Lattices, Fixed Points, and the Knaster-Tarski Theorem

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Abstract

We discuss the Knaster-Tarski fixed point theorem for complete lattices. Additionally, we look at some examples of partially ordered sets, lattices, and complete lattices, as well as some corollaries of the Knaster-Tarski theorem and other related results. Finally, we conclude by discussing improvements of the theorem, other related fixed point theorems, and applications; in particular, we focus on the special case of fixed points of closed rectangles in $\mathbb{R}^n$.

1 Introduction

We begin with some motivation for our discussion of dynamical systems.

Definition 1.1. (Dynamical System [7]). A *dynamical system* is a pair (X, f) where X is a set and $f : X \to X$ is a selfmap on X. The function f is called the *feedback function.*

Definition 1.2. (Equilibrium Solution [7]). Let (X, f) be a dynamical system. Given a point $x_0 \in X$, the sequence $(x_0, x_1, \ldots)$ defined by $x_n = f^n(x_0)$ is called a *solution* of the dynamical system. If $f(x_0) = x_0$, we say that x_0 is a *fixed point* of f, and the solution $(x_0, x_0, \ldots)$ is called an *equilbrium solution* of (X, f).

As we see, a dynamical system is a very general definition. A selfmap f on any set can be taken to form a dynamical system. Thus, it becomes natural to ask the question: when does the feedback function f have a fixed point? In other words, when does (X, f) have an equilibrium solution?

This is an essential problem in determining the behavior of dynamical systems. Sometimes, it is only possible to say very little about the system and any information that can be deduced is useful. As important as this question is to dynamical systems, the question of when a selfmap on a set is a fundamental question in general, and turns out to be a massive field, with many celebrated fixed point theorems being discovered over all areas of mathematics.

In this paper, we discuss one of the most celebrated of all: the Knaster-Tarski fixed point theorem, which is a result in order theory that deals with complete lattices. As it turns out, there are many complete lattices that are familiar to us, which allows us to apply Knaster-Tarski in a way that covers up the complicated details of order theory.

To begin, we will introduce some elementary order theory, including the concepts of partially ordered sets, lattices, and complete lattices.

2 Lattices

We begin with partially ordered sets. This idea simply generalizes the intuitive concept of the ordering of the elements in a set.

Definition 2.1. (Partially Ordered Set [1]). Let A be a nonempty set. A relation $\preccurlyeq$ on A is said to be a *partial order relation* on A if the following conditions hold for all $x, y, z \in A$:

1. $x \preccurlyeq x$;
2. if $x \preccurlyeq y$ and $y \preccurlyeq x$, then $x = y$;
3. if $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \preccurlyeq z$.

If A is equipped with a partial order $\preccurlyeq$, then $(A, \preccurlyeq)$ is said to be a *partially ordered set* or *poset.*

It is important to note that the notation $x \preccurlyeq y$ can be used interchangeably with $y \succcurlyeq x$; we will use this at certain places in this paper.

There are many common sets that are partially ordered; here we give some examples.

Example 2.2. The set of real numbers $\mathbb{R}$ equipped with the relation $\leq$ is a partially ordered set.

This is probably the most well-recognized partially ordered set, since $\mathbb{R}$ is the space we are all used to working in, and we can easily see that the relation $\leq$ satisfies all three conditions on $\mathbb{R}$.

However, there are other partially ordered sets that are less obvious or common. An example is the following.

Example 2.3. If A is any nonempty set, the power set of A, denoted $\mathcal{P}(A)$, equipped with the set inclusion relation $\subseteq$, is a partially ordered set.

This is slightly less intuitive than $\mathbb{R}$, but one can see without considerable effort that the relation $\subseteq$ satisfies all three conditions to be a partial order.

Next, we introduce the generalized concept of bounds on a partially ordered set.

Definition 2.4. ([2]). Let S be a subset of a poset $(A, \preccurlyeq)$. An element $x \in A$ is called an *upper bound* for S if $s \preccurlyeq x$ for all $s \in S$. Similarly, x is a *lower bound* for S if $s \succcurlyeq x$ for all $s \in S$.

Definition 2.5. (Join and Meet [2]). Let S be a subset of a poset $(A, \preccurlyeq)$. An element $x \in A$ is called the *supremum*, or *join* of S, denoted $\sup S = \bigvee S$, if both of the following hold:

1. x is an upper bound for S;
2. if u is an upper bound for S, then $x \preccurlyeq u$.

