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1. (a) For each of the following optimization problems, write a CVX code that solves it, if possible. Also write down the optimal value returned by CVX (corrected to 4 decimal places).

i. (10 points)

$$\begin{array}{ll} \text{Minimize} & \sqrt{x_1^2 + (x_2-5)^4 + (x_3-1)^6 + (x_4+1)^8 + 2} + |x_3 - x_4 + 7| \\ \text{Subject to} & x_1^2 + x_2^2 + x_4^2 \le 2. \end{array}$$

ii. (10 points)

$$\begin{array}{ll} \text{Minimize} & x_1^2 + 8x_2^2 + 9x_3^2 + 5x_1x_2 - x_1x_3 + 8(|x_1-1| + |x_2+3| + |x_3-5|) \\ \text{Subject to} & \begin{bmatrix} 5 & x_3^2 \\ x_3^2 & x_2+1 \end{bmatrix} \succeq 0. \end{array}$$

iii. (10 points)

$$\begin{array}{ll} \text{Minimize} & 2x_1 + 3x_2 - x_3 + \sqrt{x_1^2 + (x_2-5)^2 + 6x_3^2 - 4x_1x_3 + 1} \\ \text{Subject to} & \max\{x_1+x_2, x_3+x_2, x_1+x_3\} \le 2, \quad x_1 \ge 0, \quad x_2 \ge 0. \end{array}$$

1. (a) (i) reformulate:

$$\begin{array}{ll} \text{Min} & y + |x_3 - x_4 + 7| \\ \text{s.t.} & y^2 \geq x_1^2 + (x_2-5)^4 + (x_3-1)^6 + (x_4+1)^8 + 2, \quad y \geq 0 \\ & x_1^2 + x_2^2 + x_4^2 \leq 2 \end{array}$$

$$\Longleftrightarrow \quad \begin{array}{ll} \text{Min} & y + |x_3 - x_4 + 7| \\ \text{s.t.} & z_2 \geq (x_2-5)^2, \quad z_3 \geq (x_3-1)^3, \quad z_4 \geq (x_4+1)^2, \\ & z_5 \geq z_4^2, \quad y \geq 0 \\ & y^2 \geq x_1^2 + z_2^2 + z_3^2 + z_5^2 + 2 \\ & x_1^2 + x_2^2 + x_4^2 \leq 2 \end{array}$$

$$\Longleftrightarrow \quad \begin{array}{ll} \text{Min} & y + |x_3 - x_4 + 7| \\ \text{s.t.} & z_2 \geq (x_2-5)^2, \quad z_3 \end{array}$$

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(ii) $\Longleftrightarrow$:

$$\text{Min} \quad (x_1, x_2, x_3) \underbrace{\begin{pmatrix} 1 & 5/2 & -1/2 \\ 5/2 & 8 & 0 \\ -1/2 & 0 & 9 \end{pmatrix}}^{A :=} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + 8(|x_1 - 1| + |x_2 + 3| + |x_3 - 5|)$$

$$\text{s.t.} \quad 5(x_2+1) - x_3^4 \geq 0, \quad x_2 + 1 \geq 0$$

$$\Longleftrightarrow \quad \text{Min} \quad (x_1, x_2, x_3) A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + 8(|x_1 - 1| + |x_2 + 3| + |x_3 - 5|)$$

s.t.

$$\begin{pmatrix} 5 & y \\ y & x_2 + 1 \end{pmatrix} \succeq 0, \quad \begin{pmatrix} y & x_3 \\ x_3 & 1 \end{pmatrix} \succeq 0$$

(iii) $\Longleftrightarrow$:

$$\text{Min} \quad 2x_1 + 3x_2 - x_3 + \sqrt{\tfrac{1}{3}x_1^2 + (x_2 - 5)^2 + 6(x_3 - \tfrac{1}{3}x_1)^2 + 1}$$

$$\text{s.t.} \quad \cancel{\max\{x_1 + x_2, x_3 + x_2, x_1 + x_3\} \leq 2}, \quad x_1 \geq 0, \ x_2 \geq 0.$$

$$x_1 + x_2 \leq 2, \quad x_3 + x_2 \leq 2, \quad x_1 + x_3 \leq 2.$$

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(b) Explain whether the following optimization problems can be reformulated equivalently as an SDP problem.

