Numerically Simulating the Buckley-Leverett Equation with TCAT Capillary Pressure

Benjamin T. Shepard

October 14, 2021

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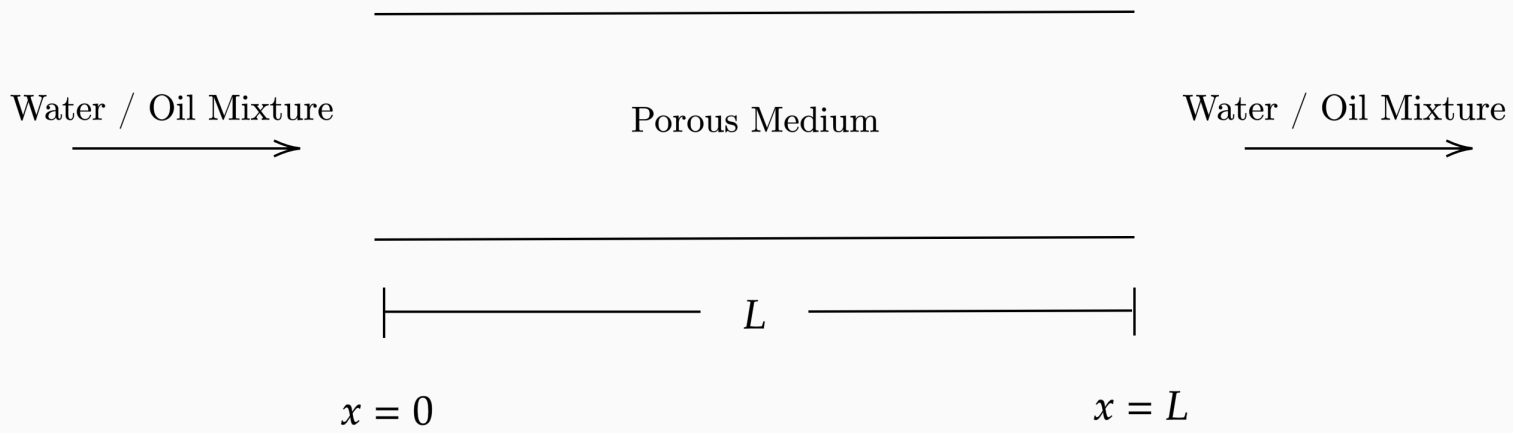

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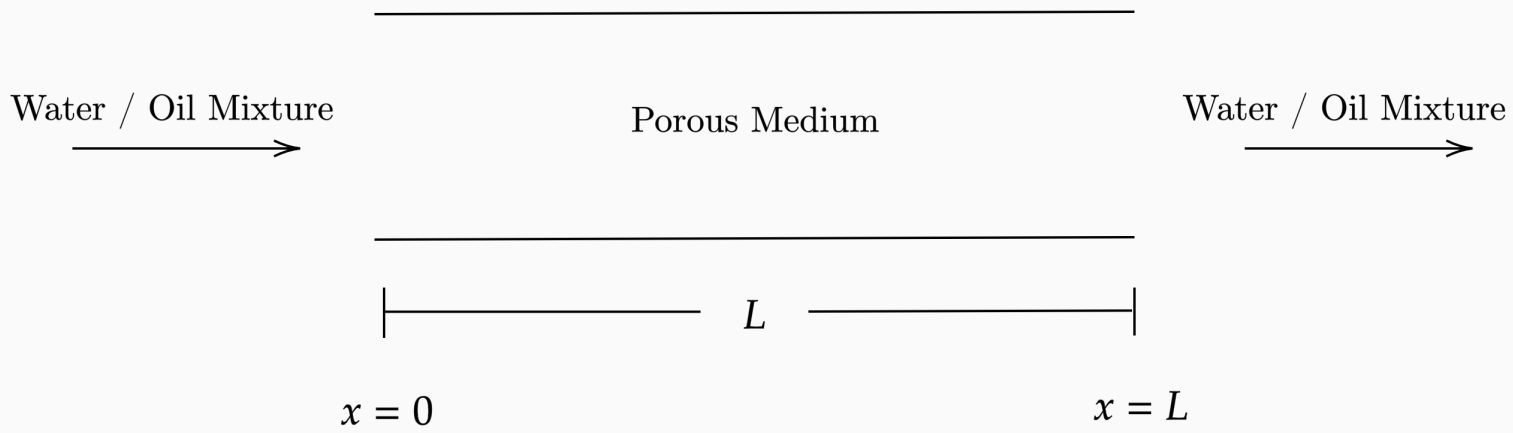

We wish to fully understand how this interaction behaves.

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Let $u(x, t)$ denote the saturation of the wetting phase at $x \in [0, L]$ and time $t \in \mathbb{R}^+$. That is, u is the volume percentage of the wetting phase in the medium, and $1 - u$ would be the percentage of the non-wetting phase.

