Belief Aggregation and Trader Compensation in Infinite Outcome Prediction Markets

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Prediction markets are incentive-compatible mechanisms for eliciting and combining the diffused, private beliefs of traders about a future uncertain event such as a political election. In this paper, we investigate the *beta-Bernoulli prediction market* algorithm that is designed to maintain an aggregated parametrized belief distribution over a bounded, continuous, real-valued random variable. One particular advantage of this algorithm over traditional markets is that it can aggregate a complete belief distribution over the underlying generative process rather than a point estimate of a probability. We experimentally assess the convergence and trader compensation characteristics of this market design by setting up a multi-agent simulation of the market ecosystem induced by it. We design a rich Bayesian trader model with explicit characterization of the heterogeneity of the trader population with respect to two attributes: how informative a trader's private observations are and how much wealth she has to express her belief through trade. A major focus of our experiments is the interplay between the arrival order of traders in the market and other trader attributes in determining their compensation. Our results in this vein suggest that early arrival can significantly dominate both wealth and informativeness in determining trader compensation.

1 INTRODUCTION

High-quality forecasts of uncertain events (the winner of an election, the likelihood of a natural disaster at a location, etc.) have always been necessary for decision-making and planning purposes in many spheres of human life. One approach towards producing such forecasts is to query experts; another is to collect and aggregate personal beliefs or partial information about the event of interest from a potentially diverse population which may include such experts. Polls and surveys are standard tools of the second approach but experience tells us that they can sometimes be messy, misleading and prone to misinterpretation. On the other hand, financial markets, where commodity prices emerge from the collective activity of a diverse trading population, have long been known to act as aggregators and disseminators of similarly diffused information and beliefs — their advantage lies in their built-in financial incentives and timely responses. *Prediction markets* [3] such as PredictIt (https://www.predictit.org/) and Iowa Electronic Markets (https://iemweb.biz.uiowa.edu/) represent a particular type of financial market that are designed with the express purpose of eliciting the personal beliefs of traders about an uncertain future event and reflecting these beliefs in the market prices. Such a market offers trade in a carefully designed bundle of contracts, or *securities*, whose monetary value at market termination is tied to the realization of the forecast event. For instance, a market designed to predict the outcome of the U.S. presidential election may issue a single security that is finally worth \$1 if a Democrat wins the race and \$0 otherwise. The price of this security over the lifetime of this market can be interpreted as the market's belief that a Democrat will win the race. There is considerable empirical evidence of such markets being at least as effective as alternative forecasting methods such as polls [4, 10, 17, 30]. There has also been a long line of theoretical research on

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prediction markets (see, e.g., [8, 16, 25]), especially after Robin Hanson introduced *market scoring rules* (MSR), a family of algorithms that act as automated market makers with many desiderata [18, 19] for prediction markets. Subsequent work [1, 7, 8] showed how such a market scoring rule (under mild regularity assumptions) can be conveniently and interpretably implemented in terms of a convex *cost function* that fully determines the *market microstructure* — the cost of purchasing and selling shares in the market securities as a function of the total number of shares sold thus far in the market. This microstructure determines traders' purchasing behaviors as well as the market's aggregation characteristics. Additionally, the cost of buying an infinitesimal number of shares of a security, called the *price* of a security, can be interpreted as the aggregated belief probability over the outcome space.

However, many issues in the design and performance evaluation of prediction markets still remain underexplored. Consider a random event with a binary outcome space, e.g. a two-candidate winner-take-all election (equivalent to the toss of a possibly biased coin). The uncertainty in the realized outcome can then be modeled as a Bernoulli distribution with an unknown (single) parameter $p \in [0, 1]$. For this forecasting task, one can use the classic *Logarithmic Market Scoring Rule* (LMSR), the most popular and best-studied member of the MSR family and the de facto standard algorithm for real-world prediction markets. An LMSR based prediction market with one security (or, equivalently, two complementary securities) is incentive-compatible for myopic, risk-neutral traders with no budget constraints [19]; i.e., the trade that maximizes such a trader's expected profit from the market also drives the market price to what this trader believes p to be. This belief itself may be based the traders' private information, which could be modeled as a small number of tosses of the same coin that determines the final, "true" outcome, and/or observations of the market's public history. Of particular interest is the market's *convergence* or *aggregation* properties. This is the phenomenon of the price (or some descriptor of the market's observable "state") getting close to the belief that would be obtained by combining the private information of all traders. A plethora of work [21, 24, 25, 27] has established the convergence and aggregation of prediction markets in general, and MSRs in particular, under various assumptions of trader behavior.

But uncertainty about the forecast variable(s) can exist on multiple levels. For our running example, not only is the outcome (the winner of the election) uncertain until market closing but the underlying process generating the outcome, represented by the Bernoulli parameter p, is itself uncertain as well. What if the market institution wanted to elicit and express uncertainty at such "deeper" levels? Is it feasible to design such a market and retain (some of the) desirable properties of MSRs? Abernethy et al. [1] introduced a family of market designs that can capture such uncertainty when it is expressed in terms of a probability distribution belonging to the well-known *exponential family* [29]. The uncertain Bernoulli parameter in our example can be modeled as a continuous random variable with support on $[0, 1]$ following a beta distribution which belongs to the exponential family and is fully described by two independent parameters (see Section 2.1 for details). Based on the principles in Abernethy et al. [1], we can implement a specific exponential family market — that we call a beta-Bernoulli market — where each trader effectively "reports" a parametric encoding of her *belief distribution* over p rather than a point probability p, and is compensated on market termination based on the revealed "true" value of p (e.g. the vote-share of the winner could serve as a proxy for p). How do we characterize convergence and belief aggregation in such a market? And, putting ourselves in the shoes of traders rather than the market institution, how does the mediation of a trading ecosystem by a such a market impact the compensation to traders for the information they inject into the market?

These questions call for a careful consideration of the attributes and circumstances of traders, especially those that might not have been taken into account in the design of the market. In addition to the information or *signal* that a trader injects into the market, there may also be a quantification of the quality of this information or strength of the signal. In this work, we model traders as Bayesian entities that compute their posterior belief distribution based on their private observations and the current market state. The size of this private binomial sample observed by a trader also relates directly to her confidence in her private information or, from the perspective of the market, the reliability of her report. We call this property the *informativeness* of a trader. Intuitively, it would be unfair if a trader of lower informativeness sustained a higher profit in expectation from the market than one of higher informativeness. Moreover, some or all traders may have a *budget* constraint that limits their trading ability, and hence their *effective* informativeness as perceived by the market. These trader attributes may interact in complex and non-intuitive ways with the microstructure. For example, in cost function-based market making, the traders arrive sequentially and directly interact with the algorithmic market maker only (a pure-dealer model), their profit being decided not by the absolute value of the information injected but by the marginal value thereof relative to current market state which depends on all previous trades. Hence, one's *position in the order of traders* substantially affects one's compensation: if a highly informative trader shows up after a large amount of information has already been incorporated into the market, can they end up booking a smaller profit than a less informative trader arriving earlier? How serious is this *sequence effect*? This is particularly important in scenarios where traders cannot fully control their arrival time owing to market regulations or circumstances external to the market. For instance, the market may emply tie-breaking to resolve simultaneous arrival or a trader may be willing to trade only after her source delivers the private observations on which she bases her belief and hence her trading decision.

Thus, our broad motivation is to develop a detailed understanding of *price formation* and *trader compensation* for an *exponential family prediction market* and a *heterogeneous* trader population. In this paper, we present our contributions to that end thus far.

1.1 Our Contributions

- We focus on a beta-Bernoulli prediction market that maintains and updates a belief distribution about the parameter of a Bernoulli event (Section 3.1), and adopt a simulation-based approach to systematically investigate its properties under various trader models. To our knowledge, this is the first systematic experimental assessment of a beta-Bernoulli market.
- We formulate an agent-based model of trading under Bayesian belief updates (Section 3.2): salient features of an agent include her *informativeness*, measured by the number of private samples observed from the true Bernoulli distribution, and her *budget*, i.e. an upper bound on the *debt* (negative wealth) she can be in when the market closes.
- We conduct and report two sets of experiments: in the first, traders with unlimited budget are divided into two types based on their asymmetric informativeness; in the other, traders of the same informativeness are divided into two types depending on whether or not they are budget-limited. In each set, we construct different trader sequences based on types (e.g. 500 high-informativeness traders followed by 500 low-informativeness traders) We measure separately the interaction of these two features with a trader's position in the sequence and how the arrival time impacts market convergence and trader compensation. In the second set, we simulate several *rounds* of forecasting, each having its own market and ending in the revelation of a different Bernoulli event but involving the same agents (reminiscent of market sequences in Beygelzimer et al. [5]). We track the evolution of traders' budgets (among other quantities) over all these rounds.

A major insight from our experimental results is that the arrival time of a trader can outweigh both informativeness and budget in terms of effect on trader compensation (or trading power). This is not a counter-intuitive finding but the value of our study lies in systematically measuring this effect and identifying conditions under which it is prominent, and laying the groundwork for further experimental (and theoretical) analysis of different exponential family prediction markets.

