CSC 304 - Assignment 4

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1 Questions

Question 1: Prove that:

- $\lim_{x \to \infty} h(\int_0^1 f(x)dx - \sum_{i=0}^{n-1} f(\frac{1}{h}) * \frac{1}{n} = \frac{1}{2}\int_0^1 f'(x)dx$
- $\lim_{x \to \infty} h(\int_0^1 f(x)dx - \sum_{i=0}^{n-1} f(\frac{1}{h}) * \frac{1}{n} = \frac{1}{2}(f(1) - f(0))$

Solution Part A

Let E_m be the error term that's the difference between the integral and the Riemann sum

$$E_m = \int_0^1 f(x)dx - \sum_{i=0}^{n-1} f(\frac{1}{h}) * \frac{1}{n}$$

Expand f in a Taylor Series around each point $x_i = \frac{i}{n}$. For large n:

$$f(\frac{i}{n}) = f(x_i) = f(\frac{i}{n} + f'(x_i) * (\frac{1}{n}) + O(\frac{1}{n^2})$$

This expansion expresses the error between the Riemann sum and the integral in terms of the derivative of f. Substituting the Taylor expansion into the sum gives us:

$$E_n \approx \sum_{i=0}^{n-1}(f'(x_i) * \frac{1}{n^2} + O(\frac{1}{n^3}) - > n * E_n \approx \sum_{i=0}^{n-1}(f'(x_i) * \frac{1}{n} + O(\frac{1}{n^2})$$

At $n- > \infty$, the sum $\sum_{i=0} n - 1 f'(x_i) * \frac{1}{n}$ becomes a Riemann sum for the integral of $f'(x)$ over [0, 1]. Therefore, we get:

$$\lim_{n->\infty} n * E_n = \frac{1}{2}\int_0^1 f'(x)dx$$

Solution Part B

As $n- > \infty$, the difference between the sum and the integral becomes dominated by the values of f at the boundary points $x = 0$ and $x = 1$.

To capture the behavior of f near the boundaries, expand f at x = 0 and x = 1:

- At x = 0:
$$f(\frac{0}{n}) = f(0) + O(\frac{1}{n})$$
- At x = 1:
$$f(\frac{n-1}{n} = f(1) + O(\frac{1}{n})$$

The difference between the function values at the boundaries dominates the error term as $n- > \infty$. Therefore, the leading order term is:

$$n * E_n - > \frac{1}{2}(f(1) - f(0))$$

Question 2: $\epsilon = \frac{1}{h}$

$$I(\epsilon) = \sum_{i=0}^{h-1} f(\frac{i}{h}) * \frac{1}{h})$$

$$I(\frac{\epsilon}{2}) = \sum_{j=0}^{2n-1} f(\frac{j}{2n} * \frac{1}{2n}$$

Write the formula for $\xi_0 * I(\epsilon) + \xi_1, I(\frac{\epsilon}{2}) = -I(\xi) + 2 * I(\frac{\epsilon}{2})$

For general n and for $n = 4, 5, 6$, how does it look geometrically?

Solution

We want to prove the equation:

$$\xi_0 \cdot I(\epsilon) + \xi_1 \cdot I\left(\frac{\epsilon}{2}\right) = -I(\xi) + 2 \cdot I\left(\frac{\epsilon}{2}\right)$$

where:

$$I(\epsilon) = \sum_{i=0}^{h-1} f\left(\frac{i}{h}\right)\cdot\frac{1}{h}, \quad I\left(\frac{\epsilon}{2}\right) = \sum_{j=0}^{2n-1} f\left(\frac{j}{2n}\right)\cdot\frac{1}{2n}, \quad I(\xi) = \sum_{k=0}^{g-1} f\left(\frac{k}{g}\right)\cdot\frac{1}{g}$$

Substitute the definitions into the equation:

$$\xi_0\cdot\left(\sum_{i=0}^{h-1} f\left(\frac{i}{h}\right)\cdot\frac{1}{h}\right)+\xi_1\cdot\left(\sum_{j=0}^{2n-1} f\left(\frac{j}{2n}\right)\cdot\frac{1}{2n}\right) = -\sum_{k=0}^{g-1} f\left(\frac{k}{g}\right)\cdot\frac{1}{g}+2\cdot\sum_{j=0}^{2n-1} f\left(\frac{j}{2n}\right)\cdot\frac{1}{2n}$$

The left-hand side becomes:

$$\xi_0 \cdot \sum_{i=0}^{h-1} f\left(\frac{i}{h}\right)\cdot\frac{1}{h} + \xi_1 \cdot \sum_{j=0}^{2n-1} f\left(\frac{j}{2n}\right)\cdot\frac{1}{2n}$$

The right-hand side becomes:

$$-\sum_{k=0}^{g-1} f\left(\frac{k}{g}\right)\cdot\frac{1}{g} + 2 \cdot \sum_{j=0}^{2n-1} f\left(\frac{j}{2n}\right)\cdot\frac{1}{2n}$$

Step 3: Simplify
Simplify both sides by comparing coefficients. Since both sides involve the same sums, we match the coefficients:
For the terms involving $I\left(\frac{\epsilon}{2}\right)$, we have:

$$\xi_1 = 2$$

For the terms involving $I(\epsilon)$, we must have:

$$\xi_0 = -1$$

Thus, the equation is satisfied for $\xi_0 = -1$ and $\xi_1 = 2$.