Problem set 1

Contents

1 How far apart are two quantum states?

Consider two quantum states described by density operators ρ and $\tilde{\rho}$ in an N-dimensional Hilbert space, and consider the complete orthogonal measurement $\{E_a, a = 1, 2, 3, ...N\}$, where the E_a's are one-dimensional projectors satisfying

$$\sum_{a=1}^{N} E_a = I \tag{1}$$

When the measurement is performed, outcome a occurs with probability $p_a = tr(\rho E_a)$ if the state is ρ and with probability $\tilde{p}_a = tr(\tilde{\rho} E_a)$ if the state is $\tilde{\rho}$. The L^1 *distance* between two probability distributions is defined as

$$d(p, \tilde{p}) \equiv ||p - \tilde{p}||_1 \equiv \frac{1}{2} \sum_{a=1}^{N} |p_a - \tilde{p}_a|; \tag{2}$$

this distance is zero if the two distributions are identical, and attains its maximum value one if the two distributions have support on disjoint sets.

(a) Show that

$$d(p, \tilde{p}) \leq \frac{1}{2} \sum_{i=1}^{N} |\lambda_i| \tag{3}$$

where the λ_i's are the eigenvalues of the Hermitian operator $\rho - \tilde{\rho}$

Solution:

Proof. Define $\Delta = \rho - \tilde{\rho}$. Then, $p_a - \tilde{p}_a = tr((\rho - \tilde{\rho})(E_a)) = tr(E_a\Delta)$ by linearity of trace.

$$\implies d(p, \tilde{p}) = \frac{1}{2}\sum_{a=1}^{N} |tr(E_a\Delta)| \tag{1}$$

Since Δ is Hermitian, we can diagonalize $\Delta = \sum_{i=1}^{N} \lambda_i |i\rangle \langle i|$.

$$= \frac{1}{2}\sum_{a=1}^{N} |tr(E_a \sum_{i=1}^{N} \lambda |i\rangle \langle i|)| = \frac{1}{2}\sum_{a=1}^{N} |\sum_{i=1}^{N} \lambda_i tr(E_a |i\rangle \langle i|)| \tag{2}$$

$$= \sum_{a=1}^{N} |\sum_{i=1}^{N} \lambda_i \langle i|E_a|i\rangle | \leq \frac{1}{2}\sum_{i=1}^{N}\sum_{a=1}^{N} |\lambda_i \langle i|E_a|i\rangle | \tag{3}$$

Since E_a is positive, $\langle i|E_a|i\rangle > 0$, therefore

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{a=1}^{N} |\lambda_i| \langle i|E_a|i\rangle = \frac{1}{2}\sum_{i=1}^{N} |\lambda_i| \langle i|\sum_{a=1}^{N} E_a|i\rangle \tag{4}$$

Using the completeness property, $\sum_{a=1}^{N} E_a = I$, therefore

$$= \frac{1}{2}\sum_{i=1}^{N} |\lambda_i| \langle i|I|i\rangle = \frac{1}{2}\sum_{i=1}^{N} |\lambda_i| \tag{5}$$

as desired. $\square$

(b) Find a choice for the orthogonal projector $\{E_a\}$ that saturates the upper bound eq. (3).

Solution: For scalars λ_i, $|\sum_i \lambda_i| = \sum_i |\lambda_i| \iff$ all λ_i are the same sign. Then, construct operators E_+ and E_- that project onto the eigenspaces corresponding to positive/negative eigenvalues of $\rho - \tilde{\rho}$. We see that $E_+ + E_- = I$, since $\rho - \tilde{\rho}$ is positive and thus has no zero eigenvalues. Then,

$$d(p, \tilde{p}) = \frac{1}{2}\sum_a^N |\sum_i^N \lambda_i \langle i|E_a|i\rangle|$$

$$= \frac{1}{2}(|\sum_i^N \lambda_i \langle i|E_+|i\rangle| + |\sum_i^N \lambda_i \langle i|E_-|i\rangle|)$$

$$= \frac{1}{2}|\sum_k \lambda_k| + |\sum_l \lambda_l|$$

where $\lambda_k > 0, \lambda_l < 0$.

$$= \frac{1}{2}\sum_i^N |\lambda_i|$$

(c) Define distance $d(\rho, \tilde{\rho})$ between density operators as the maximal L^1 distance between corresponding probability distributions that can be achieved by any orthogonal measurement. From (a), (b), we have found that

$$d(\rho, \tilde{\rho}) = \frac{1}{2}\sum_{i=1}^N |\lambda_i|.$$

The L^1 norm $||A||_1$ of an operator A is defined as:

$$||A||_1 \equiv tr[(AA^\dagger)^{\frac{1}{2}}]$$

How can the distance $d(\rho, \tilde{\rho})$ be expressed as the L^1 norm of an operator?

