Exercises from Nielsen and Chuang

Contents

1 2.1.5 Eigenvectors and eigenvalues

(2.11) Find eigenvectors, eigenvalues, and diagonal representations of Pauli X,Y,Z.

Solution: The eigenvalues of all Pauli matrices are $1, -1$. The eigenvectors are as follows:

$$|\psi\rangle_{x+} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |\psi\rangle_{x-} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$|\psi\rangle_{y+} = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle), |\psi\rangle_{y-} = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$$

$$|\psi\rangle_{z+} = |0\rangle\,, |\psi\rangle_{z-} = |1\rangle$$

A diagonal representation is defined as $A = \sum_i \lambda_i \langle i|\,|i\rangle$ where the vectors i form an orthonormal set of eigenvectors. Thus, the diagonal representations of any Pauli matrix is:

$$\sigma = |\psi\rangle_+ \langle\psi|_+ - |\psi\rangle_- \langle\psi|_-$$

(2.12) Prove the matrix $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is not diagonalizable.

Solution: The eigenvalues of the matrix are 1 with algebraic multiplicity 2 (easily seen since it is a lower triangular matrix). Then, the eigenvectors are $\begin{bmatrix} 0 \\ \alpha \end{bmatrix}$, for $\alpha \in \mathbb{C}$.
A matrix is diagonalizable iff there exists a basis of eigenvectors of the matrix. There cannot be a matrix of eigenvectors since there is only one linearly independent eigenvector of the matrix, which is less than the dimension of the matrix. Thus it cannot form a basis and it cannot be diagonalizable.

2 2.1.6 Adjoints and Hermitian Operators

(2.13) If $|w\rangle$ and $|v\rangle$ are any two vectors, show that $(|w\rangle\langle w|)^\dagger = |v\rangle\langle w|$.

Solution: $(|w\rangle\langle v|)^\dagger = \langle v|^\dagger\,|w\rangle^\dagger = |v\rangle\langle w|\,.$

(2.14) (Anti-linearity of the adjoint) Show that the adjoint operation is antilinear,

$$(\sum_i a_i A_i)^\dagger = \sum_i a_i^* A_i^\dagger$$

Solution:

$$(\sum_i a_i A_i)^\dagger = \sum_i a_i^\dagger A_i^\dagger = \sum_i a_i^* A_i^\dagger$$

where $a^\dagger = (a^*)^T = a^*$ since a is a scalar.

(2.15) Show that $(A^\dagger)^\dagger = A$

Solution:

$$(|v\rangle, A|w\rangle) = (A^\dagger |v\rangle, |w\rangle) = (|v\rangle, (A^\dagger)^\dagger |w\rangle)$$

$$\iff A = (A^\dagger)^\dagger$$

(2.16) Show any projector P satisfies $P^2 = P$

Solution: A projector is defined as $P = \sum_i |i\rangle \langle i|$

$$\implies P^2 = \sum_{ij} |i\rangle \langle i| |j\rangle \langle j| = \sum_{ij} |i\rangle \delta_{ij} \langle j| = \sum_i |i\rangle \langle i| = P.$$

(2.17) Show that a normal matrix is Hermitian iff it has real eigenvalues.

Solution:

Proof. First we prove that a normal matrix is Hermitian if it has real eigenvalues.
Suppose we have matrix A such that $A = A^\dagger$ and $AA^\dagger = A^\dagger A$.
Then, suppose A has eigenvalue $\lambda \iff Av = \lambda v$ for some nonzero v.
Then, we see that $v^\dagger A^\dagger = v^\dagger A = \lambda^\dagger v$, where the first equality is given by the fact the matrix is Hermitian.
Consider

$$Av = \lambda v$$

$$\iff v^\dagger A v = v^\dagger \lambda v$$

$$\iff \lambda^\dagger v^\dagger v = \lambda v^\dagger v \iff \lambda^\dagger = \lambda$$

$\iff \lambda$ is real. Then we prove if a normal matrix has real eigenvalues then it is Hermitian. Suppose all eigenvalues are real, then $\lambda = \lambda^\dagger$. We see that $Av = \lambda v \iff v^\dagger A^\dagger = \lambda v^\dagger$. Then, left multiplying the first equation by $v^\dagger$ and right multiplying the second equation by v yields:

$$v^\dagger A v = \lambda |v|^2 = v^\dagger A^\dagger v \iff A = A^\dagger$$

$\square$

(2.18) Show that all eigenvalues of a unitary matrix have modulus 1, that is, can be written in the form $e^{i\theta}$ for some real θ.

