In[1129]:=

```
ClearAll["Global*`"]
```

Define transition terms between road segments

See “explore-transitions-between-road-segments” file for explanations.

In[995]:=

```
capacity = 30;
```

$$\text{quadratic}[r\_] = r\left(1 - \frac{r}{\text{capacity}}\right);$$

In[997]:=

$$\sigma[x\_] = \text{LogisticSigmoid}[x];\ c = \frac{3}{8};$$

$$\text{transition1}[r\_] = a\ \sigma\left[\frac{4}{a} r\right] - \frac{a}{2};$$

$$\text{transition2}[r\_] = -c\ a\ \sigma\left[\frac{4}{a}\ (r - a)\right];$$

```
capacity = 15;
sigmoidsUnsolved[r_] = transition1[r] + transition2[r];
minimizer = Solve[sigmoidsUnsolved'[r] == 0, r, Reals][[1, 1, 2]];
asol = Solve[minimizer == capacity, a][[1, 1]];

sigmoids[r_] = sigmoidsUnsolved[r] /. asol;
```

In[1130]:=

```
b = -3; capacity = 30;
```

$$\text{before}[r\_] := r\left(1 - \frac{r}{\text{capacity}}\right);$$

$$\text{after}[r\_] := \frac{d}{r - a} - b;$$

$$\text{switchtime} = \frac{3}{4}\ \text{capacity};$$

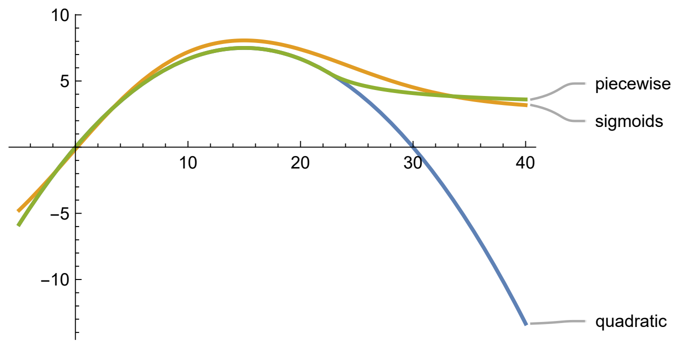
```
zsol = Solve[{before[z] == after[z], before'[z] == after'[z]}, {a, d}][[1]];

piecewise[r_] := before[r] Boole[r ≤ switchtime] + after[r] Boole[switchtime < r] /.
    (zsol /. z → switchtime);
```

In[1227]:=

```
plot = Plot[{quadratic[r], sigmoids[r], piecewise[r]}, {r, plotmin, plotmax},
  PlotLabels → {quadratic, sigmoids, piecewise}]
```

Out[1227]=



Apply Pontryagin’s Maximum Principle (PMP)

In[1136]:=

```
(*Initial conditions*)
n0 = 10; m0 = 10; l0 = 0; q0 = 10;
x0 = {n0, m0, l0, q0};

(*Parameters e.g. rates and costs*)
evoparams = {α → 1, β → 1, γ → 1, δ → 1};
costparams = {Q → 1, L → 1, U → 1};

x[t_] := {n[t], m[t], l[t], q[t]};
f[t_] := {np[t], mp[t], lp[t], qp[t]};
p[t_] := {p1[t], p2[t], p3[t], p4[t]};

ptf = Table[0, Length[p[t]]];(*state endpoints free*)
pp[H_][t_] := -D[H, {x[t]}]
H := p[t].f[t] - LL[t]
```

Quadratic transition

For short lengths of time, the solution works and makes sense...