1.2 Related Work

The prior work that is most closely related to our paper is the seminal work on exponential family markets by Abernethy et al. [1]. This paper does provide some analytical results on convergence for some trader models but traders are assumed to homogeneous and with infinite budgets. There has also been follow-up on elicitation for exponential family distributions in a non-market context [15]. There is a long line of work on convergence / aggregation in market equilibrium abstracting away from microstructure: see, e.g. Beygelzimer et al. [5], Pennock [26]; Storkey et al. [28] and references therein. Our work relates more to results on convergence with a cost function-based microstructure [21, 24, 27] although our market is designed to capture uncertainty on a different level. There has been both theoretical Devanur et al. [11] and empirical Sethi and Vaughan [27] work on budget restrictions for traders in cost function-based trading. However, in our paper, budgets are operationalized differently from prior work (Section 3.2.2). In our second set of experiments, we consider a sequence of forecast rounds and study the growth of budget over them, reminiscent of Beygelzimer et al. [5]. The main differences in our work are that we have a microstructure and have distinct characterizations of budget and informativeness. Our work also differs from the strand on *information structure* of (potentially returning and manipulative) traders and its impact on market equilibria [9] (and references therein), [2, 22] — here structure refers to correlation among traders' private information signals. We on the other hand focus on a quantification how the quality of a myopic trader's individual informativeness. Recent work on prediction markets over interval securities [14] allows traders to express their confidence by trading on an interval rather than a point estimate; however, our market is set up to elicit the entire (parametrized) posterior distribution, given that it belongs to a well-defined family and we model trader's informativeness directly in terms of the process generating her private observations.

Some other recent papers on prediction markets that deserve mention are: Dudík et al. [13] that looks at tradeoffs among forecast error components in prediction markets rather the interplay of trader attributes in determining their own compensation as well as market convergence; [23] who deal with implementation of *combinatorial* prediction markets that elicit fine-grained information of a fundamentally different nature.

2 PRELIMINARIES

2.1 The Exponential Family of Distributions

The exponential family [29] is a set of distributions whose probability density function over a random variable x can be expressed in the form

$$p(x|\eta) = \exp\{\eta \cdot \phi(x) - C(\eta)\}. \tag{1}$$

Here η are called the natural parameters of the distribution and $\phi(x)$ are the sufficient statistics. Exponential family distributions are the solution to the constrained optimization problem of finding the maximum entropy distribution with a given value of expected sufficient statistics. Many commonly used distributions like the Gaussian, Poisson, and importantly for this exposition, beta and Bernoulli distributions, are all members of this family.

The Bernoulli distribution is defined over a binary outcome space $X \in \{0, 1\}$ and is fully described by a single parameter $p \in [0, 1]$, $p = \Pr[X = 1] = \mathbb{E}[X]$. In situations where this parameter itself is unknown and we wish to maintain and update our uncertainty in it, we can model it as random variable over the parameter space $[0, 1]$. A natural choice in this case is the *beta distribution* which is, in fact, the *conjugate prior* of the Bernoulli data distribution [12] and is itself a member of the exponential family. The support of the beta distribution is $p \in [0, 1]$ and its probability density function can be specified in terms of two independent parameters $\alpha, \beta \in \mathbb{R}^+$:

$$\textsc{Beta}(p; \alpha, \beta) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha, \beta)}$$

where $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ and $\Gamma(\cdot)$ denotes the well-known *gamma function* [20]. For positive integers n, which suffices for our purposes, $\Gamma(n) = (n-1)!$. The expectation of the distribution is given by $\mathbb{E}[p] = \frac{\alpha}{\alpha+\beta}$, and $(\alpha + \beta)$ is sometimes called the pseudo-sample size or phantom sample size of the distribution. For $\alpha = \beta = 1$, the distribution is identical to the uniform distribution over the interval $[0, 1]$. Suppose a Bernoulli random variable is parameterized by $p \in [0, 1]$. Given a prior belief distribution over p that is characterized by a beta distribution with parameters α, β, and m observations $\{x_1, \ldots, x_m\}$ drawn from the true Bernoulli distribution, the Bayesian posterior belief distribution is also a beta distribution with updated parameter values

$$\alpha^{(\text{posterior})} = \alpha + \sum_{i=1}^{m} x_i; \qquad \beta^{(\text{posterior})} = \beta + m - \sum_{i=1}^{m} x_i. \tag{2}$$

2.2 Exponential Family Markets

Exponential family markets [1] use the structure of an exponential family distribution to construct a cost function-based prediction market with interesting theoretical aggregation and elicitation properties. The classical LMSR market can be seen as a special case of exponential family markets where the cost function is based on the log-partition function of the Bernoulli probability mass function.

The goal of the market maker is to elicit and aggregate beliefs of the expected value of a vector of statistics of a random variable of interest $x \in \mathcal{X}$. Based on these statistics, the market maker chooses an appropriate maximum-entropy distribution and defines the market microstructure as follows. Suppose this distribution is an exponential family distribution with natural parameter $\eta \in \mathbb{R}^d$, log-partition function $C(\eta) \in \mathbb{R}$ and sufficient statistics $\phi(x) \in \mathbb{R}^d$, where symbols have the same meaning as in Equation (1); the market maker defines its securities, payoff (i.e. cash per share that each security liquidates to at market termination), and cost function based on this distribution. Corresponding to each dimension of the sufficient statistics vector, the market maker issues a security/contract that pays off the value of the corresponding statistic for the realized outcome. In other words, if a trader holds $\delta \in \mathbb{R}^d$ shares, where each dimension δ^i corresponds to the trader's holdings of the i^{th} security, then the trader receives a payment of $\delta \cdot \phi(x)$ from the market maker from these holdings on market closing. After every trade, the vector of all traders' share holdings of all securities, called the *oustanding shares*, determines the market "state" — this vector to the natural parameter η for exponential family markets [1]. The market maker prices securities based on the log-partition function which serves as the cost function in this market — it charges the trader $C(\eta + \delta) - C(\eta)$ to purchase δ shares at market state η. Hence, the traders' profit/compensation from these holdings on market closing is $\delta \cdot \phi(x) - (C(\eta + \delta) - C(\eta))$.

Abernethy et al. [1] also define an extension of this market where the outcome space itself is the parameter space of a distribution. The market is then defined based on the conjugate prior of this distribution. The market design we study in this paper is, in fact, a specific instance of this type of market — we provide a detailed description in Section 3.1.

3 MARKET AND TRADER MODELS

In traditional LMSR markets, the goal of the market maker is to update its belief over the outcome space of some uncertain real-world event; e.g. the U.S. presidential race (before it is called) can be modeled as a Bernoulli random variable, with the outcome space $\{0, 1\}$ where 1 (resp. 0) stands for the win of the Democratic (resp. Republican) candidate. The market price at any time corresponds to the expected value of this random variable — for the above example, it would also correspond to the probability p of the Bernoulli trial.

In our model, we modify the goal of the market maker – instead of learning the expected value (a *point estimate*) of the random variable itself, we are interested in learning from traders' inputs a belief *distribution* over the parameters of probability distribution of the random variable. Our forecast event of interest is a Bernoulli random variable; we choose as our belief distribution over the outcome-generating process the *beta distribution* which is the conjugate prior for the Bernoulli data distribution [12]. The beta distribution is fully described by two parameters $\alpha > 0$ and $\beta > 0$; hence, an update to the belief distribution is equivalent to an update in the two-dimensional (α, β) parameter space.

Another feature of our model is that, at market closing, the parameter of the "true" distribution from which the realized outcome is drawn – the Bernoulli success probability p_{true}– is made public. While it is true that the revelation of the outcome is not the same as the revelation of the specifics of the outcome-generating process, there are many scenarios where we can observe a proxy for the process parameters; e.g. the vote-share of the winning candidate in a binary election can serve as such a proxy (see, e.g. Chakraborty et al. [6] for a vote-share prediction market). In fact, the market is set up to elicit and maintain a beta distribution over a continuous random variable with support on $[0, 1]$, regardless of what the variable means in the real world.

3.1 Beta-Bernoulli Market

We now specify the design of the beta-Bernoulli market for the above prediction task, based on the principles laid down in [1] for the exponential family class to which this market belongs.

Securities and payoffs. The market offers trade in two securities whose payoffs are $\ln p_{true}$ and $\ln(1 - p_{true})$ respectively when $p_{true} \in [0, 1]$ is revealed[1] and hence the market terminates.

Outstanding shares and market belief. The market maintains and continuously displays (for the trading population) the current outstanding shares $\eta = [\eta_1, \eta_2]^T$. This also corresponds to the parameters of (and hence fully describes) the market's current belief $\textsc{Beta}(p; \alpha, \beta)$ as follows: $\eta_1 = \alpha - 1$, $\eta_2 = \beta - 1$.

