1 What is a Proof?

1.1 Problem 1.1

Can't easily do a visual prolem in LaTeX, and I am too lazy to create graphics for this. I will try to use my words instead.

1.1.1 Problem 1.1 (a)

Arrange the triangles so that the "a" side of one is on the "b" side of the other, with the acute angles touching. Do this for all of them, and the "c" sides will form an outer square of size $c \times c$, and the square hole in the middle will be of size $(b - a) \times (b - a)$.

1.1.2 Problem 1.1 (b)

Move two adjacent triangles to the opposite side of the $c \times c$ sqaure, so that their "c" sides are flush, forming two $a \times b$ rectangles, touching each other and the $(b-a) \times (b-a)$ square. This forms a single shape (that looks like a 'P' turned 90° to the right. They form two squares of size $a \times a$ and $b \times b$. The $b \times b$ square includes the $(b-a) \times (b-a)$ square, one of the $b \times a$ rectangles, and a small section of the other one. The $a \times a$ square is formed with the rest of that rectangle.

1.1.3 Problem 1.1 (c)

In the case of $a = b$, The $(b - a) \times (b - a)$ rectangle becomes a degenerate 0×0 rectangle. However this does not pose a problem the proof still holds. As $a = b$, the triangles are right-angle isoceles triangles. For (a), the triangles form a square with the sides forming an "×" in the middle. For part (b), the triangles are rearranged into a $a \times 2a$ rectangle, made up of two $a \times a$ squares, which shows that $a^2 + a^2 = a^2 + b^2 = c^2$ using the preservation of area under rearrangement.

1.1.4 Problem 1.1 (d)

1. That the two acute angles of a right-angled triangle sum to 90°.
2. When putting the right angle of a right-angled triangle against a straight line, they form a retroflex right angle (For part (a), these become the corners of the square).
3. That if you put a line of length a along a line of length b, the remaining length not covered by the first line is of length $b - a$.
4. That is a shape has 4 right-angles and equal sides, it is a square. (I am not as confident in this one, because, isn't this the defintion of a square??)
5. That the acute angles from two right-angle triangles together sum to 180°.

1.2 Problem 1.2

1.2.1 Problem 1.2 (a)

$\sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1}$ is not true.

1.2.2 Problem 1.2 (b)

Assume $1 = -1$.

$$
\begin{aligned}
& 1 = -1 && (1)\\
\implies & 2 = 0 && (2)\\
\implies & 1 = 0 && (3)\\
\implies & 2 = 1 && (4)
\end{aligned}
$$

1.2.3 Problem 1.2 (c)

$$\sqrt{rs} = (rs)^{\frac{1}{2}} = r^{\frac{1}{2}}s^{\frac{1}{2}} = \sqrt{r}\sqrt{s}$$

1.3 Problem 1.3

1.3.1 Problem 1.3 (a)

$\log_{10}(1/2)$ is negative, and thus you cannot multiply both sides of the inequality without flipping the "$>$" into a "$<$".

1.3.2 Problem 1.3 (b)

"\$" and "¢" should be treated as units. As $\$1 = \$ = 100¢$, the first step is valid. However, the step $\$0.01 = (\$0.1)^2$ is invalid because the RHS expands to $\$^2 0.01$, which is not equal to $\$0.01$ as $\$^2 \neq \$$. Similarly, the step $(10¢)^2 = 100¢$ is invalid.

1.3.3 Problem 1.3 (c)

When $a = b$, $a - b$ is equal to 0, thus cancelling the $(a - b)$'s is invalid as it is dividing by zero.

1.4 Problem 1.4

While the implications going from each statement each happen to be true, this is not a valid proof as the directionality of the implicaitons should be reversed. If the proof started with $(a - b)^2 \geq 0$ and did the steps in reverse, finishing at $\frac{a+b}{2} \geq \sqrt{ab}$, then the proof would have been valid.

