AGR Assignment 2

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1 Question 1

First notice, from the definition, $v^\alpha = \frac{dx^\alpha}{d\lambda}$,

$$v^\beta \nabla_\beta v^\alpha = \underbrace{\frac{dx^\beta}{d\lambda}\left(\frac{\partial}{\partial x^\beta}\right.}_{\frac{dx^\beta}{d\lambda}\frac{\partial}{\partial x^\beta}=\frac{d}{d\lambda}}\left.\frac{dx^\alpha}{d\lambda} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\gamma}{d\lambda}\right) \tag{1}$$

$$= \frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}. \tag{2}$$

Now, if we have a geodesic parameterised by some non-affine parameter, λ, we can realise simply by the chain rule that, for some affine parameter λ',

$$\frac{d}{d\lambda} = \frac{d\lambda'}{d\lambda}\frac{d}{d\lambda'}. \tag{3}$$

and,

$$\frac{d^2}{d\lambda^2} = \frac{d\lambda'}{d\lambda}\frac{d}{d\lambda'}\left[\frac{d\lambda'}{d\lambda}\frac{d}{d\lambda'}\right] \tag{4}$$

$$= \frac{d\lambda'}{d\lambda}\left[\frac{d^2\lambda'}{d\lambda' d\lambda}\frac{d}{d\lambda'} + \frac{d\lambda'}{d\lambda}\frac{d^2}{d\lambda'^2}\right] \tag{5}$$

$$= \frac{d^2\lambda'}{d\lambda^2}\frac{d}{d\lambda'} + \left(\frac{d\lambda'}{d\lambda}\right)^2\frac{d^2}{d\lambda'^2}. \tag{6}$$

Where equation (5) to (6) is from using equation (3).

Using this we can look at the geodesic equation for a non-affine parameter.

$$v^\beta \nabla_\beta v^\alpha = \kappa v^\alpha, \tag{7}$$

where $\kappa = \kappa(\lambda)$. Using (2), we can write this as,

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = \kappa\frac{dx^\alpha}{d\lambda}. \tag{8}$$

In switching to an affine parameter λ', which we define via,

$$\frac{d\lambda'}{d\lambda} = a \exp\left(\int \kappa \, d\lambda\right), \tag{9}$$

for a some arbitrary constant. So, taking an integral wrt λ, we have,

$$\int \frac{d\lambda'(\lambda)}{d\lambda} d\lambda = a \int \exp\left(\int \kappa \, d\lambda\right) d\lambda \tag{10}$$

$$\Rightarrow \lambda'(\lambda) = a \int \exp\left(\int \kappa \, d\lambda\right) d\lambda - b \tag{11}$$

where b is an arbitrary integration constant. We can also notice that,

$$\frac{d^2\lambda'}{d\lambda^2} = \frac{d}{d\lambda}\left[a \exp\left(\int \kappa \, d\lambda\right)\right] \tag{12}$$

$$= a \frac{d \exp(\int \kappa \, d\lambda)}{d(\int \kappa \, d\lambda)} \frac{d}{d\lambda}\left[\int \kappa \, d\lambda\right] \tag{13}$$

$$= a\kappa \exp\left(\int \kappa \, d\lambda\right), \tag{14}$$

by the fundamental theorem of calculus. We can now put this all together to reparameterise the non affine geodesic, into an affine parameterisation. First we start from equation (8), and use equations (3) and (6),

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = \kappa \frac{dx^\alpha}{d\lambda} \tag{15}$$

$$\left[\frac{d^2\lambda'}{d\lambda^2} \frac{d}{d\lambda'} + \left(\frac{d\lambda'}{d\lambda}\right)^2 \frac{d^2}{d\lambda'^2}\right] x^\alpha + \Gamma^\alpha{}_{\beta\gamma} \left(\frac{d\lambda'}{d\lambda}\right) \frac{dx^\beta}{d\lambda'} \left(\frac{d\lambda'}{d\lambda}\right) \frac{dx^\gamma}{d\lambda'} = \kappa \left(\frac{d\lambda'}{d\lambda}\right) \frac{dx^\alpha}{d\lambda'}, \tag{16}$$

now we can use equation (14) and our ansatz equation (9),

$$a\kappa \exp\left(\int \kappa \, d\lambda\right) \frac{dx^\alpha}{d\lambda'} + \left(a \exp\left(\int \kappa \, d\lambda\right)\right)^2 \frac{d^2x^\alpha}{d\lambda'^2} + \left(a \exp\left(\int \kappa \, d\lambda\right)\right)^2 \Gamma^\alpha{}_{\beta\gamma} \frac{dx^\beta}{d\lambda'} \frac{dx^\gamma}{d\lambda'} = \kappa a \exp\left(\int \kappa \, d\lambda\right) \frac{dx^\alpha}{d\lambda'}. \tag{17}$$

We can notice that the blue terms cancel, in the leftover terms, we can factor out the orange terms, $\left(a \exp\left(\int \kappa \, d\lambda\right)\right)^2$, and freely divide by them, as $\exp\left(\int \kappa \, d\lambda\right) \neq 0$ in general, and we can choose $a \neq 0$. Thus we are left with,

$$\frac{d^2x^\alpha}{d\lambda'^2} + \Gamma^\alpha{}_{\beta\gamma} \frac{dx^\beta}{d\lambda'} \frac{dx^\gamma}{d\lambda'} = 0, \tag{18}$$

We can define $u^\alpha = \frac{dx^\alpha}{d\lambda'}$, and using equation (2), rewrite this as,

$$u^\beta \nabla_\beta u^\alpha = 0. \tag{19}$$

Which is the geodesic equation for an affine parameter.

