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1. Suppose X follows an exponential distribution with pdf $f_X(x) = \lambda e^{-\lambda x}$ on $x > 0$, and zero elsewhere. Find the moment generating function of X . Find the expectation of $E \frac{1}{\sqrt{X}}$.

$$\begin{aligned} \text{MGF: } M_X(t) &= E(e^{tx}) = \int_0^{+\infty} e^{tx} \cdot \lambda e^{-\lambda x} dx \\ &= \frac{\lambda}{t-\lambda} e^{(t-\lambda)x} \Big|_0^{+\infty} \\ &= \frac{\lambda}{\lambda-t} \quad (t < \lambda) \end{aligned}$$

$$E\left(\frac{1}{\sqrt{X}}\right) = \int_0^{+\infty} \frac{1}{\sqrt{x}} \cdot \lambda e^{-\lambda x} dx$$

$$= \int_0^{+\infty} 2\lambda e^{-\lambda x} d\sqrt{x}$$

$$(y := \sqrt{x}) \quad = \int_0^{+\infty} 2\lambda e^{-\lambda y^2} dy$$

$$(\lambda > 0) \quad = \int_0^{+\infty} 2\lambda e^{-(\lambda y)^2} d\lambda y$$

now we compute $\int_0^{+\infty} e^{-t^2} dt$

$$\left(\int_0^{+\infty} e^{-t^2} dt \right)^2 = \int_0^{+\infty} \int_0^{+\infty} \exp(-t^2 - p^2) dt dp$$

substitute $\begin{cases} t = r \cos \theta \\ p = r \sin \theta \end{cases}$ into the formula.

$$\int_0^{+\infty} \int_0^{\pi/2} \exp(-r^2) \left| \frac{\partial(t, p)}{\partial(r, \theta)} \right| dr d\theta = \int_0^{+\infty} \int_0^{\pi/2} \exp(-r^2) \cdot r dr d\theta$$

$$= \frac{\pi}{4}. \Rightarrow \int_0^{+\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$$

hence, $E\left(\frac{1}{\sqrt{X}}\right) = \sqrt{\pi}/2. \square$

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2. Let X_1, \dots, X_n be a random sample from the pdf $f(x|\theta) = \frac{1}{\theta-1}$ on $1 \leq x \leq \theta$ and zero elsewhere, with $\theta > 1$.

a. Find the method of moments estimator $\hat{\theta}_{MME}$ for θ .

b. Find the maximum likelihood estimator $\hat{\theta}^2_{MLE}$ for θ^2 .

c. Find the bias of $\hat{\theta}^2_{MLE}$.

$$2. (a) MME. \quad E(x) = \int_1^\theta x \cdot \frac{1}{\theta-1} dx = \frac{1}{2}(\theta+1) \quad , \quad (\theta \geq 1)$$

$$\bar{x} = \frac{1}{n}(\theta_{MME} + 1) \Rightarrow \hat{\theta}_{MME} = 2\bar{x} - 1.$$

$$(b) \hat{\theta}_{MLE} = \operatorname{argmin}_\theta L(\theta) := \operatorname{argmin}_\theta -\sum_{i=1}^n \log f(x_i|\theta)$$

$$= \operatorname{argmin}_\theta \sum_{i=1}^n (\theta-1) \quad , \quad \text{subject to } \max(x_i) \leq \theta.$$

Since $L(\theta)$ is increasing on $(1, \infty)$.

$$\hat{\theta}_{MLE} = \max\{x_i\}_{i=1}^n. \Rightarrow \hat{\theta}_{MLE} = \max\{x_i\}^2.$$

$$(c) E(\hat{\theta}_{MLE}^2) = ?$$

We compute cdf of $\hat{\theta}_{MLE}$ first.

$$P(\max\{x_i\} \leq y) = P(\max\{x_i\} \leq y) \quad (y \geq 0)$$

$$= P(X \leq y)^n \quad , \quad \text{st. } y \in [0, \infty]$$

$$\text{pdf of } \hat{\theta}_{MLE} \text{ vs } f_{\hat{\theta}_{MLE}}(y) = n \cdot P(X \leq y)^{n-1} \cdot f_X(y)$$

$$= n \cdot \left(\frac{y-1}{\theta-1}\right)^{n-1} \cdot \frac{1}{\theta-1}$$

$$E(\hat{\theta}_{MLE}^2) = \int_1^\theta y^2 \cdot n \cdot \left(\frac{y-1}{\theta-1}\right)^{n-1} \cdot \frac{1}{\theta-1} dy$$

