The Schwarz Lemma on Kähler Manifolds

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Riemann Surfaces

Classical Result

The classical Schwarz lemma states that if $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic function that fixes the origin, then $|f(z)| \leq |z|$ for every $z \in \mathbb{D}$ and $|f'(0)| \leq 1$.

Applying this to $\varphi_{f(z)} \circ f \circ \varphi_z$ where $\varphi_z(w) = \dfrac{z - w}{1 - \overline{w}z}$ yields

Theorem 1 (Schwarz-Pick)

If $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic function, then

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

Conformal Metrics

Let M be a Riemann surface with holomorphic coordinate w.

A conformal metric on M is a Riemannian metric

$$\mathrm{d}s^2 = \lambda^2(w)\,|\mathrm{d}w|^2$$

where $\lambda \in C^\infty(M)$ and $\lambda > 0$. The sectional curvature is given by

$$K_M = -\frac{\Delta \log \lambda}{\lambda^2}.$$

Writing $\mathrm{d}s^2 = E(\mathrm{d}x^2 + \mathrm{d}y^2)$, this coincides with the usual Gaussian curvature

$$K_M = -\frac{\Delta \log E}{2E}.$$

Poincaré Metric

The Poincaré metric on $\mathbb{D}$ is given by

$$\mathrm{d}\sigma^2 = \frac{4}{(1-|z|^2)^2}\,|\mathrm{d}z|^2 = \rho^2(z)\,|\mathrm{d}z|^2.$$

For $0 < R < 1$, the Poincaré metric on $\mathbb{D}_R$ is given by

$$\mathrm{d}\sigma_R^2 = \frac{4R^2}{(R^2-|z|^2)^2}\,|\mathrm{d}z|^2 = \rho_R^2(z)\,|\mathrm{d}z|^2.$$

In other words, $\rho(z) = \dfrac{2}{1-|z|^2}$ and $\rho_R(z) = \dfrac{2R}{R^2-|z|^2}$. We have

$$\Delta \log \rho_R(z) = \rho_R^2, \quad \Delta \log \rho(z) = \rho^2,$$

so both $(K_{\mathbb{D}_R}, \mathrm{d}\sigma_R^2)$ and $(K_{\mathbb{D}}, \mathrm{d}\sigma^2)$ have constant sectional curvature -1.

Furthermore, note that $\rho_R \to \rho$ as $R \to 1$.

Pullback Reformulation

Let $f\colon \mathbb{D} \to \mathbb{D}$ be holomorphic. Then $f^*\mathrm{d}\sigma^2 = \rho^2(f(z))\,|f'(z)|^2\,|\mathrm{d}z|^2$, so

$$f^*\mathrm{d}\sigma^2 \leq \mathrm{d}\sigma^2 \iff \rho(f(z))|f'(z)| \leq \rho(z) \iff \frac{2|f'(z)|}{1-|f(z)|^2} \leq \frac{2}{1-|z|^2}.$$

We can reformulate the Schwarz-Pick lemma as follows:

Theorem 2 (Schwarz-Pick)

If $f\colon \mathbb{D} \to \mathbb{D}$ is a holomorphic mapping, then $f^*\mathrm{d}\sigma^2 \leq \mathrm{d}\sigma^2$.

Applying this to f^{-1} when f is biholomorphic yields

Corollary 3

If $f \in \mathrm{Aut}(\mathbb{D})$, then f is an isometry of $\mathrm{d}\sigma^2$.

Comparison of Metrics

Let M be a Riemann surface and $f \colon \mathbb{D} \to M$ a holomorphic mapping, $f(z) = w$.

Let $\mathrm{d}s^2 = \lambda(w)^2 \, |\mathrm{d}w|^2$ be a conformal metric on M, so $\lambda \in C^\infty(M)$ and $\lambda > 0$.

Then $f|_{\mathbb{D}_R}^* \mathrm{d}s^2 = \lambda^2(f(z)) \, |f'(z)|^2 \, |\mathrm{d}z|^2 = \Lambda^2(z) \, |\mathrm{d}z|^2$ is a conformal metric on $\mathbb{D}_R$.

Also, $f|_{\mathbb{D}_R}^* \mathrm{d}s^2 \to f^* \mathrm{d}s^2$ and $\mathrm{d}\sigma_R^2 \to \mathrm{d}\sigma^2$ as $R \to 1$.

We wish to show $f^* \mathrm{d}s^2 \leq C \, \mathrm{d}\sigma^2$, for some $C \in \mathbb{R}$. Equivalently, $\Lambda^2 \leq C \rho^2$.

Claim: If $K_M \leq -k$ for some $k \geq 1$, then $\Lambda^2 \leq \dfrac{1}{k} \rho^2$.

