日期: /

1. With a random sample of size n from Exponential(λ), find the bias of $\overline{X}^2$ as an estimator for $1/\lambda^2$.

3.1.1. $f(x|\lambda) = \lambda \cdot e^{-\lambda x}$, $x > 0$, $\lambda > 0$.

$E(\bar{X}^2) = Var(\bar{X}) + E(\bar{X})^2 = \frac{1}{n} Var(X) + E(X)^2$

$= (\frac{1}{\lambda^2}) \cdot \frac{1}{n} + (\frac{1}{\lambda})^2$

$bias(\bar{X}^2) = \frac{1}{n \cdot \lambda^2}$. □

2. With a random sample of size n from Poisson(λ), find the bias of $\overline{X}^2$ as an estimator for λ^2.

3.1.2. $f(x|\lambda) = e^{-\lambda} \cdot \frac{\lambda^x}{x!}$ $x \geq 0$, $\lambda > 0$.

$E(\bar{X}^2) = Var(\bar{X}) + E(\bar{X})^2$

$= \frac{1}{n}\lambda + \lambda^2$

$bias(\bar{X}^2) = \frac{1}{n} \cdot \lambda$. □

1. Find the Fisher information $I(p)$ of the distribution Geometric(p).

3.2.1 $f(x|p) = p \cdot (1-p)^{x-1}$, $x = 1, 2, 3 \cdots$, $p \in (0,1)$.

$I(p) = -E\left(\frac{\partial^2 \log f(x|p)}{\partial p^2}\right)$

$= E\left(\frac{1}{p^2} + \frac{x-1}{(p-1)^2}\right)$ $\left(E(X) = \frac{1-p}{p}\right)$

$= \frac{1}{p^2} + \frac{1-p}{p} \cdot \frac{1}{(p-1)^2} - \frac{1}{(p-1)^2}$

$= \frac{-p^2-p+1}{p^2(1-p)^2}$.

日期: /

2. Find the Fisher information $I(\theta)$ of the distribution $\theta(1-x)^{\theta-1}$, $0 < x < 1$, $\theta > 0$, zero elsewhere.

3.2.2 $$I(\theta) = -E\left(\frac{\partial^2 \log f(x|\theta)}{\partial\theta^2}\right)$$

$$= E\left(\frac{1}{\theta^2}\right) = \frac{1}{\theta^2}$$

3. Let $X_1, \ldots, X_n$ be a random sample from $N(0,\sigma)^2$. Find the MLE $\widehat{\sigma^2}$ for σ^2. Show that $\widehat{\sigma^2}$ is unbiased. Find the variance of $\widehat{\sigma^2}$. Find the R-C lower bound for σ^2.

3.2.3. ① MLE.

$$f(x|\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

$$\hat{\sigma} = \arg\min_{\sigma} L(\sigma)$$

$$= \arg\min_{\sigma} \sum_{i=1}^{n} \log\sqrt{2\pi} + \log\sigma + \frac{x^2}{2\sigma^2}$$

$$\frac{\partial L}{\partial \sigma} = \frac{n}{\sigma} - \frac{\sum_{i=1}^{n} x_i^2}{\sigma^3},$$

$$\left.\frac{\partial L}{\partial \sigma}\right|_{\sigma=\hat{\sigma}} = 0 \Rightarrow \hat{\sigma}^2 = \frac{\sum_{i=1}^{n} x_i^2}{n}.$$

② $$E\left(\frac{\sum_{i=1}^{n} x_i^2}{n}\right) = E(X^2) = Var(X) + E(x)^2$$

$$= \sigma^2$$, so $\hat{\sigma}^2$ is unbiased of σ^2.

③ $$Var(\hat{\sigma}^2) = \frac{1}{n} Var(x^2) = \frac{1}{n}\left(E(x^4) - E(x^2)^2\right)$$

$$= \frac{1}{n}(3\sigma^4 - \sigma^4) = \frac{2}{n}\sigma^4$$

④ $$I(\sigma) = -E\left(\frac{\partial^2 \log f(x|\sigma)}{\partial\sigma^2}\right)$$

$$= E\left(-\frac{1}{\sigma^2} + 3\frac{x^2}{\sigma^4}\right)$$

$$= \frac{2}{\sigma^2}$$

R-C lower bound is $\dfrac{(2\sigma)^2}{n \cdot I(\sigma)} = \dfrac{2\sigma^4}{n}$

hence $\hat{\sigma}^2$ achieve the R-C lower bound. □

日期:

4. Let $X_1, \ldots, X_n$ be a random sample taken from density

$$f(x|\theta) = \theta(x+1)^{-(1+\theta)}, \quad 0 < x < \infty, \quad \theta > 0, \quad \text{zero elsewhere.}$$

Find the C-R lower bound for the variance of all the unbiased estimators of $1/\theta$.

3.2.4. $I(\theta) = -E\left(\frac{\partial^2 \log f(x|\theta)}{\partial \theta^2}\right)$

$= E\left(\frac{1}{\theta^2}\right) = \frac{1}{\theta^2}.$

R-C lower bound of $1/\theta = \frac{\left(-\frac{1}{\theta^2}\right)^2}{n \cdot I(\theta)}$

$= \frac{1}{n \cdot \theta^2}$. □

5. Find the Fisher information $I(\theta)$ of the distribution

$$f(x|\theta) = (\theta^2+\theta)x^{\theta-1}(1-x), \quad x \in [0,1], \quad \theta > 0, \quad \text{zero elsewhere.}$$

3.2.5. $I(\theta) = E\left(-\frac{\partial^2 \log f(x|\theta)}{\partial \theta^2}\right)$

$= E\left(\frac{1}{\theta^2} + \frac{1}{(\theta+1)^2}\right)$

$= \frac{1}{\theta^2} + \frac{1}{(\theta+1)^2}$. □