Similarly, x is called the *infimum*, or *meet* for S, denoted $\inf S = \bigwedge S$, if both of the following hold:

1. x is a lower bound for S;
2. if l is a lower bound for S, then $l \preccurlyeq x$.

As seen, the join and meet are generalized versions of the standard supremum and infimum we are used to working with in $\mathbb{R}$. With this under our belt, we can define an structure called a lattice.

Definition 2.6. (Lattice [1]). Let $(\mathcal{L}, \preccurlyeq)$ be a partially ordered set. If every two-element subset $\{x, y\} \subseteq \mathcal{L}$ has a join and a meet, then $\mathcal{L}$ is known as a *lattice*. If *every* subset $S \subseteq \mathcal{L}$ has a join and a meet, then we say that $\mathcal{L}$ is *complete*.

The concept of a lattice may seem abstract, but it turns out that many common sets are lattices.

Example 2.7. The unit interval $[0, 1]$ equipped with the usual ordering $\leq$ and the supremum and infimum defined on $\mathbb{R}$ is a complete lattice. We will revisit this example later and generalize it.

We can visualize lattices with the help of *Hasse Diagrams*, which are used to represent partially ordered sets. In the diagram, dots represent elements of the poset and lines represent two elements being ordered with respect to one another.

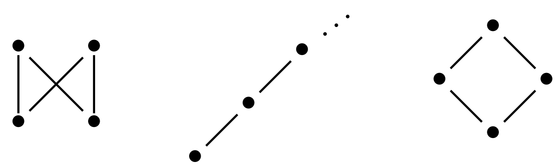

Figure 1: Hasse Digrams of partially ordered sets. The leftmost set is a poset, but not a lattice; the middle is a non-complete lattice; and the right is a complete lattice.

In the above diagram, the left poset is not a complete lattice. This can be seen easily by noting that the top two elements do not have a supremum. The middle poset is a lattice, since every two elements have a supremum and an infimum, but it is not complete, because the poset itself does not have a supremum. Finally, the right poset is a complete lattice, since every subset of these four elements has a supremum and an infimum.

Example 2.8. The non-negative integers $\mathbb{Z}^+$ equipped with the division relation $\mid$ is a complete lattice. The infimum of this lattice is $\bigwedge \mathbb{Z}^+ = 1$; it is a lower bound, since $1 \mid x$ for all $x \in \mathbb{Z}^+$, and it is the greatest lower bound since it is the only lower bound (no other $b \in \mathbb{Z}^+$ satisfies $b \mid x$ for all $x \in \mathbb{Z}^+$). Note that 1 is also the element with the least number of divisors, namely one.

The supremum of this lattice is $\bigvee \mathbb{Z}^+ = 0$; it is an upper bound since $x \mid 0$ for all $x \in \mathbb{Z}^+$, and it is the least upper bound since it is the only upper bound (no other $b \in \mathbb{Z}^+$ satisfies $x \mid b$ for all $x \in \mathbb{Z}^+$). Note that 0 also has the greatest number of divisors, namely an infinite number of them.

As we have seen, there are some sets that would not normally be complete lattices that can be made complete by using interesting and less-obvious partial orders. If $\mathbb{Z}^+$ uses the normal ordering $\leq$, it is not complete (as shown in the middle Hasse diagram above, $\sup \mathbb{Z}^+$ does not exist), but if we endow $\mathbb{Z}^+$ with the division relation, we see that it becomes complete.

This problem vanishes when considering finite sets, however.

Proposition 2.9. *Every nonempty finite lattice is complete.*

Proof. Let $\mathcal{L}$ be a nonempty finite lattice of size n. We will show by induction that every k-element subset of $\mathcal{L}$ has a sup and inf for every $2 \leq k \leq n$. The base case of $k = 2$ follows from the fact that $\mathcal{L}$ is a lattice. Now suppose that the claim holds for some $k \leq n-1$; that is, every k-element subset of $\mathcal{L}$ has a sup and inf. Consider $X = \{x_1, \ldots, x_{k+1}\} \subseteq \mathcal{L}$. By the induction step, $\sup\{x_1, \ldots, x_k\}$ exists. Set $S = \{\sup\{x_1, \ldots, x_k\}, x_{k+1}\}$ and put $s = \sup S$. Then s is an upper bound for x_{k+1} and $\sup\{x_1, \ldots, x_k\}$, so in particular it is an upper bound for $x_1, \ldots, x_k$. Hence s is an upper bound for X. Now let u be an upper bound for X. Then $x_1, \ldots, x_{k+1} \preccurlyeq u$, so $\sup\{x_1, \ldots, x_k\} \preccurlyeq u$ which implies that u is an upper bound for S. Hence $s \preccurlyeq u$, so $s = \sup X$, as desired.

