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1. Suppose X follows an exponential distribution with pdf $f_X(x) = \lambda e^{-\lambda x}$ on $x > 0$, and zero elsewhere. Find the moment generating function of X. Find the expectation of $E\frac{1}{\sqrt{X}}$.

1. MGF: $M_X(t) = E(e^{tx}) = \int_0^{+\infty} e^{tx} \cdot \lambda e^{-\lambda x}\, dx$

$$= \frac{\lambda}{t-\lambda} e^{(t-\lambda)x}\Big|_0^{+\infty}$$

$$= \frac{\lambda}{\lambda - t} \quad (t<\lambda)$$

$$E\left(\frac{1}{\sqrt{x}}\right) = \int_0^{+\infty} \frac{1}{\sqrt{x}} \cdot \lambda e^{-\lambda x}\, dx$$

$$= \int_0^{+\infty} 2\lambda e^{-\lambda x}\, d\sqrt{x}$$

$(y := \sqrt{x})$ $$= \int_0^{+\infty} 2\lambda e^{-\lambda y^2}\, dy$$

$(\lambda > 0)$ $$= \int_0^{+\infty} 2\sqrt{\lambda} e^{-(\sqrt{\lambda} y)^2}\, d\sqrt{\lambda} y$$

now we compute $\int_0^{+\infty} e^{-t^2} dt$

$$\left(\int_0^{+\infty} e^{-t^2} dt\right)^2 = \int_0^{+\infty}\int_0^{+\infty} \exp(-t^2-p^2)\, dt\, dp$$

substitute $\begin{cases} t = r\cos\theta \\ p = r\sin\theta \end{cases}$ into the formula.

$$\int_0^{+\infty}\int_0^{\pi/2} \exp(-r^2)\left|\frac{\partial(t,p)}{\partial(r,\theta)}\right| d\theta\, dr = \int_0^{+\infty}\int_0^{\pi/2} \exp(-r^2)\cdot r\, d\theta\, dr$$

$$= \frac{\pi}{4}. \quad \Rightarrow \int_0^{+\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$$

hence, $E\left(\frac{1}{\sqrt{x}}\right) = \sqrt{\lambda\pi}$. $\square$

2. Let $X_1, \ldots, X_n$ be a random sample from the pdf $f(x|\theta) = \frac{1}{\theta - 1}$ on $1 \le x \le \theta$ and zero elsewhere, with $\theta > 1$.

a. Find the method of moments estimator $\widehat{\theta}_{\mathsf{MME}}$ for θ.

b. Find the maximum likelihood estimator $\widehat{\theta^2}_{\mathsf{MLE}}$ for θ^2.

c. Find the bias of $\widehat{\theta^2}_{\mathsf{MLE}}$.

2. (a) MME. $E(X) = \int_1^\theta x \cdot \frac{1}{\theta-1}\, dx = \frac{1}{2}(\theta+1)$, $(\theta > 1)$

$\bar{X} = \frac{1}{2}(\hat{\theta}_{MME} + 1) \Rightarrow \hat{\theta}_{MME} = 2\bar{X} - 1.$

(b) $\hat{\theta}_{MLE} = \underset{\theta}{\operatorname{argmin}}\, L(\theta) := \underset{\theta}{\operatorname{argmin}} - \sum_{i=1}^{n} \log f(x_i|\theta)$

$= \underset{\theta}{\operatorname{argmin}} \sum_{i=1}^{n} (\theta - 1)$, subject to $\max(x_i) \le \theta$.

Since $L(\theta)$ is increasing on $(1, +\infty)$.

$\hat{\theta}_{MLE} = \max\{x_i\}_{i=1}^{n}. \Rightarrow \hat{\theta}^2_{MLE} = \max\{x_i\}^2.$

(c) $E(\hat{\theta}_{MLE}^2) = ?$

we compute cdf of $\hat{\theta}_{MLE}$ first,

$P(\max\{x_i\} \le y) = P(\max\{x_i\} \le y)$ $(y \ge 0)$

$= P(X \le y)^n$, s.t. $y \in [0, \theta]$

pdf of $\hat{\theta}_{MLE}$ is $f_{\hat{\theta}_{MLE}}(y) = n \cdot P(X \le y)^{n-1} \cdot f_X(y)$