Solution:

$$Uv = \lambda v \iff v^\dagger U^\dagger U v = v^\dagger |\lambda|^2 v \iff v^\dagger v = v^\dagger |\lambda|^2 v$$

$$\iff |\lambda|^2 = 1$$

(2.19) Show Pauli matrices are Hermitian and unitary.

Solution: Trivial to see Pauli matrices are equal to adjoints and to use matrix multiplication to check unitary.

(2.20) Suppose A' and A'' are matrix representations of an operator A on a vector space V with respect to two different orthonormal bases, $|v_i\rangle$ and $|w_i\rangle$. Then the elements of A' and A'' are $A'_{ij} = \langle v_i|A|v_j\rangle$ and $A''_{ij} = \langle w_i|A|w_j\rangle$. Characterize the relationship between A' and A''.

Solution: We use the change of basis theorem. Define $P = \sum_i |w_i\rangle \langle v_i| \implies P^\dagger = \sum_i |v_i\rangle \langle w_i|$. Then, the elements of $P^\dagger A' P = \sum_{ijklmn} \langle v_i |v_m\rangle \langle w_n| |A| |w_k\rangle \langle v_l| v_j\rangle = \sum_{ij} \langle w_i|A|w_j\rangle = A''$.

(2.23) Show that the eigenvalues of a projector P are all either 0 or 1.

Suppose we have projector $P = \sum_i |i\rangle \langle i|$. For any nonzero vector $|v\rangle$, then $P|v\rangle = \delta_{iv} |v\rangle \iff$ all eigenvectors for P are either 0 or 1.

(2.24) (Hermiticity of positive operators) Show that a positive operator is Hermitian.

Solution: Define B, C as follows:

$$B = \frac{1}{2}(A + A^\dagger), C = \frac{1}{2}i(A - A^\dagger)$$

We can see that $B + iC = \frac{1}{2}(A + A^\dagger) - \frac{1}{2}(A - A^\dagger) = A$ as is necessary. Then, $\langle\psi|A|\psi\rangle > 0$ for all $|\psi\rangle \iff \langle\psi|B|\psi\rangle + i\langle\psi|C|\psi\rangle$.

$$\iff \langle\psi|C|\psi\rangle = 0$$

due to the complex scalar. Therefore it must be that $A = B$, which is Hermitian.

(2.25) Show that for any operator $A, A^\dagger A$ is positive.

Solution: Suppose $A|v\rangle = |w\rangle$. Then, $\langle v| A^\dagger A |v\rangle = \langle w|w\rangle = |w|^2 > 0$.

3 2.1.7 Tensor Products

(2.26) Let $|\psi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$. Write out $|\psi\rangle^{\otimes 2}, |\psi\rangle^{\otimes 3}$ explicity, both in terms of tensor products and using the Kronecker product.

Solution: $|\psi\rangle^{\otimes 2} = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{2}\begin{bmatrix}\psi \\ \psi\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 \\ 1 \\ 1 \\ 1\end{bmatrix}$

$$|\psi\rangle^{\otimes 3} = \frac{\sqrt{2}}{4}(|000\rangle + |010\rangle + |100\rangle + |110\rangle + |001\rangle + |011\rangle + |101\rangle + |111\rangle)$$

$$= \frac{\sqrt{2}}{4}\begin{bmatrix}\psi \\ \psi \\ \psi \\ \psi\end{bmatrix} = \begin{bmatrix}1_1 \\ \vdots \\ 1_8\end{bmatrix}$$

(2.27) Calculate the matrix representation of the tensor products of Pauli operators X_1Z_2, I_1X_2, X_1I_2 . Is the tensor product commutative?

Solution:

$$X_1Z_2 = \begin{bmatrix}0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0\end{bmatrix}, I_1X_2 = \begin{bmatrix}0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0\end{bmatrix}, X_1I_2 = \begin{bmatrix}0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0\end{bmatrix}$$

The tensor product is not commutative.

(2.28) Show the tranpose, complex conjugation, and adjoint operators distribute over tensor product.

Solution: The elements of $(A \otimes B)^*$ are $(A_{ij}B)^* = A_{ij}^* B_{mn}^*$, since $(zw)* = z * w*$. This can be more rigorously expanded. The same follows for transpose and since adjoint is both conjugation and transpose, it commutes too.

