These are notes for a course (Spezialvorlesung) on Jacobi forms at Heidelberg University in Wintersemester 2024/25. The prerequisites were complex analysis and some knowledge of modular forms (the material of “Modulformen I”). Familiarity with elliptic functions was not assumed and the first lectures are a review of that subject.

The notes mostly follow *The Theory of Jacobi Forms* by Eichler and Zagier, with the caveat that I did not discuss Siegel modular forms or their Fourier–Jacobi coefficients. The later sections rely on other sources which are indicated by footnotes.

Contents

0. Introduction

Jacobi forms are a family of special functions. They generalize two major families of special functions that are characterized by their functional equations:

(1) Modular functions, which are holomorphic functions

$$f : \mathbb{H} = \{x + iy : \ x, y \in \mathbb{R}, \ y > 0\} \longrightarrow \mathbb{C}$$

that satisfy $f(\gamma\tau) = f(\tau)$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

(2) Elliptic functions, which are *meromorphic* functions

$$f : \mathbb{C} \longrightarrow \mathbb{C} \cup \{\infty\}$$

that satisfy $f(z + \omega) = f(z)$ for ω in a lattice in $\mathbb{C}$.

Both of these functional equations can be relaxed, and there are compelling reasons, both practical and theoretical, for doing it. We can pass from (1) to *modular forms*, which transform under the modular group $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ similarly to (1) but in which the functional equations include what is called a factor of automorphy. Or we can pass from elliptic functions to *quasiperiodic functions*, which transform under translations similarly to (2) with yet another factor of automorphy.
Jacobi forms are defined to encompass both constructions.

The fundamental example of a Jacobi form is the *Jacobi theta function*:

$$\theta(\tau, z) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i n z}, \quad \tau \in \mathbb{H}, \ z \in \mathbb{C}.$$

This series converges absolutely (and locally uniformly).

The function θ has fascinated mathematicians and physicists for about two centuries, and for a number of reasons. It solves a form of the heat equation:

$$\frac{\partial \theta}{\partial \tau} = \frac{1}{4\pi i} \cdot \frac{\partial^2 \theta}{\partial z^2},$$

and it is essentially the *fundamental solution* of that differential equation on the interval $z \in [0, 1]$ with periodic boundary conditions. θ and related functions are also used to evaluate elliptic integrals. We will (for the most part) not discuss these or any other physical applications in the course.

Our interest in θ is due to the functional equations it satisfies. Trivially,

$$\theta(\tau, z+1) = \theta(\tau, z),$$

and a simple reordering ($n \mapsto n+1$) in the defining series yields

$$\theta(\tau, z+\tau) = e^{-\pi i \tau - 2\pi i z}\theta(\tau, z).$$

The *theta transformation formula* is a far less obvious functional equation for θ involving the first variable:

Theorem 0.1. *The θ-function*

$$\theta(\tau, z) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i n z}, \quad \tau \in \mathbb{H},\ z \in \mathbb{C}$$

satisfies the functional equation

$$\theta\Big(-\frac{1}{\tau}, \frac{z}{\tau}\Big) = e^{-\pi i/4}\sqrt{\tau} \cdot e^{\pi i z^2/\tau}\theta(\tau, z).$$

Here, $\sqrt{\tau}$ is the branch of the square root that becomes positive as $\tau \in \mathbb{H}$ tends towards the positive real axis.

The theory of Jacobi forms is a common generalization of modular forms, doubly periodic functions, and functions like θ. It retains many of the attractive aspects of the theory of modular forms:

1. Modular forms of a fixed *weight* form finite-dimensional vector spaces. Also, Jacobi forms of fixed *weight and index* form finite-dimensional spaces.

2. Modular forms and Jacobi forms are both graded rings: the weight and index are added when Jacobi forms are multiplied.

3. Some modular forms are easy to compute (e.g. Eisenstein series). Similarly for Jacobi forms.

4. Some modular forms contain very interesting arithmetic information (e.g. theta series, or the discriminant $\Delta(\tau)$). Similarly for Jacobi forms.

1. Elliptic functions

1.1. Period lattices

Let $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be a non-constant meromorphic function.

Definition 1.1. A **period** of f is a complex number $\lambda \neq 0$ with the property

$$f(z+\lambda) = f(z) \quad \text{for all } z \in \mathbb{C}.$$

We write

$$\mathrm{Per}(f) := \{\lambda \in \mathbb{C} : \ f(z+\lambda) = f(z)\}.$$

$\mathrm{Per}(f)$ is a group under addition (since we include 0).

Lemma 1.2. $\mathrm{Per}(f)$ *is a closed and discrete subgroup of* $\mathbb{C}$.

Proof. $\mathrm{Per}(f)$ can be written as the intersection

$$\mathrm{Per}(f) = \bigcap_{z \in \mathbb{C}} \{\lambda \in \mathbb{C} : \ f(z+\lambda) - f(z) = 0\}.$$

Each function $\lambda \mapsto f(z+\lambda) - f(z)$ is meromorphic and nonconstant, so its zero set is closed and discrete. Hence $\mathrm{Per}(f)$ is an intersection of closed, discrete sets and is itself closed and discrete. $\square$

Proposition 1.3. *Suppose $G \leq \mathbb{C}$ is a closed, discrete subgroup. Then one of the following cases holds:*
(i) $G = \{0\}$; or
(ii) $G = \{n\omega : \ n \in \mathbb{Z}\}$ for some $\omega \in \mathbb{C} \backslash \{0\}$; or
(iii)

$$G = \{m\omega_1 + n\omega_2 : \ m, n \in \mathbb{Z}\}$$

for $\omega_1, \omega_2 \in \mathbb{C} \backslash \{0\}$ with the property $\tau := \frac{\omega_2}{\omega_1} \notin \mathbb{R}$. (In other words, $\{\omega_1, \omega_2\}$ is an $\mathbb{R}$-basis for $\mathbb{C}$.)
Conversely, any group of the form (i), (ii), (iii) is closed and discrete.

The proof relies on the following lemma of Jacobi:

Lemma 1.4 (Jacobi's lemma)**.** *Let $a, b, c \in \mathbb{C}$. Then*

$$\inf_{\substack{\ell,m,n\in\mathbb{Z} \\ (\ell,m,n)\neq(0,0,0)}} |\ell a + mb + nc| = 0.$$

Proof. Suppose the claim were false: that is, we could find $\delta > 0$ such that

$$|\ell a + mb + nc| > \delta \text{ for all } (\ell, m, n) \neq (0, 0, 0).$$

Then any two distinct tuples (ℓ, m, n) and $(\ell', m', n') \in \mathbb{Z}^3$ would satisfy

$$|(\ell - \ell')a + (m - m')b + (n - n')c| > \delta.$$

Therefore, for any $N \in \mathbb{N}$, the $(2N + 1)^3$ distinct points

$$\{\ell a + mb + nc : \ -N \leq \ell, m, n \leq N\}$$

would have distance at least δ from one another and (by the triangle inequality) would lie in the circle

$$\{z \in \mathbb{C} : \ |z| \leq 3N \cdot \max(|a|, |b|, |c|) =: R \cdot N\}.$$

In geometric terms we would be able to fit $(2N+1)^3$ circles of radius δ (centered at the above points) within a circle of radius $NR+\delta$ with center at the origin without overlap. For large N, we obtain a contradiction because the first set's area grows proportionally with N^3 while the second set's area is proportional to N^2. □

Proof of Proposition 1.3. G is a torsionfree abelian group. Its rank is at most two by Jacobi's lemma: if $a, b, c \in G$ were linearly independent over $\mathbb{Z}$, then the set $\mathbb{Z}a+\mathbb{Z}b+\mathbb{Z}c$ would contain nonzero numbers of arbitrary small absolute value, contradicting the fact that G is discrete. By the structure theorem for finitely generated abelian groups, either $G \cong \mathbb{Z}$ (which is case (ii)) or $G \cong \mathbb{Z}^2$ (which is case (iii)). In the latter case, writing

$$G = \{m\omega_1 + n\omega_2 : \ m, n \in \mathbb{Z}\},$$

we have $\tau = \omega_2/\omega_1 \notin \mathbb{R}$, because: supposing otherwise, define

$$a := \omega_1, \ b := \omega_2, \ c := i\omega_1.$$

By Jacobi's lemma,

$$\inf_{\substack{\ell,m,n\in\mathbb{Z} \\ (\ell,m,n)\neq(0,0,0)}} |\ell a + mb + nc|^2 = |\omega_1|^2 \cdot \inf_{(\ell,m,n)\neq(0,0,0)} \Big(|\ell + m\tau|^2 + n^2\Big) = 0,$$

which forces $\inf_{(\ell,m)\neq(0,0)} |\ell + m\tau|^2 = 0$ and therefore $0 \in \overline{G\backslash\{0\}}$. That is impossible because G is discrete. □

A closed, discrete subgroup $L \leq G$ of rank two (i.e. case (iii) in the Theorem) is called a **lattice**.

Definition 1.5. Let $L \leq \mathbb{C}$ be a lattice.
An **elliptic function** for L is a meromorphic function $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ that satisfies

$$f(z+\lambda) = f(z) \text{ for all } \lambda \in L.$$

In other words, f is either a constant or its period group is

$$\mathrm{Per}(f) = L.$$

Clearly the sum, difference, product and quotient of two functions that are L-periodic are again L-periodic. Also, the set of elliptic functions with period lattice L is closed under differentiation.

1.2. Properties of elliptic functions

Before constructing the first examples of elliptic functions, we will derive some of their general properties. Let $L \leq \mathbb{C}$ be a lattice.

Definition 1.6. A **fundamental parallelogram** for L is any set P of the form

$$P = \Big\{ a\omega_1 + b\omega_2 : \ 0 \leq a, b \leq 1 \Big\}$$

where ω_1, ω_2 is a basis of L.

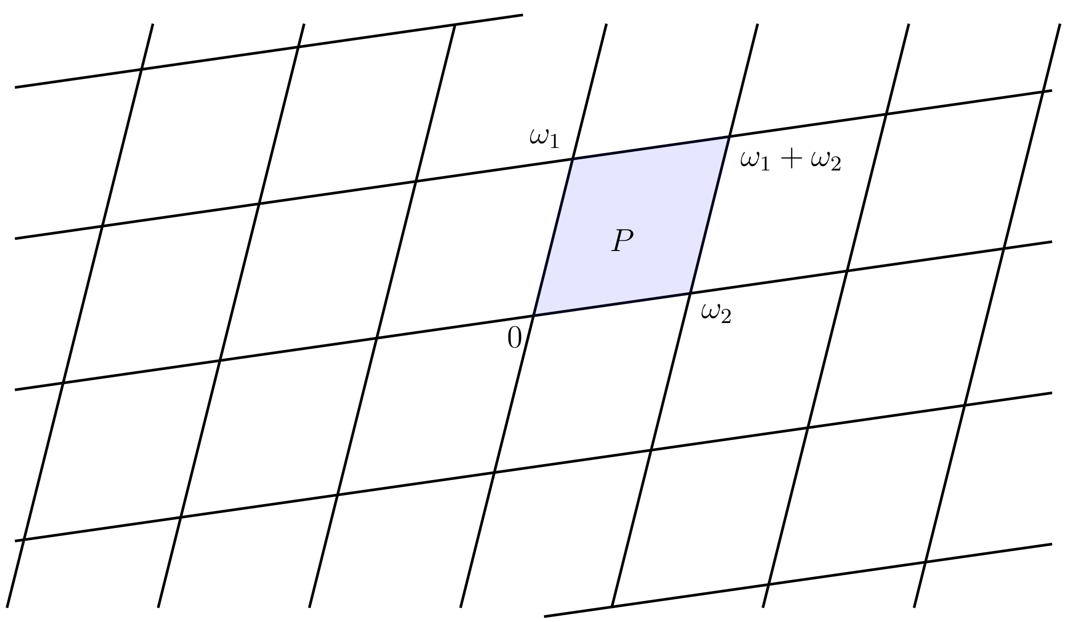


Figure 1.1: A fundamental parallelogram.

So P is a compact set, and the points of P are in bijection with cosets of $\mathbb{C}/L$ (with the exception of opposing sides, which are the same in $\mathbb{C}/L$). For the rest of this section, fix a basis $\{\omega_1, \omega_2\}$ for L and thus a parallelogram P.

An elliptic function is completely determined by its values on P. This observation quickly leads to strong restrictions for elliptic functions, known as *Liouville's theorems*:

Theorem 1.7 (Liouville's first theorem)**.** *Any holomorphic elliptic function is constant.*

Proof. For any $z \in \mathbb{C}$, we can find $\lambda \in L$ such that $z + \lambda \in P$, and then $f(z) = f(z + \lambda)$. This means that

$$f(\mathbb{C}) = f(P).$$

But P is compact, so $f(\mathbb{C})$ is also compact and therefore bounded. By Liouville's theorem (any bounded entire function is constant), f is constant. □

Theorem 1.8 (Liouville's second theorem)**.** *Let f be an elliptic function. Then*

$$\sum_{[w] \in \mathbb{C}/L} \operatorname{Res}_w(f) = 0.$$

If $[w] \in \mathbb{C}/L$ then $\operatorname{Res}_w(f)$ stands for the residue of f at any representative w. This is well-defined because

$$f(z + \lambda) = f(z), \quad \lambda \in L$$

implies that the orders and residues of f in w and any $w + \lambda$ are equal.

Proof. Choose $u \in \mathbb{C}$ such that none of the poles of f lie on the boundary of the shifted parallelogram $u + P$.

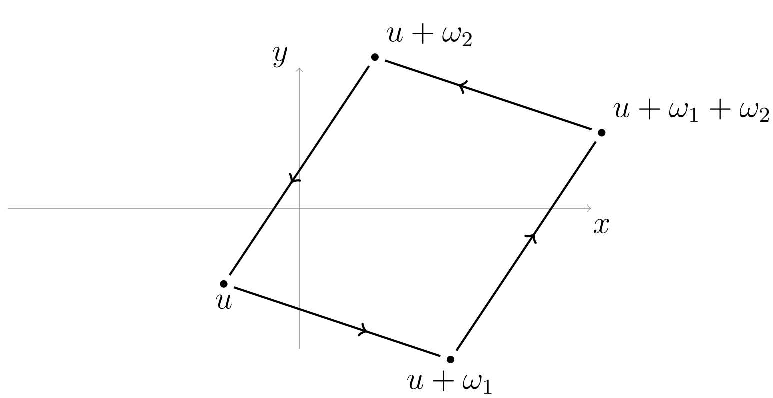


Figure 1.2: Path of integration $u + \partial P$

Using the residue theorem we obtain

$$
\begin{aligned}
&\pm 2\pi i \cdot \sum_{[w]\in\mathbb{C}/L} \mathrm{Res}_w(f) \\
&= \oint_{u+\partial P} f(z)\,\mathrm{d}z \\
&= \int_u^{u+\omega_1} f(z)\,\mathrm{d}z + \int_{u+\omega_1}^{u+\omega_1+\omega_2} f(z)\,\mathrm{d}z + \int_{u+\omega_1+\omega_2}^{u+\omega_2} f(z)\,\mathrm{d}z + \int_{u+\omega_2}^{u} f(z)\,\mathrm{d}z \\
&= \int_u^{u+\omega_1} \Big(f(z) - f(z+\omega_2)\Big)\,\mathrm{d}z - \int_u^{u+\omega_2} \Big(f(z) - f(z+\omega_1)\Big)\,\mathrm{d}z,
\end{aligned}
$$

where the sign depends on whether the basis $\{\omega_1, \omega_2\}$ of L is positively or negatively oriented. The integrands in the last line are identically zero because f is elliptic. □

Theorem 1.9 (Liouville's third theorem)**.** *Let f be an elliptic function and $a \in \mathbb{C}$. Then*

$$\sum_{[w]\in\mathbb{C}/L} \mathrm{ord}_w(f-a) = 0.$$

In words: the number of poles of f mod L is equal to its number of zeros or even the number of times it takes on the value a for any $a \in \mathbb{C}$, as long as we count with multiplicity.

Proof. Let g be the elliptic function

$$g(z) := \frac{f'(z)}{f(z)-a},$$

such that $\mathrm{Res}_w(g) = \mathrm{ord}_w(f-a)$. The claim is exactly Liouville's second theorem applied to g. □

Theorem 1.10 (Liouville's fourth theorem)**.** *Let f be an elliptic function. Then*

$$\sum_{[w]\in\mathbb{C}/L} \mathrm{ord}_w(f)\cdot[w] = [0] \in \mathbb{C}/L.$$

In other words: if $a_1, ..., a_N$ represent the zeros and $b_1, ..., b_N$ the poles of f modulo L (counted with multiplicities), then

$$a_1 + ... + a_N \equiv b_1 + ... + b_N \bmod L.$$

Proof. We follow the proof of Liouville's second theorem but integrate $z\frac{f'(z)}{f(z)}\,\mathrm{d}z$ rather than $f(z)\,\mathrm{d}z$ along the boundary of $u+P$. This yields

$$
\begin{aligned}
&\pm 2\pi i \cdot \sum_{w\in u+P} \operatorname{ord}_w(f)\cdot w \\
&= \oint z\frac{f'(z)}{f(z)}\,\mathrm{d}z \\
&= \int_u^{u+\omega_1} \Big(z\frac{f'(z)}{f(z)} - (z+\omega_2)\frac{f'(z+\omega_2)}{f(z+\omega_2)}\Big)\,\mathrm{d}z - \int_u^{u+\omega_2} \Big(z\frac{f'(z)}{f(z)} - (z+\omega_1)\frac{f'(z+\omega_1)}{z+\omega_2}\Big)\,\mathrm{d}z \\
&= \omega_1 \int_u^{u+\omega_2} \frac{f'(z)}{f(z)}\,\mathrm{d}z - \omega_2 \int_u^{u+\omega_1} \frac{f'(z)}{f(z)}\,\mathrm{d}z.
\end{aligned}
$$

By Cauchy's integral formula, $\frac{1}{2\pi i}\int_u^{u+\omega_1} \frac{f'(z)}{f(z)}\,\mathrm{d}z =: n_1$ is an integer (more precisely, it is the winding number of the curve

$$[0,1]\to\mathbb{C},\ t\mapsto f(u+t\omega_1)$$

about the origin.) Similarly, $\frac{1}{2\pi i}\int_u^{u+\omega_2} \frac{f'(z)}{f(z)}\,\mathrm{d}z =: n_2 \in \mathbb{Z}$. Altogether,

$$\sum_{w\in u+P} \operatorname{ord}_w(f)\cdot w = \pm(n_1\omega_1 + n_2\omega_2) \in L. \qquad \square$$

1.3. The Weierstrass elliptic function

Most functions f satisfy $\mathrm{Per}(f)=\{0\}$, and some well-known functions such as exp, or sin or cos, have a single period: for example,

$$\mathrm{Per}(\exp) = \{2\pi i n:\ n\in\mathbb{Z}\}.$$

The existence of doubly periodic or elliptic functions is less obvious.

Our first examples of elliptic functions will be series of the form

$$f_k(z) := \sum_{\lambda\in L} \frac{1}{(z-\lambda)^k}.$$

If we can show that f_k converges (locally uniformly), then it is elliptic by a rearrangement of the series: for $\mu\in L$,

$$f_k(z+\mu) = \sum_{\lambda\in L} \frac{1}{(z-(\lambda-\mu))^k}$$

where $\lambda-\mu$ also runs through L as λ does.

Lemma 1.11. *Let $L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \le \mathbb{C}$ be a lattice. Then*

$$\sum_{\substack{\lambda\in L\\ \lambda\neq 0}} \frac{1}{|\lambda|^a} = \sum_{\substack{m,n\in\mathbb{Z}\\ (m,n)\neq(0,0)}} \frac{1}{|m\omega_1+n\omega_2|^a}$$

converges if and only if $a>2$.

Proof. Observe that
$$\|(m,n)\| := |m\omega_1 + n\omega_2|$$
defines a norm on $\mathbb{R}^2$. Since all norms on $\mathbb{R}^2$ are equivalent, we can find constants $c, C > 0$ such that
$$c \cdot \max(|m|,|n|) \le |m\omega_1 + n\omega_2| \le C \cdot \max(|m|,|n|)$$
for all $m, n \in \mathbb{R}$. Therefore, $\sum_{\lambda \in L \setminus \{0\}} |\lambda|^{-a}$ converges if and only if the series
$$\sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{\max(|m|,|n|)^a}$$
converges.

But for each $N \in \mathbb{N}$, there are exactly $8N$ pairs (m,n) with $\max(|m|,|n|) = N$, so the latter series can be rearranged as
$$\sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{\max(|m|,|n|)^a} = \sum_{N=1}^{\infty} \frac{8}{N^{a-1}}.$$
This converges for $a - 1 > 1$ and diverges otherwise. □

Proposition 1.12. *For any $k \ge 3$, the series*
$$f_k(z) := \sum_{\lambda \in L} \frac{1}{(z-\lambda)^k}, \; z \in \mathbb{C} \setminus L$$
is an elliptic function with poles of order k exactly in the points of L. Around any $\omega \in L$, its Laurent series begins
$$f_k(z) = (z-\omega)^{-k} + O(z^0).$$
In addition, $f_k(z)$ satisfies
$$f_k(-z) = (-1)^k f(z).$$

Proof. Let $K \subseteq \mathbb{C} \setminus L$ be a compact set. Then there exists a constant $C = C_K > 0$ such that
$$|z - \lambda| > C \cdot |\lambda| \quad \text{for all } \lambda \in L \text{ and all } z \in K,$$
because: suppose $|z| \le M$ for all $z \in K$. For all $\lambda \in L$ with $|\lambda| \ge 2M$, we have
$$\Big|\frac{z}{\lambda} - 1\Big| \ge 1 - \frac{|z|}{|\lambda|} \ge \frac{1}{2}, \quad z \in K.$$
For each of the finitely many λ with $|\lambda| < 2M$, we find some constant $C(\lambda)$ with
$$\Big|\frac{z}{\lambda} - 1\Big| \ge C(\lambda) > 0, \quad z \in K,$$

since $\frac{z}{\lambda} - 1$ is nonvanishing on K. Then take

$$C_K := \min\Big(1/2, \min_{\substack{\lambda \in L \\ |\lambda| < 2M}} C(\lambda)\Big).$$

Therefore, we have the uniform majorant

$$\sum_{\lambda \in L} \frac{1}{|z-\lambda|^k} \leq \frac{1}{z^k} + C_K \sum_{\substack{\lambda \in L \\ \lambda \neq 0}} \frac{1}{|\lambda|^k}, \quad z \in K,$$

so the series is holomorphic on $\mathbb{C} \backslash L$ by the Weierstrass M-test.

About any $\omega \in L$, we can write

$$f_k(z) = (z-\omega)^{-k} + \sum_{\substack{\lambda \in L \\ \lambda \neq \omega}} \frac{1}{(z-\lambda)^k},$$

where the remaining series is holomorphic in ω. This determines the beginning of the Laurent series.

For the final claim, substitute $\lambda \mapsto (-\lambda)$ in the series to obtain

$$f_k(z) = \sum_{\lambda \in L} \frac{1}{(z+\lambda)^k} = \sum_{\lambda \in L} \frac{(-1)^k}{(-z-\lambda)^k} = (-1)^k f_k(-z). \qquad \square$$

This construction does not work when $k = 2$: the series

$$\sum_{\lambda \in L} \frac{1}{(z-\lambda)^2}$$

does not converge absolutely. But since the exponent $k = 2$ is just at the threshold of convergence, there is a workaround:

Theorem 1.13. *The* ***Weierstrass $\wp$-function***

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\lambda \in L \\ \lambda \neq 0}} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right]$$

is an elliptic function with double poles exactly in the points of L. Its Laurent series about $z = 0$ begins

$$\wp(z) = z^{-2} + O(z^2).$$

Proof. The series

$$\sum_{\substack{\lambda \in L \\ \lambda \neq 0}} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right] = \sum_{\substack{\lambda \in L \\ \lambda \neq 0}} \frac{2\lambda z - z^2}{\lambda^2 (z-\lambda)^2}$$

defines a holomorphic function on $z \in \mathbb{C}\backslash L$ by the Weierstrass M-test applied to $\sum_{\substack{\lambda\in L\\ \lambda\neq 0}} \frac{1}{\lambda^3}$, by essentially the same argument as for the elliptic functions $f_k(z)$. Hence we can differentiate the series termwise. We find

$$\wp'(z) = -\frac{2}{z^3} + \sum_{\substack{\lambda\in L\\ \lambda\neq 0}} \frac{-2}{(z-\lambda)^3} = -2f_3(z).$$

Since $\wp'(z)$ is an odd function, $\wp(z)$ is even.

The Laurent series about 0 begins

$$\wp(z) = \frac{1}{z^2} + O(z^2),$$

since the series $\sum_{\substack{\lambda\in L\\ \lambda\neq 0}} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right]$ vanishes at $z = 0$.

Finally, the fact that $\wp$ is an elliptic function follows from the following lemma. □

Lemma 1.14. *Suppose f is an even meromorphic function whose derivative is elliptic. Then f is elliptic.*

Proof. Let $\lambda \in L$ and consider the function

$$c_\lambda(z) = f(z+\lambda) - f(z).$$

By assumption,

$$\frac{d}{dz}c_\lambda(z) = f'(z+\lambda) - f'(z) = 0,$$

so c_λ is a constant. In addition, the map

$$L \to \mathbb{C},\ \lambda \mapsto c_\lambda$$

is a group homomorphism: since $c_0 = 0$ and

$$c_{\lambda+\mu} = f(z+\lambda+\mu) - f(z) = \Big(f(z+\lambda+\mu) - f(z+\mu)\Big) + \Big(f(z+\mu) - f(z)\Big) = c_\lambda + c_\mu.$$

In particular $c_{-\lambda} = -c_\lambda$. But since f is even, we have

$$c_{-\lambda} = f(z-\lambda) - f(z) = f(-z+\lambda) - f(-z) = c_\lambda(-z) = c_\lambda.$$

So $c_\lambda = 0$ for all $\lambda \in L$, which is the same as saying that f is elliptic. □

Proposition 1.15. *The complete Laurent series expansion of $\wp(z)$ about $z = 0$ is*

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty}(2n+1)G_{2n+2}(L)z^{2n} = z^{-2} + 3G_4(L)z^2 + 5G_6(L)z^4 + ...$$

where G_k is the ***Eisenstein series***

$$G_k(L) = \sum_{\substack{\lambda\in L\\ \lambda\neq 0}} \lambda^{-k}.$$

Proof. Write

$$\wp(z) = z^{-2} + \sum_{n=1}^{\infty} c_n z^{2n}.$$

(Note that only even exponents appear, since $\wp$ is even.) Then

$$c_n = \frac{1}{(2n)!}\frac{d^{2n}}{dz^{2n}}\Big|_{z=0}\Big(\wp(z) - z^{-2}\Big) = (2n+1)\sum_{\substack{\lambda\in L\\ \lambda\neq 0}}\frac{1}{(z-\lambda)^{2n+2}}\Big|_{z=0} = (2n+1)G_{2n+2}(L). \quad \square$$

1.4. The Weierstrass equation and the field of elliptic functions

Let L be a lattice with basis $\{\omega_1, \omega_2\}$.

The notation $\mathcal{E}(L)$ will be used for the set of elliptic functions for L. Clearly, $\mathcal{E}(L)$ is a field (a subfield of the field of meromorphic functions on $\mathbb{C}$): if f and g satisfy

$$f(z+\lambda) = f(z) \text{ and } g(z+\lambda) = g(z)$$

for all $\lambda \in L$, then this is also true of $f+g$ and $f \cdot g$, and of $1/f$ (if f is not identically zero). $\mathcal{E}(L)$ also contains the field $\mathbb{C}$ of constant functions.

The main result of this section is that the Weierstrass $\wp$-function and its derivative already generate the field $\mathcal{E}(L)$.

Theorem 1.16. *Every even ($f(z) = f(-z)$) elliptic function f can be written as a rational function in $\wp$.*

Proof. Suppose f is nonzero. Since f is even, the order of f in any point a equals its order in $-a$. If a is equivalent to $-a$ in $\mathbb{C}/L$ (which happens exactly when $2a = \omega$ is a period), then

$$f(z+a) = f(z+a-\omega) = f(z-a) = f(-z+a)$$

implies that the Laurent series of $f(z+a)$ about $z=0$ contains only even exponents, so f has even order in those points.

So let $2m_0, 2m_1, 2m_2, 2m_3$ be the orders of f in the respective points $0, \omega_1/2, \omega_2/2, (\omega_1+\omega_2)/2$, which represent the points a modulo L with $2a \in L$.

Observe that if $2a \notin L$, then $\wp(z) - \wp(a)$ has simple zeros precisely in the points $z = \pm a$ modulo L (it cannot have any other zeros by Liouville's third theorem) and that if $2a \in L$ then $\wp(z) - \wp(a)$ has a double zero precisely in the point a mod L.

Therefore, if

$$\pm a_1, ..., \pm a_k$$

represent the pairs of zeros (with multiplicity) a with $2a \notin L$, and if

$$\pm b_1, ..., \pm b_\ell$$

represent the pairs of poles (with multiplicity) with $2b \notin L$, then the function

$$g(z) := \frac{f(z)}{(\wp(z)-\wp(\omega_1/2))^{m_1}(\wp(z)-\wp(\omega_2/2))^{m_2}(\wp(z)-\wp(\omega_1/2+\omega_2/2))^{m_3}} \times \frac{\prod_{j=1}^{\ell}(\wp(z)-\wp(b_j))}{\prod_{i=1}^{k}(\wp(z)-\wp(a_i))}$$

is an elliptic function without zeros or poles outside of the lattice points L. By Liouville's fourth theorem, g cannot have a pole in the lattice points either. By Liouville's first theorem, g is constant, which implies that f is the rational function

$$f = \text{const} \cdot (\wp(z)-\wp(\omega_1/2))^{m_1}(\wp(z)-\wp(\omega_2/2))^{m_2}(\wp(z)-\wp(\omega_1/2+\omega_2/2))^{m_3} \times \frac{\prod_{i=1}^{k}(\wp(z)-\wp(a_i))}{\prod_{j=1}^{\ell}(\wp(z)-\wp(b_j))}$$

in $\wp$. □

Corollary 1.17. *The field of elliptic functions has the form*

$$\mathcal{E}(L) = \mathbb{C}(\wp) \oplus \wp' \cdot \mathbb{C}(\wp).$$

Proof. Any elliptic function can be decomposed into its even and odd parts as

$$f(z) = \frac{f(z)+f(-z)}{2} + \frac{f(z)-f(-z)}{2}.$$

The even part belongs to $\mathbb{C}(\wp)$ by Theorem 1.16. Since $\wp'(z)$ is an odd function, the quotient $(\frac{f(z)-f(-z)}{2})/\wp'(z)$ is even and also belongs to $\mathbb{C}(\wp)$ by Theorem 1.16. □

Since $(\wp')^2$ is even, it has a representation has a rational function of $\wp$: in particular, $\wp$ satisfies a differential equation. The precise statement is as follows.

Theorem 1.18 (Weierstrass)**.** *The $\wp$-function satisfies the differential equation*

$$\wp'(z)^2 = 4\wp(z)^3 - 60G_4(L)\wp(z) - 140G_6(L),$$

where G_4 and G_6 are the Eisenstein series

$$G_4(L) = \sum_{\omega \in L\setminus\{0\}} \omega^{-4}, \quad G_6(L) = \sum_{\omega \in L\setminus\{0\}} \omega^{-6}$$

of weights 4 and 6.

Proof. $\wp'$ has no poles outside of the lattice points. It has forced zeros at the three points $a \in \frac{1}{2}L$ with $a \notin L$, and no other zeros by Liouville's third theorem. The proof of Theorem 1.16 shows that $(\wp')^2$ has the representation

$$(\wp')^2 = (\text{constant}) \cdot (\wp(z) - \wp(\omega_1/2))(\wp(z) - \wp(\omega_2/2))(\wp(z) - \wp(\omega_1/2 + \omega_2/2)),$$

i.e. as

$$(\wp')^2 = A\wp^3 + B\wp^2 + C\wp + D$$

for some (unique!) $A, B, C, D \in \mathbb{C}$.

Using the Laurent series

$$\wp(z) = z^{-2} + 3G_4z^2 + 5G_6z^4 + O(z^6)$$

and

$$\wp'(z) = -2z^3 + 6G_4z + 20G_6z^3 + O(z^5),$$

we take $(\wp')^2 - 4\wp^3$ to cancel the leading coefficient and obtain

$$(\wp')^2 - 4\wp^3 = -60G_4z^{-2} - 140G_6;$$

and the only linear combination of $\wp^3, \wp^2, \wp, 1$ that can produce the right-hand side of that is $-60G_4\wp - 140G_6$. ☐

Corollary 1.19.
(i) $\wp''(z) = 6\wp^2 - 30G_4$.
(ii) $\wp'''(z) = 12\wp(z) \cdot \wp''(z)$.

Proof. (i) Differentiate the Weierstrass equation and divide by $2\wp'(z)$.
(ii) Differentiate (i) and divide by $\wp'$. ☐

For any $w \in \mathbb{C}$, the function

$$z \mapsto \wp(z + w)$$

is also an elliptic function, and can therefore be expressed in terms of $\wp$ and $\wp'$ alone. Any such representation is an **addition formula** for $\wp$.

Theorem 1.20 (Addition formula)**.**

$$\wp(z+w) = \frac{1}{4}\Big(\frac{\wp'(z)-\wp'(w)}{\wp(z)-\wp(w)}\Big)^2 - \wp(z) - \wp(w).$$

Proof. Assume without loss of generality that $2w$ is not a lattice point. (That case follows by a continuity argument.) Consider the function

$$f(z) := \frac{\wp'(z)-\wp'(w)}{\wp(z)-\wp(w)}.$$

This has exactly two simple poles modulo L, in $z=0$ and $z=-w$, and a short computation shows that the Laurent series about those points begin

$$f(z) = -2z^{-1} - 2\wp(w)z - \wp'(w)z^2 + (12G_4 - 2\wp^2(w))z^3 + ...$$

$$f(z-w) = 2z^{-1} + \frac{12\wp(w)}{\wp'(w)}(\wp^2(w) - 5G_4)z + O(z^2).$$

So $\frac{1}{4}f(z)^2 - \wp(z) - \wp(z+w)$ is an entire elliptic function (for fixed w, viewed as a function of z) hence constant. Since

$$\begin{aligned}\frac{1}{4}f(z)^2 - \wp(z) - \wp(z+w) &= \Big(z^{-2} + 2\wp(w) + O(z)\Big) - \Big(z^{-2} + O(z)\Big) - \Big(\wp(w) + O(z)\Big)\\ &= \wp(w) + O(z),\end{aligned}$$

the constant is $\wp(w)$. □

2. Theta functions

The Weierstrass constructions of elliptic functions suffer from very poor convergence. (Indeed the reason for the strange-looking definition of $\wp$ is the fact that the series $\sum_{\lambda \in L} \frac{1}{(z-\lambda)^2}$ fails to converge altogether.) For both practical and theoretical purposes, it is natural to construct elliptic functions as quotients f/g where f and g are entire functions and therefore *not* elliptic, but transform under translations in essentially the same way. We will study a class of functions of that type here.

2.1. Quasiperiodic functions

Definition 2.1. An entire function $f : \mathbb{C} \to \mathbb{C}$ is called **quasiperiodic** with **quasiperiod** λ if there are constants $A = A(\lambda)$, $B = B(\lambda) \in \mathbb{C}$ such that

$$f(z+\lambda) = e^{Az+B} f(z), \quad z \in \mathbb{C}.$$

The exponent $A = A(\lambda)$ is then uniquely determined from λ (as long as f is not identically zero), while $B(\lambda)$ is only determined modulo $2\pi i\mathbb{Z}$.

The quasiperiods λ of f form a group. Indeed, if f is not identically zero and λ, μ are any two of its quasiperiods, then

$$\begin{aligned} e^{A(\lambda+\mu)z+B(\lambda+\mu)} f(z) &= f(z+\lambda+\mu) \\ &= f((z+\lambda)+\mu) \\ &= e^{A(\mu)(z+\lambda)+B(\mu)} f(z+\lambda) \\ &= e^{A(\mu)(z+\lambda)+B(\mu)+A(\lambda)z+B(\lambda)} f(z), \end{aligned}$$

so A is a homomorphism on the group of quasiperiods,

$$A(\lambda+\mu) = A(\lambda) + A(\mu),$$

and B satisfies

$$B(\lambda+\mu) = B(\lambda) + B(\mu) + \lambda \cdot A(\mu) \bmod 2\pi i\mathbb{Z}.$$

Clearly $B(\lambda+\mu)$ is symmetric in λ and μ, so the right-hand side of the above equation is as well (which is less obvious): in particular,

$$\lambda A(\mu) - \mu A(\lambda) \in 2\pi i\mathbb{Z}$$

for any two quasiperiods λ, μ of f. (See also Proposition 2.5 below for a more precise statement.)

The possible quasiperiod groups are constrained by the following lemma.

Lemma 2.2. *Let f be a nonzero function. The following are equivalent:*
(i) $\omega \in \mathbb{C}$ is a quasiperiod of f;
(ii) ω is a period of $\left(\frac{f'}{f}\right)'$.

So unless $\left(\frac{f'}{f}\right)'$ is constant, the group of quasiperiods of f is either $\{0\}$, or of the form $\{n\omega : \ n \in \mathbb{Z}\}$, or is a lattice L. In the third case f is called *doubly quasiperiodic* or a *theta function* for L. (We reserve the name "theta function" for certain specific theta functions to be discussed later.)

Proof. Taking logarithmic derivatives in the identity

$$f(z+\omega) = e^{Az+B} f(z)$$

yields

$$\frac{f'(z+\omega)}{f(z+\omega)} = A + \frac{f'(z)}{f(z)}.$$

Hence $\left(\frac{f'}{f}\right)'$ is elliptic. Conversely, if $\left(\frac{f'}{f}\right)'$ is elliptic, then we have

$$\frac{f'(z+\omega)}{f(z+\omega)} = A + \frac{f'(z)}{f(z)}$$

for some constant A (depending on ω), which implies (locally)

$$\operatorname{Log} f(z+\omega) = Az + B + \operatorname{Log} f(z)$$

for some other constant B (again depending on ω) and therefore

$$f(z+\omega) = e^{Az+B} f(z). \qquad \square$$

Example 2.3. The entire functions f that have every complex number $\omega \in \mathbb{C}$ as a quasiperiod are precisely those for which $\left(\frac{f'}{f}\right)'$ is constant. Taking antiderivatives twice, this implies (locally) that

$$\operatorname{Log}(f) = az^2 + bz + c$$

for some constants a, b, c, i.e.

$$f(z) = e^{az^2+bz+c}.$$

The function $f(z) = e^{az^2+bz+c}$ is indeed quasiperiodic with respect to every $\omega \in \mathbb{C}$, and the associated exponents A, B in this case are

$$A(\omega) = 2a\omega, \quad B(\omega) = a\omega^2 + b\omega.$$

Remark 2.4. Suppose f is a doubly quasiperiodic function satisfying

$$f(z+\lambda)=e^{A(\lambda)z+B(\lambda)}f(z),\quad \lambda\in L.$$

By the above example, we can modify f by an entire function without zeros to have any fixed $\omega\in L\backslash\{0\}$ as a true period: since

$$g(z):=e^{\frac{A(\omega)}{2\omega}z^2+(\frac{B(\omega)}{\omega}-\frac{A(\omega)}{2})z}$$

also satisfies

$$g(z+\omega)=e^{A(\omega)z+B(\omega)}g(z),$$

the function

$$h(z):=e^{-\frac{A(\omega)}{2\omega}z^2+\frac{A(\omega)}{2}z-\frac{B(\omega)}{\omega}z}f(z)$$

is doubly quasiperiodic and satisfies $h(z+\omega)=h(z)$.

Doubly quasiperiodic functions satisfy the following analogue of Liouville's third theorem:

Proposition 2.5 (Legendre relation)**.** *Suppose $\{\omega_1,\omega_2\}$ is an oriented basis of L; that is, $\tau:=\omega_2/\omega_1$ belongs to the upper half-plane. Let f be a nonzero doubly quasiperiodic function for the lattice L. Then*

$$\omega_2A(\omega_1)-\omega_1A(\omega_2)=2\pi iN,$$

where N is the number of zeros of f in any fundamental parallelogram P (counted with multiplicity).

Proof. We proceed exactly as in the case of elliptic functions. Let $u\in\mathbb{C}$ be chosen such that none of the zeros of f lie on the boundary of $u+P$.

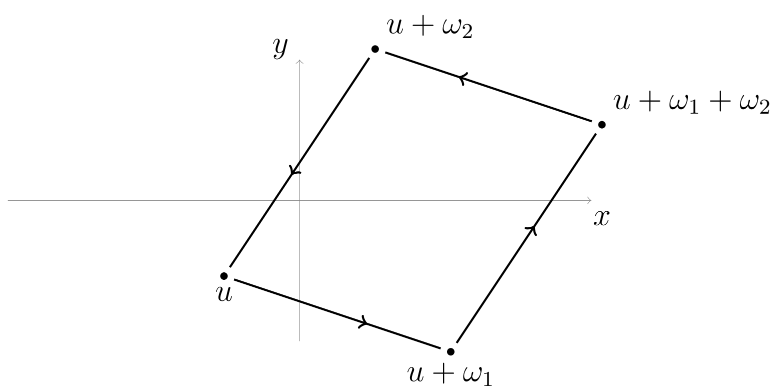


Figure 2.1: Path of integration

Integrate $\frac{f'(z)}{f(z)}\,\mathrm{d}z$ along the above contour and use the residue theorem to obtain

$$2\pi iN=\oint_{u+\partial P}\frac{f'(z)}{f(z)}\,\mathrm{d}z\quad=\int_u^{u+\omega_1}\Big(\frac{f'(z)}{f(z)}-\frac{f'(z+\omega_2)}{f(z+\omega_2)}\Big)\,\mathrm{d}z-\int_u^{u+\omega_2}\Big(\frac{f'(z)}{f(z)}-\frac{f'(z+\omega_1)}{f(z+\omega_1)}\Big)\,\mathrm{d}z.$$

Taking logarithmic derivatives in

$$f(z+\omega) = e^{A(\omega)z+B(\omega)}f(z)$$

yields

$$\frac{f'(z+\omega)}{f(z+\omega)} = A(\omega) + \frac{f'(z)}{f(z)}.$$

So the above integrals simplify to

$$\begin{aligned}
&\int_u^{u+\omega_1}\left(\frac{f'(z)}{f(z)} - \frac{f'(z+\omega_2)}{f(z+\omega_2)}\right)\mathrm{d}z - \int_u^{u+\omega_2}\left(\frac{f'(z)}{f(z)} - \frac{f'(z+\omega_1)}{f(z+\omega_1)}\right)\mathrm{d}z \\
&= \int_u^{u+\omega_1}(-A(\omega_2))\,\mathrm{d}z - \int_u^{u+\omega_2}(-A(\omega_1))\,\mathrm{d}z \\
&= \omega_2 A(\omega_1) - \omega_1 A(\omega_2).
\end{aligned}$$

□

Remark. The integer N is independent of the choice of (oriented) basis: any other basis ω_1', ω_2' of L can be represented as

$$\omega_1' = a\omega_1 + b\omega_2, \quad \omega_2' = c\omega_1 + d\omega_2$$

for some matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, and then

$$\begin{aligned}
\omega_2' A(\omega_1') - \omega_1' A(\omega_2') &= (c\omega_1 + d\omega_2)A(a\omega_1 + b\omega_2) - (a\omega_1 + b\omega_2)A(c\omega_1 + d\omega_2) \\
&= (ad-bc)\omega_2 A(\omega_1) - (ad-bc)\omega_1 A(\omega_2) \\
&= \omega_2 A(\omega_1) - \omega_1 A(\omega_2).
\end{aligned}$$

This can also be proved directly.

2.2. The Weierstrass sigma function

It follows from Lemma 2.2 that a nonzero function f is doubly quasiperiodic with respect to a lattice L if and only if $g := \left(\frac{f'}{f}\right)'$ is a nonconstant elliptic function. The poles of any such g are also tightly constrained (they can only be double poles at which g has zero residue). But we have already encountered one such g: The **Weierstrass sigma function** is a choice of f whose g is (up to sign) the Weierstrass $\wp$-function.

Let $L \leq \mathbb{C}$ be a lattice.

Definition 2.6.
The **Weierstrass σ-function** attached to L is the infinite product

$$\sigma(z) = z \cdot \prod_{\lambda \in L\setminus\{0\}} \left[\left(1 - \frac{z}{\lambda}\right)e^{z/\lambda + z^2/2\lambda^2}\right].$$

An infinite product $\prod_n a_n$ converges if and only if the series $\sum_n \mathrm{Log}(a_n)$ converges (where Log is the principal branch of the logarithm), and the notions of absolute convergence and (for functions) uniform convergence for that product and that series are equivalent. When z is confined to any compact set, since

$$\begin{aligned}\mathrm{Log}\Big((1-z/\lambda)e^{z/\lambda+z^2/2\lambda^2}\Big) &= \frac{z}{\lambda}+\frac{z^2}{2\lambda^2}+\mathrm{Log}\Big(1-\frac{z}{\lambda}\Big)\\ &= \frac{z}{\lambda}+\frac{z^2}{2\lambda^2}-\sum_{n=1}^{\infty}\frac{z^n}{n\lambda^n}\\ &\le C\cdot\frac{1}{\lambda^3}\end{aligned}$$

for some constant C and all sufficiently large $|\lambda|$, the series (and therefore the product) converges absolutely and locally uniformly and the product defines an entire function. Moreover, σ has only simple zeros and they occur precisely at the lattice points $z \in L$.

Proposition 2.7. *The Weierstrass σ-function is a doubly quasiperiodic function. It satisfies*

$$\Big(\frac{\sigma'}{\sigma}\Big)' = -\wp.$$

Proof. From

$$\mathrm{Log}\,\sigma(z) = \mathrm{Log}(z) + \sum_{\lambda\in L\setminus\{0\}}\Big[\mathrm{Log}\Big(1-\frac{z}{\lambda}\Big)+\frac{z}{\lambda}+\frac{z^2}{2\lambda^2}\Big]$$

we have

$$\begin{aligned}\frac{\sigma'(z)}{\sigma(z)} &= \frac{d}{dz}\mathrm{Log}\,\sigma(z)\\ &= \frac{1}{z}+\sum_{\lambda\in L\setminus\{0\}}\Big[\frac{1}{z-\lambda}+\frac{1}{\lambda}+\frac{z}{\lambda^2}\Big],\end{aligned}$$

hence

$$\Big(\frac{\sigma'(z)}{\sigma(z)}\Big)' = -\frac{1}{z^2}+\sum_{\lambda\in L\setminus\{0\}}\Big[-\frac{1}{(z-\lambda)^2}+\frac{1}{\lambda^2}\Big] = -\wp(z).$$ ◻

The function

$$\begin{aligned}\zeta(z) := \frac{\sigma'(z)}{\sigma(z)} &= \frac{1}{z}+\sum_{\lambda\in L\setminus\{0\}}\Big[\frac{1}{z-\lambda}+\frac{1}{\lambda}+\frac{z}{\lambda^2}\Big]\\ &= \frac{1}{z}+\sum_{\lambda\in L\setminus\{0\}}\frac{z^2}{\lambda^2(z-\lambda)}\end{aligned}$$

is called the Weierstrass ζ-function.

Fix a basis $\{\omega_1, \omega_2\}$ of L. We will determine the constants $A(\lambda), B(\lambda)$ in the identity

$$\sigma(z+\lambda) = e^{A(\lambda)z+B(\lambda)}\sigma(z).$$

Observe that $\zeta = \sigma'/\sigma$ satisfies

$$\zeta(z+\lambda) = A(\lambda) + \zeta(z).$$

If $\omega \in L$ with $\omega/2 \notin L$, then we can evaluate ζ at $\omega/2$ and we find

$$\zeta(\omega/2) = A(\omega) + \zeta(-\omega/2).$$

But ζ is an odd function (as seen by substituting $\lambda \mapsto -\lambda$ in the defining series) so

$$\zeta(\omega/2) = A(\omega) - \zeta(\omega/2)$$

and therefore

$$A(\omega) = 2\zeta(\omega/2).$$

The traditional notation is

$$\eta_i := \zeta(\omega_i/2),$$

such that $A(\omega_1) = 2\eta_1$ and $A(\omega_2) = 2\eta_2$.

Proposition 2.8. *The Weierstrass σ-function attached to the lattice $L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ satisfies*

$$\sigma(z+\omega_i) = -e^{2\eta_i z + \eta_i\omega_i}\sigma(z), \quad i = 1, 2.$$

That is: $A(\omega_i) = 2\eta_i$ and $B(\omega_i) = \eta_i\omega_i + \pi i$.

Proof. We already computed $A(\omega_i)$ so it remains to show that $B(\omega_i) = \eta_i\omega_i + \pi i$. This follows from the fact that $\sigma(z)$ is an odd function: substituting $\lambda \mapsto -\lambda$ in the defining product yields

$$\sigma(-z) = -\sigma(z).$$

But setting $z = -\omega_i/2$ in the identity

$$\sigma(z+\omega_i) = e^{2\eta_i z + B}\sigma(z)$$

yields

$$\sigma(\omega_i/2) = e^{B-\eta_i\omega_i}\sigma(-\omega_i/2) = e^{\pi i + B - \eta_i\omega_i}\sigma(\omega_i/2).$$

Hence $B = \eta_i\omega_i + \pi i \bmod 2\pi i$. $\square$

2.3. Fourier series

In this section, we consider doubly quasiperiodic functions f attached to the lattice

$$L = \mathbb{Z}\tau \oplus \mathbb{Z} = \{m\tau + n : \; m, n \in \mathbb{Z}\}$$

for some fixed $\tau \in \mathbb{H}$. Recall that we write

$$f(z+\lambda) = e^{A(\lambda)z + B(\lambda)} f(z), \quad \lambda \in L.$$

Throughout this section we make the assumption

$$f(z+1) = e^{2\pi i a} f(z), \tag{2.1}$$

i.e. $A(1) = 0$. This is more or less harmless since it becomes satisfied for any f after multiplying by e^{cz^2} for the appropriate value of c. The reason for making it is that it guarantees that f is represented by its Fourier series:

$$f(z) = \sum_{n \in \mathbb{Z}+a} c_n e^{2\pi i n z}, \quad c_n \in \mathbb{C}.$$

By the Legendre relation, f is doubly quasiperiodic if and only if

$$f(z+\tau) = e^{-2\pi i N z - 2\pi i b} f(z) \tag{2.2}$$

for some $b \in \mathbb{C}$, where $N \in \mathbb{N}$ is the number of zeros of f modulo L.

Lemma 2.9. *A Fourier series*

$$f(z) = \sum_{n \in \mathbb{Z}+a} c_n e^{2\pi i n z}$$

satisfies (2.2) if and only if its coefficients satisfy the recurrence

$$c_{n+N} = e^{2\pi i (n\tau + b)} c_n, \quad n \in \mathbb{Z} + a.$$

Proof. Any such Fourier series converges (very quickly!) as its coefficients decay at the rate of $e^{-\pi n^2 y}$ where $\tau = x + iy$. To see that f defines a theta function, compare Fourier coefficients in

$$\begin{aligned} f(z+\tau) &= \sum_{n \in \mathbb{Z}+a} c_n e^{2\pi i n (z+\tau)} \\ &= \sum_{n \in \mathbb{Z}+a} \left(c_n e^{2\pi i n \tau} \right) e^{2\pi i n z} \end{aligned}$$

and

$$\begin{aligned} e^{-2\pi i N z - 2\pi i b} f(z) &= \sum_{n \in \mathbb{Z}+a} c_n e^{-2\pi i b} e^{2\pi i (n-N) z} \\ &= \sum_{n \in \mathbb{Z}+a} \left(c_{n+N} e^{-2\pi i b} \right) e^{2\pi i n z}. \end{aligned}$$

Conversely the same calculation shows that the coefficients of a theta function satisfy that recurrence. ☐

Corollary 2.10. *The space of doubly quasiperiodic functions f satisfying (2.2) is N-dimensional.*

This is significant when $N = 1$, where the recurrence shows that any single Fourier coefficient determines the entire series:

Corollary 2.11. *Every entire function f satisfying*

$$f(z+1) = e^{2\pi i a} f(z), \quad f(z+\tau) = e^{-2\pi i z - 2\pi i b} f(z),$$

is a constant multiple of the theta function

$$\theta_{a,b}(z) := \sum_{n=-\infty}^{\infty} e^{\pi i n(n-1)\tau + 2\pi i n(a\tau+b)} \cdot e^{2\pi i(n+a)z}.$$

In terms of the "classical" theta function

$$\theta(\tau; z) := \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i n z}$$

we have

$$\theta_{a,b}(z) = e^{2\pi i a z} \theta(\tau; z + (a - 1/2)\tau + b).$$

Four particular theta functions, the **Jacobi theta functions**, play a major role in the classical theory of special functions and will appear later on in the course. Unfortunately there are many different notational conventions for them. We will use the following definitions[1]. They are multiples of $\theta_{a,b}$ for the values a and b indicated below.