Cost function. This corresponds to the beta log-partition function [29] and is also known to the traders:

$$C(\eta) = \ln \Gamma(\eta_1 + 1) + \ln \Gamma(\eta_2 + 1) - \ln \Gamma(\eta_1 + \eta_2 + 2).$$

In particular, the outstanding share vector readily gives us the market's current expectation of the unknown Bernoulli parameter $\mathbb{E}_\eta[p] = \frac{\eta_1+1}{\eta_1+\eta_2+2}$ as well as those of payoffs: $\mathbb{E}_\eta[\ln p] = \psi(\alpha) - \psi(\alpha+\beta)$ and $\mathbb{E}_\eta[\ln(1-p)] = \psi(\beta) - \psi(\alpha+\beta)$, where $\psi(y) = \frac{\mathrm{d}\Gamma(y)/\mathrm{d}y}{\Gamma(y)}$ is the *digamma function.*

In the following remarks, we address a few operational aspects of the market:

Remark 3.1. As in any cost function-based prediction market, the initial values of (η_1, η_2) should reflect the market organizer's belief about the forecast variable before obtaining any report (through orders) from the traders. If this initial belief distribution is $U[0, 1]$, which coincides with the beta distribution with $\alpha = \beta = 1$, the starting values of the

[1]For our simulations, we ensure $p_{true} \in (0, 1)$ to prevent degenerate cases.

outstanding shares conveniently work out to $\eta_1 = \eta_2 = 1 - 1 = 0$. Interestingly, the cost function also evaluates to 0 at $\eta = (0, 0)$. We use this initialization in our implementation. ♦

Remark 3.2. We know from Abernethy et al. [1] that a myopic, risk-neutral trader with belief distribution $\text{Beta}(p; \alpha', \beta')$ will move the outstanding shares to $\eta_1 = \alpha' - 1 > -1$ and $\eta_2 = \beta' - 1 > -1$ since $\alpha', \beta' > 0$. Thus, if all traders have beta beliefs, then η_1, η_2 never drop below -1 but have no upper limit. However, in general, a trader could place (sell/short-sell) orders that, if executed, would drive η_1 or η_2 to or below -1, precluding the interpretation of outstanding shares as beta belief parameters. Moroever, since the gamma function is infinite (positive/negative) for negative integers, the cost function value may become infinite for plausible (short-)sell order sizes. Thus, the market maker must actively enforce the condition $\eta > (-1, -1)$ by rejecting any orders that may lead to a violation of this constraint. As a consequence, regardless of what orders get executed and how traders decide on these orders, the market will be able to maintain a beta belief after every trade. In other words, the market "state" η (under the restriction that it is always greater than $(-1, -1)$) *can always be interpreted as the vector of defining parameters of a beta distrbution, regardless of whether a trader has a beta belief distribution* and/or is Bayesian! ♦

Remark 3.3. Another notable feature of this market is that the security payoff $(\ln p, \ln(1 - p))$ is necessarily non-positive for any realized value of the forecast variable $p \in [0, 1]$. This means that a trader who has *purchased* δ shares of security 1 (resp. 2) is *obligated to pay* the market organizer an amount $\delta|\ln p| = \delta \ln(1/p)$ (resp. $\delta|\ln(1 - p)| = \delta \ln(1/(1 - p))$) on market closure instead of being compensated! This is counter-intuitive or, at least, non-traditional in terms of the setup of prediction markets where a trader's share holdings are converted to currency at market termination. However, on the flip side, note that:

$$\frac{\partial}{\partial \eta_i} C(\eta) = \psi(\eta_i + 1) - \psi(\eta_i + \eta_{3-i} + 2) \quad \forall i \in \{1, 2\},$$

where $\psi(y)$ is the digamma function as defined above. Now, $\psi(y)$ is strictly increasing for $y \in [0, \infty]$ and $0 < \eta_i + 1 < \eta_i + \eta_{3-i} + 2$ by design (since $\eta_i + 1 > 0\ \forall i \in \{1, 2\}$); hence, $\frac{\partial}{\partial \eta_i} C(\eta) < 0$ for $\eta_i > -1$, $i \in \{1, 2\}$, i.e. $C(\eta)$ is strictly decreasing in both η_1 and η_2. Thus, the cost to the trader for purchasing $\delta > 0$ shares in either security is negative $(C((\eta_1 + \delta, \eta_2)) - C(\eta) < 0, C((\eta_1, \eta_2 + \delta)) - C(\eta) < 0)$. In other words, it is as if the market organizer *antes up* to a trader an amount determined by the cost function difference when the latter *purchases* shares and puts the trader under the above obligation of payment at market termination. This does not entail any issues formally/technically as long as a trader reasons in terms of her (uncertain) ex-post net compensation. However, the impact of this unusual interface on real-world trader behavior remains to be investigated! ♦

Remark 3.4. When the revealed value of the market's continuous forecast variable is p_{true}, a trading agent's ex-post net compensation from a trade that drove the outstanding share vector from η to $\eta' = [\eta'_1, \eta'_2]^T$ is given by

$$R(\eta, \eta', p_{true}) = [\ln p_{true}, \ln(1 - p_{true})](\eta' - \eta) - C(\eta') + C(\eta).$$

In theory, this quantity could be positively infinite (implying an infinite profit for the trader) or negatively infinite (implying an infinite loss for the trader) even when the trader's order *size* (i.e. $\eta' - \eta$) is finite. For instance, the former happens whenever $p_{true} = 0$ (resp. $p_{true} = 1$) and $\eta'_1 - \eta_1$ is positive (resp. negative) and finite; the latter happens whenever $p_{true} = 0$ (resp. $p_{true} = 1$) and $\eta'_1 - \eta_1$ is negative (resp. positive) and finite. However, for $p_{true} \in (0, 1)$, $R(\eta, \eta', p_{true})$ is finite for any finite $\eta' - \eta$, and $p_{true} \in \{0, 1\}$ is a zero-probability event under any beta distribution. ♦

3.2 Bayesian Trader Model

In our model, traders are *myopic*, i.e. on each market entry, they act as if it is their last interaction with the market, and *risk-neutral*, i.e. they trade to optimize their expectation of (any affine function) of their compensation from the market (however, this optimization may be constrained, see Section 3.2.2). Before trading, the trader formulates her own belief about p as follows: before each of her arrivals, a trader privately observes a sample of m datapoints drawn independently from the true Bernoulli distribution: $x_i \sim_{i.i.d.} \textsc{Bernoulli}(p_{true}) \quad \forall i \in \{1, \ldots, m\}$. The trader is *Bayesian*: she uses the market state η to define her Bayesian prior belief $\textsc{Beta}(\eta_1 + 1, \eta_2 + 1)$; using her sample, she constructs her Bayesian posterior belief which is a beta distribution with updated parameters as in Equations (2). Due to the incentive-compatibility of exponential family markets [1], the trader's optimal trade updates the market's state to η^{new} that corresponds to these updated parameters: $\eta_1^{new} = \eta_1 + \sum_{i=1}^m x_i$ and $\eta_2^{new} = \eta_2 + m - \sum_{i=1}^m x_i$.

Remark 3.5. Since $m > 0$, we have $\eta_i^{new} \geq \eta_i$ with $\eta_i^{new} - \eta_i \in \mathbb{Z}_+$, $i \in \{1, 2\}$, for every trade. Hence, each trader in this model can fully move the market to her belief exactly by *buying* an integral number shares (in one or both securities) with no need to (short-)sell, although the market does not disallow selling. Moreover, the market's outstanding shares also (weakly) increase with every trade, always remaining non-negative integers; consequently, the market's beta belief parameters also both (weakly) increase with each trade, always remaining positive integers. ♦

3.2.1 Trader Informativeness. We define the informativeness of a trader simply as her private sample size m as above. Since these datapoints are drawn from the true distribution, larger the sample size, the higher the quality of the posterior. Interestingly, since the market displays η at all times and updates to η are guided by above Bayesian inference, the informativeness of every trader in this model can be inferred readily as $m = \eta_1^{new} + \eta_2^{new} - (\eta_1 + \eta_2)$.

3.2.2 Trader Budget. Until now, we have implicitly assumed that every trader has an unlimited budget for trading and hence injecting her information into the market state. But, a trader may have a finite budget $B > 0$ that limits her trading ability as follows. For a budget-limited traders, we assume that she receives her private information $x_1, x_2, \cdots, x_m$ one datapoint at a time; she then decides to incorporate the largest leading sub-sequence of these datapoints that ensures that her worst-case exposure in the market for that trade respects her budget constraint. More formally, for a current market state of η, a trader chooses to trade on the posterior induced by $x_1, \ldots, x_k$, for the largest value of $k \in \{1, 2, \cdots, m\}$ such that

$$\max_p \left\{ C\left(\eta + \left[\sum_{i=1}^k x_i, k - \sum_{i=1}^k x_i\right]^T\right) - C(\eta) - \left[\sum_{i=1}^k x_i, k - \sum_{i=1}^k x_i\right] \cdot [\ln p, \ln(1-p)] \right\} \leq B.$$

In our simulations, we maximize over $p \in [\epsilon, 1 - \epsilon]$ for some small $\epsilon > 0$ to exclude the degenerate instances corresponding to sure events. If the trader returns, she retains the last $m - k$ datapoints $x_{k+1}, x_{k+2}, \cdots, x_m$ that she failed to incorporate and collects the remaining k datapoints for the new trade afresh.

4 EXPERIMENTS

4.1 Experimental Setup

We conduct two sets experiments, Set1 and Set2, on a beta-Bernoulli market as in Section 3.1, starting at $\eta = [0, 0]^T$ (i.e. no outstanding shares) or, equivalently, $\alpha = \beta = 1$ (i.e. a uniform prior distribution over the Bernoulli parameter).

4.1.1 SET1: Informativeness and Trader Ordering. In SET1, we have two types of traders with unlimited budget: HI (high informativeness) having private sample size $m = 5$, and LI (low informativeness) with $m = 1$. In each market instance, there are $n = 500$ traders of each type ($2n = 1000$ traders in total), each drawing a fresh private sample when it enters and trading only once with the market maker. The revelation at market closur, i.e. after all $2n$ traders have completed trading, has three possibilities $p_{true} \in \{0.25, 0.5, 0.75\}$, which determines each trader's payoff for that instance. For each such possibility, we compare the market performance of three different *orderings* of the trader population with respect to types:

HI first: All n HI traders arrive first, then all n LI traders.
LI first: All n LI traders arrive first, then all n HI traders.
Interleaved: LI and HI traders arrive alternately, starting with LI.

