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1. (a) ① $\theta > 0$, $f(x) = \begin{cases} \frac{1}{\theta} & \text{if } \theta < x < 2\theta \\ 0 & \text{otherwise} \end{cases}$

$E(X) = \int_{\theta}^{2\theta} \frac{1}{\theta} \cdot x \, dx = \frac{3}{2}\theta$

$\frac{3}{2}\hat{\theta}_{MME} = \overline{X}$, $\hat{\theta}_{MME} = \frac{2}{3}\overline{X}$.

② $\log L(\theta; x_1 \cdots x_n) = \begin{cases} \sum_{i=1}^{n} -\log\theta, & \text{if } \forall x_i, \text{ s.t. } \theta < x_i < 2\theta \iff \frac{x_i}{2} < \theta < x_i \\ 0 & \text{otherwise} \end{cases}$

$\frac{\partial \log L(\theta)}{\partial \theta} = \frac{-n}{\theta} < 0$,

$\hat{\theta}_{MLE} = \arg\max_{\theta} \frac{\partial \log L(\theta)}{\partial \theta} = \frac{1}{2}\max\{x_1, \cdots x_n\}$

(b) $\theta < 0$, $f(x) = \begin{cases} -\frac{1}{\theta} & \text{if } x \in (\theta, 0) \\ 0 & \text{otherwise} \end{cases}$

① $E(X) = \int_{\theta}^{0} -\frac{1}{\theta} x \, dx = \frac{1}{2}\theta$

$\hat{\theta}_{MME} = 2\overline{X}$,

② $\log L(\theta, x_1 \cdots, x_n) = \begin{cases} \sum_{i=1}^{n} -\log(-\theta) & \text{if } \theta < x_i < 0, \\ 0 & \text{otherwise} \end{cases}$

$\frac{\partial \log L(\theta)}{\partial \theta} = -\frac{n}{\theta} > 0$ is monotonically ascending.

$\hat{\theta}_{MLE} = \min(x_1, \cdots, x_n)$

(c) $\hat{\theta}_{MME} = \begin{cases} \frac{2}{3}\overline{X} & \text{if } \theta > 0 \\ 2\overline{X} & \text{if } \theta < 0 \end{cases}$

$\hat{\theta}_{MLE} = \begin{cases} \frac{1}{2}\max\{x_1 \cdots x_n\} & \text{if } \theta > 0 \\ \min\{x_1 \cdots x_n\} & \text{if } \theta < 0 \end{cases}$?

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2、 (a) $p_X(k) = \theta(1-\theta)^k$, $k=0,1,\cdots$

$= \exp\{\log\theta + k\cdot\log(1-\theta)\}$

it is exponential class

$\sum_{i=1}^{n} X_i$ is sufficient complete statistic

(b) MGF: $M_X(t) = E(e^{tx}) = \sum_{x=0}^{+\infty} e^{tx}\cdot\theta(1-\theta)^x = \sum_{x=0}^{+\infty}[e^t\cdot(1-\theta)]^x\cdot\theta$

$= \dfrac{\theta}{1-e^t(1-\theta)}$, $0<e^t(1-\theta)<1$

$X_i=k$ stands for k successive true and 1 false.

$\sum X_i = x$ stands for x times true in first $n-1+x$ trials, next is false

$P(\sum_{i=1}^{n} X_i = x) = \binom{x+n-1}{x}\theta^x(1-\theta)^n$ $x=0,1,\cdots$

(c) $\sum X_i$ is sufficient complete statistic,

$E(\sum X_i) = nE(X) = n\cdot\left(\frac{1-\theta}{\theta}\right)$

$E(X_1) = \frac{1-\theta}{\theta}$? unbiased estimator ?