Proof of Claim

Claim: If $K_M \leq -k$ for some $k \geq 1$, then $\Lambda^2 \leq \frac{1}{k}\rho^2$.

Assume that for some $k \geq 1$, we have

$$K_M = -\frac{\Delta \log \Lambda}{\Lambda^2} \leq -k.$$

Then $\Delta \log \Lambda \geq k\Lambda^2$. We already have $\Delta \log \rho_R = \rho_R^2$.

Let $U = \log \Lambda$ and $V_R = \log \rho_R$, then $\Delta U \geq k\Lambda^2$ and $\Delta V_R = \rho_R^2$.

Notice that $\Delta(U - V_R) \geq k\Lambda^2 - \rho_R^2 > 0$ in the set

$$E = \{z \in \mathbb{D}_R : k\Lambda^2(z) > \rho_R^2(z)\}.$$

Proof of Claim

Therefore, $U - V_R$ is subharmonic in $E = \{k\Lambda^2 > \rho_R^2\}$. We want $E = \emptyset$.

It has no interior maximum and its supremum is approached along $z_k \to z_0 \in \partial E$.

Observe that $\partial E \cap \partial \mathbb{D}_R = \emptyset$, since U is bounded in $\overline{\mathbb{D}}_R$, but as $|z| \to R$

$$V_R(z) = \log \rho_R(z) = \log \frac{2R}{R^2 - |z|^2} \to \infty.$$

Thus $U - V_R$ is continuous in a small disk around z_0, and

$$(U - V_R)(z_0) = \sup_{\overline{E}}(U - V_R) = \sup_{\partial E}(U - V_R) = 0.$$

This implies $U - V_R = 0$ in E, so $k\Lambda^2 = \rho_R^2$ in E.

This is a contradiction unless $E = \emptyset$. The claim is proven.

Alternative Proof of Claim

Alternatively, we may let $u_R = \dfrac{\Lambda^2}{\rho_R^2} \geq 0$ and show that $u_R \leq \dfrac{1}{k}$.

Whenever $u_R > 0$, we have

$$\frac{1}{2}\Delta \log u_R = \Delta(U - V_R) \geq k\Lambda^2 - \rho_R^2,$$

and therefore

$$u_R \leq \frac{\Delta \log u_R}{2k\rho_R^2} + \frac{1}{k},$$

so $u_R \leq \dfrac{1}{k}$ if $\log u_R$ is superharmonic in $\mathbb{D}_R$.

Notice that $u_R \geq 0$ in $\mathbb{D}_R$ and $u_R(z) \to 0$ as $|z| \to R$, since $\rho_R \to \infty$.

Thus u_R has a maximum at $z_0 \in \mathbb{D}_R$, and we may assume $u_R(z_0) > 0$.

By continuity, $u_R > 0$ near z_0, so $\log u_R$ is defined and has a maximum at z_0.

By second derivative test, $\Delta \log u_R \leq \Delta \log u_R(z_0) \leq 0$.

Ahlfors-Schwarz Lemma

We have proven: If $K_M \leq -k \leq -1$, then $f^*\mathrm{d}s^2 \leq \dfrac{1}{k}\,\mathrm{d}\sigma^2$.

Theorem 4 (Ahlfors)

Let $f\colon \mathbb{D} \to M$ be a holomorphic mapping. If $K_M \leq -1$, then $f^*\mathrm{d}s^2 \leq \mathrm{d}\sigma^2$.

Taking $(M, \mathrm{d}s^2) = (\mathbb{D}, \mathrm{d}\sigma^2)$, we recover the Schwarz-Pick lemma.

Kähler Geometry

Hermitian Metrics

We adopt the notation that repeated indices are summed over.

Let M be a complex manifold. Denote the real tangent space by $T_{\mathbb{R}}M$ and its complexification $TM = T_{\mathbb{R}}M \otimes \mathbb{C}$, the complex tangent space.

The almost complex structure $J \in \Gamma(T^*M \otimes TM)$ is integrable and splits

$$TM = T^{1,0}M \oplus \overline{T^{1,0}M},$$

where $T^{1,0}M$ is the $\sqrt{-1}$ eigenspace of J.

A Hermitian metric g on M is a smooth assignment of Hermitian inner product to the vector space $T_p^{1,0}M$, for every $p \in M$. We view $g \in \Gamma(T^{*1,0}M \otimes \overline{T^{*1,0}M})$.

Let z^i be local holomorphic coordinates on M. Then $g = g_{i\bar{j}}\,\mathrm{d}z^i \otimes \mathrm{d}\bar{z}^j$.