This implies that $\sup \mathcal{L}$ and $\inf \mathcal{L}$ exist, so vacuously $\sup \emptyset = \inf \mathcal{L}$ and $\inf \emptyset = \sup \mathcal{L}$. Finally, any singleton $\{x\} \subset \mathcal{L}$ has $\sup\{x\} = \inf\{x\} = x$. We have shown that $\sup X$ exists for every $X \subseteq \mathcal{L}$. the construction of $\inf X$ is similar, which completes the proof. ∎

Example 2.10. Recall that $(\mathbb{R}, \leq)$ is a partially ordered set. It is true that $\mathbb{R}$ is a lattice, since every $\{x, y\} \subset \mathbb{R}$ with $x < y$ has $\bigvee\{x, y\} = y$ and $\bigwedge\{x, y\} = x$. However, it is *not true* that $\mathbb{R}$ is a complete lattice. For example, $\bigvee \mathbb{R}$ does not exist since $\mathbb{R}$ is not bounded above.

This brings us to the following:

Proposition 2.11. ([1]). *Every complete lattice has a minimum and maximum. In particular, every complete lattice is bounded.*

Proof. This follows directly from the fact that if $\mathcal{L}$ is a complete lattice, then $\bigvee \mathcal{L}$ and $\bigwedge \mathcal{L}$ exist and bound $\mathcal{L}$ above and below, respectively. ∎

Proposition 2.12. *If A is any set, then $(\mathcal{P}(A), \subseteq)$ is a complete lattice.*

Proof. Recall that $(\mathcal{P}(A), \subseteq)$ is a partially ordered set. It remains to show that any subset of $\mathcal{P}(A)$ has a join and a meet. Clearly $\bigvee \mathcal{P}(A) = \max \mathcal{P}(A) = A$ and $\bigwedge \mathcal{P}(A) = \min \mathcal{P}(A) = \emptyset$. Now suppose that $X \subset \mathcal{P}(A)$. Let B be an upper bound for X; then $Y \subseteq B$ for all $Y \in X$. This implies that

$$J = \bigcup_{Y \in X} Y \subseteq B.$$

Since J is an upper bound for X, it follows that $J = \bigvee X$. The argument is similar to show that

$$\bigcap_{Y \in X} Y = \bigwedge X.$$

This completes the proof. ∎

Definition 2.13. (Lexicographic Order). Given a collection of posets $(A_1, \preccurlyeq_1), \ldots, (A_n \preccurlyeq_n)$, the *lexicographic ordering* $\rtimes$ on $A_1 \times \cdots \times A_n$ is given by $(x_1, \ldots, x_n) \rtimes (y_1, \ldots, y_n)$ if there is some $k \in \{1, \ldots, n\}$ such that $x_k \prec_k y_k$ and $x_i = y_i$ for all $i < k$.

Here we use the notation $x \prec y$ to mean that $x \preccurlyeq y$ but $x \neq y$.

Definition 2.14. (Componentwise Order). Given a collection of posets $(A_1, \preccurlyeq_1), \ldots, (A_n \preccurlyeq_n)$, the *componentwise ordering* $\leq_c$ on $A_1 \times \cdots \times A_n$ is given by $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$ if $x_k \preccurlyeq_k y_k$ for all $k \in \{1, \ldots, n\}$.

Theorem 2.15. *Suppose that L and M are complete lattices. Then $L \times M$, equipped with either the componentwise or lexicographic order, is also a complete lattice.*

Proof. First, equip $L \times M$ with the componentwise order. Let $X \subseteq L \times M$ and define

$$\begin{aligned} \ell &= \sup\{x \in L \mid (x, y) \in X\} \in L \\ m &= \sup\{y \in M \mid (x, y) \in X\} \in M. \end{aligned}$$

Since L and M are complete, ℓ and m exist. We claim that $(\ell, m) = \sup X$. It is clear that (ℓ, m) is an upper bound for X by construction. Now let (x, y) be an upper bound for X. Then given any $(z, w) \in X$, we have $(z, w) \leq_c (x, y)$, or equivalently, $z \preccurlyeq x$ and $y \preccurlyeq w$. Since $(\ell, m) \in L \times M$, this implies that $\ell \preccurlyeq x$ and $m \preccurlyeq y$, so $(\ell, m) \leq_c (x, y)$ as desired. The proof that $\inf X$ exists is similar.

