NATURAL DENSITY AND THE PRIME NUMBER THEOREM

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ABSTRACT. We discuss the density of primes and other interesting subsets of the natural numbers. Further, we discuss the Prime Number Theorem, its equivalent form, and how it is related to analytic functions such as the Riemann Zeta function. Finally, we formulate a sketch of the proof of the Prime Number Theorem and conclude by discussing Riemann's formula and approximation of the prime counting function.

CONTENTS

1. INTRODUCTION

We begin with an inquiry that may seem natural to those studying number theory, and specifically prime numbers. How often do prime numbers occur?

This may seem like a simple question at first, but it requires quite a journey to answer. There are a few different ways to answer this question. One involves the density of the primes, which is a way to quantify the proportion of primes to the natural numbers. Another way could be to define a function that outputs the number of primes below a positive real number, and to try to describe the properties of that function as best we can. We will discuss both in detail here; however, we will begin with the first idea.

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2. Density

Let us generalize our question in order to understand it more. What is the density of a certain set A in another set S containing A? In other words, how "often" do the elements in A appear in S?

If A and S are both finite, then the answer might just simply be the intuitive expression

$$\frac{|A \cap S|}{|S|}.$$

However, if A and S are infinite, this question does not make much sense, since we cannot simply divide the number of elements in A by the number of elements in S. However, we can get close by using limits to approximate the density below a certain natural number n.

In this paper, we will denote by $\mathbb{P}$ the set of all positive primes, and by $\mathcal{I}_n$ the discrete interval $\{1, \ldots, n\} = \mathbb{N} \cap [1, n]$.

Definition 2.1. Suppose that A and S are subsets of $\mathbb{N}$. The *relative density* of A in S, when it exists, is defined as

$$\delta_S(A) = \lim_{n \to \infty} \frac{A_n}{S_n},$$

where $A_n = |A \cap S \cap \mathcal{I}_n|$ and $S_n = |S \cap \mathcal{I}_n|$.

Example 2.2. Let $A = 6\mathbb{N}$ and $S = 2\mathbb{N}$ be the positive multiples of 6 and the positive evens, respectively. The relative density of A in S is $\delta_S(A) = 1/3$ since every third even number is divisible by 6.

In the special case that $S = \mathbb{N}$ this becomes the so-called natural density of A. When discussing natural density, we use the notation $A^{\complement} = \mathbb{N} \setminus A$. This is because $\mathbb{N}$ is acting as the universal set in which complements live.

Definition 2.3 ([5])**.** Let $A \subseteq \mathbb{N}$. The *natural density* of A, when it exists, is defined as

$$\delta(A) = \lim_{n \to \infty} \frac{A_n}{n}$$

where $A_n = |A \cap \mathcal{I}_n|$.

With this in mind, we can state some easily obtainable results that accompany this definition.

Proposition 2.4. *Let $A \subseteq \mathbb{N}$. If $\delta(A)$ exists, then $\delta(A^{\complement}) = 1 - \delta(A)$.*

Proof. Note that

$$A_n^{\complement} = |A^{\complement} \cap \mathcal{I}_n| = n - A_n.$$

Thus we have

$$\delta(A^{\complement}) = \lim_{n\to\infty} \frac{A_n^{\complement}}{n} = \lim_{n\to\infty} \frac{n - A_n}{n} = \lim_{n\to\infty} \frac{n}{n} - \delta(A) = 1 - \delta(A)$$

as desired. ∎

Corollary 2.5. *We have* $\delta(\mathbb{N}) = 1$.

Proof. This follows directly from the fact that $\mathbb{N}^{\complement} = \emptyset$. ∎

Corollary 2.6. *For any* $A \subseteq \mathbb{N}$*, we have* $0 \leq \delta(A) \leq 1$.

This all makes intuitive sense. We would certainly expect the density of the set of natural numbers in itself to be 1; we would also expect that any subset of $\mathbb{N}$ would have density from 0 to 1, inclusive.

Let us give some simple examples of densities of sets that are familiar. First, we state a lemma that comes in handy a lot when dealing with analysis. However, we will not prove this lemma as it is a standard exercise in many analysis textbooks.