Also $(g_{i\bar{j}})$ is a positive-definite matrix, $g_{i\bar{j}} \in C^\infty(M)$, and $g_{i\bar{j}} = g_{\bar{j}i} = \overline{g_{j\bar{i}}}$.

The inverse metric $g^{i\bar{j}}$ is defined by the property $g^{i\bar{k}}g_{\bar{k}j} = \delta_{ij}$.

Complex Connections

Let (M, g, J) be a Hermitian manifold, and identify $T_{\mathbb{R}}M$ with $T^{1,0}M$.

We view the Hermitian metric g as a Riemannian metric that is J-invariant.

Let ∇ be the Levi-Civita connection on $T_{\mathbb{R}}M$, the unique connection such that

$$\nabla g = 0, \quad T_{\nabla} = 0.$$

We extend ∇ to $TM = T_{\mathbb{R}}M \otimes \mathbb{C}$ by $\mathbb{C}$-linearity, and restrict to $T^{1,0}M$.

If $\nabla J = 0$, i.e., $\nabla_X(JY) = J(\nabla_X Y)$, then $\nabla_X : T^{1,0}M \to T^{1,0}M$.

The Chern connection $\tilde{\nabla}$ is the unique connection on $T^{1,0}M$ such that

$$\tilde{\nabla} g = 0, \quad \tilde{\nabla}^{0,1} = \bar{\partial}.$$

If $T_{\tilde{\nabla}} = 0$ or $\nabla J = 0$, then $\tilde{\nabla} = \nabla$.

Curvature Tensor

Set $\tilde{\nabla}_{\partial_{z^i}} \partial_{z^j} = \Gamma^k_{ij} \partial_{z^k}$. Then $\Gamma^k_{ij} = g^{k\bar{\ell}} \partial_{z^i} g_{j\bar{\ell}}$.

Similarly we may define $\Gamma^k_{i\bar{j}}$, $\Gamma^k_{\bar{i}j}$, $\Gamma^k_{\bar{i}\bar{j}}$, etc, with the convention $\Gamma^{\bar{k}}_{\bar{i}\bar{j}} = \overline{\Gamma^k_{ij}}$.

The Chern curvature R maps $T^{1,0}M \times \overline{T^{1,0}M} \times T^{1,0}M \times \overline{T^{1,0}M} \to \mathbb{C}$ by

$$R(U, \overline{V}, Z, \overline{W}) = R_{i\bar{j}k\bar{\ell}} U^i \overline{V^j} Z^k \overline{W^\ell},$$

where $R_{i\bar{j}k\bar{\ell}} = -g_{m\bar{\ell}} \partial_{\bar{z}^j} \Gamma^m_{ik} = -\partial_{z^i} \partial_{\bar{z}^j} g_{k\bar{\ell}} + g^{p\bar{q}} \partial_{z^i} g_{k\bar{q}} \partial_{\bar{z}^j} g_{p\bar{\ell}}$.

If we extend the Riemann curvature tensor of the Levi-Civita connection ∇ to TM by $\mathbb{C}$-linearity and require that $\nabla J = 0$, then $\Gamma^k_{ij} = \Gamma^k_{ji}$ and $\overline{\Gamma^k_{ij}} = \overline{\Gamma^k_{ji}}$ are the only nonzero terms, and it agrees with the Chern curvature tensor as above.

Kähler Metrics

If $g = g_{i\bar{j}}\,\mathrm{d}z^i \otimes \mathrm{d}\bar{z}^j$ is a Hermitian metric on M, the associated $(1,1)$-form is

$$\omega_g = \frac{\sqrt{-1}}{2}\, g_{i\bar{j}}\,\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j.$$

We say (M, g) is Kähler if $\mathrm{d}\omega_g = 0$, or equivalently $\partial_{z^k} g_{i\bar{j}} = \partial_{z^i} g_{k\bar{j}}$.

Proposition 5

A Hermitian manifold (M, g) is Kähler if and only if at each point $p \in M$ there exist holomorphic coordinates z^i such that $g_{i\bar{j}}(p) = \delta_{ij}$ and $\partial_{z^k} g_{i\bar{j}}(p) = 0$.

These are called holomorphic normal coordinates at p. We have

$$g_{i\bar{j}}(z) = \delta_{ij} + O(|z|^2)$$

in an open neighborhood of p.

Curvature Symmetries

When M is Kähler, the Levi-Civita and Chern connections agree on $T^{1,0}M$.

