1. ① $E(X)=\theta$,

$\hat{\theta}_{MME} = \bar{X}$

② $\log L(\theta, x_1 \cdots x_n) = \sum_{i=1}^{n} x_i \log\theta + (1-x_i)\log(1-\theta)$

$\frac{\partial \log L(\theta)}{\partial\theta} = \sum_{i=1}^{n} \frac{x_i}{\theta} + \frac{1-x_i}{\theta-1} = \sum_{i=1}^{n} \frac{x_i-\theta}{\theta(1-\theta)} = 0$

$\frac{\partial^2 \log L(\theta)}{\partial\theta^2} = \sum -\frac{x_i}{\theta^2} - \frac{1-x_i}{(\theta-1)^2} < 0$

so $\hat{\theta}_{MLE} = \bar{X}$

2. $f_X(x) = \frac{1}{\theta}$,　$x \in [0,\theta]$,　$\theta > 0$

(a) $E(X) = \int_0^\theta \frac{1}{\theta} x\,dx = \frac{1}{2}\theta$,

$\hat{\theta}_{MME} = 2\bar{X}$

(b) Find $\hat{\theta}_{MLE}$ first.

$L(\theta, x_1 \cdots x_n) = \begin{cases} \left(\frac{1}{\theta}\right)^n, & \text{if } 0 \le x_i \le \theta \\ 0 & \text{otherwise} \end{cases}$

montonically decrease.

$\hat{\theta}_{MLE} = \max\{X_1 \cdots X_n\}$

(c) ① $Var(2\bar{X}) = Var(\hat{\theta}_{MME}) = 4\,Var(\bar{X}) = \frac{4}{n} Var(X)$

$= \frac{4}{n} \cdot \frac{\theta^2}{12} = \frac{\theta^2}{3n}$

② $E(2\bar{X}) = \theta$,　since $MSE(\hat{\theta}_{MME}) = Var(\hat{\theta}_{MME}) + bias^2(\hat{\theta})$

$= \frac{\theta^2}{3n} \to 0$　as　$n \to \infty$.

then we have $\hat{\theta}_{MME} \xrightarrow{P} \theta$. consistent. ✓

日期: /

(d) $F_{X(n)}(x) = P(X(n) \le x) = P(X_1 \le x, X_2 \le x \cdots X_n \le x)$

$= P(X \le x)^n = (\frac{x}{\theta})^n$, if $x \in [0, \theta]$, $\theta > 0$

$\Rightarrow f_{X(n)}(x) = n \cdot (\frac{x}{\theta})^{n-1} \cdot \frac{1}{\theta}$, if $x \in [0, \theta]$, 0 otherwise.

(e) $$\frac{f(x_1;\theta) \cdots f(x_n;\theta)}{f(X(n);\theta)} = \frac{(\frac{1}{\theta})^n}{n(\frac{x}{\theta})^{n-1} \cdot \frac{1}{\theta}} = \frac{1}{n \cdot x^{n-1}}$$

independent with θ.

it is sufficient for θ

△ exponential family need the domain is independent with θ

uniform distribution is not exponential family.

3. Gamma distribution : $f_X(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \cdot x^{\alpha-1} \cdot e^{-x/\beta}$, $x > 0$

$f_X(x) = \frac{1}{\Gamma(5)\theta^5} \cdot x^4 \cdot e^{-\frac{x}{\theta}}$, $x > 0$

(a) $E(X) = \alpha \cdot \beta = 5\theta$,

$\hat{\theta}_{MME} = \frac{\bar{X}}{5}$

(b) $\log L(\theta; x_1 \cdots x_n) = \sum_{i=1}^{n} \log \frac{1}{\Gamma(5)} - 5\log\theta + 4\log x_i - \frac{x_i}{\theta}$

$\frac{\partial \log L(\theta)}{\partial \theta} = \sum_{i=1}^{n} -\frac{5}{\theta} + \frac{x_i}{\theta^2}$, $\hat{\theta}_{MLE} = \frac{1}{5}\bar{X}$

we find $\sum_{i=1}^{n} -5\theta + x_i$ maximizer is $\frac{1}{5}\bar{X}$.

so $\hat{\theta}_{MLE} = \frac{1}{5}\bar{X}$.