i. (10 points)

$$\begin{array}{ll} \text{Minimize} & |x_1+x_2+x_3-8|+\sqrt{x_1^2+x_2^2+x_3^2} \\ \text{Subject to} & x_1^2+5x_2^4+4x_1x_2^2+x_3^2 \le 1, \quad x_1 \ge 1. \end{array}$$

ii. (10 points)

$$\begin{array}{ll} \text{Minimize} & \max\{x_1+x_2^2+3, \sqrt{x_1^2+x_2^2+3}\}+|x_1+x_3-8| \\ \text{Subject to} & x_1^3+x_2^4+x_3 \le 5, \quad x_1 \ge 0. \end{array}$$

(b) (i) $\Longleftrightarrow$:

$$\begin{array}{ll} \text{Min} & z_1+z_2 \\ \text{s.t.} & y \geq x_2^2, \quad y \geq 0 \\ & x_1^2+5y^2+4x_1y+x_3^2 \leq 1, \quad x_1 \geq 1 \\ & z_1 \geq |x_1+x_2+x_3-8| \\ & z_2 \geq \sqrt{x_1^2+x_2^2+x_3^2} \end{array}$$

$\Longleftrightarrow$:

$$\begin{array}{ll} \text{Min} & z_1+z_2 \\ \text{s.t.} & y \geq x_2^2, \quad y \geq 0 \\ & (x_1 \quad y \quad x_3)\begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y \\ x_3 \end{pmatrix} \leq 1, \quad x_1 \geq 1 \\ & z_1 \geq |x_1+x_2+x_3-8|, \quad z_1 \geq 0 \\ & z_2^2 \geq x_1^2+x_2^2+x_3^2, \quad z_2 \geq 0 \end{array}$$

$\Longleftrightarrow$:

$$\begin{array}{ll} \text{Min} & z_1+z_2 \\ \text{s.t.} & \begin{pmatrix} y & x_2 \\ x_2 & 1 \end{pmatrix} \succeq 0, \quad \begin{pmatrix} 5 & -2 & 0 & x_1 \\ -2 & 1 & 0 & y \\ 0 & 0 & 1 & x_3 \\ x_1 & y & x_3 & 1 \end{pmatrix} \succeq 0, \quad x_1 \geq 1 \\ & z_1 \geq x_1+x_2+x_3-8 \geq -z_1 \\ & \begin{pmatrix} z_2 & x_1 & x_2 & x_3 \\ x_1 & & & \\ x_2 & & z_2 \cdot I_3 & \\ x_3 & & & \end{pmatrix} \succeq 0. \end{array}$$

(ii) $\Longleftrightarrow$:

$$\begin{array}{ll} \text{Min} & \max\{\ y\ , z\}+p \\ \text{s.t.} & y \geq x_1+x_2^2+3, \quad x_1 \geq 0, \quad y-x_1 \geq 0 \\ & z \geq \sqrt{x_1^2+x_2^2+3}, \quad z \geq 0 \\ & p \geq |x_1+x_3-8| \\ & x_1^3+x_2^4+x_3 \leq 5 \end{array}$$

$\Longleftrightarrow$:

$$\begin{array}{ll} \text{Min} & \frac{1}{2}(y+z+|y-z|)+p \\ \text{s.t.} & \begin{pmatrix} y-x_1 & x_2 & \sqrt{3} \\ x_2 & & \\ \sqrt{3} & & I_2 \end{pmatrix} \succeq 0, \quad x_1 \geq 0, \end{array}$$

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$$\begin{pmatrix} z & x_1 & x_2 & 3 \\ x_1 & & & \\ x_2 & & z\cdot I_3 & \\ 3 & & & \end{pmatrix} \succeq 0$$

$$p \geq x_1 + x_3 - 8 \geq -p$$

$$x_1^3 + x_2^4 + x_3 \leq 5$$

from the discussion of (i), we know objective function can be reformulated to SDP,

and now we discuss constraints $\begin{cases} x_1^3 + x_2^4 + x_3 \leq 5 \\ x_1 \geq 0 \end{cases}$