Lemma 2.7 (Squeeze Theorem [1]). *Let* $(x_n), (y_n)$ *and* (z_n) *be sequences satisfying* $x_n \leq y_n \leq z_n$ *for all* $n \in \mathbb{N}$ *and suppose that* $\lim x_n = \lim y_n = L$. *Then* $\lim z_n = L$ *as well.*

Proposition 2.8. *The density of perfect squares is zero.*

Proof. Let $A = \{k^2 \mid k \in \mathbb{N}\}$. Given $n \in \mathbb{N}$, we have

$$A_n = |\{k^2 \mid k \leq n\}| = |\{k \mid k \leq \sqrt{n}\}| = \lfloor \sqrt{n} \rfloor$$

so

$$0 \leq \lim_{n\to\infty} \frac{A_n}{n} = \lim_{n\to\infty} \frac{\lfloor \sqrt{n} \rfloor}{n} \leq \lim_{n\to\infty} \frac{\sqrt{n}}{n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

and $\delta(A) = 0$ by the Squeeze Theorem. ∎

Proposition 2.9. *The density of even and odd positive integers are both* $1/2$. *That is,*

$$\delta(2\mathbb{N}) = \delta(2\mathbb{N} + 1) = \frac{1}{2}.$$

Proof. Fix $n \in \mathbb{N}$. There are two cases. If n is even, we have

$$(2\mathbb{N})_n = |\{2, \ldots, n, \ldots\} \cap \mathcal{I}_n| = |\{2, \ldots, n\}| = \frac{n}{2}.$$

Similarly, if n is odd, then

$$(2\mathbb{N})_n = |\{2, \dots, n-1\}| = \frac{n-1}{2}.$$

Thus, we can bound $\delta(2\mathbb{N})$ by

$$\lim_{n\to\infty} \frac{n-1}{2n} \le \delta(2\mathbb{N}) \le \lim_{n\to\infty} \frac{n}{2n}.$$

However, note that

$$\lim_{n\to\infty} \frac{n-1}{2n} = \lim_{n\to\infty} \frac{n}{2n} = \frac{1}{2}$$

so $\delta(2\mathbb{N}) = 1/2$ by the Squeeze Theorem.

The fact that $\delta(2\mathbb{N}+1) = 1/2$ follows directly by noting that $(2\mathbb{N})^{\complement} = 2\mathbb{N}+1$. ■

Proposition 2.10. *Any finite set has density zero.*

Proof. Let $A \subseteq \mathbb{N}$ be finite and note that

$$A_n = |A \cap \mathcal{I}_n| \le |A|.$$

Thus we have

$$0 \le \lim_{n\to\infty} \frac{A_n}{n} \le \lim_{n\to\infty} \frac{|A|}{n} = |A| \lim_{n\to\infty} \frac{1}{n} = 0$$

so $\delta(A) = 0$ by the Squeeze Theorem. ■

Another interesting fact that we will not prove here is as follows. Recall that a positive integer n is called a *perfect number* if the sum of all proper divisors of n is n itself. We have not found very many perfect numbers (in fact, it is unknown whether or not there are infinitely many perfect numbers). Thus, perhaps unsurprisingly, we have the following, although the proof of this statement is not easy.

Theorem 2.11 (Kanold [11])**.** *The density of the set of perfect numbers is zero.*

Another interesting note about natural density are the following two facts, which was proven in 2018 by Faisant, et. al. These are interesting because they prove existence of sets that have certain natural densities, instead of proving densities about specific sets.

Theorem 2.12 (Faisant, et. al [8])**.** *Let $\alpha \in [0,1]$ and suppose that $B \subset \mathbb{N}$ is finite. Then there exists a set $A \subset \mathbb{N}$ such that $\delta(A+B) = \alpha$.*

Theorem 2.13 (Faisant, et. al [8])**.** *Let $\alpha \in [0,1]$ and $n \ge 2$. Then there is a subset $A \subset \mathbb{N}$ such that for every $k \in \mathcal{I}_n$, we have $\delta(kA) = k\alpha/n$.*

Now, restating our original question about primes in terms of density, the question we wish to answer is as follows:

Question 2.14. What is $\delta(\mathbb{P})$?