In[1399]:=

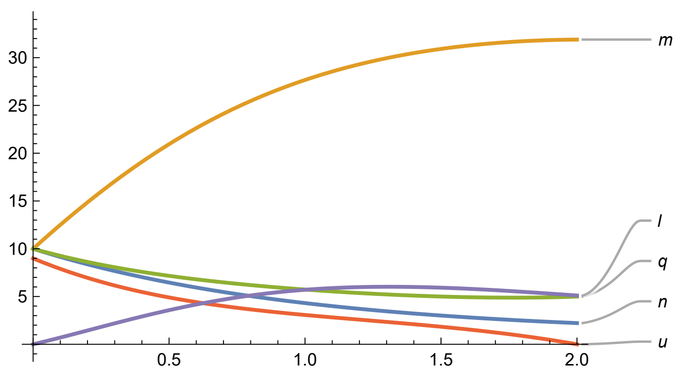
```
np[t_] = α - γ n[t];
mp[t_] = γ n[t] + δ quadratic[m[t]] + u[t];
lp[t_] = δ quadratic[m[t]];
qp[t_] = β - u[t];
LL[t_] = Q q[t]^2 - L l[t]^2 + U u[t]^2;
usol = Solve[D[H, u[t]] == 0, u[t]]

t0 = 0; tf = 2;
sol = NDSolve[
   Thread[Join[x'[t], p'[t], x[t0], p[tf]] ==
        Join[f[t], pp[H][t], x0, ptf]] /. usol /. evoparams /. costparams,
   {m, n, l, q, p1, p2, p3, p4},
   {t, t0, tf}];
Plot[Evaluate[{n[t], m[t], q[t], u[t], l[t]} /. usol /. sol /. costparams], {t, t0, tf},
 AxesOrigin → {0, 0}, PlotLabels → {n, m, q, u, l}]
```

Out[1404]=

$$\left\{\left\{u[t] \to \frac{p2[t] - p4[t]}{2\,U}\right\}\right\}$$

Out[1407]=



But for tf = 3 or 4, the solution stops making sense.

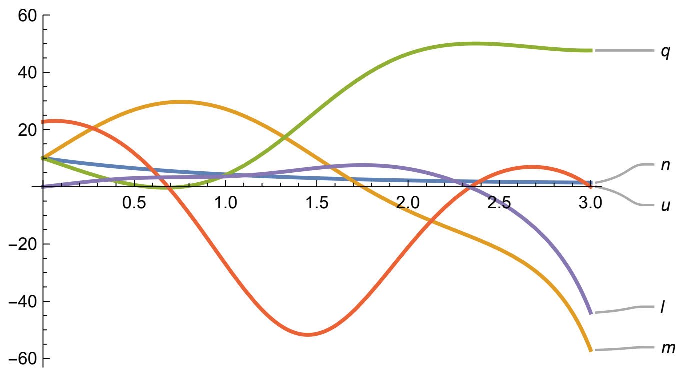
And for tf = 5 and higher, the solution fails entirely.

In[1489]:=

```
np[t_] = α - γ n[t];
mp[t_] = γ n[t] + δ quadratic[m[t]] + u[t];
lp[t_] = δ quadratic[m[t]];
qp[t_] = β - u[t];
LL[t_] = Q q[t]^2 - L l[t]^2 + U u[t]^2;
usol = Solve[D[H, u[t]] == 0, u[t]];

t0 = 0; tf = 3;
sol = NDSolve[
   Thread[Join[x'[t], p'[t], x[t0], p[tf]] ==
        Join[f[t], pp[H][t], x0, ptf]] /. usol /. evoparams /. costparams,
   {m, n, l, q, p1, p2, p3, p4},
   {t, t0, tf}];
Plot[Evaluate[{n[t], m[t], q[t], u[t], l[t]} /. usol /. sol /. costparams], {t, t0, tf},
 AxesOrigin → {0, 0}, PlotLabels → {n, m, q, u, l}]
```

Out[1497]=



Sum of sigmoids transition

This variation takes the longest to solve out of the three in this notebook attempted with PMP.

Again, for small values of tf, the solution makes sense.