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Let $p_c(u) = p_n - p_w$ denote the capillary pressure of the system, where p_n and p_w are the capillary pressures of the non-wetting and wetting phases, respectively. This does not change as the fluids interact.

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where $f(u)$ is the *fractional flow rate* and $H(u)$ is the *dissipation function*:

$$f(u) = \frac{u^2}{u^2 + 2(1-u)^2} \quad \text{and} \quad H(u) = \frac{u^2(1-u)^2}{u^2 + 2(1-u)^2}.$$

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where p_c^e is the equilibrium capillary pressure, τ_A and τ_B are dimensionless constants, and ε^{wn} and ε_{eq}^{wn} are the interfacial area and equilibrium interfacial area, respectively.

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$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \frac{\partial}{\partial x}\left[H(u)\,\frac{\partial u}{\partial x}\right] + \tau_A\,\frac{\partial}{\partial x}\left[H(u)\,\frac{\partial u}{\partial t \partial x}\right] \qquad (*)$$

and the only parameter we need to account for is τ_A.

Solution Structures

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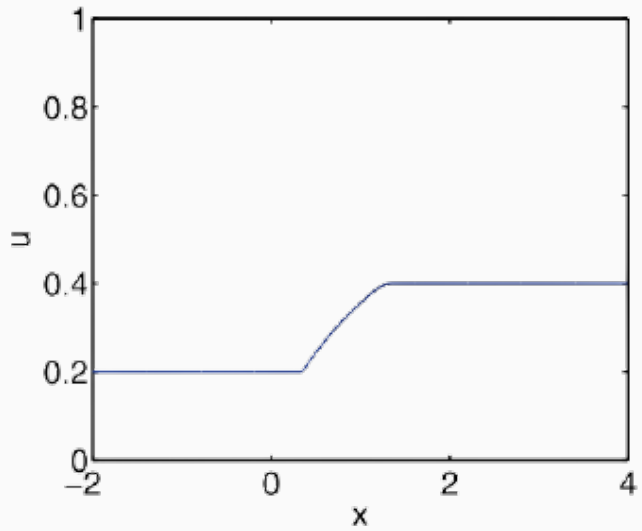

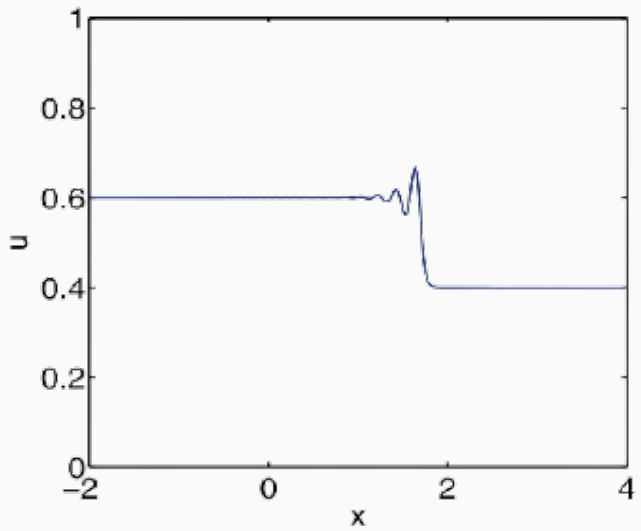

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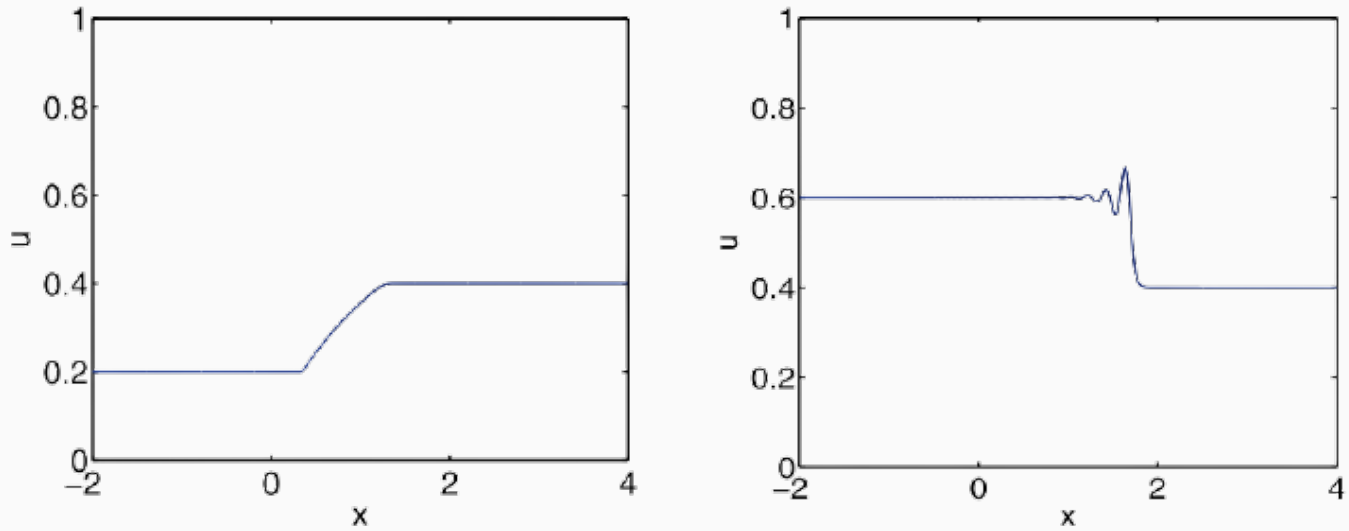

