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1. (a) For each of the following optimization problems, write a CVX code that solves it, if possible.  
Also write down the optimal value returned by CVX (corrected to 4 decimal places).

i. (10 points)

$$\begin{aligned} \text{Minimize } & \sqrt{x_1^2 + (x_2 - 5)^4 + (x_3 - 1)^6 + (x_4 + 1)^8 + 2 + |x_3 - x_4 + 7|} \\ \text{Subject to } & x_1^2 + x_2^2 + x_4^2 \leq 2. \end{aligned}$$

ii. (10 points)

$$\begin{aligned} \text{Minimize } & x_1^2 + 8x_2^2 + 9x_3^2 + 5x_1x_2 - x_1x_3 + 8(|x_1 - 1| + |x_2 + 3| + |x_3 - 5|) \\ \text{Subject to } & \begin{bmatrix} 5 \\ x_3^2 \\ x_2^2 \\ x_2 + 1 \end{bmatrix} \succeq 0. \end{aligned}$$

iii. (10 points)

$$\begin{aligned} \text{Minimize } & 2x_1 + 3x_2 - x_3 + \sqrt{x_1^2 + (x_2 - 5)^4 + (x_3 - 1)^6 + (x_4 + 1)^8 + 2}, \quad y \geq 0 \\ \text{Subject to } & \max\{x_1 + x_2, x_3 + x_4, x_1 + x_3\} \leq 2, \quad x_1 \geq 0, \quad x_2 \geq 0. \end{aligned}$$

1. (a) i) reformulate:

$$\text{Min } y + |x_3 - x_4 + 7|$$

$$\begin{aligned} \text{s.t. } & y^2 \geq x_1^2 + (x_2 - 5)^4 + (x_3 - 1)^6 + (x_4 + 1)^8 + 2, \quad y \geq 0 \\ & x_1^2 + x_2^2 + x_4^2 \leq 2 \end{aligned}$$

$$\Leftrightarrow \text{Min } y + |x_3 - x_4 + 7|$$

$$\begin{aligned} \text{s.t. } & z_2 \geq (x_2 - 5)^2, \quad z_3 \geq |x_3 - 1|^3, \quad z_4 \geq (x_4 + 1)^2, \\ & z_5 \geq z_4^2, \quad y \geq 0 \end{aligned}$$

$$\begin{aligned} & y^2 \geq x_1^2 + z_2^2 + z_3^2 + z_5^2 + 2 \\ & x_1^2 + x_2^2 + x_4^2 \leq 2 \end{aligned}$$

$$\Leftrightarrow \text{Min } y + |x_3 - x_4 + 7|$$

$$\begin{aligned} \text{s.t. } & z_2 \geq (x_2 - 5)^2, \quad p_1 \geq |x_3 - 1|, \quad p_1 \geq 0, \\ & z_3 \cdot p_1 \geq p_2^2, \quad p_2 \geq p_1^2, \quad z_3 \geq 0, \end{aligned}$$

$$\begin{aligned} & z_4 \geq (x_4 + 1)^2, \quad z_5 \geq z_4^2, \quad y \geq 0, \quad y^2 \geq x_1^2 + z_2^2 + z_3^2 + z_5^2 + 2 \\ & x_1^2 + x_2^2 + x_4^2 \leq 2. \end{aligned}$$

$$\Leftrightarrow \text{Min } y + |x_3 - x_4 + 7|$$

$$\text{s.t. } \begin{pmatrix} z_2 & x_2 - 5 \\ x_2 - 5 & 1 \end{pmatrix} \succeq 0, \quad p_1 \geq |x_3 - 1|, \quad p_1 \geq 0$$

$$\begin{pmatrix} z_3 & p_2 \\ p_2 & p_1 \end{pmatrix} \succeq 0, \quad \begin{pmatrix} p_2 & p_1 \\ p_1 & 1 \end{pmatrix} \succeq 0, \quad \begin{pmatrix} z_4 & (x_4 + 1) \\ (x_4 + 1) & 1 \end{pmatrix} \succeq 0$$