In[1516]:=

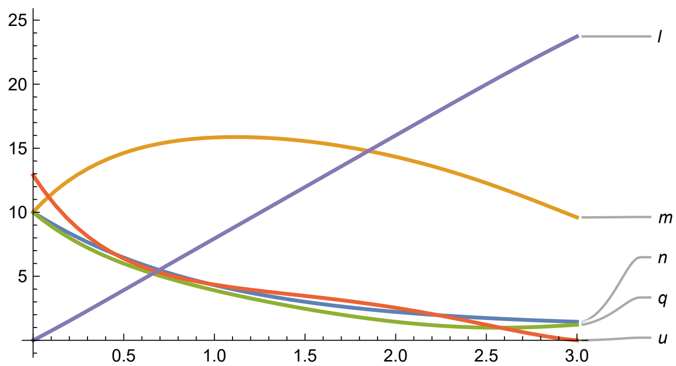
```
np[t_] = α - γ n[t];
mp[t_] = γ n[t] - δ sigmoids[m[t]] + u[t];
lp[t_] = δ sigmoids[m[t]];
qp[t_] = β - u[t];
LL[t_] = Q q[t]^2 - L l[t]^2 + U u[t]^2;
usol = Solve[D[H, u[t]] == 0, u[t]]

t0 = 0; tf = 3;
sol = NDSolve[
   Thread[Join[x'[t], p'[t], x[t0], p[tf]] ==
         Join[f[t], pp[H][t], x0, ptf]] /. asol /. usol /. evoparams /. costparams,
   {m, n, l, q, p1, p2, p3, p4},
   {t, t0, tf}];
Plot[Evaluate[{n[t], m[t], q[t], u[t], l[t]} /. usol /. sol /. costparams], {t, t0, tf},
 AxesOrigin → {0, 0}, PlotLabels → {n, m, q, u, l}]
```

Out[1521]=

$$\left\{\left\{u[t] \to \frac{p2[t] - p4[t]}{2\,U}\right\}\right\}$$

Out[1524]=



But for larger values of tf, the solution again fails.

In[1507]:=

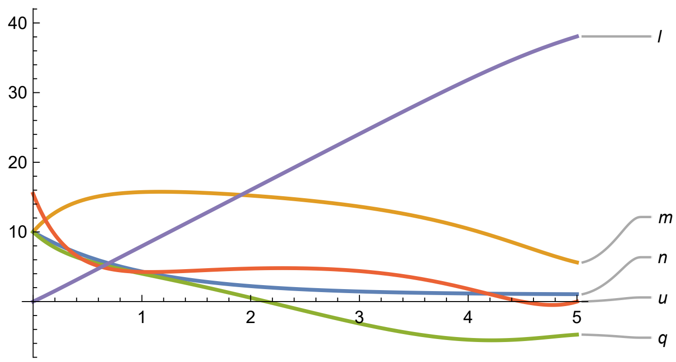
```
np[t_] = α - γ n[t];
mp[t_] = γ n[t] - δ sigmoids[m[t]] + u[t];
lp[t_] = δ sigmoids[m[t]];
qp[t_] = β - u[t];
LL[t_] = Q q[t]^2 - L l[t]^2 + U u[t]^2;
usol = Solve[D[H, u[t]] == 0, u[t]]

t0 = 0; tf = 5;
sol = NDSolve[
   Thread[Join[x'[t], p'[t], x[t0], p[tf]] ==
         Join[f[t], pp[H][t], x0, ptf]] /. asol /. usol /. evoparams /. costparams,
   {m, n, l, q, p1, p2, p3, p4},
   {t, t0, tf}];
Plot[Evaluate[{n[t], m[t], q[t], u[t], l[t]} /. usol /. sol /. costparams], {t, t0, tf},
 AxesOrigin → {0, 0}, PlotLabels → {n, m, q, u, l}]
```

Out[1512]=

$$\left\{\left\{u[t] \to \frac{p2[t] - p4[t]}{2\,U}\right\}\right\}$$

Out[1515]=



Piecewise transition

For short lengths of time, the solution works and makes sense...