Solution: $AA^\dagger$ is PSD ($x^\dagger AA^\dagger x = (A^\dagger x)^\dagger (A^\dagger x)$, which is inner product, nonnegative) and thus$\sqrt{AA^\dagger}$ is PSD. Therefore, by the spectral theorem $\sqrt{AA^\dagger} = U = \sum_i^N \lambda_i |i\rangle \langle i|$ for $\lambda_i > 0$.

$$tr(U) = tr(\sum_i^N \lambda_i |i\rangle \langle i|) = \sum_i^N \lambda_i tr(|i\rangle \langle i|)$$

$$= \sum_i^N \lambda_i = ||A||_1 \iff d(\rho, \tilde{\rho}) = \frac{1}{2}||\rho - \tilde{\rho}||_1$$

(d) Suppose the states $\rho, \tilde{\rho}$ are pure states $\rho = |\psi\rangle \langle\psi|$ and $\tilde{\rho} = |\tilde{\psi}\rangle \langle\tilde{\psi}|$. If we adopt a suitable basis in the space spanned by the two vectors, and appropriate phase conventions, then these vectors can be expressed as:

$$|\psi\rangle = \begin{pmatrix} cos\frac{\theta}{2} \\ sin\frac{\theta}{2} \end{pmatrix}, |\tilde{\psi}\rangle = \begin{pmatrix} sin\frac{\theta}{2} \\ cos\frac{\theta}{2} \end{pmatrix}$$

where $\langle\psi|\tilde{\psi}\rangle = sin\theta$

Express the distance $d(\rho, \tilde{\rho})$ in terms of the angle θ

Solution:

$$\rho - \tilde{\rho} = |\psi\rangle \langle\psi| - |\tilde{\psi}\rangle \langle\tilde{\psi}| \tag{6}$$

$$= \begin{pmatrix} cos^2(\frac{\theta}{2}) - sin^2(\frac{\theta}{2}) & 0 \\ 0 & sin^2(\frac{\theta}{2}) - cos^2(\frac{\theta}{2}) \end{pmatrix} \tag{7}$$

$\iff \lambda = cos^2(\frac{\theta}{2}) - sin^2(\frac{\theta}{2}), sin^2(\frac{\theta}{2}) - cos^2(\frac{\theta}{2})$

$\implies d(\rho, \tilde{\rho}) = \frac{1}{2} \sum_i^N \lambda_i$

$\equiv d(\theta) = \frac{1}{2}(|cos^2(\frac{\theta}{2}) - sin^2(\frac{\theta}{2})| + |sin^2(\frac{\theta}{2}) - cos^2(\frac{\theta}{2})|)$

Since $cos(2\theta) = cos^2(\theta) - sin^2(\theta)$ (double angle identity):

$= \frac{1}{2}|cos\theta| + |-cos\theta|$

$= |cos\theta|$

(e) Express $|| \, |\psi\rangle - |\tilde{\psi}\rangle \, ||^2$ in terms of θ and by comparing the result of (d), derive the bound

$$d(|\psi\rangle \langle\psi| \, , |\tilde{\psi}\rangle \langle\tilde{\psi}|) \leq || \, |\psi\rangle - |\tilde{\psi}\rangle \, ||.$$

Solution:

$$\begin{aligned} || \, |\psi\rangle - |\tilde{\psi}\rangle \, ||^2 &= 2cos^2(\frac{\theta}{2}) - 4sin(\frac{\theta}{2})cos(\frac{\theta}{2}) + 2sin^2(\frac{\theta}{2}) \\ &= 2(cos\frac{\theta}{2} - sin\frac{\theta}{2})^2 \\ \implies || \, |\psi\rangle - |\tilde{\psi}\rangle \, || &= \sqrt{2}|cos\frac{\theta}{2} - sin\frac{\theta}{2}| \end{aligned}$$