One might immediately observe that

$$\delta(\mathbb{P}) \leq 1/2,$$

which follows from the fact that the only even prime is 2.

It turns out that the density of primes is indeed zero; this essentially means that the primes are rare, which makes intuitive sense since "most" positive integers are composite.

Theorem 2.15 (Kanold, 1954 [11])**.** *The density of primes is zero. That is,* $\delta(\mathbb{P}) = 0$*.*

Proof. We first show, following [7], that the equality

$$\prod_{p \leq n,\, p \text{ prime}} p \leq 4^{n-1}$$

holds for all $n \in \mathbb{N}$ using induction. We assume the reader is familiar with binomial coefficients. Clearly, the equality holds for $n = 1$ since

$$\prod_{p \leq 2} p = 0 \leq 1.$$

Now, suppose that the equality holds for all integers less than or equal to n. If $n + 1$ is even, then there is nothing to prove; hence, suppose that $n + 1 = 2k + 1$ for some $k \in \mathbb{N}$.

Let p be a prime such that $k + 1 < p \leq 2k + 1$. Then p divides $(k + 2) \cdots (2k + 1)$ but not $k!$. Thus p divides the numerator but not the denominator of

$$\frac{(k+2)\cdots(2k+1)}{k!} = \binom{2k+1}{k}.$$

This implies that

$$\prod_{k+1<p\leq 2k+1} p \leq \binom{2k+1}{k}.$$

Since for $k \geq 1$ we have

$$2\binom{2k+1}{k} = \binom{2k+1}{k} + \binom{2k+1}{k+1} < 2^{2k+1},$$

we know that

$$\binom{2k+1}{k} < 2^{2k}.$$

Hence, by the inductive assumption, we have

$$\prod_{p\le 2k+1} p = \prod_{p\le k+1} p \prod_{k+1<p\le 2k+1} p < 4^k \binom{2k+1}{k} < 4^k \cdot 2^{2k} = 4^{2k}$$

as desired.

Now we will show that the density of primes is zero. Observe that

$$\prod_{p\le n} p \le 4^{n-1} \implies \log\left(\prod_{p\le n} p\right) \le (n-1)\log(4) < n\log(4)$$

so we have, using logarithm rules,

$$\sum_{p\le n} \log(p) < n\log(4).$$

Now, note that

$$\begin{aligned}
|\mathbb{P}\cap I_n| &= \sum_{p\le n} 1 \\
&= \sum_{p\le n} \log(p)\frac{1}{\log(p)} \\
&= \sum_{p\le\sqrt{n}} \log(p)\frac{1}{\log(p)} + \sum_{\sqrt{n}<p\le n} \log(p)\frac{1}{\log(p)} \\
&\le \sum_{p\le\sqrt{n}} \log(p)\frac{1}{\log(2)} + \sum_{\sqrt{n}<p\le n} \log(p)\frac{1}{\log(\sqrt{n})} \\
&= \frac{1}{\log(2)}\sum_{p\le\sqrt{n}} \log(p) + \frac{1}{\log(\sqrt{n})}\sum_{\sqrt{n}<p\le n} \log(p) \\
&\le \frac{1}{\log(2)}\sum_{p\le\sqrt{n}} \log(p) + \frac{1}{\log(\sqrt{n})}\sum_{p\le n} \log(p).
\end{aligned}$$

Using the fact above, we obtain

$$|\mathbb{P}\cap I_n| \le \frac{\sqrt{n}\log(4)}{\log(2)} + \frac{n\log(4)}{\log(\sqrt{n})} = 2\sqrt{n} + \frac{4n\log(2)}{\log(n)}.$$

This implies that

$$0 \le \frac{|\mathbb{P}\cap I_n|}{n} \le \frac{2\sqrt{n}}{n} + \frac{4\log(2)}{\log(n)}.$$

Since it is easily seen that

$$\lim_{n\to\infty}\left[\frac{2\sqrt{n}}{n} + \frac{4\log(2)}{\log(n)}\right] = 0,$$

the claim follows from the Squeeze Theorem. ∎

3. The Prime Number Theorem

We have seen that the density of primes is zero. We can interpret this as saying that the proportion of primes to natural numbers is zero, or equivalently, if we choose a natural number, the probability of that number being prime is zero.

