NetID: 2411245bg   Name: Zhong Qiaoyang

1. Let $X_1, \ldots, X_n$ be a random sample from a geometric distribution that has pmf $f(x|\theta) = (1-\theta)^x \theta$, $x = 0, 1, 2 \ldots$, $0 < \theta < 1$, zero elsewhere. Show that $\sum_{i=1}^{n} X_i$ is a sufficient statistic for θ.

1. $\quad f(x|\theta) = \exp\left\{ x \cdot \log(1-\theta) + \log\theta \right\}$ is exponential class

hence $\quad \sum x_i$ is sufficient statistics. $\square$

2. Let $X_1, \ldots, X_n$ be a random sample from a $\mathsf{Beta}(\theta, 5)$. Show that the product $X_1 \times \cdots \times X_n$ is a sufficient statistic for θ.

2. $\quad f(x|\theta, 5) = \dfrac{x^{\theta-1}(1-x)^4}{B(\theta, 5)}$

$L(\theta, 5) = (\prod x_i)^{\theta-1} \cdot \prod \dfrac{(1-x_i)^4}{B(\theta, 5)}$

$\prod (1-x_i)^4 \quad$ is irrelevant with θ,

hence $\quad \prod x_i$ is sufficient statistics. $\square$

3. Write the pdf

$$f(x|\theta) = \frac{1}{6\theta^4} x^3 e^{-x/\theta}, \quad 0 < x < \infty, \; 0 < \theta < \infty,$$

zero elsewhere, in the exponential form. If $X_1, X_2, ..., X_n$ is a random sample from this distribution, find a complete sufficient statistic Y_1 for θ and the unique function $\phi(Y_1)$ of this statistic that is the MVUE of θ. Is $\phi(Y_1)$ itself a complete sufficient statistic?

3. ① X is exponential form,

$f(x|\theta) = \exp\left\{ -\log(6\theta^4) + 3\log x - \dfrac{x}{\theta} \right\}$

$\Rightarrow \sum_{i=1}^{n} X_i$ is complete sufficient statistics, $Y_1 = \sum_{i=1}^{n} X_i$.

② $E(x) = \displaystyle\int_{0}^{+\infty} \dfrac{1}{6\theta^4} x^4 \cdot e^{-x/\theta} \, dx$

$= \displaystyle\int_{0}^{+\infty} -\dfrac{1}{6\theta^3} x^4 \, d\,e^{-x/\theta} = \int_{0}^{+\infty} \dfrac{4}{6\theta^3} x^3 e^{-x/\theta} \, dx$

$= \displaystyle\int_{0}^{+\infty} \dfrac{4 \times 3 \times 2 \times 1}{6} e^{-x/\theta} \, dx = 4\theta$

$$E\left(\tfrac{1}{4n}\sum_{i=1}^{n}X_i\right) = \theta \quad, \text{ is unbiased estimator.}$$

$$\phi(T_1) = E\left(\tfrac{1}{4n}\sum_{i=1}^{n}X_i \,\Big|\, \sum_{i=1}^{n}X_i\right) = \tfrac{1}{4}\bar{X},$$

$\tfrac{1}{4}\bar{X}$ is unique MVUE.

③ since $T_1 = \sum_{i=1}^{n}X_i$ is complete sufficient statistics,

$\tfrac{1}{4}\bar{X}$ is complete sufficient statistics as well. ⌐

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4. Let $X_1, \ldots, X_n$ be a random sample from the density function $f(x|\theta) = \frac{1}{\theta}e^{x/\theta}$, $x < 0$, $\theta > 0$. Determine the MVUE of θ.

4.

$$f(x|\theta) = \exp\left\{ -\log\theta + \tfrac{x}{\theta} \right\}$$

so $\sum_{i=1}^{n}X_i$ is complete sufficient statistics

$$E(X) = -\theta, \quad E(\bar{X}) = -\theta.$$

$-\bar{X}$ is unbiased estimator of θ,

$\phi(\bar{X}) = E(-\bar{X}|\bar{X}) = -\bar{X}$ is the unique MVUE for θ. ⌐

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5. Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution $[0, \theta]$, for some unknown parameter $\theta > 0$. Is $X_{(n)}$, the largest order statistic among all samples, a sufficient statistic for θ?

5. $X_{(n)} = \max\{X_1, \cdots, X_n\}$

$$P(X_{(n)} \le t) = P(X_1 \le t) \cdot P(X_2 \le t) \cdots P(X_n \le t)$$
$$= \left(\tfrac{t}{\theta}\right)^{n}, \text{ for } 0 \le t \le \theta.$$

$$f_{X_{(n)}}(t|\theta) = n \cdot \tfrac{t^{n-1}}{\theta^{n}}, \quad 0 \le t \le \theta.$$

$$\frac{f_{X_1}(x_1|\theta) \cdots f_{X_n}(x_n|\theta)}{f_{X_{(n)}}(t|\theta)} = \frac{\left(\tfrac{1}{\theta}\right)^{n}}{n \cdot \tfrac{t^{n-1}}{\theta^{n}}} = \frac{1}{n \cdot t^{n-1}} \quad \text{is independent with } \theta,$$

hence $X_{(n)}$ is sufficient. ⌐