We can see some oscillation in the solution structure; this could be an artifact of the numerical analysis used in the simulation, or a separate wave structure.

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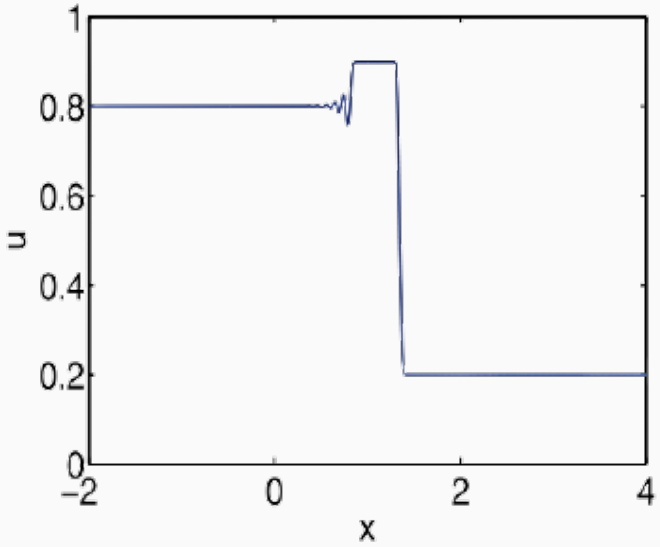

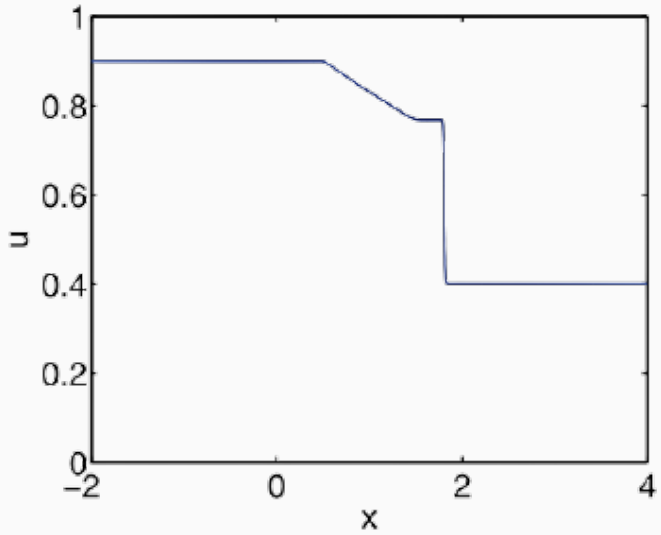

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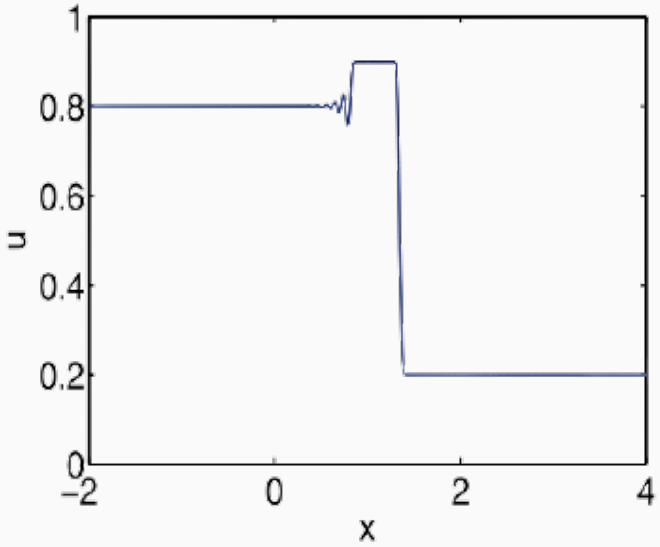

