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assume problem is:

2. (a) (1) Min $f(x) := 2(x_1)^3 + x_2$

s.t. $g_1(x) := (x_1)^2 + (x_2)^2 - 6 \le 0$

$g_2(x) := x_1 - 3x_2 + 1 \le 0$

$\nabla^2 g_1(x) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \succ 0, \quad \nabla^2 g_2(x) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

so $g_1(x)$, $g_2(x)$ are all convex.

we find a slater point $\bar{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, s.t. $g_1(\bar{x}) < 0$, $g_2(\bar{x}) < 0$

then MFCQ holds in feasible region.

(2) KKT: ① $g_1(x) = (x_1)^2 + (x_2)^2 - 6 \le 0$

$g_2(x) = x_1 - 3x_2 + 1 \le 0$

② $\nabla f(x) + \lambda_1 \nabla g_1(x) + \lambda_2 \nabla g_2(x) = 0$

$\begin{pmatrix} 6(x_1)^2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -3 \end{pmatrix} = 0$

$\lambda_1 \ge 0$, $\lambda_2 \ge 0$

③ $\lambda_1 \cdot g_1(x) = 0$, $\lambda_2 \cdot g_2(x) = 0$.

if $\lambda_1 = 0$, $\lambda_2 = 0$, $\nabla f(x) = \begin{pmatrix} 6x_1^2 \\ 1 \end{pmatrix} \ne 0$, contradictory

if $\lambda_1 = 0$, $\lambda_2 > 0$, $x_1 - 3x_2 + 1 = 0$,

$\begin{pmatrix} 6x_1^2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -3 \end{pmatrix} = 0$, $\Rightarrow \lambda_2 = \frac{1}{3}$, $x_1 = \pm\frac{\sqrt{2}}{6}$

$x = \begin{pmatrix} \frac{\sqrt{2}}{6} \\ \frac{1}{3}(\frac{\sqrt{2}}{6} + 1) \end{pmatrix}$ or $\begin{pmatrix} -\frac{\sqrt{2}}{6} \\ \frac{1}{3}(-\frac{\sqrt{2}}{6} + 1) \end{pmatrix}$,

and satisfy $g_1(x) \le 0$.

$x = \begin{pmatrix} \frac{\sqrt{2}}{6} \\ \frac{\sqrt{2}}{18} + \frac{1}{3} \end{pmatrix}$, $\begin{pmatrix} -\frac{\sqrt{2}}{6} \\ -\frac{\sqrt{2}}{18} + \frac{1}{3} \end{pmatrix}$ is stationary point.

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if $\lambda_1 > 0,\ \lambda_2 = 0, \qquad x_1^2 + x_2^2 = 6.$

$$\begin{pmatrix} 6x_1^2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} = 0, \Rightarrow x_2 = -\frac{1}{2\lambda_1},\ x_1^2 = 6 - \frac{1}{4\lambda_1^2}$$

$$6x_1^2 + 2\lambda_1 x_1 = 0 \Rightarrow x_1 = 0 \text{ or } -\frac{\lambda_1}{3}$$

so $x = \begin{pmatrix} 0 \\ -\sqrt{6} \end{pmatrix}$, $\lambda_1 = \frac{\sqrt{6}}{12}$ or $x = \begin{pmatrix} \pm\sqrt{6 - \frac{1}{108 + 6\sqrt{323}}} \\ -\frac{1}{2\sqrt{27 + \frac{3}{2}\sqrt{323}}} \end{pmatrix}$, $\lambda_1^2 = 27 + \frac{3}{2}\sqrt{323}$

check if it satisfy $g_2(x) \le 0$.

hence only $x = \begin{pmatrix} -\sqrt{6 - \frac{1}{108 + 6\sqrt{323}}} \\ -\frac{1}{2\sqrt{27 + \frac{3}{2}\sqrt{323}}} \end{pmatrix}$ is stationary

if $\lambda_1 > 0,\ \lambda_2 > 0, \qquad x_1^2 + x_2^2 = 6, \quad x_1 - 3x_2 = -1$

we solve it and get $x = \begin{pmatrix} \frac{-1+3\sqrt{59}}{10} \\ \frac{3+\sqrt{59}}{10} \end{pmatrix}$ or $x = \begin{pmatrix} \frac{-1-3\sqrt{59}}{10} \\ \frac{3-\sqrt{59}}{10} \end{pmatrix}$

and $\nabla f(x) + \lambda_1 \nabla g_1(x) + \lambda_2 \nabla g_2(x) = \begin{pmatrix} 6x_1^2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -3 \end{pmatrix} = 0$

$$\lambda_1 = -\frac{18x_1^2 + 1}{6x_1 + 2x_2}, \quad \lambda_2 = \frac{2x_1 - 12x_1^2 x_2}{6x_1 + 2x_2}$$

if $x = \begin{pmatrix} \frac{-1+3\sqrt{59}}{10} \\ \frac{3+\sqrt{59}}{10} \end{pmatrix}$, $\lambda_1 < 0$, contradictory

if $x = \begin{pmatrix} \frac{-1-3\sqrt{59}}{10} \\ \frac{3-\sqrt{59}}{10} \end{pmatrix}$, $\lambda_1 > 0$, $6x_1 + 2x_2 < 0$, $2x_1 - 12x_1^2 x_2 > 0$, $\lambda_2 < 0$ contradictory.

