AGR Assignment 1

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1 Question 1

Recall that, the commutator maps 2 vector fields to another vector field, $[\cdot,\cdot] : \Gamma(T_pM) \times \Gamma(T_pM) \to \Gamma(T_pM)$, and so we can write it as a superposition of basis vectors, (also note that to simplify notation I use $\frac{\partial}{\partial x^\mu} =: \partial_\mu$)

$$[\boldsymbol{u},\boldsymbol{v}](f) \equiv [\boldsymbol{u},\boldsymbol{v}]^\mu \partial_\mu(f). \tag{1}$$

And so can $\boldsymbol{u}$ and $\boldsymbol{v}$,

$$\boldsymbol{u} \equiv u^\mu \partial_\mu, \qquad \boldsymbol{v} \equiv v^\mu \partial_\mu. \tag{2}$$

Note that, of course, these partial derivative operators behave as normal derivative operators, specifically they are graded derivations on functions (they satisfy the Leibniz rule / product rule w.r.t functions), recall that vector components are functions.

Note the action of the commutator on vectors, $[\boldsymbol{u},\boldsymbol{v}](f) = \boldsymbol{u}(\boldsymbol{v}(f)) - \boldsymbol{v}(\boldsymbol{u}(f))$. Thus,

$$[\boldsymbol{u},\boldsymbol{v}](f) = \boldsymbol{u}(\boldsymbol{v}(f)) - \boldsymbol{v}(\boldsymbol{u}(f)) \tag{3}$$

$$[\boldsymbol{u},\boldsymbol{v}]^\mu \partial_\mu(f) = u^\mu \partial_\mu[v^\nu \partial_\nu(f)] - v^\alpha \partial_\alpha[u^\beta \partial_\beta(f)] \tag{4}$$

$$\begin{aligned} &= u^\mu \frac{\partial v^\nu}{\partial x^\mu}\partial_\nu(f) - u^\mu v^\nu \frac{\partial^2}{\partial x^\mu \partial x^\nu}(f) \\ &\quad - v^\alpha \frac{\partial u^\beta}{\partial x^\alpha}\partial_\beta(f) + v^\alpha u^\beta \frac{\partial^2}{\partial x^\alpha \partial x^\beta}(f) \end{aligned} \tag{5}$$

Where (3) $\to$ (4) is using equations (1) and (2), and (5) is simply using the graded derivation property of partial derivatives (Leibniz rule). We can see that the blue terms cancel each other out with index exchange $\beta \to \mu,\ \alpha \to$

ν, because multiplication of vector components is commutative, and partial derivatives are also commutative. So we are left with,

$$[\boldsymbol{u}, \boldsymbol{v}]^\mu \partial_\mu(f) = (u^\mu \partial_\mu[v^\nu] - v^\alpha \partial_\alpha[u^\nu])\partial_\nu(f) \tag{6}$$

$$= (u^\nu \partial_\nu[v^\mu] - v^\nu \partial_\nu[u^\mu])\partial_\mu(f), \tag{7}$$

by simply renaming dummy indices. So we can identify components,

$$[\boldsymbol{u}, \boldsymbol{v}]^\mu \equiv u^\nu \partial_\nu[v^\mu] - v^\nu \partial_\nu[u^\mu]. \tag{8}$$

As required (up to arbitrary dummy variable names).

2 Question 2

2.1 Part (a)

$k^\alpha = (1, f, 0, 0)$ in (t, r, θ, ϕ), find this in (u, r, θ, ϕ). Recall that,

$$\text{under } (t, r, \theta, \phi) \to (u, r, \theta, \phi), \tag{9}$$

$$\text{then} \qquad k^\alpha \mapsto k^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} k^\alpha \tag{10}$$

where α' denotes new coordinates and α denotes old coordinates. Thus we can calculate,

$$k^{0'} = \frac{\partial u}{\partial t} k^0 + \frac{\partial u}{\partial r} k^1 + \cancelto{0}{\frac{\partial u}{\partial \theta}} k^2 + \cancelto{0}{\frac{\partial u}{\partial \phi}} k^3. \tag{11}$$

We can realise for $u = t - r^*$, where $r^* = r + 2M \log\left(\frac{r}{2M} - 1\right)$ that $\frac{\partial u}{\partial t} = 1$, and,

$$\begin{aligned}
\frac{\partial u}{\partial r} &= -1 - 2M \left(\frac{1}{\frac{r}{2M} - 1}\right)\left(\frac{1}{2M}\right) && (12)\\
&= -1 - \left(\frac{2M}{r - 2M}\right) && (13)\\
&= -\frac{r}{r - 2M}. && (14)
\end{aligned}$$