It turns out that we can say more. For more than 100 years, mathematicians have been studying the so-called prime counting function, which "counts" the number of primes below a certain positive real number. In studying this function, it turns out that we can answer our previous question in a more precise way. Although density does tell us something about the nature of these numbers and how often they appear, we turn to the more precise question: in what *pattern* do they appear, or what is the *distribution* that they follow?

Definition 3.1 ([4])**.** The *prime counting function* $\pi : \mathbb{R}^+ \to \mathbb{N}$ is defined as the number of primes less than or equal to a given positive real number x:

$$\pi(x) := |\mathbb{P} \cap [1, x]| = |\{p \in \mathbb{P} \mid p \le x\}| = \sum_{p \le x} 1.$$

This leads to the question that we now wish to answer: can we find a closed formula for $\pi(x)$ that is meaningful? What else can we say about the function in general, and can it give us a description of what the nth prime will be?

Let us first introduce some notation.

Definition 3.2 ([4])**.** Let $f(x)$ and $g(x)$ be real-valued functions. We say that the function $f(x)$ is *asymptotically equivalent* to $g(x)$, and write $f(x) \sim g(x)$, if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

If (x_n) and (y_n) are sequences, this definition is the same; we write $x_n \sim y_n$ if

$$\lim_{x\to\infty} \frac{x_n}{y_n} = 1.$$

Now, we state the main theorem that has been the result of hundreds of years of work.

Theorem 3.3 (Prime Number Theorem)**.** *We have*

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log(x)} = 1$$

where $\log(x)$ *denotes the natural logarithm. Using asymptotic notation,* $\pi(x) \sim x/\log(x)$*.*

An immediate consequence is the following.

Theorem 3.4. *Let p_n be the nth prime. Then $p_n \sim n\log(n)$.*

Proof. We follow [2]. Clearly, $\pi(p_n) = n$ so the Prime Number Theorem implies that

$$n \sim \frac{p_n}{\log(p_n)}.$$

Therefore, $n\log(p_n) \sim p_n$ and

$$\log(n) \sim \log\left(\frac{p_n}{\log(p_n)}\right) = \log(p_n) - \log\log(p_n).$$

This implies that

$$\frac{\log(n)}{\log(p_n)} \sim \frac{\log(p_n) - \log\log(p_n)}{\log(p_n)} = 1 - \frac{\log\log(p_n)}{\log(p_n)},$$

or equivalently, since $p_n \to \infty$ as $n \to \infty$,

$$\lim_{n\to\infty} \frac{\log(n)}{\log(p_n)} = 1 - \lim_{p_n\to\infty} \frac{\log\log(p_n)}{\log(p_n)} = 1 - \lim_{p_n\to\infty} \frac{1/(p_n\log(p_n))}{1/p_n} = 1 - \lim_{p_n\to\infty} \frac{1}{\log(p_n)} = 1$$

using L'Hopital's rule in the second equality. Therefore, we have $\log(n) \sim \log(p_n)$ and hence

$$p_n \sim n\log(p_n) \sim n\log(n)$$

as claimed. ∎

The Prime Number Theorem is essentially a stronger version of the statement $\delta(\mathbb{P}) = 0$. In fact, one can use the PNT and some analysis in order to easily prove this.