(2.29) Show the tensor product of two unitary operators is unitary.

Solution: Suppose we have $U \otimes V$, where U and V are unitary $\iff UU^\dagger = I, VV^\dagger = I$. Since the adjoint distributes over tensor product, $(U \otimes V)^\dagger = U^\dagger \otimes V^\dagger \iff (U \otimes V)(U^\dagger \otimes V^\dagger) = (UU^\dagger \otimes VV^\dagger) = I^{mn}$ where m is dim U and n is dim V.

(2.30) Show that the tensor product of two Hermitian operators is Hermitian.

Solution: Suppose we have $A \otimes B$ where A, B are Hermitian. Since adjoint distributes over tensor product,

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger = A \otimes B$$

(2.31) Show the tensor product of two positive operators is positive.

Solution: Suppose we have positive operators A, B. Then $\langle\psi|A \otimes B|\psi >\rangle = \langle\psi|A|\psi\rangle \otimes \langle\psi|B|\psi\rangle$, where each side of the tensor product is greater than zero which then implies the tensor product is positive.

(2.32) Show the tensor product of two projectors is a projector.

Solution: Consider $P_1 = \sum_i |i\rangle\langle i|, P_2 = \sum_j |j\rangle\langle j|$. Then, $P_1 \otimes P_2$ is Hermitian (see exercise 2.30). Then, see $(P_1 \otimes P_2)^2 = P_1P_1 \otimes P_2P_2 = P_1 \otimes P_2$. Thus the tensor product of two projectors is a projector.

(2.33) The Hadamard operator on one qubit may be written:

$$H = \frac{1}{\sqrt{2}}[(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|]$$

Show that the Hadamard transform on n qubits, $H^{\otimes n}$ may be written as:

$$H^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x,y} (-1)^{xy} |x\rangle\langle y|$$

Write out an explicit matrix representation for $H^{\otimes 2}$.

Solution:

$$H^{\otimes 2} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

We prove by induction.

Proof. We see that $H^{\otimes 1} = \frac{1}{\sqrt{2}} \sum_{x,y} (-1)^{xy} |x\rangle \langle y| = \frac{1}{\sqrt{2}} |0\rangle \langle 0| + |1\rangle \langle 0| + |0\rangle \langle 1| - |1\rangle \langle 1| = H$. Then, assume it is true for $H^{\otimes n-1}$. We prove it is true for $H^{\otimes n}$.

$$\begin{aligned} H^{\otimes n} &= \frac{1}{\sqrt{2^{n-1}}} \sum +x, y(-1)^{xy} |x\rangle \langle y| \otimes H \\ &= \frac{1}{\sqrt{2^n}} \sum +x, y(-1)^{xy} |x\rangle \langle y| \end{aligned}$$

□

4 2.2.2 Evolution

(2.51) Verify that the Hadamard gate H is unitary.

Solution:

$$HH^T = H^T H = H^2 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I.$$

(2.52) Verify that $H^2 = I$.

Solution: See above.

(2.55) Define $U(t_1, t_2) \equiv exp[\frac{-iH(t_2-t_1)}{\hbar}]$. Show $U(t_1, t_2)$ is unitary.

Solution:

$$\begin{aligned} U^\dagger(t_1, t_2) &= exp[\frac{-i^\dagger}{\hbar} H^\dagger(t_2 - t_1)] = exp[\frac{i}{\hbar} H(t_2 - t - 1)] \\ UU^\dagger &= exp[\frac{i-i}{\hbar} H(t_2 - t_1)] = exp[0] = I \end{aligned}$$

where the last equality is from exponential operator on zero matrix.

(2.56) Use spectral decomposition to show that $K \equiv ilog(U)$ is Hermitian for any unitary U, and thus $U = exp(iK)$ for some Hermitian K.

Solution: Any eigenvalue of a unitary matrix has length 1. Since the eigenvalues are in a Hilbert space and therefore complex numbers, any eigenvalue of any unitary matrix can be represented as $\lambda_i = e^{i\theta_i}$. Then, performing the spectral decomposition of arbitrary unitary matrix U:

$$U = -i \sum_j \log(e^{i\theta_j} |j\rangle \langle j|) = -i \sum_j i\theta_j |j\rangle \langle j| = \sum_j \theta_j |j\rangle \langle j|$$

$$\implies U^\dagger = U$$

. Therefore K as defined is Hermitian for any unitary U. To show that $U = exp(iK)$, multiply both sies by i and apply the exponentiation operator to both sides.