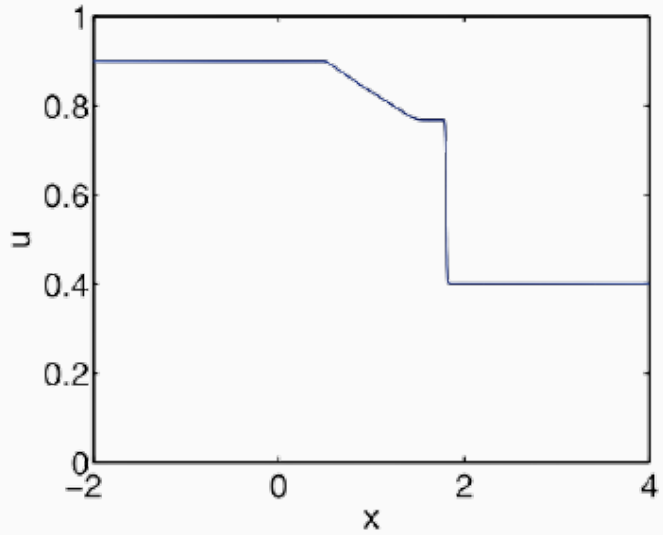

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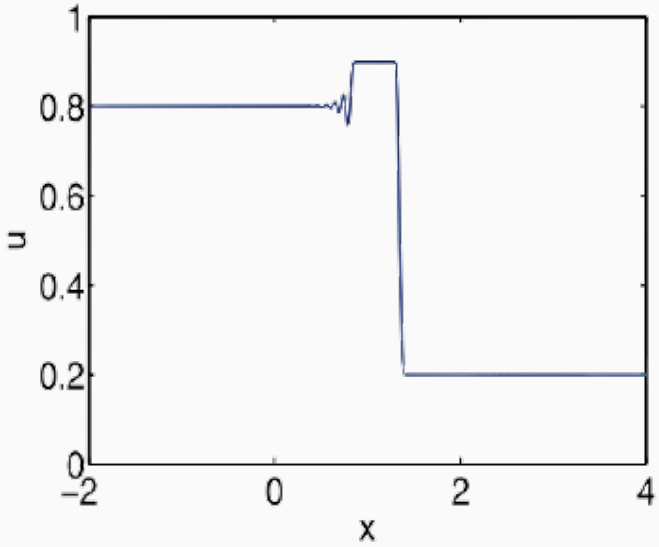