2 Question 2

First recall the action of a covariant derivative on vectors, covectors, and tensors.

$$\nabla_\alpha v^\beta = \partial_\alpha v^\beta + \Gamma^\beta{}_{\gamma\alpha} v^\gamma, \tag{20}$$

$$\nabla_\alpha \omega_\beta = \partial_\alpha \omega_\beta - \Gamma^\gamma{}_{\beta\alpha} \omega_\gamma, \tag{21}$$

$$\nabla_\alpha T^\beta{}_\sigma = \partial_\alpha T^\beta{}_\sigma + \Gamma^\beta{}_{\alpha\gamma} T^\gamma{}_\sigma - \Gamma^\gamma{}_{\alpha\sigma} T^\beta{}_\gamma. \tag{22}$$

Also recall the action of the Lie derivative on a (1,1) tensor,

$$\mathcal{L}_\xi T^\mu{}_\nu = \xi^\alpha \partial_\alpha T^\mu{}_\nu - T^\alpha{}_\nu \partial_\alpha \xi^\mu + T^\mu{}_\alpha \partial_\nu \xi^\alpha. \tag{23}$$

Now we can define a new derivative, that we will call the *covariant lie derivative*, $\mathcal{L}'$, such that we replace all partial derivatives with covariant derivatives,

$$\mathcal{L}'_\xi T^\mu{}_\nu \equiv \xi^\alpha \nabla_\alpha T^\mu{}_\nu - T^\alpha{}_\nu \nabla_\alpha \xi^\mu + T^\mu{}_\alpha \nabla_\nu \xi^\alpha \tag{24}$$

$$= \xi^\alpha \left(\partial_\alpha T^\mu{}_\nu + \Gamma^\mu{}_{\alpha\gamma} T^\gamma{}_\nu - \Gamma^\gamma{}_{\alpha\nu} T^\mu{}_\gamma\right) \tag{25}$$
$$- T^\alpha{}_\nu \left(\partial_\alpha \xi^\mu + \Gamma^\mu{}_{\alpha\gamma} \xi^\gamma\right) \tag{26}$$
$$+ T^\mu{}_\alpha \left(\partial_\nu \xi^\alpha + \Gamma^\alpha{}_{\nu\gamma} \xi^\gamma\right) \tag{27}$$

$$= \xi^\alpha \partial_\alpha T^\mu{}_\nu - T^\alpha{}_\nu \partial_\alpha \xi^\mu + T^\alpha{}_\nu \partial_\alpha \xi^\mu \tag{28}$$
$$+ \xi^\alpha \Gamma^\mu{}_{\alpha\gamma} T^\gamma{}_\nu - \xi^\alpha \Gamma^\gamma{}_{\alpha\nu} T^\mu{}_\gamma - T^\alpha{}_\nu \Gamma^\mu{}_{\alpha\gamma} \xi^\gamma + T^\mu{}_\alpha \Gamma^\alpha{}_{\nu\gamma} \xi^\gamma \tag{29}$$

We can notice that the red term cancels the first blue term by a change of dummy indices $\alpha \leftrightarrow \gamma$ noting that all of these terms commute, as they are simply components. Seeing that the orange and second blue terms cancel is similar, but also requires the fact that we are working with a Levi-Civita connection, and thus $\Gamma^\alpha{}_{[\mu\nu]} = 0$, and so $\Gamma^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{(\mu\nu)}$, is symmetric in its lower indices. Now with the $\alpha \leftrightarrow \gamma$ exchange, we can see that these terms cancel.

Now looking back at equation (23), we can see that,

$$\mathcal{L}'_\xi T^\mu{}_\nu \equiv \xi^\alpha \nabla_\alpha T^\mu{}_\nu - T^\alpha{}_\nu \nabla_\alpha \xi^\mu + T^\mu{}_\alpha \nabla_\nu \xi^\alpha \tag{30}$$

$$= \xi^\alpha \partial_\alpha T^\mu{}_\nu - T^\alpha{}_\nu \partial_\alpha \xi^\mu + T^\alpha{}_\nu \partial_\alpha \xi^\mu \tag{31}$$

$$= \mathcal{L}_\xi T^\mu{}_\nu \quad \square. \tag{32}$$

There is direct equality between our regular lie derivative, and our covariant lie derivative.