z(S(\mathit{\Gamma}, n_k))$ and, thus, $Z(W_p) \subseteq \bigcup_{k=1}^{p} Z(S(\mathit{\Gamma}, n_k))$ for all $p > 0$. In particular, $Z\left(\bigcup_{k=1}^{air(i)} S(\mathit{\Gamma}, n_k)\right) \subseteq \bigcup_{k=1}^{air(i)} Z(S(\mathit{\Gamma}, n_k))$.
(2) $\bigcup_{k=1}^{air(i)} Z(S(\mathit{\Gamma}, n_k)) = \bigcup_{k=1}^{air(i)} S(\mathit{\Gamma}, n_k + 1)$ by the definition of Z.
(3) $Z\left(\left\{s_1, ..., s_{air(i)}\right\}\right) \subseteq \bigcup_{k=1}^{air(i)} S(\mathit{\Gamma}, n_k + 1)$ because $Z\left(\left\{s_1, ..., s_{air(i)}\right\}\right) \subseteq Z\left(\bigcup_{k=1}^{air(i)} S(\mathit{\Gamma}, n_k)\right)$. That is so because if $r(\overrightarrow{a}) \in Z\left(\left\{s_1, ..., s_{air(i)}\right\}\right)$, then $\overrightarrow{a} \in \left\{s_1, ..., s_{air(i)}\right\}^{e}$ for some $e \in \mathbb{N}^{+}$. Since each $s_k \in S(\mathit{\Gamma}, n_k)$,

$\{s_1, ..., s_{air(i)}\} \subseteq \bigcup_{k=1}^{air(i)} S(\Gamma, n_k)$ and so $\{s_1, ..., s_{air(i)}\}^e \subseteq \left(\bigcup_{k=1}^{air(i)} S(\Gamma, n_k)\right)^e$; hence, $r(\overrightarrow{a}) \in Z\left(\bigcup_{k=1}^{air(i)} S(\Gamma, n_k)\right)$. We have $Z(\{s_1, ..., s_{air(i)}\}) \subseteq Z\left(\bigcup_{k=1}^{air(i)} S(\Gamma, n_k)\right)$. By (1), $Z(\{s_1, ..., s_{air(i)}\}) \subseteq \bigcup_{k=1}^{air(i)} Z(S(\Gamma, n_k))$, so by (2), $Z(\{s_1, ..., s_{air(i)}\}) \subseteq \bigcup_{k=1}^{air(i)} S(\Gamma, n_k + 1)$.

(4) $i(s_1, ..., s_{air(i)}) \in \bigcup_{k=1}^{air(i)} S(\Gamma, n_k + 1)$ because $i(s_1, ..., s_{air(i)}) \in Z(\{s_1, ..., s_{air(i)}\})$.

(5) $i(s_1, ..., s_{air(i)}) \in \sigma(\Gamma)$ because $\bigcup_{k=1}^{air(i)} S(\Gamma, n_k + 1) \subseteq \sigma(\Gamma)$.♣

Conjecture: $\sigma(\Gamma) = Con(\Gamma)$.
Conjecture: $\sigma(\Gamma)$ is a fixed point of the function z.

S can conceivably be implemented as an automated theorem prover that finds proofs of all theorems provable from Γ.

6.1. **A toy formal system.** Let the alphabet $x = \{a, b, o\}$ and set of wffs $y = \{\varphi_1, \varphi_2\} \cup \{\varphi_1 \wedge \varphi_2, \varphi_2 \wedge \varphi_1\}$ where $\varphi_1 := "a" = \{(1, a)\}$, $\varphi_2 := "b" = \{(1, b)\}$, and $\wedge := "o" = \{(1, o)\}$. In what follows, $\varphi_1 \wedge \varphi_2 := \{(1, a), (2, o), (3, b)\}$ and $\varphi_2 \wedge \varphi_1 := \{(1, b), (2, o), (3, a)\}$.

z has two inference rules: a kind of conjunction elimination and a kind of commutativity.

$$z = \{\{(\varphi_1 \wedge \varphi_2, \varphi_1), (\varphi_2 \wedge \varphi_1, \varphi_2)\}, \{(\varphi_1 \wedge \varphi_2, \varphi_2 \wedge \varphi_1), (\varphi_2 \wedge \varphi_1, \varphi_1 \wedge \varphi_2)\}\}.$$