For each market instance, we measure the per-trader average compensation of HI traders and the same quantity for LI traders. We produce averages of these quantities over 1000 market instances (the instances differing the observed random datapoints of traders) under a 3×3 experimental design (values of p_{true} and types of ordering).

4.1.2 SET2: Budgets and Trader Ordering. In this set, each trader can re-enter the market multiple times until it closes according to a stop criterion (see below). In each instance of the market, we draw the true Bernoulli parameter from a uniform distribution on $[\epsilon, 1-\epsilon]$ with $\epsilon = 0.05$ (the same ϵ as in Section 3.2.2), i.e. $p_{true} \sim U[0.05, 0.95]$. Another important difference from SET1 is that, instead of looking at a single market instance ending in one revelation, we track the overall compensation characteristics of a trader population over a sequence of $N = 25$ *rounds*, each having a market, starting at $\eta = [0, 0]^T$, for a different Bernoulli event and terminating in the revelation of the corresponding Bernoulli parameter (we assume independence among all these events). As in Beygelzimer et al. [5], the fixed population of traders can be viewed as a panel of experts responding to a series of different binary forecast questions and having their *weights* (akin to trading power — see below) readjusted by the response aggregator according to their performance history.

Each trader has at her disposal a private sample of the same size $m = 5$ on each of her entries. But, there are now two types of traders on a different basis: those with unlimited budget (UB) and those who are budget-limited (LB). We again have $n = 500$ traders of either type in the population. Each UB trader uses all her m (fresh) datapoints to determine her posterior and hence her trade on each entry in any market instance, thus acting as multi-shot versions of HI traders from SET1. At the beginning of each N-round sequence, each LB trader is endowed with an initial budget $B_0 \in (0, \infty)$ which is an experimental parameter. In our experiments, $B_0 \in \{2, 5, 10, 100\}$. Each time an LB trader has an opportunity to trade, she makes her budget-constrained trading choice as described in Section 3.2.2 with her "residual" budget B obtained by subtracting her cost of purchasing shares on her previous entry from her budget at that time. At the end of each round, an LB trader's compensation is added to her initial budget for the next round.

Bearing in mind that traders are multi-shot, we define trader orderings as follows. The first entries of all traders in any market instance are determined in a round-robin fashion based on budget-based types in one of three ways (mirroring SET1):

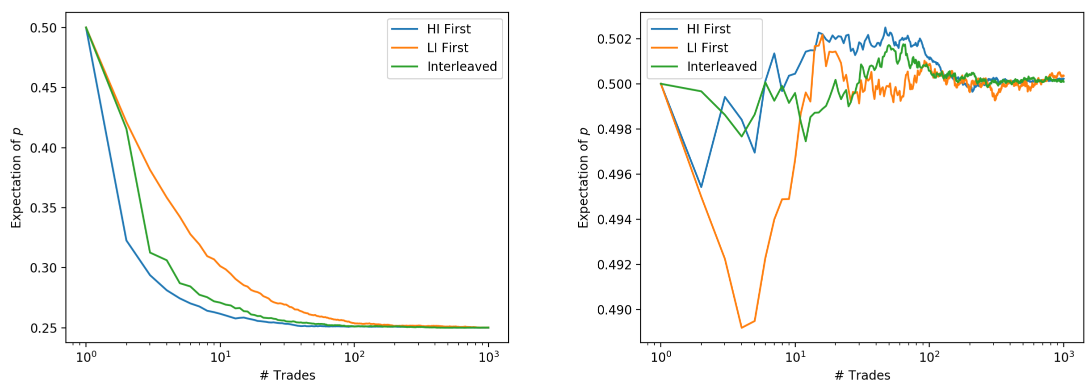
UB first: All UB traders have an opportunity to trade before LB traders in the market.
LB first: All LB traders have an opportunity to trade before UB traders in the market.
Interleaved: UB and LB traders trade alternately, starting with LB.

If the stop criterion is not met, traders re-enter in the same order as in the initial round-robin phase (with respect to their actual identities and not just types).

Stop criterion: We terminate the market either when *each* trader has individually reached a pre-determined upper bound $n_{\max}$ on the total number of opportunities to trade or when the market has converged in expectation (in the following sense), whichever comes sooner. For each trade, if η^{old} and η^{new} denote respectively the outstanding shares vector before and after the trade, we track the difference between the corresponding expected values of the Bernoulli parameter $\left|\mathbb{E}_{\eta}^{new}[p] - \mathbb{E}_{\eta}^{old}[p]\right|$ where $E_{\eta}[p] = \frac{\eta_1+1}{\eta_1+\eta_2+2}$; if this difference is at most a pre-determined small $\varepsilon > 0$, we deem the market converged in expectation. In our experiments, we choose $n_{\max} = 60$ and $\varepsilon = 0.001$. Results are again averaged 1000 simulations for each trader ordering and initial budget, each simulation being characterized by random draws for the 25 p_{true} values and random private samples of each trader.

4.2 Results

Some results omitted from this section are available in Appendix B.



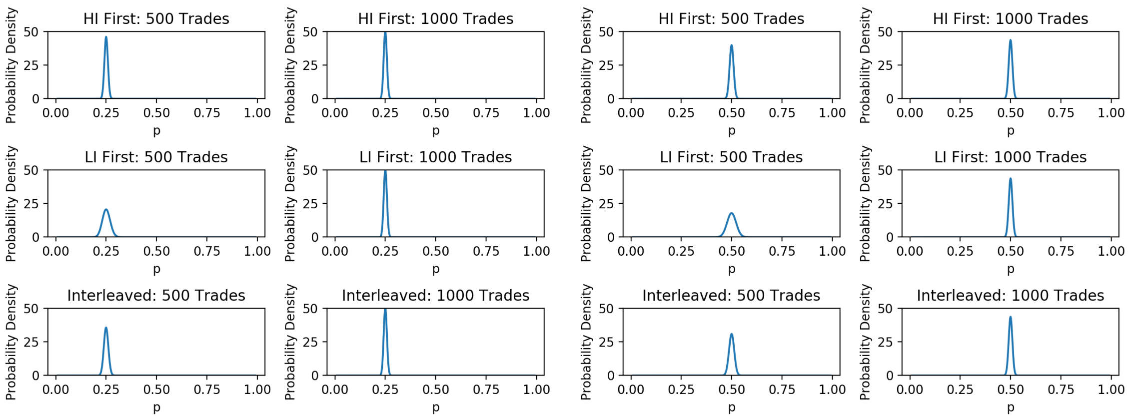
(a) $p_{true} = 0.25$ (b) $p_{true} = 0.5$

Fig. 1. The market's expectation $\mathbb{E}_{\eta}[p]$ as a function of the number of completed trades for different trader orderings, averaged 1000 simulations with the same $p_{true} \in \{0.25, 0.5\}$. Horizontal axis is logarithmic.

4.2.1 Set1. Before we discuss our main results on the difference in average compensation between HI and LI traders (Table 1), we will look at the belief evolution (and convergence) of the market for various experimental conditions (p_{true} and trader orderings). These observations are not only interesting in their own right but also help understand the measured compensation characteristics.

Figure 1 shows the market's expected value of the Bernoulli parameter $\mathbb{E}_{\eta}[p]$ as a function of the number of completed trades in the market for our choices of p_{true}, averaged over 1000 simulations; Figure 2 depicts the average posterior distribution (probability density function) after 500 trades (mid-point of a market's lifetime) and 1000 trades (market termination) for the same choices of p_{true} over 1000 simulations. The corresponding plots for $p_{true} = 0.75$ mirror those for p_{true} due to symmetry and are hence omitted.

Recall that the starting prior of the market is a uniform distribution (with expectation 0.5). In Figure 1a, we note that, on average, the market reaches steady state around the correct p_{true} in approximately 100 trades regardless of



(a) $p_{true} = 0.25$ (b) $p_{true} = 0.5$

Fig. 2. Posterior distribution (market belief) after 500 and 1000 trades for different trader orderings. This plot reports the average distribution across 1000 simulations with the same $p_{true} \in \{0.25, 0.5\}$.

trader ordering, indicating that by this point the market is saturated with information (from the private Bernoulli observations). Before this steady state, the rate of convergence gets understandably slower as we move from **LI first** to **Interleaved** to **HI first** since the amount of information injected per trade also improves in the same order. The initial coincidence of the **LI first** and **Interleaved** can also be explained by our design of making **Interleaved** start with an *LI* trader. Figure 2a offers insights into the mean as well as the spread in the market's belief: after a half of the total number of trades, the market has already developed significant confidence in its estimate of p_{true} as indicated by the concentration around the correct p_{true}, and this confidence grows over the remaining trades for all three orderings. The distributions at market closure are virtually indistinguishable for all three orderings; but the concentration after 500 trades improves in the same order as the convergence rates in Figure 1a for the same reason. In contrast, Figure 2b, where $p_{true} = 0.5$, seems to show substantial volatility in market expectation during the early part of the market's lifetime for all three orderings, with **LI first** being most volatile. However, notice that the spread in the expectations in the volatile phase is very small (around 0.5). Again, the volatility practically disappears around the 100-trade mark for all orderings. This is a natural consequence of the market starting off with expected value equal to the true parameter value and all traders having access to datapoints from the true distribution. However, as Figure 2b indicates, the market is still performing the aggregation function of improving concentration around the "correct" mean with every passing trade, although the final concentration for $p_{true} = 0.5$ appears to be less impressive than that for $p_{true} = 0.25$ across trader orderings.