$= n \cdot \left(\frac{y-1}{\theta-1}\right)^{n-1} \cdot \frac{1}{\theta-1}$

$$E(\hat{\theta}_{MLE}^2) = \int_1^\theta y^2 \cdot n \cdot \left(\frac{y-1}{\theta-1}\right)^{n-1} \cdot \frac{1}{\theta-1}\, dy$$

$$= \int_0^{\theta-1} (y+1)^2 \frac{n}{(\theta-1)^n} \cdot y^{n-1}\, dy$$

$$= \frac{n}{(\theta-1)^n}\left(\frac{1}{n+2} \cdot (\theta-1)^{n+2} + \frac{2}{n+1}(\theta-1)^{n+1} + \frac{1}{n}(\theta-1)^n\right)$$

$$= \frac{n}{n+2}\theta^2 + \frac{2n}{(n+1)(n+2)}\theta + \frac{2}{(n+1)(n+2)}$$

$$\text{bias}(\hat{\theta}^2_{MLE}) = E(\hat{\theta}_{MLE}^2) - \theta^2 = -\frac{2}{n+2}\theta^2 + \frac{2n}{(n+1)(n+2)}\theta + \frac{2}{(n+1)(n+2)}. \quad \square$$

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3. Let $X_1, \dots, X_n$ be a random sample from the pdf $f(x|\theta) = \theta(1-x)^{\theta-1}$, $0 < x < 1$, $\theta > 0$, zero elsewhere.

a. Find the method of moments estimator $\widehat{\theta}_{\mathsf{MME}}$ for θ.

b. Find the maximum likelihood estimator $\widehat{\theta}_{\mathsf{MLE}}$ for θ.

c. Compute the Fisher information $I(\theta)$.

3. (a) $E(X) = \int_0^1 x \cdot \theta(1-x)^{\theta-1}\,dx$

$= \int_0^1 (1-x)\,\theta\, x^{\theta-1}\,dx$

$= \frac{1}{\theta+1}$

$\frac{1}{\hat{\theta}+1} = \bar{X} \Rightarrow \hat{\theta}_{MME} = \frac{1}{\bar{X}} - 1.$

(b) $\hat{\theta}_{MLE} = \arg\min_{\theta} -\sum_{i=1}^{n} \log\theta + (\theta-1)\log(1-x_i).$

$L(\theta) := -\sum_{i=1}^{n} \log(\theta) + (\theta-1)\log(1-x_i)$

$\frac{\partial L(\theta)}{\partial\theta} = -\sum_{i=1}^{n} \frac{1}{\theta} + \log(1-x_i)$

$\frac{\partial L(\theta)}{\partial\theta}\Big|_{\theta=\hat{\theta}} = 0 \Rightarrow \hat{\theta}_{MLE} = \frac{n}{-\sum_{i=1}^{n}\log(1-x_i)} > 0$

(c) $I(\theta) = E\left(\frac{\partial \log f(x|\theta)}{\partial\theta}\right) = -E\left(\frac{\partial^2 \log f(x|\theta)}{\partial\theta^2}\right)$

$= E\left(\frac{1}{\theta^2}\right)$

$= \frac{1}{\theta^2}$ $\quad I(\theta) = \frac{1}{\theta^2}.$ □

4. Let $X_1, \dots, X_n$ be a random sample from the pdf $f(x|\theta) = -(\theta+\theta^2)(1+x)^{\theta-1}x$, $-1 < x < 0$, $\theta > 0$, zero elsewhere.

a. Find the method of moments estimator $\widehat{\theta}_{\mathsf{MME}}$ for θ.

b. Find the maximum likelihood estimator $\widehat{\theta}_{\mathsf{MLE}}$ for θ.

c. Compute the minimum possible variance of all the unbiased estimators for θ^2.

4. (a) $E(X) = \int_{-1}^{0} -x^2(\theta+\theta^2)(1+x)^{\theta-1}\,dx$

$= \int_0^1 -(\theta+\theta^2)(x-1)^2 x^{\theta-1}\,dx$

$= -\frac{2}{\theta+2}.$

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$$\bar{X} = -\frac{2}{\hat{\theta}_{MME}+2}$$

$$\Rightarrow \hat{\theta}_{MME} = -\frac{2}{\bar{X}} - 2$$

(b) $\hat{\theta}_{MLE} = \underset{\theta}{\arg\min}\ L(\theta)$,

$$L(\theta) := -\sum_{i=1}^{n} \log\theta + \log(\theta+1) + (\theta-1)\log(1+x_i) + \log(-x_i)$$

$$\left.\frac{\partial L(\theta)}{\partial\theta}\right|_{\theta=\hat{\theta}} = 0 \Rightarrow \sum_{i=1}^{n}\left(\frac{1}{\theta} + \frac{1}{\theta+1} + \log(1+x_i)\right) = 0$$

$$\Rightarrow \frac{n}{\theta} + \frac{n}{\theta+1} + \sum_{i=1}^{n}\log(1+x_i) = 0$$

set $c := \frac{\sum_{i=1}^{n}\log(1+x_i)}{n}$, we have $c \in (-\infty, 0)$.

