Belief Aggregation and Trader Compensation in Infinite Outcome Prediction Markets

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Introduction

- Prediction markets are incentive-compatible mechanisms designed to **elicit the personal beliefs** of traders about a future uncertain event and **aggregate those beliefs** into the market price
- Prediction markets have been empirically observed to outperform polls as they have built-in **financial incentives** and **timely responses**
- We study the impact of traders' **informativeness**, **budget**, and the **sequence** in which they trade on **aggregation** properties and trader **compensation** under a new prediction market design

Background

- The **Bernoulli probability distribution** models an uncertain binary event with success parameter p

$$f(x;p) = p^x(1-p)^{1-x}, \quad \text{for } x \in 0,1$$

- The **Beta** distribution is **conjugate prior** for the Bernoulli distribution success parameter and has pdf given by

$$f(p;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}$$

- Given a Beta prior over p and a sample of Bernoulli observations, the **Bayesian posterior update** is given by

$$\hat{\alpha} := \alpha + \sum_{i=1}^{N} x_i$$
$$\hat{\beta} := \beta + \sum_{i=1}^{N} 1 - x_i$$

- The **gamma** and **digamma** functions are used in computing statistics associated with the Beta distribution and are given below

$$\Gamma(n) = (n-1)! \qquad \psi(n) = \frac{\mathrm{d}\Gamma(n)/\mathrm{d}n}{\Gamma(n)} \qquad \text{for } n \in Z^+$$

Trader Model

- We assume traders are myopic, risk-neutral, and rational
- Trader **informativeness** is modeled by N, the size of their private sample of Bernoulli observations at each trade

$$x_1, \ldots, x_N \overset{iid}{\sim} \text{Bern}(p)$$

- Traders may also have limiting **budgets** B, modeled by an additional constraint of using the largest observation sub-sequence of length k where worst-case losses do not exceed B

$$\max_p \left\{ C\left(\eta + \left[\sum_{i=1}^{k} x_i, k - \sum_{i=1}^{k} x_i\right]^T\right) - C(\eta) - \left[\sum_{i=1}^{k} x_i, k - \sum_{i=1}^{k} x_i\right] \cdot [\ln p, \ln(1-p)] \right\} \leq B$$

- Trades proceed as follows:
  - Trader **samples from Bernoulli** distribution according to their informativeness
  - Trader **updates Beta posterior beliefs** according to $\hat{\alpha} := \alpha + \sum_{i=1}^{N} x_i$, $\hat{\beta} := \beta + \sum_{i=1}^{N} 1 - x_i$
  - Trader purchases shares such that **market posterior matches private beliefs**

Aggregation Mechanism: Beta-Bernoulli Market

- **Securities and payoffs:** The market has two securities with payoffs given by $\ln p_{true}$ and $\ln(1-p_{true})$ when $p_{true} \in [0,1]$, the true Beta forecast variable corresponding to the Bernoulli success parameter, is revealed
- **Outstanding shares and aggregated belief:** The market maintains and displays outstanding shares $\eta = [\eta_1, \eta_2]$ which also corresponds to market belief given by $\mathrm{BETA}(p;\alpha,\beta)$ where $\eta_1 = \alpha - 1, \eta_2 = \beta - 1$. This gives us expectations of the Bernoulli parameter and payoffs

$$\mathbb{E}_\eta[p] = \frac{\eta_1+1}{\eta_1+\eta_2+2} \qquad \mathbb{E}_\eta[\ln p] = \psi(\alpha) - \psi(\alpha+\beta)$$
$$\mathbb{E}_\eta[\ln(1-p)] = \psi(\beta) - \psi(\alpha+\beta)$$

- **Cost Function:** This gives us the cost of a trade $C(\eta_{new}) - C(\eta_{old})$, the prices of each security $\nabla_\eta C(\eta)$, and is given by