Table 1 shows the expected per-trader average compensation for each trader type (HI, LI) for all experimental conditions, with the expectation being estimated by averaging over 1000 market instances for each combination of p_{true} and ordering. The most important observation from this table is that, for every p_{true} considered, HI traders get less compensation on average than LI traders when LI traders arrive first; interleaving (uniformly) even with a leading LI trader does not have this effect. We return to Figures 1a and 1b for a qualitative explanation of this phenomenon: even with the low information injection rate of **LI first**, the market gets "saturated" with information long before ($\sim$ 100 trades) all LI traders have completed their trades (500 trades). Subsequent trades (by HI traders) produce minimal

p_{true}	Sequence	Avg HI ($\times 10^3$)	Avg LI ($\times 10^3$)
0.75	**HI First**	6.66 ± 0.09	0.18 ± 0.05
	LI First	**1.75 ± 0.12**	**5.07 ± 0.09**
	Interleaved	5.51 ± 0.15	1.29 ± 0.13
0.50	**HI First**	6.37 ± 0.08	0.17 ± 0.05
	LI First	**1.75 ± 0.11**	**4.78 ± 0.09**
	Interleaved	5.56 ± 0.14	0.99 ± 0.12
0.25	**HI First**	6.70 ± 0.08	0.21 ± 0.05
	LI First	**1.80 ± 0.11**	**5.04 ± 0.09**
	Interleaved	5.52 ± 0.14	1.26 ± 0.12

Table 1. Compensation of HI and LI traders for different trader orderings and p_{true} values. This table reports the compensation, averaged over all agents of the same type over 1000 simulations, with 95% confidence bounds. Each entry must be multiplied by 10^{-3} to get the correct value.

marginal improvement to aggregation quality, hence these traders tend to be compensated less (regardless of the absolute informational content of their trade). But the significantly large difference in compensation is still surprising! This experimental observation begs the more general question: Does the **LI first** ordering always result in a higher

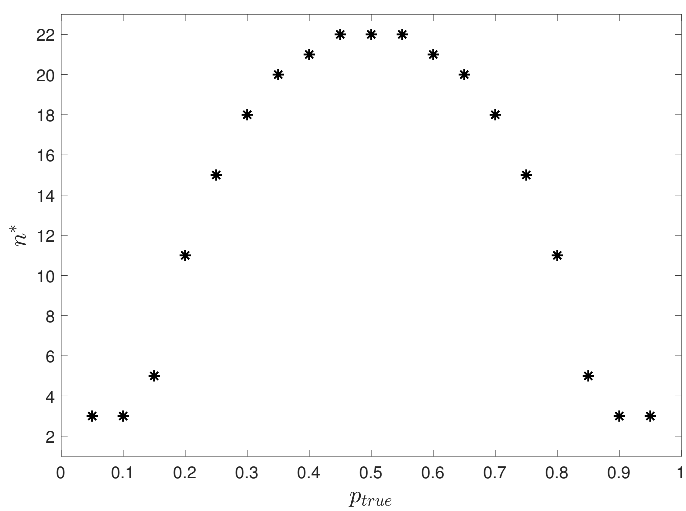


Fig. 3. Minimum value n^* of the number of traders of each kind in an **LI first** sequence for which the expected per-trader average compensation for LI traders is higher than that for HI traders vs p_{true}.

(expected per-trader average) compensation for LI traders than for HI traders, regardless of the length of the sequence (i.e. the number of traders) and the value of the Bernoulli parameter p_{true}? To answer this question, we derived closed-form expressions for the expected per-trader average compensations for LI traders (denoted by $\widehat{R}_n^{\mathrm{LI}}$) and HI traders (denoted by $\widehat{R}_n^{\mathrm{HI}}$) who trade in an **LI first** sequence of length $2n$ (i.e. n LI traders followed by n HI traders) for a fixed p_{true},[2] and compared values of these expressions for values of p_{true} ranging from 0.05 to 0.95 in steps of 0.05.

[2]The expectation is with respect to the distribution of each trader's private sample.

Let the difference between the above two expectations be $\Delta_n^{\text{HL}} := \widehat{R}_n^{\text{HI}} - \widehat{R}_n^{\text{LI}}$. We found by direct calculation that, for each p_{true} considered, there exists a value of $n \in \{1, 2, \ldots\}$, denoted by n^*, such that $\Delta_n^{\text{HL}} \geq 0$ for every $n < n^*$ and $\Delta_n^{\text{HL}} < 0$ for every $n \geq n^*$. We plot this n^* for the corresponding p_{true} in Figure 3. It appears from Figure 3 that $n^* \leq 22$ for every $p_{true} \in [0.05, 0.95]$; hence, there exist (small) values of n for which HI traders can expect to sustain a higher compensation on average than LI traders even if the former appear later in the sequence; however, for large enough values of n (e.g. $n = 500$ in our experiments), early arrival always dominates informativeness in this regard. We provide detailed derivations, calculations, and further comments in Appendix A.

4.2.2 Trader Order and Budgets. As for SET1, we begin with qualitative results on convergence and aggregation. The metric we use to characterize aggregation properties of the market is Δ^{suff}, the Euclidean distance between the final expected sufficient statistics of the market $[\mathbb{E}_\eta[\ln p], \mathbb{E}_\eta[\ln(1-p)]]^T$ and the corresponding vector induced by the revealed Bernoulli parameter p_{true}, i.e. $[\ln(p_{true}), \ln(1 - p_{true})]^T$. On average, the market seems to be a good aggregator with respect to this metric. For example, the average value of Δ^{suff} for round 25 (the round in which LB traders are least constrained by budget) over all 1000 simulations for **LB First** and $B_0 = 10$ is 0.099. For reference, the Euclidean distance between above sufficient statistics vectors for the extreme values of the support of p_{true} ($p = 0.95$ and $p = 0.05$) is 4.16, and that from either extreme to $p = 0.5$ is 2.39.

However, looking at the distribution of this metric more closely reveals interesting diversity in aggregation characteristics. Figure 5 shows select snapshots of the market's evolving belief in the final round of a "worst-case" instance with respect to the metric Δ^{suff}. Note that the market, which starts with a flat prior over the entire support, shifting its probability density in the right direction even after a single trade from an LB trader, and continues on its course. However, upon convergence, it concentrates the probability around an incorrect mean more extreme than p_{true}. This mean is still quite close to p_{true} (at a distance of 0.035). But contrast this with Figure 4 which represents the final market belief in the "best case" for Δ^{suff} (p_{true} is safely distant from the extremes). This would suggest that the closer p_{true} is to its extremes (0.05 and 0.95), the worse the aggregation performance of the market — in other words, the market performs as a better aggregrator for less certain events.

We now come to the evolution of the budget of LB traders over rounds as a result of augmentation of the initial budget for each new round with compensation earned in the previous round. As seen in Figure 6, LB traders are able to grow their budget over successive rounds, as expected (results are qualitatively similar for other values of the initial budget B_0 we considered, hence omitted). The earlier the LB traders arrive, the higher is the rate of this growth. Perhaps more interestingly, the rate of growth decreases with the size of the initial budget — with $B_0 = 5$, the budget after the final round is double its initial size for **LB First** but only 1.6 times for $B_0 = 10$ (the trend is maintained by the omitted plots). An inquiry into the cause of this observation is left for future work.

To gauge more directly the improvement in trading power induced by growing budget, we measured the number of datapoints that an LB trader is able to incorporate into her trading decision over successive rounds for an initial budget of 10. We observe that the number of entries into the market for each trader is determined primarily by convergence based on the change in expectation of p_{true}. Thus in all rounds, over all simulations, the average number of times, each trader enters the market is in the range $[10.9, 11.8]$, hence the horizontal axis in Figure 7 goes from 1 to 10, representing the entries within a particular round. The number of datapoints used per entry decreases with re-entries within a round since the residual budget decreases until market closure. As seen in Figures 7a and 7b, the first LB trader tends to be able to trade on more of her information, than the last LB trader in the sequence. All traders are able to trade on more information in round 25 than in the initial rounds due to their increased budget (see Figure 6b).

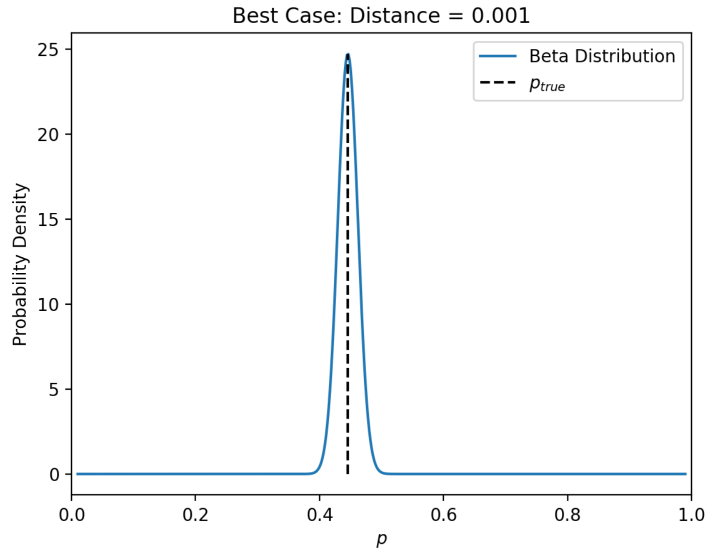


Fig. 4. Final market belief in round 25 for the simulation with the lowest Δ^{suff} ($\approx$ 0.001) across all 1000 simulations for B_0 = 10.

