AGR Assignment 3

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I use blue to set up problems.

The metric of a Schwarzschild black hole is given by,

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 (d\theta^2 + \sin^2(\theta) d\phi^2), \tag{1}$$

$$f = 1 - \frac{2M}{r}. \tag{2}$$

Our hypersurfaces, Σ_T given by level sets of,

$$T = t + 4M \left[\sqrt{\frac{r}{2M}} + \frac{1}{2} \ln \left(\frac{\sqrt{\frac{r}{2M}} - 1}{\sqrt{\frac{r}{2M}} + 1} \right) \right] \tag{3}$$

1 Question 1

Determine if hypersurfaces Σ_T are timelike, spacelike or null.

Recall that a hypersurface is spacelike, timelike or null respectively if $\nabla_\alpha T$ is timelike, spacelike or null everywhere on Σ_T. Thus we can calculate $\nabla_\alpha T$.

First things first, we can notice that the covariant derivative of a scalar function is equivalent to its partial derivative, $\nabla_\alpha T = \partial_\alpha T$. We can clearly see that $\partial_t T = 1$, and $\partial_\theta T = \partial_\phi T = 0$, as T is independent of these coordinates. We can now find,

$$\partial_r T = 4M \; \partial_r \left[\sqrt{\frac{r}{2M}} + \frac{1}{2} \ln \left(\frac{\sqrt{\frac{r}{2M}} - 1}{\sqrt{\frac{r}{2M}} + 1} \right) \right]. \tag{4}$$

Let's further break this calculation up, the first term is,

$$\partial_r \left[\sqrt{\frac{r}{2M}} \right] = \frac{1}{\sqrt{8Mr}} \tag{5}$$

and for the second term, we define $x = \sqrt{\frac{r}{2M}}$, so by (5),

$$\partial_r = \frac{\partial x}{\partial r}\partial_x \tag{6}$$

$$= \frac{1}{\sqrt{8Mr}}\partial_x. \tag{7}$$

We can then define, $w = x + 1$, so

$$\frac{\partial w}{\partial x} = 1 \;\Rightarrow\; \partial_x = \partial_w, \tag{8}$$

and our second term in these coordinates becomes, $\ln(\frac{w-2}{w}) = \ln(1 - \frac{2}{w})$. Now we define $u = 1 - \frac{2}{w}$, so,

$$\frac{\partial u}{\partial w} = \frac{2}{w^2}, \tag{9}$$

$$\partial_w = \frac{2}{w^2}\partial_u \tag{10}$$

$$= \frac{2}{(x+1)^2}\partial_u \tag{11}$$

$$= \frac{2}{\left(\sqrt{\frac{r}{2M}}+1\right)^2}\partial_u. \tag{12}$$

We can finally return to our second term, ignoring (for now) a factor $\frac{1}{2}$, and put all of this together,

$$\partial_r\left[\ln\left(\frac{\sqrt{\frac{r}{2M}}-1}{\sqrt{\frac{r}{2M}}+1}\right)\right] = \frac{1}{\sqrt{8Mr}}\partial_x\left[\ln\left(\frac{x-1}{x+1}\right)\right] \tag{13}$$

$$= \frac{1}{\sqrt{8Mr}}\partial_w\left[\ln\left(1-\frac{2}{w}\right)\right] \tag{14}$$

$$= \frac{1}{\sqrt{8Mr}}\cdot\frac{2}{\left(\sqrt{\frac{r}{2M}}+1\right)^2}\partial_u\left[\ln(u)\right] \tag{15}$$

$$= \frac{1}{\sqrt{8Mr}}\cdot\frac{2}{\left(\sqrt{\frac{r}{2M}}+1\right)^2}\cdot\frac{1}{1-\frac{2}{\sqrt{\frac{r}{2M}}+1}} \tag{16}$$

$$= \frac{1}{\sqrt{8Mr}}\cdot\frac{2}{\left(\sqrt{\frac{r}{2M}}+1\right)^2 - 2\left(\sqrt{\frac{r}{2M}}+1\right)} \tag{17}$$

$$= \frac{1}{\sqrt{8Mr}}\cdot\frac{2}{\frac{r}{2M}-1}. \tag{18}$$

Overall then,

$$\partial_r T = 4M\ \partial_r \left[\sqrt{\frac{r}{2M}} + \frac{1}{2} \ln \left(\frac{\sqrt{\frac{r}{2M}} - 1}{\sqrt{\frac{r}{2M}} + 1} \right) \right] \tag{19}$$

$$= 4M \left(\frac{1}{\sqrt{8Mr}} + \frac{1}{\sqrt{8Mr}} \cdot \frac{1}{\frac{r}{2M} - 1} \right) \tag{20}$$

$$= 4M \left(\frac{1}{\sqrt{8Mr}} \right) \cdot \left(1 + \frac{1}{\frac{r}{2M} - 1} \right) \tag{21}$$

$$= \sqrt{\frac{2M}{r}} \left(1 + \frac{1}{\frac{r}{2M} - 1} \right) \tag{22}$$

$$= \sqrt{\frac{2M}{r}} \left(\frac{\frac{r}{2M}}{\frac{r}{2M} - 1} \right) \tag{23}$$

$$= \frac{\sqrt{\frac{r}{2M}}}{\frac{r}{2M} - 1}. \tag{24}$$

So we have found that,

$$(\nabla_\alpha T) = \left(1, \frac{\sqrt{\frac{r}{2M}}}{\frac{r}{2M} - 1}, 0, 0 \right). \tag{25}$$