Theorem 3.5. *The Prime Number Theorem implies that the density of primes is zero.*

Proof. Note that $|\mathbb{P} \cap \mathcal{I}_n| = \pi(n)$. By the Prime Number Theorem, one has

$$\lim_{n\to\infty} \frac{\pi(n)\log(n)}{n} = 1.$$

In particular, there is some $N \in \mathbb{N}$ large enough and some bound $b > 1$ so that for all $n \geq N$, we have

$$\frac{\pi(n)\log(n)}{n} \leq b \iff \frac{\pi(n)}{n} \leq \frac{b}{\log(n)}.$$

Then

$$0 \leq \lim_{n\to\infty} \frac{\pi(n)}{n} \leq \lim_{n\to\infty} \frac{b}{\log(n)} = 0,$$

which implies that $\delta(\mathbb{P}) = 0$ by the Squeeze Theorem. ∎

3.1. The Logarithmic Integral Function.

Definition 3.6 ([2])**.** For $x > 1$, define the *Eulerian Logarithmic Integral function* as

$$\mathrm{Li}(x) := \int_2^x \frac{\mathrm{d}t}{\log(t)}.$$

Theorem 3.7 ([4])**.** *The Prime Number Theorem is equivalent to* $\pi(x) \sim \mathrm{Li}(x)$.

Proof. Clearly, we have

$$\lim_{x\to\infty} \frac{x}{\log(x)} = \infty.$$

Since $1/\log(t) > 1/t$ for all $t > 1$, and

$$\lim_{x\to\infty} \int_2^x \frac{\mathrm{d}t}{t} = \lim_{x\to\infty} \log(2/t) = \infty,$$

we also conclude that

$$\lim_{x\to\infty} \mathrm{Li}(x) = \infty.$$

Therefore, the limit

$$\lim_{x\to\infty} \frac{\mathrm{Li}(x)}{x/\log(x)}$$

is indeterminant; by L'Hopital,

$$\lim_{x\to\infty} \frac{\mathrm{Li}(x)}{x/\log(x)} = \lim_{x\to\infty} \frac{1/\log(x)}{(\log(x)-1)/\log^2(x)} = \lim_{x\to\infty} \frac{\log(x)}{\log(x)-1} = 1.$$

Therefore,

$$\mathrm{Li}(x) \sim \frac{x}{\log(x)},$$

from which the claim follows. ■

This is quite a remarkable observation. As x gets large, the number of primes below x gets close to $\mathrm{Li}(x)$, a function that at first glance would not seem to have anything to do with prime numbers. However, it turns out that the approximation for $\mathrm{Li}(x)$ gives us a very noticeable relationship between the two functions.

This is a slightly stronger statement than the original prime number theorem, since $\mathrm{Li}(x)$ is slightly larger than $x/\log(x)$: we will not prove this here, but it is a fact that [12]

$$\mathrm{Li}(x) \sim \frac{x}{\log(x)} \sum_{k=0}^{\infty} \frac{k!}{\log^k(x)} = \frac{x}{\log(x)} + \frac{x}{\log^2(x)} + \frac{2x}{\log^3(x)} + \cdots$$

from which we can see that the higher-ordered terms give us a larger value, one which is closer to $\pi(x)$ as x gets bigger.

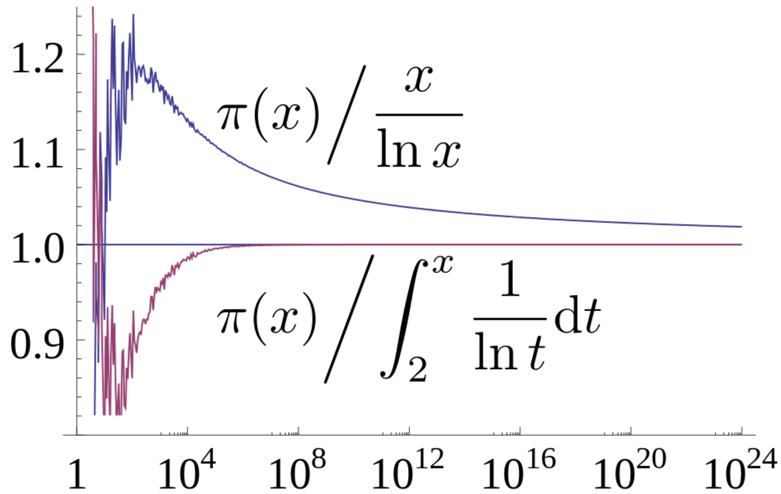

FIGURE 1. [13] Comparison of $x/\log(x)$and $\mathrm{Li}(x)$ as approximations for the prime counting function $\pi(x)$. The x-axis is on logarithmic scale, which allows us to see that $\mathrm{Li}(x)$ is a better approximation as x gets large.