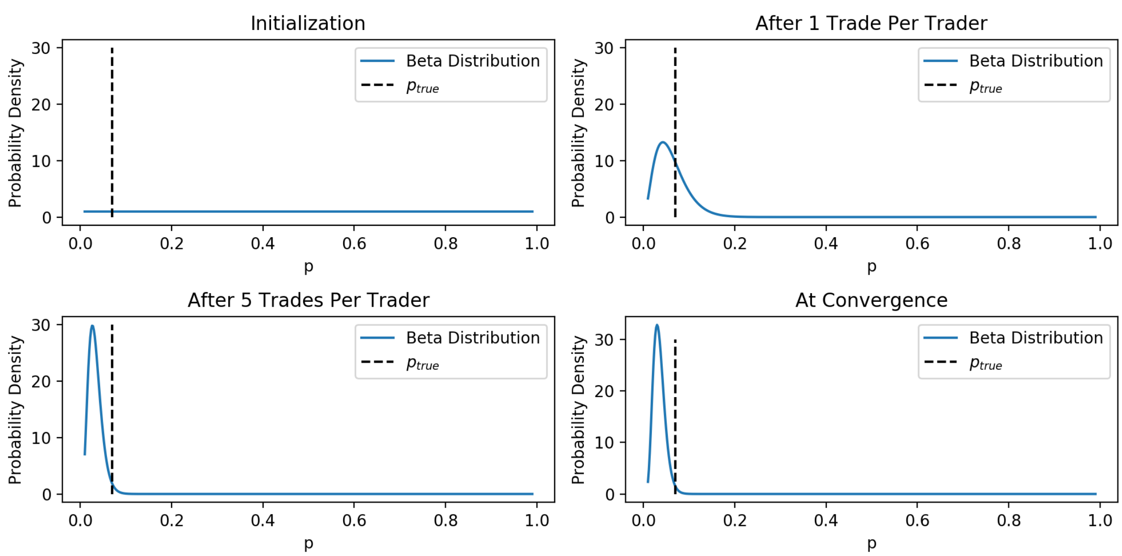
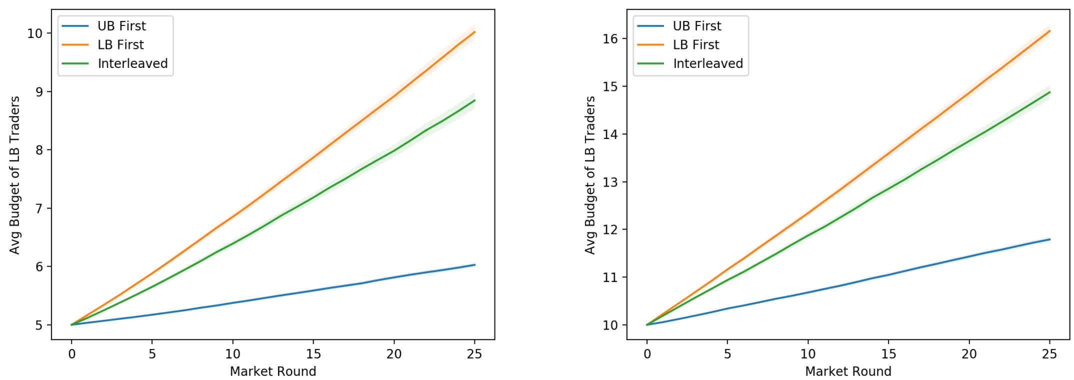
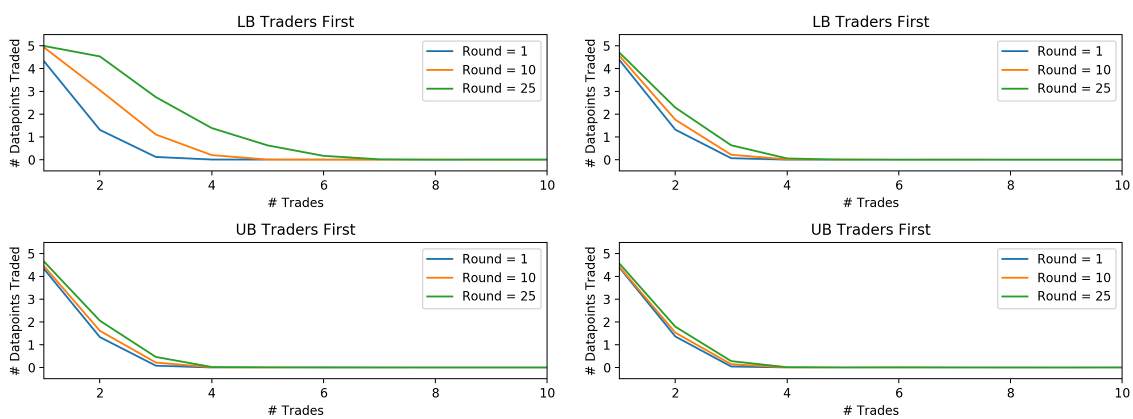


Fig. 5. Evolution of market belief in round 25 for an instance with the highest (worst) value of Δ^{suff} ($\approx$ 0.766) across all 1000 simulations for B_0 = 10 and is **LB First** ordering. Here, p_{true} is close to the lower extreme, and expectation of market belief at convergence is only at a Euclidean distance of 0.035 away from p_{true} towards 0.05.



(a) $B_0 = 5$ (b) $B_0 = 10$

Fig. 6. Change in budget of LB traders with an initial budget of B_0 over rounds $1 - 25$, averaged over 1000 simulations with $p_{true} \sim U[0.05, 0.95]$ for different trader orderings.



(a) First (b) last

Fig. 7. Average number of datapoints used by the (a) first and (b) last trader in the (sub-)sequence of multi-shot LB traders to determine a trade in the market over all 25 rounds over 1000 simulations ($p_{true} \sim \textsc{Uniform}[0.05, 0.95]$) with $B_0 = 10$ for two different trader orderings.

5 CONCLUSION AND FUTURE WORK

We have presented a pilot study on the interplay between various trader attributes under a new market design that seeks to capture the evolving uncertainty in the outcome-generating process. We have gauged the impact that arrival time can have on trader compensation under various conditions. A takeaway from our findings would be that any systematic study of inherent trader attributes in prediction markets must appropriately control for confounding effects of trader ordering. Many design issues still need to be addressed. For example, there can be several alternatives to the way traders in our model handle budgets. Instead of making the (perhaps less satisfactory but computationally tractable) choice of using the first few datapoints, they could trade on the best subset of datapoints that their budget allows, or could solve a constrained optimization problem as in [11]. As far as the arrival of traders is concerned, what differences would be observed if we had a *randomized ordering* over traders or if traders could strategize over arrival time? Overall, a more extensive exploration of the space of experimental conditions is in order to paint a more detailed picture of the parameter space and its impact. Another direction we are pursuing is a theoretical explanation of our observations.

REFERENCES

[1] Jacob Abernethy, Sindhu Kutty, Sébastien Lahaie, and Rahul Sami. 2014. Information aggregation in exponential family markets. In *Proceedings of the 15th ACM conference on Economics and Computation*. 395–412.

[2] Jerry Anunrojwong, Yiling Chen, Bo Waggoner, and Haifeng Xu. 2019. Computing Equilibria of Prediction Markets via Persuasion. In *International Conference on Web and Internet Economics*. Springer, 45–56.

[3] Kenneth J Arrow, Robert Forsythe, Michael Gorham, Robert W Hahn, Robin Hanson, Daniel Kahneman, John O Ledyard, Saul Levmore, Robert Litan, Paul R Milgrom, Forrest D. Nelson, George R. Neumann, Marco Ottaviani, Charles R. Plott, Thomas C. Schelling, Robert J. Shiller, Vernon L. Smith, Erik Snowberg, Shyam Sunder, Cass R. Sunstein, Paul C. Tetlock, Philip E. Tetlock, Hal R. Varian, Justin Wolfers, and Eric Zitzewitz. 2007. Statement on Prediction Markets. *AEI-Brookings Joint Center Related Publication* 7 (2007).

[4] J. Berg, R. Forsythe, F. Nelson, and T. Rietz. 2008. Results from a dozen years of election futures markets research. *Handbook of Experimental Economics Results* 1 (2008), 742–751.

[5] Alina Beygelzimer, John Langford, and David M Pennock. 2012. Learning performance of prediction markets with Kelly bettors. In *Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems-Volume 3*. 1317–1318.

[6] M. Chakraborty, S. Das, A. Lavoie, M. Magdon-Ismail, and Y. Naamad. 2013. Instructor Rating Markets. In *Proceedings of the 27th AAAI Conference on Artificial Intelligence*. 159–165.

[7] Yiling Chen and David M. Pennock. 2007. A utility framework for bounded-loss market makers. In *Proceedings of the 23rd Conference on Uncertainty in Artificial Intelligence*.

[8] Yiling Chen and Jennifer Wortman Vaughan. 2010. A new understanding of prediction markets via no-regret learning. In *Proceedings of the 11th ACM Conference on Electronic Commerce*. 189–198.

[9] Yiling Chen and Bo Waggoner. 2016. Informational substitutes. In *2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)*. IEEE, 239–247.