In[1372]:=

```
np[t_] = α - γ n[t];
mp[t_] = γ n[t] - δ piecewise[m[t]] + u[t];
lp[t_] = δ piecewise[m[t]];
qp[t_] = β - u[t];
LL[t_] = Q q[t]^2 - L l[t]^2 + U u[t]^2;
usol = Solve[D[H, u[t]] == 0, u[t]][[1, 1]]

t0 = 0; tf = 3;
sol = NDSolve[
   Thread[Join[x'[t], p'[t], x[t0], p[tf]] ==
        Join[f[t], pp[H][t], x0, ptf]] /. usol /. evoparams /. costparams,
   {m, n, l, q, p1, p2, p3, p4},
   {t, t0, tf},
   Method → {"PDEDiscretization" → "FiniteElement"}];
Plot[Evaluate[{n[t], m[t], q[t], u[t], l[t]} /. usol /. sol /. costparams], {t, t0, tf},
 AxesOrigin → {0, 0}, PlotLabels → {n, m, q, u, l}]
```

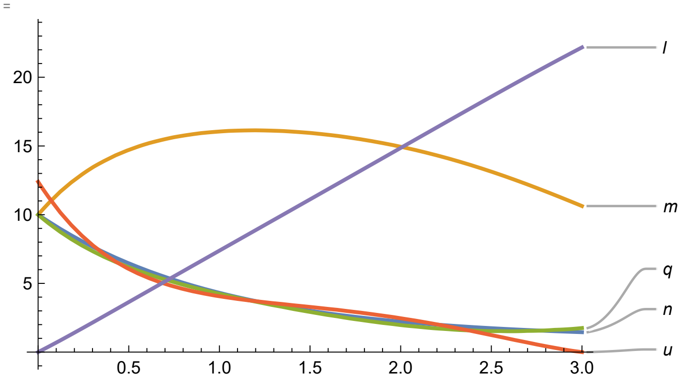
Out[1377]=

$$u[t] \to \frac{p2[t] - p4[t]}{2\,U}$$

NDSolve`FEM`InitializePDECoefficients: The PDE is convection dominated and the result may not be stable. Adding artificial diffusion may help.

NDSolve`FEM`InitializePDECoefficients: The PDE is convection dominated and the result may not be stable. Adding artificial diffusion may help.

Out[1380]=



But for longer lengths of time, we achieve negative values of q using the control, and the control itself becomes negative. It seems some numerical error is also occurring.

Upon setting tf = 10 or higher, the solution entirely fails.

In[1318]:=

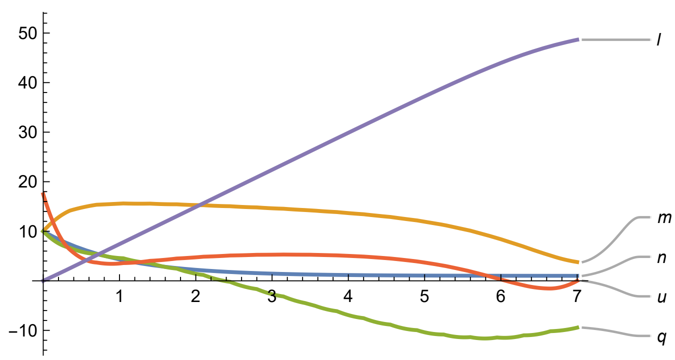
```
np[t_] = α - γ n[t];
mp[t_] = γ n[t] - δ piecewise[m[t]] + u[t];
lp[t_] = δ piecewise[m[t]];
qp[t_] = β - u[t];
LL[t_] = Q q[t]^2 - L l[t]^2 + U u[t]^2;
usol = Solve[D[H, u[t]] == 0, u[t]][[1, 1]];

t0 = 0; tf = 7;
sol = NDSolve[
   Thread[Join[x'[t], p'[t], x[t0], p[tf]] ==
        Join[f[t], pp[H][t], x0, ptf]] /. usol /. evoparams /. costparams,
   {m, n, l, q, p1, p2, p3, p4},
   {t, t0, tf},
   Method → {"PDEDiscretization" → "FiniteElement"}];
Plot[Evaluate[{n[t], m[t], q[t], u[t], l[t]} /. usol /. sol /. costparams], {t, t0, tf},
 AxesOrigin → {0, 0}, PlotLabels → {n, m, q, u, l}]
```