substitute into formula above, we have

$\frac{1}{\theta} + \frac{1}{\theta+1} + c = 0$, $\Rightarrow$ soluti. is $\theta = \frac{-(c+2) \pm \sqrt{c^2+4}}{2c}$. < 0

Since $\frac{\partial^2 L(\theta)}{\partial\theta^2} = +\frac{n}{\theta^2} + \frac{n}{(\theta+1)^2} > 0$, and we know on interval $(0, +\infty)$.

$\frac{\partial L(\theta)}{\partial\theta}$ is strictly increasing, so it has only 1 minimizer

so $\hat{\theta}_{MLE} = \frac{-(c+2) - \sqrt{c^2+4}}{2c} > 0$.

(c). using R-C lower bound theorem,

we have $\forall$ unbiased estimator $g(X_1, \cdots, X_n)$

$$Var(g(X_1, \cdots, X_n)) \geq \frac{(2\theta)^2}{n \cdot I(\theta)}$$

$$I(\theta) = -E\left(\frac{\partial^2 \log f(x|\theta)}{\partial\theta^2}\right) = E\left(\frac{1}{\theta^2} + \frac{1}{(\theta+1)^2}\right)$$

$$= \frac{1}{\theta^2} + \frac{1}{(\theta+1)^2}.$$

$Var(g(X_1 \cdots X_n)) \geq \frac{4\theta^2}{n\left(\frac{1}{\theta^2} + \frac{1}{(\theta+1)^2}\right)}$. that is the lower bound of Variance. □

5. Let $X_1, \ldots, X_n$ be a random sample from Poisson(λ).

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a. Find the maximum likelihood estimator $\widehat{\lambda}_{\mathsf{MLE}}$ for λ.

b. Compute the Fisher information $I(\lambda)$.

c. Compute the bias of $(\overline{X})^2$ as an estimator for λ^2.

d. Compute the minimum possible variance of all the unbiased estimators for $\frac{1}{\lambda}$.

5. (a) $f(x|\lambda) = e^{-\lambda} \cdot \frac{\lambda^x}{x!}$, s.t. $x \geq 0,\ \lambda > 0$

$\hat{\lambda}_{MLE} = \underset{\lambda}{\arg\min}\ L(\lambda)$

$L(\lambda) = \sum_{i=1}^{n} \lambda - x_i \log\lambda + \log(x_i!)$

$\frac{\partial L(\lambda)}{\partial \lambda} = n - \frac{n \cdot \bar{x}}{\lambda}$, $\left.\frac{\partial L(\lambda)}{\partial \lambda}\right|_{\lambda = \hat{\lambda}} = 0 \Rightarrow \hat{\lambda} = \bar{x}$.

and $\frac{\partial^2 L(\lambda)}{\partial \lambda^2} = \frac{n\bar{x}}{\lambda^2} > 0$, hence $\hat{\lambda} = \bar{x}$ is minimizer.

$\hat{\lambda}_{MLE} = \bar{x}$.

(b) $I(\lambda) = -E\left(\frac{\partial^2 \log f(x|\lambda)}{\partial \lambda^2}\right)$

$= E\left(\frac{x}{\lambda^2}\right)$

$E(x) = \sum_{i=0}^{\infty} x \cdot e^{-\lambda} \cdot \frac{\lambda^x}{x!} = \lambda$

$I(\lambda) = \frac{1}{\lambda}$.

(c) $E((\bar{X})^2) = Var(\bar{X}) + E(\bar{X})^2$

$= \frac{1}{n} Var(X) + E(X)^2$ $\qquad Var(X) = \sum_{i=0}^{\infty} [(x-1)x + x]\, e^{-\lambda} \cdot \frac{\lambda^x}{x!} - E(X)^2 = \lambda$.

$= \frac{\lambda}{n} + \lambda^2$

bias $((\bar{X})^2) = \frac{\lambda}{n}$.

(d) $\forall$ estimator $Y = g(X_1, \cdots, X_n)$, using R-C lower bound theorem.

$Var(Y) \geq \frac{(-\frac{1}{\lambda^2})^2}{n \cdot I(\lambda)}$. $I(\lambda) = \frac{1}{\lambda}$, so lower bound is $\frac{1}{n\lambda^3}$. □

$Var(Y) \geq \frac{1}{n \cdot \lambda^3}$.

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