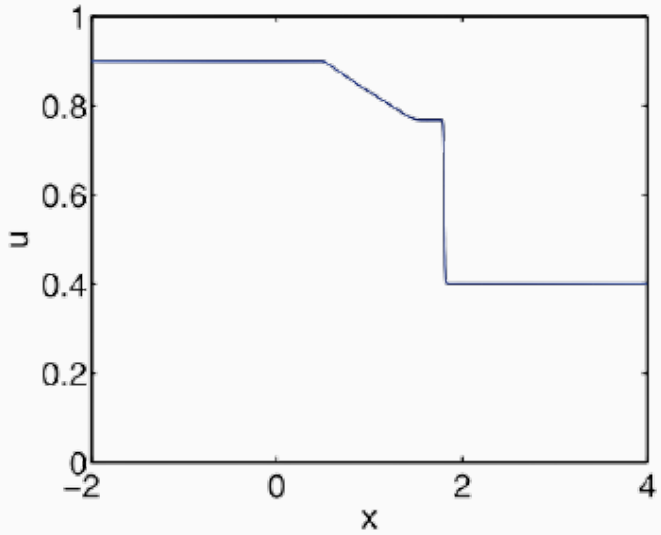

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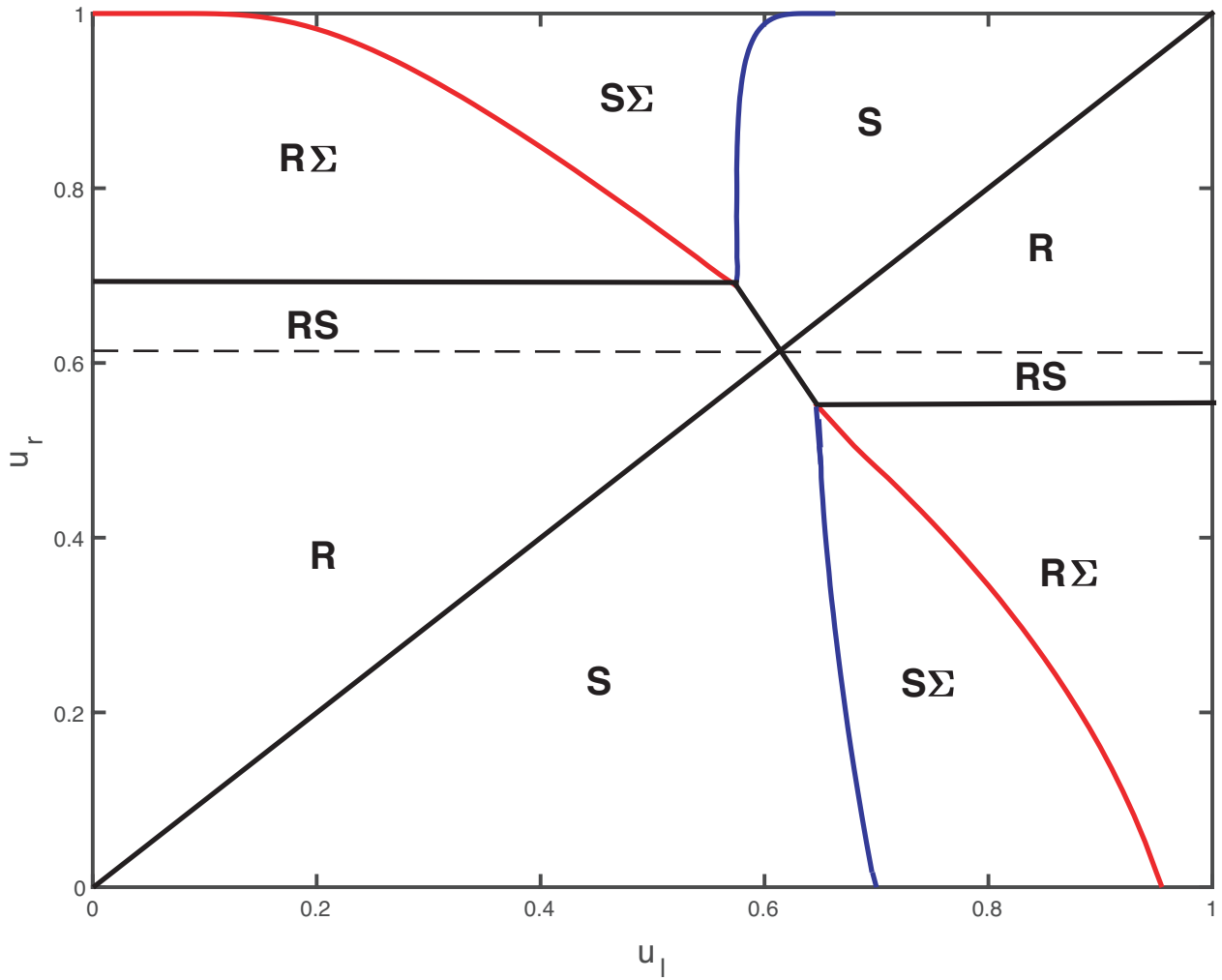
Solution Structures

Diagram of different solution structures:



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Our main focus in this research was to develop a simulation that is very fast at computing different types of solutions, yet also accurate to the previous methods.

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We can think of the saturation as a matrix in order to better understand this discretization:

$$\begin{bmatrix} u_0^0 & \cdots & u_j^0 & \cdots & u_M^0 \\ \vdots & & \vdots & & \vdots \\ u_0^n & \cdots & u_j^n & \cdots & u_M^n \\ \vdots & & \vdots & & \vdots \\ u_0^N & \cdots & u_j^N & \cdots & u_M^N \end{bmatrix}$$

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We use

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\phi}\left[g(u_j^{n+1}) - \beta(u_j^{n+1})p_j^{n+1}\right]$$

to advance the simulation by one timestep.

ϕ is the rock porosity (percentage of void space in porous medium).

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5. Check for convergence. If attained, set $u_j^{n+1} = u_j^{n+1,k}$ and go to the next timestep. Otherwise, reset iteration level k.

The formula for convergence uses an error tolerance that is predetermined by the user.