5 2.2.3 Quantum Measurement

(2.57) Suppose $\{L_l\}$ and $\{M_m\}$ are two sets of measurement operators. Show that a measurement is defined by the measurement operators $\{L_l\}$ followed by a measurement defined by the measurement operators $\{M_m\}$ is physically equivalent to a single measurement defined by measurement operators $\{N_{lm}\}$ where $N_{lm} \equiv M_m L_l$

Solution: Suppose we have vector $|\psi\rangle$ and the measurements defined above. We want to show that performing L_l and M_m is equivalent to performing measurement $N_{lm} \equiv M_m L_l$. Then, perform measurement L_l on $|\psi\rangle \rightarrow \frac{L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}} \equiv L_l |\psi\rangle$. Perform measurement M_m on $L_l |\psi\rangle \rightarrow \frac{M_m L_l|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}} \equiv M_m L_l |\psi\rangle \equiv N_{lm} |\psi\rangle$ as desired.

(2.58) Suppose we prepare a quantum system in an eigenstate $|\psi\rangle$ of some observable M, with corresponding eigenvalue m. What is the average observed value of M and the standard deviation?

Solution: $E[M] = \langle\psi|M|\psi\rangle = m \langle\psi|\psi\rangle = m$, since $M |\psi\rangle = m |\psi\rangle$. Using the identity that $\Delta_M^2 = E[M^2] - E[M]^2 = \langle\psi|M^2|\psi\rangle - m^2 = m^2 \langle\psi|\psi\rangle - m^2 = 0$.

(2.59) Suppose we have qubit in state $|0\rangle$ and we measure the observable X. What is the average value of X? What is the standard deviation of X?

Solution:

$$E[X] = \langle 0|X|0\rangle = 0$$

Since the Pauli matrices are unitary, $X^2 = I$

$$E[X^2] = \langle 0|X^2|0\rangle = \langle 0|I|0\rangle = 1$$

$$\iff \Delta_X = 1$$

(2.60) Show that $v \cdot \sigma$ has eigenvalues ± 1, and that the projectors onto the corresponding eigenspaces are given by $P_\pm = \frac{(I \pm v\cdot\sigma)}{2}$

Solution: We convert into matrix form to calculate the eigenvalues.

$$v\cdot\sigma = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3 \begin{bmatrix} v_3 & v_1 + iv_2 \\ v_1 - iv_2 & -v_3 \end{bmatrix}$$

$$det(v\cdot\sigma - \lambda) = (\lambda^2 - |v_3|^2) - |v_1|^2 + |v_2|^2 = 0 \iff \lambda^2 = |v_1|^2 + |v_2|^2 + |v_3|^3 = 1$$

$$\iff \lambda = \pm 1.$$

Then, we show the projectors are given by $P_\pm = \frac{I \pm v\cdot\sigma}{2}$. Suppose we have eigenvector $|\psi\rangle \iff v\cdot\sigma\,|\psi\rangle = |\psi\rangle$. Then, $P_+\,|\psi\rangle = \frac{1}{2}(|\psi\rangle + |\psi\rangle) = |\psi\rangle$. Then suppose we have eigenvector $|\psi\rangle \iff v\cdot\sigma\,|\psi\rangle = -\,|\psi\rangle$. Then, $P_-\,|\psi\rangle = \frac{1}{2}(|\psi\rangle - (-\,|\psi\rangle)) = |\psi\rangle$. Thus the projector to corresponding eigenspace leaves the eigenvector unchanged as expected.

(2.61) Calculate the probability of obtaining the result +1 for measurement of $v\cdot\sigma$ given that the state prior to measurement is $|0\rangle$. What is the state of the system after measurement if +1 is obtained?

Solution: $P(+1) = \langle 0|P_+|0\rangle = \frac{1+v_3}{2}$ Post measurement state: $|0\rangle' = \frac{1}{2}((1+v_3)\,|0\rangle + (v_1 - iv_2)\,|1\rangle)/\sqrt{\frac{1+v+3}{2}}$

6 2.2.6 POVM Measurements

(Nielsen 2.62) Show any measurement where the measurement operators and the POVM elements coincide is a projective measurement.