it's equivalent to: $\iff$: $\begin{cases} x_2^2 \leq p_1, \quad p_1^2 \leq p_2 \\ x_1^3 + p_2 + x_3 \leq 5, \quad x_1 \geq 0 \end{cases}$

$$\iff: \begin{cases} x_2^2 \leq p_1, \quad p_1^2 \leq p_2 \\ x_1^2 \leq q_1, \\ q_1^2 \leq x_1 \cdot (5 - p_2 - x_3), \quad x_1 \geq 0, (5 - p_2 - x_3) \geq 0 \end{cases}$$

$$\iff: \begin{cases} \begin{pmatrix} p_1 & x_2 \\ x_2 & 1 \end{pmatrix} \succeq 0, \begin{pmatrix} p_2 & p_1 \\ p_1 & 1 \end{pmatrix} \succeq 0, \begin{pmatrix} q_1 & x_1 \\ x_1 & 1 \end{pmatrix} \succeq 0 \\ \begin{pmatrix} x_1 & q_1 \\ q_1 & (5 - p_2 - x_3) \end{pmatrix} \succeq 0 \end{cases}$$

Hence, can be reformulated to

$$\text{Min} \quad \tfrac{1}{2}(y + z + c) + p$$

$$\text{s.t.} \quad c \geq y - z \geq -c$$

$$\begin{pmatrix} y - x_1 & x_2 & \sqrt{3} \\ x_2 & & \\ \sqrt{3} & & I_2 \end{pmatrix} \succeq 0, \quad \begin{pmatrix} z & x_1 & x_2 & 3 \\ x_1 & & & \\ x_2 & & z\cdot I_3 & \\ 3 & & & \end{pmatrix} \succeq 0, \quad p \geq x_1 + x_3 - 8 \geq -p$$

$$\begin{pmatrix} p_1 & x_2 \\ x_2 & 1 \end{pmatrix} \succeq 0, \quad \begin{pmatrix} p_2 & p_1 \\ p_1 & 1 \end{pmatrix} \succeq 0, \quad \begin{pmatrix} q_1 & x_1 \\ x_1 & 1 \end{pmatrix} \succeq 0$$

$$\begin{pmatrix} x_1 & q_1 \\ q_1 & (5 - p_2 - x_3) \end{pmatrix} \succeq 0. \quad \square$$

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2. Consider the following optimization problem.

$$\begin{array}{ll} \underset{x\in\mathbb{R}^2}{\text{Minimize}} & -x_1^3+2x_2^2 \\ \text{Subject to} & x_1^2+x_2^2\le 5, \\ & x_2\ge x_1^2+2. \end{array}$$

(a) (5 points) Show that the MFCQ holds at every feasible point.

(b) (25 points) Write down the KKT conditions and find all global minimizers.

2、 (a) constraints : $g_1 := x_1^2+x_2^2-5\le 0$

$g_2 := x_1^2-x_2+2\le 0$

$\nabla^2 g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \succ 0,\quad \nabla^2 g_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \succeq 0,$

g_1, g_2 both are convex.

$\exists\ (x_1, x_2) = (0.1, 2.1), \Rightarrow g_1(x_1, x_2) < 0,\ g_2(x_1, x_2) < 0$

hence, we have MFCQ holds at every point in feasible region.

(b) Min $f(x) := -x_1^3+2x_2^2$

s.t. $g_1 := x_1^2+x_2^2-5\le 0$

$g_2 := x_1^2-x_2+2\le 0$

KKT conditions: ① $g_1(x) \le 0,\ g_2(x) \le 0$

② $\nabla f(x) + \lambda_1 \cdot \nabla g_1(x) + \lambda_2 \nabla g_2(x) = 0$, $\lambda_1 \ge 0,\ \lambda_2 \ge 0$

③ $\lambda_1 \cdot g_1(x) = 0,\ \lambda_2 \cdot g_2(x) = 0$

② $\Rightarrow \begin{pmatrix} -3x_1^2 \\ 4x_2 \end{pmatrix} + \lambda_1 \cdot \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} 2x_1 \\ -1 \end{pmatrix} = 0$, $\lambda \ge 0$

if $\lambda_1 = 0,\ \lambda_2 = 0$, we have $(x_1, x_2) = (0, 0)$

$g_2(x) \le 0$ is not satisfied.

if $\lambda_1 = 0,\ \lambda_2 \ne 0$, we have $\begin{pmatrix} -3x_1^2 \\ 4x_2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2x_1 \\ -1 \end{pmatrix} = 0$, $x_1^2 - x_2 + 2 = 0$

we get solution $(x_1, x_2) = (0, 2)$, $\lambda_2 = 8$, $g_1 \le 0$, $g_2 \le 0$ satisfy.