x	$\lvert\pi(x) - \mathrm{Li}(x)\rvert$	$\lvert\pi(x) - x/\log(x)\rvert$
10^{10}	$\approx 3 \times 10^3$	$\approx 2 \times 10^7$
10^{15}	$\approx 1 \times 10^6$	$\approx 9 \times 10^{11}$
10^{20}	$\approx 2 \times 10^8$	$\approx 5 \times 10^{16}$

TABLE 1. Table of differences between $\pi(x)$ and $\mathrm{Li}(x)$ versus $x/\log(x)$. As x gets larger and larger, $\mathrm{Li}(x)$ becomes comparitively a much better approximation than $x/\log(x)$.

This uncovers the reality of natural density that we discussed eariler. The density of primes being zero gives us a way to quantify $\pi(x)$ in some way, which was not clear to us before. We saw that since

$$\lim_{n\to\infty} \frac{\pi(n)}{n} = 0,$$

the function $n \mapsto n$ grows faster than $n \mapsto \pi(n)$. This gave us a numerical way to say that the primes are rare. However, now we have two excellent approximations to $\pi(x)$, namely $x/\log(x)$ and $\mathrm{Li}(x)$. Since

$$\lim_{n\to\infty} \frac{\pi(n)}{n/\log(n)} = \lim_{n\to\infty} \frac{\pi(n)}{\mathrm{Li}(n)} = 1,$$

the functions $n \mapsto n/\log(n)$ and $n \mapsto \mathrm{Li}(n)$ grow at around the same rate as $n \mapsto \pi(n)$. As a consequence, these functions are better approximations to the prime counting function, and we can see now that the density of primes being zero was not the strongest statement that we could make; there was more to be said that was "covered up".

We can think about the comparison of the approximations $x/\log(x)$ and $\mathrm{Li}(x)$ above in the same sort of way. Even though $\pi(x)$ is asymptotically equivalent to both of these functions, there is something more to be said about $\mathrm{Li}(x)$ than $x/\log(x)$, namely that it is a much better approximation.

3.2. **Proof Sketch of the Prime Number Theorem.** Before we begin the sketch, we must define yet again another important function:

Definition 3.8. Given $x \in \mathbb{R}^+$, define

$$\theta(x) = \sum_{p \leq x,\ p \text{ prime}} \log(p).$$

We can immediately see the relation between $\theta(x)$ and $\pi(x)$, since

$$\pi(x) = \sum_{p \leq x} 1,$$

so we would expect $\theta(x)$ to be on the order of $\log(x)$ times as large as $\pi(x)$. This turns out to be correct, and useful in the proof we are about to turn to.

Now to begin a proof sketch for the Prime Number Theorem. Since this is a sketch, we will not provide all relevant details.

Theorem 3.9 ([4]). *The Prime Number Theorem is equivalent to the statement $\theta(x) \sim x$.*

We omit the proof of this theorem here. We will now prove this equivalent statement.

Theorem 3.10 ([14]). *We have $\theta(x) \sim x$.*

Proof. We follow [9] assuming that the reader is familiar with limsup and liminf. Observe that if the integral

$$\int_1^\infty \frac{\theta(t) - t}{t^2}\, \mathrm{d}t$$

converges, then

$$\lim_{x \to \infty} I(x) = \lim_{x \to \infty} \int_x^\infty \frac{\theta(t) - t}{t^2}\, \mathrm{d}t = 0.$$

It can be shown using other analytic techniques that the first integral does in fact converge, but we will omit the proof here. Now, observe that the claim is equivalent to

$$\lim_{x \to \infty} \frac{\theta(x)}{x} = 1$$

which follows from

$$\limsup_{x\to\infty} \frac{\theta(x)}{x} \le 1 \quad \text{and} \quad \liminf_{x\to\infty} \frac{\theta(x)}{x} \ge 1.$$