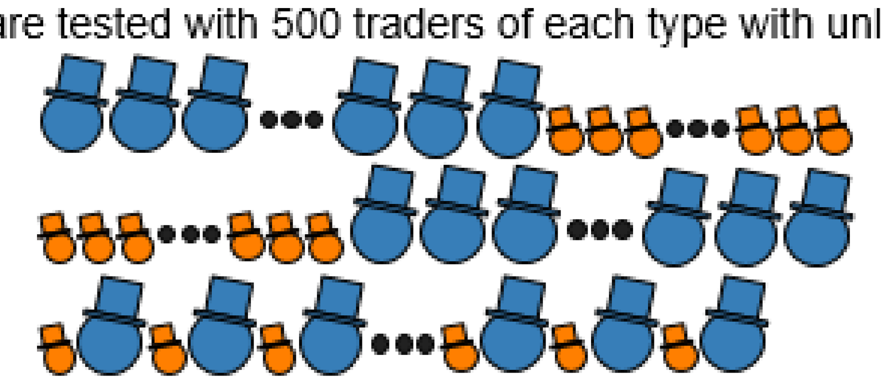
$$C(\eta) = \ln\Gamma(\eta_1+1) + \ln\Gamma(\eta_2+1) - \ln\Gamma(\eta_1+\eta_2+2)$$

Informativeness and Sequence Results

- **Experimental Setup** High Informativeness $\leftrightarrow N = 5$ Low Informativeness $\leftrightarrow N = 1$

Three sequences are tested with 500 traders of each type with unlimited budgets

1) High Info (HI) First:
2) Low Info (LI) First:
3) Interleaved:



- We report market convergence (Fig. 1), overall belief aggregation (Fig. 2) and average compensation for each trade type (Table 1)