$$\begin{pmatrix} z_5 & z_4 \\ z_4 & 1 \end{pmatrix} \succeq 0, \quad y \geq 0, \quad \begin{pmatrix} y & x_1 & z_2 & z_3 & z_5 & \sqrt{2} \\ x_1 & z_2 & z_3 & z_5 & y & 15 \end{pmatrix} \succeq 0$$

$$x_1^2 + x_2^2 + x_4^2 \leq 2. \quad \text{That is in CVX format.}$$

日期:

$$(ii) \Leftrightarrow \text{Min}_{(x_1, x_2, x_3)} A := \begin{pmatrix} 1 & 5/2 & -4/2 \\ 5/2 & 8 & 0 \\ -4/2 & 0 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + 8(|x_1 - 1| + |x_2 + 3| + |x_3 - 5|)$$

$$\text{s.t. } 5(x_2 + 1) - x_3^2 \geq 0, \quad x_2 + 1 \geq 0$$

$$\Leftrightarrow \text{Min}_{(x_1, x_2, x_3)} A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + 8(|x_1 - 1| + |x_2 + 3| + |x_3 - 5|)$$

s.t.

$$\begin{pmatrix} 5 & y \\ y & x_2 + 1 \end{pmatrix} \geq 0, \quad \begin{pmatrix} y & x_3 \\ x_3 & 1 \end{pmatrix} \geq 0$$

$$(iii) \Leftrightarrow \text{Min}_{(x_1, x_2, x_3)} 2x_1 + 3x_2 - x_3 + \sqrt{\frac{1}{3}x_1^2 + (x_2 - 5)^2 + 6(x_3 - \frac{1}{3}x_1)^2} +$$

$$\text{s.t. } \max\{x_1 + x_2, x_3 + x_2, x_1 + x_3\} \leq 2, \quad x_1 \geq 0, \quad x_2 \geq 0, \\ x_1 + x_2 \leq 2, \quad x_3 + x_2 \leq 2, \quad x_1 + x_3 \leq 2.$$

result is on next page.

日期: /

(b) Explain whether the following optimization problems can be reformulated equivalently as an SDP problem.

i. (10 points)

$$\begin{array}{ll} \text{Minimize} & |x_1 + x_2 + x_3 - 8| + \sqrt{x_1^2 + x_2^2 + x_3^2} \\ \text{Subject to} & x_1^2 + 5x_2^2 + 4x_1x_2 + x_3^2 \leq 1, \quad x_1 \geq 1. \end{array}$$

ii. (10 points)

$$\begin{array}{ll} \text{Minimize} & \max\{x_1 + x_2 + 3, \sqrt{x_1^2 + x_2^2 + 3}\} + |x_1 + x_3 - 8| \\ \text{Subject to} & x_1^2 + x_2^2 + x_3^2 \leq 5, \quad x_1 \geq 0. \end{array}$$

$$(b) \Leftrightarrow \text{Min } z_1 + z_2$$

$$\text{s.t. } y \geq x_1^*, \quad y \geq 0$$

$$x_1^* + 5y^2 + 4x_1y + x_3^2 \leq 1, \quad x_1 \geq 1$$

$$z_1 \geq |x_1 + x_2 + x_3 - 8|$$

$$z_2 \geq \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$\Leftrightarrow \text{Min } z_1 + z_2$$

$$\text{s.t. } y \geq x_1^*, \quad y \geq 0$$

$$(x_1 \quad y \quad x_3) \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y \\ x_3 \end{pmatrix} \leq 1, \quad x_1 \geq 1$$

