Ex (1.1)

1.

X	-1	0	1
P	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
X^2	1	0	1

$\rightarrow$

Y	0	1
P	$\frac{1}{3}$	$\frac{2}{3}$

2. a. $P(|X|<1) = 1-2\cdot P(X<-1) = 1-2F(-1) = \frac{1}{27}$

$P(X^2<9) = 1$

b. $P(|X|<1) = F(1) - F(-1) = \frac{2}{9}$

$P(X^2<9) = P(-3<X<3) = P(X<3) = F(3) = \frac{25}{36}$

3.

$P(C_1 \cup C_2) = P(X\in(1,2)) + P(X\in(4,5))$

$F(x) = \int f(x)\,dx + C = -\frac{1}{x} + C \quad \because F(1)=0 \quad \therefore C=1$

$\therefore F(x) = 1-\frac{1}{x} \quad x\in(1,+\infty) \quad \therefore P(C_1\cup C_2) = F(2)+F(5)-F(4) = \frac{11}{20}$

$P(C_1\cap C_2) = 0$

4. $F(x) = x^2$ if $x\in(0,1)$ $\quad P(X\geq\frac{3}{4} \mid X\geq\frac{1}{2}) = \frac{P(X\geq\frac{3}{4})}{P(X\geq\frac{1}{2})} = \frac{1-F(\frac{3}{4})}{1-F(\frac{1}{2})}$

$= \frac{1-\frac{9}{16}}{1-\frac{1}{4}} = \frac{7}{12}$

5. $F(x) = \frac{x^3}{27}$ if $x\in(0,3)$ $\quad P(X^3<a) = P(X<a^{\frac{1}{3}}) = F(a^{\frac{1}{3}}) = \frac{a}{27}$

if $a\in(0,27)$ $\therefore$ cdf of Y: $F(x) = \frac{x}{27}$, $x\in(0,27)$ $=0$, $x\leq 0$ $=1$ $x\geq 27$

$\therefore$ pdf of Y: $f(x) = \frac{1}{27}$ $x\in(0,27)$, 0 otherwise.

6. $F_X(x) = \frac{x}{\pi}+\frac{1}{2}$ if $x\in(-\frac{\pi}{2},\frac{\pi}{2})$ $\quad P(\tan X<a) = F_Y(\arctan(a))$

$= \frac{1}{\pi}\arctan(a)+\frac{1}{2} \quad \therefore f_Y(a) = \frac{1}{\pi(x^2+1)}$

this ~~a~~ is called the standard Cauchy distribution.

$\because F(x) = x^4$ if $x \in (0,1)$ $\quad F_Y(y) = P_Y(-\log(X^4) < y) = 1 - F(e^{-\frac{y}{4}})$

$= 1 - e^{-y} \quad \because e^{-\frac{y}{4}} \in (0,1) \quad y \in (0, +\infty)$

$\therefore$ cdf of Y $\quad F_Y(y) = 1 - e^{-y} \quad y \in (0, +\infty)$, $\quad 0 \quad y \in (-\infty, 0]$

$\therefore$ pdf of Y $\quad f_Y(y) = e^{-y} \quad y \in (0, +\infty)$, $\quad 0$ otherwise

$\because f(x) = \frac{1}{3} \quad x \in (-1, 2) \quad \therefore F(x) = \frac{x+1}{3}$ if $x \in (-1,2)$, $\quad 0 \quad x \in (-\infty, -1]$, $1 \quad [2, +\infty)$

$\therefore F_Y(y) = P(Y<y) = P(X^2 < y) = P(\sqrt{y} < X < \sqrt{y})$

if $y \in [4, +\infty) \quad F_Y(y) = 1$

if $y \in [1, 4) \quad F_Y(y) = F(\sqrt{y}) = \frac{\sqrt{y}+1}{3}$

if $y \in [0, 1) \quad F_Y(y) = F(\sqrt{y}) - F(-\sqrt{y}) = \frac{2}{3}\sqrt{y}$

if $y \in (-\infty, 0) \quad F_Y(y) = 0$

$$\therefore f_Y(y) = \begin{cases} \frac{1}{6\sqrt{y}} & y \in (1,4) \\ \frac{1}{3\sqrt{y}} & y \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Ex (1,2)

1. $P(0<X_1<\frac{1}{2}, \frac{1}{4}<X_2<1) = \int_0^{\frac{1}{2}}\int_{\frac{1}{4}}^{1} 4x_1x_2\,dx_2\,dx_1 = \int_0^{\frac{1}{2}} 2x_1 \cdot x_2^2\Big|_{\frac{1}{4}}^{1} dx_1$

$= \frac{15}{16}\int_0^{\frac{1}{2}} 2x_1\,dx_1 = \frac{15}{16}\cdot\frac{1}{4} = \frac{15}{64}$

$P(X_1 = X_2) = \int_0^1 \int_{x_1=x_2} 4x_1x_2\,dx_1\,dx_2 = 0$

$P(X_1<X_2) = P(X_1\le X_2) = \int_0^1\int_0^{x_2} 4x_1x_2\,dx_1dx_2 = \int_0^1 2x_2^3\,dx_2 = \frac{1}{2}\cdot x_2^4\Big|_0^1$
$= \frac{1}{2}$

2.