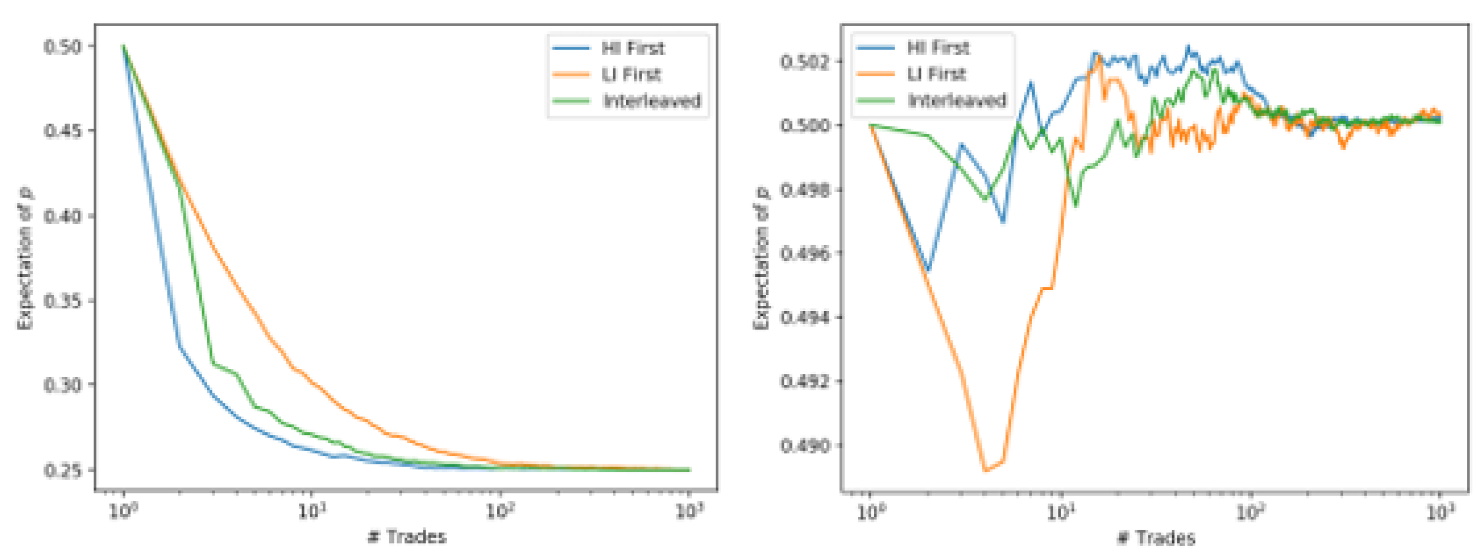


Figure 1: The market's expectation of the Bernoulli parameter as a function of the number of completed trades for different trader orderings, averaged over 1000 simulations with $p_{true} = 0.25$ (left) and $p_{true} = 0.5$ (right). Horizontal axis is logarithmic.

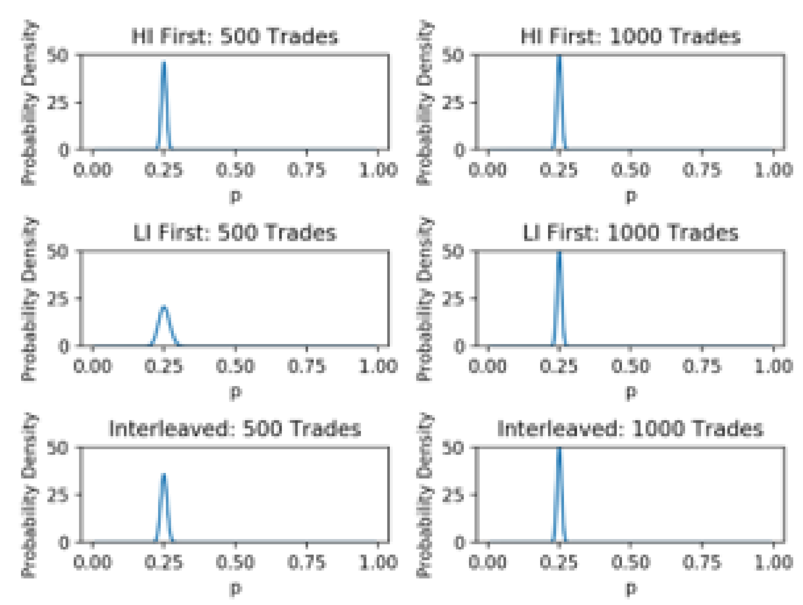


Figure 2: Posterior distribution (market belief) after 500 and 1000 trades for different trader orderings averaged across 1000 simulations with $p_{true} = 0.25$.

p_{true}	Sequence	Avg HI ($\times 10^3$)	Avg LI ($\times 10^3$)
0.75	**HI First**	6.66 ± 0.09	0.18 ± 0.05
	LI First	**1.75 ± 0.12**	**5.07 ± 0.09**
	Interleaved	5.51 ± 0.15	1.29 ± 0.13
0.50	**HI First**	6.37 ± 0.08	0.17 ± 0.05
	LI First	**1.75 ± 0.11**	**4.78 ± 0.09**
	Interleaved	5.56 ± 0.14	0.99 ± 0.12
0.25	**HI First**	6.70 ± 0.08	0.21 ± 0.05
	LI First	**1.80 ± 0.11**	**5.04 ± 0.09**
	Interleaved	5.52 ± 0.14	1.26 ± 0.12

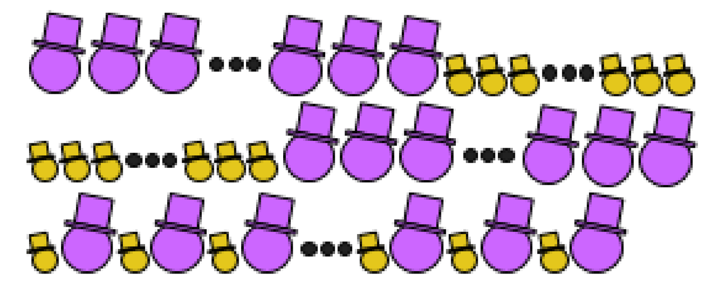
Table 1: Compensation of HI and LI traders for different trader orderings and p_{true} values. This table reports the compensation averaged over 1000 simulations with 95% confidence bounds.