Now equip $L \times M$ with the lexicographic order. Let $X \subseteq L \times M$ and (re)define

$$\begin{aligned} \ell &= \sup X_1 = \sup\{x \in L \mid (x, y) \in X\} \in L \\ m &= \inf X_2 = \inf\{b \in M \mid (\ell, b) \text{ is an upper bound for } X\} \in M. \end{aligned}$$

We claim that $(\ell, m) = \sup X$. Choose any $(x, y) \in X$. Since $x \in X_1$, we have $x \preccurlyeq \ell$. If $x \prec \ell$, then we are done. Otherwise, $x = \ell$, so $(x, y) = (\ell, y)$. Hence, for any $b \in X_2$, we have $(\ell, y) \preccurlyeq (\ell, b)$ which implies that $y \preccurlyeq b$. Thus y is a lower bound for X_2, and since $m = \inf X_2$, we have $y \preccurlyeq m$. Therefore, $(x, y) \rtimes (\ell, m)$ for all $(x, y) \in X$, so (ℓ, m) is an upper bound for X.

Now let (u, v) be an upper bound for X. Then for all $(x, y) \in X$, we have $x \prec u$ or $x = u$ and $y \preccurlyeq v$. Hence $x \preccurlyeq u$ for all $x \in X_1$, and since $\ell = \sup X_1$, we have $\ell \preccurlyeq u$. As before, if $\ell \prec u$ we are done. Otherwise, $\ell = u$, so $(u, v) = (\ell, v)$ is an upper bound for X which means that $m \preccurlyeq v$. Therefore, $(\ell, m) \rtimes (u, v)$ so $(\ell, m) = \sup X$. The proof that $\inf X$ exists is similar. ∎

Corollary 2.16. *Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a closed rectangle in $\mathbb{R}^n$. Then R is a complete lattice under both the componentwise and lexicographic orders.*

Proof. Fix $k \in \{1, \ldots, n\}$ and consider $[a_k, b_k]$. Since every $x \in [a_k, b_k]$ is vacuously an upper and lower bound for $\emptyset$, we have $\bigvee \emptyset = \min[a_k, b_k] = a_k$ and $\bigwedge \emptyset = \max[a_k, b_k] = b_k$. Now let $S \subseteq [a_k, b_k]$ be nonempty. Then S is bounded. By the completeness property of $\mathbb{R}$, this implies that $\bigvee S$ and $\bigwedge S$ exist. Thus $[a_k, b_k]$ is a complete lattice.

We proceed by induction to prove that $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is a complete lattice for each $n \in \mathbb{N}$. The base case of $n = 1$ holds by the previous argument. Now suppose that the claim holds for some $k \in \mathbb{N}$; that is, $R_k = [a_1, b_1] \times \cdots \times [a_k, b_k]$ is a complete lattice. Consider the set

$$([a_1, b_1] \times \cdots \times [a_k, b_k]) \times [a_{k+1}, b_{k+1}] = R_k \times [a_{k+1}, b_{k+1}].$$

By the previous argument, $[a_{k+1}, b_{k+1}]$ is a complete lattice, so by Theorem 2.15, and since the componentwise and lexicographic orders are both associative, $R_k \times [a_{k+1}, b_{k+1}]$ is a complete lattice under both of these orders. This completes the proof. ■

3 A Fixed Point Theorem

Throughout the rest of this paper, we will refer to $\operatorname{fix} f$ as the set of fixed points of a selfmap f.

Definition 3.1. **([1])** Let $(A, \preccurlyeq)$ be a partially ordered set. A function $f : A \to A$ is called *monotone*, or *order-preserving*, if $x \preccurlyeq y$ implies $f(x) \preccurlyeq f(y)$ for all $x, y \in A$.

With this in mind, we now state a theorem of great importance in order theory, and the one that we will be discussing for the rest of this paper.

Theorem 3.2. (The Knaster-Tarski Theorem, 1955 [3]). *Let $(\mathcal{L}, \preccurlyeq)$ be a complete lattice, and suppose that $f : \mathcal{L} \to \mathcal{L}$ is monotone. Then $(\operatorname{fix} f, \preccurlyeq)$ forms a nonempty complete lattice.*

There are a few different proofs of this theorem. Here, we follow the constructions of [5] and [6].