$$z_1 \geq |x_1 + x_2 + x_3 - 8|, \quad z_1 \geq 0$$

$$z_2 \geq x_1^2 + x_2^2 + x_3^2, \quad z_2 \geq 0$$

$$\Leftrightarrow \text{Min } z_1 + z_2$$

$$\text{s.t. } \begin{pmatrix} y & x_2 \\ x_1 & 1 \end{pmatrix} \geq 0,$$

$$\begin{pmatrix} 5 & -2 & 0 & x_1 \\ -2 & 1 & 0 & y \\ 0 & 0 & 1 & x_3 \\ x_1 & y & x_3 & 1 \end{pmatrix} \geq 0, \quad x_1 \geq 1$$

$$z_1 \geq x_1 + x_2 + x_3 - 8 \geq -z_1$$

$$\begin{pmatrix} z_2 & x_1 & x_2 & x_3 \\ x_1 & z_2 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_2 & 1 \end{pmatrix} \geq 0.$$

$$(ii) \Leftrightarrow \text{Min } \max\{y, z\} + p$$

$$\text{s.t. } y \geq x_1 + x_2 + 3, \quad x_1 \geq 0, \quad y - x_1 \geq 0$$

$$z \geq \sqrt{x_1^2 + x_2^2 + 3}, \quad z \geq 0$$

$$p \geq |x_1 + x_3 - 8|$$

$$x_1^2 + x_2^2 + x_3^2 \leq 5$$

$$\Leftrightarrow \text{Min } \frac{1}{2}(y + z + |y - z|) + p$$

$$\text{s.t. } \begin{pmatrix} y - x_1 & x_2 & \sqrt{3} \\ x_2 & z_2 & 1 \\ \sqrt{3} & 1 & 2 \end{pmatrix} \geq 0, \quad x_1 \geq 0,$$

日期: /  $\begin{pmatrix} z & x_1 & x_2 & 3 \\ x_1 & & & \\ x_2 & & & \\ 3 & & & \end{pmatrix} \geq 0$

$$p \geq x_1 + x_3 - 8 \geq p$$

$$x_1^3 + x_2^4 + x_3 \leq 5$$

from the discussion of (i), we know objective function  
can be reformulated to SDP.

and now we discuss constraints  $\begin{cases} x_1^3 + x_2^4 + x_3 \leq 5 \\ x_1 \geq 0 \end{cases}$

it's equivalent to:  $\Leftrightarrow: \begin{cases} x_2^2 \leq p_1, p_1^2 \leq p_2 \\ x_1^2 + p_2 + x_3 \leq 5, x_1 \geq 0 \end{cases}$

$$\Leftrightarrow: \begin{cases} x_2^2 \leq p_1, p_1^2 \leq p_2 \\ x_1^2 \leq q_1, \\ q_1^2 \leq x_1 \cdot (5 - p_2 - x_3), x_1 \geq 0, (5 - p_2 - x_3) \geq 0 \end{cases}$$

$$\Leftrightarrow: \begin{cases} \begin{pmatrix} p_1 & x_2 \\ x_2 & 1 \end{pmatrix} \geq 0, \begin{pmatrix} p_2 & p_1 \\ p_1 & 1 \end{pmatrix} \geq 0, \begin{pmatrix} q_1 & x_1 \\ x_1 & 1 \end{pmatrix} \geq 0 \\ \begin{pmatrix} p_1 & q_1 \\ q_1 & (5 - p_2 - x_3) \end{pmatrix} \geq 0 \end{cases}$$

Hence, can be reformulated to

$$\text{Min } \frac{1}{2}(y + z + u) + p$$

$$\text{s.t. } C \geq y - z \geq 0$$

$$\begin{pmatrix} y - x_1 & x_2 & \sqrt{3} \\ x_2 & I_2 & \\ \sqrt{3} & & \end{pmatrix} \geq 0, \quad \begin{pmatrix} z & x_1 & x_2 & 3 \\ x_1 & & & \\ x_2 & & & \\ 3 & & & \end{pmatrix} \geq 0, \quad p \geq x_1 + x_3 - 8 \geq p$$

$$\begin{pmatrix} p_1 & x_2 \\ x_2 & 1 \end{pmatrix} \geq 0, \quad \begin{pmatrix} p_2 & p_1 \\ p_1 & 1 \end{pmatrix} \geq 0, \quad \begin{pmatrix} q_1 & x_1 \\ x_1 & 1 \end{pmatrix} \geq 0$$

$$\begin{pmatrix} p_1 & q_1 \\ q_1 & (5 - p_2 - x_3) \end{pmatrix} \geq 0. \quad \square$$