[10] B. Cowgill and E. Zitzewitz. 2015. Corporate prediction markets: Evidence from Google, Ford, and Firm X. *Review of Economic Studies* 82, 4 (2015), 1309–1341.

[11] Nikhil Devanur, Miroslav Dudík, Zhiyi Huang, and David M Pennock. 2015. Budget constraints in prediction markets. In *Proceedings of the Thirty-First Conference on Uncertainty in Artificial Intelligence*. 238–247.

[12] Persi Diaconis and Donald Ylvisaker. 1979. Conjugate Priors for Exponential Families. *The Annals of Statistics* 7, 2 (1979), 269 – 281. https://doi.org/10.1214/aos/1176344611

[13] Miroslav Dudík, Sébastien Lahaie, Ryan Rogers, and Jennifer Wortman Vaughan. 2017. A decomposition of forecast error in prediction markets. In *Proceedings of the 31st International Conference on Neural Information Processing Systems*. 4374–4383.

[14] Miroslav Dudík, Xintong Wang, David M Pennock, and David M Rothschild. 2021. Log-time Prediction Markets for Interval Securities. *20th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)* (2021).

[15] Rafael Frongillo, Yiling Chen, and Ian Kash. 2015. Elicitation for aggregation. In *Proceedings of the AAAI Conference on Artificial Intelligence*, Vol. 29.

[16] Rafael M Frongillo, Nicolas D Penna, and Mark D Reid. 2012. Interpreting prediction markets: a stochastic approach. In *Advances in Neural Information Processing Systems*. 3266–3274.

[17] Andreas Graefe and J Scott Armstrong. 2011. Comparing face-to-face meetings, nominal groups, Delphi and prediction markets on an estimation task. *International Journal of Forecasting* 27, 1 (2011), 183–195.

[18] R. Hanson. 2003. Combinatorial information market design. *Information Systems Frontiers* 5, 1 (2003), 107–119.

[19] R. Hanson. 2007. Logarithmic Market Scoring Rules for Modular Combinatorial Information Aggregation. *Journal of Prediction Markets* 1, 1 (2007), 3–15.

[20] Peter D Hoff. 2009. *A first course in Bayesian statistical methods*. Vol. 580. Springer.

[21] Krishnamurthy Iyer, Ramesh Johari, and Ciamac C Moallemi. 2014. Information Aggregation and Allocative Efficiency in Smooth Markets. *Management Science* 60, 10 (2014), 2509–2524.

[22] Yuqing Kong and Grant Schoenebeck. 2018. Optimizing Bayesian information revelation strategy in prediction markets: the Alice Bob Alice case. In *9th Innovations in Theoretical Computer Science Conference (ITCS 2018)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik.

[23] Kathryn Blackmond Laskey, Wei Sun, Robin Hanson, Charles Twardy, Shou Matsumoto, and Brandon Goldfedder. 2018. Graphical model market maker for combinatorial prediction markets. *Journal of Artificial Intelligence Research* 63 (2018), 421–460.

[24] Michael Ostrovsky. 2012. Information aggregation in dynamic markets with strategic traders. *Econometrica* 80, 6 (2012), 2595–2647.

[25] D. Pennock and R. Sami. 2007. Computational Aspects of Prediction Markets. In *Algorithmic Game Theory*, N. Nisan, T. Roughgarden, E. Tardos, and V. V. Vazirani (Eds.). Cambridge University Press.

[26] David M Pennock. 1999. *Aggregating probabilistic beliefs: Market mechanisms and graphical representations.* Ph.D. Dissertation. University of Michigan.

[27] Rajiv Sethi and Jennifer Wortman Vaughan. 2016. Belief aggregation with automated market makers. *Computational Economics* 48, 1 (2016), 155–178.

[28] Amos J Storkey, Zhanxing Zhu, and Jinli Hu. 2015. Aggregation Under Bias: Rényi Divergence Aggregation and Its Implementation via Machine Learning Markets. In *Machine Learning and Knowledge Discovery in Databases*. Springer, 560–574.

[29] Martin J. Wainwright and Michael I. Jordan. 2008. Graphical Models, Exponential Families, and Variational Inference. *Foundations and Trends in Machine Learning* 1, 1-2 (2008), 1–305.

[30] Justin Wolfers and Eric Zitzewitz. 2004. Prediction Markets. *Journal of Economic Perspectives* 18, 2 (2004), 107–126.

Appendices

A THEORETICAL ANALYSIS OF COMPENSATION CHARACTERISTICS

Consider an **LI first** sequence of an arbitrary length $2n$, i.e. there are n LI traders and n HI traders in the population with all n LI traders trading first with the market, followed by the n HI traders, once each. The private sample sizes for LI and HI traders are respectively $m = 1$ and $m = N$ (here, we will derive the expression for net compensation for a general N, but $N = 5$ in our experiments). Due to independent sampling from identical Bernoulli distributions, the private sample of each HI trader follows a binomial distribution with parameters N and p_{true}, denoted by $\textsc{Bin}(N, p_{true})$.

Let $x_i^{\mathrm{LI}} \sim_{i.i.d.} \textsc{Bernoulli}(p_{true}$ and $x_i^{\mathrm{HI}} \sim_{i.i.d.} \textsc{Bin}(N, p_{true})$ denote respectively the private samples of LI traders $1, 2, \dots, n$ and HI traders $1, 2, \dots, n$, the numbers corresponding to their arrival order. Define

$$X^{\mathrm{LI}} := \sum_{i=1}^{n} x_i^{\mathrm{LI}} \sim \textsc{Bin}(n, p_{true});$$
$$X^{\mathrm{HI}} := \sum_{i=1}^{n} x_i^{\mathrm{HI}} \sim \textsc{Bin}(Nn, p_{true}).$$

Since the market starts at $\eta = (0, 0)$ and we assume that all traders are myopic and risk-neutral, the incentive-compatibility property of the market algorithm implies that the LI traders jointly move the market from $\eta = (0, 0)$ to $\eta = (X^{\mathrm{LI}}, n - X^{\mathrm{LI}})$, and the HI traders jointly move the market from $\eta = (X^{\mathrm{LI}}, n - X^{\mathrm{LI}})$ to $\eta = (X^{\mathrm{HI}} + X^{\mathrm{LI}}, Nn + n - X^{\mathrm{HI}} - X^{\mathrm{LI}})$. Moreover, since the total ex-post net compensation of the LI traders as well as that of the HI traders is a telescoping sum, the corresponding expected per-trader averages for any realized p_{true} are given by

$$\widehat{R}_n^{\mathrm{LI}}(p_{true}) = \frac{1}{n}\mathbb{E}_{X^{\mathrm{LI}} \sim \textsc{Bin}(n, p_{true})}\left[X^{\mathrm{LI}} \ln p_{true} + (n - X^{\mathrm{LI}}) \ln(1 - p_{true}) - C((X^{\mathrm{LI}}, n - X^{\mathrm{LI}}))\right];$$
$$\begin{aligned}\widehat{R}_n^{\mathrm{HI}}(N, p_{true}) = \frac{1}{n}\mathbb{E}_{X^{\mathrm{LI}} \sim \textsc{Bin}(n, p_{true}), X^{\mathrm{HI}} \sim \textsc{Bin}(Nn, p_{true})}&\Big[X^{\mathrm{HI}} \ln p_{true} + (Nn - X^{\mathrm{HI}}) \ln(1 - p_{true}) \\ &- C((X^{\mathrm{HI}} + X^{\mathrm{LI}}, Nn + n - X^{\mathrm{HI}} - X^{\mathrm{LI}})) + C((X^{\mathrm{LI}}, n - X^{\mathrm{LI}}))\Big],\end{aligned}$$

where $C((\eta_1, \eta_2)) = \ln\Gamma(\eta_1 + 1) + \ln\Gamma(\eta_2 + 1) - \ln\Gamma(\eta_1 + \eta_2 + 2)$. Recall that the amount by which the expected per-trader average net compensation for HI traders exceeds that for LI traders in an **LI first** sequence is denoted by $\Delta_n^{\mathrm{HL}}(N, p_{true}) = \widehat{R}_n^{\mathrm{HI}}(N, p_{true}) - \widehat{R}_n^{\mathrm{LI}}(p_{true})$. On simplification,

$$\begin{aligned}\Delta_n^{\mathrm{HL}}(N, p_{true}) = &- (N - 1)H(p_{true}) + \frac{1}{n}\Big[\ln\Gamma((N + 1)n + 2)) - 2\ln(\Gamma(n + 2)\Big] \\ &+ \frac{2}{n}\sum_{k=0}^{n} B_k^{n, p_{true}}\Big[\ln\Gamma(1 + k) + \ln\Gamma(1 + n - k)\Big] \\ &- \frac{1}{n}\sum_{j=0}^{Nn} B_j^{Nn, p_{true}} \sum_{k=0}^{n} B_k^{n,p}\Big[\ln\Gamma(1 + j + k) + \ln\Gamma((N + 1)n + 1 - (j + k))\Big],\end{aligned}$$

where $H(p) = -p\ln p - (1 - p)\ln(1 - p)$ is the entropy function of a Bernoulli distrbution with parameter p, and $B_i^{M,q} = \binom{M}{i} q^i (1 - q)^{M-i}$ is the binomial probability mass function with parameters M and q evaluated at $i \in \{0, 1, \dots, m\}$.