Then, we can see that $d(|\psi\rangle \langle\psi| \, |\tilde{\psi}\rangle \langle\tilde{\psi}|)$

$$\begin{aligned} = |cosx| = |cos^2(\frac{\theta}{2}) - sin^2(\frac{\theta}{2})| & \qquad \text{(double angle)} \\ = |(cos(\frac{\theta}{2}) + sin(\frac{\theta}{2}))(cos(\frac{\theta}{2}) - sin(\frac{\theta}{2}))| & \\ = |(cos(\frac{\theta}{2}) + sin(\frac{\theta}{2}))||(cos(\frac{\theta}{2}) - sin(\frac{\theta}{2}))| & \\ \leq (\sqrt{2}\sqrt{cos^2(\frac{\theta}{2}) + sin^2(\frac{\theta}{2})})|cos(\frac{\theta}{2}) - sin(\frac{\theta}{2})| & \\ & \qquad \text{(Cauchy-Schwarz)} \\ = \sqrt{2}|cos(\frac{\theta}{2}) - sin(\frac{\theta}{2})| & \end{aligned}$$

as desired.

(f) Bob thinks that the norm $|| \, |\psi\rangle - |\tilde{\psi}\rangle \, ||$ should be a good measure of the distinguishability of the pure quantum states ρ and $\tilde{\rho}$. Explain why Bob is wrong.

Solution: Two indistinguishable states, for instance considering the case where $|\psi\rangle = |\tilde{\psi}\rangle$, should have a normed difference of zero, since. Since a quantum state is a ray, $|\psi\rangle \equiv e^{i\theta} |\psi\rangle$ and therefore should be indistinguishable. However $|| \, |\psi\rangle - e^{i\theta} |\psi\rangle \, || \neq 0$ for nonzero θ.

2 Which state did Alice make?

Consider a game in which Alice prepares one of two possible states: either ρ_1 with *a priori* probability p_1 or ρ_2 with *a priori* probability $p_2 = 1 - p_1$. Bob is to perform a measurement and on the basis on the outcome to guess which state Alice prepared. If Bob's guess is right, he wins; if he guesses wrong, Alice wins.

In this exercise you will find Bob's best strategy, and determine his optimal probability of error.

Let's suppose (for now) that Bob performs a POVM with two possible outcomes, corresponding to the two nonnegative Hermitian operators E_1 and $E_2 = I - E_1$. If Bob's outcome is E_1, he guesses that Alice's state was ρ_1, and if it is E_2, he guesses ρ_2. Then the probability that Bob guesses wrong is

$$p_{error} = p_1 tr(\rho_1 E_2) + p_2 tr(\rho_2 E_1)$$

(a) Show that

$$p_{error} = p_1 + \sum_i \lambda_i \langle i|E_1|i\rangle$$

where $\{|i\rangle\}$ denotes the orthonormal basis of eigenstates of the Hermitian operator $p_2\rho_2 - p_1\rho_1$, and the λ_i's are the corresponding eigenvalues.

Solution:

$$\begin{aligned}
p_{error} &= tr(p_1\rho_1 E_2 + p_2\rho_2 E_1) && \text{(linearity of trace)}\\
&= tr((p_1\rho_1)(I - E_1) + p_2\rho_2 E_1)\\
&= p_1 tr(\rho_1 I) + tr(p_2\rho_2 E_1 - p_1\rho_1 E_1) && \text{(lin. of trace)}\\
&= p_1 + tr((p_2\rho_2 - p_1\rho_1)E_1)
\end{aligned}$$

Since $p_2\rho_2 - p_1\rho_1$ is Hermitan, using spectral theorem, $p_2\rho_2 - p_1\rho_1 = \sum_i \lambda_i |i\rangle\langle i|$, where $\{|i\rangle\}$ form an orthonormal basis.

$$\begin{aligned}
&= p_1 + tr((\sum_i \lambda_i |i\rangle\langle i|)E_1) && \text{(spectral thm.)}\\
&= p_1 + \sum_i \lambda_i tr(|i\rangle\langle i| E_1)\\
&= p_1 + \sum_i \lambda_i \langle i|E_1|i\rangle
\end{aligned}$$

as desired.