Proof. Let

$$\Gamma = \{x \in \mathcal{L} \mid x \preccurlyeq f(x)\}.$$

Note that $\bigwedge \mathcal{L} \in \Gamma$, so Γ is nonempty. Put $s = \bigvee \Gamma$. Then for every $x \in \Gamma$, we have $x \preccurlyeq s$, which implies that $f(x) \preccurlyeq f(s)$ by monotonicity of f. Also, by construction of x, we have $x \preccurlyeq f(x)$, so $f(s)$ is an upper bound for Γ. Since $s = \bigvee \Gamma$, it follows that $s \preccurlyeq f(s)$, so $s \in \Gamma$. Since f is monotone, we have $f(s) \preccurlyeq f(f(s))$ so $f(s) \in \Gamma$ which means $f(s) = s$. This implies that $\operatorname{fix} f$ is nonempty.

It remains to show that $\operatorname{fix}(f)$ is a complete lattice. It is clear that $s = \max(\operatorname{fix} f)$. Similarly, following the argument above, one can show that

$$\bigwedge \{x \in \mathcal{L} \mid f(x) \preccurlyeq x\} = \min(\operatorname{fix} f).$$

Thus, $\bigvee \operatorname{fix} f$ and $\bigwedge \operatorname{fix} f$ exist. Now, let $X \subset \operatorname{fix} f$ be nonempty. Put $j = \bigvee X$ and consider the set of upper bounds on X:

$$\Omega = \{x \in \mathcal{L} \mid j \preccurlyeq x\}.$$

If $\omega \in \Omega$, then for all $x \in X$, we have $x = f(x) \preccurlyeq f(\omega)$ and since $j \preccurlyeq \omega$, we have $j = f(j) \preccurlyeq f(\omega)$ so $f(\Omega) \subseteq \Omega$. Now, let $g = f|_\Omega$ be the restriction of f to the domain Ω. Since f is monotone, $g : \Omega \to \Omega$ is monotone. Again following the argument above, one can show $\operatorname{fix} g$ has a minimum. Since g maps Ω to Ω, and Ω is the set of upper bounds for X, this minimum must be $\bigvee X$. The construction of $\bigwedge X$ is symmetric, and this completes the proof. ■

The following corollary is encapsulated in the Knaster-Tarski theorem, but it is an extremely important result, so we state it here independently.

Corollary 3.3. *Let $\mathcal{L}$ be a complete lattice. If $f : \mathcal{L} \to \mathcal{L}$ is monotone, it has a fixed point.*

4 Generalizations and Applications

The question of whether this theorem's converse was true was resolved by Anne Davis very soon after Tarski's paper, who proved the following:

Theorem 4.1. (Davis, 1955 [4]). *Let $(\mathcal{L}, \preccurlyeq)$ be a lattice. If every monotone function $f : \mathcal{L} \to \mathcal{L}$ has a fixed point, then $\mathcal{L}$ is complete.*

Putting this together with the Knaster-Tarski Theorem gives us a full characterization of the fixed points of complete lattices.

Theorem 4.2. (Characterization of Complete Lattices [4]). *A lattice $\mathcal{L}$ is complete if and only if every monotone function $f : \mathcal{L} \to \mathcal{L}$ has a fixed point.*

This theorem implies that if there exists a monotone selfmap from a lattice to itself with no fixed point, then the lattice cannot be complete, which is quite interesting.

Example 4.3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x + 1$. Then f has no fixed points, so $\mathbb{R}$ is not complete. This is another (perhaps simpler) way to show that $\mathbb{R}$ is not a complete lattice.

4.1 Dynamical Systems

It is a consequence of the Knaster-Tarski Theorem and Corollary 2.16 that the following holds:

Theorem 4.4. *Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a closed rectangle in $\mathbb{R}^n$, and let $f : R \to R$ be monotone with respect to the componentwise or lexicographic order. Then f has a fixed point.*

Corollary 4.5. *If f is monotone, then the dynamical system (R, f) has an equilibrium solution.*

Example 4.6. Consider the complete lattice $([0,2]^2, \rtimes\!\!\!\ltimes)$ and let $f : [0,2]^2 \to [0,2]^2$ be defined by $f(1,0) = (1,0)$, $f(0,1) = (0,1)$ and for $x \notin \{(1,0),(0,1)\}$,

$$f(x) = \begin{cases} (0,0) & \text{if } x_1 + x_2 \leq 1; \\ (2,2) & \text{if } x_1 + x_2 > 1. \end{cases}$$

Then f has a fixed point; namely,

$$\operatorname{fix} f = \{(0,0),(1,0),(0,1),(2,2)\}.$$

This set forms a complete lattice (this can be easily seen since it is finite).