(i) $(a = 0, b = \tau/2)$

$$\theta_{00}(z) := \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i n z}$$

satisfies

$$\theta_{00}(z+1) = \theta_{00}(z) \quad \text{and} \quad \theta_{00}(z+\tau) = e^{-\pi i \tau - 2\pi i z} \theta_{00}(z);$$

(ii) $(a = 0, b = 1/2 + \tau/2)$

$$\theta_{01}(z) := \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i n^2 \tau + 2\pi i n z}$$

satisfies

$$\theta_{01}(z+1) = \theta_{01}(z) \quad \text{and} \quad \theta_{01}(z+\tau) = -e^{-\pi i \tau - 2\pi i z} \theta_{01}(z);$$

[1]This notation follows Mumford's *Tata lectures on theta*, up to a factor of i in the definition of θ_{11}.

(iii) $(a = 1/2, b = \tau/2)$

$$\theta_{10}(z) := \sum_{n=-\infty}^{\infty} e^{\pi i (n+1/2)^2\tau + 2\pi i (n+1/2) z}$$

satisfies

$$\theta_{10}(z+1) = -\theta_{10}(z) \quad \text{and} \quad \theta_{10}(z+\tau) = e^{-\pi i\tau - 2\pi i z}\theta_{10}(z);$$

(iv) $(a = 1/2, b = 1/2 + \tau/2)$

$$\theta_{11}(z) := \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i (n+1/2)^2\tau + 2\pi i (n+1/2) z}$$

satisfies

$$\theta_{11}(z+1) = -\theta_{11}(z) \quad \text{and} \quad \theta_{11}(z+\tau) = -e^{-\pi i\tau - 2\pi i z}\theta_{11}(z).$$

The function θ_{11} is special because it is odd: substituting $n \mapsto -n-1$ in the series implies

$$\theta_{11}(-z) = -\theta_{11}(z).$$

In particular, $\theta_{11}(0) = 0$. But by the Legendre relation, θ_{11} has only one zero modulo L. So $\theta_{11}(z)$ has only simple zeros precisely in the lattice points $z \in L$. From this we can read off the zeros of any theta function:

Proposition 2.12. *An entire function $f \neq 0$ that satisfies*

$$f(z+1) = e^{2\pi i a} f(z), \quad f(z+\tau) = e^{-2\pi i z - 2\pi i b} f(z)$$

must have simple zeros precisely in the points

$$z = (m-a)\tau + (n + 1/2 - b), \quad m, n \in \mathbb{Z}$$

and nowhere else.

For example, the classical theta function $\theta(z) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z}$ with $a = 0$ and $b = \tau/2$ has its zeros in the points $z = (m+1/2)\tau + (n+1/2)$ with $m, n \in \mathbb{Z}$.

Proof. We observed earlier that $\theta_{11}(z) = 0$ exactly when $z = m\tau + n$ with $m, n \in \mathbb{Z}$. The claim follows from that because f is a multiple of $\theta_{a,b}(z)$ and because $\theta_{a,b}$ is itself a multiple of $\theta_{11}(z + a\tau + b - 1/2)$. □

2.4. The Weierstrass sigma function revisited

For $L = \mathbb{Z}\tau \oplus \mathbb{Z}$, the Weierstrass sigma function is

$$\sigma(z) = z \cdot \prod_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \left[\left(1 - \frac{z}{m\tau + n}\right) e^{\frac{z}{m\tau+n} + \frac{z^2/2}{(m\tau+n)^2}} \right]$$

and its functional equations are $\sigma(z+1) = -e^{2\eta_1 z+\eta_1}\sigma(z)$ and $\sigma(z+\tau) = -e^{2\eta_2 z+\eta_2\tau}\sigma(z)$ where

$$\eta_1 = \frac{\zeta(z+1) - \zeta(z)}{2} = \zeta(1/2)$$

and

$$\eta_2 = \frac{\zeta(z+\tau) - \zeta(z)}{2} = \zeta(\tau/2).$$

(Here $\zeta = \sigma'/\sigma$ is the Weierstrass zeta function.)

Lemma 2.13. *The exponents η_1 and η_2 satisfy*

$$\eta_1 = \frac{1}{2}\sum_{m=-\infty}^{\infty}\Big(\sum_{\substack{n\in\mathbb{Z}\\(m,n)\neq(0,0)}}\frac{1}{(m\tau+n)^2}\Big)$$

and

$$\eta_2 = \frac{\tau}{2}\sum_{n=-\infty}^{\infty}\Big(\sum_{\substack{m\in\mathbb{Z}\\(m,n)\neq(0,0)}}\frac{1}{(m\tau+n)^2}\Big).$$

Apart from the factor of τ, the series for η_1 and η_2 look rather similar and you might think the mix-up in the indices m, n in the sums for η_1 and η_2 is an error.

It is not. The two iterated series are *not* the same (and since the underlying double series does not converge absolutely, there is no reason to think that they should be the same). We fix the following definition:

Definition 2.14. For $\tau \in \mathbb{H}$,

$$G_2(\tau) := \sum_{m=-\infty}^{\infty}\Big(\sum_{\substack{n\in\mathbb{Z}\\(m,n)\neq(0,0)}}\frac{1}{(m\tau+n)^2}\Big).$$

Convergence problems of the double series notwithstanding, G_2 converges rapidly as a single series (over m). To see why, use the partial fractions identity

$$\frac{\pi^2}{\sin^2(\pi\tau)} = \sum_{n\in\mathbb{Z}}\frac{1}{(\tau+n)^2}.$$

Replacing τ by $m\tau$ (for $m \neq 0$) yields:

$$G_2(\tau) = \sum_{n\neq 0}\frac{1}{n^2} + \sum_{m\neq 0}\Big(\sum_{n\in\mathbb{Z}}\frac{1}{(m\tau+n)^2}\Big) = \frac{\pi^2}{3} + \sum_{m\neq 0}\frac{\pi^2}{\sin^2(\pi m\tau)}. \tag{2.3}$$

Here $|\sin^2(\pi m\tau)|^{-2}$ decays exponentially in m. By writing

$$\sin^2(\pi m\tau) = \Big(\frac{e^{\pi i m\tau} - e^{-\pi i m\tau}}{2i}\Big)^2 = -\frac{1}{4}q^{-m}(1-q^m)^2,$$

we get the Fourier series:

$$G_2(\tau) = \frac{\pi^2}{3} - 8\pi^2 \sum_{m=1}^{\infty} \frac{q^m}{(1-q^m)^2} = \frac{\pi^2}{3}\Big[1 - 24\sum_{n=1}^{\infty} \sigma(n) q^n\Big],$$

where $\sigma(n) = \sum_{d|n} d$ is the sum of the divisors of $n \in \mathbb{N}$.

Proof of Lemma 2.13. To compute η_1 we write

$$\begin{aligned}
2\eta_1 &= \zeta(z+1) - \zeta(z) \\
&= \frac{1}{z+1} - \frac{1}{z} \\
&+ \sum_{m=-\infty}^{\infty} \sum_{\substack{n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \Big[\frac{1}{z+1-m\tau-n} + \frac{1}{m\tau+n} + \frac{z+1}{(m\tau+n)^2} \\
&\qquad - \frac{1}{z-m\tau-n} - \frac{1}{m\tau+n} - \frac{z}{(m\tau+n)^2}\Big] \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \Big(\frac{1}{z+1-m\tau-n} - \frac{1}{z-m\tau-n}\Big) + \sum_{m=-\infty}^{\infty} \sum_{\substack{n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^2}.
\end{aligned}$$

For any fixed m, the series

$$\sum_{n\in\mathbb{Z}} \Big(\frac{1}{z+1-m\tau-n} - \frac{1}{z-m\tau-n}\Big)$$

is a telescoping series that sums to zero. Hence

$$2\eta_1 = \sum_{m=-\infty}^{\infty} \Big(\sum_{\substack{n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^2}\Big) = G_2(\tau).$$

η_2 is dealt with similarly, writing

$$\begin{aligned}
2\eta_2 &= \zeta(z+\tau) - \zeta(z) \\
&= \frac{1}{z+\tau} - \frac{1}{z} \\
&+ \sum_{n=-\infty}^{\infty} \sum_{\substack{m\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \Big[\frac{1}{z+\tau-m\tau-n} + \frac{1}{m\tau+n} + \frac{z+\tau}{m\tau+n)^2} \\
&\qquad - \frac{1}{z-m\tau-n} - \frac{1}{m\tau+n} - \frac{z}{(m\tau+n)^2}\Big] \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \Big(\frac{1}{z+\tau-m\tau-n} - \frac{1}{z-m\tau-n}\Big) + \sum_{n=-\infty}^{\infty} \sum_{\substack{m\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{\tau}{(m\tau+n)^2} \\
&= \sum_{n=-\infty}^{\infty} \sum_{\substack{m\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{\tau}{(m\tau+n)^2}.
\end{aligned}$$

□

The calculation of the values η_1, η_2 has a significant corollary:

Theorem 2.15. *The series*

$$G_2(\tau) = \sum_{m=-\infty}^{\infty} \Big(\sum_{\substack{n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^2} \Big)$$

satisfies

$$G_2\Big(-\frac{1}{\tau}\Big) = \tau^2 G_2(\tau) - 2\pi i \tau.$$

Proof. We can write

$$\begin{aligned} G_2\Big(-\frac{1}{\tau}\Big) &= \sum_{m=-\infty}^{\infty} \Big(\sum_{\substack{n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(-m/\tau + n)^2} \Big) \\ &= \tau^2 \sum_{m=-\infty}^{\infty} \Big(\sum_{\substack{n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(n\tau - m)^2} \Big) \\ &= 2\tau \cdot \eta_2, \end{aligned}$$

and $\eta_1 = 2G_2$.

The Legendre relation for the quasiperiods of $\sigma(z)$ is

$$\eta_2 - \tau \cdot \eta_1 = -\frac{1}{2}\Big(\tau \cdot A(1) - 1 \cdot A(\tau)\Big) = -\pi i.$$

But then

$$\frac{1}{2\tau} G_2(-1/\tau) - \frac{\tau}{2} G_2(\tau) = -\pi i,$$

or equivalently $G_2(-1/\tau) = \tau^2 G_2(\tau) - 2\pi i \tau$. ☐

G_2 is not a modular form, but certain expressions in G_2 and its derivatives do define modular forms. These lead to the *Ramanujan equations* relating G_2 and its derivatives to G_4 and G_6 and their derivatives. We leave that to the problem sets.

In any case,

$$f(z) := e^{-\eta_1 z^2} \sigma(z) = e^{-\frac{1}{2} G_2(\tau) z^2} \sigma(z)$$

is a theta function that satisfies

$$f(z+1) = -f(z)$$

and

$$f(z+\tau) = -e^{2(\eta_2 - \eta_1 \tau) z + (\eta_2 \tau - \eta_1 \tau^2)} f(z) = -e^{-2\pi i z - \pi i \tau} f(z).$$

Comparing this with the transformation laws of the Jacobi theta functions, we find:

$$f(z) = \text{const} \cdot \theta_{11}(z),$$

i.e.

$$\sigma(z) = \text{const} \cdot e^{\frac{1}{2}G_2(\tau)z^2} \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i(n+1/2)^2\tau + 2\pi i(n+1/2)z}.$$

The constant can be computed by expanding both sides above as Taylor series. We have

$$\sigma(z) = z + O(z^3)$$

and $\theta_{11}(z) = \theta_{11}'(0)z + O(z^2)$. We have proved:

Theorem 2.16.

$$\theta_{11}(z) = e^{-\frac{1}{2}G_2(\tau)z^2}\theta_{11}'(0) \cdot \sigma(z).$$

θ_{11}' means the derivative with respect to z. In terms of τ,

$$\theta_{11}'(0) = \frac{d}{dz}\Big|_{z=0} \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i(n+1/2)^2\tau + 2\pi i(n+1/2)z} = 2\pi i \sum_{n=-\infty}^{\infty} (-1)^n (n+1/2) e^{\pi i(n+1/2)^2\tau}.$$

2.5. The Jacobi triple product

In this section we use the abbreviation

$$\vartheta(\tau, z) := \theta_{11}(\tau, z) = \sum_{n \in \mathbb{Z}+1/2} (-1)^{n-1/2} e^{\pi i n^2 \tau + 2\pi i n z}.$$

ϑ' will always mean the derivative with respect to z.

Taking logarithmic derivatives in Theorem 2.16,

$$\vartheta(z) = e^{-\frac{1}{2}G_2(\tau)z^2}\vartheta'(0) \cdot \sigma(z)$$

yields

$$\frac{\vartheta'(z)}{\vartheta(z)} = -G_2(\tau)z + \frac{\sigma'(z)}{\sigma(z)}$$

and therefore

$$\Big(\frac{\vartheta'(z)}{\vartheta(z)}\Big)' = -G_2(\tau) - \wp(\tau; z).$$

With the Taylor series

$$\vartheta(z) = \vartheta'(0)z + \frac{\vartheta'''(0)}{6}z^3 + O(z^5)$$

we obtain

$$\Big(\frac{\vartheta'(z)}{\vartheta(z)}\Big)' = -z^{-2} + \frac{1}{3}\frac{\vartheta'''(0)}{\vartheta'(0)} + O(z^2),$$

and on the other hand this equals

$$-\wp(\tau, z) - G_2(\tau) = -z^{-2} - G_2(\tau) + O(z^2).$$

Comparing constant coefficients gives us $\vartheta'''(0) = -3G_2(\tau)\vartheta'(0)$.

But ϑ satisfies a form of the heat equation: since

$$\vartheta''(z) = \sum_{n\in\mathbb{Z}+1/2} (-1)^{n-1/2}(2\pi in)^2 e^{\pi in^2\tau+2\pi inz}$$

and

$$\frac{\partial}{\partial\tau}\vartheta(z) = \sum_{n\in\mathbb{Z}+1/2} (-1)^{n-1/2}(\pi in^2) e^{\pi in^2\tau+2\pi inz},$$

we have

$$\frac{\partial^2}{\partial z^2}\vartheta(\tau,z) = 4\pi i\frac{\partial}{\partial\tau}\vartheta(\tau,z),$$

hence

$$-3G_2(\tau)\vartheta'(0) = \vartheta'''(0) = 4\pi i\frac{\partial}{\partial\tau}\vartheta'(0). \tag{2.4}$$

That differential equation for $\vartheta'(0)$ leads to **Jacobi's identity**:

Theorem 2.17 (Jacobi's identity)**.**

$$2\pi i(q^{1/8} - 3q^{9/8} + 5q^{25/8} - 7q^{49/8} \pm ...) = \vartheta'(0) = 2\pi i\cdot q^{1/8}\prod_{n=1}^\infty(1-q^n)^3,$$

where $q = e^{2\pi i\tau}$.

Proof. Equation 2.4 implies that the logarithmic derivative of $\vartheta(\tau,0)$ with respect to τ is

$$\frac{\partial\vartheta'(0)/\partial\tau}{\vartheta'(0)} = -\frac{3}{4\pi i}G_2(\tau).$$

Here

$$G_2(\tau) = \frac{\pi^2}{3}E_2(\tau) = \frac{\pi^2}{3}\Big(1 - 24\sum_{n=1}^\infty(\sum_{d|n} d)q^n\Big), \quad q = e^{2\pi i\tau}.$$

Since $\frac{\partial}{\partial\tau} = 2\pi iq\frac{d}{dq}$, we can write this in the form

$$\frac{d}{dq}\mathrm{Log}\,\vartheta'(0) = \frac{1}{8}q^{-1}E_2(q) = \frac{1}{8q} - 3\sum_{n=1}^\infty \sigma_1(n)q^{n-1}.$$

So

$$\begin{aligned}\mathrm{Log}\,\vartheta'(0) &= \mathrm{const} + \frac{1}{8}\mathrm{Log}(q) - 3\sum_{n=1}^\infty\frac{\sigma_1(n)}{n}q^n \\ &= \mathrm{const} + \frac{1}{8}\mathrm{Log}(q) - 3\sum_{n=1}^\infty\sum_{m=1}^\infty\frac{1}{m}q^{mn} \\ &= \mathrm{const} + \frac{1}{8}\mathrm{Log}(q) + 3\sum_{n=1}^\infty\mathrm{Log}(1-q^n)\end{aligned}$$

and

$$\vartheta'(0) = \text{const} \cdot q^{1/8} \prod_{n=1}^{\infty} (1-q^n)^3.$$

The constant is obtained by writing

$$\vartheta'(0) = 2\pi i \sum_{n \in \mathbb{Z}+1/2} (-1)^{n-1/2} n q^{n^2/2} = 2\pi i (q^{1/8} - 3q^{9/8} + 5q^{25/8} - 7q^{49/8} \pm ...)$$

and comparing coefficients of $q^{1/8}$ (or of any other exponent) on both sides. □

Theorem 2.18 (Jacobi triple product)**.** *$\vartheta(\tau, z)$ has the infinite product representation*

$$\vartheta(\tau, z) = q^{1/8}(\zeta^{1/2} - \zeta^{-1/2}) \prod_{n=1}^{\infty} (1-q^n)(1-q^n\zeta)(1-q^n\zeta^{-1}),$$

where $q = e^{2\pi i \tau}$ and $\zeta = e^{2\pi i z}$.

Proof. Denote by $F(\tau, z)$ the infinite product

$$F(\tau, z) = q^{1/8}(\zeta^{1/2} - \zeta^{-1/2}) \prod_{n=1}^{\infty} (1-q^n)(1-q^n\zeta)(1-q^n\zeta^{-1}).$$

Then $F(\tau, z+1) = -F(\tau, z)$ because substituting $z \mapsto z+1$ leaves the infinite product unchanged and it turns $\zeta^{1/2} - \zeta^{-1/2}$ into $\zeta^{-1/2} - \zeta^{1/2}$.
The substitution $z \mapsto \tau + z$ amounts to substituting $\zeta \mapsto q \cdot \zeta$, which in turn shifts n in the product:

$$\begin{aligned} F(\tau, z+\tau) &= q^{1/8}(q^{1/2}\zeta^{1/2} - q^{-1/2}\zeta^{-1/2}) \prod_{n=1}^{\infty} (1-q^n)(1-q^{n+1}\zeta)(1-q^{n-1}\zeta^{-1}) \\ &= q^{1/8}\zeta^{-1/2}q^{-1/2}(q\zeta - 1)\frac{1-\zeta^{-1}}{1-q\zeta} \prod_{n=1}^{\infty} (1-q^n)(1-q^n\zeta)(1-q^n\zeta^{-1}) \\ &= -q^{-1/2}\zeta^{-1}F(\tau, z). \end{aligned}$$

So F is doubly quasiperiodic with respect to the lattice $\mathbb{Z} \oplus \mathbb{Z}\tau$, and in fact satisfies exactly the same equations that ϑ does:

$$\vartheta(\tau, z+1) = -\vartheta(\tau, z), \quad \vartheta(\tau, z+\tau) = -q^{1/2}\zeta^{-1}\vartheta(\tau, z).$$

Therefore ϑ is a constant (with respect to z) multiple of F:

$$\vartheta(\tau, z) = C(\tau)F(\tau, z).$$

We obtain the multiple $C(\tau)$ by expanding both sides as Taylor series about $z = 0$: by Jacobi's identity,

$$\vartheta(\tau, z) = \vartheta'(0)z + O(z^3) = 2\pi i q^{1/8} \prod_{n=1}^{\infty} (1-q^n)^3 \cdot z + O(z^3).$$

Meanwhile, $\zeta^{1/2}-\zeta^{-1/2}=2i\sin(\pi z)=2\pi i z+O(z^3)$ and

$$\prod_{n=1}^{\infty}(1-q^n)(1-q^n\zeta)(1-q^n\zeta^{-1})\Big|_{z=0}=\prod_{n=1}^{\infty}(1-q^n)^3,$$

so

$$F(\tau,z)=2\pi i q^{1/8}\prod_{n=1}^{\infty}(1-q^n)^3\cdot z+O(z^3).$$

Comparing the coefficients of z shows that both sides of the claim are already equal. $\square$

Remark 2.19. It is interesting to compare the triple product formula for $\vartheta(\tau,z)$ with the identity

$$\vartheta(\tau,z)=2\pi i q^{1/8}\prod_{n=1}^{\infty}(1-q^n)^3\cdot e^{-\frac{1}{2}G_2(\tau)z^2}\sigma(\tau,z).$$

After taking the logarithmic derivative of

$$\vartheta(\tau,z)=q^{1/8}\cdot 2i\sin(\pi z)\cdot\prod_{n=1}^{\infty}(1-q^n)(1-q^ne^{2\pi i z})(1-q^ne^{-2\pi i z})$$

with respect to z we have

$$\begin{aligned}\frac{\vartheta'(\tau,z)}{\vartheta(\tau,z)}&=\pi\cot(\pi z)+2\pi i\sum_{n=1}^{\infty}\Big(-\frac{q^n\zeta}{1-q^n\zeta}+\frac{q^n\zeta^{-1}}{1-q^n\zeta^{-1}}\Big)\\&=\pi\cot(\pi z)+2\pi i\sum_{m,n=1}^{\infty}(\zeta^{-m}-\zeta^m)q^{mn},\ \text{if } |q|<|\zeta|<|q|^{-1}\end{aligned}$$

and therefore

$$\Big(\frac{\vartheta'(\tau,z)}{\vartheta(\tau,z)}\Big)'=-\frac{\pi^2}{\sin^2(\pi z)}+4\pi^2\sum_{m,n=1}^{\infty}m(\zeta^m+\zeta^{-m})q^{mn}.$$

But applying this to $\vartheta(\tau,z)=2\pi i q^{1/8}\prod_{n=1}^{\infty}(1-q^n)^3\cdot e^{-\frac{1}{2}G_2(\tau)z^2}\sigma(\tau,z)$ gives us

$$\Big(\frac{\vartheta'(\tau,z)}{\vartheta(\tau,z)}\Big)'=-G_2(\tau)-\wp(\tau,z).$$

So we get the Fourier expansion of the Weierstrass $\wp$-function with respect to τ:

$$\begin{aligned}\wp(\tau,z)&=-G_2(\tau)+\frac{\pi^2}{\sin^2(\pi z)}-4\pi^2\sum_{m,n=1}^{\infty}m(\zeta^m+\zeta^{-m})q^{mn}\\&=\frac{\pi^2}{\sin^2(\pi z)}-\frac{\pi^2}{3}-4\pi^2\sum_{m,n=1}^{\infty}m(\zeta^m-2+\zeta^{-m})q^{mn}\\&=\frac{\pi^2}{\sin^2(\pi z)}-\frac{\pi^2}{3}+\pi^2\sum_{r=1}^{\infty}\Big(\sum_{d|r}d\sin^2(\pi dz)\Big)q^r,\end{aligned}$$

or written out only in terms of $\zeta = e^{2\pi iz}$

$$-\frac{3}{\pi^2}\wp(\tau, z) = \frac{\zeta^{-1} + 10 + \zeta}{\zeta^{-1} - 2 + \zeta} + 12\sum_{r=1}^{\infty}\Big(\sum_{d|r} d(\zeta^{-d} - 2 + \zeta^{d})\Big)q^r.$$

For fixed z this q-series converges for $|q| < |\zeta| < |q|^{-1}$, i.e. for $\tau = x + iy$ in the half-plane $y > |\mathrm{im}(z)|$.

3. Jacobi forms

3.1. Motivation

Suppose $L \leq \mathbb{C}$ is a lattice.

If $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ is an elliptic function with period lattice L, then $f_\lambda(z) := f(\lambda z)$ is elliptic with period lattice $\lambda^{-1}L$ for any $\lambda \in \mathbb{C}^\times$. This is because

$$f_\lambda(z + \lambda^{-1}\omega) = f(\lambda z + \omega) = f(\lambda z)$$

for any $\omega \in L$.

More generally, if f is quasiperiodic with $f(z+\omega) = e^{A(\omega)z+B} f(z)$ then $f_\lambda(z) = f(\lambda z)$ satisfies

$$f_\lambda(z + \lambda^{-1}\omega) = f(\lambda z + \omega) = e^{\lambda A(\lambda^{-1}\omega)+B} f(\lambda z),$$

so f_λ is quasiperiodic with quasiperiod lattice $\lambda^{-1}L$ and index homomorphism λA.

We want to consider families of elliptic or quasiperiodic functions, one for each lattice, that behave in a reasonable way as the lattice varies. To that end it is useful to introduce the space

$$\Omega = \Big\{(\omega_1, \omega_2) \in \mathbb{C}^2 : \; \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \text{ is a lattice}\Big\}.$$

This is an open subset of $\mathbb{C}^2$ so we can speak of holomorphic or meromorphic functions on it.

Definition 3.1. Let $f : \Omega \to \mathbb{C} \cup \{\infty\}$ be a meromorphic function. f is **modular of weight** $k \in \mathbb{Z}$ if it satisfies:
(1) $f(\omega_1, \omega_2)$ depends only on the lattice $\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$.
(2) $f(t\omega_1, t\omega_2) = t^{-k} f(\omega_1, \omega_2)$ for every $(\omega_1, \omega_2) \in \Omega$ and $t \in \mathbb{C}^\times$.

The reason for the name is that if $f : \Omega \to \mathbb{C}$ is modular of weight k, then

$$F(\tau) := f(1, \tau), \quad \tau \in \mathbb{H}$$

is also modular of weight k under $\mathrm{SL}_2(\mathbb{Z})$: For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$F\Big(\frac{a\tau + b}{c\tau + d}\Big) = f\Big(1, \frac{a\tau + b}{c\tau + d}\Big) = (c\tau + d)^k \cdot f\Big(c\tau + d, a\tau + b\Big).$$

But the lattice spanned by $c\tau + d$ and $a\tau + b$ is the same as $\mathbb{Z} \oplus \mathbb{Z}\tau$, so this is just $(c\tau + d)^k F(\tau)$. Conversely, every function F that is modular of weight k arises in this way for a uniquely determined function f on Ω.

Naively we would like to consider functions $\phi : \Omega \times \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ that combine "modularity" and "ellipticity" (or quasi-periodicity) in the following sense:
(1) For each fixed z, $\phi(\omega_1, \omega_2; z)$ depends only on the lattice spanned by ω_1, ω_2;
(2) For each fixed ω_1, ω_2, $z \mapsto \phi(\omega_1, \omega_2; z)$ is a doubly (quasi)-periodic function with (quasi-)period lattice $\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$;
(3) ϕ is modular, i.e. homogeneous:

$$\phi(t\omega_1, t\omega_2; tz) = t^{-k}\phi(\omega_1, \omega_2; z)$$

for some integer k.

This is fine for elliptic functions. For quasiperiodic functions (2) is not naturally compatible with (1) and (3), since the exponent A will (generally) not be a lattice function and not behave correctly under scaling. We avoid that problem by considering the modified function

$$\tilde{\phi}(\omega_1, \omega_2; z) := e^{-\frac{A(\omega_1)}{2\omega_1}z^2}\phi(\omega_1, \omega_2; z).$$

This also has quasiperiod lattice $\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ and its index homomorphism $\tilde{A}$ always satisfies $\tilde{A}(\omega_1) = 0$ and $\tilde{A}(\omega_2) = -\frac{2\pi i N}{\omega_1}$, no matter which basis ω_1, ω_2 we work in, and independently of scaling. So it makes sense to ask for ϕ to be a lattice function but for $\tilde{\phi}$ to be modular.

To make this concrete, suppose we restrict to $\omega_1 = 1$ and $\omega_2 = \tau \in \mathbb{H}$; that we ask for ϕ to be 1-periodic and have index $N \in \mathbb{Z}$, and for the B-term in its quasiperiod law to be as simple as possible (which is the choice $B(\omega) = \omega A(\omega)/2$).

Then $z \mapsto f(\tau, z) := \phi(1, \tau, z)$ must satisfy the quasiperiod laws

$$f(\tau, z + 1) = f(\tau, z) \quad \text{and} \quad f(\tau, z + \tau) = e^{-2\pi i N z - \pi i N \tau} f(\tau, z).$$

For any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, modularity implies

$$f\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) = \tilde{\phi}\Big(1, \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) = (c\tau + d)^k \tilde{\phi}\Big(c\tau + d, a\tau + b, z\Big).$$

The underlying lattice function is

$$\phi(c\tau + d, a\tau + b, z) = e^{\frac{A(c\tau+d)}{c\tau+d}\cdot\frac{1}{2}z^2}\phi(c\tau + d, a\tau + b, z),$$

where $A(c\tau + d) = -2\pi i N c$. So we can write

$$\begin{aligned} f\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) &= (c\tau + d)^k e^{\pi i N \frac{cz^2}{c\tau+d}} \cdot \phi(c\tau + d, a\tau + b, z) \\ &= (c\tau + d)^k e^{\pi i N \frac{cz^2}{c\tau+d}} \phi(1, \tau, z) \\ &= (c\tau + d)^k e^{\pi i N \frac{cz^2}{c\tau+d}} f(\tau, z). \end{aligned}$$

Finally, we assume that $N = 2m$ is even in order to avoid inconvenient roots of unity ± 1. That leads to the defining functional equations:

Definition 3.2. An **unrestricted Jacobi form** of weight k and index m is a holomorphic function $f : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ that satisfies the functional equations:
(1)

$$f(\tau, z+1) = f(\tau, z) \quad \text{and} \quad f(\tau, z+\tau) = e^{-4\pi i m z - 2\pi i m \tau} f(\tau, z);$$

(2) For any matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,

$$f\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) = (c\tau+d)^k e^{2\pi i m \frac{cz^2}{c\tau+d}} f(\tau, z).$$

"Unrestricted" means there is a growth condition "at infinity" (similarly to the definition of modular forms) that is missing. We will deal with that later.

3.2. The Jacobi group

In this section, we will show that Definition 3.2 is natural in the sense that equations (1) and (2) express precisely that f is invariant under an action of some discrete group on functions on $\mathbb{H} \times \mathbb{C}$.

Lemma 3.3. *The group* $\mathrm{SL}_2(\mathbb{Z})$ *acts on* $\mathbb{H} \times \mathbb{C}$ *via*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau, z) := \Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big), \quad \tau \in \mathbb{H},\ z \in \mathbb{C}.$$

Proof. This can be checked using the standard action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ and the fact that

$$j(M;\tau) := c\tau + d, \quad \tau \in \mathbb{H},\ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

satisfies the cocycle relation

$$j(MN;\tau) = j(M; N\cdot\tau) j(N;\tau). \qquad \square$$

We would like to be able to say that $\mathrm{SL}_2(\mathbb{Z})$ acts on f defined on $\mathbb{H} \times \mathbb{C}$ via

$$f\Big|_{k,m} \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau, z) := (c\tau+d)^{-k} e^{-2\pi i m \frac{cz^2}{c\tau+d}} f(\tau, z).$$

A short computation shows that this is true if and only if the *factor of automorphy*

$$j_{k,m}(M;\tau,z) := (c\tau+d)^k e^{2\pi i m \frac{cz^2}{c\tau+d}}$$

satisfies the cocycle law

$$j_{k,m}(MN;\tau,z) = j_{k,m}(M;N\cdot(\tau,z))j_{k,m}(N;\tau,z).$$

Certainly $(c\tau+d)^k = j(M;\tau)^k$ satisfies that cocycle law: this is the factor of automorphy for modular forms. The claim for $j_{k,m}$ follows from the following lemma:

Lemma 3.4. *For $M \in \mathrm{SL}_2(\mathbb{R})$, $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$, define*

$$\alpha(M;\tau,z) := \frac{cz^2}{c\tau+d}.$$

Then α satisfies the additive cocycle law

$$\alpha(MN;\tau,z) = \alpha(M;N\cdot(\tau,z)) + \alpha(N;\tau,z).$$

Proof. By abuse of notation, write the action of $\mathrm{SL}_2(\mathbb{Z})$ as

$$N\cdot(\tau,z) = (\gamma_N(\tau),\gamma_N(z)).$$

Taking logarithms in the cocycle identity

$$j(MN;\tau) = j(M;\gamma_N(\tau))j(N;\tau)$$

and differentiating already almost leads to an additive cocycle: we have

$$\frac{j'(MN;\tau)}{j(MN;\tau)} = \frac{j'(M;\gamma_N(\tau))}{j(M;\gamma_N(\tau))}\gamma_N'(\tau) + \frac{j'(N;\tau)}{j(N;\tau)}.$$

Multiplying this by z^2 has the effect of absorbing the term $\gamma_N'(\tau)$, since

$$\gamma_N'(\tau)z^2 = \frac{z^2}{(c\tau+d)^2} = \gamma_N(z)^2$$

if $N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. So we get

$$\frac{j'(MN;\tau)}{j(MN;\tau)}z^2 = \frac{j'(M;\gamma_N(\tau))}{j(M;\gamma_N(\tau))}\gamma_N(z)^2 + \frac{j'(N;\tau)}{j(N;\tau)}z^2.$$

This is what we wanted because $\alpha(M;\tau,z) = \frac{j'(M;\tau)}{j(M;\tau)}z^2$. □

That explains the action of the modular group. We also need to investigate the quasiperiodic law under translations. For $\lambda,\mu \in \mathbb{Z}$, we define

$$(\lambda,\mu)\cdot(\tau,z) := (\tau, z+\lambda\tau+\mu).$$

This defines an action of the group $\mathbb{Z}^2$ on $\mathbb{H}\times\mathbb{C}$.

The action of $\mathbb{Z}^2$ does not commute with that of $\mathrm{SL}_2(\mathbb{Z})$. More precisely,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\lambda, \mu) \cdot (\tau, z) = \Big(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\Big)$$

while

$$(\lambda, \mu) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau, z) = \Big(\frac{a\tau + b}{c\tau + d}, \frac{z + (\lambda a + \mu c)\tau + (\lambda b + \mu d)}{c\tau + d}\Big);$$

in other words, we have

$$v \cdot M \cdot (\tau, z) = M \cdot (vM) \cdot (\tau, z)$$

for $v \in \mathbb{Z}^2$ and $M \in \mathrm{SL}_2(\mathbb{Z})$.

To express this as the action of a single group, we need the following definition:

Definition 3.5. The **Jacobi group** is the semidirect product

$$\mathcal{J} := \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2,$$

where $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{Z}^2$ via right-multiplication.

So elements of $\mathcal{J}$ are tuples (M, v) where $M \in \mathrm{SL}_2(\mathbb{Z})$ and $v \in \mathbb{Z}^2$ is a row vector, and the group operation is

$$(M, v) \cdot (N, w) := (MN, vN + w).$$

The action of $\mathcal{J}$ on $\mathbb{H} \times \mathbb{C}$ is

$$(M, v) \cdot (\tau, z) := (M, 0) \cdot (I, v) \cdot (\tau, z) = \Big(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\Big)$$

for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $v = (\lambda, \mu) \in \mathbb{Z}^2$.

We have the following generators:

Lemma 3.6. *The Jacobi group is generated by the elements $(S, 0)$, $(T, 0)$ and $(I, (0, 1))$, where*

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \textit{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Proof. It is well-known that S and T generate the group $\mathrm{SL}_2(\mathbb{Z})$; therefore, $(S, 0)$ and $(T, 0)$ generate the subgroup of tuples $(M, 0)$ where $M \in \mathrm{SL}_2(\mathbb{Z})$. All translations $(\lambda, \mu) \in \mathbb{Z}^2$ can be generated by $\zeta = (0, 1)$ and by $(1, 0) = \zeta S$, and we have

$$(I, \zeta S) = (S^{-1}, \zeta) \cdot (S, 0) = (S, 0)^{-1} \cdot (I, \zeta) \cdot (S, 0). \qquad \square$$

So for a function f to transform like a Jacobi form of weight k and index m, it is necessary and sufficient for f to satisfy the functional equations

$$f(\tau+1,z)=f(\tau,z+1)=f(\tau,z)$$

and

$$f\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big)=\tau^k e^{2\pi i m z^2/\tau} f(\tau,z).$$

Definition 3.7. Let $f:\mathbb{H}\times\mathbb{C}\to\mathbb{C}$ be a meromorphic function. The slash operator is defined by

$$f\Big|_{k,m}(M,\zeta)(\tau,z)=(c\tau+d)^{-k}e^{-2\pi i m\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}+2\pi i m\lambda^2\tau+4\pi i m\lambda z}\cdot f\Big(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\Big)$$

for $M=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z})$ and $\zeta=(\lambda,\mu)\in\mathbb{Z}^2$. It defines an action of the Jacobi group on functions f.

In other words, we extend the slash operator from $\mathrm{SL}_2(\mathbb{Z})$ to $\mathcal{J}$ by defining

$$f\Big|_{k,m}(\lambda,\mu)(\tau,z):=e^{2\pi i m(\lambda^2\tau+2\lambda z)}f(\tau,z+\lambda\tau+\mu)$$

for $(\lambda,\mu)\in\mathbb{Z}^2$, viewed as the element $(I,(\lambda,\mu))\in\mathcal{J}$.

Proof. Since we know that $|_{k,m}$ defines an action of the subgroups $\mathrm{SL}_2(\mathbb{Z})$ and $\mathbb{Z}^2$, the point is to verify that these actions are compatible with the semidirect product in the sense that

$$f\Big|_{k,m}\zeta\Big|_{k,m}M=f\Big|_{k,m}(\zeta M)$$

for every $\zeta=(\lambda,\mu)$ and $M=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z})$. Writing $\zeta M=(\tilde\lambda,\tilde\mu)$, that equation in turn follows from the identities

$$\frac{z}{c\tau+d}+\lambda\frac{a\tau+b}{c\tau+d}+\mu=\frac{z+\tilde\lambda\tau+\tilde\mu}{c\tau+d}$$

and

$$\lambda^2\frac{a\tau+b}{c\tau+d}+2\lambda\frac{z}{c\tau+d}-\frac{cz^2}{c\tau+d}=\tilde\lambda^2\tau+2\tilde\lambda z-\frac{c(z+\tilde\lambda\tau+\tilde\mu)^2}{c\tau+d}+(\tilde\lambda\tilde\mu-\lambda\mu)$$

and from the fact that $e^{2\pi i(\tilde\lambda\tilde\mu-\lambda\mu)}=1$. $\square$

Remark 3.8. The slash operator does not define an action of the Lie group $\mathrm{SL}_2(\mathbb{R})\ltimes\mathbb{R}^2$ on functions! The problem is that the last step of the proof, $e^{2\pi i(\tilde\lambda\tilde\mu-\lambda\mu)}=1$ no longer holds when λ,μ are arbitrary reals. The correct notion of *real Jacobi group* is the semidirect product

$$\mathcal{J}_{\mathbb{R}}:=\mathrm{SL}_2(\mathbb{R})\ltimes\mathrm{Heis}_{\mathbb{R}},$$

where $\mathrm{Heis}_{\mathbb{R}}$ is the Heisenberg group, which is a central extension of $\mathbb{R}^2$ by $\mathbb{R}$: the underlying set is $\mathbb{R}^2 \times \mathbb{R}$ and the group operation is

$$(\lambda, t) \cdot (\mu, u) = (\omega(\lambda, \mu) + t + u)$$

where

$$\omega\Big((a, b), (c, d)\Big) = ad - bc.$$

The action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathrm{Heis}_{\mathbb{R}}$ from the right is by ignoring the second component:

$$(\zeta, t) \cdot M = (\zeta M, t).$$

One can show that the map

$$\mathcal{J}_{\mathbb{R}} \longrightarrow \mathrm{GL}_4(\mathbb{R}),$$

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, ((\lambda, \mu), t)\right) \mapsto \begin{pmatrix} a & 0 & b & a\mu - b\lambda \\ \lambda & 1 & \mu & t \\ c & 0 & d & c\mu - d\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a faithful representation. It identifies $\mathcal{J}_{\mathbb{R}}$ with the subgroup of

$$\mathrm{Sp}_4(\mathbb{R}) = \Big\{M \in \mathrm{GL}_4(\mathbb{R}) : \; M^T J M = J\Big\}$$

(where $J = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$) of matrices of the form

$$\begin{pmatrix} * & 0 & * & * \\ * & 1 & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This is another interpretation of the group law in $\mathcal{J}_{\mathbb{R}}$.

Remark 3.9. The transformation law under $(-I, 0)$ is:

$$f(\tau, -z) = (-1)^k f(\tau, z).$$

Unlike the case of modular forms for $\mathrm{SL}_2(\mathbb{Z})$, this does not imply that $f = 0$ when k is odd; and indeed nonzero Jacobi forms of odd weight do exist. For example, the function

$$\wp'(\tau, z) = -2 \sum_{(m,n) \neq (0,0)} \frac{1}{(z - m\tau - n)^3}$$

is a (meromorphic) Jacobi form of weight 3 and index 0.

3.3. Theta transformation formula

The notation, the name and their presence in the earlier lectures suggest that the Jacobi theta functions $\theta(\tau, z)$ might transform in an orderly way under the action of the Jacobi group. This is true.
Recall that ϑ stands for the odd Jacobi theta function

$$\vartheta(\tau, z) = \sum_{n \in \frac{1}{2}+\mathbb{Z}} (-1)^{n-1/2} q^{n^2/2} \zeta^n, \quad q = e^{2\pi i \tau},\ \zeta = e^{2\pi i z}.$$

Theorem 3.10 (Theta transformation formula)**.**
For every $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$*, there is an eighth root of unity* $\chi(M)$ *such that*

$$\vartheta\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) = \chi(M) \cdot \sqrt{c\tau + d} \cdot e^{\pi i c z^2/(c\tau + d)} \vartheta(\tau, z).$$

Here $\tau \mapsto \sqrt{c\tau + d}$ is the branch of the square root that maps $\mathbb{H}$ into $\mathbb{H}$.

Proof. Essentially we want to show that $\vartheta|_{1/2,1/2} M = \chi(M)\vartheta$ for every $M \in \mathrm{SL}_2(\mathbb{Z})$. But since we defined the slash action only for integral k and m, it is better to apply it to the square ϑ^2. (This avoids some technicalities involving multiplier systems.)
For any $\zeta \in \mathbb{Z}^2$ and any $M \in \mathrm{SL}_2(\mathbb{Z})$, we have the equation

$$(I, \zeta M^{-1}) \cdot (M, 0) = (M, 0) \cdot (I, \zeta)$$

in the Jacobi group. So

$$\Big(\vartheta^2\Big|_{1,1} M\Big)\Big|_{1,1} \zeta = \Big(\vartheta^2\Big|_{1,1} (\zeta M^{-1})\Big)\Big|_{1,1} M.$$

But the quasiperiodic law for ϑ^2,

$$\vartheta^2(\tau, z + \lambda\tau + \mu) = e^{2\pi i (\lambda^2 \tau + 2\lambda z)} \vartheta^2(\tau, z)$$

implies that $\vartheta^2|_{1,1}\zeta = \vartheta^2$. So $\vartheta^2|_{1,1} M$ is also quasiperiodic of index 1, just as ϑ^2 is.

Also,

$$\vartheta^2|_{1,1} M = (c\tau + d)^{-k} e^{-2\pi i m c z^2/(c\tau + d)} \vartheta^2\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big)$$

has double zeros exactly in the lattice points $z \in \mathbb{Z} \oplus \tau\mathbb{Z}$, exactly as ϑ^2 does. So $\frac{\vartheta^2|_{1,1} M}{\vartheta^2}$ is a holomorphic elliptic function and therefore a constant (with respect to z). Hence we can write

$$\vartheta\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) = \alpha(M; \tau) \cdot e^{\pi i c z^2/(c\tau + d)} \vartheta(\tau, z)$$

with a factor $\alpha(M; \tau)$ that depends only on $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and on τ.

To compute the multiple $\alpha(M;\tau)$, we write $\vartheta(\tau, z) = \vartheta'(\tau, 0)z + O(z^3)$, where

$$\vartheta'(\tau, 0) = 2\pi i q^{1/8} \prod_{n=1}^{\infty}(1-q^n)^3$$

by Jacobi's identity. Then

$$\vartheta\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) = \vartheta'\Big(\frac{a\tau+b}{c\tau+d}, 0\Big)\frac{z}{c\tau+d} + O(z^3).$$

The claim follows from the following lemma. □

Lemma 3.11. *The function*

$$f(\tau) := \vartheta'(\tau, 0) = 2\pi i q^{1/8} \prod_{n=1}^{\infty}(1-q^n)^3$$

satisfies

$$f\Big(\frac{a\tau+b}{c\tau+d}\Big) = \chi(M)(c\tau+d)^{3/2} f(\tau)$$

for every $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *where* $\chi(M)$ *is an eighth root of unity.*

This proves the theorem because comparing coefficients of z^1 in

$$\vartheta\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) = \vartheta'\Big(\frac{a\tau+b}{c\tau+d}, 0\Big)\frac{z}{c\tau+d} + O(z^3);$$

$$\alpha(M;\tau)\cdot e^{\pi i c z^2/(c\tau+d)}\vartheta(\tau, z) = \alpha(M;\tau)\vartheta'(\tau,0)z + O(z^3)$$

yields $\alpha(M;\tau) = \chi(M)\sqrt{c\tau+d}$.

Proof. It is enough to prove this when $M = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ or $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The case $M = T$ is trivial. For $M = S$, consider that the logarithmic derivative is

$$\begin{aligned} \frac{f'}{f} &= \frac{(q^{1/8})'}{q^{1/8}} + \sum_{n=1}^{\infty} \frac{((1-q^n)^3)'}{(1-q^n)^3} \\ &= \frac{\pi i}{4} - 6\pi i \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} \\ &= \frac{\pi i}{4}\Big[1 - 24\sum_{n=1}^{\infty}\Big(\sum_{d|n} d\Big)q^n\Big] \\ &= \frac{3}{4\pi} i G_2(\tau). \end{aligned}$$

So $g(\tau) := f(-1/\tau)$ satisfies

$$\begin{aligned}\frac{g'}{g}(\tau) &= \frac{1}{\tau^2}\frac{f'}{f}(-1/\tau) \\ &= \frac{1}{\tau^2}\cdot\frac{3}{4\pi}i\Big(\tau^2 G_2(\tau) - 2\pi i\tau\Big) \\ &= \frac{f'}{f}(\tau) + \frac{3}{2\tau}.\end{aligned}$$

But this forces

$$f\Big(-\frac{1}{\tau}\Big) = g(\tau) = \mathrm{const}\cdot\tau^{3/2}f(\tau).$$

Setting $\tau = i$ shows that the constant is $i^{-3/2} = e^{-3\pi i/4}$. □

Remark 3.12. The map $M \mapsto \chi(M)$ is *not* a character of $\mathrm{SL}_2(\mathbb{Z})$, because $S^4 = I$ but $\chi(S)^4 \neq 1$.

Corollary 3.13. *(i)*

$$\phi_{-2,1}(\tau, z) := \Big(2\pi i\frac{\vartheta(\tau,z)}{\vartheta'(\tau,0)}\Big)^2$$

is a meromorphic Jacobi form of weight -2 *and index* 1.
(ii)

$$\phi_{-1,2}(\tau, z) := 2\pi i\frac{\vartheta(\tau,2z)}{\vartheta'(\tau,0)}$$

is a meromorphic Jacobi form of weight -1 *and index* 2.

Proof. Both claims follow from the theta transformation formula

$$\vartheta\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) = \chi(M)\cdot\sqrt{c\tau+d}\cdot e^{\pi icz^2/(c\tau+d)}\vartheta(\tau,z)$$

and the formula

$$\vartheta'\Big(\frac{a\tau+b}{c\tau+d}, 0\Big) = \chi(M)\cdot\sqrt{(c\tau+d)^3}\vartheta'(\tau,0)$$

with the same root of unity $\chi(M)$. □

Corollary 3.14. *The theta function*

$$\theta(\tau,z) := \theta_{00}(\tau,z) = \sum_{n=-\infty}^{\infty} e^{\pi in^2\tau + 2\pi inz}$$

satisfies the theta transformation formula

$$\theta\Big(-\frac{1}{\tau}, \frac{z}{\tau}\Big) = \sqrt{\frac{\tau}{i}}\cdot e^{\pi iz^2/\tau}\theta(\tau,z).$$

Unlike ϑ, θ does not transform under the full modular group, even with a multiplier system: for example, $\theta(\tau+1,z)$ is not a multiple of $\theta(\tau,z)$. The subgroup of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ under which θ does transform correctly (with multiplier) is the *theta group*

$$\Gamma^\theta = \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \; a+c \equiv b+d \equiv 1 \bmod 2 \Big\},$$

which is a subgroup of index 3. (The proof for general $M \in \Gamma^\theta$ is not significantly more difficult than for $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.)