Thus the Riemann curvature tensor of a Kähler manifold has the symmetries

$$R_{i\bar{\jmath}k\bar{\ell}} = R_{k\bar{\jmath}i\bar{\ell}} = R_{i\bar{\ell}k\bar{\jmath}} = R_{k\bar{\ell}i\bar{\jmath}}.$$

The Ricci tensor is then $\mathrm{Ric} = R_{i\bar{\jmath}}\,\mathrm{d}z^i \otimes \mathrm{d}\bar{z}^j$, where $R_{i\bar{\jmath}} = g^{k\bar{\ell}} R_{k\bar{\ell}i\bar{\jmath}}$.

Ricci and Bisectional Curvature

The Ricci curvature in the direction $v \in T^{1,0}M$ is defined as

$$\frac{\mathrm{Ric}(v,\bar{v})}{g(v,\overline{v})} = \frac{R_{i\bar{j}}\, v^i\, \overline{v^j}}{|v|^2}.$$

We say that $\mathrm{Ric} \leq C$ if $R_{i\bar{j}} v^i \overline{v^j} \leq C|v|^2$ for every $v \in T^{1,0}M$.

The holomorphic bisectional curvature of M in the directions $v, w \in T^{1,0}M$ is

$$\frac{R(v,\overline{v},w,\overline{w})}{g(v,\overline{v})g(w,\overline{w})} = \frac{R_{i\bar{j}k\bar{\ell}}\, v^i\, \overline{v^j}\, w^k\, \overline{w^\ell}}{|v|^2|w|^2}.$$

We say that $B \leq C$ if $R_{i\bar{j}k\bar{\ell}}\, v^i\, \overline{v^j}\, w^k\, \overline{w^\ell} \leq C|v|^2|w|^2$ for every $v, w \in T^{1,0}M$.

Complex Laplacian

When M is Kähler, the complex Laplacian of $f \in C^2(M)$ is given in local holomorphic coordinates as

$$\Delta_{\bar{\partial}} f = g^{i\bar{j}} \partial_{z^i} \partial_{\bar{z}^j} f,$$

and coincides with the real Laplacian, $\Delta f = \Delta_{\bar{\partial}} f$.

Chern-Lu Analysis

Setup / Notation

Let (M, g) and (N, h) be Hermitian manifolds, $f\colon M \to N$ a holomorphic mapping.

Let z^i and w^α be local holomorphic coordinates on M and N, respectively. Then

$$g = g_{i\bar{j}}\, \mathrm{d}z^i \otimes \mathrm{d}\bar{z}^j, \quad h = h_{\alpha\bar{\beta}}\, \mathrm{d}w^\alpha \otimes \mathrm{d}\bar{w}^\beta.$$

We adopt the notation that $i, j, k, \ldots$ correspond to coordinates on M and $\alpha, \beta, \gamma, \ldots$ correspond to coordinates on N, for example $\partial_k = \partial_{z^k}$ and $\partial_\gamma = \partial_{w^\gamma}$.

We denote $f^\alpha(z) = w^\alpha$, $f_i^\alpha = \partial_i f^\alpha$, and $f_{ij}^\alpha = \partial_i \partial_j f^\alpha$. Then $\overline{f_i^\alpha} = \partial_{\bar{\imath}} \overline{f^\alpha}$.

We wish to answer the question: how is f^*h related to g?

In the 1-dimensional case, we saw $f^*h \leq Cg$, where $C > 0$ is a constant relating the negative curvature bounds of M and N. This will also happen in higher dimensions.

Associated Tensor Bundle

Recall f is holomorphic $\iff \mathrm{d}f\colon T^{1,0}M \to T^{1,0}N \iff \bar{\partial} f = 0.$

We can view $\partial f \in \Gamma(E)$, where $E = T^{*1,0}M \otimes f^*(T^{1,0}N)$.

Consider the holomorphic frame $\mathrm{d}z^i \otimes \partial_\alpha$ of E. This means $\mathrm{d}z^i|_p \otimes \partial_\alpha|_{f(p)}$ is a holomorphic function on $E_p = T_p^{*1,0}M \otimes T_{f(p)}^{1,0}N$ for every $p \in M$, and the map $p \mapsto \mathrm{d}z^i|_p \otimes \partial_\alpha|_{f(p)}$ is smooth. Then

$$\partial f|_p = f_i^\alpha(p)\,\mathrm{d}z^i|_p \otimes \partial_\alpha|_{f(p)} \quad \text{or} \quad \partial f = f_i^\alpha\,\mathrm{d}z^i \otimes \partial_\alpha.$$

A Hermitian metric on E is $\eta \in \Gamma(E^* \otimes \overline{E^*})$ so that $\eta(p)$ is a Hermitian inner product on $E_p \otimes \overline{E_p}$ for each $p \in M$. The induced Hermitian metric on E is

$$\eta(\varphi \otimes U, \overline{\psi \otimes V}) = g(\varphi, \overline{\psi})\,h(U, \overline{V})$$

for $\varphi, \psi \in \Gamma(T^{*1,0}M)$ and $U, V \in \Gamma(f^*(T^{1,0}N))$.