NDSolve`FEM`InitializePDECoefficients: The PDE is convection dominated and the result may not be stable. Adding artificial diffusion may help.

NDSolve`FEM`InitializePDECoefficients: The PDE is convection dominated and the result may not be stable. Adding artificial diffusion may help.

Out[1326]=



Explore variations on *u* in cost and evolution

evolution: u^2
cost: u^2
PMP yields $u = 0$

In[∘]:=
```
np[t_] = α - γ n[t];
mp[t_] = γ n[t] - δ sigmoids[m[t]] + u[t]^2;
lp[t_] = δ sigmoids[m[t]];
qp[t_] = β - u[t]^2;
LL[t_] = Q q[t]^2 - L l[t]^2 + U u[t]^2;
usol = Solve[D[H, u[t]] == 0, u[t]]
```

Out[∘]=
```
{{u[t] → 0}}
```

evolution: u^2
cost: u
Solution fails

In[∘]:=
```
np[t_] = α - γ n[t];
mp[t_] = γ n[t] - δ sigmoids[m[t]] + u[t]^2;
lp[t_] = δ sigmoids[m[t]];
qp[t_] = β - u[t]^2;
LL[t_] = Q q[t]^2 - L l[t]^2 + U u[t];
usol = Solve[D[H, u[t]] == 0, u[t]]
```

Out[∘]=

$$\left\{\left\{u[t] \to \frac{U}{2\,(p2[t] - p4[t])}\right\}\right\}$$

In[∘]:=
```
t0 = 0; tf = 1;
sol = NDSolve[
    Thread[Join[x'[t], p'[t], x[t0], p[tf]] ==
          Join[f[t], pp[H][t], x0, ptf]] /. asol /. usol /. evoparams /. costparams,
    {m, n, l, q, p1, p2, p3, p4},
    {t, t0, tf}];
```

Power: Infinite expression $\frac{1}{0.^2}$ encountered.

evolution: *u*
cost: *u*
Bang-bang

In[∘]:=
```
np[t_] = α - γ n[t];
mp[t_] = γ n[t] - δ sigmoids[m[t]] + u[t];
lp[t_] = δ sigmoids[m[t]];
qp[t_] = β - u[t];
LL[t_] = Q q[t]^2 - L l[t]^2 + U u[t];
usol = Solve[D[H, u[t]] == 0, u[t]]
```
Out[∘]=
```
{}
```

evolution: u^2
cost: no *u*
PMP yields $u = 0$

In[∘]:=
```
np[t_] = α - γ n[t];
mp[t_] = γ n[t] - δ sigmoids[m[t]] + u[t]^2;
lp[t_] = δ sigmoids[m[t]];
qp[t_] = β - u[t]^2;
LL[t_] = Q q[t]^2 - L l[t]^2;
usol = Solve[D[H, u[t]] == 0, u[t]]
```
Out[∘]=
```
{{u[t] → 0}}
```

evolution: *u*
cost: no *u*
Bang-bang

In[∘]:=
```
np[t_] = α - γ n[t];
mp[t_] = γ n[t] - δ sigmoids[m[t]] + u[t];
lp[t_] = δ sigmoids[m[t]];
qp[t_] = β - u[t];
LL[t_] = Q q[t]^2 - L l[t]^2;
usol = Solve[D[H, u[t]] == 0, u[t]]
```
Out[∘]=
```
{}
```