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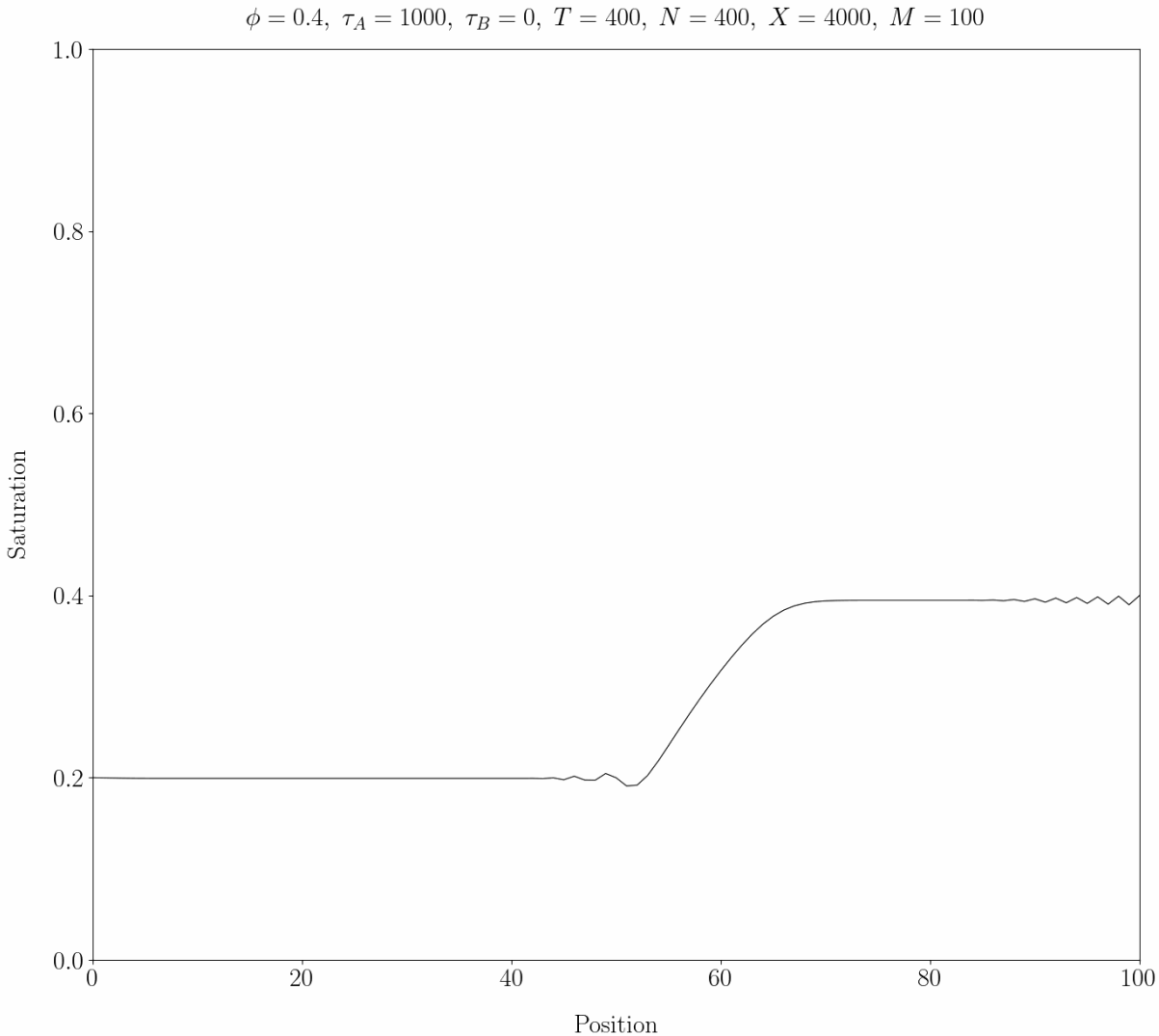
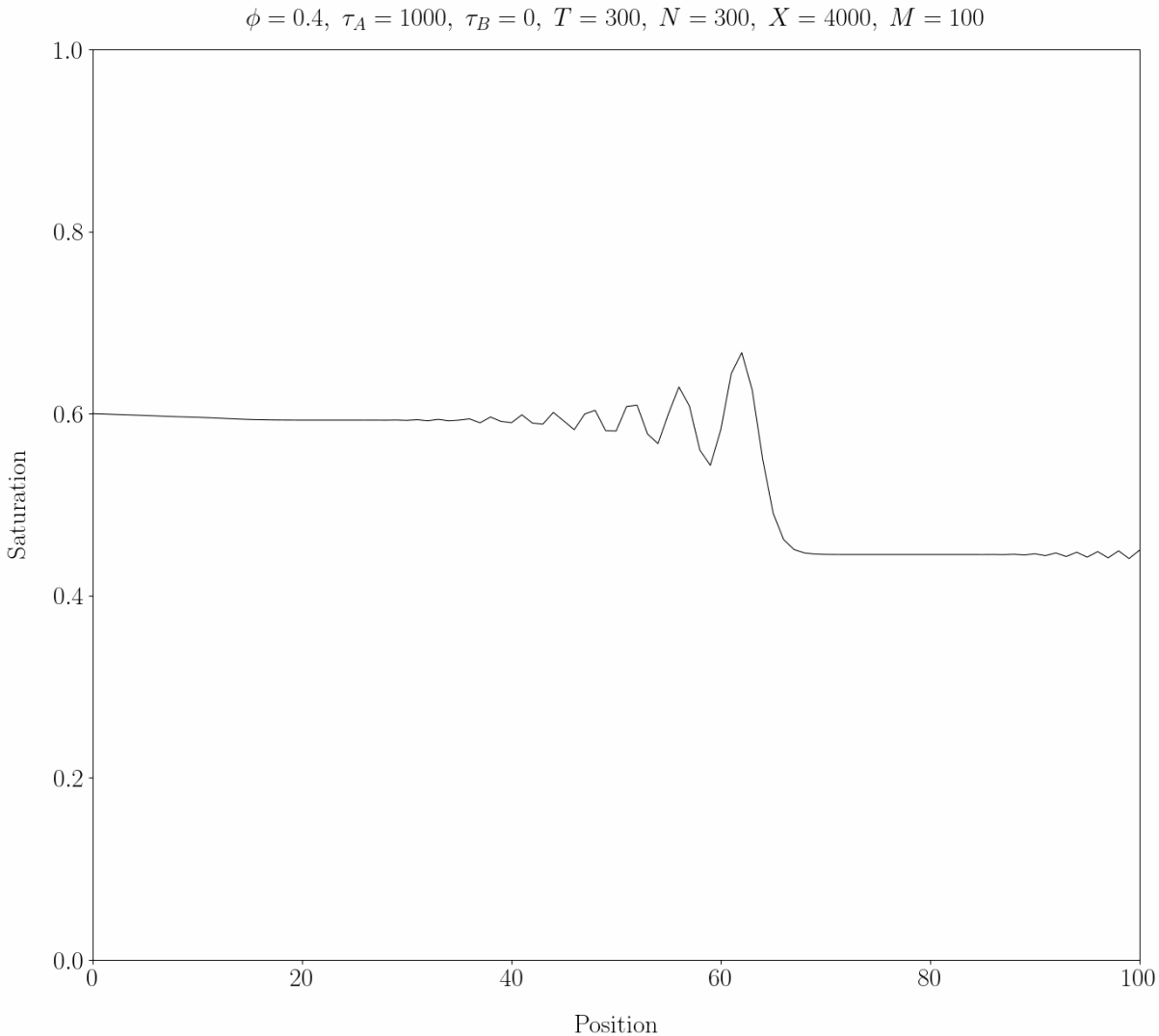