For $M \in \Gamma(2)$, the multiplier is trivial: we have

$$\theta\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) = \sqrt{c\tau+d} \cdot e^{\pi i c z^2/(c\tau+d)} \theta(\tau,z).$$

Proof. Using

$$\vartheta(\tau,z) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i (n+1/2)^2 \tau + 2\pi i (n+1/2) z}$$

we obtain

$$\vartheta(\tau, z-\tau/2-1/2) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i (n^2-1/4)\tau + 2\pi i (n+1/2)(z-1/2)} = -ie^{-\pi i \tau/4 + \pi i z} \theta(\tau,z),$$

so

$$\theta(\tau,z) = ie^{\pi i \tau/4 - \pi i z} \vartheta(\tau, z-\tau/2-1/2).$$

From the theta transformation formula we obtain

$$\begin{aligned} \theta\Big(-\frac{1}{\tau}, \frac{z}{\tau}\Big) &= ie^{\pi i(-1/4\tau) - \pi i z/\tau} \vartheta\Big(-\frac{1}{\tau}, \frac{z}{\tau} + \frac{1}{2\tau} - \frac{1}{2}\Big) \\ &= ie^{\pi i(-1/4\tau)-\pi i z/\tau} \cdot e^{-3\pi i/4} \sqrt{\tau} e^{\pi i (z-\tau/2+1/2)^2/\tau} \vartheta(\tau, z-\tau/2+1/2) \\ &= e^{-3\pi i/4} \cdot e^{\pi i \tau/4 - \pi i z} \cdot \sqrt{\tau} e^{\pi i z^2/\tau} \vartheta(\tau, z-\tau/2+1/2) \\ &= \sqrt{\frac{\tau}{i}} e^{\pi i z^2/\tau} \theta(\tau,z). \end{aligned}$$

□

3.4. The theta decomposition

The quasiperiod law for an (unrestricted) Jacobi form of index m is

$$f(\tau, z+1) = f(\tau,z), \quad f(\tau, z+\tau) = e^{-4\pi i m z - 2\pi i m \tau} f(\tau,z).$$

By Lemma 2.9, this is equivalent to its Fourier series

$$f(\tau,z) = \sum_{n \in \mathbb{Z}} c_n(\tau) e^{2\pi i n z}$$

satisfying the recurrence

$$c_{n+2m} = e^{2\pi i(m+n)\tau} c_n. \tag{3.1}$$

A particularly simple basis $\Theta_{m,j}$, $j = 0, 1, ..., 2m-1$ of the space of theta functions satisfying (3.1) is obtained by setting $c_j = e^{\pi i \frac{j^2}{2m}\tau}$ and $c_i = 0$ if $i \not\equiv j \bmod 2m$. That choice of c_j determines $c_{j+2mN} = e^{\pi i \frac{(j+2mN)^2}{2m}\tau}$ for every $N \in \mathbb{Z}$, so

$$\Theta_{m,j} = \sum_{\substack{r \in \mathbb{Z} \\ r \equiv j \bmod 2m}} e^{\pi i \frac{r^2}{2m}\tau + 2\pi i r z} = \sum_{\substack{r \in \mathbb{Z} \\ r \equiv j \bmod 2m}} q^{r^2/4m} \zeta^r.$$

Definition 3.15. The **theta decomposition** of a Jacobi form f is its representation as a linear combination

$$f(\tau, z) = \sum_{j \in \mathbb{Z}/2m\mathbb{Z}} h_j(\tau) \Theta_{m,j}(\tau, z).$$

Since $f(\tau+1, z) = f(\tau, z)$ but $\Theta_{m,j}(\tau+1, z) = e^{\pi i j^2/2m} \Theta_{m,j}(\tau, z)$, we have

$$h_j(\tau+1) = e^{-\pi i j^2/2m} h_j(\tau).$$

So $h_j(\tau)$ itself has a Fourier series of the form

$$h_j(\tau) = \sum_{n \equiv -j^2 \,(\mathrm{mod}\, 4m)} a_j(n) q^{n/4m}, \quad a_j(n) \in \mathbb{C}.$$

Lemma 3.16. *The theta functions $\Theta_{m,j}$ satisfy the theta transformation formula*

$$\Theta_{m,j}\Big(-\frac{1}{\tau}, \frac{z}{\tau}\Big) = \sqrt{\frac{\tau}{2mi}} e^{2\pi i m z^2/\tau} \cdot \sum_{a \in \mathbb{Z}/2m} e^{-\pi i j a/m} \Theta_{m,a}(\tau, z).$$

Proof. We will show that this follows from the transformation law (Corollary 3.14) for

$$\theta = \theta_{00} = \sum_{n \in \mathbb{Z}} q^{n^2/2} \zeta^n.$$

For $a \in \mathbb{Z}/2m\mathbb{Z}$, write

$$\theta\Big(\frac{\tau}{2m}, z + \frac{a}{2m}\Big) = \sum_{r \in \mathbb{Z}} e^{\pi i \frac{r^2}{2m}\tau + 2\pi i r(z + a/2m)} = \sum_{r \in \mathbb{Z}} e^{\pi i a r/m} q^{r^2/4m} \zeta^r.$$

Using the identity

$$\sum_{a \in \mathbb{Z}/2m\mathbb{Z}} e^{\pi i a(r-j)/m} = \begin{cases} 2m: & r \equiv j \,(\mathrm{mod}\, 2m); \\ 0: & \text{otherwise}; \end{cases}$$

we extract the coefficients $r \equiv j\,(2m)$ with the linear combination

$$\Theta_{m,j}(\tau, z) = \frac{1}{2m} \sum_{a \in \mathbb{Z}/2m} e^{-\pi i j a/m} \theta\Big(\frac{\tau}{2m}, z + \frac{a}{2m}\Big).$$

The theta transformation formula for θ yields

$$\begin{aligned}
\Theta_{m,j}\Big(-\frac{1}{\tau}, \frac{z}{\tau}\Big) &= \frac{1}{2m} \sum_{a \in \mathbb{Z}/2m} e^{-\pi i j a/m} \theta\Big(-\frac{1}{2m\tau}, \frac{z}{\tau} + \frac{a}{2m}\Big) \\
&= \frac{1}{2m} \sum_{a \in \mathbb{Z}/2m} e^{-\pi i j a/m} \cdot \sqrt{\frac{2m\tau}{i}} e^{\pi i (2mz + a\tau)^2/(2m\tau)} \theta\Big(2m\tau, 2mz + a\tau\Big) \\
&= \sqrt{\frac{\tau}{2mi}} e^{2\pi i m z^2/\tau} \sum_{a \in \mathbb{Z}/2m} e^{-\pi i j a/m} \cdot e^{\pi i (a^2/2m)\tau + 2\pi i a z} \sum_{r \in \mathbb{Z}} e^{2\pi i m r^2 \tau + 2\pi i r (2m) z + 2\pi i a r \tau} \\
&= \sqrt{\frac{\tau}{2mi}} e^{2\pi i m z^2/\tau} \sum_{a \in \mathbb{Z}/2m} e^{-\pi i j a/m} \cdot \sum_{r \in \mathbb{Z}} e^{\pi i (2mr + a)^2 \tau + 2\pi i (2mr + a) z} \\
&= \sqrt{\frac{\tau}{2mi}} e^{2\pi i m z^2/\tau} \sum_{a \in \mathbb{Z}/2m} e^{-\pi i j a/m} \Theta_{m,a}(\tau, z).
\end{aligned}$$

□

Theorem 3.17. *Let f be a Jacobi form of weight k and index m, with theta decomposition*

$$f(\tau, z) = \sum_{j \in \mathbb{Z}/2m} h_j(\tau) \Theta_{m,j}(\tau, z).$$

The coefficients h_j transform under the modular group by

$$h_j(\tau + 1) = e^{-\pi i j^2/(2m)} h_j(\tau);$$

$$h_j\Big(-\frac{1}{\tau}\Big) = \tau^{k-1/2} \cdot \sqrt{\frac{i}{2m}} \sum_{a \in \mathbb{Z}/2m} e^{\pi i j a/m} h_a(\tau).$$

Proof. Only the second identity needs a proof. Compare coefficients of $\Theta_{m,a}$ in

$$f\Big(-\frac{1}{\tau}, \frac{z}{\tau}\Big) = \tau^k e^{2\pi i m z^2/\tau} f(\tau, z) = \tau^k e^{2\pi i m z^2/\tau} \sum_{a \in \mathbb{Z}/2m} h_a(\tau) \Theta_{m,a}(\tau, z)$$

and

$$\begin{aligned}
f\Big(-\frac{1}{\tau}, \frac{z}{\tau}\Big) &= \sum_{j \in \mathbb{Z}/2m} h_j(-1/\tau) \Theta_{m,j}\Big(-\frac{1}{\tau}, \frac{z}{\tau}\Big) \\
&= \sqrt{\frac{\tau}{2mi}} e^{2\pi i m z^2/\tau} \cdot \sum_{j \in \mathbb{Z}/2m} \sum_{a \in \mathbb{Z}/2m} h_j(-1/\tau) e^{-\pi i j a/m} \Theta_{m,a}(\tau, z)
\end{aligned}$$

to see that

$$\tau^k h_a(\tau) = \sqrt{\frac{\tau}{2mi}} \sum_{j \in \mathbb{Z}/2m} e^{-\pi i j a/m} h_j(-1/\tau).$$

The claim follows either by Fourier inversion, or by substituting $\tau \mapsto -1/\tau$ and using the fact that $f(\tau, -z) = (-1)^k f(\tau, z)$ implies $(-1)^k h_j = h_{-j}$. □

Theorem 8.21 implies that the *vector* $H(\tau) = (h_j(\tau))_{j=0,...,2m-1}$ transforms like a modular form of weight $k-1/2$ for $\mathrm{SL}_2(\mathbb{Z})$. In other words, for any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, there is a matrix $\rho(M)$ such that

$$H(M \cdot \tau) = (c\tau + d)^{k-1/2} \rho(M) H(\tau).$$

ρ is the simplest case of what is called the *Weil representation* of $\mathrm{SL}_2(\mathbb{Z})$.
Warning: ρ is not a true representation of $\mathrm{SL}_2(\mathbb{Z})$ (similarly to the "character" χ of ϑ); one can show that $\rho(S)^4 = -I$ for every index m, while $S^4 = I$. What is true however is that ρ defines a projective representation, i.e. a homomorphism from $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{PGL}_{2m}(\mathbb{C})$, and in fact into $\mathrm{GL}_{2m}(\mathbb{C})/\{\pm 1\}$. We will not explore this further.

Example 3.18. When $m = 1$ and a Jacobi form of index 1 is written

$$f(\tau, z) = h_0(\tau)\Theta_{1,0}(\tau, z) + h_1(\tau)\Theta_{1,1}(\tau, z),$$

the transformation law for the vector $H = (h_0, h_1)^T$ is

$$H(\tau + 1) = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} H(\tau)$$

and

$$H(-1/\tau) = \tau^{k-1/2} \cdot \begin{pmatrix} (1+i)/2 & (1+i)/2 \\ (1+i)/2 & -(1+i)/2 \end{pmatrix} H(\tau).$$

Example 3.19. Let's work out the decompositions of the forms $\phi_{-2,1}$ and $\phi_{-1,2}$ from Corollary 3.13. Since

$$\vartheta(\tau, z) = q^{1/8}(\zeta^{1/2} - \zeta^{-1/2}) + q^{9/8}(\zeta^{3/2} - \zeta^{-3/2}) + q^{25/8}(\zeta^{5/2} - \zeta^{-5/2}) + ...$$

$$\frac{1}{2\pi i}\vartheta'(\tau, 0) = q^{1/8} - 3q^{9/8} + 5q^{25/8} - 7q^{49/8} \pm ...$$

we obtain

$$\begin{aligned} \phi_{-2,1}(\tau, z) = \Big(2\pi i \frac{\vartheta(\tau, z)}{\vartheta'(\tau, 0)}\Big)^2 &= \zeta^{-1} - 2 + \zeta \\ &+ (-2\zeta^{-2} + 8\zeta^{-1} - 12 + 8\zeta - 2\zeta^2)q \\ &+ (\zeta^{-3} - 12\zeta^{-2} + 39\zeta^{-1} - 56 + 39\zeta - 12\zeta^2 + \zeta^3)q^2 + ... \end{aligned}$$

From the coefficients of ζ^0 and ζ^1 we read off the q-series

$$h_0(\tau) = -2 - 12q - 56q^2 \pm ...$$

$$h_1(\tau) = q^{-1/4} + 8q^{3/4} + 39q^{7/4} \pm ...$$

For $\phi_{-1,2}$ we have the Fourier series

$$\begin{aligned}\phi_{-1,2}(\tau, z) = 2\pi i \frac{\vartheta(\tau, 2z)}{\vartheta'(\tau, 0)} &= -\zeta^{-1} + \zeta \\ &+ (\zeta^{-3} - 3\zeta^{-1} + 3\zeta - \zeta^3)q \\ &+ (3\zeta^{-3} - 9\zeta^{-1} + 9\zeta - 3\zeta^3)q^2 + ...\end{aligned}$$

The terms h_0 and h_2 vanish and the terms h_1 and h_3 can be read off the coefficients of ζ^1 and ζ^{-1} respectively. We have

$$h_1(\tau) = q^{1/8} + 3q^{7/8} + 9q^{15/8} + ...$$

$$h_3(\tau) = -q^{-1/8} - 3q^{7/8} - 9q^{15/8} - ...$$

Notice that h_1 and h_3 are $\pm 2\pi i$ times the series expansion of

$$\vartheta'(\tau, 0)^{-1} = \frac{1}{2\pi i}(q^{1/8} - 3q^{9/8} + 5q^{25/8} - 7q^{49/2} \pm ...)^{-1}$$

which has weight $-1 - 1/2 = -3/2$ as predicted by Theorem 8.21.

We finish this section by discussing two useful properties of the (projective) representation ρ.

Proposition 3.20. *For every $M \in \mathrm{SL}_2(\mathbb{Z})$, $\rho(M)$ is unitary:*

$$\overline{\rho(M)^T} = \rho(M)^{-1}.$$

Proof. If we can prove this for $M = S$ and $M = T$ then it will be true for any M (despite ρ not being a true homomorphism), because $\rho(MN) = \pm\rho(M)\rho(N)$ for any M, N.
We have

$$\rho(T) = \mathrm{diag}(e^{-\pi i j^2/(2m)},\ j = 0, 1, ..., 2m-1),$$

which is certainly unitary. $\rho(S)$ is the matrix

$$\rho(S)_{a,j} = \sqrt{\frac{i}{2m}} e^{\pi i j a/m}, \quad a, j = 0, 1, ..., 2m-1,$$

so the (a, b)-entry in $\overline{\rho(S)^T}\rho(S)$ is

$$(\overline{\rho(S)^T}\rho(S))_{a,b} = \frac{1}{2m}\sum_{j=0}^{2m-1} e^{-\pi i j a/m} e^{\pi i j b/m} = \begin{cases} 1: & a = b; \\ 0: & \text{otherwise.} \end{cases}$$

That means $\overline{\rho(S)^T} = \rho(S)^{-1}$. ∎

Finally it is good to know that ρ is trivial on a finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. We will show specifically that ρ acts trivially on the principal congruence subgroup

$$\Gamma(4m) = \Big\{ M \in \mathrm{SL}_2(\mathbb{Z}) : \ M - I \in 4m\mathbb{Z}^{2\times 2} \Big\}.$$

In particular, if the Jacobi form f of weight k and index m has theta decomposition

$$f(\tau, z) = \sum_{j \in \mathbb{Z}/2m} h_j(\tau) \Theta_{m,j}(\tau, z)$$

then each $h_j(\tau)$ is a modular form of weight $k - 1/2$ and level $4m$, i.e.

$$h_j\Big(\frac{a\tau + b}{c\tau + d}\Big) = (c\tau + d)^{k-1/2} h_j(\tau),$$

where $(c\tau + d)^{k-1/2}$ involves the branch of the square root of $\tau \mapsto c\tau + d$ that maps $\mathbb{H}$ into $\mathbb{H}$.

Theorem 3.21 (Hecke–Schoeneberg)**.** *Suppose* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(4m)$. *Then*

$$\Theta_{m,j}\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) = \sqrt{(c\tau + d)} e^{2\pi i m c z^2/(c\tau + d)} \Theta_{m,j}(\tau, z)$$

for every $j \in \mathbb{Z}/2m$. *In particular* $\rho(M) = I$.

Actually the theorem of Hecke–Schoeneberg applies to more general theta functions; this is only a special case.

Proof. We have

$$\begin{aligned} \Theta_{m,j}\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) &= \frac{1}{2m} \sum_{u \in \mathbb{Z}/2m} e^{-\pi i j u/m} \theta\Big(\frac{a\tau + b}{2m(c\tau + d)}, \frac{z}{c\tau + d} + \frac{u}{2m}\Big) \\ &= \frac{1}{2m} \sum_{u \in \mathbb{Z}/2m} e^{-\pi i j a/m} \theta\Big(\frac{a\frac{\tau}{2m} + b/2m}{2mc\frac{\tau}{2m} + d}, \frac{z + u(c\tau + d)/2m}{2mc(\tau/2m) + d}\Big). \end{aligned}$$

Since $\begin{pmatrix} a & b/2m \\ 2mc & d \end{pmatrix} \in \Gamma(2)$, the theta transformation formula for θ yields

$$\begin{aligned} &\theta\Big(\frac{a\frac{\tau}{2m} + b/2m}{2mc\frac{\tau}{2m} + d}, \frac{z + u(c\tau + d)/2m}{2mc(\tau/2m) + d}\Big) \\ &= \sqrt{(c\tau + d)} e^{\pi i (2mc)(z + u(c\tau + d)/2m)^2/(c\tau + d)} \theta\Big(\frac{\tau}{2m}, z + uc\frac{\tau}{2m} + \frac{ud}{2m}\Big). \end{aligned}$$

Here we can write

$$\theta\Big(\frac{\tau}{2m}, z + uc\frac{\tau}{2m} + \frac{ud}{2m}\Big) = e^{-\pi i (uc)^2(\tau/2m) - 2\pi i uc(z + \frac{ud}{2m})} \theta\Big(\frac{\tau}{2m}, z + \frac{ud}{2m}\Big).$$

After simplifying

$$\frac{2mc\Big(z+\frac{u(c\tau+d)}{2m}\Big)^2}{c\tau+d}-(uc)^2\frac{\tau}{2m}-2uc\Big(z+\frac{ud}{2m}\Big)=\frac{2mcz^2}{c\tau+d}-\frac{cd}{2m}u^2,$$

and using $e^{-\pi i\frac{cd}{2m}u^2}=1$ (since $c\in 4m\mathbb{Z}$) as well as

$$\theta\Big(\frac{\tau}{2m},z+\frac{ud}{2m}\Big)=\theta\Big(\frac{\tau}{2m},z+\frac{u}{2m}\Big)$$

(since $d\in 1+2m\mathbb{Z}$), we obtain

$$\Theta_{m,j}\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big)=\sqrt{(c\tau+d)}e^{2\pi imcz^2/(c\tau+d)}\Theta_{m,j}(\tau,z).$$ □

3.5. Weak and holomorphic Jacobi forms

Let f be an unrestricted Jacobi form of weight k and index m.

Definition 3.22. (i) f is a **weak Jacobi form** if its Fourier series is holomorphic at $q=0$: i.e. if it has the form

$$f(\tau,z)=\sum_{n=0}^{\infty}\Big(\sum_{r\in\mathbb{Z}}c(n,r)\zeta^r\Big)q^n.$$

(ii) Suppose f has theta decomposition

$$f(\tau,z)=\sum_{j\in\mathbb{Z}/2m}h_j(\tau)\Theta_{m,j}(\tau,z).$$

f is a **holomorphic Jacobi form** if the Fourier series of each h_j is holomorphic at $q=0$: i.e.

$$h_j(\tau)=\sum_{\substack{n\in\mathbb{N}_0\\ n\equiv -j^2\ (\mathrm{mod}\ 4m)}}a_j(n)q^{n/4m}.$$

(iii) f is a **Jacobi cusp form** if each h_j vanishes at $q=0$: i.e.

$$h_j(\tau)=\sum_{\substack{n\geq 1\\ n\equiv -j^2\ (\mathrm{mod}\ 4m)}}a_j(n)q^{n/4m}.$$

The $\mathbb{C}$-vector spaces of weak Jacobi forms, holomorphic Jacobi forms and Jacobi cusp forms of weight k and index m will be labeled

$$J_{k,m}^{\mathrm{w}},\quad J_{k,m},\quad J_{k,m}^{\mathrm{cusp}}.$$

Example 3.23. $\phi_{-2,1} = (2\pi i\vartheta(\tau,z)/\vartheta'(\tau,0))^2$ is a weak Jacobi form, because its q-series expansion begins

$$\phi_{-2,1} = (\zeta^{-1} - 2 + \zeta) + O(q).$$

It is not a holomorphic Jacobi form, because its theta decomposition is

$$h_0(\tau) = -2 - 12q - 56q^2 \pm ...$$

$$h_1(\tau) = q^{-1/4} + 8q^{3/4} + 39q^{7/4} \pm ...$$

and h_1 is not holomorphic at $q = 0$.

$\Delta(\tau)\phi_{-2,1}$ is a Jacobi cusp form of weight 10, since the coefficients in its theta decomposition are $\Delta(\tau)h_0(\tau)$ and $\Delta(\tau)h_1(\tau)$ and both vanish at $q = 0$.

Remark 3.24. In Definition 3.22 (ii) it is equivalent to require each h_j to be a holomorphic modular form (of level $\Gamma(4m)$). By itself, the condition of holomorphy at $q = 0$ only implies that h_j is holomorphic at the cusp ∞, and $\Gamma(4m)$ has many other cusps. But for any matrix $M \in \mathrm{SL}_2(\mathbb{Z})$, we can write

$$h_j\Big|_{k-1/2} M = \rho(M) \cdot (h_1, ..., h_{2m})^T$$

and observe the right-hand side is also bounded as $q \to 0$. This shows that h_j is holomorphic at the other cusps of $\Gamma(4m)$ as well.
Similarly, in part (iii) of Definition 3.22 it is equivalent to ask for each h_j to be a cusp form.

For many purposes it is convenient to have an alternative form of Definition 3.22:

Lemma 3.25. *Suppose f has Fourier series $f(\tau,z) = \sum_{n,r\in\mathbb{Z}} c(n,r)q^n\zeta^r$.*
(i) f is holomorphic if and only if $c(n,r) = 0$ whenever $r^2 > 4mn$.
(ii) f is a Jacobi cusp form if and only if $c(n,r) = 0$ whenever $r^2 \geq 4mn$.
(iii) If f is a weak Jacobi form, then $c(n,r) = 0$ whenever $r^2 > 4mn + m^2$.

Proof. (i), (ii) If $f(\tau,z) = \sum_{n,r} c(n,r)q^n\zeta^r$ then the components $h_j(\tau)$ are given by

$$h_j(\tau) = \sum_{n\in\mathbb{Z}} c(n,r)q^{n-r^2/4m},$$

for *any* (fixed) $r \in \mathbb{Z}$ with $r \equiv j \bmod 2m$. So all h_j contain only non-negative exponents if and only if $c(n,r) = 0$ whenever $n - r^2/4m < 0$, and all h_j contain only positive exponents if and only if $c(n,r) = 0$ whenever $n - r^2/4m \leq 0$.
(iii) If we write the theta decomposition in the form

$$f(\tau,z) = \sum_{j=-m+1}^{m} h_j(\tau)\Theta_{m,j}(\tau,z)$$

then the theta series $\Theta_{m,j}$ has Fourier expansion beginning

$$q^{j^2/4m}\zeta^j + (\text{higher powers of } q)$$

(except in the case $j = m$, where it begins $q^{j^2/4m}(\zeta^j + \zeta^{-j})$). Since there is no cancellation (the powers of ζ are distinct), this sums to a q-series without negative coefficients if and only if all h_j have the form

$$h_j(\tau) = q^{-j^2/4m} + (\text{higher powers of } q),$$

i.e. if $c(n, r) = 0$ whenever $n - r^2/4m < -j^2/4m$ for the representative $j \in \{-m+1, ..., m\}$ with $j \equiv r \bmod 2m$.
In particular, we have $c(n, r) = 0$ whenever $n - r^2/4m < -m^2/4m = -m/4$, or equivalently $r^2 > 4mn + m^2$. □

Therefore the Fourier series of a holomorphic Jacobi form begins with exponent $n \geq 0$ (so $J_{k,m} \subseteq J^{\mathrm{w}}_{k,m}$) and the q^0-term can only be a constant. And the Fourier series of a Jacobi cusp form begins in exponent $n \geq 1$. This also shows that the Fourier series of a weak Jacobi form is of the form

$$f(\tau, z) = \sum_{n=0}^{\infty} p_n(\zeta) q^n,$$

where p_n is a Laurent polynomial:

$$p_n(\zeta) = a_{-N}\zeta^{-N} + a_{-N+1}\zeta^{-N+1} + ... + a_{N-1}\zeta^{N-1} + a_N\zeta^N$$

for some $N \in \mathbb{N}$.

Remark 3.26. To check whether a Jacobi form f is holomorphic (or a cusp form) it is *not* sufficient to look at the constant term in its q-expansion. A counterexample is the form

$$f(\tau, z) := \frac{\vartheta^{10}(\tau, z)}{\eta^6(\tau)} = (2\pi i)^2 \frac{\vartheta^{10}(\tau, z)}{\vartheta'(\tau, 0)^2}$$

of weight 2 and index 5, where $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n)$. The Fourier expansion of f begins

$$f(\tau, z) := (\zeta^{-1/2} - \zeta^{1/2})^{10} q - 2(\zeta^{-1/2} - \zeta^{1/2})^{10}(5\zeta^{-1} - 2 + 5\zeta)q^2 + O(q^3).$$

This is not holomorphic because the coefficient $c(1, 5)$ of $q\zeta^5$ is nonzero.

If a weak Jacobi form f fails to be holomorphic then it has nonzero coefficients $c(n, r)$ with $4mn < r^2 \leq 4mn + m^2$. In particular, $r \leq m$ and $n < r^2/4m \leq m/4$. So one needs exactly the first $\lceil m/4 \rceil - 1$ terms of the q-expansion of f to decide whether or not f is holomorphic. Similarly one needs the first $\lfloor m/4 \rfloor$ terms to decide whether f is a cusp form.

Using Lemma 3.25 one can prove the following (without which the notions above would be dubious):

Proposition 3.27. *(i) Let $f \in J^{\mathrm{w}}_{k_1,m_1}$ and $g \in J^{\mathrm{w}}_{k_2,m_2}$ be weak Jacobi forms. Then $fg \in J^{\mathrm{w}}_{k_1+k_2,m_1+m_2}$ is a weak Jacobi form.*
(ii) Let $f \in J_{k_1,m_1}$ and $g \in J_{k_2,m_2}$ be holomorphic Jacobi forms. Then

$$fg \in J_{k_1+k_2,m_1+m_2}$$

is a holomorphic Jacobi form.
(iii) Suppose $f \in J_{k_1,m_1}$ and $g \in J_{k_2,m_2}$ are holomorphic Jacobi forms and either f or g is a cusp form. Then fg is a cusp form.

Here m_1 or m_2 are allowed to be zero, in which case f or g is a modular form for $\mathrm{SL}_2(\mathbb{Z})$ in the usual sense.

Proof. (i) The Fourier series of fg is the product of the Fourier series of f and g, hence also supported on non-negative exponents.
(ii) Write $f(\tau, z) = \sum_{n,r} c(n,r)q^n\zeta^r$ and $g(\tau, z) = \sum_{n,r} d(n,r)q^n\zeta^r$. Then

$$f(\tau,z)g(\tau,z) = \sum_{n,r\in\mathbb{Z}} \Big(\sum_{\substack{n_1+n_2=n \\ r_1+r_2=r}} c(n_1,r_1)d(n_2,r_2) \Big) q^n\zeta^r.$$

If both $c(n_1, r_1)$ and $d(n_2, r_2)$ are nonzero then $r_1^2 \le 4m_1n_1$ and $r_2^2 \le 4m_2n_2$, hence

$$\begin{aligned} 4(m_1+m_2)n - r^2 &= 4(m_1+m_2)(n_1+n_2) - (r_1+r_2)^2 \\ &= (4m_1n_1 - r_1^2) + (4m_2n_2 - r_2)^2 + 4(m_1n_2 + m_2n_1) - 2r_1r_2 \\ &\ge 4(m_1n_2 + m_2n_1) - 2\sqrt{4m_1n_1}\cdot\sqrt{4m_2n_2} \\ &= 4(m_1n_2 + m_2n_1 - 2\sqrt{m_1m_2n_1n_2}) \\ &= 4(\sqrt{m_1n_2} - \sqrt{m_2n_1})^2 \ge 0. \end{aligned}$$

So the Fourier series for fg contains only terms (n, r) with $r^2 \le 4(m_1 + m_2)n$.
(iii) follows from an argument similar to (ii). $\square$

Remark 3.28. It is possible for fg to be a cusp form even if neither f nor g is a cusp form (which does not happen for classical modular forms!). For example define

$$f = \theta_{00}^8 + \theta_{01}^8 + \theta_{10}^8 - \theta_{11}^8$$

and $g = \theta_{11}^8$, both of which are holomorphic Jacobi forms (and not cusp forms!) of weight 4 and index 4. But fg is a Jacobi cusp form of weight 8 and index 8.

Finally, we will show that (holomorphic) Jacobi forms only exist in nonnegative weight:

Theorem 3.29. *Let f be a holomorphic Jacobi form of weight k and index m. Then $k \geq 0$.*
If $k = 0$, then f is constant (and therefore $m = 0$).

This is another hint that weak Jacobi forms do not capture the correct notion of "holomorphic", since there do exist weak Jacobi forms of negative weight.

Proof. For $m < 0$, there are no entire doubly-quasiperiodic functions, and certainly no holomorphic Jacobi forms, and if $m = 0$ then any such function is constant (at least, as a function of the elliptic variable z). Therefore assume $m > 0$.

Let $H(\tau) = (h_j(\tau))$ be the vector-valued modular form attached to f, such that

$$f(\tau, z) = \sum_{j \in (\mathbb{Z}/2m)} h_j(\tau) \Theta_{m,j}(\tau, z).$$

Then $H(M \cdot \tau) = (c\tau + d)^{k-1/2} \rho(M) H(\tau)$ for every $M \in \mathrm{SL}_2(\mathbb{Z})$. Since $\rho(M)$ is unitary, the ℓ^2-norm $\|H\|$ satisfies

$$\|H(M \cdot \tau)\|^2 = |c\tau + d|^{2k-1} \|H(\tau)\|,$$

and the function $y^{k-1/2} \|H\|^2$ (where $\tau = x + iy$) is invariant under $\mathrm{SL}_2(\mathbb{Z})$.

The fact that H is holomorphic at $q = 0$ (in the sense that all its components are holomorphic) implies that $\|H\|^2$ is bounded on the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$. If $k \leq 0$, then $y^{k-1/2}$ is also bounded, so by $\mathrm{SL}_2(\mathbb{Z})$-invariance $y^{k-1/2} \|H\|^2$ is bounded on the upper half-plane, say $y^{k-1/2} \|H\|^2 \leq C$.

This leads to a contradiction when we compute Fourier coefficients. If

$$H(\tau) = \sum_n v_n e^{2\pi i n \tau}, \quad v_n \in \mathbb{C}^{2m},$$

then for any $y > 0$, we can bound

$$\begin{aligned} \|v_n\| &= \Big\| \int_{0+iy}^{1+iy} H(\tau) e^{-2\pi i n \tau} \, \mathrm{d}\tau \Big\| \\ &\leq \int_0^1 \|H(x + iy)\| e^{-2\pi n y} \, \mathrm{d}x \\ &\leq \sqrt{C} \cdot y^{1/4 - k/2} e^{-2\pi n y}. \end{aligned}$$

Since $k \leq 0$, this upper bound tends to 0 in the limit $y \to 0$ so we have $v_n = 0$ for all n. Hence $H \equiv 0$ and $f \equiv 0$ identically. □

4. Jacobi Eisenstein series

The Jacobi forms we encountered so far were defined in terms of theta functions. It is possible to build up the theory of Jacobi forms more in the spirit of a typical course on classical modular forms, where Eisenstein series are the protagonists. In this chapter we consider the "Jacobi" analogue of the Eisenstein series.

4.1. Jacobi Eisenstein series

Recall that the normalized Eisenstein series E_k for $\mathrm{SL}_2(\mathbb{Z})$ can be defined by

$$E_k(\tau) = \sum_{M \in \Gamma_\infty \backslash \Gamma} 1\Big|_k M(\tau) = \sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1 \\ c>0 \text{ or } c=0,\, d=1}} (c\tau + d)^{-k},$$

where $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : \ n \in \mathbb{Z}\}$ is the subgroup of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ that stabilizes the constant function 1 under the $|_k$-action.

Definition 4.1. Let $k \geq 4$ be even and $m \in \mathbb{N}$. The **Jacobi Eisenstein series** of weight k and index m is the series

$$E_{k,m}(\tau, z) := \sum_{\gamma \in \mathcal{J}_\infty \backslash \mathcal{J}} 1\Big|_{k,m} \gamma(\tau, z),$$

where $\mathcal{J} = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ is the Jacobi group and $\mathcal{J}_\infty$ is the stabilizer of the constant function 1 under the $|_{k,m}$-action.

If $\gamma = (M, (\lambda, \mu)) \in \mathcal{J}$ then

$$1\Big|_{k,m} \gamma(\tau, z) = (c\tau + d)^{-k} e^{-2\pi i m \frac{c(z+\lambda\tau+\mu)^2}{c\tau+d} + 2\pi i m \lambda^2 \tau + 4\pi i m \lambda z},$$

and that equals 1 if and only if $c = 0$ and $d = \pm 1$ and $\lambda = 0$. So the cosets $\gamma \in \mathcal{J}_\infty \backslash \mathcal{J}$ are represented by coprime pairs (c, d) with $c > 0$ or $(c, d) = (0, 1)$ and by tuples

$(\lambda, 0) \in \mathbb{Z}$. This leads to

$$\begin{aligned} E_{k,m}(\tau,z) &= \sum_{M \in \Gamma_\infty \backslash \Gamma} \Big(\sum_{\lambda \in \mathbb{Z}} 1 \Big|_{k,m} (\lambda, 0) \Big) \Big|_{k,m} M \\ &= \sum_{M \in \Gamma_\infty \backslash \Gamma} \Big(\sum_{\lambda \in \mathbb{Z}} e^{2\pi i m (\lambda^2 \tau + 2\lambda z)} \Big) \Big|_{k,m} M \\ &= \sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1 \\ c>0 \text{ or } c=0,\, d=1}} (c\tau + d)^{-k} e^{-2\pi i m \frac{cz^2}{c\tau+d}} \sum_{\lambda \in \mathbb{Z}} e^{2\pi i m \lambda^2 \frac{a\tau+b}{c\tau+d} + 4\pi i m \lambda \frac{z}{c\tau+d}}. \end{aligned}$$

Here $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is any matrix in $\mathrm{SL}_2(\mathbb{Z})$ with the bottom row (c, d). The choice of a, b does not matter: a different choice would only replace M by $T^n M$ for some $n \in \mathbb{Z}$, i.e. $\frac{a\tau+b}{c\tau+d}$ by $\frac{a\tau+b}{c\tau+d} + n$. But $e^{2\pi i m \lambda^2 n} = 1$.

Lemma 4.2. *$E_{k,m}$ converges absolutely and locally uniformly for $k \geq 4$.*

Proof. Write
$$\Theta_{m,0}(\tau, z) = \sum_{\lambda \in \mathbb{Z}} e^{2\pi i m(\lambda^2 \tau + 2\lambda z)} = \theta(2m\tau, 2mz)$$
in the notation of Section 3.4. Then
$$E_{k,m} = \sum_{M \in \Gamma_\infty \backslash \Gamma} \Theta_{m,0} \Big|_{k,m} M.$$

For any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4m)$, i.e. for which $c \equiv 0 \bmod 4m$, the matrix $\begin{pmatrix} a & 2mb \\ c/2m & d \end{pmatrix}$ belongs to $\Gamma(2)$, and the theta transformation formula implies

$$\begin{aligned} \Theta_{m,0}\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) &= \theta\Big(\frac{2ma\tau + 2mb}{c\tau+d}, \frac{2mz}{c\tau+d}\Big) \\ &= \theta\Big(\frac{a(2m\tau) + 2mb}{\frac{c}{2m}(2m\tau) + d}, \frac{2mz}{\frac{c}{2m}(2m\tau)+d}\Big) \\ &= \sqrt{c\tau+d} \cdot e^{\pi i \frac{c}{2m}(2mz)^2/(c\tau+d)} \theta(2m\tau, 2mz) \\ &= \sqrt{c\tau+d} \cdot e^{2\pi i m c z^2/(c\tau+d)} \Theta_{m,0}(\tau, z). \end{aligned}$$

Since $\Gamma_0(4m)$ contains Γ_∞, one can rewrite $\sum_{A \in \Gamma_\infty \backslash \Gamma} = \sum_{M \in \Gamma_0(4m) \backslash \Gamma} \sum_{N \in \Gamma_\infty \backslash \Gamma_0(4m)}$ (by factoring $A = NM$). If we only sum over $M \in \Gamma_0(4m)$, we have

$$F := \sum_{M \in \Gamma_\infty \backslash \Gamma_0(4m)} \Theta_{m,0} \Big|_{k,m} = \Theta_{m,0} \cdot \sum_{\substack{c,d \text{ coprime} \\ c \equiv 0\ (4m) \\ c>0 \text{ or } c=0,\, d=1}} (c\tau+d)^{1/2-k}$$

The latter series converges absolutely and locally uniformly because $k-1/2 > 2$. Finally

$$E_{k,m} = \sum_{M \in \Gamma_0(4m)\backslash\Gamma} F\Big|_{k,m} M$$

which is a finite sum. $\square$

4.2. Fourier decomposition of the Eisenstein series

Theorem 4.3. *$E_{k,m}$ is a holomorphic Jacobi form and the Fourier coefficients of $E_{k,m}$ are rational numbers. The coefficient $c(n,r)$ of $q^n\zeta^r$ depends only on $4mn - r^2$.*

The proof leads to a sort of formula. It is not exactly a closed expression for the coefficient of $q^n\zeta^r$, but for any given n and r it is straightforward to work out what that coefficient is. Make yourself comfortable: this is going to take a while.

Proof. Split the series as

$$\sum_{\substack{c,d\in\mathbb{Z} \\ \gcd(c,d)=1 \\ c>0 \text{ or } c=0,\, d=1}} (c\tau+d)^{-k} e^{-2\pi i m \frac{cz^2}{c\tau+d}} \sum_{\lambda\in\mathbb{Z}} e^{2\pi i m \lambda^2 \frac{a\tau+b}{c\tau+d} + 4\pi i m \lambda \frac{z}{c\tau+d}} = \sum_{\lambda\in\mathbb{Z}} q^{m\lambda^2}\zeta^{2m\lambda} + f(\tau,z),$$

where $f(\tau,z)$ counts only the contributions from pairs (c,d) with $c \neq 0$:

$$f(\tau,z) = \sum_{c=1}^{\infty} \sum_{\substack{d\in\mathbb{Z} \\ \gcd(c,d)=1}} (c\tau+d)^{-k} e^{-2\pi i m \frac{cz^2}{c\tau+d}} \sum_{\lambda\in\mathbb{Z}} e^{2\pi i m\lambda^2 \frac{a\tau+b}{c\tau+d} + 4\pi i m\lambda \frac{z}{c\tau+d}}.$$

This still satisfies $f(\tau+1,z) = f(\tau,z+1) = f(\tau,z)$ so it has a Fourier decomposition

$$f(\tau,z) = \sum_{n=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} a_{n,r} e^{2\pi i(n\tau+rz)},$$

in which the coefficients $a_{n,r}$ are given by the integral formula

$$a_{n,r} = \int_w^{w+1} \int_0^1 f(\tau,z) e^{-2\pi i(n\tau+rz)}\, \mathrm{d}z\, \mathrm{d}\tau,$$

for any basepoint $w \in \mathbb{H}$.

Replacing d by $d+c$ in

$$(c\tau+d)^{-k} e^{-2\pi i m \frac{cz^2}{c\tau+d}} e^{2\pi i m\lambda^2 \frac{a\tau+b}{c\tau+d} + 4\pi i m\lambda \frac{z}{c\tau+d}}$$

is the same as replacing M by MT and therefore τ by $\tau+1$. So rather than summing over all d and integrating from w to $w+1$, we might as well sum over a system of representatives for d (modulo c) and integrate from $w-\infty$ to $w+\infty$:

$$a_{n,r} = \sum_{c=1}^{\infty} \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^\times} \sum_{\lambda\in\mathbb{Z}} \int_{w-\infty}^{w+\infty} \int_0^1 (c\tau+d)^{-k} e^{-2\pi i m \frac{cz^2}{c\tau+d} + 2\pi i m\lambda^2 \frac{a\tau+b}{c\tau+d} + 4\pi i m\lambda \frac{z}{c\tau+d} - 2\pi i(n\tau+rz)} \,\mathrm{d}z\,\mathrm{d}\tau.$$

After applying the identity

$$\lambda^2 \frac{a\tau+b}{c\tau+d} + 2\lambda \frac{z}{c\tau+d} - \frac{cz^2}{c\tau+d} = -\frac{c(z-\lambda/c)^2}{c\tau+d} + \frac{a\lambda^2}{c}$$

and substituting $\tau \mapsto \tau - d/c$, we obtain

$$a_{n,r} = \sum_{c=1}^{\infty} c^{-k} \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^\times} e^{2\pi i(am\lambda^2+nd)/c} \sum_{\lambda\in\mathbb{Z}} \int_{w-\infty}^{w+\infty} \tau^{-k} \int_0^1 e^{-2\pi i m \frac{(cz-\lambda)^2}{c^2\tau} - 2\pi i(n\tau+rz)} \,\mathrm{d}z\,\mathrm{d}\tau.$$

Replacing λ by $\lambda+c$ in $e^{-2\pi i m \frac{(cz-\lambda)^2}{c^2\tau} - 2\pi i(n\tau+rz)}$ amounts to replacing z by $z+1$, so we can write

$$a_{n,r} = \sum_{c=1}^{\infty} c^{-k} \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^\times} e^{2\pi i(am\lambda^2+nd)/c} \sum_{\lambda\in\mathbb{Z}/c} \int_{w-\infty}^{w+\infty} \tau^{-k} \int_{-\infty}^{\infty} e^{-2\pi i m \frac{(cz-\lambda)^2}{c^2\tau} - 2\pi i(n\tau+rz)} \,\mathrm{d}z\,\mathrm{d}\tau.$$

Substituting $z \mapsto z + \lambda/c$ simplifies that to

$$a_{n,r} = \sum_{c=1}^{\infty} c^{-k} \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^\times} e^{2\pi i(am\lambda^2 - r\lambda + nd)/c} \sum_{\lambda\in\mathbb{Z}/c} \int_{w-\infty}^{w+\infty} \tau^{-k} \int_{-\infty}^{\infty} e^{-2\pi i m z^2/\tau - 2\pi i(n\tau+rz)} \,\mathrm{d}z\,\mathrm{d}\tau.$$

Using the integral

$$\int_{-\infty}^{\infty} e^{-ax^2-bx}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}} e^{b^2/4a},$$

(which is also valid for complex a, b as long as $\mathrm{Re}[a] > 0$), we find that the inner integral is

$$\int_{-\infty}^{\infty} e^{-2\pi i m \frac{z^2}{\tau} - 2\pi i(n\tau+rz)} \,\mathrm{d}z = i^{-1/2} \sqrt{\frac{\tau}{2m}} e^{\pi i r^2 \frac{\tau}{2m} - 2\pi i n\tau}.$$

So

$$a_{n,r} = i^{-1/2} \frac{1}{\sqrt{2m}} \sum_{c=1}^{\infty} \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^\times} \sum_{\lambda\in\mathbb{Z}/c} c^{-k} e^{2\pi i(am\lambda^2 - r\lambda + nd)/c} \int_{w-\infty}^{w+\infty} \tau^{-k+1/2} e^{-2\pi i(n-r^2/4m)\tau} \,\mathrm{d}\tau. \tag{4.1}$$

The integral in (4.1) no longer depends on any of the sums (over c, d or λ) so we have split $a_{n,r}$ into the product of a series and an integral. We compute them separately. □

Lemma 4.4. *For any real $s > 0$ and any $w \in \mathbb{H}$,*

$$\int_{w-\infty}^{w+\infty} \tau^{-s} e^{-i\tau}\, \mathrm{d}\tau = 2\pi \cdot \frac{e^{-\pi i s/2}}{\Gamma(s)}.$$

Here $\Gamma(s)$ is the Gamma function for which there are several standard definitions. (The integral above is essentially Hankel's representation of $\Gamma(s)^{-1}$.) I will use Gauss's limit definition,

$$\Gamma(s) = \lim_{n\to\infty} \frac{n^s \cdot (n-1)!}{s(s+1)...(s+n-1)} = \lim_{n\to\infty} \binom{s+n-1}{n}^{-1} \cdot n^{s-1}.$$

Proof. Since $e^x = \lim_{n\to\infty}(1+x/n)^n$, we can write

$$\int_{w-\infty}^{w+\infty} \tau^{-s} e^{-i\tau}\, \mathrm{d}\tau = \lim_{n\to\infty} \int_{w-n}^{w+n} \tau^{-s}(1+i\tau/n)^{-n}\, \mathrm{d}\tau.$$

This is justified by the dominated convergence theorem. For $n \in \mathbb{N}$, the integrand has a pole of order n in $\tau = in$, and around that point we have the Laurent series

$$(\tau+in)^{-s}(1+i(\tau+in)/n)^{-n} = \left((in)^{-s} \cdot \sum_{k=0}^{\infty} \binom{s+k-1}{k} (i\tau/n)^k\right) \cdot (i/n)^{-n}\tau^{-n}.$$

So the residue in $\tau = in$ is

$$\mathrm{Res}_{\tau=in}\Big(\tau^{-s}(1+i\tau/n)^{-n}\Big) = (in)^{-s}(i/n)^{-1}\binom{s+n-2}{n-1}.$$

For all large enough n, the line segment $[w-n, w+n]$ can be completed with a circular arc to form a closed contour γ_n around the pole at in as in the following figure:

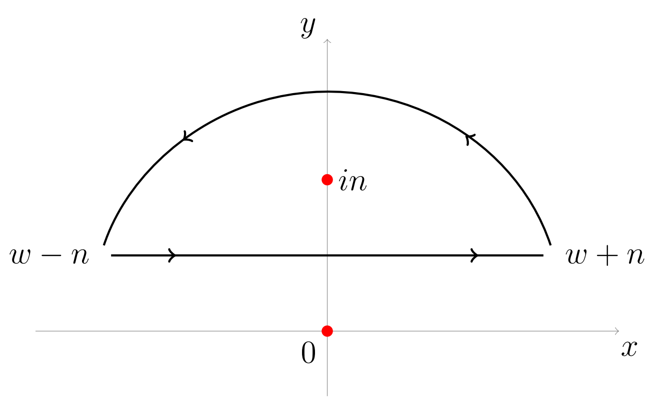


Figure 4.1: Path of integration.

Using the residue theorem we obtain

$$\oint_{\gamma_n} \tau^{-s}(1+i\tau/n)^{-n} \mathrm{d}\tau = 2\pi i \cdot (in)^{-s}(i/n)^{-1}\binom{s+n-2}{n-1}.$$

In the limit $n \to \infty$, the integral along the upper arc tends to zero and since

$$\binom{s+n-1}{n} \Big/ \binom{s+n-2}{n-1} \to 1$$

we have

$$\int_{w-\infty}^{w+\infty} \tau^{-s} e^{-i\tau}\,\mathrm{d}\tau = \lim_{n\to\infty} 2\pi \cdot i^{-s} n^{1-s} \binom{s+n-1}{n} = 2\pi \cdot \frac{i^{-s}}{\Gamma(s)}. \qquad \square$$

Proof, continued. If $n - r^2/4m > 0$ then replacing τ by $\frac{\tau}{2\pi(n-r^2/4m)}$ and using the Lemma yields

$$\int_{w-\infty}^{w+\infty} \tau^{-k+1/2} e^{-2\pi i(n-r^2/4m)\tau}\,\mathrm{d}\tau = \Big(2\pi(n-r^2/4m)\Big)^{k-3/2} \cdot 2\pi \cdot \frac{i^{-k+1/2}}{\Gamma(k-1/2)}.$$

On the other hand, if $n - r^2/4m \le 0$, then the integrand tends uniformly to 0 as $\mathrm{Im}[w] \to \infty$ so the integral is simply 0. So

$$\int_{w-\infty}^{w+\infty} \tau^{-k+1/2} e^{-2\pi i(n-r^2/4m)\tau}\,\mathrm{d}\tau = \begin{cases} \frac{(2\pi)^{k-1/2}(n-r^2/4m)^{k-3/2} i^{-k+1/2}}{\Gamma(k-1/2)} : & 4mn - r^2 > 0; \\ 0 : & 4mn - r^2 \le 0. \end{cases}$$

Now we compute the series in Equation (4.1). For any fixed c, the double sum

$$\sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} \sum_{\lambda \in \mathbb{Z}/c\mathbb{Z}} e^{2\pi i(am\lambda^2 - r\lambda + nd)/c}$$

simplifies if we observe that $d\lambda$ runs through $\mathbb{Z}/c\mathbb{Z}$ just as λ does (because $\gcd(c,d) = 1$) and that a is the inverse of d mod c (because $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$). We obtain

$$\sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} \sum_{\lambda \in \mathbb{Z}/c\mathbb{Z}} e^{2\pi i(am\lambda^2 - r\lambda + nd)/c} = \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} \sum_{\lambda \in \mathbb{Z}/c\mathbb{Z}} e^{2\pi i d(m\lambda^2 - r\lambda + n)/c}.$$

$\sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} e^{2\pi i dN/c}$ is the Ramanujan sum of elementary number theory:

$$\sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} e^{2\pi i dN/c} = \sum_{u \mid \gcd(c,N)} u \cdot \mu(c/u),$$

where μ is the Möbius function. Therefore

$$\begin{aligned} &\sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} \sum_{\lambda \in \mathbb{Z}/c\mathbb{Z}} e^{2\pi i d(m\lambda^2 - r\lambda + n)/c} \\ &= \sum_{\lambda \in \mathbb{Z}/c\mathbb{Z}} \sum_{\substack{u \mid c \\ u \mid (m\lambda^2 - r\lambda + n)}} u\mu(c/u) \\ &= \sum_{u \mid c} u\mu(c/u) \cdot \#\{\lambda \in \mathbb{Z}/c\mathbb{Z} :\ m\lambda^2 - r\lambda + n \equiv 0 \bmod u\}. \end{aligned}$$

Whether $m\lambda^2 - r\lambda + n \equiv 0 \bmod u$ is true or false depends only on the remainder class of λ in $\mathbb{Z}/u\mathbb{Z}$ (and not in $\mathbb{Z}/c\mathbb{Z}$!). So we have

$$\#\{\lambda \in \mathbb{Z}/c\mathbb{Z} : \ m\lambda^2 - r\lambda + n \equiv 0 \bmod u\} = \frac{c}{u} \cdot \#\{\lambda \in \mathbb{Z}/u\mathbb{Z} : \ m\lambda^2 - r\lambda + n \equiv 0 \bmod u\}.$$

Denoting the latter numbers by

$$N_{n,r,m}(u) := \#\Big\{\lambda \in \mathbb{Z}/u\mathbb{Z} : \ m\lambda^2 - r\lambda + n \equiv 0 \bmod u\Big\},$$

we have:

$$\sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} \sum_{\lambda \in \mathbb{Z}/c\mathbb{Z}} e^{2\pi i (am\lambda^2 - r\lambda + nd)/c} = c \cdot \sum_{u|c} \mu(c/u) N_{n,r,m}(u).$$

This is a Dirichlet convolution so the Dirichlet series factors:

$$\begin{aligned} &\sum_{c=1}^\infty \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} \sum_{\lambda \in \mathbb{Z}/c\mathbb{Z}} c^{-k} e^{2\pi i (am\lambda^2 - r\lambda + nd)/c} \\ &= \sum_{c=1}^\infty c^{1-k} \sum_{u|c} \mu(c/u) N_{n,r,m}(u) \\ &= \Big(\sum_{c=1}^\infty c^{1-k} \mu(c) \Big) \cdot \Big(\sum_{c=1}^\infty c^{1-k} N_{n,r,m}(c) \Big) \\ &= \frac{1}{\zeta(k-1)} \sum_{c=1}^\infty \frac{N_{n,r,m}(c)}{c^{k-1}}. \end{aligned}$$

The function $N_{n,r,m}(c)$ is multiplicative because a number λ solves $m\lambda^2 - r\lambda + n \equiv 0 \bmod ab$ (a, b coprime) if and only if it does so mod a and mod b. So we have an Euler product

$$\sum_{c=1}^\infty \frac{N_{n,r,m}(c)}{c^{k-1}} = \prod_p \Big(\sum_{j=0}^\infty \frac{N_{n,r,m}(p^j)}{p^{j(k-1)}} \Big).$$

Lemma 4.5. *For any prime p, the series*

$$\sum_{j=0}^\infty \frac{N_{n,r,m}(p^j)}{p^{js}} = R(p^{-s})$$

is a rational expression in p^{-s}.

Proof. Any solution $\lambda \in (\mathbb{Z}/p^n\mathbb{Z})$ of $m\lambda^2 - r\lambda + n \equiv 0$ determines a solution $\lambda \bmod p^k$ of that equation for all $k \leq n$. Conversely one can ask how many "lifts" of a solution $\lambda \in (\mathbb{Z}/p^n\mathbb{Z})$ to $\lambda \in (\mathbb{Z}/p^{n+1}\mathbb{Z})$ exist. Under certain conditions, Hensel's lemma guarantees that such a lift exists and is unique; for example, this is true if the derivative $2m\lambda - r$ at that solution is nonzero in $\mathbb{Z}/p^m\mathbb{Z}$ for any $m < n/2$.

Since $4mn - r^2 \neq 0$, in particular, Hensel's lemma always applies for large enough modulus p^N, and the series ultimately simplifies to

$$\sum_{j \leq N} \frac{N_{n,r,m}(p^j)}{p^{js}} + \sum_{j=N+1}^{\infty} \frac{C}{p^{js}}$$

with a number C that no longer depends on j. The first summand is a polynomial in p^{-s} and the second is the geometric series $Cp^{-(N+1)s} \cdot \frac{1}{1-p^{-s}}$. $\square$

If p is a prime that divides neither $2m$ nor $r^2 - 4mn$,

$$m\lambda^2 - r\lambda + n \equiv 0 \iff (2m\lambda - r)^2 + 4mn - r^2 \equiv 0.$$

So

$$N_{n,r,m}(p^j) = \#\{\lambda \in \mathbb{Z}/p^j\mathbb{Z} : \lambda^2 \equiv r^2 - 4mn \bmod p^j\}.$$

Solutions of this equation can only exist for any p^j ($j \geq 1$) if they exist modulo p, and in fact Hensel's lemma guarantees that solutions mod p^j lift uniquely to solutions mod p^{j+1} for all $j \geq 1$. So

$$N_{n,r,m}(p^j) = N_{n,r,m}(p) = \begin{cases} 2 : & r^2 - 4mn \text{ is a quadratic residue mod } p; \\ 0 : & r^2 - 4mn \text{ is a quadratic nonresidue mod } p. \end{cases}$$

The resulting series are

$$\sum_{j=0}^{\infty} \frac{N_{n,r,m}(p^j)}{p^{js}} = \begin{cases} (1+p^{-s})/(1-p^{-s}) : & r^2 - 4mn \text{ is a quadratic residue mod } p; \\ 1 : & r^2 - 4mn \text{ is a quadratic nonresidue mod } p. \end{cases}$$

Let $\chi = \chi_{4m^2(r^2-4mn)}$ be the quadratic character defined by

$$\chi(p) = \left(\frac{4(r^2 - 4mn)}{p} \right) = \begin{cases} 0 : & r^2 - 4mn \equiv 0 \bmod p \text{ or } p|2m; \\ 1 : & r^2 - 4mn = \square \bmod p; \\ -1 : & r^2 - 4mn \neq \square \bmod p. \end{cases}$$

Its Dirichlet series has Euler factors

$$\sum_{j=0}^{\infty} \frac{\chi(p^j)}{p^{js}} = \begin{cases} 1 : & \chi(p) = 0; \\ 1/(1-p^{-s}) : & \chi(p) = 1; \\ 1/(1+p^{-s}) : & \chi(p) = -1. \end{cases}$$

Since this differs from $\sum_{j=0}^{\infty} \frac{N_{n,r,m}(p^j)}{p^{js}}$ only by the factor $1 + p^{-s} = \frac{1-p^{-2s}}{1-p^{-s}}$, we have:

$$\sum_{c=1}^{\infty} \frac{N_{n,r,m}(c)}{c^s} = \frac{\zeta(s)}{\zeta(2s)} \sum_{c=1}^{\infty} \frac{\chi(c)}{c^s} \times \prod_{\substack{p|2m \\ \text{or} \\ p|(r^2-4mn)}} (\text{rational expression in } p^{-s}).$$

So we obtain

$$\sum_{c=1}^{\infty} \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} \sum_{\lambda \in \mathbb{Z}/c\mathbb{Z}} c^{-k} e^{2\pi i (am\lambda^2 - \tau\lambda + nd)/c} = \frac{L(k-1, \chi_{r^2-4mn})}{\zeta(2k-2)} \prod_{\substack{p \mid 2m \\ \text{or} \\ p \mid (r^2-4mn)}} (\text{rational number}).$$

Altogether the formula is

$$a_{n,r} = \frac{(2\pi)^{k-1/2}(n - r^2/4m)^{k-3/2} i^{-k}}{\sqrt{2m} \cdot \Gamma(k-1/2)} \cdot \frac{L(k-1,\chi)}{\zeta(2k-2)} \prod_{\substack{p \mid 2m \\ \text{or} \\ p \mid (r^2-4mn)}} \alpha_p, \tag{4.2}$$

with the rational numbers

$$\alpha_p = \frac{1}{1+p^{1-k}} \sum_{j=0}^{\infty} p^{-j(k-1)} \#\{\lambda \in \mathbb{Z}/p^j \,:\, m\lambda^2 - r\lambda + n \equiv 0\}.$$

Finally, observe that:
(1) $\Gamma(k-1/2)$ is a rational multiple of $\sqrt{\pi}$;
(2) $\zeta(2k-2)$ is a rational multiple of π^{2k-2};
(3) $L(k-1, \chi_{r^2-4mn})$ is a rational multiple of $\frac{\pi^{k-1}}{\sqrt{4mn-r^2}}$.
So $a_{n,r}$ is a rational number. It is clear from the formula (and from the definition of the "rational numbers") in Equation (4.2) that $a_{n,r}$ depends only on $4mn - r^2$. □

4.3. Examples

The results of the preceding section can be simplified further. We have not used the functional equations of $\zeta(s)$ or of $L(s,\chi)$, or the value of $\Gamma(s)$ at half-integers, or any number of facts about quadratic equations modulo prime powers (for example, quadratic reciprocity).