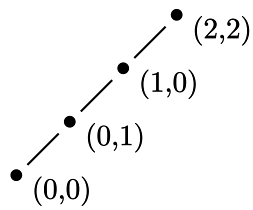

Figure 2: A Hasse diagram of the complete lattice (fix f, $\rtimes$).

Example 4.7. Let $f : [0,1] \to [0,1]$ be monotone. By The Knaster-Tasrski Theorem, f has a fixed point, even if it is discontinuous. The below figure shows an arbitrary discontinuous yet monotone function on the unit interval.

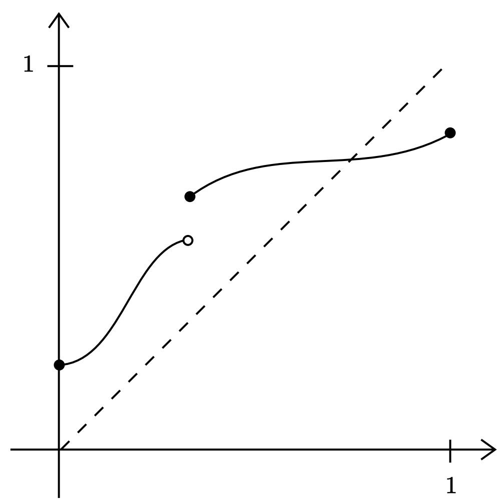

Figure 3: A simple discontinuous, monotone function on the unit interval.

There is something to be said about the geometric nature of this example. If a function on an interval is monotone, *and* maps the interval to itself, then it must cross the $y = x$ line at least once. For example, the monotone function $f(x) = 1 - x$ maps the unit interval to itself, and has the fixed point $f(1/2) = 1/2$. However, the function $g(x) = x - 1$ does not map the unit interval to itself, so it is not "forced" to cross the $y = x$ line, which of course it does not: $x - 1 = x$ has no solution.

Example 4.8. Consider a function f mapping the unit square to itself, i.e. $f : [0,1]^2 \to [0,1]^2$. Recall that $[0,1]^2$ is a complete lattice under the lexicographic partial order. Thus, f is monotone if every $(x,y), (p,q) \in [0,1]^2$ satisfying $(x,y) \rtimes (p,q)$ implies $f(x,y) \rtimes f(p,q)$.

In other words, if $x < p$ or $x = p$ and $y \leq q$, then $f(x) < f(p)$ or $f(x) = f(p)$ and $f(y) \leq f(q)$. Geometrically, this means that f is monotone if f "preserves the distance" between every pair of points in $[0,1]^2$.

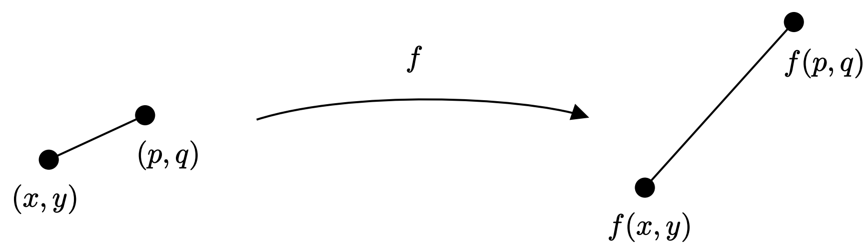

Figure 4: A cartoon of what a monotone function does to points in $[0,1]^2$.

Recall that $[0,1]^2$ is a complete lattice under this ordering, and so the Knaster-Tarski Theorem says that f has a fixed point. This is quite interesting; if we have a selfmap on the unit square that shrinks or stretches the distance between pairs of points a positive amount, then there is at least one point in the space that does not move.

Example 4.9. Consider $f : [0,1]^2 \to [0,1]^2$ given by

$$f(x,y) = \begin{cases} (x, (y+1)/2) & \text{if } x < 1/2; \\ ((x+1)/2, (y+1)/2) & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

We can see that f maps the unit square to itself, since the both output coordinates are always between 0 and 1. If we graph a sort of crude cartoon of where points get sent to under f, we obtain:

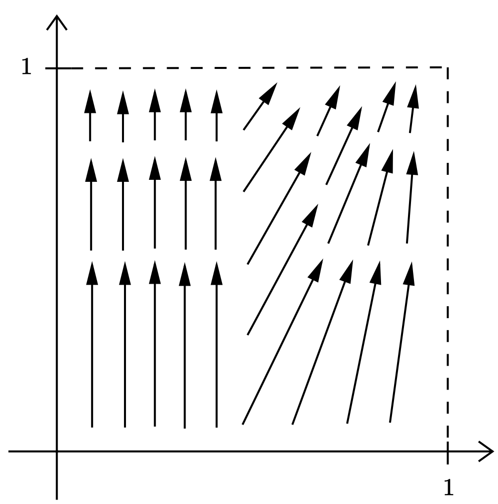

Figure 5: A cartoon of where points get sent to under f.