We can find the square of this,

$$(\nabla_\alpha T)^2 = g^{\alpha\beta} \nabla_\alpha T\ \nabla_\beta T. \tag{26}$$

We can conveniently notice from our line element that our metric is diagonal, thus $g^{\mu\mu} = \frac{1}{g_{\mu\mu}}$. This can be immediately seen by noticing that this means the metric is only non zero for a repeated index, and so any $\alpha \neq \beta$,

$$g^{\alpha\mu} g_{\mu\beta} = g^{\alpha\beta} g_{\beta\beta} = 0,\ \Rightarrow\ g^{\alpha\beta} = 0, \tag{27}$$

whereas if $\alpha = \beta$,

$$g^{\alpha\mu} g_{\mu\alpha} = 1,\ \Rightarrow\ g^{\alpha\alpha} = \frac{1}{g_{\alpha\alpha}}. \tag{28}$$

by the definition of the inverse metric $g^{\alpha\beta}$, $g^{\alpha\mu}g_{\mu\beta} = \delta^\alpha_\beta$. we can thus see that,

$$
\begin{aligned}
(\nabla_\alpha T)^2 &= g^{tt} + g^{rr}\left(\frac{\sqrt{\frac{r}{2M}}}{\frac{r}{2M}-1}\right)^2 && (29)\\
&= -\frac{1}{1-\frac{2M}{r}} + \left(1-\frac{2M}{r}\right)\cdot\left(\frac{\frac{r}{2M}}{\left(\frac{r}{2M}-1\right)^2}\right) && (30)\\
&= -\frac{1}{1-\frac{2M}{r}} + \frac{\frac{r}{2M}-1}{\left(\frac{r}{2M}-1\right)^2} && (31)\\
&= \frac{1}{\frac{2M}{r}-1} + \frac{1}{\frac{r}{2M}-1} && (32)\\
&= \frac{\frac{r}{2M}-1+\frac{2M}{r}-1}{\left(\frac{2M}{r}-1\right)\cdot\left(\frac{r}{2M}-1\right)} && (33)\\
&= \frac{-2+\frac{r}{2M}+\frac{2M}{r}}{2-\frac{r}{2M}-\frac{2M}{r}} && (34)\\
&= -1. && (35)
\end{aligned}
$$

Thus, as we are working with a mostly plus metric signature, $(\nabla_\alpha T)$ is timelike everywhere, and thus Σ_T are spacelike hypersurfaces.

And calculate their unit normal n^α

Recall the definition of the unit normal,

$$
n_\alpha = \frac{\epsilon\nabla_\alpha T}{\sqrt{|g^{\mu\nu}\nabla_\mu T\ \nabla_\nu T|}}. \tag{36}
$$

Where $\epsilon = +1$, if Σ_T is timelike, and $\epsilon = -1$ if Σ_T is spacelike, thus, in our case $\epsilon = -1$. The denominator is clearly 1, by our previous calculation of $(\nabla_\alpha T)^2 = -1$, $|-1| = 1$, $\sqrt{1} = 1$. Thus we see,

$$
\begin{aligned}
(n_\alpha) &= -(\nabla_\alpha T) && (37)\\
&= \left(-1, \frac{\sqrt{\frac{r}{2M}}}{1-\frac{r}{2M}}, 0, 0\right). && (38)
\end{aligned}
$$

So, as $n^\alpha = g^{\alpha\beta}n_\beta$,

$$
\begin{aligned}
n^t &= g^{tt}n_t && (39)\\
&= -f^{-1}\cdot(-1) && (40)\\
&= f^{-1} && (41)
\end{aligned}
$$

$$n^r = g^{rr} n_r \tag{42}$$

$$= \left(1 - \frac{2M}{r}\right) \cdot \left(\frac{\sqrt{\frac{r}{2M}}}{1 - \frac{r}{2M}}\right) \tag{43}$$

$$= \frac{\sqrt{\frac{r}{2M}} - \sqrt{\frac{2M}{r}}}{1 - \frac{r}{2M}} \tag{44}$$

$$= \sqrt{\frac{2M}{r}} \cdot \left(\frac{\frac{r}{2M} - 1}{1 - \frac{r}{2M}}\right) \tag{45}$$

$$= -\sqrt{\frac{2M}{r}} \tag{46}$$

Clearly $n^\theta = n^\phi = 0$, so,

$$(n^\alpha) = \left(f^{-1}, -\sqrt{\frac{2M}{r}}, 0, 0\right) \tag{47}$$

2 Question 2

Using $(y^i) = (r, \theta, \phi)$ as coordinates on each Σ_T, write the embeddings $x^\alpha(y^i)$.