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3. (a) $H_0: \theta = 1/2$ null hypothesis, simple hypothesis

$H_1: \theta < \frac{1}{2}$ alternative hypothesis, composite hypothesis

	Ground Truth H_0	H_1
Reject H_0	type 1 error	true
Accept H_0	true	type 2 error

(b) set $\theta' = \frac{1}{2}$, $\theta'' < \frac{1}{2}$

$$\frac{L(\theta'; x_1 \cdots x_n)}{L(\theta''; x_1 \cdots x_n)} = \frac{\prod_{i=1}^{n} \theta'^{x_i} (1-\theta')^{1-x_i}}{\prod_{i=1}^{n} \theta''^{x_i} (1-\theta'')^{1-x_i}} = \prod_{i=1}^{n} \left(\frac{\theta'}{\theta''} \cdot \frac{1-\theta''}{1-\theta'}\right)^{x_i} \cdot \left(\frac{1-\theta'}{1-\theta''}\right) \le k$$

$\Leftrightarrow$ since $\frac{1-\theta'}{1-\theta''} < 1$, $\sum_{i=1}^{n} x_i \cdot \log\left(\frac{\theta'}{\theta''} \cdot \frac{1-\theta''}{1-\theta'}\right) \le k_1$

$\Leftrightarrow$ $\frac{\theta'}{\theta''} > 1$, $\sum_{i=1}^{n} x_i \le c$.

$C = \{x_i \mid \sum_{i=1}^{n} x_i \le c\}$ is critical region, when $\alpha = P(\sum_{i=1}^{n} x_i \le c \mid \theta = 1/2)$

(c) power function: $\gamma_C(\theta) = P(\sum_{i=1}^{n} x_i \le c \mid \theta) = P(\{x_1 \cdots x_n\} \in C \mid \theta)$

$$= \binom{n}{c} \theta^{c} (1-\theta)^{n-c}, \quad \text{if } c = 0, 1, \cdots, n$$

$$= \binom{n}{\lfloor c \rfloor} \theta^{\lfloor c \rfloor} (1-\theta)^{n-\lfloor c \rfloor}, \quad \text{if } 0 < c < n$$

(d) $\alpha = P(\{x_1 \cdots x_n\} \in C \mid \theta = 1/2) = P(\sum_{i=1}^{n} x_i \le c \mid \theta = 1/2)$

$$= \binom{n}{c} \left(\tfrac{1}{2}\right)^n \quad \text{if } c = 0, 1, \cdots, n$$

$$= \binom{n}{\lfloor c \rfloor} \left(\tfrac{1}{2}\right)^n \quad \text{if } 0 < c < n.$$

(e) likelihood ratio test: $\hat{\theta} = \arg\max_{\theta} L(\theta; x_1 \cdots x_n) = 1/2$

$$\hat{\Omega} = \arg\max_{\theta \le \frac{1}{2}} L(\theta; x_1 \cdots x_n)$$

$$= \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{1-x_i}$$

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$$\log L(\theta; x_1 - x_n) = (\sum_{i=1}^{n} x_i)\log\theta + (\sum_{i=1}^{n} 1-x_i)\log(1-\theta)$$

$$\frac{\partial \log L}{\partial\theta} = \frac{\sum x_i}{\theta} + \frac{n-\sum x_i}{\theta-1} = \frac{n\theta - \sum x_i}{\theta(\theta-1)} = 0$$

$$\Rightarrow \theta = \bar{x},$$

$$\frac{\partial^2 \log L}{\partial\theta^2} = -\frac{\sum x_i}{\theta^2} - \frac{n-\sum x_i}{(\theta-1)^2} \le 0,$$

so $\hat{\Omega} = \arg\max_{\theta\le 1/2} L(\theta; x_1 \cdots x_n) = \min(\bar{x}, 1/2) \le 1/2$

critical region $\bar{C} = \left\{ \frac{L(\hat{\theta})}{L(\hat{\Omega})} \le k \right\}$

$$= \left\{ \frac{\prod (\frac{1}{2})^{x_i} (\frac{1}{2})^{1-x_i}}{\prod (\hat{\Omega})^{x_i} (\hat{\Omega})^{1-x_i}} \le k \right\}$$

$= \left\{ \sum_{i=1}^{n} x_i \le c \right\}$ is the same as UMP test.