From this figure, we can see what f does to points in the unit square. If the point's x coordinate is less than $1/2$, the point gets shifted upwards by $(x+1)/2$. If the x coordinate is between $1/2$ and 1, the point gets shifted diagonally, $(x+1)/2$ horizontally and $(y+1)/2$ vertically.

This map is clearly monotone with respect to the lexicographic ordering; if we choose two points in the unit square, (x,y) and (p,q), where $(x,y) \rtimes (p,q)$, then $f(p,q)$ will turn out to be either to the right of or upwards from $f(x,y)$. Thus, by Theorem 4.4, f has a fixed point; indeed, points get too "cramped" near $y = 1$ to move, and it turns out that the fixed points of f are exactly those with x coordinate between 0 and 1/2 and y coordinate equal to 1, and also the point $(1, 1)$, i.e.

$$\operatorname{fix} f = \{(x, 1) \mid 0 \leq x < 1/2\} \cup \{(1, 1)\}.$$

One can work this out algebraically. To see this function in the context of dynamical systems, below are two solutions of $([0, 1]^2, f)$:

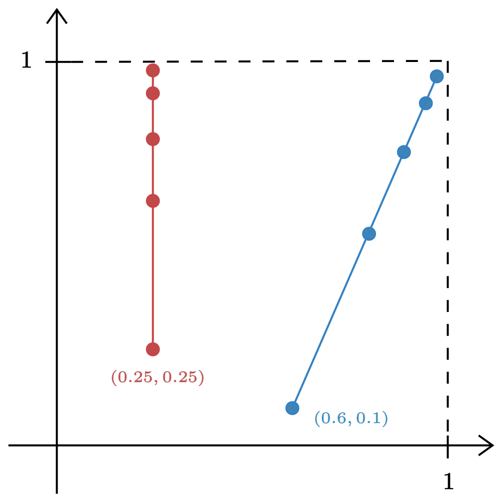

Figure 6: Two solutions of $([0, 1]^2, f)$.

Looking at this figure, it seems as though all solutions of the dynamical system $([0, 1]^2, f)$ will converge to a fixed point of f.

Note that Theorem 4.4 is a somewhat stronger result than Theorem 2.2.14/16 in [7], which states the following:

Theorem 4.10. **([7]).** *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be Lipschitz, and $[a, b] \subseteq A$. The following hold:*

1. *If $f(a) \geq a$ and $f(b) \leq b$, then f has a fixed point on $[a, b]$.*
2. *If $[a, b] \subseteq f([a, b])$, then f has a fixed point on $[a, b]$.*

This theorem is useful as well in determining fixed points of interval maps. However, Knaster-Tarski provides a different, simpler restriction, and is extendable to n dimensions as we saw in Theorem 4.4.

4.2 Other Applications

Tarski's Theorem also helps us prove an important result in set theory:

Theorem 4.11. **(Cantor-Schröder-Bernstein Theorem [1]).** *Let A and B be sets. There exists a bijection $h : A \hookrightarrow\!\!\!\!\rightarrow B$ if and only if there exist injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$.*

Proof. We follow the proof in [1] closely. First suppose that there exists a bijection $h : A \hookrightarrow\!\!\!\!\rightarrow B$; then h is also an injection from A to B. Also, h^{-1} exists and is an injection from B to A.

Now suppose that $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$ are injections. Note that $(\mathcal{P}(A), \subseteq)$ is a complete lattice, as previously shown. For any $S \subseteq A$, define the map $\psi : \mathcal{P}(A) \to \mathcal{P}(A)$ by

$$\psi(X) = A \setminus g(B \setminus f(X)).$$

Suppose that $X \subseteq Y \subseteq A$; then we have the following string of implications:

$$\Big[f(X) \subseteq f(Y)\Big] \Rightarrow \Big[B \setminus f(X) \subseteq B \setminus f(Y)\Big] \Rightarrow \Big[g(B \setminus f(X)) \subseteq g(B \setminus f(Y))\Big]$$
$$\Rightarrow \Big[A \setminus g(B \setminus f(X) \subseteq A \setminus g(B \setminus f(Y)\Big] \Rightarrow \Big[\phi(X) \subseteq \phi(Y)\Big].$$