Chern-Lu Formula

Let $F = f^*h$. Then F is a positive semi-definite $(1,1)$-tensor on M.

We have $|\partial f|^2 = \eta(\partial f, \overline{\partial f}) = g^{i\bar{j}} h_{\alpha\bar{\beta}} f_i^\alpha \overline{f_j^\beta} = g^{i\bar{j}} F_{i\bar{j}} = \operatorname{tr}_g F \in C^\infty(M)$.

Furthermore, $\overline{F_{i\bar{j}}} = h_{\beta\bar{\alpha}} f_j^\beta \overline{f_i^\alpha} = F_{j\bar{\imath}}$, so F is Hermitian.

Theorem 6 (Chern-Lu)

Let $f : M \to N$ be a holomorphic mapping between a Kähler manifold (M, g) and a Hermitian manifold (N, h). Then at each point of M we have

$$\Delta_{\bar{\partial}} |\partial f|^2 \geq R_{i\bar{j}} f_i^\alpha \overline{f_j^\alpha} - S_{\alpha\bar{\beta}\gamma\bar{\delta}} f_i^\alpha \overline{f_i^\beta} f_j^\gamma \overline{f_j^\delta},$$

where R is the curvature tensor of M and S is the curvature tensor of N.

Proof of Chern-Lu Formula

For simplicity, we assume N is Kähler. Fix $p \in M$.

We can find holomorphic normal coordinates z^i at p and w^α at $f(p)$ so that

$$g_{i\bar{\jmath}}(p) = \delta_{ij}, \quad \partial_k g_{i\bar{\jmath}}(p) = 0, \quad h_{\alpha\bar{\beta}}(f(p)) = \delta_{\alpha\beta}, \quad \partial_\gamma h_{\alpha\bar{\beta}}(f(p)) = 0.$$

Set $F_{i\bar{\jmath}} = h_{\alpha\bar{\beta}} f_i^\alpha \overline{f_j^\beta}$ and $u = g^{i\bar{\jmath}} F_{i\bar{\jmath}} = |\partial f|^2$.

Since $\Delta_{\bar{\partial}} u = \partial_k \partial_{\bar{k}} u = u_{k\bar{k}}$ at p, we compute

$$\begin{aligned}
u_{k\bar{k}} &= \partial_k \partial_{\bar{k}} g^{i\bar{\jmath}} F_{i\bar{\jmath}} + g^{i\bar{\jmath}} \partial_k \partial_{\bar{k}} F_{i\bar{\jmath}} + \partial_{\bar{k}} g^{i\bar{\jmath}} \partial_k F_{i\bar{\jmath}} + \partial_k g^{i\bar{\jmath}} \partial_{\bar{k}} F_{i\bar{\jmath}} \\
&= \partial_k \partial_{\bar{k}} g^{i\bar{\jmath}} F_{i\bar{\jmath}} + g^{i\bar{\jmath}} \partial_k \partial_{\bar{k}} F_{i\bar{\jmath}} \\
&= \partial_k \partial_{\bar{k}} g^{i\bar{\jmath}} h_{\alpha\bar{\beta}} f_i^\alpha \overline{f_j^\beta} + g^{i\bar{\jmath}} \partial_k \partial_{\bar{k}} F_{i\bar{\jmath}} \\
&= \partial_k \partial_{\bar{k}} g^{i\bar{\jmath}} f_i^\alpha \overline{f_j^\alpha} + g^{i\bar{\jmath}} \partial_k \partial_{\bar{k}} F_{i\bar{\jmath}}.
\end{aligned}$$

Proof of Chern-Lu Formula

By the chain rule $\partial_k h_{\alpha\bar\beta} = \partial_\gamma h_{\alpha\bar\beta} f_k^\gamma$, so

$$\begin{aligned}
\partial_k \partial_{\bar k} F_{i\bar j} &= \partial_k \partial_{\bar k}(h_{\alpha\bar\beta} f_i^\alpha \overline{f_j^\beta}) \\
&= \partial_k(\partial_{\bar k} h_{\alpha\bar\beta} f_i^\alpha \overline{f_j^\beta} + h_{\alpha\bar\beta} f_i^\alpha \overline{f_{jk}^\beta}) \\
&= \partial_k \partial_{\bar k} h_{\alpha\bar\beta} f_i^\alpha \overline{f_j^\beta} + \partial_{\bar k} h_{\alpha\bar\beta} f_i^\alpha \overline{f_{jk}^\beta} + \partial_{\bar k} h_{\alpha\bar\beta} f_{ik}^\alpha \overline{f_j^\beta} + h_{\alpha\bar\beta} f_{ik}^\alpha \overline{f_{jk}^\beta} \\
&= \partial_\gamma \partial_{\bar\delta} h_{\alpha\bar\beta} f_i^\alpha \overline{f_j^\beta} f_k^\gamma \overline{f_k^\delta} + f_{ik}^\alpha \overline{f_{jk}^\alpha}.
\end{aligned}$$