But in practice the Jacobi Eisenstein series already succumbs to Equation (4.2). Here are a few computations.

1. $k = 4, m = 1$**.** The simplest Jacobi Eisenstein series has weight 4 and index 1. It has a Fourier series

$$\begin{aligned} E_{4,1}(\tau, z) = 1 &+ (\zeta^{-2} + C_3\zeta^{-1} + C_4 + C_3\zeta + \zeta^2)q \\ &+ (C_4\zeta^{-2} + C_7\zeta^{-1} + C_8 + C_7\zeta + C_4\zeta^2)q^2 + O(q^3), \end{aligned}$$

where $C_\Delta = c(n,r)$ for any n, r satisfying $4n - r^2 = \Delta$.

The Fourier coefficients involve values at odd integers of L-functions attached to quadratic Dirichlet characters. In weight 4 we need the values $L(3, \chi_{-\Delta})$. We have:

$$L(3, \chi_{-12}) = \sum_{n=0}^{\infty} \frac{1}{(6n+1)^3} - \sum_{n=0}^{\infty} \frac{1}{(6n+5)^3} = \frac{\pi^3}{18\sqrt{3}};$$

$$L(3,\chi_{-4}) = \sum_{n=0}^{\infty} \frac{1}{(4n+1)^3} - \sum_{n=0}^{\infty} \frac{1}{(4n+3)^3} = \frac{\pi^3}{32};$$

$$L(3,\chi_{-28}) = \frac{4}{49\sqrt{7}}\pi^3;$$

$$L(3,\chi_{-8}) = \frac{3}{64\sqrt{2}}\pi^3;$$

et cetera. The Gamma value is $\Gamma(4-1/2) = \frac{15}{8}\sqrt{\pi}$ and the zeta value is $\zeta(2\cdot 4-2) = \frac{\pi^6}{945}$. So the formula of Equation (4.2) without the factors α_p,

$$b(\Delta) := \frac{(2\pi)^{k-1/2}(\Delta/4m)^{k-3/2}i^{-k}}{\sqrt{2m}\cdot\Gamma(k-1/2)} \cdot \frac{L(k-1,\chi_{-\Delta})}{\zeta(2k-2)},$$

has values

$$b(3) = 63, \quad b(4) = 126, \quad b(7) = 504, \quad b(8) = 756.$$

Now we have to solve some quadratic equations:

(i) $\Delta = 3$. We can take $n = r = 1$, and the quadratic equation becomes

$$m\lambda^2 - r\lambda + n = \lambda^2 - \lambda + 1 \equiv 0.$$

This has no zeros modulo 2 (and therefore any power of 2), so

$$\alpha_2 = \frac{1}{1+2^{-3}} \cdot 1 = \frac{8}{9}.$$

It has one solution modulo 3 but no solution modulo 9 (or any higher power of 3), so

$$\alpha_3 = \frac{1}{1+3^{-3}} \cdot (1 + 1\cdot 3^{-3}) = 1.$$

We obtain

$$C_3 = 63 \cdot \frac{8}{9} = 56.$$

(ii) $\Delta = 4$. We can take $n = 1$ and $r = 0$ and the quadratic equation becomes

$$\lambda^2 + 1 \equiv 0.$$

This has a solution mod 2 but not mod any higher power of 2, so $\alpha_2 = \frac{1}{1+2^{-3}}\cdot(1+2^{-3}) = 1$. Therefore $C_4 = 126$.

(iii) $\Delta = 7$. Take $n = 2$ and $r = 1$ and the quadratic equation becomes

$$\lambda^2 - \lambda + 2 \equiv 0.$$

This has two solutions mod 2 that lift to two solutions mod every power of 2 (applying Hensel's lemma), so

$$\alpha_2 = \frac{1}{1+2^{-3}} \cdot \Big(1 + \sum_{j=1}^{\infty} 2\cdot (2^j)^{-3}\Big) = \frac{8}{7}.$$

The equation has one solution mod 7 and no solutions mod 49, so

$$\alpha_7 = \frac{1}{1+7^{-3}}(1+7^{-3}) = 1.$$

We obtain $C_7 = 504 \cdot \frac{8}{7} = 576$.
(iv) $\Delta = 8$. Take $n = 2$ and $r = 0$ and the equation is $\lambda^2 + 2 \equiv 0$. This has one solution mod 2 and no solutions mod 4, so $\alpha_2 = \frac{1}{1+2^{-3}}(1+2^{-3}) = 1$. We obtain $C_8 = 756$.

So

$$\begin{aligned} E_{4,1}(\tau, z) = 1 &+ (\zeta^{-2} + 56\zeta^{-1} + 126 + 56\zeta + \zeta^2)q \\ &+ (126\zeta^{-2} + 576\zeta^{-1} + 756 + 576\zeta + 126\zeta^2)q^2 + O(q^3). \end{aligned}$$

2. $k = 6, m = 1$**.** Again we have

$$\begin{aligned} E_{6,1}(\tau, z) = 1 &+ (\zeta^{-2} + C_3\zeta^{-1} + C_4 + C_3\zeta + \zeta^2)q \\ &+ (C_4\zeta^{-2} + C_7\zeta^{-1} + C_8 + C_7\zeta + C_4\zeta^2)q^2 + O(q^3), \end{aligned}$$

where $C_\Delta = c(n, r)$ for any n, r satisfying $4n - r^2 = \Delta$. Using the L-values

$$L(5, \chi_{-12}) = \frac{11}{1944\sqrt{3}}\pi^5, \quad L(5, \chi_{-4}) = \frac{5}{1536}\pi^5,$$

$$L(5, \chi_{-28}) = \frac{62}{7203\sqrt{7}}\pi^5, \quad L(5, \chi_{-8}) = \frac{19}{4096\sqrt{2}}\pi^5,$$

we find that the values of

$$b(\Delta) := \frac{(2\pi)^{k-1/2}(\Delta/4m)^{k-3/2} i^{-k}}{\sqrt{2m}\,\Gamma(k-1/2)} \cdot \frac{L(k-1, \chi_{-\Delta})}{\zeta(2k-2)}$$

are

$$b(3) = -\frac{363}{4}, \quad b(4) = -330, \quad b(7) = -4092, \quad b(8) = -7524.$$

The quadratic equations in the rational numbers α_p do not depend on the weight, so the solution counts are the same as for $E_{4,1}$. So
(i) $\Delta = 3$: we have

$$\alpha_2 = \frac{1}{1+2^{-5}} \cdot 1 = \frac{32}{33}$$

and $\alpha_3 = 1$, so $C_3 = -\frac{363}{4} \cdot \frac{32}{33} = -88$.
(ii) $\Delta = 4$: we have $\alpha_2 = 1$ and $C_4 = -330$.
(iii) $\Delta = 7$: we have

$$\alpha_2 = \frac{1}{1+2^{-5}} \cdot \Big(1 + \sum_{j=1}^{\infty} 2 \cdot (2^j)^{-5}\Big) = \frac{32}{31}$$

and $C_7 = -4092 \cdot \frac{32}{31} = -4224$.
(iv) $\Delta = 8$: we have $\alpha_2 = 1$ and $C_8 = -7524$.

So

$$E_{6,1}(\tau,z) = 1 + (\zeta^{-2} - 88\zeta^{-1} - 330 - 88\zeta + \zeta^2)q \\ + (-330\zeta^{-2} - 4224\zeta^{-1} - 7524 - 4224\zeta - 330\zeta^2)q^2 + O(q^3).$$

By similar calculations one obtains

$$E_{8,1}(\tau,z) = 1 + (\zeta^{-2} + 56\zeta^{-1} + 366 + 56\zeta + \zeta^2)q \\ + (366\zeta^{-2} + 14016\zeta^{-1} + 33156 + 14016\zeta + 366\zeta^2)q^2 + O(q^3),$$

as well as the first Eisenstein series with nonintegral coefficients in weight 10,

$$E_{10,1}(\tau,z) = 1 + (\zeta^{-2} - \frac{860776}{43867}\zeta^{-1} - \frac{9947070}{43867} - \frac{860776}{43867}\zeta + \zeta^2)q \\ + (-\frac{9947070}{43867}\zeta^{-2} - \frac{1159757568}{43867}\zeta^{-1} - \frac{3601586268}{43867} - \frac{1159757568}{43867}\zeta + \zeta^2)q^2 + O(q^3).$$

3. $k = 4, m = 2$**.** This Eisenstein series is

$$E_{4,2}(\tau,z) = 1 + (C_4\zeta^{-2} + C_7\zeta^{-1} + C_8 + C_7\zeta + C_4\zeta^2)q \\ + (\zeta^{-4} + C_7\zeta^{-3} + C_{12}\zeta^{-2} + C_{15}\zeta^{-1} + C_{16} + C_{15}\zeta + C_{12}\zeta^2 + C_7\zeta^3 + \zeta^4)q^2 + O(q^3),$$

where $C_\Delta = c(n,r)$ for any n,r with $8n - r^2 = \Delta$. Using the same L-values as the case $(k = 4, m = 1)$ we obtain the following values for

$$b(\Delta) := \frac{(2\pi)^{k-1/2}(\Delta/4m)^{k-3/2}i^{-k}}{\sqrt{2m}\Gamma(k-1/2)} \cdot \frac{L(k-1,\chi_{-\Delta})}{\zeta(2k-2)}:$$

$$b(4) = \frac{63}{4}, \quad b(7) = 63, \quad b(8) = \frac{189}{2},$$
$$b(12) = 252, \quad b(15) = 441, \quad b(16) = 504.$$

Now we count solutions of quadratic equations: (i) $\Delta = 4$: take $n = 1$ and $r = 2$ so the equation is

$$2\lambda^2 - 2\lambda + 1 \equiv 0.$$

This has no solutions modulo any 2^n because $2\lambda^2 - 2\lambda + 1$ is odd, so we have

$$\alpha_2 = \frac{1}{1+2^{-3}} \cdot 1 = \frac{8}{9}$$

and therefore $C_4 = \frac{63}{4} \cdot \frac{8}{9} = 14$.
(ii) $\Delta = 7$: take $n = r = 1$. The equation $2\lambda^2 - \lambda + 1 \equiv 0$ has only one solution mod any power of 2, so

$$\alpha_2 = \frac{1}{1+2^{-3}} \cdot \sum_{j=0}^{\infty} (2^j)^{-3} = \frac{64}{63}.$$

There is a unique solution mod 7 and none mod 49, so $\alpha_7 = 1$. We obtain $C_7 = 63 \cdot \frac{64}{63} = 64$. Repeating this procedure for $\Delta = 8, 12, 15, 16$ yields

$$E_{4,2}(\tau,z) = 1 + (14\zeta^{-2} + 64\zeta^{-1} + 84 + 64\zeta + 14\zeta^2)q \\ + (\zeta^{-4} + 64\zeta^{-3} + 280\zeta^{-2} + 448\zeta^{-1} + 574 + 448\zeta + 280\zeta^2 + 64\zeta^3 + \zeta^4)q^2 + O(q^3).$$

4.4. Eisenstein series and cusp forms

The Jacobi Eisenstein series $E_{k,m}$ is clearly not a cusp form as it has a nonzero constant term. For *squarefree index* m and weights $k \geq 3$ it accounts for all non-cusp forms in the following sense:

Proposition 4.6. *Let $m \in \mathbb{N}$ be squarefree and $k \geq 3$.*
(i) If k is odd, then every holomorphic Jacobi form is a cusp form: $J_{k,m} = J_{k,m}^{\text{cusp}}$.
(ii) If k is even, then the space of holomorphic Jacobi forms splits as

$$J_{k,m} = \mathbb{C} \cdot E_{k,m} \oplus J_{k,m}^{\text{cusp}}.$$

Proof. Let f be a Jacobi form. f can only fail to be a cusp form by having nonzero Fourier coefficients $c(n,r)$ with $r^2 = 4mn$, and if m is squarefree then any solution to this equation has $r \equiv 0 \bmod 2m$. In other words, in the theta decomposition

$$f(\tau, z) = \sum_{j=0}^{2m-1} h_j(\tau)\Theta_{m,j}(\tau, z),$$

the forms $h_1, ..., h_{2m-1}$ vanish at $q = 0$ automatically, and the only condition is that the q-series of h_0 vanishes at $q = 0$.
(i) In odd weight, the identity $h_{2m-j} = (-1)^k h_m$ forces $h_0 \equiv 0$ identically.
(ii) In even weight, the Jacobi Eisenstein series has a theta decomposition whose $\Theta_{m,0}$-coefficient does not vanish at $q = 0$. So subtracting off some multiple of $E_{k,m}$ from f produces a cusp form. □

What about the non-squarefree case? If we write $m = df^2$ where d is squarefree and $f \in \mathbb{N}$, then all of the reminader classes $r = 2df \cdot b$, $b = 0, 1, 2, ..., f-1$ yield solutions to $r^2 = 4mn$. Conversely, if $m = df^2$ and $r^2 \equiv 0 \bmod 4m$, then r must be a multiple of $2df$.

But for any such b and any such solution $r^2 = 4mn$ (i.e. $r = 2df \cdot b$ and $n = r^2/4m = d$), we can construct an "Eisenstein series"

$$E_{k,m,b}(\tau, z) := \sum_{\gamma \in \mathcal{J}_\infty \backslash \mathcal{J}} \left(q^{r^2/4m}\zeta^r + (-1)^k q^{-r^2/4m}\zeta^{-r}\right)\Big|_{k,m} \gamma(\tau, z).$$

This is well-defined, because $\mathcal{J}_\infty$ *almost* stabilizes $q^{r^2/4m}\zeta^r$: it is invariant under

$$(\tau, z) \mapsto (\tau + 1, z), \quad (\tau, z) \mapsto (\tau, z + 1),$$

and the action $(\tau, z) \mapsto (\tau, -z)$ sends $q^{r^2/4m}\zeta^r$ to $(-1)^k q^{-r^2/4m}\zeta^{-r}$.

Imitating the proof of Theorem 4.3 shows that the tuple $(c, d) = (0, 1)$ contributes the theta functions $\Theta_{m,b} + (-1)^k \Theta_{m,-b}$ to $E_{k,m,b}$ and that the remainder of the Fourier series is supported on exponents $q^n \zeta^r$ with $4mn - r^2 > 0$: in other words, in the theta decomposition

$$E_{k,m,b}(\tau, z) = \sum_{j \in \mathbb{Z}/2m} h_j(\tau)\Theta_{m,j}(\tau, z),$$

only the components h_b and h_{-b} are nonvanishing at $q = 0$.

This means that given any Jacobi form of weight $k \geq 3$, one can subtract off some linear combination of $E_{k,m,b}$ to make the constant terms of all components of its theta decomposition vanish. We obtain the following:

Theorem 4.7. *Let $m = df^2$ with d squarefree, and let $k \geq 3$. Then there is a decomposition*

$$J_{k,m} = J_{k,m}^{\mathrm{Eis}} \oplus J_{k,m}^{\mathrm{cusp}},$$

where the Eisenstein space *$J_{k,m}^{\mathrm{Eis}}$ is spanned by Jacobi Eisenstein series $E_{k,m,b}$, $1 \leq b < f/2$ (if k is odd) or $0 \leq b \leq f/2$ (if k is even). In particular,*

$$\dim J_{k,m}^{\mathrm{Eis}} = \begin{cases} \lfloor f/2 \rfloor + 1 : & k \text{ even;} \\ \lfloor (f-1)/2 \rfloor : & k \text{ odd.} \end{cases}$$

Computing the Fourier expansion of $E_{k,m,b}$ with $b \neq 0$ is more difficult than computing $E_{k,m,0}$ so we will not do it.

Remark 4.8. This motivates the choice of the non-cusp forms f and g in Remark 3.28: up to constant multiples, they are the Jacobi Eisenstein series $E_{4,4,0}$ and $E_{4,4,1}$, respectively.

Remark 4.9. The Eisenstein spaces in weight $k = 2$ are also known completely: $J_{2,m}^{\mathrm{Eis}}$ is spanned by the linear combinations $\sum_\chi \chi(b) E_{k,m,b}$, where χ runs through the non-principal Dirichlet characters modulo f with $\chi(-1) = 1$. However the proof of this is outside the scope of these lectures.

5. The algebra of Jacobi forms

In this chapter we work out structure theorems for weak and holomorphic Jacobi forms. The main goal will be to compute dimensions, but along the way we will also develop better methods of calculating the Fourier coefficients of Jacobi forms.

5.1. Jacobi forms and power series

The simplest way to show that two Jacobi forms (of the same weight and index) are equal is to check that their difference vanishes in $z = 0$ to order $2m$. That works because a Jacobi form (and indeed any doubly quasiperiodic function of that index) can have only $2m$ zeros, counting multiplicities, within any fundamental domain for $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$.

It is inconvenient to check that directly because that amounts to comparing Taylor coefficients of Jacobi forms about $z = 0$, and those coefficients are *not* generally modular forms. On the other hand one would like to generalize the fact (cf. the first chapter) that the power series coefficients of $\wp(\tau; z)$ about $z = 0$ are simple multiples of the Eisenstein series G_k and therefore actually are modular forms.

As a substitute there is the following lemma. Recall that $G_2(\tau)$ is the nonmodular Eisenstein series

$$G_2(\tau) = \sum_{m=-\infty}^{\infty} \Big(\sum_{\substack{n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^2} \Big) = \frac{\pi^2}{3} \Big(1 - 24 \sum_{n=1}^{\infty} \sum_{d \mid n} dq^n \Big).$$

Lemma 5.1. *Suppose f is a Jacobi form of weight k and index m. Then the function*

$$\tilde f(\tau, z) := e^{mG_2(\tau)z^2} \cdot f(\tau, z)$$

satisfies

$$\tilde f\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) = (c\tau + d)^k \tilde f(\tau, z).$$

Proof. Observe that $f/\phi_{-2,1}^m$ satisfies

$$\frac{f}{\phi_{-2,1}^m}\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) = (c\tau + d)^{k+2m} \frac{f}{\phi_{-2,1}^m}(\tau, z).$$

But $\phi_{-2,1}(\tau,z) = (2\pi i\vartheta(\tau,z)/\vartheta'(\tau,0))^2$, and by the results of Section 2.4 we have

$$\frac{\vartheta(\tau,z)}{\vartheta'(\tau,0)} = e^{-G_2(\tau)z^2/2}\sigma(\tau,z)$$

with the Weierstrass σ-function

$$\sigma(\tau,z) = z\prod_{(m,n)\neq(0,0)}\Big(1-\frac{z}{m\tau+n}\Big)e^{\frac{z}{m\tau+n}+\frac{z^2/2}{(m\tau+n)^2}}.$$

So we can write

$$\begin{aligned}\tilde{f}(\tau,z) &= e^{mG_2(\tau)z^2}f(\tau,z)\\ &= \Big(\frac{\sigma(\tau,z)^2}{2\pi i\phi_{-2,1}(\tau,z)}\Big)^m f(\tau,z)\\ &= (2\pi i)^{-m}\sigma(\tau,z)^{2m}\cdot\frac{f}{\phi_{-2,1}^m}.\end{aligned}$$

For any $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, a rearrangement of that product implies

$$\begin{aligned}&\sigma\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big)\\ &= \frac{z}{c\tau+d}\prod_{(m,n)\neq(0,0)}\Big(1-\frac{z}{(ma+nc)\tau+(mb+nd)}\Big)e^{\frac{z}{(ma+nc)\tau+(mb+nd)}+\frac{z^2/2}{((ma+nc)\tau+(mb+nd))^2}}\\ &= (c\tau+d)^{-1}\sigma(\tau,z),\end{aligned}$$

i.e. σ is modular of weight -1. Therefore

$$\tilde{f}\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big) = (c\tau+d)^k\tilde{f}(\tau,z).$$ □

Theorem 5.2. *(i) Let f be a weak Jacobi form of weight k and index m, and write the Taylor series of $\tilde{f}$ as*

$$\tilde{f}(\tau,z) = e^{mG_2(\tau)z^2}f(\tau,z) = \sum_{n=0}^{\infty}a_n(\tau)z^n.$$

Then each $a_n(\tau)$ is a (holomorphic) modular form of weight $k+n$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$.
(ii) If f is a holomorphic Jacobi form, then each $a_n(\tau)$ $(n\geq 1)$ is a cusp form.

Proof. Writing out the Taylor expansion

$$\tilde{f}(\tau,z) = \sum_{n=0}^{\infty}a_n(\tau)z^n$$

and comparing coefficients of z^n in

$$\tilde{f}\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big) = \sum_{n=0}^{\infty} a_n\Big(\frac{a\tau+b}{c\tau+d}\Big)(c\tau+d)^{-n}z^n$$

and

$$(c\tau+d)^k\tilde{f}(\tau,z) = \sum_{n=0}^{\infty} a_n(\tau)(c\tau+d)^k z^n$$

shows that each a_n satisfies

$$a_n\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^{k+n}a_n(\tau).$$

The q-coefficients of the exponential $e^{mG_2(\tau)z^2}$ only appear in nonnegative exponents because this is true for $G_2(\tau)$, so if f is weak then the q-expansion of each $a_n(\tau)$ involves only nonnegative exponents. If f is a holomorphic Jacobi form, then its q^0-term is a constant so all derivatives (with respect to z) are 0. This means that the constant term in any $a_n(\tau)$ with $n \geq 1$ is zero. $\square$

It is therefore natural to compare the Taylor coefficients (up to order $2m$) of the modified functions $\tilde{f}$ rather than f. We have

$$\begin{aligned}\tilde{f}(\tau,z) &= \Big(\sum_{a=0}^{\infty}\frac{m^aG_2(\tau)^a}{a!}z^{2a}\Big)\cdot\Big(\sum_{b=0}^{\infty}\frac{1}{b!}f^{(b)}(\tau,0)z^b\Big)\\ &= \sum_{n=0}^{\infty}\Big(\sum_{\substack{a,b\geq 0\\ 2a+b=n}}\frac{m^aG_2(\tau)^af^{(b)}(\tau,0)}{a!b!}\Big)z^n.\end{aligned}$$

Since the derivatives involve powers of $2\pi i$, it simplifies things to substitute $z \mapsto z/(2\pi i)$ and write:

$$\begin{aligned}\tilde{f}\Big(\tau,\frac{z}{2\pi i}\Big) &= \sum_{n=0}^{\infty}\Big(\sum_{\substack{a,b\geq 0\\ 2a+b=n}}\frac{m^aG_2(\tau)^af^{(b)}(\tau,0)}{(2\pi i)^{2a}a!(2\pi i)^bb!}\Big)z^n\\ &= \sum_{n=0}^{\infty}\Big(\sum_{\substack{a,b\geq 0\\ 2a+b=n}}\Big(-\frac{m}{12}\frac{1}{a!b!}E_2(\tau)\Big)^aD_z^bf(\tau,0)\Big)z^n,\end{aligned}$$

where $D_z = \frac{1}{2\pi i}\frac{\partial}{\partial z}$. Here we have used $G_2(\tau) = \frac{\pi^2}{3}E_2(\tau)$.

Definition 5.3. The **Taylor expansion map** is

$$\mathcal{T} : J^{\text{weak}}_{k,m} \longrightarrow \bigoplus_{\substack{0 \le n \le 2m \\ n \equiv k \,(\mathrm{mod}\, 2)}} M_{k+n}(\mathrm{SL}_2(\mathbb{Z})),$$

defined by sending $f \in J^{\text{weak}}_{k,m}$ to the Taylor coefficients

$$c_n = \sum_{\substack{a,b \ge 0 \\ 2a+b=n}} \Big(-\frac{m}{12} \frac{1}{a!b!} E_2(\tau) \Big)^a D_z^b f(\tau, 0), \quad 0 \le n \le 2m,\ n \equiv k \,(\mathrm{mod}\, 2)$$

of $\tilde{f}(\tau, z/2\pi i)$.

These are not the development coefficients as defined by Eichler–Zagier but they are related to them. (See the next section.)

Our observation at the beginning of this section implies that the Taylor expansion map is injective. Theorem 5.2 implies that it sends $J_{k,m}$ into the space

$$M_k \oplus \bigoplus_{\substack{1 \le n \le 2m \\ n \equiv k \,(\mathrm{mod}\, 2)}} S_{k+n}.$$

In fact, if k is odd, then every weak Jacobi form of weight k is an odd function and has forced zeros at the 2-torsion points $1/2, \tau/2, (\tau+1)/2$. So in odd weight, to show two weak Jacobi forms are equal it suffices to check whether their difference vanishes in $z = 0$ to order at least $2m - 3$. Therefore the modified development map

$$\mathcal{T} : J^{\text{weak}}_{k,m} \longrightarrow \bigoplus_{\substack{1 \le n \le 2m-3 \\ n \equiv k \,(\mathrm{mod}\, 2)}} M_{k+n}(\mathrm{SL}_2(\mathbb{Z}))$$

is already injective and maps $J_{k,m}$ into $\bigoplus_n S_{k+n}(\mathrm{SL}_2(\mathbb{Z}))$.

Example 5.4. Since $\mathcal{T}$ defines an injection

$$\mathcal{D} : J_{8,1} \longrightarrow M_8 \oplus S_{10} = M_8 \oplus \{0\},$$

$J_{8,1}$ is (at most) one-dimensional. But the Jacobi forms $E_{8,1}$ and $E_4 \cdot E_{4,1}$ both belong to $J_{8,1}$ and have the same constant coefficient. So $E_{8,1} = E_4 E_{4,1}$.

5.2. Development coefficients

The point of this section is to improve on the map $\mathcal{T}$ by defining certain differential operators that map Jacobi forms to modular forms (without involving the series $G_2(\tau)$). Actually the map $\mathcal{T}$ is sufficient for our main goal (the dimension formula) but the operators we will discuss here are very useful for computations and are interesting in their own right.

The basic observation is that any f has a theta decomposition

$$f(\tau, z) = \sum_{j \in \mathbb{Z}/2m} h_j(\tau)\Theta_{m,j}(\tau, z)$$

and that each series

$$\Theta_{m,j} = \sum_{r \equiv j \, (2m)} q^{r^2/4m}\zeta^r$$

satisfies

$$\frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial z^2}\Theta_{m,j} = \sum_{r \equiv j \, (2m)} r^2 q^{r^2/4m}\zeta^r = \frac{4m}{2\pi i}\frac{\partial}{\partial \tau}\Theta_{m,j}.$$

So the product rule yields

$$\frac{1}{(2\pi i)^2}\Big(\frac{\partial^2}{\partial z^2} - 8\pi i m\frac{\partial}{\partial \tau}\Big)(f) = -\frac{4m}{2\pi i}\sum_{j \in \mathbb{Z}/2m} h_j'(\tau)\Theta_{m,j}(\tau, z).$$

Unfortunately $h_j'(\tau)$ is not a modular form: for the vector-valued function $H(\tau)$, differentiating

$$H\Big(\frac{a\tau + b}{c\tau + d}\Big) = (c\tau + d)^{k-1/2}\rho(M)H(\tau)$$

yields

$$H'\Big(\frac{a\tau + b}{c\tau + d}\Big) = (k - 1/2)c(c\tau + d)^{k+1/2}\rho(M)H(\tau) + (c\tau + d)^{k+3/2}\rho(M)H'(\tau).$$

We correct for this by applying $\frac{\partial}{\partial z}$: from

$$f\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) = (c\tau + d)^k e^{2\pi i m c z^2/(c\tau+d)} f(\tau, z),$$

we find

$$\begin{aligned}\frac{\partial}{\partial z} f\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) &= (c\tau + d)^k(4\pi i m c z)e^{2\pi i m c z^2/(c\tau+d)} f(\tau, z) \\ &\quad + (c\tau + d)^{k+1}e^{2\pi i m c z^2/(c\tau+d)}\frac{\partial}{\partial z}f(\tau, z),\end{aligned}$$

such that $\frac{1}{z}\frac{\partial}{\partial z}$ gives (up to a constant multiple) exactly the correction that makes f transform correctly. Written out more carefully, we have the following lemma:

Lemma 5.5. *Define the* ***modified heat operator*** *of weight k and index m by*

$$L_{k,m} := \frac{1}{(2\pi i)^2}\Big[8\pi i m\frac{\partial}{\partial \tau} - \frac{2k - 1}{z}\frac{\partial}{\partial z} - \frac{\partial^2}{\partial z^2}\Big].$$

For any $M \in \mathrm{SL}_2(\mathbb{Z})$ and any holomorphic function f on $\mathbb{H} \times \mathbb{C}$,

$$L_{k,m}\Big(f\Big|_{k,m} M\Big) = (L_{k,m}f)\Big|_{k+2,m} M.$$

Note however that $L_{k,m}$ breaks quasiperiodicity due to the z in the denominator of $\frac{2k-1}{z}\frac{\partial}{\partial z}$. In particular it does not map Jacobi forms to Jacobi forms.

If f is even, the function $\frac{\partial}{\partial z}f$ is odd and therefore $\frac{2k-1}{z}\frac{\partial}{\partial z}f$ is still holomorphic. Since $L_{k,m}f$ is then again even, we can apply $L_{k+2,m}$ to it.

Starting with a Jacobi form f of even weight k (which is therefore an even function) and index m, we obtain a sequence of modular forms of weights $k+2N$ for $\mathrm{SL}_2(\mathbb{Z})$ by defining

$$\mathcal{D}_{2N}(f) := \frac{(2N)!}{N!(-4)^N}\Big(L_{k+2N-2,m}...L_{k+2,m}L_{k,m}f\Big)\Big|_{z=0}.$$

(The multiple $\frac{(2N)!}{N!(-4)^N}$ turns out to make the result nicer.)

Definition 5.6. $\mathcal{D}_{2N}f$ is the $(2N)$**-th development coefficient** of f.

Explicitly, if $f(\tau,z)=\sum_{n=0}^{\infty}a_{2n}(\tau)z^{2n}$ then

$$L_{k,m}f = (2\pi i)^{-2}\sum_{n=0}^{\infty}\Big(8\pi i m a'_{2n}(\tau) - 4(n+1)(n+k)a_{2n+2}(\tau)\Big)z^{2n}.$$

By induction, one can show that

$$\begin{aligned}&L_{k+2N-2,m}...L_{k,m}f\\ &= (2\pi i)^{-2N}\sum_{n=0}^{\infty}\Big[\sum_{j=0}^{N}(-4)^{N-j}(8\pi i m)^j\binom{N}{j}\frac{(n+N-j)!(n+k+2N-2-j)!}{n!(n+k+N-2)!}\frac{d^j}{d\tau^j}a_{2(n+N-j)}(\tau)\Big]z^{2n}.\end{aligned}$$

So

$$\mathcal{D}_{2N}(f) = (2\pi i)^{-2N}\sum_{j=0}^{N}(-4)^{-j}(8\pi i m)^j\frac{(2N)!}{N!j!}\frac{(k+2N-2-j)!}{(k+N-2)!}\frac{d^j}{d\tau^j}a_{2(N-j)}(\tau).$$

In terms of the Fourier expansion of f, if

$$f(\tau,z) = \sum_{n,r}c(n,r)q^n\zeta^r$$

then

$$a_j(\tau) = \frac{(2\pi i)^j}{j!}\sum_{n=0}^{\infty}\Big(\sum_r r^j c(n,r)\Big)q^n.$$

So

$$(8\pi i m)^j\frac{d^j}{d\tau^j}a_{2(N-j)}(\tau) = \frac{(2\pi i)^{2N}}{(2N-2j)!}(4m)^j\sum_{n=0}^{\infty}\Big(\sum_r r^{2(N-j)}n^j c(n,r)\Big)q^n.$$

This leads to the expansion

$$\mathcal{D}_{2N}(f) = \sum_{n=0}^{\infty}\Big(\sum_{j=0}^{N}(-1)^{-j}\frac{(2N)!}{j!(2N-2j)!}\frac{(k+N-2+(N-j))!}{(k+N-2)!}\sum_r r^{2(N-j)}(mn)^j c(n,r)\Big)q^n.$$

For example, we have:

$$\mathcal{D}_0(f) = \sum_{n=0}^{\infty}\Big(\sum_r c(n,r)\Big)q^n = f(\tau,0);$$

$$\mathcal{D}_2(f) = \sum_{n=0}^{\infty}\Big(\sum_r (kr^2-2mn)c(n,r)\Big)q^n;$$

$$\mathcal{D}_4(f) = \sum_{n=0}^{\infty}\Big(\sum_r [(k+2)(k+1)r^4 - 12(k+1)r^2mn + 12m^2n^2]c(n,r)\Big)q^n;$$

$$\mathcal{D}_6(f) = \sum_{n=0}^{\infty}\Big(\sum_r [(k+4)(k+3)(k+2)r^6 - 30(k+3)(k+2)r^4mn + 180(k+2)r^2m^2n^2 - 120m^3n^3]c(n,r)\Big)q^n;$$

etc.

Example 5.7. (i) The equations

$$\mathcal{D}_0(E_{4,1}) = E_4 = 1 + 240q + 2160q^2 + 6720q^3 + ...$$

and

$$\mathcal{D}_2(E_{4,1}) = 0 \in S_6$$

are already enough to (recursively) determine the entire Fourier expansion of $E_{4,1}$. Write

$$\begin{aligned} E_{4,1}(\tau,z) = 1 &+ (\zeta^{-2} + C_3\zeta^{-1} + C_4 + C_3\zeta + \zeta^2)q \\ &+ (C_4\zeta^{-2} + C_7\zeta^{-1} + C_8 + C_7\zeta + C_4\zeta^2)q^2 \\ &+ (C_3\zeta^{-3} + C_8\zeta^{-2} + C_{11}\zeta^{-1} + C_{12} + C_{11}\zeta + C_8\zeta^2 + C_3\zeta^3)q^3 + ... \end{aligned}$$

Comparing coefficients of q^1 in $\mathcal{D}_0(E_{4,1})$ and E_4 and in $\mathcal{D}_2(E_{4,1})$ and 0 yields

$$2 + 2C_3 + C_4 = 240, \quad 2\cdot 14 + 2\cdot 2C_3 - 2C_4 = 0,$$

i.e. $2C_3 + C_4 = 238$ and $4C_3 - 2C_4 = -28$, and therefore $C_3 = 56$ and $C_4 = 126$. Doing this for the coefficients of q^2 yields

$$2\cdot 126 + 2C_7 + C_8 = 2160 \quad \text{and} \quad 24\cdot 126 + 0\cdot C_7 - 4\cdot C_8 = 0,$$

i.e. $2C_7 + C_8 = 1908$ and $4C_8 = 3024$, hence $C_7 = 576$ and $C_8 = 756$. With q^3,

$$2\cdot 56 + 2\cdot 756 + 2C_{11} + C_{12} = 6720 \quad \text{and} \quad 60\cdot 56 + 20\cdot 756 - 4C_{11} - 6C_{12} = 0,$$

hence $2C_{11} + C_{12} = 5096$ and $4C_{11} + 6C_{12} = 18480$, and therefore $C_{11} = 1512$ and $C_{12} = 2072$.

(ii) Applying this to

$$f := E_6 E_{4,1} = 1 + (\zeta^{-2} + 56\zeta^{-1} - 378 + 56\zeta + \zeta^2)q \\ + (-378\zeta^{-2} - 27648\zeta^{-1} - 79380 - 27648\zeta - 378\zeta^2)q^2 + O(q^3),$$

we get

$$\mathcal{D}_0(f) = 1 - 264q - 135432q^2 \pm ...$$

and

$$\mathcal{D}_2(f) = 1728q - 41472q^2 \pm ...$$

hence $\mathcal{D}_0(f) = E_{10}$ and $\mathcal{D}_2(f) = 1728\Delta$.

Remark 5.8. If f has odd weight k, then $\frac{1}{z}f$ is holomorphic and transforms under $\mathrm{SL}_2(\mathbb{Z})$ (but not under lattice translations) like a Jacobi form of weight $k+1$ and index m. For these functions, the development coefficients are defined by

$$\mathcal{D}_{2N+1}(f) := \frac{(2N)!}{N!(-4)^N}\Big(L_{k+2N-1,m} ... L_{k+3,m} L_{k+1,m}(\frac{1}{z}f)\Big)\Big|_{z=0}.$$

So if

$$f(\tau, z) = \sum_{n,r} c(n,r) q^n \zeta^r,$$

then the development coefficients of f are:

$$\mathcal{D}_1(f) = \sum_n \Big(\sum_r r c(n,r)\Big) q^n;$$

$$\mathcal{D}_3(f) = \sum_n \Big(\sum_r [(k+1)r^3 - 6mnr] c(n,r)\Big) q^n;$$

$$\mathcal{D}_5(f) = \sum_n \Big(\sum_r [(k+3)(k+2)r^5 - 20(k+2)r^3 mn + 60 r m^2 n^2] c(n,r)\Big) q^n;$$

et cetera.

5.3. The ring of weak Jacobi forms

Since the product of two weak Jacobi forms is again a weak Jacobi form where weights and indices are added, the set of weak Jacobi forms is naturally a bi-graded ring:

$$\mathbf{J}^{\mathrm{weak}} := \bigoplus_{k \in \mathbb{Z}} \bigoplus_{m=0}^{\infty} J^{\mathrm{weak}}_{k,m}.$$

In this section we will describe $\mathbf{J}^{\mathrm{weak}}$ completely in terms of generators and relations. Recall that $\phi_{-2,1}$ and $\phi_{-1,2}$ are the weak Jacobi forms

$$\phi_{-2,1}(\tau, z) = \Big(\frac{2\pi i \vartheta(\tau, z)}{\vartheta'(\tau, 0)}\Big)^2 \\ = (\zeta^{-1} - 2 + \zeta) + (-2\zeta^{-2} + 8\zeta^{-1} - 12 + 8\zeta - 2\zeta^2)q + O(q^2)$$

and

$$\begin{aligned}\phi_{-1,2}(\tau,z) &= 2\pi i\frac{\vartheta(\tau,2z)}{\vartheta'(\tau,0)} \\ &= (-\zeta^{-1}+\zeta)+(\zeta^{-3}-3\zeta^{-1}+3\zeta-\zeta^{3})q+O(q^2),\end{aligned}$$

and that $\phi_{-2,1}$ has weight -2 and index 1 and that $\phi_{-1,2}$ has weight -1 and index 2.

Let $\phi_{0,1}$ be the form

$$\phi_{0,1}(\tau,z) := -\frac{3}{\pi^2}\wp(\tau,z)\phi_{-2,1}(\tau,z),$$

where $\wp$ is the Weierstrass elliptic function. This is holomorphic because the double zeros of $\phi_{-2,1}$ for z in lattice points cancel out the poles of $\wp$. Using the Jacobi triple product and the resulting Fourier series for $\wp$ (see Remark 2.19) one can compute the Fourier expansion of $\phi_{0,1}$. It begins

$$\phi_{0,1}(\tau,z) = (\zeta^{-1}+10+\zeta)+(10\zeta^{-2}-64\zeta^{-1}+108-64\zeta+10\zeta^{2})q+O(q^2).$$

Theorem 5.9. *Let E_4, E_6 be the normalized Eisenstein series of weights 4 and 6:*

$$E_4(\tau) = 1+240\sum_{n=1}^{\infty}\Big(\sum_{d|n} d^3\Big)q^n;$$

$$E_6(\tau) = 1-504\sum_{n=1}^{\infty}\Big(\sum_{d|n} d^5\Big)q^n.$$

Then $E_4, E_6, \phi_{-2,1}, \phi_{0,1}$ are algebraically independent, and they generate the graded subring of weak Jacobi forms of even weight:

$$\bigoplus_{k\in 2\mathbb{Z}}\bigoplus_{m=0}^{\infty} J_{k,m}^{\mathrm{weak}} = \mathbb{C}[E_4(\tau), E_6(\tau), \phi_{-2,1}(\tau,z), \phi_{0,1}(\tau,z)].$$

Proof. First we prove that $E_4, E_6, \phi_{0,1}, \phi_{-2,1}$ are algebraically independent:
It is not hard to see that Jacobi forms of different weight or index cannot be linearly dependent. So suppose we have some polynomial relation of the form

$$\sum_{a+b=m} f_{a,b}\phi_{0,1}^a\phi_{-2,1}^b = 0 \in J_{k,m}^{\mathrm{weak}}$$

where $f_{a,b}\in\mathbb{C}[E_4,E_6]$ is a modular form of weight $k+2b$, and m can be chosen to be minimal. Setting $z=0$ and using $\phi_{-2,1}(\tau,0)=0$ shows that the $(a,b)=(m,0)$ term $f_{m,0}=0$ vanishes. But then we can divide the equation by $\phi_{-2,1}$ to get a relation of the form

$$\sum_{a+b=m-1} f_{a,b}\phi_{0,1}^a\phi_{-2,1}^b = 0 \in J_{k,m-1}^{\mathrm{weak}}.$$

This contradicts the choice of m.

To prove that $E_4, E_6, \phi_{-2,1}, \phi_{0,1}$ generate all weak Jacobi forms we use induction on m:
(i) Any weak Jacobi form of index $m = 0$ is constant with respect to z, so it belongs to $\mathbb{C}[E_4, E_6]$.
(ii) Let $m \geq 1$ and let $f \in J_{k,m}^{\text{weak}}$ for some weight k. Then the function $g(\tau) := f(\tau, 0)$ transforms like a modular form of weight k, so it belongs to $\mathbb{C}[E_4, E_6]$. Since $\phi_{0,1}(\tau, 0) = 12$, the function

$$f(\tau, z) - \frac{g(\tau)\phi_{0,1}(\tau, z)^m}{12^m}$$

is a weak Jacobi form of weight k and index m that vanishes at $z = 0$, and (since it is even) has a double zero there. So it has double zeros at all lattice points $z \in \mathbb{Z} \oplus \tau\mathbb{Z}$. Then

$$\frac{f(\tau, z) - g(\tau)\phi_{0,1}(\tau, z)^m/12^m}{\phi_{-2,1}(\tau, z)}$$

is a well-defined, weak Jacobi form of weight $k + 2$ and index $m - 1$, and belongs to $\mathbb{C}[E_4, E_6, \phi_{-2,1}, \phi_{0,1}]$ by induction. Therefore $f \in \mathbb{C}[E_4, E_6, \phi_{-2,1}, \phi_{0,1}]$. □

Corollary 5.10. *The ring of weak Jacobi forms is generated by the forms $E_4, E_6, \phi_{-2,1}, \phi_{0,1}, \phi_{-1,2}$ modulo the single relation*

$$\phi_{-1,2}^2 = \phi_{-2,1} \cdot \frac{\phi_{0,1}^3 - 3E_4\phi_{0,1}\phi_{-2,1}^2 + 2E_6\phi_{-2,1}^3}{432}.$$

Proof. $\phi_{-1,2}$ has only simple zeros in the points $\frac{1}{2}\mathbb{Z} \oplus \frac{\tau}{2}\mathbb{Z}$. Since any odd-weight Jacobi form f vanishes in those points due to the identity $f(\tau, -z) = -f(\tau, z)$, it follows that every weak Jacobi form of odd weight is a multiple of $\phi_{-1,2}$: so

$$\mathbf{J}^{\text{weak}} = \mathbf{J}_{2*,m}^{\text{weak}} \oplus \phi_{-1,2} \cdot \mathbf{J}_{2*,m}^{\text{weak}}.$$

This implies that $\mathbf{J}^{\text{weak}}$ is generated by $E_4, E_6, \phi_{-2,1}, \phi_{0,1}, \phi_{-1,2}$ and that the only defining relation is the representation of $\phi_{-1,2}^2$ in $\mathbb{C}[E_4, E_6, \phi_{-2,1}, \phi_{0,1}]$. One can either compute this directly (using Fourier series) or observe that $\phi_{-1,2}$ is (a multiple of) $\phi_{-2,1}^2 \cdot \wp'$ and that the relation is just the Weierstrass equation

$$(\wp')^2 = 4\wp^3 - 60G_4\wp - 140G_6.$$ □

So we can express the dimensions of $J_{k,m}^{\text{weak}}$ as a generating function:

Corollary 5.11. $\dim J_{k,m}^{\text{weak}}$ *is the coefficient of $t^k u^m$ in the following series:*

$$\sum_{m=0}^{\infty} \sum_{k=-\infty}^{\infty} (\dim J_{k,m}^{\text{weak}}) t^k u^m = \frac{1 + u^2/t}{(1 - t^4)(1 - t^6)(1 - u)(1 - u/t^2)}.$$

Finally, the dimension formula implies that the development map $\mathcal{D}$ is actually an isomorphism (on weak Jacobi forms):

Corollary 5.12. *For every $m \in \mathbb{N}_0$ and even $2k \in \mathbb{Z}$, the development map*

$$\mathcal{D} : J_{2k,m}^{\text{weak}} \longrightarrow \bigoplus_{n=0}^{m} M_{k+2n}$$

is an isomorphism of $\mathbb{C}$-vector spaces.

Similarly, for odd $2k+1 \in \mathbb{Z}$, the map

$$\mathcal{D} : J_{2k,m}^{\text{weak}} \longrightarrow \bigoplus_{n=1}^{m-1} M_{k+2n-1}$$

is an isomorphism of $\mathbb{C}$-vector spaces. So we have

$$\dim J_{k,m}^{\text{weak}} = \begin{cases} \sum_{n=0}^{m} \dim M_{k+2n} : & k \text{ even;} \\ \sum_{n=1}^{m-1} \dim M_{k+2n-1} : & k \text{ odd.} \end{cases}$$

Proof. We know that $\mathcal{D}$ is injective so it is enough to compare dimensions. Using

$$\sum_{k=0}^{\infty} \dim M_k t^k = \frac{1}{(1-t^4)(1-t^6)},$$

we find

$$\begin{aligned}
&\sum_{m=0}^{\infty} \sum_{k\in\mathbb{Z}} \dim\Big(\bigoplus_{n=0}^{m} M_{k+n}\Big) t^k u^m \\
&= \sum_{k=0}^{\infty} \dim M_k \cdot \Big(\sum_{m=0}^{\infty}\sum_{n=0}^{m} t^{k-2n} u^m\Big) \\
&= \sum_{k=0}^{\infty} \dim M_k \cdot t^k \sum_{m=0}^{\infty} \frac{1-t^{-2m-2}}{1-t^{-2}} u^m \\
&= \frac{1}{1-t^{-2}} \sum_{k=0}^{\infty} \dim M_k \cdot t^k \cdot \Big(\frac{1}{1-u} + \frac{t^{-2}}{1-u/t^2}\Big) \\
&= \frac{1}{(1-t^{-2})(1-t^4)(1-t^6)} \cdot \Big(\frac{(1-u/t^2) - t^{-2}(1-u)}{(1-u)(1-u/t^2)}\Big) \\
&= \frac{1}{(1-t^4)(1-t^6)(1-u)(1-u/t^2)} = \sum_{k,m} (\dim J_{2k,m}^{\text{weak}}) t^{2k} u^m.
\end{aligned}$$

□

m \ k	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12
1											1	0	1	0	1	0	2	0	2	0	2	0	3	0	3
2									1	0	1	1	2	0	2	1	3	1	3	1	4	1	4	2	5
3							1	0	1	1	2	1	3	1	3	2	4	2	5	2	5	3	6	3	7
4					1	0	1	1	2	1	3	2	4	2	4	3	6	3	6	4	7	4	8	5	9
5			1	0	1	1	2	1	3	2	4	3	5	3	6	4	7	5	8	5	9	6	10	7	11
6	1	0	1	1	2	1	3	2	4	3	5	4	7	4	7	6	9	6	10	7	11	8	12	9	14

Figure 5.1: $\dim J^{\mathrm{w}}_{k,m}$ for $1 \le m \le 6$ and $-12 \le k \le 12$.

5.4. Holomorphic Jacobi forms

The situation for holomorphic Jacobi forms is very different from weak Jacobi forms:

Proposition 5.13. *The graded ring* $\mathbf{J}$ *of Jacobi forms is not finitely generated.*

This is a little reminiscent of the fact that the ring of cusp forms for $\mathrm{SL}_2(\mathbb{Z})$ is not finitely generated.

Proof. Suppose $\{f_1, ..., f_n\}$ is any finite set of (nonconstant) Jacobi forms. Let M be the largest index of any f_i. Since all f_i have weight at least 1, it follows that every monomial in $\{f_1, ..., f_n\}$ of index at least $5M$ has weight at least 5. But then the Eisenstein series $E_{4,5M}$ is not contained in the ring generated by $f_1, ..., f_n$.

Since that applies to any finite set of Jacobi forms, it follows that $\mathbf{J}$ is not finitely generated. $\square$

In weights $k \ge 3$, the formula for $\dim J^{\mathrm{weak}}_{k,m}$ leads quickly to a formula for $\dim J_{k,m}$:

Theorem 5.14. *Let $k \ge 3$ and $m \in \mathbb{N}$, and let $N_{k,m}$ be the number of tuples (n, r) with $0 < r \le m$ (if k is even) or $0 < r < m$ (if k is odd) and $0 \le n < r^2/4m$. Then*
$$\dim J_{k,m} = \dim J^{\mathrm{weak}}_{k,m} - N_{k,m}.$$

Note that $N_{k,m}$ depends only on the parity of k.

Proof. Let $f(\tau, z) = \sum_{n,r} c(n, r) q^n \zeta^r \in J^{\mathrm{weak}}_{k,m}$. By Lemma 3.25, all coefficients $c(n, r)$ with $r^2 > 4mn + m^2$ are zero. The condition for f to be holomorphic is then precisely the vanishing of $c(n, r)$ with $4mn + m^2 \ge r^2 > 4mn$, i.e. $0 < r \le m$ and $0 \le n < r^2/4m$. If k is odd, then the coefficients of a weak Jacobi form with $r = m$ automatically vanish as well, so f is already holomorphic if $c(n, r) = 0$ whenever $0 < r < m$ and $0 \le n < r^2/4m$.

Therefore we have a map
$$\varphi : J^{\mathrm{weak}}_{k,m} \longrightarrow \mathbb{C}^{N_{k,m}}, \quad f \mapsto (c(n, r))_{n,r}$$

whose kernel is $J_{k,m}$. The claim will follow as soon as we show that φ is surjective. Since $k \geq 3$, we can construct weak Jacobi forms by modifying the Eisenstein series:

$$P_{k,m;n,r} := \frac{1}{2} \sum_{\gamma \in \mathcal{J}_\infty \backslash \mathcal{J}} \Big(q^n \zeta^r + (-1)^k q^n \zeta^{-r} \Big)\Big|_{k,m} \gamma(\tau, z), \quad 0 < r \leq m,\ 0 \leq n < r^2/4m.$$

(These are called Jacobi Poincaré series.) The general Fourier coefficients of $P_{k,m;n,r}$ are complicated, but the coefficients with $n - r^2/4m \leq 0$ are very simple: one can show (by computations similar to Section 4.2) that $\varphi(P_{k,m;n,r})$ is the tuple with 1 in the (n,r)-entry and 0 otherwise. $\square$

The situation in weights $k \in \{1, 2\}$ is far less obvious. Certainly Theorem 5.14 is no longer correct as stated: for example, in weight $k = 2$ and index $m = 6$ it predicts $\dim J_{2,6} = -1$.
The dimensions in low weight were computed by Skoruppa. We only sketch a rough idea of the proof.