$(x_1, x_2) = (0, 2)$ is stationary point.

if $\lambda_1 \ne 0,\ \lambda_2 = 0$, we have $\begin{pmatrix} -3x_1^2 \\ 4x_2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} = 0$, $x_1^2 + x_2^2 = 5$

$4x_2 + 2\lambda_1 x_2 = 0$, we get $x_2 = 0$ or $\lambda_1 = -2$.

if $x_2 = 0$, we have $x_1 = \pm\sqrt{5}$, but $g_2(x) \le 0$ is not satisfied.

if $\lambda_1 = -2$, we have $x_1 = 0$ or $-\frac{4}{3}$, $x_2 = \sqrt{5}$ or $\frac{\sqrt{29}}{3}$

only $(x_1, x_2) = (0, \sqrt{5})$ satisfy $g_2(x) \le 0$.

hence $(0, \sqrt{5})$ is stationary point.

if $\lambda_1 \ne 0,\ \lambda_2 \ne 0$, we have $x_1^2 + x_2^2 = 5$, $x_2 = x_1^2 + 2$

we have solution $x_1^2 = \frac{-5 + \sqrt{29}}{2}$, $x_2 = \frac{-1 + \sqrt{29}}{2}$

$\begin{pmatrix} 2x_1 & 2x_1 \\ 2x_2 & -1 \end{pmatrix}$ is full rank, so $\left(\pm\sqrt{\frac{-5+\sqrt{29}}{2}},\ \frac{-1+\sqrt{29}}{2} \right)$ is stationary points.

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we find the minimal point is $(0,2)$, $\text{Min } f(x) = 8$.

$(0,2)$ is global minimizer. $\square$

3. Consider the following optimization problem, where n is an integer greater than 2024:

$$\underset{x \in \mathbb{R}^n}{\text{Minimize}} \quad \frac{1}{2}\sum_{i=1}^{n}(x_i - i)^2$$

$$\text{Subject to} \quad 1 + \sum_{i=1}^{n} i x_i \le 0.$$

You may leave the term S_n in your answer, where $S_n := \sum_{i=1}^{n} i^2$.

(a) (10 points) For each $c > 0$, define

$$q_c(x) := \frac{1}{2}\sum_{i=1}^{n}(x_i - i)^2 + \frac{c}{2}\left(1 + \sum_{i=1}^{n} i x_i\right)_+^2$$

Argue that q_c is convex and find the global minimizer of q_c.

3. (a) ① $\nabla q_c(x) = \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_i - i \\ \vdots \\ x_n - n \end{pmatrix} + c \cdot \left(1 + \sum_{i=1}^{n} i x_i\right)_+ \cdot \begin{pmatrix} 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ n \end{pmatrix}$

since $\frac{\partial^2}{\partial x_i^2}(x_i - i)^2 = 2 > 0$, we know $(x_i - i)^2$ is convex,

so $\frac{1}{2}\sum_{i=1}^{n}(x_i - i)^2$ is convex.

assume that $y := 1 + \sum_{i=1}^{n} i x_i$, y is linear affine of x.

and y_+^2 is convex, $\Rightarrow$ we have $\frac{c}{2}\left(1 + \sum_{i=1}^{n} i x_i\right)_+^2$, $c > 0$ is convex

Hence, $q_c(x)$ is convex.

Lemma: $g(y) = y_+^2$ is convex.

Proof: $\forall \lambda \in (0,1)$, if $y_1, y_2 < 0$, we have

$\lambda \cdot g(y_1) + (1-\lambda) g(y_2) = g(\lambda y_1 + (1-\lambda) y_2) = 0$

if $y_1, y_2 \ge 0$, since y^2 is convex, the formula above still holds.

if $y_1 \ge 0$, and $y_2 < 0$, $\lambda g(y_1) + (1-\lambda) g(y_2) = \lambda g(y_1) = \lambda y_1^2$

$g(\lambda y_1 + (1-\lambda) y_2) \le g(\lambda y_1) = \lambda^2 y_1^2 \le \lambda y_1^2$.