We obtain

$$\begin{aligned}
u_{k\bar k} &= \partial_k \partial_{\bar k} g^{i\bar j} f_i^\alpha \overline{f_j^\alpha} + g^{i\bar j} \partial_k \partial_{\bar k} F_{i\bar j} \\
&= \partial_k \partial_{\bar k} g^{i\bar j} f_i^\alpha \overline{f_j^\alpha} + \partial_\gamma \partial_{\bar\delta} h_{\alpha\bar\beta} f_i^\alpha \overline{f_i^\beta} f_k^\gamma \overline{f_k^\delta} + f_{ik}^\alpha \overline{f_{ik}^\alpha}.
\end{aligned}$$

Proof of Chern-Lu Formula

So far, $u_{k\bar{k}} = \partial_k \partial_{\bar{k}} g^{i\bar{j}} f_i^\alpha \overline{f_j^\alpha} + \partial_\gamma \partial_{\bar{\delta}} h_{\alpha\bar{\beta}} f_i^\alpha \overline{f_i^\beta} f_k^\gamma \overline{f_k^\delta} + f_{ik}^\alpha \overline{f_{ik}^\alpha}$ at p.

Lemma 7

We have $\partial_k \partial_{\bar{k}} g^{i\bar{j}} = -g^{i\bar{r}} g^{s\bar{j}} \partial_k \partial_{\bar{k}} g_{s\bar{r}}$, so $\partial_k \partial_{\bar{k}} g^{i\bar{j}} = -\partial_k \partial_{\bar{k}} g_{i\bar{j}}$ at p.

Lemma 8

We have $S_{\alpha\bar{\beta}\gamma\bar{\delta}} = -\partial_\gamma \partial_{\bar{\delta}} h_{\alpha\bar{\beta}}$ at $f(p)$ and $R_{i\bar{j}k\bar{\ell}} = -\partial_k \partial_{\bar{\ell}} g_{i\bar{j}}$ at p.

It follows that $\partial_k \partial_{\bar{k}} g^{i\bar{j}} = g^{i\bar{r}} g^{s\bar{j}} R_{k\bar{k}s\bar{r}} = R_{k\bar{k}i\bar{j}} = R_{i\bar{j}}$ at p, and

$$\begin{aligned} u_{k\bar{k}} &= R_{i\bar{j}} f_i^\alpha \overline{f_j^\alpha} - S_{\alpha\bar{\beta}\gamma\bar{\delta}} f_i^\alpha \overline{f_i^\beta} f_k^\gamma \overline{f_k^\delta} + f_{ik}^\alpha \overline{f_{ik}^\alpha} \\ &\geq R_{i\bar{j}} f_i^\alpha \overline{f_j^\alpha} - S_{\alpha\bar{\beta}\gamma\bar{\delta}} f_i^\alpha \overline{f_i^\beta} f_k^\gamma \overline{f_k^\delta}. \end{aligned}$$

Complex Bochner Formula

When $f: M \to \mathbb{C}$ is a holomorphic function, we recover the following:

Theorem 9

If M is a Kähler manifold and $f: M \to \mathbb{C}$ is a holomorphic function, then

$$\Delta_{\bar{\partial}} |\partial f|^2 = \mathrm{Ric}(\partial f, \overline{\partial f}) + |\nabla \partial f|^2.$$

General Schwarz Lemma

On Compact Manifolds

Let (M, g) be Kähler, (N, h) Hermitian, and $f \colon M \to N$ holomorphic.

Assume that $\mathrm{Ric}_M \geq k_1$ and $B_N \leq k_2 < 0$. This means

$$R_{i\bar{j}} v^i \overline{v^j} \geq k_1 |v|^2, \quad S_{\alpha\bar{\beta}\gamma\bar{\delta}}\, w^\alpha \overline{w^\beta} \zeta^\gamma \overline{\zeta^\delta} \leq k_2 |w|^2 |\zeta|^2$$

for every $v \in T^{1,0}M$ and $w, \zeta \in T^{1,0}N$.

If M is compact, $u = g^{i\bar{j}} F_{i\bar{j}} = g^{i\bar{j}} h_{\alpha\bar{\beta}} f_i^\alpha \overline{f_j^\beta}$ attains a maximum at $p \in M$.