$P(Z\le z) = P(X+Y\le z) = \int_0^z\int_0^{z-x} e^{-x-y}\,dy\,dx = \int_0^z e^{-x}(1 - e^{-(z-x)})\,dx$

$= \int_0^z e^{-x} - e^{-z}\,dx = -e^{-x}\Big|_0^z - e^{-z}x\Big|_0^z = 1-e^{-z}-ze^{-z} \quad z\in(0,+\infty)$

$\therefore P(Z\le 0) = 0 \qquad P(Z\le 6) = 1-e^{-6}-6e^{-6} = 1-7e^{-6}$

pdf of Z $\quad f_Z(z) = ze^{-z}$

3.

$f(x,y)=1 \quad Z=XY \quad \because x,y\in(0,1) \quad \therefore z\in(0,1)$

$\cancel{P(Z\le z) = \int_0^1\int_0^{\frac{z}{x}} 1\,dy\,dx = \int_0^1 y\Big|_0^{\frac{z}{x}} dx = \int_0^1 \frac{z}{x}dx} =$

$P(Z\le z) = \int_0^1\int_0^z 1\,dx\,dy + \int_z^1\int_0^{\frac{z}{x}} 1\,dy\,dx = \int_0^1 z\,dy + \int_z^1 \frac{z}{x}\,dx$

$\therefore F(z) = z - z(-\ln z) = (1-\ln z)z \qquad z\in(0,1)$

$= 0 \quad z\le 0, \quad 1 \quad z\ge 1$

$\therefore f(z) = -\ln z \quad z\in(0,1) \qquad 0$ otherwise.

4.

$P(2X+3Y<1) = \int_0^{\frac{1}{2}}\int_0^{\frac{1-2x}{3}} 6(1-x-y)\,dy\,dx = \int_0^{\frac{1}{2}}\left(2-4x-2x+4x^2-\frac{(1-2x)^2}{3}\right)dx$

$=\frac{1}{3}\int_0^{\frac{1}{2}}(5-14x+8x^2)\,dx = \frac{13}{36}$

$g(X,Y) = XY+2X^2$

$E(g(x,y)) = \iint g(x,y)f(x,y)\,dx\,dy = \int_0^1\int_0^{1-x}(xy+2x^2)\,6(1-x-y)\,dy\,dx$

$= 6\cdot\int_0^1\int_0^{1-x}(xy-3x^2y-xy^2+2x^2-2x^3)\,dy\,dx$

$= 6\cdot\int_0^1\left(\frac{x}{6}+\frac{x^2}{2}-\frac{3}{2}x^3+\frac{5}{6}x^4\right)dx = \left(\frac{x^2}{2}+x^3-\frac{9x^4}{4}+x^5\right)\Big|_0^1$

$=\frac{1}{4}$

5.

$X_2 \backslash X_1$	0	1
0	$\frac{1}{9}$	$\frac{2}{9}$
1	$\frac{2}{9}$	$\frac{4}{9}$

$(X_1,X_2)\quad (Y_1,Y_2) = (X_1-X_2,\ X_1+X_2)$

$(0,0)\to(0,0)$

$(1,0)\to(1,0)$

$(0,1)\to(-1,1)$

$(1,1)\to(0,1)$

$P = \begin{cases}\frac{1}{9}\\ \frac{2}{9}\\ \frac{2}{9}\\ \frac{4}{9}\end{cases}$

$\therefore Y_1 = \begin{cases}0 & \frac{5}{9}\\ 1 & \frac{2}{9}\\ -1 & \frac{2}{9}\end{cases}$

6.

$P_{Y_1Y_2}(y_1,y_2) = P(Y_1=y_1, Y_2=y_2) = \frac{y_1}{36} \quad y_2=1,2,3 \;,\; y_1=y_2, 2y_2, 3y_2$

$= 0 \quad$ otherwise

$P_{Y}(y_1) = \frac{y_1}{36} \quad y_1 = 1,4,9$

$= \frac{y_1}{18} \quad y_1 = 2,3,6.$

7. $x_1 = Y_1/2 \quad x_2 = Y_1/2 + Y_2 \quad J = \begin{vmatrix} 1/2 , & 1 \\ 0, & 1 \end{vmatrix} = \frac{1}{2}$

$\therefore f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2}h(x_1,x_2) = \frac{1}{2}h(\frac{y_1}{2}, y_2+\frac{y_1}{2}) = e^{-y_1-y_2} \quad y_1, y_2 \in (0,+\infty)$

$= 0 \quad$ otherwise.

8.

$x_1 = Y_1Y_2 \quad x_2 = Y_2 \quad J = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = y_2$

$\because \; 0 < x_1 < x_2 < 1 \;$ then $\; 0 < y_1 y_2 < y_2 < 1 \quad \therefore y_1, y_2 \in (0,1)$

$\therefore f_{Y_1,Y_2}(y_1,y_2) = 8y_1y_2^3 \quad y_1,y_2 \in (0,1)$

$f_{Y_1}(y_1) = \int_0^1 8y_1y_2^3 dy_2 = 2y_1 \quad y_1 \in (0,1)$

9.