Theorem 5.15 (Skoruppa)**.** *(i) There are no nonzero holomorphic Jacobi forms of weight 1 and any index:*

$$J_{1,m} = \{0\},\ m \in \mathbb{N}.$$

(ii) In weight 2*,*

$$\dim J_{2,m} = \dim J_{2,m}^{\text{weak}} - N_{2,m} + \#\{\text{divisors } d|m \textit{ with } d^2 \nmid m \textit{ and } d < m/d\}.$$

Proof. Both claims involve counting vector-valued modular forms of weight 1/2:
(i) If $f \in J_{1,m}$ has the decomposition

$$f(\tau) = \sum_{j \in \mathbb{Z}/2m} h_j(\tau) \Theta_{m,j}(\tau, z),$$

then each of the functions h_j is a holomorphic modular form of weight 1/2 and level $\Gamma(4m)$. The Serre–Stark basis theorem implies that any such modular form is a linear combination of functions $\theta_{\chi,r} = \sum_{n=-\infty}^{\infty} \chi(n) q^{rn^2}$ for some appropriate numbers $r \in \mathbb{N}$ and Dirichlet characters χ. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on this space via the slash operator $|_{1/2,\rho_m}$ and the possibilities for $H = (h_j)_{j \in \mathbb{Z}/2m}$ are exactly the invariants of that action. In this case there are no such invariants.
(ii) The correction term $\#\{\text{divisors } d|m \text{ with } d^2 \nmid m \text{ and } d < m/d\}$ measures a space of "dual" cusp forms of weight 1/2. The image of φ in the proof of Theorem 5.14 consists exactly of tuples that are orthogonal to those cusp forms in an appropriate sense, so this is the number that must be added on to get $\dim J_{2,m}$. $\square$

$m \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0	1	0	0	0	1	0	1	0	1	0	1	0	2	0	1	0	2	0	2	0	2	0	2	0	3
1	0	0	0	0	1	0	1	0	1	0	2	0	2	0	2	0	3	0	3	0	3	0	4	0	4
2	0	0	0	0	1	0	1	0	2	0	2	1	3	0	3	1	4	1	4	1	5	1	5	2	6
3	0	0	0	0	1	0	2	0	2	1	3	1	4	1	4	2	5	2	6	2	6	3	7	3	8
4	0	0	0	0	2	0	2	1	3	1	4	2	5	2	5	3	7	3	7	4	8	4	9	5	10
5	0	0	0	0	1	1	2	1	3	2	4	3	5	3	6	4	7	5	8	5	9	6	10	7	11
6	0	0	0	0	1	0	2	1	3	2	4	3	6	3	6	5	8	5	9	6	10	7	11	8	13
7	0	0	0	0	2	1	3	2	4	3	6	4	7	5	8	6	10	7	11	8	12	9	14	10	15
8	0	0	0	0	2	1	3	2	5	3	6	5	8	5	9	7	11	8	12	9	14	10	15	12	17
9	0	0	0	1	2	2	4	3	5	5	7	6	9	7	10	9	12	10	14	11	15	13	17	14	19
10	0	0	0	0	2	1	3	3	5	4	7	6	9	7	10	9	13	10	14	12	16	13	18	15	20

Figure 5.2: $\dim J_{k,m}$ for $m \leq 10$ and $k \leq 24$.

m	25	37	43	49	50	53	57	58	61	64	65	67	73	74	75	77	79
$\dim J_{2,m}$	1	1	1	2	1	1	1	1	1	1	1	2	2	1	1	1	1
$\dim J_{2,m}^{\mathrm{cusp}}$	0	1	1	0	0	1	1	1	1	0	1	2	2	1	0	1	1

Figure 5.3: Jacobi forms of weight 2 and index $m \leq 80$

5.5. Modules of Jacobi forms

In the previous sections, we observed that the graded ring structure of $\mathbf{J}$ is rather opaque: it is not even finitely generated. A different approach to understanding $\mathbf{J}$ is to view

$$\mathbf{M} := M_*(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]$$

as the underlying (graded) ring and to consider $\mathbf{J}$ as a graded $\mathbf{M}$-module.

The main theorem is then

Theorem 5.16. *$\mathbf{J}$, $\mathbf{J}^{\mathrm{weak}}$ and $\mathbf{J}^{\mathrm{cusp}}$ are free $\mathbf{M}$-modules.*

More precisely, $\mathbf{J}$ splits as a direct sum

$$\mathbf{J} = \bigoplus_{m=1}^{\infty} J_{*,m},$$

where $J_{*,m}$ consists of Jacobi forms of index m (and any weight), and each $J_{*,m}$ is itself a graded $\mathbf{M}$-module. Theorem 5.16 follows easily from the following stronger result:

Theorem 5.17. *For any fixed index m, each of*

$$J_{*,m}, \quad J_{*,m}^{\mathrm{weak}}, \quad J_{*,m}^{\mathrm{cusp}}$$

is a free $\mathbf{M}$*-module of rank* $2m$*.*

I.e. there is a basis $f_1, ..., f_{2m}$ of Jacobi forms of index m and some weights $k_1, ..., k_{2m}$ such that every Jacobi form f of weight k and index m can be written *uniquely* in the form

$$f = h_1 f_1 + ... + h_{2m} f_{2m}$$

with modular forms $h_i \in M_{k-k_i}(\mathrm{SL}_2(\mathbb{Z}))$.

Proof. We will first show that $J_{*,m}$ is free. The $\mathbb{C}[E_4, E_6]$-basis of $J_{*,m}$ will be constructed by induction on the weight: we begin with $\emptyset$, and suppose that for some k we have already found a set $\{f_1, ..., f_r\}$ of Jacobi forms of weights $k_1, ..., k_r$ with the following property: every Jacobi form f of weight $< k$ can be written uniquely in the form $f = \sum_i h_i f_i$ with modular forms h_i. (Clearly $\emptyset$ works for $k = 1$.) Then $\{f_1, ..., f_r\}$ remain linearly independent in weight k: suppose there is a relation

$$\sum_{i=1}^r h_i f_i = 0, \quad h_i \in M_{k-k_i}$$

in weight k. Since each h_i has strictly positive weight, we can further decompose

$$h_i = \alpha_i E_4 + \beta_i E_6, \quad \text{where } \alpha_i \in M_{k-k_i-4},\ \beta_i \in M_{k-k_i-6}.$$

Then

$$E_4 \cdot \Big(\sum_{i=1}^r \alpha_i f_i \Big) + E_6 \cdot \Big(\sum_{i=1}^r \beta_i f_i \Big) = 0.$$

At the point $\tau = i$, we have

$$E_4(i) \cdot \Big(\sum_{i=1}^r \alpha_i f_i \Big)(i, z) = 0$$

because $E_6(i) = 0$. This implies that the quotient $\frac{\sum_{i=1}^r \alpha_i f_i}{E_6}$ is holomorphic; its Fourier series also satisfies the vanishing condition (because E_6 has constant term 1) so it is a true Jacobi form ϕ of weight $k - 10$. So we have the identity

$$\phi = \frac{\sum_{i=1}^r \alpha_i f_i}{E_6} = -\frac{\sum_{i=1}^r \beta_i f_i}{E_4}.$$

By the induction assumption, we have a unique representation

$$\phi = \sum_{i=1}^r \gamma_i f_i, \quad \gamma_i \in M_{k-k_i-10}.$$

The uniqueness forces $\alpha_i = E_6\gamma_i$ and $\beta_i = -E_4\gamma_i$, hence

$$h_i = \alpha_i E_4 + \beta_i E_6 = (E_6 E_4 - E_4 E_6)\gamma_i = 0$$

for all i.

Now $\bigoplus_{i=1}^r f_i M_{k-k_i}$ is a subspace of $J_{k,m}$, and can be extended to all of $J_{k,m}$ by choosing a basis $f_{r+1}, ..., f_s$ of a complement. We have

$$J_{k,m} = \bigoplus_{i=1}^s f_i M_{k-k_i}$$

because $M_0 = \mathbb{C}$. The set $\{f_1, ..., f_s\}$ now satisfies the induction hypothesis in weight $k+1$. The fact that $J_{*,m}^{\text{weak}}$ and $J_{*,m}^{\text{cusp}}$ are free follows with the same argument (although in the weak forms case, we must start in weight $k = -2m$).

Finally we have to compute the rank. For $J_{*,m}^{\text{weak}}$, this is a consequence of the ring structure

$$\mathcal{J}^{\text{weak}} = \mathbb{C}[E_4, E_6, \phi_{-2,1}, \phi_{0,1}, \phi_{-1,2}]/(\phi_{-1,2}^2 = ...) :$$

a basis of $J_{*,m}^{\text{weak}}$ is given by the monomials in $\phi_{0,1}, \phi_{-2,1}, \phi_{-1,2}$ of index m in which at most one copy of $\phi_{-1,2}$ appear, and the number of those monomials is $2m$. Now the identity

$$\Delta^N \cdot J_{*,m}^{\text{weak}} \subseteq J_{*,m}^{\text{cusp}} \subseteq J_{*,m} \subseteq J_{*,m}^{\text{weak}}$$

(N sufficiently large) implies that $J_{*,m}$ and $J_{*,m}^{\text{cusp}}$ also have rank $2m$. $\square$

Example 5.18. $J_{*,1}$, $J_{*,1}^{\text{weak}}$, $J_{*,1}^{\text{cusp}}$ have the following bases:

$$J_{*,1} = \mathbf{M}E_{4,1} \oplus \mathbf{M}E_{6,1}, \quad J_{*,1}^{\text{weak}} = \mathbf{M}\phi_{-2,1} \oplus \mathbf{M}\phi_{0,1}, \quad J_{*,1}^{\text{cusp}} = \mathbf{M}\Delta\phi_{-2,1} \oplus \mathbf{M}\Delta\phi_{0,1}.$$

Remark 5.19. The weights of any $\mathbb{C}[E_4, E_6]$-basis of $J_{*,m}$ are uniquely determined. If we label them $k_1, ..., k_{2m}$, then as a formal power series,

$$\sum_{k=0}^{\infty} \Big(\dim J_{k,m}\Big) t^k = \frac{\sum_{j=1}^{2m} t^{k_j}}{(1-t^4)(1-t^6)}.$$

The first few polynomials $P_m(t) := \sum_{j=1}^{2m} t^{k_j}$ are:

$$P_1(t) = t^4 + t^6, \quad P_2(t) = t^4 + t^6 + t^8 + t^{11}, \quad P_3(t) = t^4 + 2t^6 + t^8 + t^9 + t^{11}.$$

6. Hecke theory

6.1. The Petersson norm

The natural norm on modular forms f of weight k is defined in terms of the associated invariant function $y^{k/2}|f(x+iy)|$ and the invariant metric $\frac{\mathrm{d}x\otimes\mathrm{d}y}{y^2}$ on $\mathbb{H}$. In this section we define the natural norm on Jacobi forms.

Lemma 6.1. *If f is a Jacobi form of weight k and index m, then*

$$\tilde{f}(\tau,z) := y^{k/2}e^{-2\pi m v^2/y}|f(\tau,z)|$$

satisfies $\tilde{f}(\tau+1,z)=\tilde{f}(\tau,z+1)=\tilde{f}(\tau,z)$ and

$$\tilde{f}\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big)=\tilde{f}(\tau,z).$$

In other words, $\tilde{f}$ transforms like a Jacobi form of weight 0 and index 0.

Proof. The identities for $\tau\mapsto\tau+1$ and $z\mapsto z+1$ are immediate. If $\tau=x+iy$ and $z=u+iv$ then, in terms of the new variables

$$-\frac{1}{\tau}=\frac{x-iy}{x^2+y^2}=:\tilde{x}+i\tilde{y}$$

and

$$\frac{z}{\tau}=\frac{ux+vy}{x^2+y^2}+i\frac{vx-uy}{x^2+y^2}=:\tilde{u}+i\tilde{v},$$

we have

$$\mathrm{Im}\Big[\frac{z^2}{\tau}\Big]=\frac{v^2}{y}-\frac{\tilde{v}^2}{\tilde{y}}.$$

Therefore

$$\begin{aligned}\tilde{f}\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big)&=\tilde{y}^{k/2}e^{-2\pi m\tilde{v}^2/y}\Big|\tau^k e^{-2\pi i m z^2/\tau}f(\tau,z)\Big|\\&=|\tau|^k\tilde{y}^{k/2}e^{-2\pi m(\tilde{v}^2/y+\mathrm{Im}[z^2/\tau])}|f(\tau,z)|\\&=y^{k/2}e^{-2\pi m v^2/y}|f(\tau,z)|=\tilde{f}(\tau,z).\end{aligned}$$

□

Definition 6.2. Let f be a Jacobi cusp form of weight k and index m, with invariant function
$$\tilde{f}(\tau, z) = y^{k/2} e^{-2\pi m v^2/y} |f(\tau, z)|.$$
The **Petersson norm** $\|f\|$ is defined by

$$\begin{aligned}\|f\|^2 &:= \int_X \tilde{f}(\tau, z)^2 \, \frac{\mathrm{d}x \, \mathrm{d}y \, \mathrm{d}u \, \mathrm{d}v}{y^3} \\ &= \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \int_{\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau)} |f(\tau, z)|^2 e^{-4\pi m v^2/y} y^{k-3} \, \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}y,\end{aligned}$$

where X is (the closure of a) fundamental domain for the action of $\mathcal{J}$ on $\mathbb{H} \times \mathbb{C}$.

One can take $X = \{(\tau, a\tau + b) : \ \tau \in \mathcal{F}, \ a, b \in [0, 1]\}$, where
$$\mathcal{F} = \{x + iy : \ x^2 \leq 1/4 \text{ and } x^2 + y^2 \geq 1\}$$
is the closure of the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$.

$y^{-3} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}u \, \mathrm{d}v$ is the natural invariant metric on $\mathbb{H} \times \mathbb{C}$: it is the product of the hyperbolic metric $y^{-2} \, \mathrm{d}x \, \mathrm{d}y$ on $\mathbb{H}$ and the unique translation-invariant metric $y^{-1} \, \mathrm{d}u \, \mathrm{d}v$ on $\mathbb{C}$ that makes $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ have volume 1.

The Petersson inner product is induced from the Petersson norm in the usual way: for cusp forms f, g of weight k and index m,
$$\langle f, g \rangle := \int_X f(\tau, z) \overline{g(\tau, z)} e^{-4\pi m v^2/y} y^{k-3} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}u \, \mathrm{d}v.$$

This reduces to the inner product for modular forms (of half-integral weight) in the following sense - which also yields the proof that $\|f\|$ is finite:

Proposition 6.3. *Suppose f and g are Jacobi cusp forms of weight k and index m, with theta decompositions*
$$f(\tau, z) = \sum_{j \in \mathbb{Z}/2m} f_j(\tau) \Theta_{m,j}(\tau, z), \quad g(\tau, z) = \sum_{j \in \mathbb{Z}/2m} g_j(\tau) \Theta_{m,j}(\tau, z).$$
Then
$$\langle f, g \rangle = \frac{1}{\sqrt{4m}} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \Big(\sum_{j \in \mathbb{Z}/2m} f_j(\tau) \overline{g_j(\tau)} \Big) y^{k-5/2} \, \mathrm{d}x \, \mathrm{d}y.$$

The right-hand side can be viewed as the natural inner product of the vector-valued cusp forms $F(\tau) = (f_j)_{j \in \mathbb{Z}/2m}$ and $G(\tau) = (g_j)_{j \in \mathbb{Z}/2m}$ of weight $k - 1/2$. Note that the function $y^{k-1/2} \sum_{j \in \mathbb{Z}/2m} f_j(\tau) \overline{g_j(\tau)} = y^{k-1/2} F^T \overline{G}$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant because the representation ρ that F transforms with is unitary.

Proof. We have

$$\langle f,g\rangle = \int_{\mathcal{F}}\int_{\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau} \sum_{j_1,j_2\in\mathbb{Z}/2m} f_{j_1}(\tau)\overline{g_{j_2}(\tau)}\cdot\Theta_{m,j_1}(\tau,z)\overline{\Theta_{m,j_2}(\tau,z)}\, e^{-4\pi m v^2/y} y^{k-3}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}y.$$

For any fixed indices j_1, j_2, substituting $z = a\tau + b$, the interior integral is

$$\begin{aligned}&\int_{\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau} \Theta_{m,j_1}(\tau,z)\overline{\Theta_{m,j_2}(\tau,z)} e^{-4\pi m v^2/y}\,\frac{\mathrm{d}u\,\mathrm{d}v}{y}\\ &= \sum_{\substack{r_1\in\mathbb{Z}\\ r_1\equiv j_1\ (\mathrm{mod}\ 2m)}} \sum_{\substack{r_2\in\mathbb{Z}\\ r_2\equiv j_2\ (\mathrm{mod}\ 2m)}} \int_0^1\int_0^1 e^{\pi i\frac{r_1^2}{2m}\tau + 2\pi i r_1(a\tau+b)}\cdot\overline{e^{\pi i\frac{r_2^2}{2m}\tau+2\pi i r_2(a\tau+b)}}\cdot e^{-4\pi m a^2 y}\,\mathrm{d}a\,\mathrm{d}b.\end{aligned}$$

The integral over b is zero unless $r_1 = r_2 =: r$, so the entire integral is zero unless $j_1 = j_2 =: j$ in which case it is

$$\begin{aligned}&\sum_{\substack{r\in\mathbb{Z}\\ r\equiv j\ (2m)}} \int_0^1 e^{\pi i\frac{r^2}{2m}\tau + 2\pi i r a\tau - \pi i\frac{r^2}{2m}\overline{\tau} - 2\pi i r a\overline{\tau} - 4\pi m a^2 y}\,\mathrm{d}a\\ &= \sum_{\substack{r\in\mathbb{Z}\\ r\equiv j\ (2m)}} \int_0^1 e^{-4\pi m(a+r/2m)^2 y}\,\mathrm{d}a\\ &= \int_{-\infty}^{\infty} e^{-4\pi m a^2 y}\,\mathrm{d}a = \frac{1}{\sqrt{4my}}.\end{aligned}$$

Hence

$$\begin{aligned}\langle f,g\rangle &= \int_{\mathcal{F}} \sum_{j\in\mathbb{Z}/2m} f_j(\tau)\overline{g_j(\tau)}\cdot\frac{1}{\sqrt{4my}} y^{k-2}\,\mathrm{d}x\,\mathrm{d}y\\ &= \frac{1}{\sqrt{4m}}\int_{\mathcal{F}}\Big(\sum_{j\in\mathbb{Z}/2m} f_j(\tau)\overline{g_j(\tau)}\Big) y^{k-5/2}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

□

6.2. The U_N operator

For $N \in \mathbb{N}$, define a map U_N on functions on $\mathbb{H}\times\mathbb{C}$ by

$$U_N(\tau,z) := f(\tau, Nz).$$

It is not difficult to see that if $f(\tau+1,z) = f(\tau,z+1) = f(\tau,z)$ then $U_N f$ is also 1-periodic in both variables. If f transforms like a Jacobi form of weight k and index m, then the calculation

$$\begin{aligned}U_N f\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big) &= f\Big(-\frac{1}{\tau},\frac{Nz}{\tau}\Big)\\ &= \tau^k e^{-2\pi i m N^2 z^2/\tau} f(\tau, Nz)\end{aligned}$$

shows that $U_N f$ transforms like a Jacobi form of weight k and index $N^2 m$. So the rule U_N defines maps

$$U_N : J^{\mathrm{w}}_{k,m} \to J^{\mathrm{w}}_{k,N^2m},$$
$$U_N : J_{k,m} \to J_{k,N^2m},$$
$$U_N : J^{\mathrm{cusp}}_{k,m} \to J^{\mathrm{cusp}}_{k,N^2m}.$$

If $f(\tau,z) = \sum_{n,r} c(n,r) q^n \zeta^r$ then

$$U_N f(\tau,z) = \sum_{n,r} c(n,r) q^n \zeta^{Nr} = \sum_{n \in \mathbb{Z}} \sum_{\substack{r \in \mathbb{Z} \\ r \equiv 0 \, (\mathrm{mod}\, N)}} c(n, r/N) q^n \zeta^r.$$

Through the Petersson inner product, U_N induces an adjoint map $U_N^* : J^{\mathrm{cusp}}_{k,N^2m} \to J^{\mathrm{cusp}}_{k,m}$ which is more interesting:

Proposition 6.4. *The adjoint operator of U_N is*

$$U_N^* f(\tau,z) = \frac{1}{N^2} \sum_{a,b \in \mathbb{Z}/N\mathbb{Z}} e^{2\pi i m(a^2\tau + 2az)} f\Big(\tau, \frac{z + a\tau + b}{N}\Big), \quad f \in J^{\mathrm{cusp}}_{k,N^2m}.$$

For $f(\tau,z) = \sum_{n,r} c(n,r) q^n \zeta^r \in J^{\mathrm{cusp}}_{k,N^2m}$,

$$U_N^* f(\tau,z) = \sum_{n,r} \Big(\frac{1}{N} \sum_{a \in \mathbb{Z}/N\mathbb{Z}} c(n - ra + ma^2, N(r - 2am)) \Big) q^n \zeta^r.$$

Proof. Morally, U_N^* should be more or less the map given by substituting $z \mapsto z/N$; however, $f(\tau, z/N)$ transforms only under the translations $z \mapsto z + N\tau$ and $z \mapsto z + N$. We get around that problem by averaging: let f be a Jacobi cusp form of index $N^2 m$, and define $f_N(\tau,z) := f(\tau, z/N)$ and

$$\begin{aligned} g(\tau,z) &:= \sum_{\zeta \in \mathbb{Z}^2/N\mathbb{Z}^2} f_N \Big|_{k,m} \zeta \\ &= \sum_{a,b \in \mathbb{Z}/N\mathbb{Z}} e^{2\pi i m(a^2\tau + 2az)} f\Big(\tau, \frac{z + a\tau + b}{N}\Big). \end{aligned}$$

Then the identities

$$f_N|_{k,m}\zeta|_{k,m}M = f_N|_{k,m}(\zeta M), \quad f_N|_{k,m}\zeta|_{k,m}\eta = f_N|_{k,m}(\zeta + \eta)$$

(where $\zeta, \eta \in \mathbb{Z}^2$ and $M \in \mathrm{SL}_2(\mathbb{Z})$) shows that $g|_{k,m}M = g$ for all $M \in \mathcal{J}$.

g turns out to be a cusp form (see the Fourier expansion computed later on). For any $\varphi \in J_{k,m}^{\text{cusp}}$ we have

$$\begin{aligned}
\langle \varphi, g\rangle &= \int_{\mathcal{F}} \int_{\mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z}} \sum_{a,b\in\mathbb{Z}/N\mathbb{Z}} \overline{e^{2\pi i m(a^2\tau+2az)}} \varphi(\tau,z) \overline{f\Big(\tau, \frac{z+a\tau+b}{N}\Big)} e^{-4\pi m v^2/y} y^{k-3}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}y \\
&= N^2 \int_{\mathcal{F}} \int_{\mathbb{C}/N^{-1}\mathbb{Z}\oplus N^{-1}\tau\mathbb{Z}} \sum_{a,b\in\mathbb{Z}/N\mathbb{Z}} \overline{e^{2\pi i m(a^2\tau+2aNz)}} \varphi(\tau,Nz) \overline{f\Big(\tau, z+\frac{a\tau+b}{N}\Big)} \\
&\qquad\quad \times e^{-4\pi N^2 m v^2/y} y^{k-3}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}y \\
&= N^2 \int_{\mathcal{F}} \int_{\mathbb{C}/\mathbb{Z}\oplus N^{-1}\tau\mathbb{Z}} \sum_{a\in\mathbb{Z}/N\mathbb{Z}} \overline{e^{2\pi i m(a^2\tau+2aNz)}} \varphi(\tau,Nz) \overline{f(\tau, z+a\tau/N)} e^{-4\pi N^2 m v^2/y} y^{k-3}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Use the substitution $z \mapsto z - a\tau/N$ and write

$$\varphi(\tau, N(z-a\tau/N)) = \varphi(\tau, Nz - a\tau) = e^{2\pi i m(a^2\tau+2aNz)}\varphi(\tau,Nz)$$

and

$$\overline{e^{2\pi i m(a^2\tau+2aN(z-a\tau/N))}} \cdot e^{-4\pi N^2 m(v-ay/N)^2/y} = e^{2\pi i m(a^2\tau+2aNz)} \cdot e^{-4\pi N^2 m v^2/y}$$

to see that this equals

$$\begin{aligned}
& N^2 \int_{\mathcal{F}} \int_{\mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z}} \varphi(\tau,Nz)\overline{f(\tau,z)} e^{-4\pi N^2 m v^2/y} y^{k-3}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}y \\
&= N^2 \langle U_N\varphi, f\rangle.
\end{aligned}$$

Hence $g = N^2 \cdot U_N^* f$.

To work out the Fourier coefficients, we write

$$\begin{aligned}
U_N^* f(\tau,z) &= \frac{1}{N^2} \sum_{a,b\in\mathbb{Z}/N\mathbb{Z}} q^{ma^2}\zeta^{2am} \sum_{n,r} c(n,r) q^{n+ar/N} \zeta^{r/N} e^{2\pi i b r/N} \\
&= \frac{1}{N} \sum_{a\in\mathbb{Z}/N\mathbb{Z}} q^{ma^2}\zeta^{2am} \sum_{\substack{n,r\in\mathbb{Z} \\ r\equiv 0\,(N)}} c(n,r) q^{n+ar/N}\zeta^{r/N} \\
&= \frac{1}{N} \sum_{n,r} c(n,Nr) \sum_{a\in\mathbb{Z}/N} q^{ma^2+n+ar}\zeta^{2am+r}
\end{aligned}$$

and substitute first $r \mapsto r - 2am$ and then $n \mapsto n - ra + ma^2$. □

The formula for U_N^* also defines maps that will still be denoted by

$$U_N^* : J_{k,N^2m}^{\mathrm{w}} \to J_{k,m}^{\mathrm{w}}, \quad U_N^* : J_{k,N^2m} \to J_{k,m},$$

but these are no longer the adjoint of U_N with respect to any natural inner product.

Remark 6.5. It follows from Proposition 6.4 that $U_N^* U_N = \mathrm{id}$ is the identity on $J_{k,m}$, so we do not get any nontrivial "Hecke operators" on $J_{k,m}$ by combining U_N and U_N^*. This might be expected as the definition of U_N is rather simple.

6.3. Double coset operators

The operators U_N are in fact true Hecke operators in the sense of being averaging operators induced from decomposing double cosets into one-sided cosets. We will consider the latter notion more generally.

Let $\mathbb{Z}^{2\times 2}_{>0}$ be the set of integral (2×2)-matrices with positive determinant. There is a double-coset decomposition given by elementary divisors (or Smith normal form):

$$\mathbb{Z}^{2\times 2}_{>0} = \bigcup_{\substack{d_1,d_2>0\\ d_1|d_2}} \Gamma \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \Gamma, \quad \Gamma = \mathrm{SL}_2(\mathbb{Z}).$$

Each double coset splits as a disjoint, finite union of right cosets:

$$\Gamma \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \Gamma = \bigcup_{i=1}^r \Gamma\gamma_i.$$

If $N = d_1 d_2$ then one choice of the representatives γ_i is

$$\gamma_i = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

where $a, d > 0$, $ad = N$, $\gcd(a,b,d) = d_1$ and $b \in \{0,...,d-1\}$.

Suppose f is a Jacobi form of weight k and index m, that

$$\alpha = \Gamma \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \Gamma$$

is a double coset with $\det(\alpha) = d_1 d_2 = N$, and that $\Delta_\alpha = \{\gamma_1, ..., \gamma_r\}$ are the right cosets that make up α. Then the slash action

$$F = f\Big|\alpha := \sum_{i=1}^r f\Big|_{k,m}\Big(\frac{1}{\sqrt{N}}\gamma_i\Big)$$

is well-defined (independent of the choice of representatives γ_i) and F transforms correctly under $\mathrm{SL}_2(\mathbb{Z})$ in the sense that

$$F\Big|_{k,m} A = F \quad \text{for all } A \in \mathrm{SL}_2(\mathbb{Z});$$

this is because right-multiplication by A simply permutes the classes of Δ_α. However, the quasiperiod lattice of F is now $\frac{1}{\sqrt{N}}\mathbb{Z} \oplus \frac{\tau}{\sqrt{N}}\mathbb{Z}$ rather than $\mathbb{Z}\oplus\tau\mathbb{Z}$. To obtain a Jacobi form we substitute $z \mapsto \sqrt{N}\cdot z$, which is formally the operator $U_{\sqrt{N}}$ and therefore multiplies the index by N.

Definition 6.6. Let $\alpha = \Gamma \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \Gamma$ be a double coset, $\det(\alpha) = N$. The Hecke operator $\langle \alpha \rangle$ is defined by

$$\langle \alpha \rangle f(\tau, z) := N^{k-1} \sum_{M \in \Delta_\alpha} (c\tau + d)^{-k} e^{-2\pi i m N c z^2/(c\tau+d)} \cdot f\Big(\frac{a\tau + b}{c\tau + d}, \frac{Nz}{c\tau + d}\Big),$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. It maps (unrestricted) Jacobi forms of index m into (unrestricted) Jacobi forms of index $m \cdot N$.

Remark 6.7. Suppose $\alpha = \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}$. (Note that $\det(\alpha) = N^2$, not N!) This commutes with everything and Δ_α consists only of one coset, represented by $M = \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}$. So

$$\langle \alpha \rangle f(\tau, z) = N^{k-2} f(\tau, Nz),$$

i.e. $\langle \alpha \rangle = N^{k-2} U_N$.

To work out the action on Fourier coefficients it is easier to use the operators

$$V_N = \sum_{\det(\alpha) = N} \langle \alpha \rangle.$$

So

$$V_N f = N^{k-1} \sum_{\gamma \in \Delta_N} \Big(f|_{k,m} \frac{1}{\sqrt{N}} \gamma \Big)(\tau, \sqrt{N} \cdot z).$$

Here $\Delta_N = \bigcup_{\det(\alpha)=N} \Delta_\alpha$ is represented by matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $a, d > 0$, $ad = N$ and $b \in \{0, ..., d-1\}$, without any constraint on the g.c.d. of the entries.

Proposition 6.8. *Suppose f is an unrestricted Jacobi form of weight k and index m with Fourier series*

$$f(\tau, z) = \sum_{n,r} c(n, r) q^n \zeta^r.$$

Then

$$V_N f(\tau, z) = \sum_{n,r} \Big(\sum_{a | \gcd(n,r,N)} a^{k-1} c\Big(\frac{Nn}{a^2}, \frac{r}{a}\Big)\Big) q^n \zeta^r.$$

Proof. Use the representatives $M = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ for Δ_N, where $N = ad$ and $0 \le b < d$. We have

$$
\begin{aligned}
V_N f(\tau, z) &= N^{k-1} \sum_{\substack{ad=N \\ a,d>0}} \sum_{b \in \mathbb{Z}/d\mathbb{Z}} d^{-k} f\Big(\frac{a\tau + b}{d}, \frac{Nz}{d}\Big) \\
&= \frac{1}{N} \sum_{a|N} a^k \sum_{b \in \mathbb{Z}/(N/a)} \sum_{n,r} c(n,r) q^{n(a^2/N)} \zeta^{ar} e^{2\pi i b n/(N/a)} \\
&= \sum_{a|N} a^{k-1} \sum_{n,r} c(Nn/a, r) q^{an} \zeta^{ar} \\
&= \sum_{n,r} \sum_{a|\gcd(N,n,r)} a^{k-1} c\Big(\frac{Nn}{a^2}, \frac{r}{a}\Big) q^n \zeta^r. \qquad \square
\end{aligned}
$$

The expression for the Fourier coefficients of $V_N f$ yields:

Corollary 6.9. *V_N defines linear maps*

$$J_{k,m}^{\mathrm{weak}} \to J_{k,Nm}^{\mathrm{weak}}, \quad J_{k,m} \to J_{k,Nm}, \quad J_{k,m}^{\mathrm{cusp}} \to J_{k,Nm}^{\mathrm{cusp}}.$$

Example 6.10. We apply the operator V_2 to $E_{4,1}$. Recall that the coefficients $c(\Delta) = c(n,r)$ (where $4n - r^2 = \Delta$) are given by

$$c(0) = 1, \quad c(3) = 56, \quad c(4) = 126, \quad c(7) = 576,$$

$$c(8) = 756, \quad c(11) = 1512, \quad c(12) = 2072,$$

etc. Then

$$V_2 E_{4,1}(\tau, z) = b(0,0) + \Big(\sum_{r \in \mathbb{Z}} b(1,r)\zeta^r\Big) q + O(q^2),$$

where:

$$b(0,0) = \sum_{a|2} a^3 c(0,0) = 9;$$

$$b(1,2) = c(2,2) = 126, \quad b(1,1) = c(2,1) = 576, \quad b(1,0) = c(2,0) = 756,$$

i.e.

$$V_2 E_{4,1}(\tau, z) = 9 + (126\zeta^{-2} + 576\zeta^{-1} + 756 + 576\zeta + 126\zeta^2)q + O(q^2).$$

Since $\dim J_{4,2} = 1$ we must have $V_2 E_{4,1} = 9 \cdot E_{4,2}$. (Compare this with the Fourier coefficients of $E_{4,2}$ that were worked out in Section 4.3.)

Note that $E_{4,1}$ and $E_{4,2}$ are theta functions attached to the E_7 and D_7 root lattices, respectively, so $V_2(E_{4,1}) = E_{4,2}$ has an interpretation in terms of counting vectors of a given length: e.g. the 756 vectors in E_7 of length two are nine times the number 84 of roots in D_7.

We can recover all double-coset operators $\langle \alpha \rangle$ from V_N and U_N by Möbius inversion as follows: from

$$\Delta_N = \bigcup_{d^2|N} d \cdot \Big\{ M \in \Delta_{\begin{pmatrix} 1 & 0 \\ 0 & N/d^2 \end{pmatrix}} \Big\},$$

it follows that

$$V_N f = \sum_{d^2|N} d^{k-2} U_d \langle \begin{pmatrix} 1 & 0 \\ 0 & N/d^2 \end{pmatrix} \rangle f$$

and therefore

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix} \rangle f = \sum_{d^2|N} \mu(d) d^{k-2} U_d V_{N/d^2} f.$$

More generally if $\alpha = \Gamma \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \Gamma$ then

$$\langle \alpha \rangle = d_1^{k-2} U_{d_1} \langle \begin{pmatrix} 1 & 0 \\ 0 & d_2/d_1 \end{pmatrix} \rangle.$$

It follows that all $\langle \alpha \rangle$ preserve weak, holomorphic and cusp Jacobi forms.

As in the case of Hecke operators for $\mathrm{SL}_2(\mathbb{Z})$, we have the following important fact:

Theorem 6.11. *The Hecke algebra is commutative: for any double cosets α, β,*

$$\langle \alpha \rangle \langle \beta \rangle = \langle \beta \rangle \langle \alpha \rangle.$$

In particular, for any $N_1, N_2 \in \mathbb{N}$ we have

$$U_{N_1} U_{N_2} = U_{N_2} U_{N_1}, \quad U_{N_1} V_{N_2} = V_{N_2} U_{N_1}, \quad V_{N_1} V_{N_2} = V_{N_2} V_{N_1}.$$

Multiplication in the Hecke algebra can be defined either by composition of the associated operators, or abstractly by writing

$$(\Gamma \alpha \Gamma) \cdot (\Gamma \beta \Gamma) = \sum_{\delta} n_\delta \Gamma \delta \Gamma,$$

where n_δ is the number of pairs $\Gamma \alpha_i \in \Gamma \alpha \Gamma$ and $\Gamma \beta_j \in \Gamma \beta \Gamma$ for which $\Gamma \alpha_i \beta_j = \Gamma \delta$.

Proof. This follows from the fact that

$$(\Gamma \alpha \Gamma) \cdot (\Gamma \beta \Gamma) = (\Gamma \beta \Gamma) \cdot (\Gamma \alpha \Gamma).$$

One way to see that is by using the fact that α and α^T have the same elementary divisors, i.e. $\Gamma \alpha \Gamma = (\Gamma \alpha \Gamma)^T$, while $\alpha \mapsto \alpha^T$ is an anti-homomorphism (i.e. order-reversing): so

$$\begin{aligned} (\Gamma \alpha \Gamma) \cdot (\Gamma \beta \Gamma) &= \Big((\Gamma \alpha \Gamma) \cdot (\Gamma \beta \Gamma) \Big)^T \\ &= (\Gamma \beta \Gamma)^T \cdot (\Gamma \alpha \Gamma)^T \\ &= (\Gamma \beta \Gamma) \cdot (\Gamma \alpha \Gamma). \end{aligned}$$

□

Finally, we compare the actions of U_N and V_N with the development coefficients D_ν. Recall that D_ν was defined by repeated application of the modified heat operator $L_{k,m}$

and then setting $z=0$, and that $L_{k,m}$ is equivariant with respect to $\mathrm{SL}_2(\mathbb{R})$ (not just $\mathrm{SL}_2(\mathbb{Z})$!),

$$L_{k,m}\Big(f\Big|_k M\Big) = (L_{k,m}f)\Big|_{k+2} M.$$

Since V_N is a sum of actions by elements of $\mathrm{SL}_2(\mathbb{R})$, followed by the substitution $z \mapsto z\sqrt{N}$ (which does not matter, since we set $z=0$ when we apply D_ν), and since U_N is just scaling by N, we have:

Proposition 6.12. *Let f be a weak Jacobi form of weight k and index m. For any $\nu \in \mathbb{N}_0$,*

$$D_\nu V_N(f) = T_N D_\nu(f)$$

where T_N is the Nth Hecke operator on $M_{k+\nu}(\mathrm{SL}_2(\mathbb{Z}))$, and

$$D_\nu U_N(f) = N^\nu f.$$

Corollary 6.13. *The composition of V_N-operators is given by*

$$V_{N_1}V_{N_2}f = \sum_{d|\gcd(N_1,N_2)} d^{k-1} U_d V_{N_1N_2/d^2}.$$

Proof. Since the family of maps $(D_\nu)_{\nu\geq 0}$ is injective, it is enough to check that

$$D_\nu(V_{N_1}V_{N_2}f) = \sum_{d|\gcd(N_1,N_2)} d^{k-1} D_\nu\Big(U_d V_{N_1N_2/d^2} f\Big).$$

By Proposition 6.12 this is equivalent to checking that

$$T_{N_1}T_{N_2}(D_\nu f) = \sum_{d|\gcd(N_1,N_2)} d^{k+\nu-1} T_{N_1N_2/d^2}\Big(D_\nu f\Big).$$

But this is just the formula for the composition of T_N on modular forms. □

6.4. Hecke operators

Now that we have obtained some non-trivial maps from $J_{k,m}$ into J_{k,mN^2}, the plan is to project the image back down to $J_{k,m}$ via the averaging map U_N^*.

Theorem 6.14. *The operators*

$$U_N^* \circ V_{N^2} : J_{k,m}^{\mathrm{cusp}} \longrightarrow J_{k,m}^{\mathrm{cusp}}$$

are self-adjoint.

Proof. Let f, g be any cusp forms of weight k and index m. By definition,

$$\langle U_N^* V_{N^2} f, g\rangle = \langle V_{N^2} f, U_N g\rangle,$$

with the inner product on the right taking place in $J_{k,N^2m}^{\mathrm{cusp}}$. Written out, this is

$$\langle V_{N^2} f, U_N g\rangle = (N^2)^{k/2-1} \int_X \sum_{\gamma \in \Delta_{N^2}} \Big(f\Big|_{k,m} \frac{1}{N}\gamma\Big)(\tau, Nz)\overline{g}(\tau, Nz) e^{-4\pi m N^2 v^2/y} y^k \,\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}u\,\mathrm{d}v}{y^3}$$

where X is any fixed fundamental domain for $\mathcal{J}$ on $\mathbb{H} \times \mathbb{C}$.

We move the sum over

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta_{N^2}$$

out of the integral and apply the substitution

$$(\tau, z) = \Big(\frac{d\tau' - b}{-c\tau' + a}, \frac{Nz'}{-c\tau' + a}\Big),$$

i.e. we act by $N\gamma^{-1} \in \mathrm{SL}_2(\mathbb{Q})$. Note that $\delta := N^2\gamma^{-1}$ is an integral matrix of determinant N^2 whenever γ is.

But δ is actually a left-coset whereas γ is a right-coset. The key fact is that one can choose the representatives $\gamma_1, ..., \gamma_r$ of Δ_{N^2} such that they simultaneously represent the left and right cosets of $\{M \in \mathbb{Z}^{2\times 2} : \det(M) = n\}$ by $\mathrm{SL}_2(\mathbb{Z})$: that is, $\Delta_{N^2} = \{\gamma_i\}$ where

$$\{M \in \mathbb{Z}^{2\times 2} : \det(M) = N^2\} = \bigcup_{i=1}^r \mathrm{SL}_2(\mathbb{Z})\gamma_i = \bigcup_{i=1}^r \gamma_i \mathrm{SL}_2(\mathbb{Z}).$$

Namely if we pick right- and left-coset representatives such that

$$\bigcup_i \bigcup_{\substack{d_1|d_2 \\ d_1d_2=N^2}} \alpha_i \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \mathrm{SL}_2(\mathbb{Z}) = \bigcup_j \bigcup_{\substack{d_1|d_2 \\ d_1d_2=N^2}} \mathrm{SL}_2(\mathbb{Z}) \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \beta_j,$$

then the matrices $\gamma_i := \alpha_i \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \beta_i$ are a system of representatives of that sort.

For these representatives γ_i, the matrices $\delta_i = N^2\gamma_i^{-1}$ also represent the different right-cosets in Δ_{N^2}. After applying the Möbius transformation $\frac{1}{N}\delta_i$ and observing that

$$f\Big|_{k,m} \frac{1}{N}\gamma_i\Big|_{k,m} \frac{1}{N}\delta_i = f,$$

we obtain

$$\begin{aligned}\langle V_{N^2} f, U_N g\rangle &= (N^2)^{k/2-1} \int_X \sum_{\delta \in \Delta_{N^2}} f(\tau, Nz)\Big(\overline{g}\Big|_{k,m} \frac{1}{N}\delta\Big)(\tau, Nz) e^{-4\pi m N^2 v^2/y} y^k \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}u\,\mathrm{d}v}{y^3} \\ &= \langle U_N f, V_{N^2} g\rangle.\end{aligned}$$

So $\langle U_N^* V_{N^2} f, g\rangle = \langle f, U_N^* V_{N^2} g\rangle$. ∎

More generally, the same proof shows that

$$\Big(U_N^*\langle\alpha\rangle\Big)^* = U_N^*\langle N^2\alpha^{-1}\rangle$$

for any double coset α of determinant N^2 taken by itself. But $N^2\alpha^{-1}$ belongs to the same double coset as α, so all $U_N^*\langle\alpha\rangle$ are self-adjoint. Alternatively, this fact follows from Theorem 6.14 and the formula $\langle\begin{pmatrix}1 & 0\\ 0 & N^2\end{pmatrix}\rangle = \sum_{d|N}\mu(d)d^{k-2}U_dV_{(N/d)^2}$ and $U_N^*U_N = 1$: we have

$$U_N^*\langle\begin{pmatrix}1 & 0\\ 0 & N^2\end{pmatrix}\rangle = \sum_{d|N}\mu(d)d^{k-2}U_{N/d}^*V_{(N/d)^2}.$$

Warning: the fact that U_{N_1} and V_{N_2} commute does *not* imply that $U_{N_1}^*$ and V_{N_2} commute (even when both operators are defined), and in general they do not. So the following commutativity is nontrivial:

Proposition 6.15. *Suppose $\ell, N \in \mathbb{N}$ are coprime and $f \in J_{k,mN^2}$. Then*

$$U_N^*V_\ell f = V_\ell U_N^* f.$$

Proof. Suppose f has Fourier series

$$f(\tau,z) = \sum_{n,r} c(n,r)q^n\zeta^r.$$

Then the coefficient of $q^n\zeta^r$ in $V_\ell U_N^* f(\tau,z)$ is

$$\sum_{d|\gcd(n,r,\ell)} d^{k-1}\Big[\frac{1}{N}\sum_{a\in\mathbb{Z}/N\mathbb{Z}} c\Big(\frac{\ell n}{d^2} - \frac{ra}{d} + ma^2, N\Big(\frac{r}{d} - 2am\Big)\Big)\Big],$$

which we can write in the form

$$\frac{1}{N}\sum_{d|\gcd(n,r,\ell)} d^{k-1}\sum_{a\in\mathbb{Z}/N} c\Big(\frac{\ell n - rad + ma^2d^2}{d^2}, \frac{Nr - 2amd}{d}\Big).$$

Replacing ad by $a\ell$, (which also runs through $\mathbb{Z}/N$ because ℓ/d is coprime to N), we have

$$\frac{1}{N}\sum_{a\in\mathbb{Z}/N}\sum_{d|\gcd(n,r,\ell)} d^{k-1}c\Big(\frac{\ell(n - ra + \ell ma^2)}{d^2}, \frac{Nr - 2am\ell}{d}\Big).$$

This is exactly the coefficient of $q^n\zeta^r$ in $U_N^*V_\ell f$. □

For reasons that will become clear later, the Hecke operators T_N are defined to be neither $U_N^*V_{N^2}$ nor any single $U_N^*\langle\alpha\rangle$, but certain linear combinations:

Definition 6.16. Let $N \in \mathbb{N}$. The Nth Hecke operator on $J_{k,m}^{\mathrm{cusp}}$ is defined by

$$T_N = \sum_{a^2 | N} U_N^* \langle \begin{pmatrix} a^2 & 0 \\ 0 & (N/a^2)^2 \end{pmatrix} \rangle.$$

So

$$\begin{aligned} T_N &= \sum_{a^2|N} a^{2k-4} U_{N/a^2}^* \langle \begin{pmatrix} 1 & 0 \\ 0 & (N/a^2)^2 \end{pmatrix} \rangle \\ &= \sum_{a^2|N} \sum_{d|(N/a^2)} \mu(d) d^{k-2} a^{2k-4} U_{N/(da^2)}^* V_{(N/(da^2))^2}. \end{aligned}$$

Example 6.17. For a prime p, we have

$$T_{p^\ell} = \sum_{j=0}^{\ell} (-p^{k-2})^j (U_{p^{\ell-j}})^* V_{p^{2\ell-2j}}, \quad \ell \in \mathbb{N}_0.$$

So

$$\begin{aligned} T_p f &= U_p^* V_{p^2} f - p^{k-2} f; \\ T_{p^2} f &= U_{p^2}^* V_{p^4} f - p^{k-2} U_p^* V_{p^2} f + p^{2k-4} f; \end{aligned}$$

etc.

Using our earlier results it is not hard to reduce to the case of prime power index:

Proposition 6.18. *Suppose N_1, N_2 are coprime. Then*

$$T_{N_1} T_{N_2} = T_{N_2} T_{N_1} = T_{N_1 N_2}.$$

In particular if N has prime factorization $N = p_1^{\ell_1} ... p_r^{\ell_r}$ then $T_N = T_{p_1^{\ell_1}} ... T_{p_r^{\ell_r}}$.

Proof. By Proposition 6.15 and Corollary 6.13 we have

$$(U_a^* V_{a^2})(U_b^* V_{b^2}) = U_a^* U_b^* V_{a^2} V_{b^2} = U_{ab}^* V_{(ab)^2}$$

for any divisors $a|N_1$ and $b|N_2$. Taking the appropriate linear combinations, we get

$$T_{N_1} T_{N_2} = T_{N_1 N_2}.$$ □

Expressing the prime-power-index operators T_{p^ℓ} in terms of T_p is more difficult. We will first work out the effect of T_{p^ℓ} on Fourier series.

Theorem 6.19. *Suppose p is a prime and $\ell \in \mathbb{N}$. Let $f \in J_{k,m}$ have Fourier series*
$$f(\tau, z) = \sum_{n,r} c(n,r) q^n \zeta^r = \sum_{D} \sum_{\gamma \in \mathbb{Z}/2m} c_\gamma(D) q^n \zeta^r,$$
where $c_\gamma(D) = c(n,r)$ for any numbers n, r that satisfy $4mn - r^2 = D$ and $r \equiv \gamma \bmod 2m$. Then the coefficient of $q^n \zeta^r$ in $T_{p^\ell} f$ is
$$\sum_{b=0}^{2\ell} p^{(k-2)b} c_{p^{\ell-b} r}\Big(p^{2\ell-2b}(4mn - r^2)\Big) N_{p^\ell}(n, r; p^b)$$
where
$$N_{p^\ell}(n, r; p^b) = \sum_{\substack{i+j=b \\ i,j \geq 0 \\ i+2j \leq 2\ell}} (-1)^j \cdot \#\{a \in \mathbb{Z}/p^i : \; n - ra + ma^2 = p^{\ell-j}(r - 2am) = 0\},$$
and where $c_\gamma(D) = 0$ if D or γ is not integral.

When $\ell = 1$, we have:
$$N_p(n, r, p^0) = 1;$$
$$N_p(n, r, p^1) = -1 + \#\{a \in \mathbb{Z}/p : \; n - ra + ma^2 = 0\};$$
$$N_p(n, r, p^2) = \#\{a \in \mathbb{Z}/p^2 : \; n - ra + ma^2 = p(r - 2am) = 0\}.$$
That is valid for any p. But if we suppose $p \nmid 2m$, then
$$N_p(n, r, p^1) = \chi_{4mn - r^2}(p) = \left(\frac{4mn - r^2}{p}\right) \in \{-1, 0, 1\}$$
is the quadratic reciprocity symbol, and the term $b = 2$ appears exactly when $p | r$ and $p^2 | n$, in which case $n - ra + ma^2 = 0$ has only the p solutions $a \in p\mathbb{Z} \bmod p^2$. So the coefficient of $q^n \zeta^r$ in $T_p f$ for $p \nmid (2m)$ is
$$c(p^2 n, pr) + p^{k-2} \chi_{4mn - r^2}(p) \cdot c(n, r) + p^{2k-3} c\Big(\frac{n}{p^2}, \frac{r}{p}\Big),$$
where the convention is that $c(n, r) = 0$ if either n or r is nonintegral. More generally:

Corollary 6.20. *Suppose N is coprime to m and let $f \in J_{k,m}$ have Fourier series as in Theorem 6.19. Then*

$$T_N f(\tau, z) = \sum_{D,\gamma} b_\gamma(D) q^n \zeta^r,$$

where $b_\gamma(D) = b(n,r)$ for any n, r with $D = 4mn - r^2$ and $r \equiv \gamma \bmod 2m$ are given by

$$b_\gamma(D) = \sum_{\substack{a \mid N^2 \\ (N/a)^2 D \in \Delta}} \varepsilon_D(a) a^{k-2} \cdot c_{a^* N\gamma}((N/a)^2 D),$$

where Δ are the integers that are 0 or 3 mod 4 and where a^* is the inverse of a mod $2m$, and where ε_D is defined as follows: if $D = -D_0 f^2$ where $D_0 < 0$ is a fundamental discriminant,

$$\varepsilon_D(n) = \begin{cases} \chi_{D_0}(n_0) \cdot g & \text{if } n = n_0 g^2 \text{ with } g \mid f \text{ and } \gcd(n_0, f/g) = 1; \\ 0; & \text{if } \gcd(n, f^2) \text{ is not a square.} \end{cases}$$

And if $D = 0$ then

$$\varepsilon_D(n) = \begin{cases} r: & n = r^2,\ r \geq 0; \\ 0: & \text{otherwise.} \end{cases}$$

In particular, if $D = D_0$ is itself a fundamental discriminant, then $\varepsilon = \chi_{D_0}$ is the quadratic character attached to D_0.