Suppose to the contrary that the first condition does not hold. Then there is some $\lambda > 1$ for which

$$\limsup_{x\to\infty} \frac{\theta(x)}{x} > \lambda > 1.$$

Note that since $\lambda > 0$, we have

$$\lim_{x\to\infty} I(x) = 0 \implies \lim_{x\to\infty} I(\lambda x) = 0 \implies \lim_{x\to\infty} (I(\lambda x) - I(x)) = 0$$

and so

$$(*) \qquad \liminf_{x\to\infty} (I(\lambda x) - I(x)) = 0.$$

Going back to our assumption, this means that there are arbitrarily large values of x for which

$$\frac{\theta(x)}{x} > \lambda \iff \theta(x) > \lambda x.$$

Observe that $\theta(x)$ is increasing. Hence, for these values of x, we have

$$I(\lambda x) - I(x) = \int_x^{\lambda x} \frac{\theta(t) - t}{t^2}\, \mathrm{d}t \ge \int_x^{\lambda x} \frac{\lambda x - t}{t^2}\, \mathrm{d}t.$$

Evaluating this last integral, we obtain

$$\int_x^{\lambda x} \frac{\lambda x - t}{t^2}\, \mathrm{d}t = \left[-\frac{\lambda x}{t} - \log(t)\right]_x^{\lambda x} = -1 - \log(\lambda x) - (-\lambda - \log(x)) = \lambda - 1 - \log(\lambda) > 0.$$

Thus,

$$\liminf_{x\to\infty} (I(\lambda x) - I(x)) > 0,$$

contradicting $(*)$. Thus we have

$$\limsup_{x\to\infty} \frac{\theta(x)}{x} \le 1.$$

The proof that

$$\liminf_{x\to\infty} \frac{\theta(x)}{x} \ge 1$$

is similar. These two facts imply that

$$\lim_{x\to\infty} \frac{\theta(x)}{x} = 1,$$

as desired. ■

This concludes the sketch of the proof of the Prime Number Theorem.

4. Riemann's Formula

Now that we have seen some approximations for $\pi(x)$, one might now ask: is there a meaningful closed form for $\pi(x)$?

It turns out that the answer is yes, and it is highly nontrivial. In 1859, Riemann proposed a closed formula that gives a description of $\pi(x)$ as related to other functions we have been discussing thus far. Before we discuss this formula, we need to define a few more functions:

Definition 4.1 (Möbius Function [4])**.** Let $n \in \mathbb{N}$ and denote k by the number of prime factors of n. The *Möbius function* $\mu : \mathbb{N} \to \mathbb{N}$ is defined as follows:

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is squarefree;} \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4.2 (Riemann Zeta Function)**.** Let $s \in \mathbb{C}$. For $\Re(s) > 1$, define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

We now state Riemann's analytic formula.

Theorem 4.3 (Riemann's Formula [3])**.** *Let p be a positive prime and*

$$\Pi(x) = \sum_{p^k \leq x} \frac{1}{k}.$$

We then have

$$\Pi(x) = \mathrm{Li}(x) - \sum_{\rho} \mathrm{Li}(x^{\rho}),$$

where ρ indexes over all nontrivial zeros of $\zeta(s)$. As a consequence of the fact that

$$\Pi(x) = \sum_{k=1}^{\infty} \frac{\pi(x^{1/k})}{k},$$

we also have

$$\pi(x) = \mathrm{R}(x) - \sum_{\rho} \mathrm{R}(x^{\rho}) \quad \text{where} \quad \mathrm{R}(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{Li}(x^{1/n}).$$

Clearly, this formula is very complicated. To see what's going on here, Hutama [10] implemented Riemann's Formula in Python, which shows how the sum in the explicit formula gets more and more accurate with the help of the Riemann Zeta function.