Thus ψ is monotone. Since $\mathcal{P}(A)$ is a complete lattice, by the Knaster-Tarski Theorem, ψ has a fixed point, i.e. there is a set S such that $\psi(S) = S$. Then $S = A \setminus g(B \setminus f(S))$, so $a \in A \setminus S$ implies $a \in g(B \setminus f(S))$. Now, define $h : A \to B$ as

$$h(x) = \begin{cases} f(x) & \text{if } x \in S; \\ g^{-1}(x) & \text{if } x \notin S \end{cases}$$

where $g^{-1}(x)$ is the left inverse of g. We claim that h is a bijection. First, let $a_1, a_2 \in A$ satisfy $h(a_1) = h(a_2)$. If $a_1, a_2 \in S$, then $h(a_1) = f(a_1) = h(a_2) = f(a_2)$ and since f is injective, $a_1 = a_2$.

If instead, without loss of generality, $a_1 \in S$ but $a_2 \notin S$, then $a_2 \in g(B \setminus f(S))$ so there is an element $b \in g(B \setminus f(S))$ such that $a_2 = h(b)$, and hence $h(a_2) = g^{-1}(a_2) = g^{-1}(g(b)) = b$ so $h(a_2) \notin f(S)$. But since $a_1 \in S$, we have $h(a_2) = f(a_2) \in f(S)$ so this case is impossible.

Finally, if $a_1, a_2 \notin S$, then there are $b_1, b_2 \in g(B \setminus f(S))$ such that $a_1 = h(b_1)$ and $a_2 = h(b_2)$ and so $h(a_1) = g^{-1}(g(b_1)) = b_1$ and $h(a_2) = g^{-1}(g(b_2)) = b_2$. Since $h(a_1) = h(a_2)$, we have $b_1 = b_2$ so $a_1 = g(b_1) = g(b_2) = a_2$. Thus, h is injective.

Now let $b \in B$. If $b \in f(S)$, then there is some $a \in A$ such that $b = f(a)$. Thus $h(a) = b$. If instead $b \notin f(S)$, then $g(b) \in g(B \setminus f(S))$ so $g(b) \notin A \setminus g(B \setminus f(S)) = \psi(S) = S$. Hence $h(g(b)) = g^{-1}(g(b)) = b$ so setting $a = g(b)$ yields an element $a \in A$ such that $h(a) = b$. Thus, h is surjective, which completes the proof. ∎

Another theorem of great importance in fixed point theory is Brouwer's theorem, which has a similar corollary to the Knaster-Tarski theorem.

Theorem 4.12. (Brouwer's Fixed Point Theorem [7]). *Suppose that $S \subseteq \mathbb{R}^n$ is a compact and convex set (that is, given any $x, y \in S$, the straight line segment connecting x and y is in S). If $f : S \to S$ is continuous, then f has a fixed point.*

Corollary 4.13. *Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ be a closed rectangle in $\mathbb{R}^n$. If $f : R \to R$ is continuous, then it has a fixed point.*

Tarski's theorem, as we saw, provides the same result with a different restriction, namely that f is monotone. This restriction is sometimes lighter, and sometimes stronger. Monotonicity and continuity are, of course, not mutually exclusive, but each theorem gives a different perspective.

5 Conclusion

In this paper, we discussed partially ordered sets, lattices, and complete lattices. We saw that some sets that are very familiar to most mathematicians are complete lattices. We also saw that the Knaster-Tarski theorem implies that every selfmap on a complete lattice must have a fixed point. As we saw, this gives us a way to characterize fixed points on dynamical systems. Specifically, we looked at the complete lattice $R = [a_1, b_1] \times \cdots \times [a_n, b_n] \in \mathbb{R}^n$ and concluded that any monotone selfmap on R must have a fixed point, and that the dynamical system (R, f) must have an equilibrium solution if f is monotone. This is a different restriction than what most theorems in the field of dynamical systems usually use, namely that the map f is continuous. One such example is Brouwer's fixed point theorem. Finally, we looked at some implications of the Knaster-Tarski theorem, in particular, that it provides a relatively simple proof of the Cantor-Schröder-Bernstein theorem.

As seen here, the Knaster-Tarski theorem is a celebrated theorem that is widely used in many areas of mathematics. Although it is purely an order-theoretic result, it can be used to prove various theorems in dynamical systems.

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