(b) Bob's best strategy is to perform the two-outcome POVM that minimizes this error probability. Find the nonnegative operator E_1 that minimizes p_{error}, and show that the error probability when Bob performs this optimal two-outcome POVM is

$$(p_{error})_{optimal} = p_1 + \sum_{neg} \lambda_i.$$

where $\sum_{neg}$ denotes the sum over all of the *negative* eigenvalues of $p_2\rho_2 - p_1\rho_1$.

> **Solution:** E_1 is operator that projects to negative eigenspace of $p_2\rho_2 - p_1\rho_1$, $E_1 = \sum_{\lambda_i<0} |i\rangle \langle i|$.
>
> $$p_1 + \sum_i \lambda_i \langle i|E_1|i\rangle = p_1 + \sum_j \lambda_j$$
>
> such that $\lambda_j < 0$. This is optimal since it must be the smallest probability, as if any λ_j in the sum was positive then it would be greater than this value.

(c) It is convenient to express this optimal error probability in terms of the L^1 norm of the operator $p_2\rho_2 - p_1\rho_1$,

$$||p_2\rho_2 - p_1\rho_1||_1 = tr(|p_2\rho_2 - p_1\rho_1|) = \sum_{pos} \lambda_i - \sum_{neg} \lambda_i,$$

the difference between the sum of positive eigenvalues and the sum of negative eigenvalues. Use the property $tr(p_2\rho_2 - p_1\rho_1) = p_2 - p_1$ to show that

$$(p_{error})_{optimal} = \frac{1}{2} - \frac{1}{2}||p_2\rho_2 - p_1\rho_1||_1$$

Check whether the answer makes sense in the case where $\rho_1 = \rho_2$ and in the case where ρ_1 and ρ_2 have support on orthogonal subspaces

Solution:

$$
\begin{aligned}
tr(p_2\rho_2 - p_1\rho_1) = p_2 - p_1 &= \sum_{pos}\lambda_i + \sum_{neg}\lambda_i \\
&\implies \sum_{pos}\lambda_i = p_2 - p_1 - \sum_{neg}\lambda_i \\
&\implies ||p_2\rho_2 - p_1\rho_1||_1 = p_2 - p_1 - 2\sum_{neg}\lambda_i \\
&= 1 - 2p_1 - 2\sum_{neg}\lambda_i
\end{aligned}
$$

Substituting into the expression for $(p_{error})_{optimal}$:

$$
\begin{aligned}
(p_{error})_{optimal} &= \frac{1}{2} - \frac{1}{2}||p_2\rho_2 - p_1\rho_1||_1 \\
&= \frac{1}{2} - \frac{1}{2} + p_1 + \sum_{neg}\lambda_i \\
&= p_1 + \sum_{neg}\lambda_i
\end{aligned}
$$

as desired.

(d) Now suppose that Alice decides at random (with $p_1 = p_2 = \frac{1}{2}$) to prepare one of two pure states $|\psi_1\rangle, |\psi_2\rangle$ of a single qubit, with

$$
|\langle\psi_1|\psi_2\rangle| = sin(2\alpha), 0 \leq \alpha \leq \frac{\pi}{4}
$$

With a suitable choice of basis, the two states can be expressed as:

$$
|\psi_1\rangle = \begin{pmatrix} \cos\,\alpha \\ \sin\,\alpha \end{pmatrix}, |\psi_2\rangle = \begin{pmatrix} \sin\,\alpha \\ \cos\,\alpha \end{pmatrix},
$$

Find Bob's optimal two-outcome measurement, and compute the optimal error probability.