(c) $I(\theta) = -E\left(\frac{\partial^2 \log f(x|\theta)}{\partial\theta^2}\right)$

$= -E\left(\frac{5}{\theta^2} - \frac{2x}{\theta^3}\right)$　　　$\frac{1}{\Gamma(5)\cdot\theta^5}\cdot x^4\cdot e^{-x/\theta}$

$= -\frac{5}{\theta^2} + \frac{2}{\theta^3}\cdot E(X)$　　　$\frac{1}{\Gamma(2)\cdot\beta^2}\cdot x^{2-1}\cdot e^{-x/\beta}$

$= \frac{5}{\theta^2}$

$2\beta^2$　$5\cdot\theta^2$

(d) $Var(\hat{\theta}_{MLE}) = Var(\frac{1}{5}\bar{X}) = \frac{1}{25}\cdot\frac{1}{n}Var(X) = \frac{1}{25}\cdot\frac{1}{n}\cdot 5\cdot\theta^2 = \frac{1}{5}\cdot\frac{1}{n}\cdot\theta^2$

R-C lower bound $= \frac{1}{nI(\theta)} = \frac{1}{n\cdot\frac{5}{\theta^2}} = \frac{1}{5n}\cdot\theta^2 = Var(\hat{\theta}_{MLE})$

efficient.

$\left(\frac{1}{x}\right)\theta^x$

4. (a) $b(1,\theta)$　　$p_X(x) = \theta^x(1-\theta)^{1-x}$, $x=0,1$

$= \begin{cases}\theta & \text{if } x=1\\ 1-\theta & \text{if } x=0\end{cases}$

$p_X(x) = \exp\{x\cdot\log\theta + (1-x)\log(1-\theta)\}$　　θ^2 lower

$= \exp\{\log(1-\theta) + x\cdot\log\frac{\theta}{1-\theta}\}$

hence $\sum_{i=1}^{n} x_i$ is complete sufficient.

(b) MVUE of θ,

$E(\bar{X}) = E(X) = \theta$,

MVUE is $E(\bar{X}|\sum_{i=1}^{n}x_i) = \bar{X}$

日期: /

(c) since Σx_i is sufficient complete

$\bar{X}$ is unique MVUE of θ.

θ^2 expo ?

5. (a) reject H_0 if $Y = \Sigma x_i \le c$

$\forall\ \theta' = \frac{1}{3},\ \theta'' < \frac{1}{3}$

$$\frac{L(\theta')}{L(\theta'')} = \frac{\prod_{i=1}^{n} \theta'^{x_i}(1-\theta')^{1-x_i}}{\prod \theta''^{x_i}(1-\theta'')^{1-x_i}} = \left(\frac{\theta'}{\theta''}\right)^{\Sigma x_i} \cdot \left(\frac{1-\theta'}{1-\theta''}\right)^{4-\Sigma x_i}$$

$$= \left(\frac{1-\theta'}{1-\theta''}\right)^4 \cdot \left(\frac{\theta'}{\theta''} \cdot \frac{1-\theta''}{1-\theta'}\right)^{\Sigma x_i} \le k \qquad \frac{\theta'}{\theta''} \cdot \frac{1-\theta''}{1-\theta'} > 1$$

$\iff \sum_{i=1}^{4} x_i \le k_1$

we find $\sum_{i=1}^{4} x_i \le c$ is the critical region in UMP test.

$\alpha = P\left(\sum_{i=1}^{4} x_i \le c \,\middle|\, \theta = \frac{1}{3}\right)$

(b) $\sum_{i=1}^{4} x_i \sim \text{Binomial}(4, \theta)$

$P_{\Sigma x_i}(x) = \binom{4}{x} \theta^x (1-\theta)^{4-x}$

$\alpha = P\left(\sum_{i=1}^{4} x_i \le 0 \,\middle|\, \theta = \frac{1}{3}\right) = \left(\frac{2}{3}\right)^4$

(c) $\alpha = P\left(\sum_{i=1}^{4} x_i \le 2 \,\middle|\, \theta = \frac{1}{3}\right) = \left(\frac{2}{3}\right)^4 + 4 \cdot \frac{1}{3} \cdot \left(\frac{2}{3}\right)^3 + 6 \cdot \left(\frac{1}{3}\right)^2 \cdot \left(\frac{2}{3}\right)^2 = \frac{8}{9}$

日期:　　　/