If $\Gamma = \{\varphi_1 \wedge \varphi_2\}$, then let's explore the conjecture $\varphi_2 \in Con(\Gamma)$. $S(1) = \{\varphi_1 \wedge \varphi_2\}$, $S(2) = \{\varphi_1, \varphi_2 \wedge \varphi_1\}$, $S(3) = \{\varphi_2, \varphi_1 \wedge \varphi_2\}$, $S(4) = \{\varphi_1, \varphi_2, \varphi_2 \wedge \varphi_1\}$, $S(5) = \{\varphi_2, \varphi_1 \wedge \varphi_2\} = S(3)$, $S(6) = S(4)$, $S(7) = S(3)$, for $n \geq 3$:

$$S(n) = \begin{cases} \{\varphi_2, \varphi_1 \wedge \varphi_2\} & \text{if} \quad \text{n odd} \\ \{\varphi_1, \varphi_2, \varphi_2 \wedge \varphi_1\} & \text{if} \quad \text{n even} \end{cases}.$$

Since $\sigma(\Gamma) = Con(\Gamma)$, $Con(\Gamma) = \bigcup_{n=1}^{\infty} S(n) = \{\varphi_1, \varphi_2, \varphi_1 \wedge \varphi_2, \varphi_2 \wedge \varphi_1\}$. Therefore, $|Con(\Gamma)| = 4$.

7. Metatheorems

Conjecture: If $F_1 = (x, y, z_1)$ is a formal system with $|z_1| \geq 5$, then there is a formal system $F_2 = (x, y, z_2)$ such that for all $\Gamma \subseteq y$, $Con_{F_1}(\Gamma) = Con_{F_2}(\Gamma)$ and $|z_2| = 4$.

Definition: Let r be a unary inference rule. $\gamma \in \Gamma$ is said to be a periodic point of r if for some $m \geq 1$, $\gamma = r^m(\gamma)$.

Conjecture: If z contains inference rules such that there exist periodic points in Γ, then it is possible that (i) $\left|Con_{(x,y,z)}(\Gamma)\right| < \aleph_0$ and (ii) there exists an $m \geq 1$ such that $V(\Gamma, m) = \emptyset$ and for all $n > m$, $V(\Gamma, n) = \emptyset$.

Conjecture: If there are no periodic points in Γ under any inference rules in z, then (i) $\left|Con_{(x,y,z)}(\Gamma)\right| = \aleph_0$ and (ii) $(\forall m \geq 1)(V(\Gamma, m) \neq \emptyset)$.

Corollary: In first order logic, if $\Gamma \neq \emptyset$, then (i) $|Con_{FOL}(\Gamma)| = \aleph_0$ and (ii) $(\forall m \geq 1)(V(\Gamma, m) \neq \emptyset)$.

Conjecture: If γ is a periodic point of $r \in z$, then $Con_{(x,y,\{r\})}(\{\gamma\}) = \{\gamma, r(\gamma), \ldots, r^m(\gamma)\}$ and $V(\Gamma, m+1) = \emptyset$.

Conjecture: Let a formal system $F = (x, y, z)$ be given. Let z' be a proper subset of z. In the presence of universal quantification, conjunction, and implication, there exists a map $\lambda \in [z \to y]$ such that $Con_{(x,y,z)}(\mathit{\Gamma}) = Con_{(x,y,z-z')}(\mathit{\Gamma} \cup \lambda(z'))$.

Specifically, if $\mu \in z'$ we define $\lambda(\mu)$ to be the following, where μ has arity m:

$$\lambda(\mu) = (\forall \alpha_1) \ldots (\forall \alpha_m) \left(\left(\bigwedge_{i=1}^{m} \alpha_i \right) \to \mu(\alpha_1, ..., \alpha_m) \right).$$

8. Conclusion

Once the document can be concluded, its conclusion will go here.

9. References

[1] S. Cook, R. Reckhow, The Relative Efficiency of Propositional Proof Systems, The Journal of Symbolic Logic, Volume 44, Number 1, 1979.