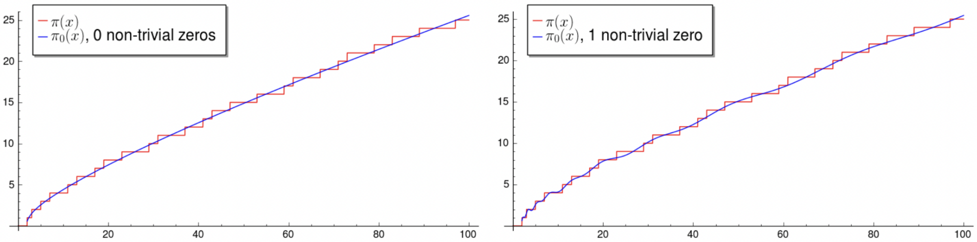

Figure 6: **Left**: $\pi_0(x)$ with 0 non-trivial zeros of the zeta function.
Right: $\pi_0(x)$ with the first non-trivial zero of the zeta function.

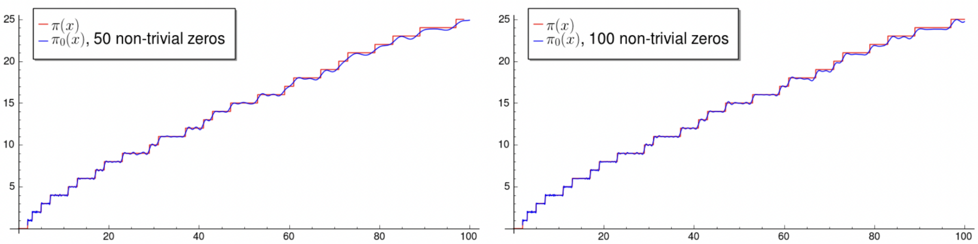

Figure 9: **Left**: $\pi_0(x)$ with the first 50 non-trivial zeros of the zeta function.
Right: $\pi_0(x)$ with the first 100 non-trivial zeros of the zeta function.

(Note: In the above figure, $\pi_0(x)$ denotes the approximation for $\pi(x)$ using the specified number of nontrivial zeros.)

As we include more zeros ρ, the right-hand side gets closer to the exact value of $\pi(x)$. Theoretically, in the limit that ρ hits every zero of $\zeta(s)$, this formula is exact [3]. Note that in order to evaluate the formula for $\pi_0(x)$ accurately enough, one must assume the Riemann Hypothesis, which is precisely what Hutama did here.

Before concluding, we mention one more theorem that connects the Riemann Zeta function to the evaluation of $\pi(x)$.

Theorem 4.4 ([6])**.** *The Riemann Hypothesis (that is, that all nontrivial zeros ρ of $\zeta(s)$ lie on the line $1/2 + it$ for $t \in \mathbb{R}$), is equivalent to the statement that for every $\varepsilon > 0$ the relative error in the approximation $\pi(x) \sim \mathrm{Li}(x)$ is less than*

$$\frac{1}{x^{1/2+\varepsilon}}$$

for sufficiently large x.

5. Conclusion

As of now, our original question has mostly been answered: how often do primes occur? We have seen that there are multiple ways to answer this question. First, we discussed how the natural density of primes was zero, which can be interpreted as that the primes are "rare" or are a "small" subset of $\mathbb{N}$. Next, we discussed the prime counting function. We talked about the Prime Number Theorem and a (more accurate) equivalent form that relates to the function $\mathrm{Li}(x)$. In doing so, we observed that the natural density of primes being zero was indeed not as strong of a statement. Finally, we concluded by discussing Riemann's closed form, which gives us an exact formula for $\pi(x)$ in terms of some non-elementary functions such as $\mu(x)$, $\zeta(s)$ and $\mathrm{Li}(x)$. This showed us that the Riemann Zeta function has important implications to the study of prime numbers and the approximation of $\pi(x)$ given a certain number of nontrivial zeros.

Now that we have adequately explored the prime counting function, there is still yet another way to answer our original question, namely, by considering gaps between primes. This is a very large topic in and of itself, and its exploration will allow us to gain even more insight as to how the primes are spread out.

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