In holomorphic normal coordinates at p, we have $u = F_{i\bar{i}} > 0$.

We can choose holomorphic coordinates at $f(p)$ so that $h_{\alpha\bar{\beta}}(f(p)) = \delta_{\alpha\bar{\beta}}$.

Then at p, we have $u = |f_i^\alpha|^2$, and by the Chern-Lu formula

$$0 \geq \Delta u = \Delta_{\bar{\partial}}\, u \geq R_{i\bar{j}} f_i^\alpha \overline{f_j^\alpha} - S_{\alpha\bar{\beta}\gamma\bar{\delta}}\, f_i^\alpha \overline{f_i^\beta} f_j^\gamma \overline{f_j^\delta}.$$

On Compact Manifolds

By the curvature assumptions, we obtain

$$\begin{aligned}0 &\geq R_{i\bar{j}} f_i^\alpha \overline{f_j^\alpha} - S_{\alpha\bar{\beta}\gamma\bar{\delta}} f_i^\alpha \overline{f_i^\beta} f_j^\gamma \overline{f_j^\delta} \\ &= \mathrm{Ric}(f_i^\alpha \partial_i, \overline{f_j^\alpha \partial_j}) - S(f_i^\alpha \partial_\alpha, \overline{f_i^\beta \partial_\beta}, f_j^\gamma \partial_\gamma, \overline{f_j^\delta \partial_\delta}) \\ &\geq k_1 |f_i^\alpha|^2 - k_2 |f_i^\alpha|^2 |f_k^\gamma|^2 \\ &= k_1 u - k_2 u^2.\end{aligned}$$

As long as $u \neq 0$, we obtain $k_1 \leq k_2 u \leq 0$ and $u \leq u(p) \leq \dfrac{k_1}{k_2}$.

On Compact Manifolds

Theorem 10 (Yau)

Suppose (M, g) is a compact Kähler manifold and (N, h) is a Hermitian manifold. Assume that $\mathrm{Ric}_M \geq k_1$ and $B_N \leq k_2 < 0$. Then $k_1 \leq 0$, and any non-constant holomorphic mapping $f : M \to N$ satisfies

$$|\partial f|^2 \leq \frac{k_1}{k_2} \quad \text{and} \quad f^* h \leq \frac{k_1}{k_2} g.$$

We now want to weaken the assumption that M is compact.

Generalized Maximum Principle

We say that a Riemannian manifold M satisfies the generalized maximum principle if for every $\varphi \in C^2(M)$ bounded from above, there exists $\{p_k\} \subset M$ so that

$$\lim_{k\to\infty} |\nabla\varphi(p_k)| = 0, \quad \limsup_{k\to\infty} \Delta\varphi(p_k) = 0, \quad \lim_{k\to\infty} \varphi(p_k) = \sup\varphi.$$

Theorem 11 (Yau)

Let M be a complete Riemannian manifold with Ricci curvature bounded below. Then M satisfies the generalized maximum principle.

Removal of Compactness Assumption

In our setting M is Kähler with $\mathrm{Ric} \geq k_1$ and N is Hermitian with $B \leq k_2 < 0$.

Then M satisfies the generalized maximum principle. Set

$$\varphi = \frac{1}{(u+1)^{1/2}} \geq 0.$$

By Omori-Yau theorem, for every $\varepsilon > 0$ we can find $p_\varepsilon \in M$ so that

$$|\nabla\varphi(p_\varepsilon)| < \varepsilon, \quad \Delta\varphi(p_\varepsilon) > -\varepsilon, \quad \varphi(p_\varepsilon) < \inf\varphi + \varepsilon.$$

Notice $\varphi(p_\varepsilon) \to \inf\varphi$ as $\varepsilon \to 0$, so $u(p_\varepsilon) \to \sup u$.

Claim: We have $\sup u < \infty$, and $k_1 \leq k_2 \sup u$.

Proof of Claim

Claim: We have $\sup u < \infty$, and $k_1 \leq k_2 \sup u$.

Recall $\varphi = (u+1)^{-1/2}$ and $\Delta\varphi = g^{i\bar{j}}\varphi_{i\bar{j}}$.

We compute $\varphi_{i\bar{j}} = \dfrac{3}{4}\varphi^5 u_i u_{\bar{j}} - \dfrac{1}{2}\varphi^3 u_{i\bar{j}}$, so $\Delta\varphi = \dfrac{3}{4}\varphi^5|\nabla u|^2 - \dfrac{1}{2}\varphi^3\Delta u$.

Since $|\nabla\varphi|^2 = g^{i\bar{j}}\varphi_i\varphi_{\bar{j}} = \dfrac{1}{4}\varphi^6|\nabla u|^2$, we obtain $\Delta u = \dfrac{6}{\varphi^4}|\nabla\varphi|^2 - \dfrac{2}{\varphi^3}\Delta\varphi$.