Proof. Since both $N \mapsto T_N$ and this formula for $b_\gamma(D)$ are multiplicative in N, we can assume without loss of generality that $N = p^\ell$ is a prime power. The claim follows from Theorem 6.19 by counting solutions to $n - ra + ma^2$ mod prime powers. $\square$

Proof of Theorem 6.19. Using

$$T_{p^\ell} = \sum_{j=0}^{\ell} (-p^{k-2})^j (U_{p^{\ell-j}})^* V_{p^{2\ell-2j}}$$

and $U_{p^j}^* U_{p^j} = 1$, we can rewrite $T_{p^\ell} f$ as

$$T_{p^\ell} f = U_{p^\ell}^* \circ \sum_{j=0}^{\ell} (-p^{k-2})^j U_{p^j} V_{p^{2\ell-2j}} f =: U_{p^\ell}^* g.$$

The coefficient of $q^n\zeta^r$ in g is

$$\begin{aligned}
c_g(n,r) &= \sum_{j:p^j|r}(-p^{k-2})^j \sum_{a|\gcd(n,r/p^j,p^{2\ell-2j})} a^{k-1}c\Big(\frac{p^{2\ell-2j}n}{a^2},\frac{r}{p^j a}\Big)\\
&= \sum_{j=0}^{\nu_p(r)}(-p^{k-2})^j \sum_{i=0}^{\min(\nu_p(n),\nu_p(r)-j,2\ell-2j)} (p^{k-1})^i c\Big(p^{2\ell-2j-2i}n, p^{-j-i}r\Big)\\
&= \sum_{b=0}^{\nu_p(r)}\Big(\sum_{\substack{i+j=b\\ i+2j\le 2\ell\\ i\le\nu_p(n)}}(-p^{k-2})^j(p^{k-1})^i\Big)\cdot c\Big(p^{2\ell-2b}n,p^{-b}r\Big).
\end{aligned}$$

So the coefficient of $q^n\zeta^r$ in $T_{p^\ell}f$ is

$$\begin{aligned}
&p^{-\ell}\sum_{a\in\mathbb{Z}/p^\ell} c_g\Big(n-ra+ma^2, p^\ell(r-2am)\Big)\\
&= p^{-\ell}\sum_{i+2j\le 2\ell}(-p^{k-2})^j(p^{k-1})^i\cdot \sum_{\substack{a\in\mathbb{Z}/p^\ell\\ p^i|(n-ra+ma^2)\\ p^i|p^{\ell-j}(r-2am)}} c\Big(p^{2\ell-2i-2j}(n-ra+ma^2), p^{\ell-i-j}(r-2am)\Big).
\end{aligned}$$

Since the conditions $p^i|(n-ra+ma^2)$ and $p^i|p^{\ell-j}(r-2am)$ depend only on the remainder class of a mod p^i, we can rewrite the inner sum as a sum over $a\in\mathbb{Z}/p^\ell$, multiplied by $p^{\ell-i}$. So the coefficient simplifies to

$$\sum_{i+2j\le 2\ell}(-p^{k-2})^j(p^{k-2})^i \sum_{\substack{a\in\mathbb{Z}/p^i\\ n-ra+ma^2\equiv 0\\ p^{\ell-j}(r-2am)\equiv 0}} c\Big(p^{2\ell-2i-2j}(n-ra+ma^2), p^{\ell-i-j}(r-2am)\Big).$$

The numbers $\tilde{n} := p^{2\ell-2i-2j}(n-ra+ma^2)$ and $\tilde{r} := p^{\ell-i-j}(r-2am)$ satisfy

$$4m\tilde{n}-\tilde{r}^2 = p^{2\ell-2i-2j}\Big(4m(n-ra+ma^2)-(r-2am)^2\Big) = p^{2\ell-2i-2j}(4mn-r^2).$$

So using the notation $c_\gamma(D)=c(n,r)$ if $4mn-r^2=D$ and $r\equiv\gamma \bmod 2m$, and writing $b=i+j$, the sum becomes

$$\sum_{b\le 2\ell} p^{(k-2)b}c_{p^{\ell-b}r}\Big(p^{2\ell-2b}(4mn-r^2)\Big)\sum_{\substack{i+j=b\\ i+2j\le 2\ell}}(-1)^j\#\{a\in\mathbb{Z}/p^i:\ n-ra+ma^2=p^{\ell-j}(r-2am)=0\}.$$

□

Theorem 6.21. *Suppose N_1, N_2 are coprime to m. Then*

$$T_{N_1}T_{N_2} = \sum_{d|\gcd(N_1,N_2)} d^{2k-3}T_{N_1N_2/d^2}.$$

Proof. Suppose f has Fourier series

$$f(\tau, z) = \sum_{n,r} c(n,r) q^n \zeta^r.$$

With $D = 4mn - r^2$ and γ with $r \equiv \gamma \bmod 2m$, write $c_\gamma(D) = c(n,r)$. By Corollary 6.20, the coefficient of $q^n \zeta^r$ in $T_{N_1} T_{N_2} f$ is

$$b_\gamma(D) = \sum_{\substack{a_2 | N_2^2 \\ (N_2/a_2)^2 D \in \Delta}} \sum_{\substack{a_1 | N_1^2 \\ (N_1/a_1)^2 (N_2/a_2)^2 D \in \Delta}} \varepsilon_D(a_1) \varepsilon_D(a_2) a_1^{k-2} a_2^{k-2} \cdot b_{a_1^* a_2^* N_1 N_2 \gamma}\Big((N_1/a_1)^2 (N_2/a_2)^2 D \Big).$$

Now sort all such pairs according to their gcd $d = \mathrm{gcd}(a_1, a_2)$, and write $N := N_1 N_2$:

$$\begin{aligned} & b_\gamma(D) \\ = & \sum_{d | \mathrm{gcd}(N_1, N_2)} \sum_{\mathrm{gcd}(a_1, a_2) = d} \varepsilon_D(a_1 a_2) d^{2k-4} (a_1 a_2 / d^2)^{k-2} b_{a_1^* a_2^* N \gamma}\Big((\frac{N}{d^2})^2 / (a_1 a_2 / d^2) \cdot D \Big) \\ = & \sum_{d | \mathrm{gcd}(N_1, N_2)} \sum_{\mathrm{gcd}(a_1/d, a_2/d) = 1} \varepsilon_D(a_1 a_2 / d^2) d^{2k-3} (a_1 a_2 / d^2)^{k-2} b_{(a_1 a_2 / d^2)^* (N/d^2) \gamma}\Big((\frac{N}{d^2})^2 / (a_1 a_2 / d^2) \cdot D \Big). \end{aligned}$$

The product $a := a_1 a_2 / d^2$ runs exactly once through the divisors of $(N/d^2)^2$ for which $(\frac{N/d^2}{a})^2 D = \frac{N^2}{a_1^2 a_2^2} D \in \Delta$, so the sum simplifies to

$$\sum_{d | \mathrm{gcd}(N_1, N_2)} d^{2k-3} \sum_{\substack{a | (N/d^2)^2 \\ (N/(ad^2))^2 D \in \Delta}} \varepsilon_D(a) a^{k-2} b_{a^* (N/d^2) \gamma}\Big(\frac{(N/d^2)^2}{a^2} D \Big).$$

This is just the coefficient of $q^n \zeta^r$ in $\sum_{d | \mathrm{gcd}(N_1, N_2)} d^{2k-3} T_{N/d^2} f$. □

6.5. Eisenstein series and Hecke operators

We end the discussion of Hecke operators by studying their action on Eisenstein series.

Suppose $m = df^2$ where d is squarefree and $f \in \mathbb{N}$. Recall that for $k \geq 3$, the Eisenstein space is the span of the Jacobi Eisenstein series $E_{k,m,b}$, which are characterized by the fact that their "singular coefficients" (n, r) with $4mn - r^2 = 0$ are nonzero if $r \equiv \pm b \cdot 2df \bmod 2m$, and 0 otherwise.

The Eisenstein space has an intrinsic characterization with respect to the Petersson inner product:

> **Proposition 6.22.** *A Jacobi form $f \in J_{k,m}$ belongs to the Eisenstein space if and only if $\langle f, g \rangle = 0$ for every cusp form g.*

Proof. (i) Suppose that g is a cusp form. Then

$$\langle E_{k,m,b}, g\rangle = \int_X E_{k,m,b}(\tau,z)\overline{g(\tau,z)}e^{-4\pi mv^2/y}y^{k-3}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}y,$$

where X is a fundamental domain for the Jacobi group $\mathcal{J}$ on $\mathbb{H}\times\mathbb{C}$ and where $\tau = x+iy$, $z = u+iv$. But $E_{k,m,b}$ was itself defined as a series

$$E_{k,m,b} = \sum_{\gamma\in\mathcal{J}_\infty\backslash\mathcal{J}} \frac{1}{2}\Big(q^n\zeta^r + (-1)^k q^n\zeta^{-r}\Big)\Big|_{k,m}\gamma,$$

where (n,r) is any solution to $4mn - r^2 = 0$ and $r \equiv b\cdot 2df \mod 2m$. Instead of integrating over $\mathcal{J}\backslash(\mathbb{H}\times\mathbb{C})$ and summing over $\mathcal{J}_\infty\backslash\mathcal{J}$, we can simply integrate over $\mathcal{J}_\infty\backslash(\mathbb{H}\times\mathbb{C})$, so

$$\langle E_{k,m,b}, g\rangle = \int_{\mathcal{J}_\infty\backslash(\mathbb{H}\times\mathbb{C})} \frac{1}{2}\Big(q^n\zeta^r + (-1)^k q^n\zeta^{-r}\Big)\overline{g(\tau,z)}e^{-4\pi mv^2/y}y^{k-3}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}y.$$

Since $\mathcal{J}_\infty$ is generated by the translations $\tau\mapsto\tau+1$ and $z\mapsto z+1$ and by the map $z\mapsto -z$, a fundamental domain for $\mathcal{J}_\infty$ on $\mathbb{H}\times\mathbb{C}$ is given by the product of two strips

$$X_\infty = \{(x+iy, u+iv):\ -1/2 \le \mathrm{Re}[x], \mathrm{Re}[u] \le 1/2,\ y>0,\ v>0\}.$$

Now expanding the Fourier series for $g(\tau,z)$ and carrying out the integral over x and over u shows that the integral $\int_{\mathcal{J}_\infty\backslash(\mathbb{H}\times\mathbb{C})}$ picks out Fourier coefficients of g with $4mn - r^2 = 0$. But these vanish identically because g is a cusp form.

(ii) Conversely, suppose f is orthogonal to all cusp forms and write $f = e+g$ where $e\in J_{k,m}^{\mathrm{Eis}}$ and $g\in J_{k,m}^{\mathrm{cusp}}$. Then $0 = \langle f,g\rangle = \langle g,g\rangle$ implies $g=0$ and therefore $f\in J_{k,m}^{\mathrm{Eis}}$. ☐

Proposition 6.23. *The Hecke operators V_N and U_N preserve the Eisenstein space.*

Proof. (somewhat sketchy) Take a right-coset decomposition

$$\{M\in\mathbb{Z}^{2\times 2}:\ \det(M) = N\} = \bigcup_i \Gamma\alpha_i, \quad \Gamma = \mathrm{SL}_2(\mathbb{Z})$$

and consider the group

$$\tilde{\Gamma} := \mathrm{SL}_2(\mathbb{Z})\cap\bigcap_i \alpha_i^{-1}\mathrm{SL}_2(\mathbb{Z})\alpha_i$$

(which contains the principal congruence subgroup $\Gamma(N)$). Then

$$V_N f = N^{k/2-1}\sum_i f\Big|_{k,m}\Big(\frac{1}{\sqrt{N}}\alpha_i\Big)(\tau,\sqrt{N}z)$$

and each term $f|_{k,m}(\frac{1}{\sqrt{N}}\alpha_i)(\tau, \sqrt{N}z)$ transforms correctly under $\tilde{\Gamma}$ by construction, and indeed under the preimage

$$\tilde{\mathcal{J}} := \{\gamma = (M, \zeta) \in \mathcal{J} : \ M \in \tilde{\Gamma}\}$$

of $\tilde{\Gamma}$ in the Jacobi modular group.

If $f = E_{k,m,b}$ is an Eisenstein series and g is any cusp form of index Nm, then we can write

$$\langle V_N, E_{k,m,b}, g\rangle = \int_{\mathcal{J}\backslash(\mathbb{H}\times\mathbb{C})} V_N E_{k,m,b}\overline{g}e^{-4\pi my^2/v}y^k \,\mathrm{d}\mu$$

with the invariant volume form $\mathrm{d}\mu = y^{-3}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}y$. This is the same as

$$\frac{1}{[\mathcal{J} : \tilde{\mathcal{J}}]}\int_{\tilde{\mathcal{J}}\backslash(\mathbb{H}\times\mathbb{C})} V_N E_{k,m,b}\overline{g}e^{-4\pi mv^2/y}y^k \,\mathrm{d}\mu.$$

Since each term $e_i := E_{k,m,b}|_{k,m}(\frac{1}{\sqrt{N}}\alpha_i)$ that makes up $V_N E_{k,m,b}$ transforms under $\tilde{\mathcal{J}}$, this decomposes as a sum of Petersson inner products

$$\langle V_N E_{k,m,b}, g\rangle = N^{k/2-1}\frac{1}{[\mathcal{J} : \tilde{\mathcal{J}}]}\sum_i \langle e_i, g\rangle$$

with respect to Jacobi forms under the subgroup $\tilde{\mathcal{J}}$. Each individual product $\langle e_i, g\rangle$ can be viewed (after substituting by the inverse Möbius transformation to $\frac{1}{\sqrt{N}}\alpha_i$) as the inner product of $E_{k,m,b}$ against a cusp form on some finite-index subgroup of $\mathcal{J}$, which is zero. So $V_N E_{k,m,b}$ is orthogonal to all cusp forms, hence $V_N E_{k,m,b} \in J^{\mathrm{Eis}}_{k,Nm}$.
For U_N, one can show directly from the definition that

$$U_N E_{k,m,b} = \sum_{\substack{\gamma\in\mathbb{Z}/Nf \\ \gamma\equiv b \bmod f}} E_{k,N^2m,\gamma},$$

if $m = df^2$ with d squarefree. □

Corollary 6.24. *Let $m = df^2$ with d squarefree, and $k \geq 4$ even. Then*

$$V_m E_{k,1} = \sum_{b\in\mathbb{Z}/f\mathbb{Z}} \sigma_{k-1}\Big(d\cdot \gcd(f, b)^2\Big)E_{k,m,b}.$$

In particular, if m is squarefree then $V_m E_{k,1} = \sigma_{k-1}(m)E_{k,m}$.

Proof. Since $V_m E_{k,1}$ belongs to the Eisenstein space, we can identify it once we compute its Fourier coefficients $b(n, r)$ with $4mn - r^2 = 0$. These are given by the formula:

$$b(n, r) = \sum_{a|\gcd(n,r,m)} a^{k-1}c\Big(\frac{mn}{a^2}, \frac{r}{a}\Big)$$

where $c(n,r)$ is the coefficient of $q^n\zeta^r$ in $E_{k,1}$. But $4mn - r^2 = 0$ implies that

$$4\Big(\frac{mn}{a^2}\Big) - \Big(\frac{r}{a}\Big)^2 = 0$$

so $c(mn/a^2, r/a) = 1$, i.e.

$$b(n,r) = \sum_{a|\gcd(n,r,m)} a^{k-1} = \sigma_{k-1}\Big(\gcd(n,r,m)\Big).$$

Since we can assume without loss of generality that $r = b \cdot 2df$, we have

$$\gcd(n,r,m) = d \cdot \gcd(f^2, 2bf, b^2) = d \cdot \gcd(f,b)^2.$$

So

$$V_m E_{k,1} = \sum_{b \in \mathbb{Z}/f\mathbb{Z}} \sigma_{k-1}\Big(d \cdot \gcd(f,b)^2\Big) E_{k,m,b}. \qquad \square$$

Suppose $m = df^2$ with d squarefree. The series $E_{k,m}$ can be recovered from the above remarks by Möbius inversion, using the fact that

$$\sum_{a|f} \mu(a)\sigma_{k-1}\Big(d \cdot \gcd(f/a,b)^2\Big) = 0 \quad \text{if } f \nmid b,$$

which holds for arbitrary $d \in \mathbb{N}$. We obtain

$$\begin{aligned} \sum_{a|f} \mu(a) U_a V_{m/a^2} E_{k,1} &= \sum_{b \in \mathbb{Z}/f} \sum_{a|f} \mu(a)\sigma_{k-1}\Big(d \cdot \gcd(f/a,\gamma)^2\Big) \cdot E_{k,m,b} \\ &= \sum_{a|f} \mu(a)\sigma_{k-1}(df^2/a^2) \cdot E_{k,m} \\ &= \Big(\sum_{a^2|m} \mu(a)\sigma_{k-1}(m/a^2)\Big) \cdot E_{k,m}. \end{aligned}$$

The convolution simplifies as

$$\sum_{a^2|m} \mu(a)\sigma_{k-1}(m/a^2) = m^{k-1} \cdot \prod_{\substack{p|m \\ \text{prime}}} (1 + p^{1-k}).$$

So all $E_{k,m}$ for any index can be expressed via Hecke operators in terms of $E_{k,1}$ alone.

Finally, we have:

Proposition 6.25. *Suppose ℓ is coprime to m. Then $E_{k,m}$ is an eigenfunction of T_ℓ with*

$$T_\ell E_{k,m} = \sigma_{2k-3}(\ell) E_{k,m}.$$

When ℓ is not coprime to m this can fail in several ways. For example, $T_2 E_{4,18} = 37 \cdot E_{4,18}$ has the wrong eigenvalue, and $T_3 E_{4,18} = 262 E_{4,18} + 2E_{4,18,1}$ is not a multiple of $E_{4,18}$ at all. Nevertheless T_ℓ maps the Eisenstein space into itself.

Proof. Since $E_{k,m}$ is a linear combination of images of $E_{k,1}$ under operators $U_d V_{m/d^2}$ (where $d^2|m$) and all of these commute with T_ℓ, it is enough to show this for $E_{k,1}$. Since T_ℓ is self-adjoint and preserves the space of cusp forms, and $E_{k,1}$ is characterized by orthogonality to all cusp forms, it follows that $E_{k,1}$ is an eigenfunction of T_ℓ. By Corollary 6.20 the coefficient of $(n, r) = (0, 0)$ in $T_\ell E_{k,1}$ is

$$b(0) = \sum_{a^2|\ell^2} a \cdot (a^2)^{k-2} = \sigma_{2k-3}(\ell). \qquad \square$$

7. The Shimura lift

Since the spaces $J_{10,1}^{\mathrm{cusp}}$ and $J_{12,1}^{\mathrm{cusp}}$ of Jacobi cusp forms of weights 10 and 12 and of index 1 are one-dimensional, the forms $f_{10} = \Delta\phi_{-2,1} \in J_{10,1}^{\mathrm{cusp}}$ and $f_{12} = \Delta\phi_{0,1} \in J_{12,1}^{\mathrm{cusp}}$ will inevitably be eigenforms of all Hecke operators. A computation reveals:

$$T_2 f_{10} = -528 f_{10}, \quad T_3 f_{10} = -4284 f_{10}, \quad T_5 f_{10} = -1025850 f_{10}, \quad T_7 f_{10} = 3225992 f_{10}$$

and

$$T_2 f_{12} = -288 f_{12}, \quad T_3 f_{12} = -128844 f_{12}, \quad T_5 f_{12} = 21640950 f_{12}, \quad T_7 f_{12} = -768078808 f_{12}.$$

On the other hand, the (more or less unique) cusp forms of weights 18 and 22 have q-series that begin

$$\Delta(\tau)E_6(\tau) = q - 528q^2 - 4284q^3 + 147712q^4 - 1025850q^5 + 2261952q^6 + 3225992q^7 \pm ...$$

and

$$\Delta(\tau)E_{10}(\tau) = q - 288q^2 - 128844q^3 - 2014208q^4 + 21640950q^5 + 37107072q^6 - 768078808rq^7 \pm ...$$

The fact that the T_p-eigenvalues of Jacobi forms coincide with q^p-coefficients of modular forms is not at all an accident. It is a special case of a general correspondence, discovered by Shimura, between modular forms of half-integral weight $k - 1/2$ and modular forms of integral weight $2k - 2$.

In this section we will roughly follow Kohnen[1] and Gross–Kohnen–Zagier [2] but in less generality. On an abstract level the reason this lift exists is that $\mathrm{SL}_2(\mathbb{Z})$ can be viewed (more or less) as an orthogonal group acting on binary quadratic forms, and indeed quadratic forms play a leading role in the following notes.

7.1. Zagier's modular forms

Let $\Delta > 0$ with $\Delta \equiv 0$ or 1 mod 4. Zagier[3] introduced the following functions:

[1]W. Kohnen. *Fourier coefficients of modular forms of half-integral weight.* Math. Ann. 271 (1985), 237–268.

[2]B. Gross, W. Kohnen and D. Zagier. *Heegner points and derivatives of L-series, II.* Math. Ann. 278 (1987), 497–562.

[3]Appendix 2 of D. Zagier. *Modular forms associated to real-quadratic fields.* Invent. Math. 30, 1–46 (1975)

$$f_{k,\Delta}(w) := \sum_{\substack{a,b,c\in\mathbb{Z}\\ b^2-4ac=\Delta}} \frac{1}{(aw^2+bw+c)^k}.$$

This is only meaningful if k is even: otherwise the summands cancel out in pairs.

Proposition 7.1. *Suppose $k \geq 2$. Then $f_{k,\Delta}$ converges for all $w \in \mathbb{H}$ and it defines a cusp form of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z})$.*

Proof. We can factor the summands as

$$|aw^2+bw+c| = a\cdot|w+(b+\sqrt{\Delta})/2a|\cdot|w-(b+\sqrt{\Delta})/2a|.$$

None of these terms has a zero in $\mathbb{H}$ because $\Delta > 0$. If w is confined to any compact set K in $\mathbb{H}$ then the above factorization shows that $\inf_{w\in K}|aw^2+bw+c|$ grows at most linearly in a and quadratically in $|b|$, so $f_{k,\Delta}$ can be majorized by the series

$$\sum_{a,b=1}^{\infty}(ab^2)^{-k} = \zeta(k)\zeta(2k)$$

for $k \geq 2$. In particular it converges uniformly on K. Moreover each summand $(aw^2+bw+c)^{-k}$ individually tends to zero as $\mathrm{Im}[w]\to\infty$.

For any $M = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, and any aw^2+bw+c, substituting $w \mapsto M\cdot w$ yields

$$\begin{aligned}&a\Big(\frac{\alpha w+\beta}{\gamma w+\delta}\Big)^2 + b\frac{\alpha w+\beta}{\gamma w+\delta}+c\\ =\;&\frac{(a\alpha^2+b\alpha\gamma+c\gamma^2)w^2+(2a\alpha+b\beta\gamma+b\alpha\delta+2c\gamma\delta)w+a\beta^2+b\beta\delta+c\gamma^2}{(\gamma w+\delta)^2},\end{aligned}$$

where the numerator is another polynomial $\tilde{a}w^2+\tilde{b}w+\tilde{c}$ with

$$\tilde{b}^2-4\tilde{a}\tilde{c} = (b^2-4ac)(\alpha\delta-\beta\gamma)^2 = \Delta.$$

Therefore we have $f_{k,\Delta}(M\cdot w) = (\gamma w+\delta)^{2k}f_{k,\Delta}(w)$. □

Remark 7.2. $f_{k,\Delta}$ is also well-defined for $\Delta = 0$ as long as we exclude the term $a = b = c = 0$. In this case, one can factor out the g.c.d. of a, b, c to obtain

$$f_{k,0}(w) = \zeta(k)\sum_{\substack{b^2-4ac=0\\ \gcd(a,b,c)=1}} \frac{1}{(aw^2+bw+c)^k}.$$

The fact that $b = \pm 2\sqrt{ac}$ is integral then implies that $a = m^2$ and $c = n^2$ for some (coprime) m, n, in which case $aw^2+bw+c = (\pm mw \pm n)^2$, so

$$f_{k,0}(w) = \zeta(k)\sum_{\substack{m,n\in\mathbb{Z}\\ \gcd(m,n)=1}} \frac{1}{(mw+n)^{2k}} = \zeta(k)E_{2k}(w)$$

is a multiple of the Eisenstein series.
Even if $\Delta < 0$, the series $f_{k,\Delta}$ is well-defined, but in this case each polynomial aw^2+bw+c that occurs in the denominator in the defining series has a root in $\mathbb{H}$ and therefore $f_{k,\Delta}$ has poles.

Remark 7.3. The series $f_{k,\Delta}$ can be written as $\sum_Q Q(w,1)^{-k}$, where the sum runs through all (indefinite) binary quadratic forms

$$Q(X,Y) = aX^2 + bXY + cY^2$$

of discriminant Δ. There is a natural notion of equivalence (often called *proper* equivalence) whereby two forms Q_1, Q_2 are called equivalent if

$$Q_2(X,Y) = Q_1(\alpha X + \beta Y, \gamma X + \delta Y)$$

for some matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. In other words, if one associates to

$$Q_1(X,Y) = aX^2 + bXY + cY^2$$

the Gram matrix

$$M_1 = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix},$$

such that $Q_1(X,Y) = (X,Y)M_1(X,Y)^T$, then the corresponding matrix for Q_2 is $M_2 = A^T M_1 A$.

A minor variation of the above proof shows that the series

$$f_{k,\Delta,\mathcal{A}} = \sum_{Q \in \mathcal{A}} Q(w,1)^{-k}$$

over any fixed equivalence class $\mathcal{A}$ define cusp forms of weight $2k$. These can also yield nontrivial series for odd k since quadratic forms Q are not generally $\mathrm{SL}_2(\mathbb{Z})$-equivalent to their negatives $-Q$. We will pursue that thought further in the next section.

Suppose $\Delta > 0$ and write $f_{k,\Delta}$ as a Fourier series:

$$f_{k,\Delta}(w) = \sum_{n=1}^{\infty} a_n e^{2\pi i n w}.$$

Then the coefficients are given by the integrals

$$\begin{aligned} a_n &= \int_{0+i}^{1+i} f_{k,\Delta}(w) e^{-2\pi i n w} \, \mathrm{d}w \\ &= \sum_{b^2-4ac=\Delta} \int_{0+i}^{1+i} (aw^2+bw+c)^{-k} e^{-2\pi i n w} \, \mathrm{d}w. \end{aligned}$$

Substituting $w \mapsto w + \lambda$ (with $\lambda \in \mathbb{Z}$) has the effect of replacing $aw^2 + bw + c$ by

$$a(w+\lambda)^2 + b(w+\lambda) + c = aw^2 + (b + 2a\lambda)w + (c + b\lambda + a\lambda^2);$$

and as we vary λ, that runs through all integer polynomials $\tilde{a}w^2 + \tilde{b}w + \tilde{c}$ with $\tilde{a} = a$ and $\tilde{b} \equiv b \,(\mathrm{mod}\, 2a)$ and $\tilde{b}^2 - 4\tilde{a}\tilde{c} = \Delta$. Considering first the tirms with $a \neq 0$, we can write $c = \frac{b^2-\Delta}{4a}$ and are left with

$$\sum_{\substack{b \,(\mathrm{mod}\, 2a) \\ b^2 \equiv \Delta \,(\mathrm{mod}\, 4a)}} \int_{-\infty+i}^{\infty+i} \Big(aw^2 + bw + \frac{b^2-\Delta}{4a}\Big)^{-k} e^{-2\pi i n w}\, \mathrm{d}w.$$

After $w \mapsto w - b/2a$ that becomes

$$e^{\pi i n b/a} \int_{-\infty+i}^{\infty+i} \frac{e^{-2\pi i n w}}{(aw^2 - \Delta/4a)^k}\, \mathrm{d}w.$$

To compute the integral we use the folllowing lemma without proof, which is a formula for a particular inverse Laplace transform:

Lemma 7.4. *Let $k > 0$ and $a \in \mathbb{R}_{>0}$. Then for any $C > 0$,*

$$\frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{e^{ts}}{(s^2+a^2)^k}\, \mathrm{d}s = \frac{\sqrt{\pi}}{\Gamma(k)} \cdot (t/2a)^{k-1/2} J_{k-1/2}(at),$$

where $J_\alpha(x)$ is the Bessel J-function

$$J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} (x/2)^{2m+\alpha}.$$

As an aside, note that the Bessel J-function at half-integer indices α simplifies to elementary functions: for $k = 1$ we have

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \sin(x),$$

and formulas for $J_{k-1/2}$ at higher values of k are obtained by differentiating the left-hand side of the above lemma with respect to a. In any case we will not need this.

Writing $w = -is$ we obtain

$$\begin{aligned}
\int_{-\infty+i}^{\infty+i} \frac{e^{-2\pi i n w}}{(aw^2 - \Delta/4a)^k}\, \mathrm{d}w &= (2\pi i) \cdot -i|a|^{-k} \int_{1-i\infty}^{1+i\infty} \frac{e^{2\pi n s}}{(s^2 + \Delta/4a^2)^k}\, \mathrm{d}s \\
&= 2\pi |a|^{-k} \frac{\sqrt{\pi}}{(k-1)!} \Big(\frac{2\pi n |a|}{\sqrt{\Delta}}\Big)^{k-1/2} J_{k-1/2}\Big(\frac{\pi n \sqrt{\Delta}}{|a|}\Big) \\
&= \frac{2^{k+1/2}\pi^{k+1} n^{k-1/2}}{\Delta^{k/2-1/4}\sqrt{|a|} \cdot (k-1)!} J_{k-1/2}\Big(\frac{\pi n \sqrt{\Delta}}{|a|}\Big).
\end{aligned}$$

Finally, the terms with $a = 0$ appear only if $\Delta = b^2$ is a perfect square. In that case use the formula

$$\sum_{n=-\infty}^{\infty} (z+n)^{-k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n z}$$

to see that

$$\sum_{b^2=\Delta} \sum_{c=-\infty}^{\infty} (bw+c)^{-k} = 2 \cdot \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n \sqrt{\Delta} w}.$$

Altogether, (after combining the terms for a and $-a$),

Theorem 7.5. *If Δ is not a square, then*

$$f_{k,\Delta}(w) = \sum_{n=1}^{\infty} c_n e^{2\pi i n w}$$

where the Fourier coefficients are

$$c_n = \frac{2^{k+3/2}\pi^{k+1} n^{k-1/2}}{\Delta^{k/2-1/4}(k-1)!} \cdot \sum_{a=1}^{\infty} \Big(\sum_{\substack{b \bmod 2a \\ b^2 \equiv \Delta \bmod 4a}} e^{\pi i n b/a} \Big) a^{-1/2} J_{k-1/2}\Big(\frac{\pi n \sqrt{\Delta}}{a}\Big). \qquad (7.1)$$

If Δ is a square then c_n is given by (7.1) unless $n^2 = d^2 \cdot \Delta$ for some $d \in \mathbb{N}$, in which case it is (7.1) plus $2 \cdot \frac{(2\pi i)^k}{(k-1)!} d^{k-1}$.

It is apparently impossible to simplify this Fourier series much further. For example, with $k = 6$ we have:

$$\begin{aligned} f_{6,1}(w) &\approx -1153.593453q + 27686.242873q^2 - 290705.550167q^3 \pm ... \\ &= -1153.593453...\Delta(w); \end{aligned}$$

$$\begin{aligned} f_{6,4}(w) &\approx 31.54357q - 757.04570q^2 + 7948.97989q^3 \pm ... \\ &= 31.54357...\Delta(w); \end{aligned}$$

$$\begin{aligned} f_{6,5}(w) &\approx -19.81066q + 475.45591q^2 - 4992.28701q^3 \pm ... \\ &= -19.81066...\Delta(w). \end{aligned}$$

7.2. Genus characters and modular forms

The functions $f_{k,\Delta}$ described above have a significant role in the Shimura correspondence, but not in the context that we are working in. The obvious problem is that their weight is off: $f_{k,\Delta}$ only produces cusp forms in weights divisible by 4, while the Shimura correspondence is meant to lift Jacobi forms of weight $k \in 2\mathbb{Z}$ (and index 1) to weight $2k-2 \equiv 2 \pmod 4$.

To construct nonzero forms of weight 2 mod 4 of this type, we need to find $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes $\mathcal{A}$ for which forms $Q \in \mathcal{A}$ are not equivalent to their negatives $-Q$, and we have to assign the classes $\mathcal{A}$ and $-\mathcal{A}$ different signs ± 1 in a consistent way. Gauss's genus theory of binary quadratic forms explains how to do this. We have to use some results from algebraic number theory in this section (mostly without proof).

The $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of *primitive* binary quadratic forms, (where a form $aX^2+bXY+cY^2$ is called primitive if $\gcd(a,b,c)=1$) of any fixed discriminant Δ form a finite, abelian group $\mathcal{C}_\Delta$ with the group operation given by the Gauss composition law. In Dirichlet's formulation: any primitive forms Q_1, Q_2 of the same discriminant are properly equivalent to forms

$$f = aX^2 + bXY + cY^2, \quad g = a'X^2 + bXY + c'Y^2$$

with coprime a, a' and the same middle term b, (these are sometimes called concordant forms), and then the composition $[Q_1] \cdot [Q_2]$ is represented by $h = aa'X^2 + bXY + c''Y^2$ where c'' is the integer that satisfies $b^2 - 4aa'c'' = \Delta$.

If Δ is not a perfect square, then a useful point of view is to associate, to any binary quadratic form $aX^2 + bXY + cY^2$ of discriminant Δ, the ideal

$$I = \Big(a, \frac{b+\sqrt{\Delta}}{2}\Big)$$

in the ring $R = \mathbb{Z}\Big[\frac{b+\sqrt{\Delta}}{2}\Big]$. (If Δ is a fundamental discriminant then this is the ring of integers of $\mathbb{Q}(\sqrt{\Delta})$.) Ideals I, J of R are called equivalent if there exist $a, b \in R$ such that $a \cdot I = b \cdot J$, and are called narrowly equivalent if a and b can be chosen totally positive (i.e. positive in all real embeddings). Then the $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of quadratic forms of discriminant Δ correspond exactly to the narrow-sense equivalence classes of nonzero ideals of $\mathbb{Z}[\sqrt{\Delta}]$, and through this bijection the Gauss composition becomes multiplication of ideals.

In the trivial case that $\Delta = b^2$ is a square, the distinct classes of primitive quadratic forms of discriminant Δ are represented by $aX^2 + bXY$ with $a \in (\mathbb{Z}/b\mathbb{Z})^\times$, and Dirichlet's form of the composition law shows that the class group is exactly $(\mathbb{Z}/b\mathbb{Z})^\times$.

Let us say that a form $Q = aX^2 + bXY + cY^2$ *properly represents* $n \in \mathbb{Z}$ if there are coprime integers $\lambda, \mu \in \mathbb{Z}$ such that $n = Q(\lambda, \mu)$.

Proposition 7.6. *Let Δ be a discriminant and suppose $n \in \mathbb{Z}$ is coprime to Δ. The following are equivalent:*
(i) There exists a primitive binary quadratic form Q of discriminant $b^2 - 4ac = \Delta$ that properly represents n;
(ii) Δ is a square mod $4n$.

Proof. If $\Delta = b^2 \bmod 4n$ then we simply take the form $Q(X,Y) = nX^2 + bXY + \frac{b^2-\Delta}{4n}Y^2$ with $Q(1,0) = n$. Conversely, if $Q(X,Y)$ properly represents n, say $Q(\lambda, \mu) = n$, then after conjugating by any matrix $M = \begin{pmatrix} \lambda & * \\ \mu & * \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we obtain a quadratic form $\tilde{Q}(X,Y) = nX^2 + \tilde{b}XY + \tilde{c}Y^2$ of the same discriminant $\Delta = \tilde{b}^2 - 4n\tilde{c}$ whose coefficient of X^2 is n. In particular $\Delta \equiv \tilde{b}^2$ is a square mod $4n$. □

It is not hard to see that equivalent quadratic forms represent the same numbers. Whether or not a number can be represented by a single quadratic form is a more difficult question when there are multiple classes with that discriminant.

Example 7.7. When $\Delta = 21$ there are two (proper) equivalence classes, represented by
$$Q = X^2 + XY - 5Y^2 \quad \text{and} \quad -Q = -X^2 - XY + 5Y^2.$$
After reducing modulo 3 they become $X^2 + XY + Y^2$ and $2(X^2 + XY + Y^2)$. As $X, Y \in \mathbb{Z}/3\mathbb{Z}$ run through possible values the first form only evaluates to 0 or 1 and the second form to 0 or 2. So Q and $-Q$ do not represent the same numbers.

This motivates the following definition:

Definition 7.8. Two integral binary quadratic forms of the same discriminant belong to the same **genus** if they represent the same numbers modulo n for all $n \in \mathbb{N}$.

More generally, suppose $\Delta = D \cdot D'$ is a fundamental discriminant that itself factors as a product of two discriminants D, D' (so $D, D' \equiv 0, 1 \bmod 4$). Then by Gauss's theory of genera there is a well-defined character
$$\chi = \chi_D = \chi_{D'} : \mathcal{C}_\Delta \to \{\pm 1\}$$
for which $\chi(Q) = \left(\frac{D}{n}\right) = \left(\frac{D'}{n}\right)$ for any integer n that is coprime to Δ and properly represented by Q. Moreover χ is nontrivial (both values ± 1 do occur), and two forms Q_1, Q_2 belong to the same genus if and only if $\chi(Q_1) = \chi(Q_2)$ for every such χ.

Definition 7.9. A **genus character** is a character $\chi_D : \mathcal{C}_\Delta \to \{\pm 1\}$ of the class group attached to a splitting of $\Delta = DD'$ into two discriminants as described above.

To simplify things later on, we extend χ_D to imprimitive forms by defining

$$\chi_D(nQ) = \left(\frac{D}{n}\right) \cdot \chi_D(Q).$$

The relevant case will be when $\Delta = DD'$ splits into a product of two *negative* discriminants. Then

$$\left(\frac{D}{-n}\right) = -\left(\frac{D}{n}\right),$$

so a quadratic form Q of discriminant Δ and its negative never represent the same integers mod D. (In particular, quadratic forms Q and $-Q$ of discriminant Δ are never $\mathrm{SL}_2(\mathbb{Z})$-equivalent.) Therefore the genus character χ_D satisfies $\chi_D(-Q) = -\chi_D(Q)$. This is exactly what we need to construct modular forms of weight 2 mod 4:

Definition 7.10. Suppose $\Delta = DD'$ splits as a product of two negative discriminants and let χ be the associated genus character. For odd $k \geq 3$, define the cusp form

$$f_{k,D,D'}(w) := \sum_{b^2-4ac=\Delta} \frac{\chi([a,b,c])}{(aw^2+bw+c)^k} \in S_{2k}(\mathrm{SL}_2(\mathbb{Z})),$$

where $[a,b,c]$ is the quadratic form $aX^2 + bXY + cY^2$.

The Fourier expansion of $f_{k,D,D'}$ can be computed similarly to $f_{k,\Delta}$. Suppose D is a fundamental discriminant (this is the only case we need). We have

$$f_{k,D,D'}(w) = \sum_{n=1}^{\infty} a_n e^{2\pi i n w}$$

where

$$a_n = \sum_{b^2-4ac=\Delta} \chi([a,b,c]) \int_{0+i}^{1+i} (aw^2+bw+c)^{-k} e^{-2\pi i n w}\, \mathrm{d}w,$$

and substituting $w \mapsto w + \lambda$ (for $\lambda \in \mathbb{Z}$) has the effect of replacing the form $[a,b,c]$ by the equivalent form $[a, b+2a\lambda, c+b\lambda+a\lambda^2]$. So the forms $[a,b,c]$ with $a \neq 0$ contribute

$$a_n = 2\sum_{a=1}^{\infty} \sum_{\substack{b \,(\mathrm{mod}\, 2a) \\ b^2 \equiv \Delta \bmod 4a}} \chi\Big(\Big[a, b, \frac{b^2-\Delta}{4a}\Big]\Big) \int_{-\infty+i}^{\infty+i} \Big(aw^2 + bw + \frac{b^2-\Delta}{4a}\Big)^{-k} e^{-2\pi i n w}\, \mathrm{d}w,$$

and the integral can be evaluated exactly as in the previous section. In the case that Δ is a perfect square, (and therefore $D' = D \cdot f^2$ for some $f \in \mathbb{N}$), we also have the forms $[0,b,c] = bXY + cY^2$ which properly represent c and therefore have $\chi([0,b,c]) = \left(\frac{D}{c}\right)$

(using the extension of χ to imprimitive forms). So these contribute the series

$$\begin{aligned}
&\sum_{b^2=\Delta}\sum_{c=-\infty}^{\infty}\left(\frac{D}{c}\right)(bw+c)^{-k}\\
&=|D|^{-k}\sum_{a\in\mathbb{Z}/|D|}\left(\frac{D}{a}\right)\sum_{b^2=\Delta}\sum_{c=-\infty}^{\infty}\Big(\frac{bw+a}{|D|}-c\Big)^{-k}\\
&=2\cdot\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}n^{k-1}\Big(\frac{1}{|D|}\sum_{a\in\mathbb{Z}/D}\left(\frac{D}{a}\right)e^{2\pi ina/|D|}\Big)e^{2\pi infw}.
\end{aligned}$$

The sum over a is a quadratic Gauss sum. Since we supposed D to be a fundamental discriminant,

$$\frac{1}{|D|}\sum_{a\in\mathbb{Z}/D}\left(\frac{D}{a}\right)e^{2\pi ina/|D|}=\frac{i}{\sqrt{|D|}}\cdot\left(\frac{D}{n}\right).$$

Putting all of this together we get the Fourier coefficients:

Theorem 7.11. *Let $k\geq 3$ be odd. Suppose $\Delta=DD'$ where D is a negative fundamental discriminant and let $\chi=\chi_D$ be the genus character. If Δ is not a square, then $f_{k,D,D'}$ has Fourier series*

$$f_{k,D,D'}(w)=\sum_{n=1}^{\infty}c_ne^{2\pi inw}$$

where

$$c_n=\frac{2^{k+3/2}\pi^{k+1}n^{k-1/2}}{\Delta^{k/2-1/4}(k-1)!}\tag{7.2}$$

$$\times\sum_{a=1}^{\infty}\Big(\sum_{\substack{b\,mod\,2a\\ b^2\equiv\Delta\,mod\,4a}}\chi([a,b,(b^2-\Delta)/4a])e^{\pi inb/a}\Big)a^{-1/2}J_{k-1/2}\Big(\frac{\pi n\sqrt{\Delta}}{a}\Big).\tag{7.3}$$

If Δ is a square then c_n is given by (7.2) unless $Dn^2=D'd^2$ for some $d\in\mathbb{N}$, in which case it is (7.2) plus

$$2i\cdot\frac{(2\pi i)^k}{\Delta^{(k-1)/2}(k-1)!\sqrt{|D|}}\left(\frac{D}{d}\right)n^{k-1}.$$

7.3. Poincaré series

The Bessel J-functions appear often in formulas for the Fourier coefficients of modular forms, due to their close relationship with the representation theory of SL_2. But usually the coefficients of a modular form of weight k are expressed in terms of the Bessel function J_{k-1}, not the Bessel function $J_{k/2-1/2}$ as in the Fourier series (Theorem 7.11) for

$f_{k,D,D'}$. This discrepancy suggests that we try to identify the coefficients of the weight $2k$ form $f_{k,D,D'}$ with those of some modular forms of weight $k+1/2$, or equivalently, with Jacobi forms of weight $k+1$.

Let $m \in \mathbb{N}$ be an index. For any tuple (n_0, r_0) with $4mn_0 - r_0^2 > 0$, there is a linear functional

$$\varphi_{n_0,r_0} : J_{k,m}^{\mathrm{cusp}} \longrightarrow \mathbb{C}, \quad \sum_{n,r} c(n,r)q^n\zeta^r \mapsto c(n_0, r_0).$$

Since $J_{k,m}^{\mathrm{cusp}}$ is a Hilbert space with respect to $\langle -, - \rangle$, there is a unique cusp form f_{n_0,r_0} with the property

$$\langle f, f_{n_0,r_0} \rangle = \varphi_{n_0,r_0}(f) \quad \text{for all cusp forms } f \in J_{k,m}^{\mathrm{cusp}}.$$

Theorem 7.12. *Suppose $k \geq 3$. The series*

$$P_{k,m;n_0,r_0} = \frac{1}{2} \sum_{\gamma \in \mathcal{J}_\infty \backslash \mathcal{J}} \Big(q^{n_0}\zeta^{r_0} + (-1)^k q^{n_0}\zeta^{-r_0} \Big)\Big|_{k,m}\gamma$$

converges and defines a Jacobi cusp form of weight k and index m. It extracts Fourier coefficients with respect to the Petersson inner product: if $f = \sum_{n,r} c(n,r)q^n\zeta^r$ is another Jacobi cusp form of that weight and index then

$$\langle f, P_{k,m;n_0,r_0} \rangle = c(n_0, r_0) \cdot \frac{m^{k-2}\Gamma(k-3/2)}{2\pi^{k-3/2}} \cdot (4mn_0 - r_0^2)^{3/2-k}.$$

Proof. The series is majorized by the Eisenstein series and therefore converges by the same proof. It defines a cusp form by the same argument that applies to the Eisenstein series, noting that the terms $\gamma = (\begin{pmatrix} a & b \\ c & d \end{pmatrix}; \zeta)$ with $c = 0$ and $d = 1$ do not yield a non-cuspidal contribution in this case because $4mn_0 - r_0^2 > 0$.

For any cusp form f as above, we have

$$\begin{aligned} &\langle f, P_{k,m;n_0,r_0} \rangle \\ &= \frac{1}{2} \int_{\mathcal{J}_\infty \backslash (\mathbb{H} \times \mathbb{C})} f(\tau, z) \overline{\Big(q^{n_0}\zeta^{r_0} + (-1)^k q^{n_0}\zeta^{-r_0} \Big)} e^{-4\pi m v^2/y} y^{k-3} \,\mathrm{d}u \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}y \\ &= \frac{1}{2} \sum_{n,r} c(n,r) \int_0^\infty \int_0^\infty \Big[\int_{-1/2}^{1/2} \int_{-1/2}^{1/2} q^n\zeta^r \overline{(q^{n_0}\zeta^{r_0} + (-1)^k q^{n_0}\zeta^{-r_0})} \,\mathrm{d}u \,\mathrm{d}x \Big] e^{-4\pi m v^2/y} y^{k-3} \,\mathrm{d}v \,\mathrm{d}y. \end{aligned}$$

The double integral over $\mathrm{d}u, \mathrm{d}x$ is zero unless $n = n_0$ and $r = \pm r_0$, in which case it is $e^{-4\pi n_0 y - 4\pi r_0 v}$ or $(-1)^k e^{-4\pi n_0 y + 4\pi r_0 v}$, respectively. But $c(n_0, r_0) = (-1)^k c(n_0, -r_0)$, so we can rewrite this as

$$\begin{aligned}
\langle f, P_{k,m;n_0,r_0}\rangle &= c(n_0,r_0)\int_0^\infty \int_0^\infty e^{-4\pi n_0 y}\Big(e^{-4\pi r_0 v}+e^{4\pi r_0 v}\Big)e^{-4\pi m v^2/y}y^{k-3}\,\mathrm{d}v\,\mathrm{d}y \\
&= c(n_0,r_0)\int_0^\infty e^{-4\pi n_0 y}y^{k-3}\int_{-\infty}^\infty e^{-4\pi m v^2/y-4\pi r_0 v}\,\mathrm{d}v\,\mathrm{d}y \\
&= c(n_0,r_0)\int_0^\infty y^{k-3}e^{-4\pi(n_0-r_0^2/4m)y}\sqrt{\frac{y}{4m}}\,\mathrm{d}y \\
&= \frac{\Gamma(k-3/2)}{(4\pi)^{k-3/2}\sqrt{4m}}\cdot(n_0-r_0^2/4m)^{3/2-k}c(n_0,r_0). \qquad \square
\end{aligned}$$

By the formula for $\langle f, P_{k,m;n_0,r_0}\rangle$, it is impossible for a cusp form f to be orthogonal to all Poincaré series, so we immediately have:

Corollary 7.13. *The series $P_{k,m;n_0,r_0}$ span $J_{k,m}^{\mathrm{cusp}}$.*

The formula for $\langle f, P_{k,m;n_0,r_0}\rangle$ also implies that $P_{k,m;n_0,r_0}$ depends only on the discriminant $D_0 = 4mn_0 - r_0^2$ and on the remainder of r_0 mod $2m$. So we also use the notation $P_{k,m;D_0;r_0}$.

Corollary 7.14. *Suppose N is coprime to m and let $D_0 = 4mn_0 - r_0^2$. Then*

$$T_N P_{k,m;D_0;r_0} = N^{2k-3}\sum_{\substack{a|N^2 \\ (N/a)^2 D_0\in\Delta}} a^{1-k}\epsilon_{D_0}(a)P_{k,m;(N/a)^2D_0;a^*Nr_0}.$$

Proof. T_N is self-adjoint, so for any cusp form $f=\sum_{n,r}c(n,r)q^n\zeta^r$ we have

$$\begin{aligned}
\langle f, T_N P_{k,m;n_0,r_0}\rangle &= \langle T_N f, P_{k,m;n_0,r_0}\rangle \\
&= \frac{m^{k-2}\Gamma(k-3/2)}{2^{k-3/2}}\cdot D_0^{3/2-k}b(n_0,r_0)
\end{aligned}$$

where $b(n,r)$ are the Fourier coefficients of $T_N f$. Since N is coprime to m, we have the formula

$$b(n_0,r_0) = \sum_{\substack{a|N^2 \\ (N/a)^2 D_0\in\Delta}} \varepsilon_{D_0}(a)a^{k-2}c_{a^*Nr_0}((N/a)^2D_0).$$

So we can write

$$\begin{aligned}
\langle f, T_N P_{k,m;n_0,r_0}\rangle &= \sum_{\substack{a|N^2 \\ (N/a)^2 D_0\in\Delta}} \varepsilon_{D_0}(a)a^{k-2}D_0^{3/2-k}\Big(\frac{N^2}{a^2}D_0\Big)^{k-3/2}\cdot\langle f, P_{k,m;(N/a)^2D_0;a^*Nr_0}\rangle \\
&= \Big\langle f, N^{2k-3}\sum_{\substack{a|N^2 \\ (N/a)^2 D_0\in\Delta}} a^{1-k}\varepsilon_{D_0}(a)P_{k,m;(N/a)^2D_0;a^*Nr_0}\Big\rangle. \qquad \square
\end{aligned}$$

In particular, if D_0 is a fundamental discriminant, then $(N/a)^2 D_0$ belongs to Δ if and only if $d = N/a \in \mathbb{Z}$. In this case, $P_{k,m;d^2D_0;dr_0}$ is represented by $P_{k,m;d^2n_0;dr_0}$. So we have

$$T_N P_{k,m;n_0;r_0} = N^{k-2}\left(\frac{D_0}{N}\right) \cdot \sum_{d|N} \left(\frac{D_0}{d}\right) d^{k-1} P_{k,m;d^2n_0;dr_0}.$$

As promised, the Bessel J-function features prominently in the Fourier expansion of these Jacobi forms:

Theorem 7.15. *Suppose $D = 4mn - r^2 > 0$ and $D' = 4mn_0 - r_0^2 > 0$. The coefficient of $q^n \zeta^r$ in $P_{k,m;n_0,r_0}$ is*

$$\frac{1}{2}(\delta(n,r) + \delta(n,-r)) + a_{n,r},$$

where

$$\delta(n,r) = \begin{cases} 1: & D = D_0 \text{ and } r \equiv r_0 \bmod 2m; \\ 0: & \text{otherwise}; \end{cases}$$

and where $a_{n,r}$ is the series

$$a_{n,r} = i^k \frac{\pi}{\sqrt{2m}} (D/D')^{k/2-3/4} \sum_{\substack{c \in \mathbb{Z} \\ c \neq 0}} \mathrm{sgn}(c)^k J_{k-3/2}\Big(\frac{\pi\sqrt{DD'}}{|c|}\Big) H_{m,c}(n,r,n_0,r_0)$$

in which

$$\begin{aligned} &H_{m,c}(n,r,n_0,r_0) \\ &:= |c|^{-3/2} \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} \sum_{\lambda \in \mathbb{Z}/c} \exp\Big(2\pi i \frac{a}{c}(n_0 + r_0\lambda + m\lambda^2) - 2\pi i \frac{\lambda r}{c} - 2\pi i \frac{nd}{c} - \pi i \frac{r_0 r}{cm}\Big) \end{aligned}$$

is a "Kloosterman-type" sum.