Now we can substitute in,

$$\begin{aligned}
k^{0'} &= 1 - \frac{r}{r - 2M}\left(1 - \frac{2M}{r}\right) && (15)\\
&= 1 - \frac{1}{1 - \frac{2M}{r}}\left(1 - \frac{2M}{r}\right) && (16)\\
&= 1 - 1 = 0 && (17)
\end{aligned}$$

Now we can find,

$$\begin{aligned}
k^{1'} &= \cancelto{0}{\frac{\partial r}{\partial t}} k^0 + \frac{\partial r}{\partial r} k^1 + \cancelto{0}{\frac{\partial r}{\partial \theta}} k^2 + \cancelto{0}{\frac{\partial r}{\partial \phi}} k^3 && (18)\\
&= f && (19)
\end{aligned}$$

$$k^{2'} = 0 \tag{20}$$

$$k^{3'} = 0 \tag{21}$$

(20) and (21) are simply because the only non-zero derivative terms have $k^2 = 0$ and $k^3 = 0$ respectively. Thus, we have,

$$(k^{\alpha'}) = (0, f, 0, 0) \tag{22}$$

Or equivalently,

$$(k^{\alpha'}) = f\frac{\partial}{\partial r} =: f\partial_1 \tag{23}$$

2.2 Part (b)

Recall that, for some smooth function, $f \in C^\infty$, and some vector field, $\boldsymbol{v} \in \Gamma(TM)$, that $df(\boldsymbol{v}) = \boldsymbol{v}(f)$. Thus using this relation, in our specific case, in (u, r, θ, ϕ) coordinates,

$$\begin{aligned} du(\boldsymbol{k}) &= \boldsymbol{k}(u) && (24)\\ &= f\partial_1(u) = 0 && (25) \end{aligned}$$

as u is independent of r in this coordinate system, its derivative w.r.t. r is vanishing. Alternatively in (t, r, θ, ϕ) coordinates,

$$\begin{aligned} du(\boldsymbol{k}) &= \boldsymbol{k}(u) && (26)\\ &= \left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial r}\right)\left[t - r - 2M\log\left(\frac{r}{2M} - 1\right)\right] && (27) \end{aligned}$$

If we look back at our calculation for $k^{0'}$, in (15), (16), and (17), we can see it is exactly equivalent to this calculation. So as we expect, as this is a scalar, and so should not transform under coordinate transformations, $du(k) = 0$.

Then we can find $dv(\boldsymbol{k})$, we work in (t, r, θ, ϕ) coordinates, and using our result (14),

$$\begin{aligned} dv(\boldsymbol{k}) &= \boldsymbol{k}(v) && (28)\\ &= \left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial r}\right)\left[t + r + 2M\log\left(\frac{r}{2M} - 1\right)\right] && (29)\\ &= 1 + \left(1 - \frac{2M}{r}\right)\frac{r}{r - 2M} && (30)\\ &= 1 + \frac{r}{r - 2M} - \frac{2M}{r - 2M} && (31)\\ &= 2 && (32) \end{aligned}$$

We can also work in (u, r, θ, ϕ) coordinates for fun. Notice that $t = u + r^*$, so $v = t + r^* = u + 2r^*$. Now recall that the exterior derivative is a linear operator, thus,

$$
\begin{aligned}
dv(\boldsymbol{k}) &= \cancelto{0}{du(\boldsymbol{k})} + 2dr^*(\boldsymbol{k}) && (33)\\
&= 2dr^*(\boldsymbol{k}) && (34)\\
&= 2\boldsymbol{k}(r^*) && (35)\\
&= 2f\partial_1 \left(r + 2M\log\left(\frac{r}{2M} - 1\right)\right) && (36)\\
&= 2\left(1 - \frac{2M}{r}\right)\frac{r}{r-2M} && (37)\\
&= 2 \cdot 1 = 2 && (38)
\end{aligned}
$$

Yippee! As expected for these scalars, it is invariant under coordinate transformations.

3 Additional (unmarked)

S symmetric tensor, A antisymmetric tensor, then,

$$S^{a_1 \dots a_n} A_{a_1 \dots a_n} \overset{\text{sym}}{=} -S^{a_n \dots a_1} A_{a_n \dots a_1} \tag{39}$$

$$\overset{dum}{=} -S^{a_1 \dots a_n} A_{a_1 \dots a_n} \tag{40}$$

$$\Rightarrow S^{a_1 \dots a_n} A_{a_1 \dots a_n} = 0 \tag{41}$$