It follows from the Chern-Lu formula that at p_ε we have

$$k_1 u - k_2 u^2 \leq \Delta u < 6\varepsilon^2(u+1)^2 + 2\varepsilon(u+1)^{3/2}.$$

Taking $\varepsilon \to 0$, if u was not bounded we obtain a contradiction.

As long as $u \neq 0$, we have $k_1 \leq k_2 \sup u$. The claim is proved.

On Complete Manifolds

Therefore, $\sup u \leq \dfrac{k_1}{k_2}$ and $k_1 \leq 0$.

Theorem 12 (Yau)

Suppose (M, g) is a complete Kähler manifold and (N, h) is a Hermitian manifold. Assume that $\mathrm{Ric}_M \geq k_1$ and $B_N \leq k_2 < 0$. Then $k_1 \leq 0$, and any non-constant holomorphic mapping $f : M \to N$ satisfies

$$|\partial f|^2 \leq \frac{k_1}{k_2} \quad \text{and} \quad f^* h \leq \frac{k_1}{k_2} g.$$

Liouville Theorems

Corollary 13

Let M be a complete Kähler manifold with $\mathrm{Ric} \geq 0$ and N a Hermitian manifold with $B \leq k_2 < 0$. If $f : M \to N$ is a holomorphic mapping, then f is constant on every connected component of M.

Since $\mathbb{D}$ admits a Kähler metric of constant negative sectional curvature, we have

Corollary 14

Let M be a connected, complete Kähler manifold with non-negative Ricci curvature. Then every bounded holomorphic function $M \to \mathbb{C}$ is a constant.

From this we recover the usual Liouville theorem.

Holomorphic Sectional Curvature

The holomorphic sectional curvature of M in the direction $v \in T^{1,0}M$ is

$$H(v) = B(v, v) = \frac{R_{i\bar{j}k\bar{\ell}}\, v^i\, \overline{v^j}\, v^k\, \overline{v^\ell}}{|v|^4}.$$

We say that $H \leq C$ if $R_{i\bar{j}k\bar{\ell}}\, v^i\, \overline{v^j}\, v^k\, \overline{v^\ell} \leq C|v|^4$ for every $v \in T^{1,0}M$.

In general, a bound for H is a weaker condition than a bound for B.

If N is Kähler, we may replace $B \leq k_2 < 0$ by $H \leq k_2 < 0$.

Royden's Trick

Lemma 15 (Royden)

Let S be a symmetric bi-hermitian form on N in the sense that

$$S(v_1, \overline{v_2}, v_3, \overline{v_4}) = S(v_3, \overline{v_2}, v_1, \overline{v_4}), \quad S(v_1, \overline{v_2}, v_3, \overline{v_4}) = S(v_2, \overline{v_1}, v_4, \overline{v_3}).$$

Suppose there exists $K \le 0$ so that $S(v, \overline{v}, v, \overline{v}) \le K|v|^4$ for all $v \in T^{1,0}N$. If $v_1, \ldots, v_r \in T^{1,0}N$ are orthogonal, then

$$S(v_i, \overline{v_i}, v_j, \overline{v_j}) \le \frac{r+1}{2r} K \left(\sum_i |v_i|^2 \right)^2 .$$

When N is Kähler and $H_N \le k_2 < 0$, applying this lemma to $v_i = f_i^\alpha \partial_\alpha$ yields

$$S_{\alpha\bar{\beta}\gamma\bar{\delta}} f_i^\alpha \overline{f_i^\beta} f_j^\gamma \overline{f_j^\delta} \le \frac{r+1}{2r} k_2 |\partial f|^4,$$

where r is the rank of ∂f.

Sectional Curvature Bound

It follows from the Chern-Lu formula that

$$\Delta_{\bar{\partial}} |\partial f|^2 \geq k_1 |\partial f|^2 - \frac{r+1}{2r} k_2 |\partial f|^4.$$

Through the same argument as before, we obtain

Theorem 16 (Royden)

Suppose (M, g) is a complete Kähler manifold and (N, h) is also a Kähler manifold. Assume that $\mathrm{Ric}_M \geq k_1$ and $H_N \leq k_2 < 0$. Then $k_1 \leq 0$, and any non-constant holomorphic mapping $f : M \to N$ satisfies

$$|\partial f|^2 \leq \frac{2r}{r+1} \frac{k_1}{k_2} \quad \text{and} \quad f^* h \leq \frac{2r}{r+1} \frac{k_1}{k_2} g,$$

where r is the rank of ∂f.

Thank You!

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