日期: /

2. Consider the following optimization problem.

$$\begin{array}{ll} \text{Minimize}_{x \in \mathbb{R}^2} & -x_1^3 + 2x_2^2 \\ \text{Subject to} & x_1^2 + x_2^2 \leq 5, \\ & x_2 \geq x_1^2 + 2. \end{array}$$

- (a) (5 points) Show that the MFCQ holds at every feasible point.  
(b) (25 points) Write down the KKT conditions and find all global minimizers.

assume

$$2. \quad (a) \quad g_1 := x_1^2 + x_2^2 - 5 \leq 0,$$

$$g_2 := x_1^2 - x_2 + 2 \leq 0.$$

$$\nabla g_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \geq 0, \quad \nabla g_2 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \geq 0$$

we find a slater point,  $x = \begin{pmatrix} 0 \\ 2.1 \end{pmatrix}$  satisfy  $g_1(x) < 0, g_2(x) < 0$ .

Hence MFCQ holds on feasible region.

$$(b) \quad \text{Min } f(x) := -x_1^3 + 2x_2^2$$

$$\text{s.t. } g_1 := x_1^2 + x_2^2 - 5 \leq 0$$

$$g_2 := x_1^2 - x_2 + 2 \leq 0$$

日期:

KKT conditions : ①  $g_1(x) \leq 0, g_2(x) \leq 0$

②  $\nabla f(x) + \lambda_1 \cdot \nabla g_1(x) + \lambda_2 \cdot \nabla g_2(x) = 0, \lambda_1 \geq 0, \lambda_2 \geq 0$

③  $\lambda_1 \cdot g_1(x) = 0, \lambda_2 \cdot g_2(x) = 0$

$$② \Rightarrow \begin{pmatrix} -3x_1^2 \\ 4x_2 \end{pmatrix} + \lambda_1 \cdot \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} 2x_1 \\ -1 \end{pmatrix} = 0, \lambda \geq 0$$

if  $\lambda_1 = 0, \lambda_2 = 0$ , we have  $(x_1, x_2) = (0, 0)$

$g_2(x) \leq 0$  is not satisfied.

if  $\lambda_1 \geq 0, \lambda_2 > 0$ , we have  $\begin{pmatrix} -3x_1^2 \\ 4x_2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2x_1 \\ -1 \end{pmatrix} = 0, x_1^2 - x_2 + 2 = 0$

we get solution  $(x_1, x_2) = (0, 2), \lambda_2 = 8, g_1 \leq 0, g_2 \leq 0$  satisfy.

$(x_1, x_2) = (0, 2)$  is stationary point.

if  $\lambda_1 \neq 0, \lambda_2 = 0$ , we have  $\begin{pmatrix} -3x_1^2 \\ 4x_2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} = 0, x_1^2 + x_2^2 = 5$

$4x_2 + 2\lambda_1 x_1 = 0$ , we get  $x_2 = 0$  or  $\lambda_1 = -2$ .

if  $x_2 = 0$ , we have  $x_1 = \pm\sqrt{5}$ , but  $g_2(x) \leq 0$  is not satisfied.

if  $\lambda_1 = -2$ , it's contradictory with  $\lambda \geq 0$ .

if  $\lambda_1 \neq 0, \lambda_2 \neq 0$ , we have  $x_1^2 + x_2^2 = 5, x_2 = x_1^2 + 2$

we have solution  $x_1^2 = \frac{-5 + \sqrt{59}}{2}, x_2 = \frac{-1 + \sqrt{59}}{2}$