4. UMP def: $\forall \mu'' \ne 0,\ \mu' = 0$

$\alpha = P(\{x_1 \cdots x_n\} \in C \mid \mu')$

$\forall A$, s.t. $P(\{x_1 - x_n\} \in A \mid \mu') = \alpha$

$P(\{x_1 \cdots x_n\} \in C \mid \mu'') \ge P(\{x_1 \cdots x_n\} \in A \mid \mu'')$

$\forall \mu'' \ne 0,\ \mu' = 0$

$\alpha = P(\sqrt{n}|\bar{x}| > z(\alpha/2) \mid \mu' = 0)$

$\forall A$, s.t. $P(\{x_1 \cdots x_n\} \in A \mid \mu') = \alpha$

$P(\sqrt{n}|\bar{x}| > z(\alpha/2) \mid \mu'') \ge P(\{x_1 \cdots x_n\} \in A \mid \mu'')$?

$\sqrt{n}(\bar{x} - \mu) \sim N(0,1)$

$P(\sqrt{n}|\bar{x}| > z(\alpha/2) \mid \mu = 0) = P(\sqrt{n}\bar{x} > z(\alpha/2)) + P(\sqrt{n}\bar{x} < -z(\alpha/2)) = \alpha.$

$P(\sqrt{n}|\bar{x}| > z(\alpha/2) \mid \mu \ne 0) = P(\sqrt{n}(\bar{x}-\mu) > z(\alpha/2) - \mu\sqrt{n} \mid \mu \ne 0) + P(\sqrt{n}(\bar{x}-\mu) < -z(\alpha/2) - \mu\sqrt{n} \mid \mu \ne 0)$

we need to find A. A = ?

Find a UMP test.

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$$f_X(x) = \frac{1}{\sqrt{2\pi}} \cdot \exp\left\{-\frac{(x-\mu)^2}{2}\right\}$$

$$\frac{L(\mu';x_1 \cdots x_n)}{L(\mu'';x_1 - x_n)} = \frac{(\frac{1}{\sqrt{2\pi}})^n \cdot \exp\left\{-\sum_{i=1}^{n}\frac{(x_i-\mu')^2}{2}\right\}}{(\frac{1}{\sqrt{2\pi}})^n \cdot \exp\left\{-\sum_{i=1}^{n}\frac{(x_i-\mu'')^2}{2}\right\}}$$

$$= \exp\left\{\sum_{i=1}^{n}\frac{(x_i-\mu'')^2}{2} - \frac{(x_i-\mu')^2}{2}\right\}$$

$$= \exp\left\{\sum_{i=1}^{n}(\mu'-\mu'')\left(x_i - \frac{\mu'+\mu''}{2}\right)\right\} \leq k$$

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$\Longleftrightarrow$ if $\mu' > \mu''$, $\sum_{i=1}^{n} x_i \leq c$

so that $A := \left\{\sum_{i=1}^{n} x_i \leq c\right\}$, $\alpha = P\left(\sum_{i=1}^{n} x_i \leq c \,\middle|\, \mu = 0\right) = P\left(\sum_{i=1}^{n} x_i \leq -z(\alpha)\right)$

$P\left(\sqrt{n}|\bar{x}| > z(\alpha/2) \,\middle|\, \mu''\right) \neq P\left(\sum_{i=1}^{n} x_i \leq -z(\alpha) \,\middle|\, \mu''\right)$, RHS is best test when $\mu'' < 0$

hence $P\left(\sqrt{n}|\bar{x}| > z(\alpha/2) \,\middle|\, \mu''\right) < P\left(\sum_{i=1}^{n} x_i \leq -z(\alpha) \,\middle|\, \mu''\right)$.