(b) assume $n = 1$, $g_1(x) = x^3 - 1$, $g_2 := x^3 + 1$

if $g_1(x) = 0$, $x = 1$, contradictory

if $g_2(x) = 0$, $x = -1$, $g_1(x) \ne 0$, $\lambda \cdot \nabla g_2(x) = -\lambda 3x^2 = 0 \Rightarrow \lambda = 0$

$g_1(x) + g_2(x) = 2x^3$, if $g_1(x) + g_2(x) = 0$, $x = 0$, $\nabla g_1(x) + \nabla g_2(x) = 0$, MFCQ does not hold

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Hence we set $g_1(x) = \begin{pmatrix} x_1^3 - 1 \\ x_2^3 - 1 \\ \vdots \\ x_n^3 - 1 \end{pmatrix}$, $g_2(x) = \begin{pmatrix} x_1^3 + 1 \\ x_2^3 + 1 \\ \vdots \\ x_n^3 + 1 \end{pmatrix}$

it's the counterexample.

3. (a) $\iff$ Min $(y_1^2)^{1.5} + z_1$

s.t. $y_1^2 \geq x_1^2 + x_2^2 + x_3^4 + 1$, $y_1 \geq 0$,

$z_1 \geq 7x_1 - x_2$, $z_1 \geq 0$

$z_2 \geq |x_2|$, $2 \geq \dfrac{x_2^2 + x_3^2}{x_1 + 1} + z_2$, $x_1 \geq 1$

$\iff$ Min $y_2 + z_1$

s.t. $y_2 \geq y_1^3$,

$y_1^2 \geq x_1^2 + x_2^2 + y_3^2 + 1$, $y_3 \geq x_3^2$, $y_1 \geq 0$

$z_1 \geq 7x_1 - x_2$, $z_1 \geq 0$

$z_2 \geq x_2 \geq -z_2$,

$(2 - z_2)(x_1 + 1) \geq x_2^2 + x_3^2$, $x_1 \geq 1$

$\iff$ Min $y_2 + z_1$

s.t. $y_2 \cdot y_1 \geq y_4^2$, $y_4 \geq y_1^2$, $y_1 \geq 0$

$y_1^2 \geq x_1^2 + x_2^2 + y_3^2 + 1$, $y_3 \geq x_3^2$

$z_1 \geq 7x_1 - x_2$, $z_1 \geq 0$, $z_2 \geq x_2 \geq -z_2$

$(2 - z_2)(x_1 + 1) \geq x_2^2 + x_3^2$, $x_1 \geq 1$

$\iff$ Min $y_2 + z_1$

s.t. $\begin{pmatrix} y_2 & y_4 \\ y_4 & y_1 \end{pmatrix} \succeq 0$, $\begin{pmatrix} y_4 & y_1 \\ y_1 & 1 \end{pmatrix} \succeq 0$

$\begin{pmatrix} y_1 \cdot I_4 & \begin{pmatrix} x_1 \\ x_2 \\ y_3 \\ 1 \end{pmatrix} \\ (x_1 \ x_2 \ y_3 \ 1) & y_1 \end{pmatrix} \succeq 0$, $\begin{pmatrix} y_3 & x_3 \\ x_3 & 1 \end{pmatrix} \succeq 0$

$z_1 \geq 7x_1 - x_2$, $z_1 \geq 0$, $z_2 \geq x_2 \geq -z_2$

$\begin{pmatrix} (2 - z_2) \cdot I_2 & \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \\ (x_2 \ x_3) & x_1 + 1 \end{pmatrix} \succeq 0$, $x_1 \geq 1$

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(b) $C \succ 0$.

Primal: $\min \ tr(CX)$

s.t. $tr(A_1X) = b_1$

$tr(A_2X) = b_2$

$X \succeq 0$

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

dual: $\max \ b^T y$

s.t. $C - y_1 \cdot A_1 + y_2 A_2 \succeq 0$.

(2) ① find slater point of primal prob.

$\bar{X} \triangleq \begin{pmatrix} 2 & & & 0 \\ & 1 & & \\ & & 2 & \\ 0 & & & 5/2 \end{pmatrix}$, satisfy $tr(A_1\bar{X}) = b_1$, $tr(A_2\bar{X}) = b_2$

② find slater point of dual

$\bar{y} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $C \succ 0$.

hence $V_p = V_d$, and $tr(C\bar{X}) \geq V_p = V_d \geq b^T \bar{y}$

V_p, V_d is finite, then V_p, V_d is attainable.

