Johns Hopkins
Engineering for Professionals

605.767 Applied Computer Graphics

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Module 6D
B-Splines

B-spline Curves

- B-splines are another common class of spline curves
- 2 major advantages over Bezier curves
  - Allow local control over the shape of the curve or surface
    - Bezier curves: control points most heavily influence the part of the curve closest to it but they also have some effect on the rest of the curve
  - Degree of the polynomial can be set independently
    - Does not depend on the number of control points as in Bezier curves
    - Cannot use Bezier curve to approximate n points without either increasing the degree of the curve or using multiple curve segments
- Disadvantage: more complex than Bezier curves

B-spline Curves (cont.)

- Expression for calculating coordinate positions along a B-spline curve using blending function B:

$$p(t) = \sum_{i=0}^{n} B_{i,d}(t) p_i \qquad t_{\min} \le t \le t_{\max}, 2 \le d \le n+1$$

  - p_i are the set of n+1 control points
- Blending functions $B_{i,d}$ are polynomials of degree d
  - 2 <= d <= n+1 (number of control points)
- Local control is achieved by defining the blending functions over d **subintervals** of the total range of t
  - The selected set of endpoints t_j is referred to as the **knot vector**
  - Can choose any value for subinterval endpoints
    - Non decreasing: $t_j <= t_{j+1}$
  - t_{min} and t_{max} depend on number of control points, the value of d, and how we set up the knot vector

B-spline Curve Properties

- Polynomial curve has degree d-1 and exhibits C^{d-2} continuity over the range of t
- For n+1 control points the curve is described with n+1 blending functions
- Each blending function $B_{i,d}$ is defined over d subintervals of the range of t, starting at knot value t_i
- Range of t is divided into n+d subintervals by the n+d+1 values of the knot vector
- With knot values $\{t_0, t_1, \ldots, t_{n+d}\}$ the B-spline curve is defined only in the interval from knot value t_{d-1} up to knot value t_{n+1}
- Each section of the spline curve between two successive knot values is influenced by d control points

B-spline Curve Properties (cont)

- One control point can affect the shape of at most d curve sections
  - Exhibits local control
- B-spline curves follow the shape of the control polygon and are constrained within the convex hull formed by the control points
  - Like Bezier curves
- This is derived from the fact that for any value of t in the interval t_{d-1} to t_{n+1}, the sum of all blending functions is 1

$$\sum B_{i,d}(t) = 1$$

- Curve is transformed by applying combination of linear transformations to its control point representation

Cox-deBoor Recursive Functions

- B-spline blending functions can be defined by the **Cox-deBoor recursive formula**
  - Able to generate uniform or non-uniform B-splines of any degree using a single recursive formula

$$B_{i,1}(t) = \begin{cases} 1 & t_i \leq t < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$B_{i,d}(t) = \frac{t - t_i}{t_{i+d-1} - t_i} B_{i,d-1}(t) + \frac{t_{i+d} - t}{t_{i+d} - t_{i+1}} B_{i+1,d-1}(t)$$

Uniform B-splines

- Uniform B-splines are formed by specifying a constant spacing between knot values
  - Often knot values are normalized to the range [0, 1]
    - { 0.0, 0.2, 0.4, 0.6, 0.8, 1.0 }
  - Sometimes set uniform knot values starting at 0 with an interval of 1
    - { 0, 1, 2, 3, 4, 5, 6, 7 }
- Uniform B-splines have **periodic** blending functions
  - For given values of n and d all blending functions have the same shape
  - Each successive blending function is a shifted version of the previous:

$$B_{i,d}(t) = B_{i+1,d}(t + \Delta t) = B_{i+2,d}(t + 2\Delta t)$$

    - where Δt is the knot interval

Cubic, Uniform B-splines

- Cubic, uniform B-splines are commonly used in computer graphics
  - Useful for generating certain closed curves
- Cubics
  - d = 4: blending function is described over 4 subintervals
    - Each blending function spans four subintervals of the range of t
    - Polynomial degree is 3
  - Have C^2 curve continuity
  - $$p(t) = \sum_{k=0}^{3} B_{i-3+k}(t) p_{i-3+k}$$
- Blending functions for t normalized to the range 0 to 1:
  - $B_i = 1/6\, t^3$
  - $B_{i-1} = 1/6(-3t^3 + 3t^2 + 3t + 1)$
  - $B_{i-2} = 1/6(3t^3 - 6t^2 + 4)$
  - $B_{i-3} = 1/6(1 - t)^3$

Cubic B-splines (cont.)

- Cubic B-spline curve is a series of n-2 curve segments
  - Label Q_3, Q_4, ...,Q_n
  - Determined from a set of n+1 control points, p_0, p_1, ..., p_n where ($n >= 3$)
    - Q_3 is defined by $p_0p_1p_2p_3$ which are scaled by $B_0B_1B_2B_3$
    - Q_4 is defined by $p_1p_2p_3p_4$ which are scaled by $B_1B_2B_3B_4$
    - Q_5 is defined by $p_2p_3p_4p_5$ which are scaled by $B_2B_3B_4B_5$
  - Each curve segment is defined by 4 control points
  - Each control point influences 4 curve segments
- Curve does not interpolate (pass through) any of the control points
  - Can force the curve to interpolate control points by repeating control points
    - Increases the influence of the control point on the curve
    - Leads to loss of continuity (may not be serious at the end points)