NOT UMP. $\square$

5. (a) $\forall\ \sigma'^2 = 1,\ \sigma''^2 > 1$ $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\}$

$$\frac{L(\sigma'^2)}{L(\sigma''^2)} = \frac{(\frac{1}{\sqrt{2\pi}})^n \cdot (\frac{1}{\sigma'})^n \cdot \exp\left\{\sum_{i=1}^{n}\frac{-(x_i-\mu)^2}{2\sigma'^2}\right\}}{(\frac{1}{\sqrt{2\pi}})^n \cdot (\frac{1}{\sigma''})^n \cdot \exp\left\{\sum_{i=1}^{n}\frac{-(x_i-\mu)^2}{2\sigma''^2}\right\}}$$

$$= \left(\frac{\sigma''}{\sigma'}\right)^n \cdot \exp\left\{\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\sigma''^2} - \frac{(x_i-\mu)^2}{2\sigma'^2}\right\} \leq k$$

$$\Longleftrightarrow \sum_{i=1}^{n}(x_i-\mu)^2\left(\frac{1}{2\sigma''^2} - \frac{1}{2\sigma'^2}\right) \leq k_1$$

$$\Longleftrightarrow \sum_{i=1}^{n}(x_i-\mu)^2 \geq c$$

$\alpha = P\left(\sum_{i=1}^{n}(x_i-\mu)^2 \geq c \,\middle|\, \sigma^2 = 1\right)$ 、

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(b) $\bar{\mu} \triangleq \underset{\mu}{\operatorname{argmax}}\ L(\mu, 6^2=1)$

$\hat{\mu}, \hat{6}^2 \triangleq \underset{\mu, 6^2 \geq 1}{\operatorname{argmax}}\ L(\mu, 6^2)$

$L(\mu, 6^2=1) = (\frac{1}{\sqrt{2\pi}})^n \cdot \exp\{\sum_{i=1}^n -\frac{(x_i-\mu)^2}{2}\}$

$\frac{\partial}{\partial\mu}\left(-\sum_{i=1}^n \frac{(x_i-\mu)^2}{2}\right) = \sum_{i=1}^n (x_i-\mu)$, $\frac{\partial^2}{\partial\mu^2}\left(-\sum_{i=1}^n \frac{(x_i-\mu)^2}{2}\right) = -n < 0$

$\bar{\mu} = \bar{X}$,

$\log L(\mu, 6^2) = n\log\frac{1}{\sqrt{2\pi}} - n\log 6 + \sum_{i=1}^n -\frac{(x_i-\mu)^2}{26^2}$.

$\frac{\partial L}{\partial\mu} = \sum_{i=1}^n \frac{(x_i-\mu)}{6^2}$, $\frac{\partial L}{\partial 6} = -\frac{n}{6} + \frac{\sum(x_i-\mu)^2}{6^3}$

$\Rightarrow \hat{\mu} = \bar{X}$, $\hat{6} = \frac{\sum_{i=1}^n (x_i-\hat{\mu})^2}{n} = \frac{\sum_{i=1}^n (x_i-\bar{X})^2}{n}$

since $\hat{6} \geq 1$, $\hat{6} = \max\{1, \frac{\sum(x_i-\bar{X})^2}{n}\}$

$\frac{L(\bar{\mu}, \bar{6}^2=1)}{L(\hat{\mu}, \hat{6})} = \frac{(\frac{1}{\sqrt{2\pi}\bar{6}})^n \cdot \exp\{-\sum_{i=1}^n \frac{(x_i-\bar{\mu})^2}{2\bar{6}^2}\}}{(\frac{1}{\sqrt{2\pi}\hat{6}})^n \exp\{-\sum \frac{(x_i-\hat{\mu})^2}{2\hat{6}^2}\}}$

$= (\frac{\hat{6}}{6})^n \cdot \exp\{\sum_{i=1}^n \frac{(x_i-\hat{\mu})^2}{2\hat{6}^2} - \frac{(x_i-\bar{\mu})^2}{2\bar{6}^2}\} \leq k$

$\Leftrightarrow \sum_{i=1}^n (x_i-\bar{X})^2 \geq c$

$\alpha = \max_{\mu} P(\sum_{i=1}^n (x_i-\bar{X})^2 \geq c \mid 6^2=1, \mu)$. $\square$

6. (a)

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