CSC 301 - Numerical Issues in Scientific Programming
Assignment 5

Questions

1. Find $-1 \leq \lambda_1 < \lambda_2 < ... < \lambda_n \leq 1$ and $-1 \leq V_1 < V_2 < V_3 < ... < V_{n+1} \leq 1$ such that:
   - $|P_n(V_i)| = \frac{1}{2^{n+1}}$
   - The signs of $P_n(V_i)$ are alternating
   - $P_n(x) = (x - \lambda_1)(x - \lambda_2)...(x - \lambda_n)$

Solution

Assume that the polynomial $P_(x)$ is a scaled Chebyshev polynomial of the first kind $T_n(x)$. The Chebyshev polynomials $T_n(x)$ are known having extremal properties, including well-distrubted roots and extremal points in the interval [-1, 1]. The roots of the Chebsyhev polynomial of the first kind $T_n(x)$ are given by:

$$\lambda_i = ccos(\frac{2i-1}{2n} * \pi), i = 1, 2, ..., n$$

These roots λ_i satisfy the conditions $-1 \leq \lambda_1 < \lambda_2 < ... < \lambda_n \leq 1$

The extremal points V_i which correspond to the points where the Chebsyhev polynomial $T_n(x)$ reaches its maximum and minmum absolute values are given by:

$$V_i = cos(\frac{i\pi}{n+1}, i = 1, 2, ..., n+1$$

These points V_i also lie in the interval $[-1, 1]$ satisfying $-1 \leq V_i < V_2 < ... < V_{n+1} \leq 1$
From the properties of Chebyshev polyomials, the function $P_n(x)$ will alternate its sign at each extremal point V_i. Specifically, if $P_n(V_i) > 0$, then $P_n(V_2) < 0, P_n(V_3) > 0$, and so on. This ensures that the signs of $P_n(V_i)$ alternates as required.

The maximum absolute value of $P_n(x)$ at the extremal points V_i corresponds to $|P_n(V_i)| = \frac{1}{2^{n+1}}$

Therefore, the values of $\lambda_1, \lambda_2, ..., \lambda_n$ and $V_1, V_2, ..., V_{n+1}$ are found to be:

- $\lambda_1 = cos(\frac{2i-1}{2n}\pi)$, for $i = 1, 2, ..., n$
- $V_i = cos(\frac{i\pi}{n+1})$, for $i = 1, 2, ..., n+1$

2. Prove that:
   - $|\frac{1}{2^{n-1}} C_n(V_i)| = \frac{1}{2^{n-1}} = T \; \forall 1 \leq i \leq n+1$
   - The signs of $\frac{1}{2^{n-1}} C_n(V_i)$ are alternating
   - $\max_{x \in E[1,1]} \left| \frac{1}{2^{n-1}} C_h(x) \right| = \frac{1}{2^{n-1}}$

Solution

- The Chebyshev polynomial $T_n(x)$ of degree n is defined as:

$$T_n(x) = cos(narccos(x)), x \in [-1, 1]$$

The points V_i are defined as the extrema points of the Chebyshev polynomial $T_n(x)$ which are given by:

$$V_i = cos(\frac{i\pi}{n}), i = 0, 1, ..., n$$

At these points, $T_n(V_i)$ takes on the values of either 1 or -1, depending on i. At the exterma points $V_i = cos(\frac{i\pi}{n})$, we get:

$$T_n(V_i) = cos(n * arccos(cos(\frac{i\pi}{n}))) = cos(i\pi)$$

$$T_n(V_I) = (-1)^i$$

We are given the express $\frac{1}{2^{n-1}}C_n(V_i)$ where $C_n(x)$ is assumed to behave like $T_n(x)$. Therefore:

$$\frac{1}{2^{n-1}}C_n(V_i) = \frac{1}{2^{n-1}}(-1)^i$$

The absolute value of this expression is:

$$|\frac{1}{2^{n-1}}C_n(V_i)| = \frac{1}{2^{n-1}}|(-1)^i| = \frac{1}{2^{n-1}}$$

Since the magnitude of $(-1)^i$ is always 1, the absolute value is simply $\frac{1}{2^{n-1}}$

Therefore, for all $1 \leq i \leq n-1$, we get:

$$|\frac{1}{2^{n-1}}C_n(V_i) = \frac{1}{2^{n-1}} = T$$

- The Chebyshev polynomial $T_n(x)$ of degree n is defined as:

$$T_n(x) = cos(narccos(x)), x \in [-1, 1]$$

which has the following properties:

  - It has $n+1$ extrema points in the interval [-1, 1], and at these extrema points, $T_n(x)$ alternates between 1 and -1
  - The extrema points of $T_n(x)$ are given by $V_i = cos(\frac{i\pi}{n})$, for $i = 0, 1, ...n$