$\begin{pmatrix} 2x_1 & 2x_2 \\ 2x_2 & -1 \end{pmatrix}$  is full rank, so  $(\pm\sqrt{\frac{-5 + \sqrt{59}}{2}}, \frac{-1 + \sqrt{59}}{2})$  is stationary points.

$$\text{and } \nabla f(x) + \lambda_1 \nabla g_1(x) + \lambda_2 \nabla g_2(x) = \begin{pmatrix} -3x_1^2 + \lambda_1 2x_1 + \lambda_2 \cdot 2x_1 \\ 4x_1 + \lambda_1 2x_2 - \lambda_2 \end{pmatrix} = 0.$$

日期 and we don't need to test if it satisfies  $\lambda_1, \lambda_2 \geq 0$ ,  
 since if  $x = (\pm \sqrt{\frac{-5+\sqrt{17}}{2}}, \frac{-1+\sqrt{17}}{2})$ ,  $f(x)$  is greater than  $f(0,2)$ , they cannot be global minimizer.  
 we find the minimal point is  $(0,2)$ ,  $\min f(x) = 8$ .  
 $(0,2)$  is global minimizer.  $\square$

3. Consider the following optimization problem, where  $n$  is an integer greater than 2024:

$$\underset{x \in \mathbb{R}^n}{\text{Minimize}} \quad \frac{1}{2} \sum_{i=1}^n (x_i - i)^2$$

$$\text{Subject to} \quad 1 + \sum_{i=1}^n i x_i \leq 0.$$

You may leave the term  $S_n$  in your answer, where  $S_n := \sum_{i=1}^n i^2$ .

(a) (10 points) For each  $c > 0$ , define

$$q_c(x) := \frac{1}{2} \sum_{i=1}^n (x_i - i)^2 + \frac{c}{2} \left( 1 + \sum_{i=1}^n i x_i \right)_+$$

Argue that  $q_c$  is convex and find the global minimizer of  $q_c$ .

$$3. \text{ (a) } \nabla q_c(x) = \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_n - n \end{pmatrix} + c \cdot \left( 1 + \sum_{i=1}^n i x_i \right)_+ \cdot \begin{pmatrix} 1 \\ \vdots \\ n \end{pmatrix}$$

since  $\frac{\partial^2}{\partial x_i^2} (x_i - i)^2 = 2 > 0$ , we know  $(x_i - i)^2$  is convex,

so  $\frac{1}{2} \sum_{i=1}^n (x_i - i)^2$  is convex.

assume that  $y := 1 + \sum_{i=1}^n i x_i$ ,  $y$  is linear affine of  $x$ .

and  $y^+$  is convex,  $\Rightarrow$  we have  $\frac{c}{2} \left( 1 + \sum_{i=1}^n i x_i \right)_+^2$ ,  $c > 0$  is convex

Hence,  $q_c(x)$  is convex.

Lemma:  $g(y) = y^2$  is convex.

Proof:  $\forall \lambda \in (0,1)$ , if  $y_1, y_2 < 0$ , we have

$$\lambda \cdot g(y_1) + (1-\lambda) g(y_2) = g(\lambda y_1 + (1-\lambda) y_2) = 0$$

if  $y_1, y_2 \geq 0$ , since  $y^2$  is convex, the formula above still holds.

if  $y_1 > 0$ , and  $y_2 < 0$ ,  $\lambda g(y_1) + (1-\lambda) g(y_2) = \lambda g(y_1) = \lambda y_1^2$

$$g(\lambda y_1 + (1-\lambda) y_2) \leq g(\lambda y_1) = \lambda^2 y_1^2 \leq \lambda y_1^2.$$

Hence  $g(y) = y^2$  is convex.