Proof. $P_{k,m;n_0,r_0}$ can be expanded as a Fourier series in almost exactly the same way as the Jacobi Eisenstein series $E_{k,m}$. Using the coset representatives for $\mathcal{J}_\infty \backslash \mathcal{J}$ from that computation, we have

$$\begin{aligned} P_{k,m;n_0,r_0}(\tau,z) &= \sum_{M \in \Gamma_\infty \backslash \Gamma} \Big(\sum_{\lambda \in \mathbb{Z}} e^{2\pi i m(\lambda^2\tau + 2\lambda z)} \cdot \Big[q^{n_0 + r_0\lambda}\zeta^{r_0} + (-1)^k q^{n_0 - r_0\lambda}\zeta^{-r_0} \Big] \Big) \Big|_{k,m} M \\ &= \frac{1}{2} \sum_{\lambda \in \mathbb{Z}} \Big(q^{n_0 + r_0\lambda + m\lambda^2} \zeta^{r_0 + 2m\lambda} + (-1)^k q^{n_0 + r_0\lambda + m\lambda^2} \zeta^{-r_0 + 2m\lambda} \Big) \\ &+ \sum_{c=1}^{\infty} \sum_{\substack{d \in \mathbb{Z} \\ \gcd(c,d)=1}} (c\tau + d)^{-k} e^{-2\pi i m \frac{cz^2}{c\tau+d}} \sum_{\lambda \in \mathbb{Z}} e^{2\pi i m \lambda^2 \frac{a\tau+b}{c\tau+d} + 4\pi i m \lambda \frac{z}{c\tau+d}} \\ &\qquad \times \frac{1}{2}\Big(e^{2\pi i (n_0 + \lambda r_0)\frac{a\tau+b}{c\tau+d} + 2\pi i r_0 \frac{z}{c\tau+d}} + (-1)^k e^{2\pi i (n_0 - \lambda r_0)\frac{a\tau+b}{c\tau+d} - 2\pi i r_0 \frac{z}{c\tau+d}} \Big). \end{aligned}$$

Here a, b are any numbers such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to $\mathrm{SL}_2(\mathbb{Z})$.

The first summand is already a Fourier series, and it contributes $\frac{1}{2}(\delta(n,r)+\delta(n,-r))$ to the formula for the Fourier coefficients. If we write the second summand above as

$$\sum_{n,r} a_{n,r} q^n \zeta^r$$

then we have:

$$a_{n,r} = \frac{1}{2} \sum_{c \neq 0} c^{-k} \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} \sum_{\lambda \in \mathbb{Z}} \int_{w-\infty}^{w+\infty} \int_0^1$$
$$(\tau + d/c)^{-k} e^{-2\pi i m \frac{cz^2}{c\tau+d} + 2\pi i (n_0 + r_0\lambda + m\lambda^2) \frac{a\tau+b}{c\tau+d} + 2\pi i (r_0 + 2m\lambda) \frac{z}{c\tau+d} - 2\pi i (n\tau + rz)} \, \mathrm{d}z \, \mathrm{d}\tau$$

for any basepoint $w \in \mathbb{H}$.

We can write

$$\frac{a\tau+b}{c\tau+d} = \frac{a}{c} - \frac{1/c}{c\tau+d}$$

and substitute $\tau \mapsto \tau - d/c$ to obtain

$$a_{n,r} = \frac{1}{2} \sum_{c \neq 0} c^{-k} \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} \sum_{\lambda \in \mathbb{Z}} e^{2\pi i \frac{a}{c}(n_0 + r_0\lambda + m\lambda^2) + 2\pi i \frac{d}{c} n}$$
$$\int_{w-\infty}^{w+\infty} \int_0^1 \tau^{-k} e^{-2\pi i m \frac{z^2}{\tau} - 2\pi i (n_0 + r_0\lambda + m\lambda^2) \frac{1}{c^2\tau} + 2\pi i (r_0 + 2m\lambda) \frac{z}{c\tau} - 2\pi i (n\tau + rz)} \, \mathrm{d}z \, \mathrm{d}\tau.$$

The effect of substituting $z \mapsto z+1$ is to replace

$$-m\frac{z^2}{\tau} - (n_0 + r_0\lambda + m\lambda^2)\frac{1}{c^2\tau} + (r_0 + 2m\lambda)\frac{z}{c\tau} - rz$$

by

$$\begin{aligned} & -m\frac{(z+1)^2}{\tau} - (n_0 + r_0\lambda + m\lambda^2)\frac{1}{c^2\tau} + (r_0 + 2m\lambda)\frac{z+1}{c\tau} - r(z+1) \\ = & -m\frac{z^2}{\tau} - (n_0 + r_0\lambda + m\lambda^2 - r_0 c - 2m\lambda c + mc^2)\frac{1}{c^2\tau} + (r_0 + 2m\lambda - 2mc)\frac{z}{c\tau} - rz - r; \end{aligned}$$

so, up to addition by an integer, it is the same as substituting $\lambda \mapsto \lambda - c$. Therefore we can write

$$a_{n,r} = \frac{1}{2} \sum_{c \neq 0} c^{-k} \Big(\sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} \sum_{\lambda \in \mathbb{Z}/c} e^{2\pi i \frac{a}{c}(n_0 + r_0\lambda + m\lambda^2) + 2\pi i \frac{d}{c} n} \Big)$$
$$\times \int_{w-\infty}^{w+\infty} \int_{-\infty}^{\infty} \tau^{-k} e^{-2\pi i m \frac{z^2}{\tau} - 2\pi i (n_0 + r_0\lambda + m\lambda^2) \frac{1}{c^2\tau} + 2\pi i (r_0 + 2m\lambda) \frac{z}{c\tau} - 2\pi i (n\tau + rz)} \, \mathrm{d}z \, \mathrm{d}\tau.$$

The integral over z is a special case of the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-ax^2 - bx} \, \mathrm{d}x = \sqrt{\frac{\pi}{a}} \cdot e^{b^2/4a}, \quad (\mathrm{Re}[a] > 0).$$

So

$$\int_{w-\infty}^{w+\infty}\int_{-\infty}^{\infty}\tau^{-k}e^{-2\pi im\frac{z^2}{\tau}-2\pi i(n_0+r_0\lambda+m\lambda^2)\frac{1}{c^2\tau}+2\pi i(r_0+2m\lambda)\frac{z}{c\tau}-2\pi i(n\tau+rz)}\,\mathrm{d}z\,\mathrm{d}\tau$$
$$=\int_{w-\infty}^{w+\infty}\tau^{-k}e^{-2\pi i(n_0+r_0\lambda+m\lambda^2)\frac{1}{c^2\tau}-2\pi in\tau}\sqrt{\frac{\tau}{2im}}e^{\frac{\pi i(cr\tau-2\lambda m-r_0)^2}{2mc^2\tau}}\,\mathrm{d}\tau$$
$$=i^{-1/2}\frac{1}{\sqrt{2m}}e^{-2\pi i\frac{\lambda r}{c}-\pi i\frac{r_0r}{cm}}\int_{w-\infty}^{w+\infty}\tau^{-k+1/2}\exp\Big(\frac{\pi i}{2}(r^2-4mn)\tau+\frac{\pi i}{2c^2\tau}(r_0^2-4mn_0)\Big)\,\mathrm{d}\tau.$$

The integral over τ can be calculated by means of Schläfli's integral for the Bessel J-function. For $\mathrm{Re}[b]>0$,

$$\int_{w-\infty}^{w+\infty}\tau^{-\nu}\exp\Big(-2\pi ia\tau-2\pi i\frac{b}{\tau}\Big)\,\mathrm{d}\tau=\begin{cases}-2\pi i\cdot\Big(-i\sqrt{\frac{a}{b}}\Big)^{\nu-1}\cdot J_{\nu-1}\Big(4\pi\sqrt{ab}\Big): & \mathrm{Re}[a]>0;\\ 0: & \mathrm{Re}[a]\le 0,\end{cases}$$

where J is the Bessel J-function. With $a=D/4>0$ and $b=D'/4c^2$ we have

$$\int_{w-\infty}^{w+\infty}\tau^{-k+1/2}\exp\Big(-\frac{\pi i}{2}D\tau-\frac{\pi i}{2c^2\tau}D'\Big)\,\mathrm{d}\tau=2\pi(-i)^{k-1/2}(D/D')^{k/2-3/4}|c|^{k-3/2}\cdot J_{k-3/2}\Big(\frac{\pi\sqrt{DD'}}{|c|}\Big).$$

Altogether,

$$a_{n,r}=\frac{\pi}{\sqrt{2m}}i^k(D/D')^{k/2-3/4}\sum_{c\neq 0}J_{k-3/2}\Big(\frac{\pi\sqrt{DD'}}{|c|}\Big)$$
$$\times\,\mathrm{sgn}(c)^{-k}|c|^{-3/2}\sum_{d\in(\mathbb{Z}/c\mathbb{Z})^\times}\sum_{\lambda\in\mathbb{Z}/c}\exp\Big(2\pi i\frac{a}{c}(n_0+r_0\lambda+m\lambda^2)-2\pi i\frac{\lambda r}{c}-2\pi i\frac{nd}{c}\Big)\exp\Big(-\pi i\frac{r_0r}{cm}\Big).$$

□

Remark 7.16. The expression for $P_{k,m;n_0,r_0}$ as a Fourier series is also well-defined when $k=2$, and it converges to a Jacobi cusp form that satisfies the characterization with respect to the Petersson inner product: for any $f=\sum_{n,r}c(n,r)q^n\zeta^r\in J_{2,m}^{\mathrm{cusp}}$,

$$\langle f,P_{2,m;n_0,r_0}\rangle=c(n_0,r_0)\cdot\frac{1}{2\sqrt{4mn_0-r_0^2}}.$$

This is not at all obvious.

7.4. The holomorphic kernel

Let $D_0<0$ be a negative fundamental discriminant.

Definition 7.17. Let k be an odd integer. For $\tau,w\in\mathbb{H}$ and $z\in\mathbb{C}$, define the series[a]

$$\Omega_{2k,D_0}(\tau,z,w):=\sum_{D<0}\sum_{\substack{n,r\in\mathbb{Z}\\4n-r^2=-D}}(DD_0)^{k-1/2}f_{k,D_0,D}(w)q^n\zeta^r.$$

[a] This is $D_0^{k-1/2}$ times the function in G-K-Z.

The series that defines Ω_{2k,D_0} can be thought of as an infinite linear combination of the modular forms $f_{k,D_0,D} \in S_{2k}$. It converges (with respect to any metric; S_{2k} is finite-dimensional, so all Hausdorff linear topologies on it are the same), since for any fixed N the Fourier coefficient of $e^{2\pi i N w}$ in $(DD_0)^{k-1/2} f_{k,D_0,D}(w)$ grows at most polynomially in D. Clearly Ω_{2k,D_0} transforms like a modular form of weight $2k$ with respect to the variable w.

With respect to the variables τ and z, the notation is meant to suggest that Ω_{2k,D_0} behaves like a Jacobi form of index one. This turns out to be true. More precisely:

Theorem 7.18.

$$\Omega_{2k,D_0}(\tau, z, w) = \frac{(2i)^{k+1}\pi^k |D_0|^{k-1/2}}{(k-1)!} \cdot \sum_{N=1}^{\infty} T_N P_{k+1,1;D_0}(\tau, z) e^{2\pi i N w}.$$

In particular, Ω_{2k,D_0} satisfies the weight $(k+1)$ Jacobi transformation law

$$\Omega_{2k,D_0}\Big(-\frac{1}{\tau}, \frac{z}{\tau}, w\Big) = \tau^{k+1} e^{-2\pi i z^2/\tau} \cdot \Omega_{2k,D_0}(\tau, z, w).$$

It seems to be difficult to prove the Jacobi transformation law for Ω_{2k,D_0} with respect to its (τ, z)-variables directly (in any case, I do not know how to do it). We will basically follow Gross–Kohnen–Zagier, specialized to index 1, and prove the claim by matching up Fourier coefficients of Ω_{2k,D_0} with certain linear combinations of Jacobi Poincaré series. In particular the proof is somewhat technical.

We can write

$$\begin{aligned}
&\Omega_{2k,D_0}(\tau, z, w) \\
&= \frac{2^{k+3/2}\pi^{k+1}}{(k-1)!} \sum_{N=1}^{\infty} N^{k-1/2} \sum_{D<0} \sum_{4n-r^2=D} (DD_0)^{k/2-1/4} \qquad (7.4) \\
&\quad \times \sum_{a=1}^{\infty} \Big(\sum_{\substack{b \bmod 2a \\ b^2 \equiv DD_0 \bmod 4a}} \chi([a, b, (b^2 - DD_0)/4a]) e^{\pi i N b/a} \Big) a^{-1/2} J_{k-1/2}\Big(\frac{\pi N \sqrt{DD_0}}{a} \Big) q^n \zeta^r e^{2\pi i N w} \\
&+ 2i \sum_{N=1}^{\infty} \sum_{\delta | N} (N|D_0|/\delta)^k \frac{(2\pi i)^k}{(k-1)! \sqrt{|D_0|}} \Big(\frac{D_0}{\delta} \Big) N^{k-1} \sum_{4n-r^2=(N/\delta)^2 D_0} q^n \zeta^r e^{2\pi i N w},
\end{aligned}$$

with the sum in the final row accounting for the correction to the coefficient formula when $D\delta^2 = D_0 N^2$ and $\Delta = DD_0 = (ND_0/\delta)^2$.

Proof. Since D_0 is a fundamental discriminant, the right-hand side of the claim is given (formally) by

$$\begin{aligned}
&\sum_{N=1}^{\infty} T_N P_{k+1,1;D_0}(\tau,z) e^{2\pi i N w} \\
&= \sum_{N=1}^{\infty} N^{k-1} \left(\frac{D_0}{N}\right) \sum_{\delta | N} \left(\frac{D_0}{\delta}\right) \delta^k P_{k+1,1;\delta^2 D_0} e^{2\pi i N w}.
\end{aligned}$$

Recall that we can write

$$P_{k+1,1;D_0} = \sum_{4n-r^2=D_0} q^n \zeta^r + \sum_{n,r} a_{n,r}(D_0) q^n \zeta^r$$

with the coefficients

$$\begin{aligned}
a_{n,r}(D_0) &= \pi\sqrt{2} \cdot i^{k+1} (D/D_0)^{k-1/2} \sum_{c=1}^{\infty} J_{k-1/2}\Big(\frac{\pi\sqrt{DD_0}}{|c|}\Big) \\
&\quad \times |c|^{-3/2} \sum_{d \in (\mathbb{Z}/c)^\times} \sum_{\lambda \in \mathbb{Z}/c} \exp\Big(2\pi i \frac{a}{c}(n_0 + r_0\lambda + \lambda^2) - 2\pi i \frac{\lambda r}{c} - 2\pi i \frac{nd}{c} - \pi i \frac{r r_0}{c}\Big).
\end{aligned}$$

So

$$\begin{aligned}
&\sum_{N=1}^{\infty} T_N P_{k+1,1;D_0}(\tau,z) e^{2\pi i N w} \\
&= \sum_{N=1}^{\infty} \sum_{\delta|N} N^{k-1} \delta^k \left(\frac{D_0}{N\delta}\right) \Big[\sum_{4n-r^2=\delta^2 D_0} q^n \zeta^r e^{2\pi i N w} + \sum_{n,r} a_{n,r}(\delta^2 D_0) q^n \zeta^r e^{2\pi i N w} \Big] \\
&= \sum_{N=1}^{\infty} \sum_{\delta|N} N^{2k-1} \delta^{-k} \left(\frac{D_0}{\delta}\right) \Big[\sum_{4n-r^2=(N/\delta)^2 D_0} q^n \zeta^r e^{2\pi i N w} + \sum_{n,r} a_{n,r}((N/\delta)^2 D_0) q^n \zeta^r e^{2\pi i N w} \Big].
\end{aligned}$$

If we first look only at the first summands,

$$\sum_{N=1}^{\infty} \sum_{\delta|N} N^{2k-1} \delta^{-k} \left(\frac{D_0}{\delta}\right) \Big[\sum_{4n-r^2=(N/\delta)^2 D_0} q^n \zeta^r e^{2\pi i N w} \Big]$$

and multiply by the constant factor $2i \cdot \frac{(2\pi i)^k}{(k-1)!} |D_0|^{k-1/2}$, we get exactly the "correction term"

$$2i \sum_{N=1}^{\infty} \sum_{\delta|N} (N|D_0|/\delta)^k \frac{(2\pi i)^k}{(k-1)!\sqrt{|D_0|}} \left(\frac{D_0}{\delta}\right) N^{k-1} \sum_{4n-r^2=(N/\delta)^2 D_0} q^n \zeta^r e^{2\pi i N w}.$$

The coefficient of $e^{2\pi i N w}$ in the remaining part of the series is

$$\begin{aligned}
&\sum_{\delta|N} N^{2k-1} \delta^{-k} \left(\frac{D_0}{\delta}\right) a_{n,r}((N/\delta)^2 D_0) \\
&= N^{k-1/2} \pi\sqrt{2} i^{k+1} \sum_{\delta|N} \left(\frac{D_0}{\delta}\right) \delta^{-1/2} (D/D_0)^{k/2-1/4} \sum_{c=1}^{\infty} J_{k-1/2}\Big(\frac{N\pi\sqrt{DD_0}}{c\delta}\Big) \\
&\qquad \times H_{1,c}(n,r,n_0,r_0).
\end{aligned}$$

To match this (after multiplication by the factor $2i \cdot \frac{(2\pi i)^k}{(k-1)!}|D_0|^{k-1/2}$) with the result of Equation (7.4),

$$\begin{aligned} &\frac{2^{k+3/2}\pi^{k+1}}{(k-1)!} N^{k-1/2}(DD_0)^{k/2-1/4} \\ &\quad \times \sum_{a=1}^{\infty} \sum_{\substack{b \bmod 2a \\ b^2 \equiv DD_0 \bmod 4a}} \chi([a,b,(b^2-DD_0)/4a]) e^{\pi i N b/a} a^{-1/2} J_{k-1/2}\Big(\frac{\pi N \sqrt{DD_0}}{a}\Big), \end{aligned}$$

we write $a = c\delta$; then it is enough to show the following identity:

$$\sum_{\substack{b \bmod 2a \\ b^2 \equiv DD_0 \bmod 4a}} \chi([a,b,(b^2-DD_0)/4a]) e^{\pi i N b/a} a^{-1/2} = \sum_{\delta \mid (a,N)} \left(\frac{D_0}{\delta}\right) \delta^{-1/2} H_{1,a/\delta}(n,r,n_0,r_0).$$

This follows from the following technical lemma, which we cite without proof[4]. □

Lemma 7.19. *The genus character can be expressed as the following Gauss sum. Suppose $D_0 = r_0^2 - 4n_0$ and $D = r^2 - 4n$ and that $b^2 \equiv DD_0$ mod $4a$. Then*

$$\begin{aligned} &\chi([a,b,(b^2-DD_0)/4a]) \\ &= \frac{1}{a^2} \sum_{\delta \mid a} \left(\frac{D_0}{\delta}\right) \delta \sum_{d \in (\mathbb{Z}/(a/\delta))^\times} \sum_{\lambda,\mu \in (\mathbb{Z}/(a/\delta))} \\ &\qquad \exp\Big(2\pi i \frac{d}{a/\delta}(\lambda^2 + r_0\lambda\mu + n_0\mu^2 + r\lambda + \frac{r_0 r - b}{2}\mu + n)\Big). \end{aligned}$$

7.5. The Shimura lift

Let D be a fixed fundamental discriminant and let $k \geq 3$ be an odd integer.

For any fixed $w \in \mathbb{H}$, the kernel function $\Omega_{2k,D}(\tau, z, w)$ defines a Jacobi cusp form of weight $k+1$ and index 1. In particular if $f \in J_{k+1,1}^{\mathrm{cusp}}$ then the Petersson inner product

$$\langle f, \Omega_{2k,D}(\tau, z, w)\rangle$$

is well-defined, and is still a function of w.

Since the inner product $\langle f, \Omega_{2k,D}(\tau, z, w)\rangle$ involves taking the complex conjugate of Ω_{2k}, the function $w \mapsto \langle f, \Omega_{2k,D}(\tau, z, w)\rangle$ actually transforms like the complex conjugate of a modular form. So it is natural to replace w by $-\overline{w}$. This leads to the following definition:

Definition 7.20. The **D-th Shimura lift** is the map

$$\mathcal{S}_D : J_{k+1,1}^{\mathrm{cusp}} \longrightarrow S_{2k}(\mathrm{SL}_2(\mathbb{Z})), \quad f \mapsto \langle f, \Omega_{2k,D}(\tau, z, -\overline{w})\rangle.$$

[4] This is a special case of Proposition 2 of section 1.2 of Gross–Kohnen–Zagier.

Using the Fourier expansion of $\Omega_{2k,D}(\tau,z,w)$ with respect to w, we immediately obtain the following formula:

Theorem 7.21. *Suppose f is a Jacobi eigenform with eigenvalues λ_N and Fourier coefficients $c_f(D)$ (i.e. $c_f(D)$ is the coefficient of any $q^n\zeta^r$ with $r^2-4n=D$). Then*

$$\mathcal{S}_D(f) = 4^{1-k}(-i)^{k+1}\binom{2k-2}{k-1}\pi\cdot c_f(D)\cdot\sum_{N=1}^\infty \lambda_N e^{2\pi i N w}.$$

Proof. Recall that

$$\Omega_{2k,D}(\tau,z,w) = \frac{(2i)^{k+1}\pi^k}{(k-1)!}|D|^{k-1/2}\cdot\sum_{N=1}^\infty T_N P_{k+1,1;D}(\tau,z)e^{2\pi i N w}.$$

Since f is an eigenform and all T_N are self-adjoint, we have

$$\langle f, T_N P_{k+1,1;D}\rangle = \langle T_N f, P_{k+1,1;D}\rangle = \lambda_N c_f(D)\cdot\frac{\Gamma(k-1/2)}{2\pi^{k-1/2}}|D|^{1/2-k}.$$

Therefore

$$\begin{aligned}\mathcal{S}_D(f) &= \frac{(-2i)^{k+1}\pi^k|D|^{k-1/2}}{(k-1)!}\cdot\sum_{N=1}^\infty\langle f, T_N P_{k+1,1;D}\rangle e^{2\pi i N w}\\ &= \frac{(-2i)^{k+1}\pi^k D^k}{(k-1)!\sqrt{|D|}}\cdot\frac{\Gamma(k-1/2)}{2\pi^{k-1/2}}|D|^{1/2-k}\cdot c_f(D)\cdot\sum_{N=1}^\infty\lambda_N e^{2\pi i N w}.\end{aligned}$$ ☐

Since $\Gamma(k-1/2) = \frac{(2k-2)!}{4^{k-1}(k-1)!}\sqrt{\pi}$, this simplifies to

$$4^{1-k}(-i)^{k+1}\frac{(2k-2)!}{(k-1)!^2}\pi\cdot c_f(D)\cdot\sum_{N=1}^\infty\lambda_N e^{2\pi i N w}.$$

Corollary 7.22. *Suppose f is a Jacobi eigenform of weight $k+1$ and index 1 with eigenvalues λ_N. Then $\sum_{N=1}^\infty \lambda_N e^{2\pi i N w}$ is a cusp form for $\mathrm{SL}_2(\mathbb{Z})$ of weight $2k$.*

Proof. This almost follows immediately from the theorem, but we still have to show that there is some fundamental discriminant D for which $c_f(D)\neq 0$. If all $c_f(D)$, D a fundamental discriminant, were zero then the formula

$$T_N f(\tau,z) = \sum_D b_f(D)q^n\zeta^r,$$

where

$$0 = b_f(D) = \sum_{\substack{a|N^2\\ (N/a)^2 D\in\Delta}} \varepsilon_D(a)a^{k-1}c_f((N/a)^2 D)$$

for all fundamental discriminants, implies that $c_f(D)=0$ for all non-fundamental discriminants as well. So f is identically zero, which is a contradiction. ☐

Corollary 7.23. *Every Shimura lift*

$$\mathcal{S}_D : J_{k+1,1}^{\mathrm{cusp}} \longrightarrow S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$$

defines a map that sends a Jacobi eigenform of weight $k+1$ and index 1 to a classical eigenform of weight $2k$ with the same eigenvalues. For each k there is a linear combination of the $\mathcal{S}_D$ that is an isomorphism.

8. Jacobi forms and lattices

8.1. Integral lattices

Definition 8.1. An **integral lattice** is a free $\mathbb{Z}$-module L of finite rank, together with an integer-valued, symmetric, nondegenerate bilinear form

$$\langle -, - \rangle : L \times L \longrightarrow \mathbb{Z}.$$

By choosing a basis we may assume without any loss of generality that $L = \mathbb{Z}^n$ with bilinear form

$$\langle x, y \rangle = x^T S y$$

for some matrix S (the Gram matrix of the bilinear form in the basis). The conditions integral, symmetric, nondegenerate are equivalent to $S \in \mathbb{Z}^{n \times n}$ satisfying $S^T = S$ and $\det(S) \neq 0$.

The lattice L is called **even** if $\langle x, x \rangle \in 2\mathbb{Z}$ for every $x \in L$ (equivalently, if its Gram matrix in any basis has even numbers on the diagonal). In this case the **quadratic form** attached to L is

$$Q_L : L \longrightarrow \mathbb{Z}, \quad Q_L(x) = \frac{1}{2}\langle x, x \rangle.$$

By the polarization identity

$$\langle x, y \rangle = Q_L(x + y) - Q_L(x) - Q_L(y)$$

for every $x, y \in L$. The Gram matrix with respect to any basis is recovered as the Hessian matrix of Q_L in those coordinates.

The dual lattice of L is

$$L' = \{y \in L \otimes \mathbb{Q} : \ \langle x, y \rangle \in \mathbb{Z} \text{ for every } x \in L\}.$$

(L' is usually not integral with respect to $\langle -, - \rangle$.) There is a natural identification of $\mathbb{Z}$-modules

$$L' \cong \mathrm{Hom}(L, \mathbb{Z})$$

under which $y \in L'$ corresponds to the map $x \mapsto \langle x, y \rangle$. Clearly $L \subseteq L'$ is a subgroup; the index is denoted by

$$\det(L) = |L'/L|$$

and it is $|\det(S)|$ for any Gram matrix S for L. (In the exceptional case $L = \{0\}$ with basis $\emptyset$ we define $\det(L) = 1$.)

For a fixed lattice L (integral or not) and $N \neq 0$ we write $L(N)$ for the $\mathbb{Z}$-module L equipped with the bilinear form

$$(x, y) \mapsto N \cdot \langle x, y \rangle.$$

The lattice L is called N-modular if $L' = L(N)$; if $N = 1$ (by far the most important case) it is called unimodular.

After changing coefficients to $\mathbb{R}$, it is always possible to find an orthogonal basis $v_1, ..., v_n$ where $Q_L(v_1) = ... = Q_L(v_r) = 1$ and $Q_L(v_{r+1}) = ... = Q_L(v_n) = -1$. The integers r and $s = n - r$ are uniquely determined and are called the *signature* of L.

L is positive-definite if its signature is $(n, 0)$ with $n \in \mathbb{N}_0$.

8.2. Jacobi forms of lattice index

Let L be a positive definite even lattice and let $L_{\mathbb{C}} = L \otimes \mathbb{C}$ be the $\mathbb{C}$-vector space spanned by L.

Experience shows that many interesting Jacobi forms come in families indexed by the vectors of a lattice. So we make the following definition:

Definition 8.2. An unrestricted **Jacobi form** of weight k and lattice index L is a holomorphic function

$$f : \mathbb{H} \times L_{\mathbb{C}} \longrightarrow \mathbb{C}$$

with the following property: for any lattice vector $v \in L$, the function

$$f_v : \mathbb{H} \times \mathbb{C} \longrightarrow \mathbb{C}, \quad f_v(\tau, z) := f(\tau, v \otimes z)$$

is an unrestricted Jacobi form of weight k and index $Q_L(v)$.

Recall that unrestricted means without any vanishing condition for Fourier coefficients.

Example 8.3. (Unrestricted) Jacobi forms f of index $m \in \mathbb{N}$ are exactly the same as (unrestricted) Jacobi forms of lattice index L_m (and the same weight), where L_m is the lattice $\mathbb{Z}$ with the quadratic form

$$Q_{L_m}(x) = m \cdot x^2.$$

The defining property 8.2 is that $f(\tau, nz)$ must be a Jacobi form of index mn^2 for every $n \in \mathbb{Z}$, which is true: these are the images of f under the operator U_n.

Before going any further we state the following lemma for completeness, which implies that f is determined uniquely by its evaluations f_v along lattice vectors:

Lemma 8.4. *Suppose $f : L_{\mathbb{C}} \to \mathbb{C}$ is a holomorphic function with the property $f(v \otimes z) \equiv 0$ for every lattice vector $v \in L$. Then $f \equiv 0$ identically.*

Proof. f is zero on $L \otimes \mathbb{Q}$ since every element of $L \otimes \mathbb{Q}$ is already a pure tensor i.e. of the form $v \otimes z$ with $v \in L$ and $z \in \mathbb{Q}$. By continuity it is zero on $L \otimes \mathbb{R}$. By the identity theorem it is identically zero on $L \otimes \mathbb{C}$. □

Proposition 8.5. *Let $f : \mathbb{H} \times L_{\mathbb{C}} \to \mathbb{C}$. The following are equivalent:*
(i) f is an unrestricted Jacobi form of weight k and lattice index L;
(ii) f satisfies the transformation laws

$$f\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) = (c\tau + d)^k e^{\pi i c\langle z,z\rangle/(c\tau+d)} f(\tau, z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

and

$$f(\tau, z + \lambda\tau + \mu) = e^{-\pi i\tau\langle\lambda,\lambda\rangle - 2\pi i\langle\lambda,z\rangle} f(\tau, z), \quad \lambda, \mu \in L.$$

Proof. By the Lemma, condition (ii) is equivalent to each evaluation f_v along $v \otimes \mathbb{C}$ satisfying the transformation laws

$$f_v\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) = (c\tau + d)^k e^{2\pi i Q_L(v) cz^2/(c\tau+d)} f_v(\tau, z)$$

and

$$f_v(\tau, z + \lambda\tau + \mu) = e^{-2\pi i Q_L(v)\lambda^2\tau - 4\pi i Q_L(v)z} f(\tau, z), \quad \lambda, \mu \in \mathbb{Z},$$

where z now belongs to $\mathbb{C}$. In other words, it is equivalent to f_v being an (unrestricted) Jacobi form of index $Q_L(v)$ for every $v \in L$. □

The transformation laws $f(\tau + 1, z) = f(\tau, z)$ and $f(\tau, z + \mu) = f(\tau, z)$ (for $\mu \in L$) imply that f has a Fourier decomposition with respect to both variables, with the expansion with respect to z running over $\mathrm{Hom}(L, \mathbb{Z}) = L'$:

$$f(\tau, z) = \sum_{n\in\mathbb{Z}} \sum_{r\in L'} c_f(n, r) e^{2\pi i n\tau + 2\pi i\langle r,z\rangle}, \quad c_f(n, r) \in \mathbb{C}.$$

It is convenient to use the notation

$$f(\tau, z) = \sum_{n\in\mathbb{Z}} \sum_{r\in L'} c_f(n, r) q^n \zeta^r,$$

where $q = e^{2\pi i\tau}$ and where ζ^r formally stands for $e^{2\pi i\langle r,z\rangle}$.

Remark 8.6. For Jacobi forms in the usual sense (index $m \in \mathbb{N}$) this is a slightly different way of writing the Fourier series. The lattice $L_m = \mathbb{Z}$ with quadratic form mx^2 has bilinear form $\langle r, z\rangle = 2mrz$ and dual lattice $L_m' = \frac{1}{2m}\mathbb{Z}$. So the expansion above becomes

$$f(\tau, z) = \sum_{n\in\mathbb{Z}} \sum_{r\in\frac{1}{2m}\mathbb{Z}} c_f(n, r) q^n e^{2\pi i\cdot(2mrz)}.$$

Definition 8.7. Let $f(\tau, z) = \sum_{n \in \mathbb{Z}} \sum_{r \in L'} c_f(n, r) q^n \zeta^r$ be an unrestricted Jacobi form of lattice index L.
(i) f is a weak Jacobi form if $c_f(n, r) = 0$ whenever $n < 0$.
(ii) f is a (holomorphic) Jacobi form if $c_f(n, r) = 0$ whenever $n < Q_L(r)$.
(iii) f is a Jacobi cusp form if $c_f(n, r) = 0$ whenever $n \leq Q_L(r)$.

Note that the Fourier series of f_v is just

$$f_v(\tau, z) = \sum_{n \in \mathbb{Z}} \sum_{r \in L'} c_f(n, r) q^n \zeta^{\langle r, v \rangle} = \sum_{n \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} \Big(\sum_{\langle r, v \rangle = \ell} c_f(n, r) \Big) q^n \zeta^\ell.$$

If f satisfies (i) then clearly each f_v is a weak Jacobi form.

Suppose f satisfies (ii) and $v \neq 0$. Suppose (n, ℓ) is an index such that $4Q_L(v)n - \ell^2 < 0$, and $r \in L'$ with $\langle r, v \rangle = \ell$. By the Cauchy–Schwarz inequality,

$$4Q_L(v)n < \ell^2 = \langle r, v \rangle^2 \leq \langle r, r \rangle \cdot \langle v, v \rangle = 4Q_L(v)Q_L(r),$$

and therefore $n - Q_L(r) > 0$. Therefore $c_f(n, r) = 0$ for all such r, so f_v is a holomorphic Jacobi form.
Similarly, if f satisfies (iii) then every f_v is a Jacobi cusp form.

Jacobi forms behave well with respect to embeddings of lattices:

Proposition 8.8. *Suppose L and M are positive-definite even lattices and that $\iota : L \to M$ is an isometric embedding, i.e. $Q_L(x) = Q_M(\iota(x))$ for every $x \in L$. If f is an unrestricted Jacobi form of weight k and lattice index M, then*

$$\iota^* f(\tau, z) := f(\tau, \iota z)$$

is an unrestricted Jacobi form of weight k and lattice index L. If f is weak / holomorphic / cusp then $\iota^* f$ is also.

Proof. For any $v \in L$ the "evaluation along v" of $\iota^* f$ is just

$$(\iota^* f)_v = f_{\iota v},$$

hence an unrestricted Jacobi form of index $Q_M(\iota v) = Q_L(v)$. The vanishing conditions for the Fourier coefficients of $\iota^* f$ follow easily from those of f. □

In terms of Fourier series, remember that ι induces a dual map

$$\iota^* : M' \cong \mathrm{Hom}(M, \mathbb{Z}) \longrightarrow \mathrm{Hom}(L, \mathbb{Z}) \cong L'$$

defined by $\langle \iota^* x, y \rangle = \langle x, \iota y \rangle$ for all $x \in M'$ and $y \in L$. We have

$$\iota^* f(\tau, z) = \sum_{n \in \mathbb{Z}} \sum_{r \in L'} \Big(\sum_{\substack{\ell \in M' \\ \iota^* \ell = r}} c_f(n, \ell) \Big) q^n \zeta^r.$$

Example 8.9. In particular the orthogonal group

$$\mathrm{O}(L) = \{\text{linear maps } \sigma : L \to L \text{ such that } Q_L \sigma = Q_L\}$$

of L acts on Jacobi forms of that lattice index by

$$\sigma^* f(\tau, z) := f(\tau, \sigma z),$$

and if $f(\tau, z) = \sum_{n,r} c_f(n,r) q^n \zeta^r$ then

$$\sigma^* f(\tau, z) = \sum_{n,r} c_f(n, \sigma^{-1} r) q^n \zeta^r.$$

8.3. Properties of Jacobi forms of lattice index

Many of the properties of Jacobi forms carry over to the general setting of Jacobi forms of lattice index (often by slightly modifying the proof). We will mostly omit proofs here.

Let L be an even integral lattice.

Proposition 8.10. *For every integer k, the space $J^{\mathrm{weak}}_{k,L}$ of weak Jacobi forms of lattice index L is finite-dimensional. It contains the subspaces $J_{k,L}$ and $J^{\mathrm{cusp}}_{k,L}$ of Jacobi forms and Jacobi cusp forms.*
If $k < \frac{1}{2}\mathrm{rank}(L)$ then $J_{k,L} = J^{\mathrm{cusp}}_{k,L} = \{0\}$.

The point is that Jacobi forms of weight k can be identified with certain vector-valued modular forms of weight $k - \frac{1}{2}\mathrm{rank}(L)$, which do not exist if that number is negative. There do exist nontrivial examples of Jacobi forms of weight exactly $k = \frac{1}{2}\mathrm{rank}(L)$.

Viewing Jacobi forms as modules over the graded ring of modular forms, we have the following:

Proposition 8.11. *The $\mathbb{C}[E_4, E_6]$-modules*

$$J^{\mathrm{weak}}_{*,L} = \bigoplus_{k \in \mathbb{Z}} J^{\mathrm{weak}}_{k,L}$$

as well as $J_{*,L}$ and $J^{\mathrm{cusp}}_{*,L}$ are free with $\det(L)$ generators.

This implies (but is not implied by) the fact that there are Laurent polynomials $P^{\mathrm{weak}}(t) \in \mathbb{Z}[t, t^{-1}]$ and $P(t) \in \mathbb{Z}[t]$ such that

$$\sum_{k \in \mathbb{Z}} \dim J^{\mathrm{weak}}_{k,L} \cdot t^k = \frac{P^{\mathrm{weak}}(t)}{(1-t^4)(1-t^6)} \quad \text{and} \quad \sum_{k=0}^{\infty} \dim J_{k,L} \cdot t^k = \frac{P(t)}{(1-t^4)(1-t^6)}.$$

One major difference between regular Jacobi forms and lattice-index Jacobi forms is that the latter do not have an obvious ring structure. But there are various senses in which lattice-index Jacobi forms can be multiplied.

The following definition is natural:

Definition 8.12. Let L and M be positive-definite even lattices and $k, \ell \in \mathbb{Z}$. Then there are maps

$$\otimes : J^{\mathrm{weak}}_{k,L} \otimes J^{\mathrm{weak}}_{\ell,M} \longrightarrow J^{\mathrm{weak}}_{k+\ell,L\oplus M}$$

defined by

$$(f \otimes g)(\tau, z \oplus w) = f(\tau, z) \cdot g(\tau, w).$$

If f and g are holomorphic Jacobi forms then $f \otimes g$ is also a holomorphic Jacobi form. If in addition either f or g is a cusp form then $f \otimes g$ is a cusp form.

Proposition 8.13. [a] *$\otimes$ defines an isomorphism of $\mathbb{C}[E_4, E_6]$-modules*

$$\otimes : J^{\mathrm{weak}}_{*,L} \otimes_{\mathbb{C}[E_4,E_6]} J^{\mathrm{weak}}_{*,M} \xrightarrow{\sim} J^{\mathrm{weak}}_{*,L\oplus M}.$$

[a] Theorem 2.4 of Wang, H. and Williams, B. *On weak Jacobi forms of rank two.* J. Algebra 634 (2023), 722–754

In particular the Laurent polynomials $P^{\mathrm{weak}}_L(t)$ satisfy $P^{\mathrm{weak}}_{L\oplus M} = P^{\mathrm{weak}}_L \cdot P^{\mathrm{weak}}_M$.

Proposition 8.14. *For any even integer k with $k > 2 + \frac{1}{2}\mathrm{rank}(L)$, the Eisenstein series*

$$E_{k,L}(\tau, z) := \sum_{\substack{c,d\in\mathbb{Z} \\ \gcd(c,d)=1 \\ c>0 \text{ or } c=0,d=1}} (c\tau + d)^{-k} e^{-2\pi i c Q_L(z)/(c\tau+d)} \sum_{\lambda \in L} e^{2\pi i \frac{a\tau+b}{c\tau+d} Q_L(\lambda) + 2\pi i \langle \lambda, z\rangle/(c\tau+d)}$$

converges absolutely and defines a (holomorphic) Jacobi form of weight k and lattice index L. All of its Fourier coefficients are rational numbers. It satisfies $\sigma^* E_{k,L} = E_{k,L}$ for every $\sigma \in \mathrm{O}(L)$. The "singular Fourier coefficients" (meaning $c_f(n, r)$ where $n = Q_L(r)$) are 1 if $r \in L$ and 0 if $r \notin L$.

There is a formula for the Fourier coefficients of $E_{k,L}$[1] but they can sometimes be computed in a more elementary way by evaluating along lattice vectors.

Example 8.15. Let $L = A_2$ be the lattice $\mathbb{Z}^2$ with Gram matrix $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. The dual lattice L' consists of vectors $(a/3, b/3)$ with $a = -b \bmod 3$. There is one vector $r \in L'$ of norm 0 (the zero vector); there are 6 vectors

$$r = \pm(1/3, 2/3), (2/3, 1/3), (1/3, -1/3) \in L'$$

[1] J.H. Bruinier and M. Kuss, *Eisenstein series attached to lattices and modular forms on orthogonal groups*, Manuscripta Math. 106 (2001), 443–459

with $\langle r,r\rangle = 2/3$; and 6 vectors with

$$r = \pm(1,0), (0,1), (1,1) \in L'$$

with $\langle r,r\rangle = 2$ (i.e. roots), and in these cases the vectors of the same norm are equivalent under $\mathrm{O}(L)$. Due to the vanishing condition on the Fourier coefficients, the Eisenstein series $E_{4,L}$ has Fourier expansion of the form

$$E_{4,L}(\tau,z) = 1 + \Big(\sum_{\langle r,r\rangle=2} \zeta^r + A\cdot \sum_{\langle r,r\rangle=2/3} \zeta^r + B\Big)q + O(q^2)$$

for some constants A and B.

Suppose we fix a root $r_0 \in L$ of A_2. Then the inner products $\langle r_0, r\rangle$ with the six vectors with $\langle r,r\rangle = 2/3$ are $-1,-1,0,0,1,1$ and the inner products $\langle r_0,r\rangle$ with the roots $\langle r,r\rangle = 2$ are $-2,-1,-1,1,1,2$. So evaluating $E_{4,L}$ along r_0 yields

$$E_{4,L}(\tau, r_0\otimes z) = 1 + \Big(\zeta^{-2} + 2\zeta^{-1} + 2\zeta + \zeta + A\cdot(2\zeta^{-1} + 2 + 2\zeta) + B\Big)q + O(q^2).$$

Since this is a Jacobi form of weight 4 and index $Q_L(r_0) = 1$, it must be exactly the Jacobi Eisenstein series

$$E_{4,1}(\tau,z) = 1 + \Big(\zeta^{-2} + 56\zeta^{-1} + 126 + 56\zeta + \zeta^2\Big)q + O(q^2).$$

Comparing coefficients yields $2 + 2A = 56$ and $2A + B = 126$, i.e. $A = 27$ and $B = 72$, so

$$E_{4,L}(\tau,z) = 1 + \Big(\sum_{\langle r,r\rangle=2} \zeta^r + 27\cdot \sum_{\langle r,r\rangle=2/3} \zeta^r + 72\Big)q + O(q^2).$$

Proposition 8.16. *(i) For every $N \in \mathbb{N}$, the Hecke U-operator*

$$U_N : J^{\mathrm{weak}}_{k,L} \longrightarrow J^{\mathrm{weak}}_{k,L(N^2)}, \quad U_N f(\tau, z) := f(\tau, Nz)$$

is well-defined. It maps holomorphic and cusp Jacobi forms to holomorphic and cusp Jacobi forms.
(ii) For every $N \in \mathbb{N}$, the Hecke V-operator

$$V_N : J^{\mathrm{weak}}_{k,L} \longrightarrow J^{\mathrm{weak}}_{k,L(N)},$$

$$V_N f(\tau, z) := N^{k-1} \sum_{M \in \Delta_N} (c\tau + d)^{-k} e^{-2\pi i c Q_L(z)/(c\tau+d)} f\Big(\frac{a\tau + b}{c\tau + d}, \frac{Nz}{c\tau + d}\Big),$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ runs through representatives of $\{\det(M) = N\}$ modulo $\mathrm{SL}_2(\mathbb{Z})$, is well-defined. It maps holomorphic and cusp Jacobi forms to holomorphic and cusp Jacobi forms. If

$$f(\tau, z) = \sum_{n \in \mathbb{Z}} \sum_{r \in L'} c_f(n, r) q^n \zeta^r$$

then

$$V_N f(\tau, z) = \sum_{n,r} \Big(\sum_{a | (n,r,N)} a^{k-1} c_f\Big(\frac{Nn}{a^2}, \frac{r}{a}\Big)\Big) q^n \zeta^r,$$

where $a|(n, r, N)$ means that $n/a, N/a \in \mathbb{N}$ and $r/a \in L'$.

There are also self-adjoint Hecke operators T_N that map $J_{k,L}$ into itself but the convention for them depends strongly on whether $\mathrm{rank}(L)$ is even or odd.

8.4. Theta functions

Let L be a positive-definite even integral lattice.

The theta function

$$\theta_L(\tau) = \sum_{r \in L} q^{\langle r,r \rangle/2}$$

is the generating series whose coefficients count lattice vectors of a given norm, and it is known to be modular with respect to a subgroup of $\mathrm{SL}_2(\mathbb{Z})$: if N is the *level* of L,

$$N = \min\{N \in \mathbb{N} : \ L'(N) \text{ is an even integral lattice}\}$$

then θ_L is a modular form of weight $\frac{1}{2}\mathrm{rank}(L)$ for the subgroup $\Gamma_1(N)$.

We consider the "Jacobi" versions of these functions.

Definition 8.17. For $\tau \in \mathbb{H}$ and $z \in L \otimes \mathbb{C}$,

$$\theta_L(\tau, z) := \sum_{r \in L} q^{\langle r,r\rangle/2} \zeta^r = \sum_{r \in L} e^{\pi i \langle r,r\rangle \tau + 2\pi i \langle r,z\rangle}.$$

More generally, for cosets $\gamma \in L'/L$,

$$\theta_{L,\gamma}(\tau, z) := \sum_{r \in \gamma + L} q^{\langle r,r\rangle/2} \zeta^r = \sum_{r \in \gamma + L} e^{\pi i \langle r,r\rangle \tau + 2\pi i \langle r,z\rangle}.$$

The quadratic form $Q_L : L \to \mathbb{Z}$, $x \mapsto \frac{\langle x,x\rangle}{2}$ descends to a quadratic form

$$Q_L : L'/L \longrightarrow \mathbb{Q}/\mathbb{Z}, \quad x + L \mapsto Q_L(x) + \mathbb{Z};$$

in other words the exponents of q in $\theta_{L,\gamma}$ all have the same fractional part. That is what we need for the translation-invariance part of modularity:

$$\theta_{L,\gamma}(\tau + 1, z) = e^{2\pi i Q_L(\gamma)} \cdot \theta_{L,\gamma}(\tau, z).$$

Since r runs through $L + \gamma \subseteq L'$, we have $\langle r, \mu\rangle \in \mathbb{Z}$ for every $\mu \in L$, hence

$$\theta_{L,\gamma}(\tau, z + \mu) = \theta_{L,\gamma}(\tau, z).$$

Finally, substituting $z \mapsto z + \lambda\tau$ with $\lambda \in L$ leads to

$$\begin{aligned}
\theta_{L,\gamma}(\tau, z + \lambda\tau) &= \sum_{r \in \gamma + L} e^{\pi i \langle r, r + 2\lambda\rangle \tau + 2\pi i \langle r,z\rangle} \\
&= \sum_{r \in \gamma + L} e^{\pi i \langle r - \lambda, r + \lambda\rangle \tau + 2\pi i \langle r - \lambda, z\rangle} \quad (r \mapsto r - \lambda) \\
&= e^{-\pi i \langle \lambda,\lambda\rangle \tau - 2\pi i \langle \lambda, z\rangle} \cdot \theta_{L,\gamma}(\tau, z).
\end{aligned}$$

That covers the elementary transformation laws of $\theta_{L,\gamma}$. The behavior under

$$(\tau, z) \mapsto (-1/\tau, z/\tau)$$

is not as obvious and it will be useful to first introduce some notation. Let e_γ be formal basis elements attached to the cosets $\gamma \in L'/L$, such that

$$\mathbb{C}[L'/L] = \mathrm{span}(e_\gamma : \ \gamma \in L'/L).$$

We define the *discrete Fourier transform* on the space $\mathbb{C}[L'/L]$ to be the linear map $\mathcal{F}$ with

$$\mathcal{F} \cdot e_\gamma = \frac{1}{\sqrt{\det L}} \sum_{\beta \in L'/L} \exp\Big(-2\pi i \langle \gamma, \beta\rangle \Big) e_\beta.$$

So

$$\begin{aligned}
\mathcal{F}^2 \cdot e_\gamma &= \frac{1}{\det L} \sum_{\alpha \in L'/L} \sum_{\beta \in L'/L} \exp\big(-2\pi i \langle \gamma, \beta\rangle - 2\pi i \langle \beta, \alpha\rangle \big) e_\alpha \\
&= e_{-\gamma},
\end{aligned}$$

since $\sum_{\beta \in L'/L} \exp(-2\pi i \langle \gamma, \beta \rangle - 2\pi i \langle \beta, \alpha \rangle)$ is a character sum that vanishes unless $\langle \gamma + \alpha, \beta \rangle \equiv 0$ identically in β, i.e. if $\gamma + \alpha \in (L')' = L$.

Proposition 8.18 (Poisson summation for lattices)**.** *Suppose $h : L \otimes \mathbb{R} \to \mathbb{C}$ is a Schwartz function (smooth with rapidly decreasing derivatives of all orders) with Fourier transform*

$$\hat{h}(y) := \int_{L \otimes \mathbb{R}} h(x) e^{-2\pi i \langle x, y \rangle} \, \mathrm{d}x.$$

Here $\mathrm{d}x$ is the Haar measure that gives any fundamental domain of L the volume $\sqrt{\det L}$. Then

$$\sum_{x \in L'} h(x) e_{x+L} = \mathcal{F} \cdot \Big(\sum_{y \in L'} \hat{h}(y) e_{y+L} \Big).$$

Proof. Consider the function

$$f : L \otimes \mathbb{R} \longrightarrow \mathbb{C}[L'/L], \quad f(x) := \sum_{r \in L'} h(x + r) e_{r+L}.$$

Then $f(x + \mu) = f(x)$ for every $\mu \in L$, so f has a Fourier series

$$f(x) = \sum_{\mu \in L'} c(\mu) e^{2\pi i \langle \mu, x \rangle}, \quad c(\mu) \in \mathbb{C}[L'/L]$$

in which the coefficients are integrals

$$\begin{aligned} c(\mu) &= \frac{1}{\sqrt{\det L}} \int_{(L \otimes \mathbb{R})/L} f(x) e^{-2\pi i \langle \mu, x \rangle} \, \mathrm{d}x \\ &= \frac{1}{\sqrt{\det L}} \sum_{\gamma \in L'/L} \sum_{r \in L} \int_F h(x + \gamma + r) e_\gamma e^{-2\pi i \langle \mu, x \rangle} \, \mathrm{d}x \quad (F \text{ a fundamental domain for } L) \\ &= \frac{1}{\sqrt{\det L}} \sum_{\gamma \in L'/L} e^{2\pi i \langle \mu, \gamma \rangle} e_\gamma \int_{L \otimes \mathbb{R}} h(x) e^{-2\pi i \langle \mu, x \rangle} \, \mathrm{d}x \\ &= \frac{1}{\sqrt{\det L}} \sum_{\gamma \in L'/L} \hat{h}(-\mu) e^{2\pi i \langle \mu, \gamma \rangle} e_\gamma. \end{aligned}$$

Evaluating f at $x = 0$ yields

$$\begin{aligned} \sum_{r \in L'} h(r) e_{r+L} &= \sum_{\mu \in L'} c(\mu) = \sum_{\mu \in L'} c(-\mu) \\ &= \frac{1}{\sqrt{\det(L)}} \sum_{\mu \in L'} \sum_{\gamma \in L'/L} \hat{h}(\mu) e^{-2\pi i \langle \mu, \gamma \rangle} e_\gamma \\ &= \mathcal{F} \cdot \Big(\sum_{\mu \in L'} \hat{h}(\mu) e_{\mu+L} \Big). \end{aligned}$$

□

Proposition 8.19 (Theta transformation formula)**.** *The theta function*

$$\Theta_L : \mathbb{H} \times L_{\mathbb{C}} \longrightarrow \mathbb{C}[L'/L],$$

$$\Theta_L(\tau, z) := \sum_{\gamma \in L'/L} \theta_{L,\gamma}(\tau, z) e_\gamma$$

satisfies the formula

$$\Theta_L\Big(-\frac{1}{\tau}, \frac{z}{\tau}\Big) = \tau^{\mathrm{rank}(L)/2} e^{\pi i \langle z,z\rangle/\tau} \cdot e^{-\pi i \mathrm{rank}(L)/4} \cdot \mathcal{F}\Theta_L(\tau, z)$$

where $\mathcal{F}$ is the discrete Fourier transform on $\mathbb{C}[L'/L]$.