We used MATLAB to evaluate $\Delta_n^{\mathrm{HL}}(N, p_{true})$ for $N = 5$ and $p_{true} \in \{0.05, 0.10, \dots, 0.95\}$ at $n = 1, 2, 3, \dots$. For each p_{true}, we found that there exists a value of n, denoted by $n^*(p_{true})$, such that $\Delta_n^{\mathrm{HL}}(5, p_{true}) \geq 0$ for every $n < n^*$ and $\Delta_n^{\mathrm{HL}}(N, p_{true}) < 0$ for every $n \geq n^*$. The results are provided in Table 2.

p_{true}	$n^*(p_{true})$	$\Delta^{\mathrm{HL}}_{n^*-1}(5, p_{true}) \times 10^3$	$\Delta^{\mathrm{HL}}_{n^*}(5, p_{true}) \times 10^3$
0.05	3	8.65	−61.02
0.10	3	19.16	−26.45
0.15	5	4.53	−2.97
0.20	11	1.25	−1.17
0.25	15	0.76	−0.91
0.30	18	0.25	−1.00
0.35	20	0.22	−0.83
0.40	21	0.61	−0.38
0.45	22	0.40	−0.50
0.50	22	0.66	−.25
0.55	22	0.40	−0.50
0.60	21	0.61	−0.38
0.65	20	0.22	−0.83
0.70	18	0.25	−1.00
0.75	15	0.76	−0.91
0.80	11	1.25	−1.17
0.85	5	4.53	−2.97
0.90	3	19.16	−26.45
0.95	3	8.65	−61.02

Table 2. The second column gives values of the half-length of the **LI first** sequence at which the LI traders surpass HI traders in terms of expected per-trader average net compensation, corresponding to different values of the "ground truth". The last two columns provide the difference in the above expectations (HI − LI) just before and at the above "switch"; each entry must be multiplied by 10^{-3} to get the correct value. Notice that the values are symmetric about $p_{true} = 0.5$, as expected from the design of the market security and cost function.

B SUPPLEMENTAL FIGURES AND TABLES

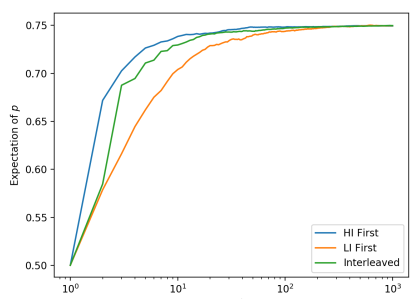


Fig. 8. The market's expectation $\mathbb{E}_\eta[p]$ as a function of the number of completed trades for different trader orderings, averaged 1000 simulations with $p_{true} = 0.75$. Horizontal axis is logarithmic.

Figures 8 and 9 show that with $p_{true} = 0.75$ we see results that are symmetric to using $p_{true} = 0.25$ in Figures 1a and 2a. This confirms the intuition that given initial state $\alpha = \beta = 1$ corresponding to $\mathbb{E}_{\alpha,\beta}[p] = 0.5$, we expect equivalent results but with swapped values of α and β for p_{true} an equal distance above and below 0.5.

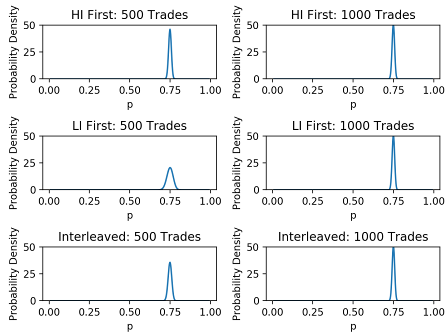


Fig. 9. Posterior distribution (market belief) after 500 and 1000 trades for different trader orderings. This plot reports the average distribution across 1000 simulations with $p_{true} = 0.75$.

Table 3 shows the effects of trader sequence within the LI and HI groups of traders. We see significant differences between the average compensation of the first and last ten traders of the informativeness group that enters the market first. For example, the first ten HI traders can expect 175 times the compensation of the last 10 HI traders when HI traders enter the market first. Likewise, the first ten LI traders can expect 34 times the compensation of the last 10 LI traders when LI traders enter the market first. Disparities between first and last ten traders of the same informativeness are less pronounced for the group that does not enter the market first. Overall, this shows that individual trader compensation is not solely driven by whether LI or HI traders are first, but it is also significantly affected by a trader's position relative to other traders of the same informativeness.

$P(X = 1)$	Sequence	First 10 LI ($\times 10^{-3}$)	First 10 HI ($\times 10^{-3}$)	Final 10 LI ($\times 10^{-3}$)	Final 10 HI ($\times 10^{-3}$)
0.75	High Info First	0.06 ± 0.38	140.45 ± 68.25	0.38 ± 0.36	0.80 ± 0.70
	Low Info First	64.58 ± 16.02	5.44 ± 2.29	1.87 ± 0.89	0.18 ± 0.78
	Interleaved	32.20 ± 21.87	0.53 ± 0.59	-0.08 ± 0.40	0.18 ± 0.78
0.50	High Info First	-0.06 ± 0.34	125.09 ± 48.68	0.36 ± 0.43	0.07 ± 1.19
	Low Info First	53.56 ± 13.89	4.55 ± 2.37	0.93 ± 0.88	0.67 ± 1.31
	Interleaved	18.88 ± 8.10	1.07 ± 1.23	0.13 ± 0.15	0.67 ± 1.31
0.25	High Info First	0.44 ± 0.33	136.69 ± 74.00	-0.08 ± 0.45	1.55 ± 0.76
	Low Info First	64.54 ± 16.39	5.56 ± 1.91	1.02 ± 1.02	1.13 ± 0.90
	Interleaved	29.92 ± 24.06	0.45 ± 0.74	-0.12 ± 0.30	1.13 ± 0.90

Table 3. Compensation of the first and last 10 HI and LI traders for different trader orderings and p_{true} values. This table reports the compensation averaged over all agents of the same type over 1000 simulations with 95% confidence bounds.

We can also see in Figure 10 that for a sufficient number of traders, LI compensation exceeds that of HI traders when LI traders are first. The opposite is true for this market when there are very few traders. The transition, the point where the expected per-trader compensation of HI traders falls below that of LI traders is experimentally found to be at $N = 14$ traders of each type.

In our runs comparing compensation to LI and HI traders with different sequences and unlimited budget, there were two sources of randomness: sequence of sampled data points and sequence of traders. To get a representation

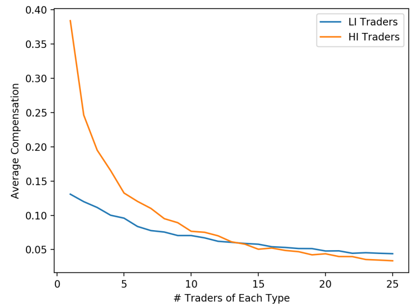


Fig. 10. Average compensation for HI and LI traders for the LI traders first sequence as we vary n, the number of each type of trader in the market. Compensation is averaged over 1000 simulations with $p_{true} = 0.75$.

of expected compensation without being fixed to a predetermined subset of trader sequences, we can average over both sources of randomness. To do this, we uniformly sampled with replacement 100 of the $\binom{1000}{500}$ possible trader sequences of 500 LI and HI traders. Likewise, we sampled 100 sequences of the 3000 datapoints drawn from the true data distribution. With this setup, the trader in position 1 has access to the first k data points, where $k \in 1, 5$ is determined by whether the trader is LI or HI, and subsequent traders see the subsequent data points in the sampled data point sequence. Each data point sequence was then paired with each trader sequence and LI and HI compensations were recorded. Finally, the compensations over these $10,000$ simulations were averaged, yielding average LI compensation of 1.17×10^{-3} and average HI compensation of 5.68×10^{-3} for $p_{true} = 0.75$

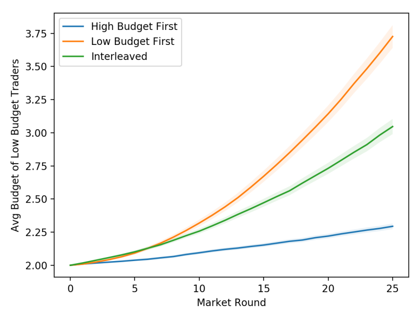


Fig. 11. Change in budget of LB traders with an initial budget of 2 over rounds $1 - 25$, averaged over 1000 simulations with $p_t rue \sim U[0.05, 0.95]$ for different trader orderings.

Figures 11 and 12 provide supplemental budget evolution plots with more extreme initial budgets of LB traders. For a very low initial budget of $B_0 = 2$, we no longer see the linear relationship between budget and round number but instead see that the per-round increase in budget is growing with market round. This is interesting as it suggests that traders are rewarded for contributing information to the market and are able to more significantly affect the market in future rounds. with a very large budget of 100, traders are much closer to the UB case. We unsurprisingly see a linear relationship between budget and market round in this case as the increase in budget each round has little effect on the amount of information traders are able to inject into the market.

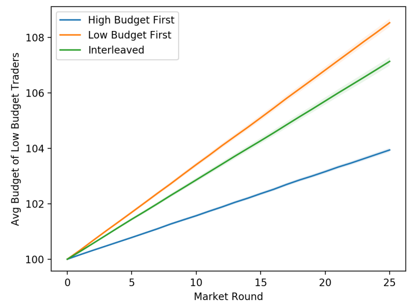


Fig. 12. Change in budget of LB traders with an initial budget of 100 over rounds $1-25$, averaged over 1000 simulations with $p_true \sim U[0.05, 0.95]$ for different trader orderings.