(3) we swap row 2 and row 3, column 2 and column 3.

in X, the Problem is

$\min \ tr(\bar{C} X)$

s.t. $x_{11} + x_{22} = 4$

$2x_{33} + 2x_{34} + 2x_{44} = 7$, $X \succeq 0$

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$\bar{C}=\begin{pmatrix} \begin{smallmatrix}3&1\\1&3\end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix}1&1\\1&5\end{smallmatrix}\end{pmatrix}$, the Min value keep stable.

exactly they are the same problem.

consider $C_1 := \begin{pmatrix}3&1\\1&3\end{pmatrix}$, $x_{11}+x_{22}=4$, we set $X_1 := \begin{pmatrix}2&-2\\-2&2\end{pmatrix} \succeq 0$

consider $C_2 := \begin{pmatrix}1&1\\1&5\end{pmatrix}$, $tr(\bar{C}X) = tr(C_1X_1) + tr(C_2X_2)$

if we set $X_2 = \begin{pmatrix}7/2&0\\0&0\end{pmatrix}$, $tr(\bar{C}X) = 11.5$, then $v_p \le 11.5$.

if we set $X_2 = \begin{pmatrix}4&-0.6\\-0.6&0.1\end{pmatrix} \succeq 0$,

$tr(C_2X_2) = 3.3$

$tr(\bar{C}\cdot\bar{X}) = tr(C_1X_1) + tr(C_2X_2) = 8 + 3.3 = 11.3$

hence $v_p \le 11.3$. □

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4. (a) $\det\left[\begin{pmatrix}5 & -1\\ -1 & 1\end{pmatrix} - \lambda\cdot I_2\right] = \lambda^2 - 6\lambda + 4 = 0$

$\lambda = 3 \pm \sqrt{5}$

eigenvalues are $1,\ 3\pm\sqrt{5}$

(b) $\frac{1}{2}x^T B x \geq \frac{1}{2}\lambda_{\min}(B)\,\|x\|_2^2 = \frac{1}{2}(3-\sqrt{5})\,\|x\|_2^2$

hence, $\|x^k\|_2^2 \leq \frac{1}{2}x^{kT}Bx^k \leq f(x^k) \leq f(x^{k-1}) - \alpha_{k-1}\sigma\cdot\nabla f(x^{k-1})^T\nabla f(x^{k-1})$

$\leq f(x^{k-1}) \leq \cdots \leq f(x^0)$

$\{x^k\}$ is bounded.

(c) assume $g(y) \triangleq \sqrt{y^2+3}$, $y := x_1 - x_2 + 3x_3$

$\nabla_x^2 g(y(x)) = \begin{pmatrix}1\\-1\\3\end{pmatrix}\frac{3}{(y^2+3)^{3/2}}(1,-1,3)$

$\|\nabla^2 f(x)\|_2 \leq \left\|\begin{pmatrix}1\\-1\\3\end{pmatrix}\right\|_2^2\cdot\left\|\frac{3}{(y^2+3)^{3/2}}\right\|_2 + \|B\|_2$

$\leq 11\cdot\frac{1}{\sqrt{3}} + (3+\sqrt{5}) := L$

$\alpha \leq \frac{2}{L} < 0.17$, so $\{x^k\}$ accumulation point is stationary.

5. assume that $y_+ \triangleq \max\{y, 0\}$,

$g(y_+) \triangleq (y_+)^2 = \max\{y,0\}^2$

$g'(y_+) = 2\cdot y_+ = 2\cdot\max\{y,0\}$

$\Rightarrow \nabla q_c(x) = 2x + \frac{c}{2}\cdot\begin{pmatrix}0\\-1\\2\end{pmatrix}\cdot 2\max\{y,0\}$

$= 0$

$y := 6 - x_2 + 2x_3.$

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$$\Rightarrow \quad 2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + c\cdot\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \max\{y, 0\} = 0$$

if $y \leq 0$, $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

$y = 6 > 0$, contradictory.

if $y > 0$. $2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -c\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} y$

$$\Rightarrow x_1 = 0,\ x_2 = \frac{c}{2}y,\ x_3 = -cy$$

$$y = 6 - x_2 + 2x_3 \Rightarrow y = \frac{12}{5c+2}$$

$$\Rightarrow x_1 = 0,\ x_2 = \frac{6c}{5c+2},\ x_3 = \frac{-12c}{5c+2}$$

Since q_c is convex, $x = \begin{pmatrix} 0 \\ \frac{6c}{5c+2} \\ \frac{-12c}{5c+2} \end{pmatrix}$ is global minimizer.

$\square$