Proof. Fix $\tau \in \mathbb{H}$ and $z \in L \otimes \mathbb{C}$ and let h be the function

$$h : L \otimes \mathbb{R} \longrightarrow \mathbb{C}, \quad h(x) := e^{\pi i \langle x,x\rangle \tau + 2\pi i \langle x,z\rangle},$$

such that $\Theta_L(\tau, z) = \sum_{x \in L'} h(x) e_{x+L}$. The Fourier transform of h is

$$\hat{h}(y) = \int_{L\otimes\mathbb{R}} e^{\pi i \langle x,x\rangle\tau + 2\pi i\langle x,z\rangle - 2\pi i \langle x,y\rangle} \,\mathrm{d}x.$$

We complete the square by writing $x = u + \frac{y-z}{\tau}$ such that

$$\langle x, x\rangle \tau + 2\langle x, z-y\rangle = \langle u,u\rangle\tau - \frac{1}{\tau}\langle y-z, y-z\rangle$$

and then

$$\hat{h}(y) = e^{-\pi i \frac{1}{\tau}\langle y-z,y-z\rangle} \cdot \int_{L\otimes\mathbb{R}} e^{\pi i \langle u,u\rangle \tau} \,\mathrm{d}u.$$

Over $\mathbb{R}$, $\langle -, -\rangle$ is diagonalizable and the normalization of $\mathrm{d}u$ is such that the integral breaks up upon diagonalization into a product of simple Gauss integrals:

$$\begin{aligned}
\int_{L\otimes\mathbb{R}} e^{\pi i \langle u,u\rangle\tau}\,\mathrm{d}u &= \prod_{j=1}^{\mathrm{rank}(L)} \Big(\int_{-\infty}^{\infty} e^{\pi i x_j^2 \tau}\,\mathrm{d}x_i\Big) \\
&= \prod_{j=1}^{\mathrm{rank}(L)} \frac{1}{\sqrt{\tau/i}} \\
&= e^{\pi i \mathrm{rank}(L)/4} \cdot \tau^{-\mathrm{rank}(L)/2}.
\end{aligned}$$

Using Poisson summation for $\mathbb{C}[L'/L]$-valued functions we obtain

$$\begin{aligned}
\Theta_L(\tau, z) &= \mathcal{F} \cdot \Big(\sum_{y \in L'} \hat{h}(y) e_{y+L}\Big) \\
&= e^{\pi i \mathrm{rank}(L)/4} \tau^{-\mathrm{rank}(L)/2} \mathcal{F} \cdot \Big(\sum_{y\in L'} e^{-\pi i \frac{1}{\tau}\langle y-z,y-z\rangle} e_{y+L}\Big) \\
&= e^{\pi i \mathrm{rank}(L)/4} \tau^{-\mathrm{rank}(L)/2} e^{-\pi i \langle z,z\rangle/\tau} \mathcal{F} \cdot \Theta_L\Big(-\frac{1}{\tau}, \frac{z}{\tau}\Big).
\end{aligned}$$

The claim follows after substituting $(\tau, z) \mapsto (-1/\tau, z/\tau)$ and using $\Theta_L(\tau, -z) = \Theta_L(\tau, z)$. $\square$

In other words, Θ_L satisfies

$$\Theta_L(\tau+1, z) = \rho_L(T)\Theta_L(\tau, z), \quad \Theta_L\Big(-\frac{1}{\tau}, \frac{z}{\tau}\Big) = \rho_L(S)\Theta_L(\tau, z),$$

where $\rho_L(T)$ and $\rho_L(S)$ are the linear automorphisms of $\mathbb{C}[L'/L]$ defined by

$$\rho_L(T)e_\gamma = e^{2\pi i Q_L(\gamma)}e_\gamma, \quad \gamma \in L'/L$$

and

$$\rho_L(S)e_\gamma = e^{-\pi i \cdot (\mathrm{rank}(L)/4)}\mathcal{F}e_\gamma = \frac{e^{-\pi i \cdot (\mathrm{rank}(L)/4)}}{\sqrt{\det L}} \sum_{\beta \in L'/L} e^{-2\pi i \langle \gamma, \beta \rangle} e_\beta.$$

One can show that these fit together to a representation ρ_L of $\mathrm{Mp}_2(\mathbb{Z})$ (if this were not the case then a function transforming like Θ_L as above could not exist). Together with the behavior of Θ_L under lattice translations, we have:

Theorem 8.20. *Θ_L is a Jacobi form of weight $\frac{1}{2}\mathrm{rank}(L)$ and index L with respect to the multiplier system ρ_L. In other words*

$$\Theta_L\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) = (\sqrt{c\tau+d})^{\mathrm{rank}(L)} e^{\pi i c \langle z, z\rangle/(c\tau+d)} \cdot \rho\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau+d}\Big)\Theta_L(\tau, z)$$

for each $(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau+d})$ *in* $\mathrm{Mp}_2(\mathbb{Z})$, *and*

$$\Theta_L(\tau, z + \lambda\tau + \mu) = e^{-\pi i \langle \lambda, \lambda \rangle \tau - 2\pi i \langle \lambda, z \rangle}\Theta_L(\tau, z)$$

for any $\lambda, \mu \in L$.

The functions Θ_L occur in a generalization of the theta decomposition of Jacobi forms:

Theorem 8.21. *Let $\varphi \in J^{\mathrm{ur}}_{k,L}$ be an unrestricted Jacobi form of weight k and index L. Then there are uniquely determined functions $f_\gamma : \mathbb{H} \to \mathbb{C}$, $\gamma \in L'/L$ such that*
$$\varphi(\tau, z) = \sum_{\gamma \in L'/L} f_\gamma(\tau)\theta_{L,\gamma}(\tau, z)$$
and the vector function
$$F(\tau) = \sum_{\gamma \in L'/L} f_\gamma(\tau) e_\gamma$$
satisfies
$$F\Big(\frac{a\tau + b}{c\tau + d}\Big) = (\sqrt{c\tau + d})^{2k - \mathrm{rank}(L)} \overline{\rho}\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d}\Big) F(\tau) \tag{8.1}$$
for each $(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d})$ in $\mathrm{Mp}_2(\mathbb{Z})$. Conversely, if $F = (f_\gamma)_{\gamma \in L'/L}$ satisfies (8.1) then
$$\varphi(\tau, z) = \sum_{\gamma \in L'/L} f_\gamma(\tau)\theta_{L,\gamma}(\tau, z)$$
is an unrestricted Jacobi form.

Moreover, φ is a holomorphic Jacobi form if and only if F is bounded at ∞ and φ is a cusp form if and only if F vanishes at ∞.

Bear in mind however that restricting this equation along lines through lattice vectors $z \in v \cdot \mathbb{C}$ does not produce the theta decomposition of the index $Q_L(v)$ Jacobi forms
$$\varphi_v(\tau, z) = \varphi(\tau, v \otimes z).$$
The relationship between the modular forms $f_\gamma(\tau)$ and the theta decomposition of φ_v is not entirely trivial.

The proof of Theorem 8.21 is similar to theta decomposition in index $m \in \mathbb{N}$, so we omit it. Note that the Fourier expansion of F is
$$f_\gamma(\tau) = \sum_{n \in \mathbb{Z} - Q_L(\gamma)} c_\gamma(n) q^n,$$
where $c_\gamma(n) = c(n + \langle r, r\rangle/2, r)$ for any vector $r \in L + \gamma$ and where
$$\varphi(\tau, z) = \sum_{n \in \mathbb{Z}} \sum_{r \in L'} c(n, r) q^n \zeta^r.$$

8.5. Unimodular lattices

The results of the previous section have a special meaning in the case that L is a *unimodular* even positive-definite lattice: $L' = L$, as in this case the theta transformation formula does not involve vector-valued functions and multiplier systems.

Such lattices do exist. The most famous example is the E_8 root lattice, the maximal lattice containing $\mathbb{Z}^8$ that is even with respect to the quadratic form $Q(x_1, ..., x_8) = \sum x_i^2$. (Strictly speaking there are several such lattices but they are all equivalent.) Of course the rank zero lattice $\{0\}$ is also unimodular.

Even integral lattices that are unimodular are often also called Type II unimodular lattices (Type I meaning odd unimodular lattices).

For unimodular lattices the theta decomposition reduces to the following proposition:

Proposition 8.22. *Suppose L is an even positive-definite unimodular lattice.*
(i) The theta function

$$\theta_L(\tau, z) = \sum_{r \in L} q^{\langle r,r\rangle/2} \zeta^r, \quad \tau \in \mathbb{H},\ z \in L \otimes \mathbb{C}$$

is a holomorphic Jacobi form of weight $\frac{1}{2}\mathrm{rank}(L)$ and index L.
(ii) The holomorphic and weak Jacobi forms of index L are precisely the products of θ_L with modular forms:

$$J_{k,L} = J_{k,L}^{\mathrm{weak}} = \{f(\tau) \cdot \theta_L(\tau, z) : \ f \in M_{k-\mathrm{rank}(L)/2}(\mathrm{SL}_2(\mathbb{Z}))\}$$

for every $k \in \mathbb{Z}$.

In particular there are no weak Jacobi forms of unimodular index and negative weight. This is in stark contrast to the situation for Jacobi forms of index $m \in \mathbb{N}$ (or most lattice indices for that matter).

Proof. (i) Since L is even unimodular and positive-definite, its rank is a multiple of 8. Proof of that claim: suppose not. By passing from L to $L \oplus L$ if necessary we can assume that $\mathrm{rank}(L)$ is even; then the theta transformation formula does not involve square roots of $c\tau + d$ and ρ_L is a true representation (in fact a character) of $\mathrm{SL}_2(\mathbb{Z})$. By construction, the action of $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ through ρ_L is by

$$\rho_L(T)e_0 = e_0, \quad \rho_L(S)e_0 = e^{-\pi i \mathrm{rank}(L)/4} e_0,$$

where e_0 is the single basis element of $\mathbb{C}[L'/L] \cong \mathbb{C}$. But then $S^4 = (TS)^3 = I$ implies

$$\rho(S)^4 e_0 = \rho(S)^3 e_0 = \rho(I) e_0$$

and therefore $\rho(S)e_0 = e_0$; in other words $e^{-\pi i \cdot \mathrm{rank}(L)/4} = 1$.

So the representation ρ_L is trivial and the theta transformation formula simply says

$$\theta_L\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) = (c\tau + d)^{\mathrm{rank}(L)/2} e^{\pi i \langle z,z\rangle/\tau} \cdot \theta_L(\tau, z).$$

(ii) The vector-valued modular form attached to $\varphi \in J_{k,L}$ is $F(\tau) = f(\tau) \cdot e_0$, and its associated representation is $\overline{\rho_L}$ which is also trivial. So

$$f\Big(\frac{a\tau + b}{c\tau + d}\Big) = (c\tau + d)^{\mathrm{rank}(L)/2}.$$

Both growth conditions (holomorphic or weak Jacobi form) are equivalent to the Fourier series of f beginning in exponent q^0, i.e. $f \in M_{k-\mathrm{rank}(L)/2}$. $\square$

Example 8.23. With $L = E_8$, we have the theta series

$$\theta_{E_8}(\tau, z) = 1 + \underbrace{\Big(\sum_{\langle r,r\rangle=2} \zeta^r\Big)}_{240\,\text{terms}} q + \underbrace{\Big(\sum_{\langle r,r\rangle=4} \zeta^r\Big)}_{2160\,\text{terms}} q^2 + O(q^3) \in J_{4,E_8}.$$

For any fixed root $v \in E_8$, the restricted function $\theta_{E_8}(\tau, v \otimes z)$ is a Jacobi form of weight 4 and index 1 and therefore equals the Eisenstein series $E_{4,1}$:

$$\begin{aligned} &1 + (\zeta^{-2} + 56\zeta^{-1} + 126 + 56\zeta + \zeta^2)q \\ &\quad + (126\zeta^{-2} + 576\zeta^{-1} + 756 + 576\zeta + 126\zeta^2)q^2 + O(q^3). \end{aligned}$$

The coefficients of $\zeta^0 q^n$ in $E_{4,1}$ therefore count numbers of vectors in the E_8 lattice orthogonal to v with a given norm. In other words they count vectors in the E_7 root lattice of norm n.

Letting v instead have $\langle v, v\rangle = 4$ or $\langle v, v\rangle = 6$ gives a similar interpretation of the coefficients of the Jacobi Eisenstein series $E_{4,2}$ and $E_{4,3}$ as enumerating vectors in the D_7 and A_7 root lattices according to their norms.

Example 8.24. Suppose $L \neq \{0\}$ is a positive-definite even unimodular lattice of rank $N \in 8\mathbb{N}$ and let $v \in L$ be a vector, and define

$$f(\tau, z) := \theta_L(\tau, v \otimes z) \in J_{\frac{N}{2},\, \frac{1}{2}\langle v,v\rangle}.$$

So

$$f(\tau, z) = \sum_{n=0}^{\infty} \sum_r c(n, r) q^n \zeta^r, \quad c(n, r) = \#\{x \in L :\ \langle x, x\rangle = 2n,\ \langle x, v\rangle = r\}.$$

Then the first development coefficients of f are,

$$\begin{aligned} \mathcal{D}_0 f &= \sum_{n=0}^{\infty} \sum_r c(n, r) q^n \\ &= \sum_{x \in L} q^{\langle x,x\rangle/2} = \theta_L(\tau); \end{aligned}$$

$$\begin{aligned} \mathcal{D}_2 f &= \sum_{n=0}^{\infty} \sum_r (kr^2 - \langle v, v\rangle n) c(n, r) q^n, \quad k = \frac{1}{2}\mathrm{rank}(L) \\ &= \frac{1}{2} \sum_{x \in L} \Big(N\langle x, v\rangle^2 - \langle x, x\rangle\langle v, v\rangle\Big) q^{\langle x,x\rangle/2}; \end{aligned}$$

$$\begin{aligned}\mathcal{D}_4 f &= \sum_{n=0}^{\infty}\sum_r [(k+2)(k+1)r^4 - 12(k+1)r^2mn + 12m^2n^2]c(n,r)q^n \\ &= \frac{1}{4}\sum_{x\in L}\Big((N+4)(N+2)\langle x,v\rangle^4 - 6(N+2)\langle x,v\rangle^2\langle v,v\rangle\langle x,x\rangle + 3\langle v,v\rangle^2\langle x,x\rangle^2\Big)q^{\langle x,x\rangle/2},\end{aligned}$$

and $\mathcal{D}_\nu f$ is a modular form of weight $\frac{N}{2}+\nu$ (and a cusp form if $\nu \neq 0$).

These are generalized theta functions of the form

$$\theta_{L;P}(\tau) = \sum_{x\in L} P(x)q^{\langle x,x\rangle/2}$$

where P is a homogeneous polynomial (depending on the choice of v). The definition of $\mathcal{D}_\nu$ is such that the polynomials P that appear in this way are *spherical* for L, which means that in orthonormal coordinates x_i they are annihilated by the Laplace operator $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$, and which is the condition for the theta function $\theta_{L;P}$ to transform correctly under $\mathrm{SL}_2(\mathbb{Z})$. (A direct proof that these P are spherical is given in Theorem 7.2 in Eichler–Zagier.)

8.6. Root systems and Jacobi forms

Although the notion of Jacobi forms makes sense for any positive-definite lattice index, the most geometrically interesting cases are lattices attached to root systems. These lead to a significant generalization of the Jacobi triple product. We will follow Borcherds[2] and Gritsenko–Skoruppa–Zagier[3]

Let V be a finite-dimensional inner product space.

[2]Section 6 of Borcherds, R. *Automorphic forms on* $\mathrm{O}_{s+2,2}(\mathbf{R})$ *and infinite products.* Invent. Math. 120, 161–213 (1995)

[3]Sections 10 and 11 of Gritsenko, V. and Skoruppa, N.P. and Zagier, D. *Theta blocks.* J. Eur. Math. Soc., in press.

Definition 8.25. A **root system** $R \subseteq V$ is a set of vectors with the following properties:
(i) $\mathrm{span}(R) = V$;
(ii) No multiples of $r \in R$ belong to r other than $\pm r$;
(iii) For any roots $r, s \in R$, the number

$$\beta(r, s) := 2 \cdot \frac{\langle r, s \rangle}{\langle r, r \rangle} \in \mathbb{Z}$$

is integral;
(iv) For any root $r \in R$, the reflection

$$\sigma_r : V \to V,\ x \mapsto x - \frac{\langle x, r \rangle}{\langle r, r \rangle} r$$

maps R into R.

Root systems appear in a number of classification problems throughout mathematics; for example, they occur in the classification of semisimple Lie algebras, in the classification of du Val singularities of algebraic surfaces, and via the McKay correspondence in the classification of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$.

In most cases what we are actually interested in is a system of *positive roots*: a subset $R_+ \subseteq R$ that is closed under addition (i.e. if x and y are positive roots and $x+y$ is a root at all, then it is also positive) and that contains exactly one of each pair of roots $\pm r$. Such a set can be obtained by defining

$$R_+ = \{r \in R : \ \langle r, v \rangle > 0\}$$

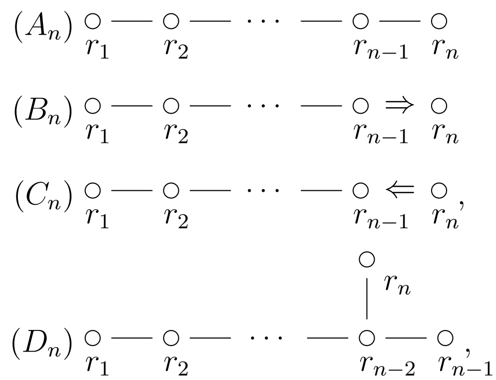
for a vector $v \in V$ that is not orthogonal to any root; the choice is not unique. Among a fixed set of positive roots, the *simple roots* Δ are those which cannot be written as the sum of two or more positive roots.

The *Dynkin diagram* is the partially directed graph whose vertices are the set of simple roots Δ and where $x, y \in \Delta$ are connected by

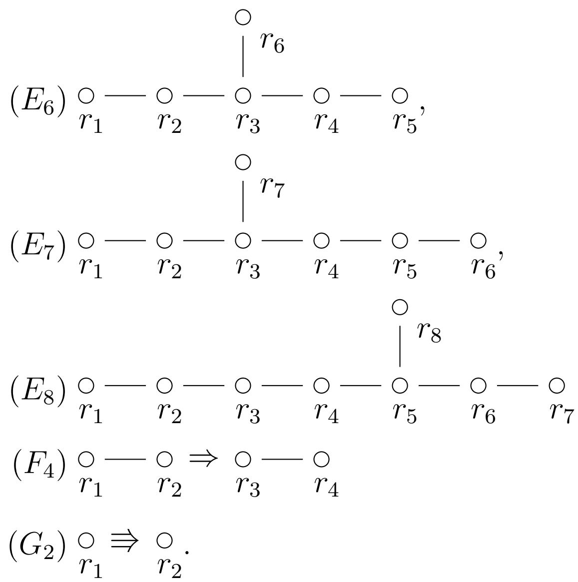
$$4 \frac{\langle x, y \rangle}{\langle x, x \rangle \cdot \langle y, y \rangle} = \beta(x, y)\beta(y, x) \in \mathbb{N}_0$$

edges. If in addition x is longer than y i.e. $\langle x, x \rangle > \langle y, y \rangle$ then any edges between x and y are directed from x towards y.

A root system R is called *reducible* if it can be partitioned as $R = R_1 \cup R_2$ with R_1, R_2 nonempty and $\langle x, y \rangle = 0$ for all $x \in R_1$, $y \in R_2$; otherwise it is *irreducible*. In this case the Dynkin diagram of R (with respect to any system of simple roots) is the disjoint union of those of R_1 and R_2. The Dynkin diagrams of irreducible root systems consist of four infinite families



and five exceptional examples



The *Weyl group* W_R is the group generated by the reflections σ_r along the roots $r \in R$.

The *root lattice* L_R associated to the root system R is the integral lattice in V spanned by the roots, with the bilinear form rescaled such that the shortest simple root r has $\langle r, r\rangle = 2$.

For an irreducible root system R and any system of positive roots R_+, define the number

$$h^\vee := \frac{1}{\operatorname{rank}(R)} \sum_{r \in R_+} \langle r, r\rangle.$$

(If the norm is rescaled such that the *longest* root has $\langle r, r\rangle = 2$, then $h^\vee$ is the *dual Coxeter number* of the root system.)

R	$\lvert R\rvert$	$\lvert W_R\rvert$	L_R	$h^\vee$	C	Simple Lie algebra
A_n	$n(n+1)$	$(n+1)!$	A_n	$n+1$	$n+1$	$\mathfrak{sl}_{n+1}$
B_n	$2n^2$	$2^n \cdot n!$	$A_1^{\oplus n}$	$2n-1$	$4n-2$	$\mathfrak{so}_{2n+1}$
C_n	$2n^2$	$2^n \cdot n!$	D_n	$n+1$	$2n+2$	$\mathfrak{sp}_{2n}$
D_n	$2n(n-1)$	$2^{n-1} \cdot n!$	D_n	$2n-2$	$2n-2$	$\mathfrak{so}_{2n}$
E_6	72	51840	E_6	12	12	$\mathfrak{e}_6$
E_7	126	2903040	E_7	18	18	$\mathfrak{e}_7$
E_8	240	696729600	E_8	30	30	$\mathfrak{e}_8$
F_4	48	1152	D_4	9	18	$\mathfrak{f}_4$
G_2	12	12	A_2	4	12	$\mathfrak{g}_2$

Table 8.1: Data for irreducible Dynkin diagrams. The index C is defined below.

Lemma 8.26. *Let $R \subseteq V$ be an irreducible root system with system of positive roots R_+. For any $x \in V$,*

$$\sum_{r \in R} \langle x, r\rangle^2 = 2 \sum_{r \in R_+} \langle x, r\rangle^2 = 2h^\vee \cdot \langle x, x\rangle.$$

Proof. Consider the function

$$f : V \longrightarrow \mathbb{R}, \quad x \mapsto \sum_{r \in R_+} \langle x, r\rangle^2 = \frac{1}{2} \sum_{r \in R} \langle x, r\rangle^2.$$

Then f is constant on the unit sphere $S = \{x : \ \langle x, x\rangle = 1\}$, because: suppose not. The vectors $x \in S$ where f takes its maximal value are precisely the eigenvectors for the maximal eigenvalue of a Gram matrix of the quadratic form f and therefore span a proper subspace of V. Since f is invariant under the Weyl group W_R, that subspace is invariant under W_R also. However the Weyl group of an irreducible root system acts on the ambient space V without proper invariant subspaces, which is a contradiction.

It follows that

$$\sum_{r \in R_+} \langle x, r\rangle^2 = C \cdot \langle x, x\rangle$$

for some constant C. To compute C we let $e_1, ..., e_n$ be any orthonormal basis of V; then

$$C \cdot \mathrm{rank}(R) = \sum_{i=1}^n C\langle e_i, e_i\rangle = \sum_{i=1}^n \sum_{r \in R_+} \langle e_i, r\rangle^2.$$

By the Pythagorean theorem $\sum_i \langle e_i, r\rangle^2 = \langle r, r\rangle$ for each root r, hence

$$\sum_{i=1}^n \sum_{r \in R_+} \langle e_i, r\rangle^2 = \sum_{r \in R_+} \langle r, r\rangle^2 = h^\vee \cdot \mathrm{rank}(R),$$

so $C = h^\vee$. $\square$

More generally, following Borcherds a multiset of positive vectors (vectors may occur with multiplicities) R in an even integral lattice L is called a *vector system of index* C if
(i) R spans L;
(ii) Each of $+r, -r$ occurs in R with equal multiplicity, for any $r \in L$;
(iii) The identity

$$\sum_{r \in R} \langle x, r \rangle^2 = 2C \cdot \langle x, x \rangle$$

holds for all $x \in L \otimes \mathbb{R}$.
(More rigorously, R is a set of vectors together with a multiplicity function $c : R \to \mathbb{N}_0$.) By abuse of notation the vectors $r \in R$ will still be called roots. A set R satisfying (i)-(iii) (or more precisely the nonzero vectors in R) is also called a *eutactic star*.

So an irreducible root system, scaled such that the shortest root has norm $\langle r, r \rangle = 2$, is a vector system of index

$$C = r \cdot h^\vee$$

where $h^\vee$ is the dual Coxeter number and $r \in \{1, 2, 3\}$ is the highest number of edges between two vertices in its Dynkin diagram. (Reducible root systems are not generally vector systems.) More generally, the weights of an irreducible representation of a simple Lie algebra form a vector system.

For any vector $v \in V$ not orthogonal to any nonzero $r \in R$, one obtains a splitting of R into positive and negative vectors and some number of copies of the zero vector:

$$R = R_+ \cup R_- \cup \{0, ..., 0\}, \quad \text{where } R_+ = \{r \in R : \ \langle r, v \rangle > 0\}.$$

The vector system identity has the following bilinear variant:

Lemma 8.27. *Suppose $R \subseteq L$ is a vector system of index C. Then*

$$\sum_{r \in R} \langle x, r \rangle \langle y, r \rangle = 2C \cdot \langle x, y \rangle$$

for all $x, y \in V$.

Proof. This is because

$$\begin{aligned} \langle x, y \rangle &= \frac{\langle x + y, x + y \rangle - \langle x, x \rangle - \langle y, y \rangle}{2} \\ &= C \sum_{r \in R} \langle x + y, r \rangle^2 - C \sum_{r \in R} \langle x, r \rangle^2 - C \sum_{r \in R} \langle y, r \rangle^2 \\ &= C \cdot \sum_{r \in R} \langle x, r \rangle \langle y, r \rangle. \end{aligned}$$

□

Lemma 8.28. *Suppose $R \subseteq L$ is a vector system. Then the index C is an integer.*

Proof. Let N be the g.c.d. of all inner products $\langle \lambda, \mu \rangle$ where $\lambda, \mu \in L$. Then

$$\langle x, r\rangle\langle y, r\rangle + \langle x, -r\rangle\langle y, -r\rangle \in 2N^2\mathbb{Z}$$

for every $x, y \in L$ and every nonzero root r. Taking the sum over all nonzero roots we obtain

$$2C\langle x, y\rangle \in 2N^2\mathbb{Z}$$

for every $x, y \in L$. By definition of N this means $2CN \in 2N^2\mathbb{Z}$ and therefore $C \in N\mathbb{Z}$. $\square$

Vector systems (and root systems) are related to weak Jacobi forms as follows:

Theorem 8.29. *Suppose L is a positive-definite even lattice and*

$$\varphi(\tau, z) = \sum_{n=0}^{\infty} \sum_{r \in L'} c(n, r) q^n \zeta^r \in J_{0,L}^{\text{weak}}$$

is a weak Jacobi form of weight 0. Then the coefficients $c(0, r)$ satisfy the identity

$$\sum_{r \in L'} c(0, r)\langle x, r\rangle^2 = 2C \cdot \langle x, x\rangle, \quad x \in L \otimes \mathbb{R}$$

with the number

$$C := \frac{1}{24} \sum_{r \in L'} c(0, r).$$

In particular if L has level N and all $c(0, r) \in \mathbb{N}_0$, then the rescaled dual lattice $L'(N)$, where each $r \in L'$ is counted with multiplicity $c(0, r)$, is a vector system R_φ in $L \otimes \mathbb{R}$ of index $N^2 \cdot C$.

Note that this vector system is finite since $c(0, r) = 0$ for all but finitely many $r \in L'$. In fact if $c(0, r) \neq 0$ and $x \in r + L$ is any other vector in the same L-coset, then the quasi-periodicity of φ implies

$$c(\langle x, x\rangle - \langle r, r\rangle, x) = c(0, r) \neq 0,$$

which (since φ is a weak Jacobi form) forces $\langle x, x\rangle \geq \langle r, r\rangle$. So R_φ consists at most of vectors $r \in L'$ which have minimal length within their coset $r + L$.

Proof. The proof follows Gritsenko[4]. Both sides of the claim define holomorphic functions of $x \in L \otimes \mathbb{C}$ so by the identity theorem 8.4 it is enough to prove this for lattice vectors $x = v \in L$. Consider the pullback function

$$\varphi_v(\tau, z) = \varphi(\tau, v \otimes z) = \sum_{n=0}^{\infty} \sum_{b \in \mathbb{Z}} \Big(\sum_{\substack{r \in L' \\ \langle r, v\rangle = b}} c(n, r) \Big) q^n \zeta^b$$

[4] Proposition 2.2 of V. Gritsenko, *24 faces of the Borcherds modular form* Φ_{12}, arXiv:1203.6503

which is a weak Jacobi form of weight 0 and index $\frac{1}{2}\langle v, v\rangle$. The modified Taylor coefficients (in the sense of section 5.1) are given by

$$c_n = \sum_{\substack{a,b\geq 0\\ 2a+b=n}} \Big(- \frac{\langle v, v\rangle}{24} \cdot \frac{1}{a!b!} E_2(\tau)\Big)^a D_z^b \varphi_v(\tau, 0)$$

where $D_z = \frac{1}{2\pi i}\frac{\partial}{\partial z}$, and they define modular forms of weight n for $\mathrm{SL}_2(\mathbb{Z})$. In particular c_2 is zero. But

$$c_2(\tau) = -\frac{\langle v, v\rangle}{24} E_2(\tau)\varphi_v(\tau, 0) + D_z^2 \varphi_v(\tau, 0)$$

and the constant term in its Fourier series is

$$0 = -\frac{\langle v, v\rangle}{24} \sum_{r\in L'} c(0, r) + \sum_{r\in L'} \langle r, v\rangle^2 c(0, r),$$

which is exactly what we wanted to prove. □

Example 8.30. The simplest example has $L = \mathbb{Z}$ with quadratic form x^2. The level is $N = 4$. In this case we have the weak Jacobi form

$$\phi_{0,1} \in J_{0,1}^{\text{weak}} = J_{0,L}^{\text{weak}}$$

of weight 0 and index 1 whose Fourier expansion (as a Jacobi form of lattice index) begins

$$\begin{aligned}\phi_{0,1}(\tau, z) &= \sum_{n=0}^{\infty} \sum_{r\in\frac{1}{2}\mathbb{Z}} c(0, r) q^n \zeta^r \\ &= (\zeta^{-1/2} + 10 + \zeta^{1/2}) + (10\zeta^{-1} - 64\zeta^{-1/2} + 108 - 64\zeta^{1/2} + 10\zeta)q + O(q^2).\end{aligned}$$

The vector system attached to $\phi_{0,1}$ consists of $\{\pm 1/2\}$ and 10 copies of the zero vector. After rescaling and throwing out the zero vector we have the A_1 root system.

The following theorem shows that conversely one can construct Jacobi forms out of vector systems. Recall that $\vartheta = \theta_{11}$ is the odd theta function

$$\vartheta(\tau, z) = \sum_{n\in\mathbb{Z}+1/2} (-1)^{n-1/2} e^{\pi i n^2\tau + 2\pi i n z} = \sum_{n\in\mathbb{Z}+1/2} (-1)^{n-1/2} q^{n^2/2} \zeta^n$$

and it satisfies the Jacobi triple product in the form

$$\vartheta(\tau, z) = q^{1/8}(\zeta^{1/2} - \zeta^{-1/2}) \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n\zeta)(1 - q^n\zeta^{-1}).$$

Let $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty}(1 - q^n)$ be the Dedekind eta function, such that $\eta^3 = \frac{1}{2\pi i}\vartheta'(\tau, 0)$.

Theorem 8.31. *Let $R \subseteq L$ be a vector system of index C and let $R_+ \subseteq R$ be a system of positive vectors. Let ρ be the* ***Weyl vector***

$$\rho := \frac{1}{2} \sum_{r \in R_+} r.$$

Then

$$\begin{aligned}\Psi_R(\tau, z) &= q^{(\# R_+)/12} \zeta^{-\rho} \prod_{r>0} \Big[(1-\zeta^r) \prod_{n=1}^{\infty} (1-q^n \zeta^r)(1-q^n \zeta^{-r}) \Big] \\ &= \pm \prod_{r \in R_+} \frac{\vartheta(\tau, \langle z, r \rangle)}{\eta(\tau)}\end{aligned}$$

is a weak Jacobi form of weight 0 and index $L'(C)$ for some character.

As always L' is the dual lattice. Note that $L'(C)$ might not be an even lattice, in which case Ψ_R transforms under lattice translations with a character as well.

Proof. Since $\frac{\vartheta(\tau+1,z)}{\eta(\tau+1)} = e^{\pi i/6} \frac{\vartheta(\tau,z)}{\eta(\tau)}$, we have

$$\Psi_R(\tau+1, z) = e^{\pi i (\# R_+)/6} \Psi_R(\tau, z).$$

Also, the theta transformation formula implies

$$\frac{\vartheta(-1/\tau, z/\tau)}{\eta(-1/\tau)} = e^{-\pi i/2} \cdot e^{\pi i z^2/\tau} \frac{\vartheta(\tau, z)}{\eta(\tau)}$$

and therefore

$$\begin{aligned}\Psi_R(-1/\tau, z/\tau) &= e^{-\pi i (\# R_+)/2} \cdot e^{\pi i \sum_{r \in R_+} \langle z, r \rangle^2/\tau} \Psi_R(\tau, z) \\ &= e^{-\pi i (\# R_+)/2} \cdot e^{\pi i C \langle z, z \rangle/\tau} \Psi_R(\tau, z).\end{aligned}$$

(In the last line we use the vector system identity.)

Finally we check the quasiperiod laws. For $\lambda \in L'$ we have

$$\begin{aligned}\Psi_R(\tau, z+\lambda\tau) &= \prod_{r \in R_+} \frac{\vartheta(\tau, \langle z, r \rangle + \langle \lambda, r \rangle \tau)}{\eta(\tau)} \\ &= \prod_{r \in R_+} (-1)^{\langle \lambda, r \rangle} e^{-\pi i \langle \lambda, r \rangle^2 \tau - 2\pi i \langle \lambda, r \rangle \langle z, r \rangle} \cdot \frac{\vartheta(\tau, z)}{\eta(\tau)} \\ &= (-1)^{\langle \lambda, 2\rho \rangle} e^{-\pi i C \langle \lambda, \lambda \rangle^2 \tau - 2\pi i C \langle \lambda, z \rangle} \cdot \Psi_R(\tau, z)\end{aligned}$$

as well as

$$\begin{aligned}\Psi_R(\tau, z+\lambda) &= \prod_{r\in R_+} \frac{\vartheta(\tau, \langle z, r\rangle + \langle \lambda, r\rangle)}{\eta(\tau)} \\ &= \prod_{r\in R_+} (-1)^{\langle \lambda, r\rangle} \frac{\vartheta(\tau, \langle z, r\rangle)}{\eta(\tau)} \\ &= (-1)^{\langle \lambda, 2\rho\rangle} \Psi_R(\tau, z).\end{aligned}$$

□

8.7. The Macdonald identities

In the previous section we constructed an infinite product Ψ_R attached to any vector system R and showed that it transforms like a Jacobi form of weight zero (up to some character). Now we work out in more detail what happens when R is an irreducible root system.

Let $R \subseteq V$ be an irreducible root system in the ambient vector space V with Weyl group W_R, and let C be the index of the associated vector system. Let L be the lattice generated by C times the coroots $r^\vee = \frac{2}{\langle r, r\rangle} r$ for $r \in R$:

$$L = \sum_{r\in R} \frac{2}{\langle r, r\rangle} C\mathbb{Z} \cdot r;$$

this is an integral (but not necessarily even) lattice with respect to the inner product

$$(x, y) := \frac{1}{C}\langle x, y\rangle$$

due to the identity

$$(x, y) = \frac{1}{C}\langle x, y\rangle = \frac{1}{C} \cdot \frac{1}{2C} \sum_{r\in R} \langle x, r\rangle \langle y, r\rangle,$$

the fact that $\langle x, r\rangle$ and $\langle y, r\rangle \in C\mathbb{Z}$ for each root r (as this is true if x or y is C times a coroot) and because roots $r \in R$ come in $\pm$ pairs.

Let $L_{\mathrm{ev}} \subseteq L$ be the even sublattice

$$L_{\mathrm{ev}} = \{x \in L : \ (x, x) \in 2\mathbb{Z}\};$$

this is either L itself (if L is even) or has index two in it (if L is odd).

Let $\rho = \frac{1}{2} \sum_{r\in R_+} r$ be the Weyl vector of R with respect to any system of positive roots.

Definition 8.32. Define the theta function

$$\Theta_R(\tau, z) := \sum_{\lambda \in \rho + L_{\mathrm{ev}}} q^{(\lambda,\lambda)/2} \sum_{g \in W} \mathrm{sgn}(g) \zeta^{g\lambda}, \quad \tau \in \mathbb{H},\ z \in L \otimes \mathbb{C}.$$

Here $\zeta^{g\lambda}$ means $e^{2\pi i (g\lambda, z)}$. The sign of $g \in W$ is the determinant of g as a map on V; it is $(-1)^n$ if g is a product of n reflections.

In other words,

$$\Theta_R = \sum_{g \in W} \mathrm{sgn}(g) \Theta_{L_{\mathrm{ev}},\rho}(\tau, gz)$$

is the theta function of L_{ev} attached to the coset of ρ, symmetrized over the Weyl group. Note that ρ belongs to the dual $(L_{\mathrm{ev}})'$ because: 2ρ certainly does (being an integer combination of roots) and

$$\begin{aligned} 2 \cdot (\rho, x) &= \sum_{r \in R_+} (r, x) \\ &\equiv \sum_{r \in R_+} (r, x)^2 \pmod 2 \\ &= \frac{1}{C^2} \sum_{r \in R_+} \langle r, x \rangle^2 \\ &= \frac{1}{C} \langle x, x \rangle = (x, x) \end{aligned}$$

is even for each $x \in L_{\mathrm{ev}}$ by the definition of L_{ev}. So $\Theta_{L_{\mathrm{ev}},\rho}$ is a well-defined theta function in the sense of the previous lectures.

Theorem 8.33. *Let R be an irreducible root system of rank N. Then Θ_R satisfies the transformation rules*
(i) $\Theta_R(\tau + 1, z) = e^{\pi i (\rho,\rho)} \Theta_R(\tau, z)$;
(ii) $\Theta_R(-1/\tau, z/\tau) = e^{-3\pi i(\rho,\rho)} \cdot \tau^{N/2} e^{\pi i (z,z)/\tau} \Theta_R(\tau, z)$;
(iii) $\Theta_R(\tau, z + \lambda\tau) = e^{-\pi(\lambda,\lambda)\tau - 2\pi i (\lambda, z)} \Theta_R(\tau, z), \quad \lambda \in L_{\mathrm{ev}}$;
(iv) $\Theta_R(\tau, z + \lambda) = \Theta_R(\tau, z), \quad \lambda \in L_{\mathrm{ev}}$.

Proof. (i), (iii) and (iv) are more or less trivial.
Point (ii) is essentially the theta transformation formula, but the fact that that formula reproduces Θ_R and not a sum involving all of the shifted theta series $\Theta_{L_{\mathrm{ev}},\gamma}$ ($\gamma \in L'_{\mathrm{ev}}/L_{\mathrm{ev}}$) depends on a property of the Weyl vector ρ and the action of the Weyl group (in the terminology of Gritsenko–Skoruppa–Zagier, the eutactic star R defined by an irreducible root system is *extremal*) and it is not trivial at all. See sections 10 and 11 of Gritsenko–Skoruppa–Zagier. □

Properties (i)-(iv) are the defining equations for a (holomorphic) Jacobi form of weight $N/2$ and lattice index L_{ev}, together with a character. Since the Dedekind eta function satisfies $\eta(\tau+1) = e^{\pi i/12}\eta(\tau)$ and $\eta(-1/\tau) = e^{-\pi i/4}\tau^{1/2}\eta(\tau)$, that character is the $12(\rho,\rho)$-th power of η's multiplier system.

By contrast, Theorem 8.31 shows that $\eta^N \cdot \Psi_R$ transforms like a Jacobi form of weight $N/2$ and lattice index L_{ev} (which is the even sublattice of the index indicated there) and the $(N+2\#R_+)$-th power of η's multiplier system.

The multiplier systems are the same by the following fact from Lie algebra theory:

Lemma 8.34 (Freudenthal–de Vries "strange formula"). *The Weyl vector satisfies*

$$(\rho,\rho) = \frac{N+2\#R_+}{12}.$$

If R is the root system of the simple Lie algebra $\mathfrak{g}$ then $N+2\#R_+$ is (by the root space decomposition) the dimension $\dim\mathfrak{g}$.

The construction of Θ_R is such that for any root $r \in R$ with attached reflection σ_r,

$$\Theta_R(\tau,\sigma_r z) = -\Theta_R(\tau,z).$$

So if z belongs to the hyperplane orthogonal to r then $\sigma_r z = z$ and therefore $\Theta_R(\tau,z) = 0$. By quasiperiodicity, $\Theta_R(\tau,z)$ vanishes more generally whenever $\langle r,z\rangle$ belongs to the lattice $\mathbb{Z}\oplus\mathbb{Z}\tau$.

The hyperplanes $r^\perp$ are pairwise distinct as r runs through a system of positive roots because no nontrivial multiples of r belong to R_+. Since $\vartheta(\tau,\langle z,r\rangle)$ has simple zeros exactly on the hyperplane $r^\perp$ and its translations, it follows that for any fixed $\tau\in\mathbb{H}$, the function

$$\frac{\Theta_R(\tau,z)}{\prod_{r\in R_+}\vartheta(\tau,\langle z,r\rangle)}$$

and therefore

$$\frac{\Theta_R(\tau,z)}{\Psi_R(\tau,z)}$$

is holomorphic in z. Since Θ_R and Ψ_R have the same transformation under lattice translations $z\mapsto z+\lambda\tau$ and $z\mapsto z+\lambda$ (for $\lambda\in L_{\mathrm{ev}}$) it follows that Θ_R/Ψ_R is a constant in z (but still depends on τ). Due to the behavior under $\mathrm{SL}_2(\mathbb{Z})$ it follows further that

$$\frac{\Theta_R(\tau,z)}{\eta^N(\tau)\Psi_R(\tau,z)}$$

is a modular function that is holomorphic for $\tau\in\mathbb{H}$, i.e. it belongs to the ring $\mathbb{C}[j]$, where j is the j-invariant.

Finally note that $\Theta_R(\tau,z)$ has q-expansion beginning in exponent $q^{(\rho,\rho)/2}$, i.e. ρ has minimal length in its L_{ev}-coset. This is related to extremality of the eutactic star R.

By the Freudenthal–de Vries strange formula, $q^{(\rho,\rho)/2} = q^{N/24+(\#R_+)/12}$ is exactly the leading exponent in the q-expansion of $\eta^N(\tau)\Psi_R(\tau,z)$. Therefore the quotient is actually a constant (which turns out to be 1). Altogether:

Theorem 8.35 (Macdonald identities)**.** *Let R be an irreducible root system with system of positive roots R_+ and Weyl vector $\rho = \frac{1}{2}\sum_{r\in R_+} r$. Then*

$$\eta(\tau)^{\mathrm{rank}(R)-\#R_+}\prod_{r\in R_+}\vartheta(\tau,(r,z)) = \sum_{\lambda\in\rho+L_{\mathrm{ev}}} q^{(\lambda,\lambda)/2}\sum_{g\in W}\det(g)\zeta^{g\lambda}.$$

Remark 8.36. Macdonald's identities include other identities attached to affine root systems which are not contained in the above result but also have Jacobi form interpretations. For example the Watson quintuple product

$$\begin{aligned}\eta(\tau)\frac{\vartheta(\tau,2z)}{\vartheta(\tau,z)} &= q^{1/24}(\zeta^{1/2}+\zeta^{-1/2})\prod_{n=1}^{\infty}(1-q^n)(1+q^n\zeta)(1+q^n\zeta^{-1})(1-q^{2n-1}\zeta^2)(1-q^{2n-1}\zeta^{-2})\\ &= \sum_{n=1}^{\infty}\chi(n)q^{n^2/24}(\zeta^{n/2}+\zeta^{-n/2}),\end{aligned}$$

where $\chi(n) = \left(\frac{n}{12}\right)$ is 1 if $n\equiv\pm1 \bmod 12$ and -1 if $n\equiv\pm5 \bmod 12$ and 0 otherwise, is the Macdonald identity attached to the affine root system of type BC_1.

Example 8.37. For $R = A_1$ viewed as $\mathbb{Z}$ with inner product $\langle x,x\rangle = 2x^2$, we have $\mathrm{rank}(R) = \#R_+ = 1$, and $L_{\mathrm{ev}} = 2\mathbb{Z}$. As positive roots take $R_+ = \{1\}$ with Weyl vector $\rho = 1/2$. The index $C = 2$ and the Weyl group is $W = \{\pm1\}$. The Macdonald identity is

$$\vartheta(\tau,z) = \sum_{\lambda\in\frac{1}{2}+2\mathbb{Z}} q^{\lambda^2/2}(\zeta^\lambda-\zeta^{-\lambda}).$$

Example 8.38. Let $R = A_2$ viewed as follows. Let $V\subseteq\mathbb{R}^3$ be the space of vectors whose entries sum to zero together with the Euclidean norm, and let

$$R = \pm(1,-1,0), \pm(0,1,-1), \pm(1,0,-1).$$

The choices with sign + form a system of positive roots with Weyl vector $\rho = (1,0,-1)$. The vector system has index $C = 3$. The lattice L_{ev} consists of vectors (a,b,c) with $a+b+c = 0$ and $a\equiv b\equiv c\equiv 0 \bmod 3$ together with the Euclidean inner product divided by 3, and the Weyl group is the group of permutations. Therefore the Macdonald identity for A_2 is

$$\begin{aligned}&\eta(\tau)^{-1}\vartheta(\tau,z_1-z_2)\vartheta(\tau,z_2-z_3)\vartheta(\tau,z_1-z_3)\\ = &\sum_{\substack{a,b,c\in\mathbb{Z}^3\\ a+b+c=0\\ (a,b,c)\equiv(1,0,-1) \bmod 3}} q^{(a^2+b^2+c^2)/6}\Big(\zeta_1^a\zeta_2^b\zeta_3^c+\zeta_1^b\zeta_2^c\zeta_3^a+\zeta_1^c\zeta_2^a\zeta_3^b-\zeta_1^a\zeta_2^c\zeta_3^b-\zeta_1^b\zeta_2^a\zeta_3^c-\zeta_1^c\zeta_2^b\zeta_3^a\Big)\end{aligned}$$

where ζ_j means $e^{2\pi i z_j}$.

8.8. Theta blocks

The fact that the left-hand side of the Macdonald identity for an irreducible root system R,

$$\eta(\tau)^{\mathrm{rank}(R)-\#R_+} \cdot \prod_{r \in R_+} \vartheta(\tau, \langle r, z \rangle)$$

defines a *holomorphic* Jacobi form leads to a powerful way to construct Jacobi forms of scalar index and low weight.

Another way to state it is as follows:

Theorem 8.39. *Let R be a root system of rank n attached to the semisimple complex Lie algebra $\mathfrak{g}$, let $R_+ \subseteq R$ be a system of positive roots and $\Delta = \{\alpha_1, ..., \alpha_n\} \subseteq R_+$ the system of simple roots. For any $r \in R_+$ let $\gamma_{r,i} \in \mathbb{N}_0$, $i = 1, ..., n$ be its coordinates with respect to Δ, such that*

$$r = \sum_{i=1}^n \gamma_{r,i} \alpha_i.$$

Then the function

$$\Phi_R(\tau, z) := \eta(\tau)^{n-\#R_+} \prod_{r \in R_+} \vartheta\Big(\tau, \sum_{i=1}^n \gamma_{r,i} z_i\Big)$$

defines a ***holomorphic*** *Jacobi form of weight $n/2$, with multiplier system the $N = \dim \mathfrak{g} = n + 2\#R_+$-th power of the η function's multiplier system, and of lattice index equal to $L = \mathbb{Z}^n$ with quadratic form*

$$\langle (x_1, ..., x_n), (x_1, ..., x_n) \rangle = \sum_{r \in R_+} \Big(\sum_{i=1}^n \gamma_{r,i} z_i \Big)^2.$$

Note that the irreducible case easily implies the general theorem, because if R splits orthogonally as $R_1 \cup R_2$ then $\Phi_R = \Phi_{R_1} \cdot \Phi_{R_2}$.

The practical aspect is that one can restrict along lattice vectors to produce lots of *holomorphic* Jacobi forms of scalar index, all having product expansions (due to the Jacobi triple product), and sometimes of quite low weight. This is the method of **theta blocks** of Gritsenko–Skoruppa–Zagier.

To construct Jacobi forms without a multiplier system, we have to restrict to semisimple Lie algebras whose dimension is a multiple of 24.

Among the semisimple Lie algebras of rank four, which produce Jacobi forms of weight two, there are exactly four whose dimension is a multiple of 24 (and in all four cases the dimension is exactly 24): namely A_4, $A_1 \oplus B_3$, $A_1 \oplus C_3$ and $B_2 \oplus G_2$. (Note

R	Simple Lie algebra	Dimension
A_n	$\mathfrak{sl}_{n+1}$	$n(n+2)$
B_n	$\mathfrak{so}_{2n+1}$	$n(2n+1)$
C_n	$\mathfrak{sp}_{2n}$	$n(2n+1)$
D_n	$\mathfrak{so}_{2n}$	$n(2n-1)$
E_6	$\mathfrak{e}_6$	78
E_7	$\mathfrak{e}_7$	133
E_8	$\mathfrak{e}_8$	248
F_4	$\mathfrak{f}_4$	52
G_2	$\mathfrak{g}_2$	14

Table 8.2: Dimensions of simple Lie algebras

that $B_2 = C_2$.) These yield the following four families of holomorphic Jacobi forms of weight two (where ϑ_n stands for $\vartheta(\tau, nz)$ for $n \in \mathbb{N}$):

(i) ($R = A_4$)

$$f(\tau, z) := \eta^{-6}\vartheta_a\vartheta_{a+b}\vartheta_{a+b+c}\vartheta_{a+b+c+d}\vartheta_b\vartheta_{b+c}\vartheta_{b+c+d}\vartheta_c\vartheta_{c+d}\vartheta_d \in J_{2,m}$$

where $m = 2a^2 + 3ab + 2ac + ad + 3b^2 + 4bc + 2bd + 3c^2 + 3cd + 2d^2$;

(ii) ($R = A_1 \oplus B_3$)

$$f(\tau, z) := \eta^{-6}\vartheta_a\vartheta_b\vartheta_{b+c}\vartheta_{b+c+d}\vartheta_{b+c+2d}\vartheta_{b+2c+2d}\vartheta_c\vartheta_{c+d}\vartheta_{c+2d}\vartheta_d \in J_{2,m}$$

where $m = \frac{a^2+5b^2+10bc+10bd+10c^2+20cd+15d^2}{2}$. (Note that m can be half-integral, in which case f transforms with a character under lattice translations.)

(iii) ($R = A_1 \oplus C_3$)

$$f(\tau, z) := \eta^{-6}\vartheta_a\vartheta_b\vartheta_{b+c}\vartheta_{b+c+d}\vartheta_{b+2c+d}\vartheta_{2b+2c+d}\vartheta_c\vartheta_{c+d}\vartheta_{2c+d}\vartheta_d \in J_{2,m}$$

where $m = \frac{1}{2}a^2 + 4b^2 + 8bc + 4bd + 8c^2 + 8cd + 3d^2$.

(iv) ($R = B_2 \oplus G_2$)

$$f(\tau, z) := \eta^{-6}\vartheta_a\vartheta_{a+b}\vartheta_{a+2b}\vartheta_b\vartheta_c\vartheta_{c+d}\vartheta_{c+2d}\vartheta_{c+3d}\vartheta_{2c+3d}\vartheta_d \in J_{2,m}$$

where $m = \frac{3}{2}a^2 + 3ab + 3b^2 + 4c^2 + 12cd + 12d^2$.

Example 8.40. Taking $a = b = c = d = 1$ in the theta block attached to A_4 produces the Jacobi form

$$f(\tau, z) = \eta^{-6}\vartheta_1^4\vartheta_2^3\vartheta_3^2\vartheta_4 \in J_{2,25}.$$

This is a one-dimensional space spanned by a Jacobi Eisenstein series $E_{2,25,\chi}$ attached to a primitive Dirichlet character χ mod 5, which implies the product formula

$$\begin{aligned} E_{2,25,\chi}(\tau, z) &= \frac{\vartheta^4(\tau, z)\vartheta^3(\tau, 2z)\vartheta^2(\tau, 3z)\vartheta(\tau, 4z)}{\eta^6(\tau)} \\ &= q(\zeta^{-1} - \zeta)^3(\zeta^{-2} - \zeta^2)(\zeta^{-1} + 1 + \zeta)^2(\zeta^{-1} - 2 + \zeta)^3 \\ &\quad \times \prod_{n=1}^{\infty} \Big[(1 - q^n)^4(1 - q^n\zeta)^4(1 - q^n\zeta^{-1})^4(1 - q^n\zeta^2)^3(1 - q^n\zeta^{-2})^3 \\ &\qquad \times (1 - q^n\zeta^3)^2(1 - q^n\zeta^{-3})^2(1 - q^n\zeta^4)(1 - q^n\zeta^{-4})\Big]. \end{aligned}$$

Example 8.41. Taking $a = b = c = 1$ and $d = 2$ in the theta block attached to A_4 produces the Jacobi cusp form

$$f(\tau, z) = \eta^{-6}\vartheta_1^3\vartheta_2^3\vartheta_3^2\vartheta_4\vartheta_5 \in J_{2,37} = J_{2,37}^{\text{cusp}}$$

of weight 2 and index 37.