日期: ②  $\nabla q^*(\infty) = \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_i - i \\ \vdots \\ x_n - n \end{pmatrix} + c \cdot \left( 1 + \sum_{i=1}^n ix_i \right) + \begin{pmatrix} 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ n \end{pmatrix} = 0$

we have  $x_i - i + c \cdot \left( 1 + \sum_{i=1}^n ix_i \right) + i = 0$ , for  $\forall i = 1, 2, \dots, n$

$$x_i = i \left[ 1 - c \cdot \left( 1 + \sum_{i=1}^n ix_i \right) \right]$$

if  $1 + \sum_{i=1}^n ix_i < 0$ , we have  $x_i = i$ ,

$$1 + \sum_{i=1}^n ix_i = 1 + S_n > 0, \text{ it's contradiction.}$$

if  $1 + \sum_{i=1}^n ix_i \geq 0$ , define  $z := 1 - c \cdot (1 + \sum_{i=1}^n ix_i)$ ,  $y = \sum_{i=1}^n ix_i$

$$\sum_{i=1}^n ix_i = \sum_{i=1}^n i^2 \cdot z, \text{ that is } y = S_n \cdot (1 - c(1 + y))$$

$$y = \frac{S_n(1-c)}{1+cS_n}, \quad z = \frac{1-c}{1+cS_n}$$

$$1 + \sum_{i=1}^n ix_i = \frac{1+S_n}{1+cS_n} \geq 0, \text{ satisfy the assumption.}$$

we get  $x_i = i \cdot \frac{1-c}{1+cS_n}$ , for  $i = 1, 2, \dots, n$ ,

is the global minimizer, since convexity.

日期:

(b) (10 points) For each  $\mu > 0$ , define

$$f_\mu(x) := \frac{1}{2} \sum_{i=1}^n (x_i - i)^2 - \mu \ln \left( -1 - \sum_{i=1}^n ix_i \right).$$

Argue that  $f_\mu$  is convex and find the global minimizer of  $f_\mu$ . You may use without proof the fact that the function  $t \mapsto -\ln(-1-t)$  is convex (as an extended real-valued function).

(b) ①  $(x_i - i)^2$  is convex. since  $\frac{\partial^2}{\partial x_i^2} (x_i - i)^2 = 2 > 0$ .

so  $\frac{1}{2} \sum_{i=1}^n (x_i - i)^2$  is convex.

assume  $t := \sum_{i=1}^n ix_i$ ,  $t$  is affine projection of  $x$ .

and  $\mu > 0$ ,  $-\mu \ln(-1-t)$  is convex.

it implies  $-\mu \ln(-1 - \sum_{i=1}^n ix_i)$  is convex.

Hence,  $f_\mu(x)$  is convex.

$$\textcircled{2} \quad \nabla f_\mu(x) = \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_n - n \end{pmatrix} - \frac{\mu}{1 + \sum_{i=1}^n ix_i} \cdot \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix} = 0$$

we have  $x_i - i - \frac{\mu}{1 + \sum_{i=1}^n ix_i} \cdot i = 0 \quad \text{for } i = 1, 2, \dots, n$ .

assume that  $y := \sum_{i=1}^n ix_i$ ,

$$xi = i + \frac{\mu}{1+y} \cdot i = i \left( 1 + \frac{\mu}{1+y} \right)$$

$$y = \sum_{i=1}^n ix_i = s_n \cdot \left( 1 + \frac{\mu}{1+y} \right), \Rightarrow y = \frac{s_n - 1 \pm \sqrt{(s_n - 1)^2 + 4(1+\mu)s_n}}{2}$$

$$xi = i \left( 1 + \frac{\mu}{1+y} \right), \forall i = 1, 2, \dots, n, \text{ and } y < -1$$

we get  $\boxed{y = \frac{s_n - 1 - \sqrt{(s_n - 1)^2 + 4(1+\mu)s_n}}{2} < -1 \text{ holds for } \forall \mu > 0}$

the global minimizer is  $xi = i + \frac{\mu}{1+y} \cdot i$ , for  $i = 1, 2, \dots, n$ .  $\square$

日期: /