Solution:

Proof. Suppose the measurement operators and the POVM elements coincide. Then, for all M_m, $M_m = E_m$. Since $E_m = M_m^\dagger M_m = M_m$, we have idempotence (prop 1). Then, since $\sum_m E_m = I \iff \sum_m M_m = I$ and therefore we have completeness.
Then, consider $\sum_m M_m = I \implies M_n \sum_m M_m = M_n$

$$\implies M_n^2 + \sum_{m \neq n} M_n M_m = M_n \iff M_n^2 + \sum_{m \neq n} M_n M_m = M_n$$

$$\sum_{m \neq n} M_n M_m = 0 \iff \sum_{m \neq n} \langle \psi | M_n M_m | \psi \rangle = 0$$

Since 0 is the sum of all strictly positive operators, it must be that each $M_n M_m$ is zero. Thus we have $M_m M_n = \delta_m n M_m$. Therefore if measurement operators and POVM elements coincide, then it is a projective measurement. □

(2.63) Suppose a measurement is described by measurement operators $\{M_m\}$. Show there exist unitary operators U_m such that $M_m = U_m \sqrt{E_m}$ where E_m is the POVM associated to the measurement.

Solution:

Proof. Suppose we are given measurement operator M_m. Then, using polar composition there exists unitary operator U such that $M_m = U_m J_m$, where J_m is a positive operator. A positive operator always has a positive square root $\iff \exists E_m s.t. J_m = \sqrt{E_m}$, satisfying first condition for POVM (positive).
See that $M_m^\dagger = \sqrt{E_m}^\dagger U_m^\dagger = \sqrt{E_m} U_m^\dagger$

$$\implies M_m^\dagger M_m = \sqrt{E_m} U_m^\dagger U \sqrt{E_m} = E_m$$

Since $\sum_m M_m^\dagger M_M = \sum_m E_m = I$, the second condition is met. Since $M_M^\dagger M_M = E_M \iff p(m) = \langle \psi | E_m | \psi \rangle$, satisfying the third condition. Therefore we see a there exists a unitary operator U such that $M_m = U_m \sqrt{E_m}$ where E_m is the POVM associated to the measurement. □

(2.64) Suppose Bob is given a quantum state chosen from a set $|\psi\rangle, \ldots, |\psi_m\rangle$ of linearly independent states. Construct a POVM $\{E_1, E_2, \ldots E_{m+1}\}$ such that if outcome E_i occurs, $1 \leq i \leq m$, then Bob must know with certainty that he was given the state $|i\rangle$.

Each $E_i = |\psi_i\rangle \langle\psi_i|$ for $1 \leq i \leq m$, where $E_{m+1} = I - \sum_i E_i$. This will yield the probability 1 when applied to $|\psi_i\rangle$ and 0 otherwise. We show this meets the criteria for POVM. We see that each E_i is positive semi-definite, since as a projector its only eigenvalues are 0 and 1. We can also clearly see that adding all of the POVM operators equals the identity.

7 2.2.8 Composite Systems

(2.66) Show the expected value of the observable $X_1 Z_2$ for two qubit system measured in state $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ (2.68) Show the state $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ is entangled.

Solution: We show the state is entangled by showing that there is no possible a_0, a_1, b_0, b_1 such that $|\psi\rangle = (a_0|0\rangle + a_1|1\rangle)(b_0|0\rangle + b_1|1\rangle)$ Suppose we can find valid a_0, a_1, b_0, b_1.
Then, $\frac{1}{\sqrt{2}}|00\rangle + 1\sqrt{2}|11\rangle = a_0b_0|00\rangle + a_0b_1|01\rangle + a_1b_0|10\rangle + a_1b_1|11\rangle$ Since $|00\rangle, ..., |11\rangle$ form a basis for $H^{\otimes 2}$, this writing for $|\psi\rangle$ is unique $\iff \frac{1}{\sqrt{2}} = a_0b_0 = a_1b_1$ and $a_0b_1 = a_1b_0 = 0$. Then, we see that:

$$a_0b_0 \times a_1b_1 = a_0a_1b_0b_1 = \frac{1}{2} \neq a_0b_1 \times a_1b_0 = 0$$

Therefore there are no valid a_0, a_1, b_0, b_1.

8 2.3 Superdense Coding

(2.69) (2.70)

9 2.4.2 Density Operator Properties

(2.71)

10 2.4.3 Reduced Density Operator