Simulation Results: Lax Shock



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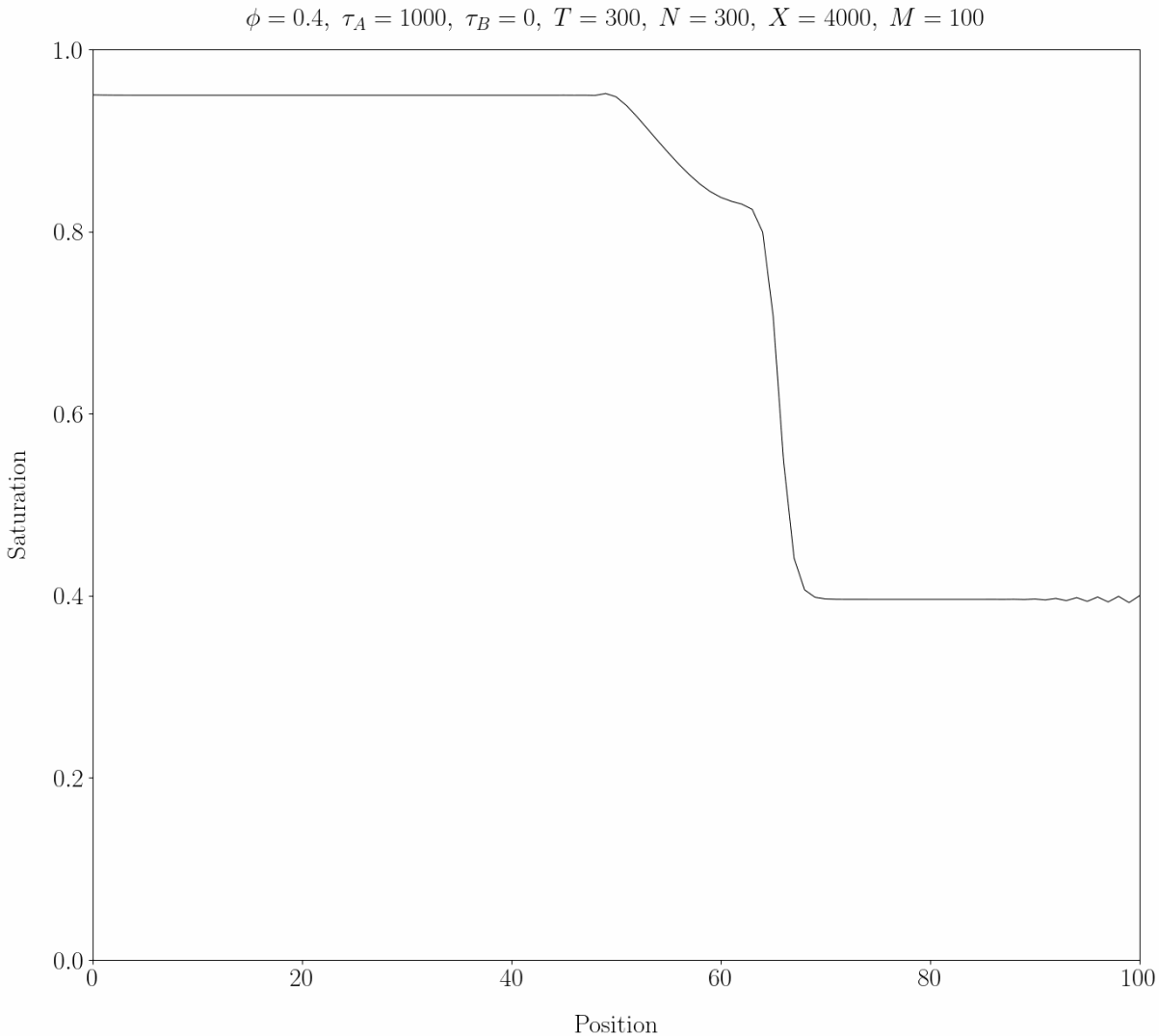
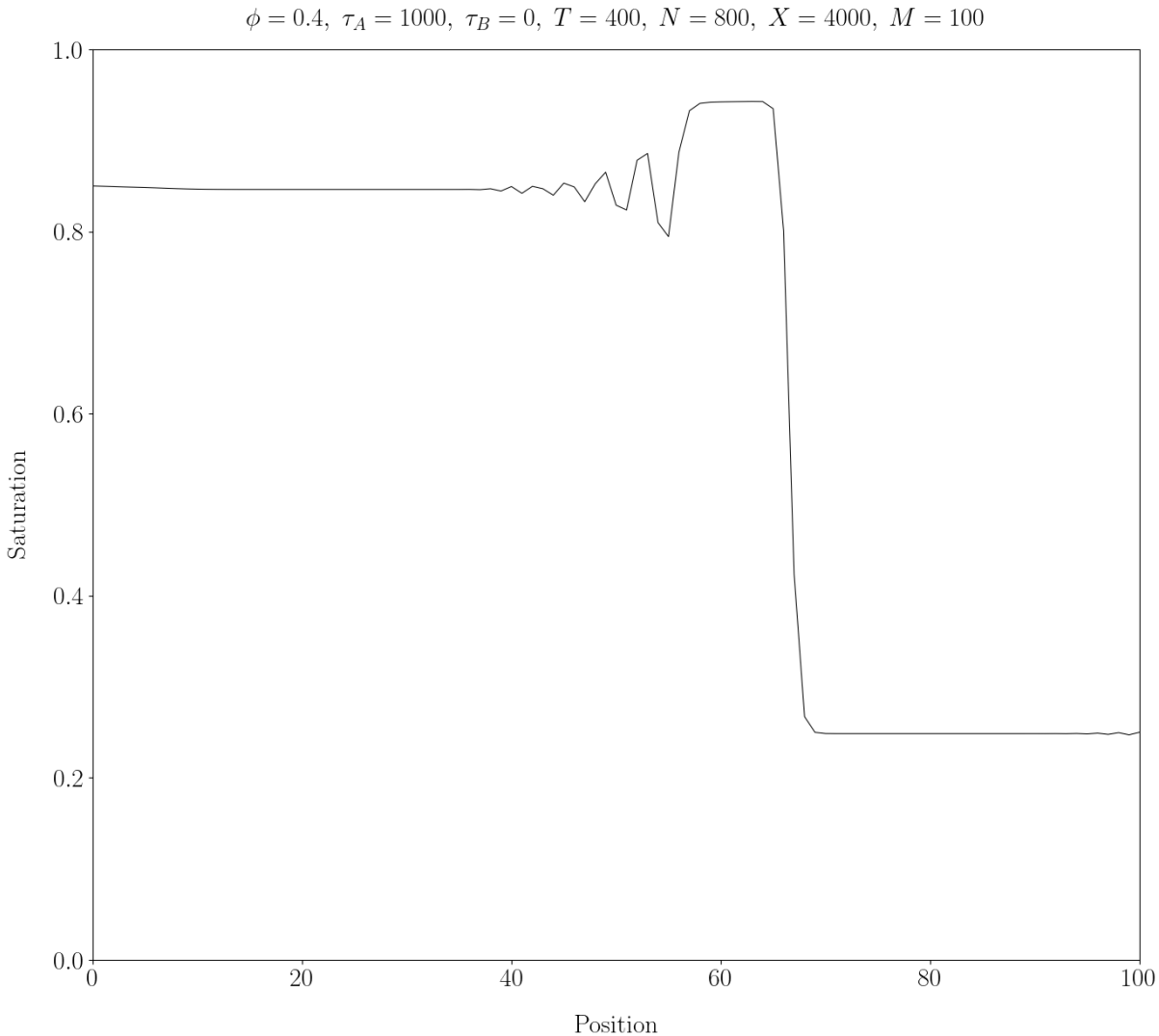


Simulation Results: Double Shock



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This is about $20,000$ times faster than the previous simulation.

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The program also does not like very large values of τ_A. This may have to do with other parameters not being callibrated correctly.

Future work will also use the current program to compare real data with simulated data in order to get a more accurate value of τ_A.

References

[1] Kimberly Spayd. “Generalizing the Modified Buckley–Leverett Equation with TCAT Capillary Pressure”. In: *Euro. Jnl of Applied Mathematics* 29 (2018), pp. 338–351.

[2] Eduardo Abreu and Jardel Vieira. “Computing Numerical Solutions of the Pseudo-parabolic Buckley–Leverett Equation with Dynamic Capillary Pressure”. In: *Mathematics and Computers in Simulation* 137 (2017), pp. 29–48.