$$= \int_0^{\theta-1} (y+1)^2 \cdot \frac{n}{(\theta-1)^n} \cdot y^{n-1} dy$$

$$= \frac{n}{(\theta-1)^n} \left(\frac{1}{n+2} \cdot (\theta-1)^{n+2} + \frac{2}{n+1} \cdot (\theta-1)^{n+1} + \frac{1}{n} \cdot (\theta-1)^n \right)$$

$$= \frac{n}{n+2} \theta^2 + \frac{2n}{(n+1)(n+2)} \theta + \frac{2}{(n+1)(n+2)}$$

$$\text{bias}(\hat{\theta}_{MLE}^2) = E(\hat{\theta}_{MLE}^2) - \theta^2 = -\frac{2}{n+2} \theta^2 + \frac{2n}{(n+1)(n+2)} \theta + \frac{2}{(n+1)(n+2)}. \quad \square$$

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3. Let X_1, \dots, X_n be a random sample from the pdf $f(x|\theta) = \theta(1-x)^{\theta-1}$, $0 < x < 1$, $\theta > 0$, zero elsewhere.

- Find the method of moments estimator $\hat{\theta}_{\text{MME}}$ for θ .
- Find the maximum likelihood estimator $\hat{\theta}_{\text{MLE}}$ for θ .
- Compute the Fisher information $I(\theta)$.

3. (a) $E(X) = \int_0^1 x \cdot \theta(1-x)^{\theta-1} dx$

$$= \int_0^1 (1-x) \theta x^{\theta-1} dx$$

$$= \frac{1}{\theta+1}$$

$$\frac{1}{\theta+1} = \bar{x} \Rightarrow \hat{\theta}_{\text{MME}} = \frac{1}{\bar{x}} - 1.$$

(b) $\hat{\theta}_{\text{MLE}} = \arg \min_{\theta} - \sum_{i=1}^n \log \theta + (\theta-1) \log(1-x_i)$,

$$L(\theta) := - \sum_{i=1}^n \log(\theta) + (\theta-1) \log(1-x_i)$$

$$\frac{\partial L(\theta)}{\partial \theta} = - \sum_{i=1}^n \frac{1}{\theta} + \log(1-x_i)$$

$$\frac{\partial L(\theta)}{\partial \theta} \Big|_{\theta=\theta} = 0 \Rightarrow \hat{\theta}_{\text{MLE}} = \frac{n}{\sum_{i=1}^n \log(1-x_i)} \geq 0$$

(c) $I(\theta) = E\left(\frac{\partial \log f(x|\theta)}{\partial \theta}\right) = -E\left(\frac{\partial^2 \log f(x|\theta)}{\partial \theta^2}\right)$

$$= E\left(\frac{1}{\theta^2}\right)$$

$$I(\theta) = \frac{1}{\theta^2}. \quad \square$$

4. Let X_1, \dots, X_n be a random sample from the pdf $f(x|\theta) = -(\theta+\theta^2)(1+x)^{\theta-1}x$, $-1 < x < 0$, $\theta > 0$, zero elsewhere.

- Find the method of moments estimator $\hat{\theta}_{\text{MME}}$ for θ .
- Find the maximum likelihood estimator $\hat{\theta}_{\text{MLE}}$ for θ .
- Compute the minimum possible variance of all the unbiased estimators for θ^2 .

4. (a) $E(X) = \int_{-1}^0 -x^2(\theta+\theta^2)(1+x)^{\theta-1} dx$

$$= \int_0^1 -(\theta+\theta^2)(1-x)^2 x^{\theta-1} dx$$

$$= -\frac{2}{\theta+2}.$$

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$$\bar{x} = -\frac{2}{\theta_{MME} + 2}$$

$$\Rightarrow \hat{\theta}_{MME} = -\frac{2}{\bar{x}} - 2$$

$$(b) \hat{\theta}_{MLE} = \arg \min_{\theta} L(\theta),$$

$$L(\theta) := -\sum_{i=1}^n \log \theta + \log(\theta+1) + (\theta-1) \log(1+x_i) + \log(-x_i)$$

$$\left. \frac{\partial L(\theta)}{\partial \theta} \right|_{\theta=\hat{\theta}} = 0 \Rightarrow \sum_{i=1}^n \left(\frac{1}{\theta} + \frac{1}{\theta+1} + \log(1+x_i) \right) = 0$$

$$\Rightarrow \frac{n}{\theta} + \frac{n}{\theta+1} + \sum_{i=1}^n \log(1+x_i) = 0$$

$$\text{set } c := \frac{\sum_{i=1}^n \log(1+x_i)}{n}, \text{ we have } c \in (-\infty, 0).$$

substitute into formula above, we have

$$\frac{1}{\theta} + \frac{1}{\theta+1} + c = 0, \Rightarrow \text{soluti. } \Leftrightarrow \theta = \frac{-(c+2) \pm \sqrt{c^2+4}}{2c}. \quad \text{so}$$

$$\text{since } \frac{\partial^2 L(\theta)}{\partial \theta^2} = +\frac{n}{\theta^2} + \frac{n}{(\theta+1)^2} > 0, \text{ and we know on interval } (0, +\infty).$$