Of course we have that,

$$x^r(y^i) = x^r(r) = r \tag{48}$$
$$x^\theta(y^i) = x^\theta(\theta) = \theta \tag{49}$$
$$x^\phi(y^i) = x^\phi(\phi) = \phi \tag{50}$$

For any given Σ_T, we have that $T = c$, for some constant $c \in \mathbb{R}$. This motivates us to define, for a given Σ_T,

$$x^t(y^i) = x^t(r) = c - 4M\left[\sqrt{\frac{r}{2M}} + \frac{1}{2}\ln\left(\frac{\sqrt{\frac{r}{2M}} - 1}{\sqrt{\frac{r}{2M}} + 1}\right)\right]. \tag{51}$$

And calculate the tangent vectors e_i^α to the surfaces.

Recall the definition of the tangent vectors,

$$e_i^\alpha = \frac{\partial x^\alpha}{\partial y^i}. \tag{52}$$

We can clearly see that a lot of these will be 0. e_θ^t, e_ϕ^t, e_θ^r, e_ϕ^r e_r^θ, e_ϕ^θ, e_r^ϕ, e_θ^ϕ, will all be vanishing. Some are trivial, e_r^r, e_θ^θ, e_ϕ^ϕ, are all 1. Thus, the only tricky one would be,

$$e_r^t = \frac{\partial x^t}{\partial r}. \tag{53}$$

However, we have done basically this calculation in question 1, equations (19) to (24). The only differences being that there is a $\partial_r[c] = 0$ instead of $\partial_r[t] = 0$, and a factor of (-1), which can be factored out of the derivative. Thus,

$$e_r^t = -\frac{\sqrt{\frac{r}{2M}}}{\frac{r}{2M} - 1} \tag{54}$$

$$= \frac{\sqrt{\frac{r}{2M}}}{1 - \frac{r}{2M}}. \tag{55}$$

So, tangent vectors to all hypersurfaces Σ_T have the same form, regardless of the specific Σ_T we are talking about.

3 Question 3

Calculate the induced metric, h_{ij} on Σ_T in y^i.

Recall that the induced metric, h_{ij}, is given by,

$$h_{ij} = e_i^\alpha e_j^\beta g_{\alpha\beta}. \tag{56}$$

Again we use the fact that $g_{\alpha\beta}$ is diagonal, and so the induced metric is as well. We can thus find non-zero components of the induced metric,

$$\begin{aligned} h_{\theta\theta} &= e_\theta^\alpha e_\theta^\alpha g_{\alpha\alpha} && (57)\\ &= e_\theta^\theta e_\theta^\theta g_{\theta\theta} && (58)\\ &= r^2, && (59) \end{aligned}$$

and similarly,

$$\begin{aligned} h_{\phi\phi} &= e_\phi^\alpha e_\phi^\alpha g_{\alpha\alpha} && (60)\\ &= e_\phi^\phi e_\phi^\phi g_{\phi\phi} && (61)\\ &= r^2 \sin^2(\theta). && (62) \end{aligned}$$

Now for the more tricky induced metric component,

$$\begin{aligned} h_{rr} &= e_r^\alpha e_r^\alpha g_{\alpha\alpha} && (63)\\ &= (e_r^t)^2 g_{tt} + (e_r^r)^2 g_{rr} && (64)\\ &= \left(\frac{\sqrt{\frac{r}{2M}}}{1-\frac{r}{2M}}\right)^2 \cdot \left(\frac{2M}{r} - 1\right) + \frac{1}{1-\frac{2M}{r}} && (65)\\ &= \frac{1-\frac{r}{2M}}{\left(1-\frac{r}{2M}\right)^2} + \frac{1}{1-\frac{2M}{r}} && (66)\\ &= \frac{1}{1-\frac{r}{2M}} + \frac{1}{1-\frac{2M}{r}} && (67)\\ &= \frac{1-\frac{2M}{r}+1-\frac{r}{2M}}{1-\frac{r}{2M}-\frac{2M}{r}+1} && (68)\\ &= 1. && (69) \end{aligned}$$

Thus our induced metric is given by,

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2). \tag{70}$$

What is the curvature of this induced metric? Explain (no calculation required)

Thus, the curvature on these hypersurfaces is flat?? As this is the metric of flat space in spherical coordinates? This seems strange at first, as we are ultimately in Schwarzchild spacetime, but of course the hypersurfaces within this spacetime having 0 curvature is somewhat intuitive after a little thought.