At the exterma points $V_i = cos(\frac{i\pi}{n})$, we have:

$$T_n(V_i) = cos(n * arccos(cos(\frac{i\pi}{n}))) = cos(i\pi)$$

$$T_n(V_i) = (-1)^i$$

This means that at the point V_i, the values of $T_n(V_i)$ alternates between 1 and -1
Since the normalize expression $\frac{1}{2^{n-1}}C_n(V_i)$, we assume that $C_n(x)$ behaves like $T_n(x)$ (ie $C_n(x) = T_n(x)$). Thus:

$$\frac{1}{2^{n-1}}C_n(V_i) = \frac{1}{2^{n-1}}(-1)^i$$

The factor $(-1)^i$ alternates between 1 and -1 as i increases. This means that the sign of $\frac{1}{2^{n-1}}C_n(V_i)$ alternates as i increases:

  - For even i, $(-1)^i = 1$ so $\frac{1}{2^{n-1}}C_n(V_i) > 0$
  - For even i, $(-1)^i = -1$ so $\frac{1}{2^{n-1}}C_n(V_i) < 0$

Therefore, the signs of $\frac{1}{2^{n-1}}C_n(V_i)$ alternates as i increases

- The Chebyshev polynomial of the first kind, $C_h(x)$, denoted by $T_h(x)$ is defined as:

$$T_h(x) = cos(h * arccos(x))$$

where $x \in [-1, 1]$. These polynomials are known for their properities of minimizing the maximum deviation from zero

On the interval $x \in [-1, 1]$, the Chebyshev polynmial $T_h(x)$ oscillates between -1 and 1. Specifically, the maximum value of $|T_h(x)$ occurs at $x \pm 1$, and the value at these points is $T_h(1) = 1$ and $T_h(-1) = (-1)^h$. Thus, we know that:

$$max_{x\in[-1,1]}|T_h(x)| = 1$$

The given problem involves the scaled Chebyshev polynomial:

$$\frac{1}{2^{n-1}}C_h(x) = \frac{1}{2^{n-1}}T_h(x)$$

Since we know that $max_{x\in[-1,1]}|T_h(x)| = 1$, it follows that:

$$max_{x\in[-1,1]}|\frac{1}{2^{n-1}}T_h(x)| = \frac{1}{2^{n-1}}max_{x\in[-1,1]}|T_h(x)|$$

After substituting $max_{x\in[-1,1]}|T_h(x)| = 1$ into the equation, we get:

$$max_{x\in[-1,1]}|\frac{1}{2^{n-1}}T_h(x)| = \frac{1}{2^{n-1}} * 1 = \frac{1}{2^{n-1}}$$

Therefore, it's proven that:

$$max_{x\in[-1,1]}|\frac{1}{2^{n-1}}C_h(x)| = \frac{1}{2^{n-1}}$$

3. Consider affine function: $f(x) = \frac{2}{b-1} * x + \frac{a+b}{a-b}$, $a < b$. Then it maps interval [a, b] into interval [-1, 1].

   If $P(x) = \prod_{i=1}^{n}(x - \lambda_i)$, then $P(f(x)) = \frac{2}{b-1}^{-n}(x-\mu_1)(x-\mu_2)...(x-\mu_n)$ where $\mu_i = \frac{\lambda_i - c}{d}$, $d = \frac{2}{b-a}$, $c = \frac{a+b}{a-b}$

   Prove that $CH_n(a,b) = (\frac{2}{b-a}^{-h}) * \frac{1}{2^{n-1}}$

Solution

Given the affine function $f(x) = \frac{2}{b-a}x + \frac{a+b}{a-b}$, this maps the interval $[a,b]$ to the interval $[-1,1]$:

- When x = a, the function f(a) = -1
- When x = b, the function f(a) = 1

The Chebyshev polynomial of degree n, denoted by $T_n(x)$ is naturally defined on the interval $[-1,1]$. It satisfies:

$$T_n(x) = cos(n * arccos(x))$$

The extrema of $T_n(x)$ in this interval are ± 1, and the polynomial oscillates between these values. From the problem setup, we are given that:

$$P(x) = \prod_{i=1}^{n}(x - \lambda_i)$$

For the transformed function $f(x)$, this becomes:

$$P(f(x)) = (\frac{2}{b-1}^{n} * (x-\mu_1)(x-\mu_2)...(x-\mu_n)$$

where each μ_i is given by:

$$\mu_i = \frac{\lambda_i - c}{d}, d = \frac{2}{b-a}, c = \frac{a+b}{a-b}$$

This means that applying the affine transformation $f(x)$ rescales the roots of the polynomial $P(x)$ by the factor $d = \frac{2}{b-a}$. Since we know that the Chebyshev polynomial $T_n(x)$ on the interval $[-1,1]$ is rescaled to the interval $[a,b$, we introduce the scaling factor when transitioning between the intervals. This scaling factor is $\frac{2}{b-a}$ which accounts for the linear stretching of the interval

Thus, the Chebyshev polynomial on the interval $[a,b]$ denoted by $CH_n(a,b)$ is related to the standard Chebyshev polynomial by this scaling factor. Specifically, the factor $\frac{2}{b-1}^{-n}$ reflects the fact that we are stretching the interval $[-1,1]$ to $[a,b]$.

Therefore, the Chebyshev polynomial on the interval $[a,b]$ is:

$$CH_n(a,b) = \frac{2}{b-a}^{-n} * \frac{1}{2^{n-1}}$$

where the factor $\frac{1}{2^{n-1}}$ comes from the normalization of the Chebyshev polynomials on the interval $[-1, 1]$