Hence $g(y) = y_+^2$ is convex.

② $$\nabla q_c(x) = \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_i - i \\ \vdots \\ x_n - n \end{pmatrix} + c\cdot\left(1+\sum_{i=1}^n i x_i\right)_+ \cdot \begin{pmatrix} 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ n \end{pmatrix} = 0$$

we have $x_i - i + c\cdot\left(1+\sum_{i=1}^n i x_i\right)_+ \cdot i = 0$, for $\forall\, i = 1, 2, \cdots, n$

$$x_i = i\left[1 - c\cdot\left(1+\sum_{i=1}^n i x_i\right)_+\right]$$

if $1+\sum_{i=1}^n i x_i < 0$, we have $x_i = i$,

$1+\sum_{i=1}^n i x_i = 1 + S_n > 0$, it's contradiction,

if $1+\sum_{i=1}^n i x_i \geq 0$, define $z := 1 - c\cdot\left(1+\sum_{i=1}^n i x_i\right)$, $y = \sum_{i=1}^n i x_i$

$\sum_{i=1}^n i x_i = \sum_{i=1}^n i^2 \cdot z$, that is $y = S_n \cdot (1 - c(1+y))$

$y = \dfrac{S_n(1-c)}{1+cS_n}$, $z = \dfrac{1-c}{1+cS_n}$

$1+\sum_{i=1}^n i x_i = \dfrac{1+S_n}{1+cS_n} \geq 0$, satisfy the assumption.

we get $x_i = i\cdot\dfrac{1-c}{1+cS_n}$, for $i = 1, 2, \cdots, n$,

is the global minimizer, since convexity.

(b) (10 points) For each $\mu > 0$, define

$$f_\mu(x) := \frac{1}{2}\sum_{i=1}^{n}(x_i - i)^2 - \mu \ln\left(-1 - \sum_{i=1}^{n} i x_i\right).$$

Argue that f_μ is convex and find the global minimizer of f_μ. You may use without proof the fact that the function $t \mapsto -\ln(-1-t)$ is convex (as an extended real-valued function).

(b) ① $(x_i - i)^2$ is convex, since $\frac{\partial^2}{\partial x_i^2}(x_i - i)^2 = 2 > 0$.

so $\frac{1}{2}\sum_{i=1}^{n}(x_i - i)^2$ is convex.

assume $t := \sum_{i=1}^{n} i x_i$, t is affine projection of x.

and $\mu > 0$, $-\mu \ln(-1-t)$ is convex.

it implies $-\mu \ln\left(-1 - \sum_{i=1}^{n} i x_i\right)$ is convex.

Hence, $f_\mu(x)$ is convex.

② $\nabla f_\mu(x) = \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_i - i \\ \vdots \\ x_n - n \end{pmatrix} - \frac{\mu}{1 + \sum_{i=1}^{n} i x_i} \cdot \begin{pmatrix} 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ n \end{pmatrix} = 0$

we have $x_i - i - \frac{\mu}{1 + \sum_{i=1}^{n} i x_i} \cdot i = 0$ for $i = 1, 2, \cdots n$.

assume that $y := \sum_{i=1}^{n} i x_i$,

$x_i = i + \frac{\mu}{1+y} \cdot i = i\left(1 + \frac{\mu}{1+y}\right)$

$y = \sum_{i=1}^{n} i x_i = S_n \cdot \left(1 + \frac{\mu}{1+y}\right)$, $\Rightarrow$ $y = \frac{S_n - 1 \pm \sqrt{(S_n - 1)^2 + 4(1+\mu)S_n}}{2}$

$x_i = i\left(1 + \frac{\mu}{1+y}\right)$, $\forall\, i = 1, 2, \cdots, n$, and $y < -1$

we get $\boxed{y = \frac{S_n - 1 - \sqrt{(S_n - 1)^2 + 4(1+\mu)S_n}}{2}} < -1$ holds for $\forall\, \mu > 0$.

the global minimizer is $x_i = i + \frac{\mu}{1+y} \cdot i$, for $i = 1, 2, \cdots n$. $\square$

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