Home Work 1
Com S 311
Due: Jan 24, 11:59PM

Late Submission Due: Jan 25, 11:59PM (25% penalty)

Outcomes. Be comfortable with proof techniques and use them to prove program correctness. There are 4 problems and each problem is worth 50 points.

1. *Fibonacci numbers* are defined recursively as follows:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \end{aligned}$$

   Show the following property of Fibonacci numbers by induction.

   For every $n \geq 1$,

$$F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n \times F_{n+1}.$$

   Your proof must use mathematical induction; otherwise you will receive zero credit.

2. Refer to the definition of Full Binary Tree from the notes. For a Full Binary Tree T, we use $n(T), h(T)$, and $i(T)$ to refer to number of nodes, height, and number of internal nodes (non-leaf nodes) respectively. Note that the height of a tree with single node is 1 (not zero). Using structural induction, prove the following:

   (a) For every Full Binary Tree T, $n(T) \geq 2h(T) - 1$.

   (b) For every Full Binary Tree T, $i(T) = (n(T) - 1)/2$

   Your proof must use structural induction; otherwise you will receive zero credit.

3. Consider the following program, where a and n are positive integers.

```
Input: a, n

x = a; m = n; y = 1;
while (m > 1) {
   if m is even
        x = x*x;
        m = m/2;
   else if m is odd
         y = x*y;
         x = x*x;
         m = (m-1)/2;
}

Output x*y
```

Let x_i, y_i, and m_i denote the value of the variables x, y, and m at the start of the ith iteration. Using induction show the following

$$\forall i \quad a^n = x_i^{m_i} \times y_i$$

Your proof must use induction. Otherwise you will not receive any credit.

4. Consider the following problem. Given a sorted array a of consisting of distinct integers and an integer T, determine if there exist two integers in the array (possibly the same integer) whose sum equals T. Consider the following algorithm:

```
Input: Array a, Integer T.
left = 0;
right = length of the array-1;
while (left <=right){

  x = a[left] + a[right];

  if (x==T)
        return true;
  if (x <  T)
     left++;
  if (x > T)
     right--;

}

return false;
```

Show that the above program is correct by proving the following:

At the start of the ith iteration the following conditions hold:

- $left \leq right$
- If there exist indices ℓ and j such that $a[\ell] + a[j]$ equals T, then $left \leq \ell \leq j \leq right$.

Your proof must use induction.

GUIDE LINES:

- Please write your recitation number, time and TA name.
- It is important to know whether your really know! For each problem, if you write the statement "I do not know how to solve this problem" (and nothing else), you will receive 20% credit for that problem. If you do write a solution, then your grade could be anywhere between 0% to 100%. To receive this 20% credit, you must explicitly state that you do not know how to solve the problem.
- You must work on the homework problems on your own. You should write the final solutions alone, without consulting any one. Your writing should demonstrate that you understand the proofs completely.
- When proofs are required, you should make them both clear and rigorous. Do not hand waive.
- If you hand writing is not legible, then your homework will not be graded.
- Any concerns about grading should be made within one week of returning the homework.
- **Please submit your HW via Canvas. If you type your solutions, then please submit pdf version. If you hand-write your solutions, then please scan your solutions and submit a pdf version. Please make sure that the quality of the scan is good, and your hand writing is legible. Name your file must be** $YourNetID$**-**$HW1.pdf$**. For example, if your net id is** `potterh`**, then the file name should be** $potterh$**-**$HW1.pdf$**. HW's submitted in incorrect format (non pdf, incorrect file name etc) will incur a penalty of 20%**