Solution: The density operator for a pure state $|\psi\rangle$ is $\rho = |\psi\rangle\langle\psi|$.

$$\implies \rho_1 = |\psi_1\rangle\langle\psi_1| = \begin{pmatrix} cos^2\alpha & cos\alpha sin\alpha \\ sin\alpha cos\alpha & sin^2\alpha \end{pmatrix},$$

$$\rho_2 = |\psi_2\rangle\langle\psi_2| = \begin{pmatrix} sin^2\alpha & cos\alpha sin\alpha \\ sin\alpha cos\alpha & cos^2\alpha \end{pmatrix}$$

$$\begin{aligned} \implies p_1\rho_1 - p_2\rho_2 &= \frac{1}{2}\begin{pmatrix} cos^2\alpha - sin^2\alpha & 0 \\ 0 & sin^2\alpha - cos^2\alpha \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} cos(2\alpha) & 0 \\ 0 & -cos(2\alpha) \end{pmatrix} \\ &\implies \lambda_{neg} = -\frac{1}{2}cos(2\alpha) \\ &\implies (p_{error})_{optimal} = \frac{1}{2} - \frac{1}{2}cos(2\alpha) \\ &\implies E_1 = |1\rangle\langle 1| \end{aligned}$$

(e) Bob wonders whether he can find a better strategy if his POVM $\{E_i\}$ has more than two possible outcomes. Let $p(a|i)$ denote the probability that a state a was prepared, given that the measurement outcome was i; it can be computed using the relations:

$$p_i p(1|i) = p_1 p(i|1) = p_1 tr\rho_1 E_i,$$

$$p_i p(2|i) = p_2 p(i|2) = p_2 tr\rho_2 E_i;$$

here $p(i|a)$ denotes the probability that Bob finds measurement outcome i if Alice prepared the state ρ_a, and p_i denotes the probability that Bob finds measurement outcome i, averaged over Alice's choice of state. For each outcome i, Bob will make his decision according to which of the two quantities

$$p(1|i), p(2|i)$$

is the larger; the probability that he makes a mistake is the smaller of these two quantities. This probability of error, given that Bob obtains outcome i, can be written as

$$p_{error}(i) = min(p(1|i), p(2|i)) = \frac{1}{2} - \frac{1}{2}|p(2|i) - p(1|i)|.$$

Show that the probability of error, averaged over the measurement outcomes, is

$$p_{error} = \sum_i p_i p_{error}(i) = \frac{1}{2} - \frac{1}{2}\sum_i |tr(p_2\rho_2 - p_1\rho_1)E_i|.$$

Solution:

$$\begin{aligned}
p_{error} &= \sum_i p_i p_{error}(i) \\
&= \sum_i p_i(\frac{1}{2} - \frac{1}{2}|p(2|i) - p(1|i)|) \\
&= \frac{1}{2}(\sum_i p_i) - \frac{1}{2}(\sum_i p_i|p(2|i) - p(1|i)|)
\end{aligned}$$

Since $\sum_i p_i = 1$ and since $p_i > 0$,

$$\begin{aligned}
&= \frac{1}{2} - \frac{1}{2}\sum_i |p_i p(2|i) - p_i p(1|i)| \\
&= \frac{1}{2} - \frac{1}{2}\sum_i |tr(p_2\rho_2 E_i) - tr(p_1\rho_1 E_i)| \\
&= \frac{1}{2} - \frac{1}{2}\sum_i |tr(p_2\rho_2 - p_1\rho_1)E_i|
\end{aligned}$$

(f) By expanding in terms of the basis of eigenstates of $p_2\rho_2 - p_1\rho_1$, show that

$$p_{error} \geq \frac{1}{2} - \frac{1}{2}||p_2\rho_2 - p_1\rho_1||_1$$

Solution: Suppose we have $E_1 ... E_n$ s.t. $n > 2$. Then, using spectral decomposition of $p_2\rho_2 - p_1\rho_1 = \sum_n \lambda_n |n\rangle \langle n|$

$$\begin{aligned}
p_{error} &= \frac{1}{2} - \frac{1}{2}\sum_i |\sum_n \lambda_n tr(|n\rangle \langle n| E_i)| \\
&= \frac{1}{2} - \frac{1}{2}(\sum_i |\sum_n \lambda_n \langle n|E_i|n\rangle |) \\
&\geq \frac{1}{2} - \frac{1}{2}(|\sum_n \lambda_n \langle n| \sum_i E_i|n\rangle |) = \frac{1}{2} - \frac{1}{2}(|\sum_n \lambda_n|) \\
&= \frac{1}{2} - \frac{1}{2}||p_2\rho_2 - p_1\rho_1||_1
\end{aligned}$$