$\frac{\partial L(\theta)}{\partial \theta}$ is strictly increasing, so it has only 1 minimizer

$$\text{so } \hat{\theta}_{MLE} = \frac{-(c+2) - \sqrt{c^2+4}}{2c}, \text{ and } c := \frac{\sum_{i=1}^n \log(1+x_i)}{n}.$$

(C). using R-C lower bound theorem,

we have A unbiased estimator $g(x_1, \dots, x_n)$

$$\text{Var}(g(x_1, \dots, x_n)) \geq \frac{(2\theta)^2}{n \cdot L(\theta)}$$

$$L(\theta) = -E\left(\frac{\partial^2 \log f(x|\theta)}{\partial \theta^2}\right) = E\left(\frac{1}{\theta^2} + \frac{1}{(\theta+1)^2}\right)$$

$$= \frac{1}{\theta^2} + \frac{1}{(\theta+1)^2}.$$

$$\text{Var}(g(x_1, \dots, x_n)) \geq \frac{4\theta^2}{n\left(\frac{1}{\theta^2} + \frac{1}{(\theta+1)^2}\right)}. \text{ that is the lower bound of Variance. } \square$$

5. Let X_1, \dots, X_n be a random sample from $\text{Poisson}(\lambda)$.

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- Find the maximum likelihood estimator $\hat{\lambda}_{\text{MLE}}$ for λ .
- Compute the Fisher information $I(\lambda)$.
- Compute the bias of $(\bar{X})^2$ as an estimator for λ^2 .
- Compute the minimum possible variance of all the unbiased estimators for λ .

5. (a) $f(x|\lambda) = e^{-\lambda} \cdot \frac{\lambda^x}{x!}$, s.t. $x \geq 0$, $\lambda > 0$

$$\hat{\lambda}_{\text{MLE}} = \arg \min_{\lambda} L(\lambda)$$

$$L(\lambda) = \sum_{i=1}^n \lambda - x_i \log \lambda + \log(x_i!)$$

$$\frac{\partial L(\lambda)}{\partial \lambda} = n - \frac{n \cdot \bar{x}}{\lambda}, \quad \left. \frac{\partial^2 L(\lambda)}{\partial \lambda^2} \right|_{\lambda=\hat{\lambda}} = 0 \Rightarrow \hat{\lambda} = \bar{x}.$$

and $\frac{\partial^2 L(\lambda)}{\partial \lambda^2} = \frac{n \bar{x}}{\lambda^2} > 0$, hence $\hat{\lambda} = \bar{x}$ is minimizer.

$$\hat{\lambda}_{\text{MLE}} = \bar{x}.$$

(b) $I(\lambda) = -E\left(\frac{\partial^2 \log f(x|\lambda)}{\partial \lambda^2}\right)$

$$= E\left(\frac{\lambda}{\bar{x}^2}\right)$$

$$E(X) = \sum_{i=0}^{\infty} x \cdot e^{-\lambda} \cdot \frac{\lambda^x}{x!} = \lambda$$

$$I(\lambda) = \frac{1}{\lambda}.$$

(c) $E((\bar{X})^2) = \text{Var}(\bar{X}) + E(\bar{X})^2$

$$= \frac{1}{n} \text{Var}(X) + E(X)^2$$

$$\text{Var}(X) = \sum_{i=0}^{\infty} [(x-i)x + x] e^{-\lambda} \cdot \frac{\lambda^x}{x!} - E(X)^2 = \lambda.$$

$$= \frac{\lambda}{n} + \lambda^2$$

$$\text{bias}((\bar{X})^2) = \frac{\lambda}{n}.$$

(d) A estimator $T = g(X_1, \dots, X_n)$, using R-C lower bound theorem,

$$\text{Var}(T) \geq \frac{(-\frac{1}{\lambda^2})^2}{n \cdot I(\lambda)}, \quad I(\lambda) = \frac{1}{\lambda}, \quad \text{so lower bound is } \frac{1}{n \lambda^3}. \quad \square$$

$$\text{Var}(T) \geq \frac{1}{n \cdot \lambda^3}.$$

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