$f_{X_1}(x_1) = \int_0^\infty e^{-x_1-x_2} dx_2 = e^{-x_1} \quad \therefore f_{X_2}(x_2) = e^{-x_2}.$

$\therefore$ pdf of Y $f_Y(y) = \int_0^y f_{X_1}(x_1)\cdot f_{X_2}(y-x_1)\,dx_1 = \int_0^y e^{-y} dx_1 = ye^{-y}$

$\therefore f_Y(y) = ye^{-y} \quad y \in (0,+\infty) \quad 0$ otherwise

MGF: $M_Y(t) = E(e^{ty}) = \int_0^{+\infty} y\cdot e^{-(1-t)y} dy$

if $t \geq 1$ donot exist $\quad$ if $t<1$ $M_Y(t) = \int_0^{+\infty} (1-t)ye^{-(1-t)y} dy \frac{1}{1-t}$

let $(1-t)y = z$, $M_Y(t) = \int_0^{+\infty} ze^{-z} dz \frac{1}{(1-t)^2} = \frac{1}{(1-t)^2}$

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10.

$E(Y|X=x) = \int_x^{+\infty} y\cdot f_{Y|X}(Y|x)\,dy$

$f_{Y|X}(Y|x) = \frac{f(x,y)}{f_Y(x)} = \frac{2e^{-x-y}}{2\int_x^{+\infty}2e^{-x-y}dy} = \frac{2e^{-x-y}}{2e^{-x}\cdot(-e^{-y}|_0^{+\infty})} = e^{x-y}$

$E(Y|X=x) = \int_x^{+\infty} y\cdot e^{x-y}dy = e^{x}\int_x^{+\infty} ye^{-y}dy = e^{x}(-(y+1)e^{-y}|_x^{+\infty})$

$= x+1$

11.

$X,Y\sim N(0,1)$ $f_X(x)=f_Y(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

$P(X^2<a) = P(-\sqrt{a}<x<\sqrt{a}) = F(\sqrt{a})-F(-\sqrt{a}) = F_{X^2}(a)$ if $a\geq 0$

$\therefore f_{X^2}(a) = F'_{X^2}(a) = \frac{1}{2\sqrt{a}}f(\sqrt{a}) - (-\frac{1}{2\sqrt{a}})f(-\sqrt{a}) = \frac{1}{\sqrt{a}}\cdot f(\sqrt{a}) = \frac{1}{\sqrt{2\pi a}}e^{-\frac{a}{2}}$

$= 0$ otherwise.

PS: $f_{X^2}(a) = \frac{1}{\sqrt{2\pi a}}e^{-\frac{a}{2}} = \frac{1}{2^{\frac12}\Gamma(\frac12)}a^{(\frac12-1)}e^{-\frac{a}{2}}$ is pdf of χ^2_1

$\therefore X^2, Y^2 \sim \chi^2_1$

$\because M_{Z}(t)$ of $Z\sim\chi^2_k$ is $M_Z(t) = (1-2t)^{-k/2}$ $(2t<1)$

$\therefore M_{X^2+Y^2}(t) = M_{X^2}(t)M_{Y^2}(t) = \cancel{(1-2t)^{-\frac12}}$ $(1-2t)^{-\frac12}\cdot(1-2t)^{-\frac12}$

$= (1-2t)^{-1} = (1-2t)^{-\frac22}$ is the MGF of χ^2_2

$\therefore (X^2+Y^2)\sim\chi^2_2$

$= 0$ otherwise

$\therefore$ pdf of (X^2+Y^2) is $f_{X^2+Y^2} = \frac{1}{2^{\frac22}\Gamma(\frac22)}\cdot x^{\frac22-1}e^{-\frac{x}{2}} = \frac{1}{2}e^{-\frac{x}{2}}$ $x\geq 0$

12. $X, Y \sim N(0,1)$ $X \perp\!\!\!\perp Y$ $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$F_{2X}(z) = P(2X < z) = P(X < \frac{z}{2}) = F_X(\frac{x}{2})$

$\therefore f_{2X}(x) = F_X(\frac{x}{2})' = \frac{1}{2} f_X(x) = \frac{1}{\sqrt{2\pi} \cdot 2} e^{-\frac{x^2}{2}}$ $\therefore 2X \sim N(0,4)$

$f(z) = \int_{-\infty}^{+\infty} f(x) f(z-x) dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2} - \frac{(z-x)^2}{2}} dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}[2x^2 - 2xz + z^2]} dx$

$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(\sqrt{2}x - \frac{\sqrt{2}}{2} z)^2} dx \cdot e^{-\frac{z^2}{4}}$ let $w = \sqrt{2}x$

$= \frac{1}{2\sqrt{\pi}} e^{-\frac{z^2}{4}} \cdot \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(w - \frac{\sqrt{2}}{2} z)^2}{2}} dw = \frac{1}{2\sqrt{\pi}} e^{-\frac{z^2}{4}}$

$\therefore Z \sim N(0,2)$

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