1.5 Problem 1.5

This is the Unexpected Hanging Paradox, and there is no consensus to its nature. My opinion is that the "it will be a surprise" element of the statement is equivalent to "the day cannot be deduced from this statement". But, this is self-referencial, leading to a paradox in a manner similar to the Liar Paradox.

1.6 Problem 1.6

We use proof by contradiction.
Suppose "$log_7 n$ is either an integer or irrational" is false. This is equivalent to "$log_7 n$ is a rational number and not an integer".

$$
\begin{aligned}
log_7 n &= \frac{p}{q} \text{ where coprime } p, q \in \mathbb{Z}^+ && \text{definition of rational number} && (5)\\
q log_7 n &= p && \text{multiply both sides by } q \text{ as } q \in \mathbb{Z}^+ \implies q \neq 0 && (6)\\
log_7 n^q &= p && y log_b x = log_b x^y && (7)\\
7^{log_7 n^q} &= 7^p && && (8)\\
n^q &= 7^p && b^{log_b x} = x && (9)
\end{aligned}
$$

Note that the prime-factorisation of the RHS must be equal to the prime-factorisation of the RHS. Thus n^q is made up entirely of 7's. As exponentiation by q does not change which primes are included in the factorisation, n must thus be entirely made of 7's as well, i.e, n is a power of 7. Thus, $n = 7^m$, where $m \in \mathbb{N}$. Thus, $n^q = (7^m)^q = 7^{mq} = 7^p$, since $(a^b)^c = a^{(bc)}$. Thus $mq = p$, since $x^a = x^b \implies a = b$.

Note that either $m = 0$, or q is a factor of p (since $mq = p$ and $m, q, p \in \mathbb{Z}^+$). If q is a factor of p, then it is either 1, p, or a non-trivial factor of p (i.e., a factor that isn't 1 or p).

All of these cases lead to contradictions. If $m = 0$, then $mq = 0q = 0 = p$, since $0x = 0$. But $p = 0$ contradicts $p \in \mathbb{Z}^+$. If $q = 1$, then $log_7 n = \frac{p}{q} = \frac{p}{1} = p$, but this contradicts $log_7 n$ being not an integer, as p is an integer. If $q = p$, then $log_7 n = \frac{p}{q} = \frac{p}{p} = 1$, which is a contradiction with $log_7 n$ not being an integer as 1 is an integer. If q is a non-trivial factor of p, then this contradicts q and p being coprime, as p and q share a common factor, namely q.

Thus $log_7 n$ is either an integer or irrational. ■

1.7 Problem 1.7

Case 1: $r < s$
Since $r < s$, $max(r, s) = s$, and $min(r, s) = r$. Thus, $max(r, s) + min(r, s) = s + r = r + s$.

Case 2: $r > s$
Since $r > s$, $max(r, s) = r$, and $min(r, s) = s$. Thus $max(r, s) + min(r, s) = r + s$.

Case 3: $r = s$
Since $r = s$, $max(r, s) = max(r, r) = r$, and $min(r, s) = min(s, s) = s$. Thus $max(r, s) + min(r, s) = r + s$.

Thus $max(r, s) + min(r, s) = r + s$ for all real numbers r, s, as it is true in all cases. ■

1.8 Problem 1.8

Note that $\sqrt{2}$ is irrational.

Case 1: $\sqrt{2}^{\sqrt{2}}$ is irrational
Consider the following.

$$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^{2} = 2$$

As $\sqrt{2}^{\sqrt{2}}$ is irrational, $\sqrt{2}$ is irrational, and 2 is rational, an irrational number can be raised to an irrational power to obtain a rational result.

Case 2: $\sqrt{2}^{\sqrt{2}}$ is rational
Since $\sqrt{2}$ is irrational, and $\sqrt{2}^{\sqrt{2}}$ is rational, an irrational number can be raised to an irrational power to obtain a rational result.

Thus it holds in all cases. ■