Budget and Sequence Results

- **Experimental Setup** Unlimited Budget $\leftrightarrow B_0 = \infty$ Limited Budget $\leftrightarrow B_0 \in (0,\infty)$

Three sequences are tested with 500 traders of each type with constant informativeness ($N = 5$ per trade) and initial budgets B_0. There are **25 successive market instances** where agents trade, receive compensation, and update their budgets before proceeding to the next round.

1) Unlimited Budget (UB) First:
2) Limited Budget (LB) First:
3) Interleaved:



- We report the change in average LB trader budget over market rounds

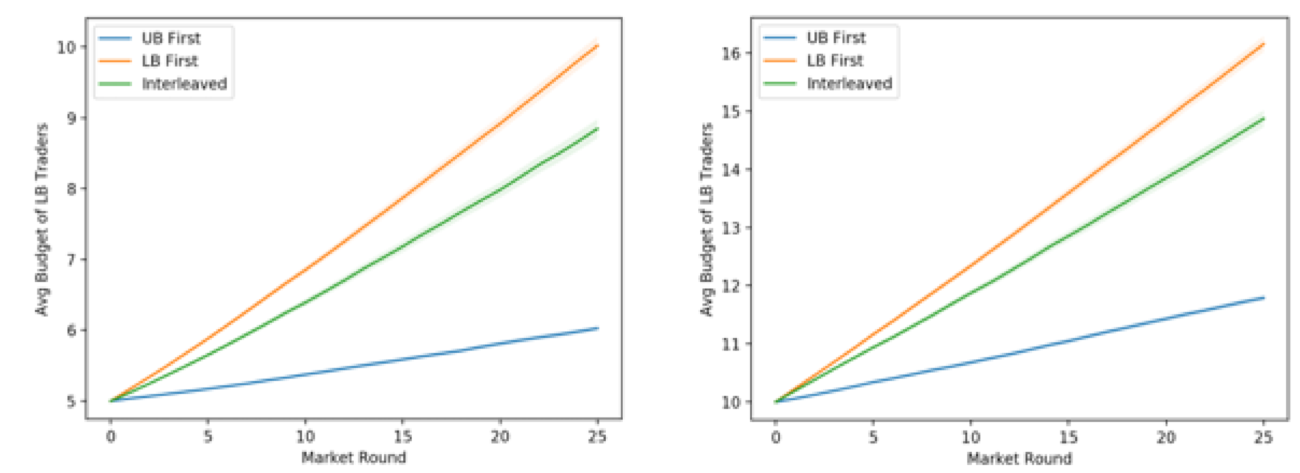


Figure 3: Change in budget of LB traders with an initial budget of $B_0 = 5$ (left) and $B_0 = 10$ (right) over rounds 1 – 25, averaged over 1000 simulations with $p_{true} \sim U[0.05, 0.95]$ for different trader orderings.

Conclusions

1. The expected **compensation** of a trader depends not only on her **informativeness** but also strongly on the **sequence** (Table 1)
2. Notably, **LI traders receive higher compensation** on average than HI traders when they **trade first**. This phenomenon underscores the advantage gained by injecting information into the market early as **higher marginal value of information** leads to **higher compensation**
3. Traders with the same overall informativeness induce the same **aggregation characteristics** at convergence **regardless of sequence**, but number of trades required to reach **steady-state** is largely impacted by **trader ordering** (Figs. 1-2)
4. All LB agents can trade on **more information** in round 25 than in the initial rounds due to **increased budget regardless of sequence**, though rate of change in budgets is affected by trader ordering (Fig. 3)
5. Future work includes further experiments with other **trader models** as well as theoretical analysis to quantify the impact of the **sequencing-informativeness-budget interplay**

Significant References

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