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1. Let $X_1, \ldots, X_n$ be a random sample from the pdf $f(x;\theta) = \theta x^{-2}$, $0 < \theta \le x < \infty$. Find the method of moments estimator (MME) of θ

2.| $E(x) = \int_\theta^\infty x \cdot \theta \cdot x^{-2}\, dx = \theta \cdot \ln x \Big|_\theta^\infty = +\infty$, divergent.

$E(x^2) = \int_\theta^\infty \theta\, dx = +\infty$, divergent.

can't use MME.

2. Let $X_1, \ldots, X_n$ be a random sample from the pdf $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, $\theta > 0$, zero elsewhere. Find the method of moments estimator (MME) of θ.

2、 $E(x) = \int_0^1 x \cdot \theta x^{\theta-1}\, dx = \frac{\theta}{\theta+1} x^{\theta+1} \Big|_0^1 = \frac{\theta}{\theta+1}$

MME: $\frac{\theta}{\theta+1} = \bar{x}$, $\Rightarrow \hat{\theta} = \frac{\bar{x}}{1-\bar{x}}$.

3. Let $X_1, \ldots, X_n$ be a random sample from the pdf $f(x;\theta) = \theta(1+x)^{-(1+\theta)}$, $0 < x < \infty$, $\theta > 1$, zero elsewhere. Find the method of moments estimator (MME) of $1/\theta$.

3. $E(x) = \int_0^\infty x \cdot \theta \cdot (1+x)^{-(1+\theta)}\, dx$

$= \int_0^{+\infty} x \cdot \theta \cdot (-\frac{1}{\theta})\, d(1+x)^{-\theta}$

$= -x(1+x)^{-\theta} \Big|_0^{+\infty} + \int_0^{+\infty} (1+x)^{-\theta}\, dx$

$(\theta > 1)$

$= 0 + \frac{1}{\theta-1}$

MME: $\bar{x} = \frac{1}{\hat{\theta}-1}$, $\Rightarrow \hat{\theta} = \frac{1+\bar{x}}{\bar{x}} \Rightarrow \frac{1}{\hat{\theta}} = \frac{\bar{x}}{1+\bar{x}}$. □

4. Let $X_1, \ldots, X_n$ be a random sample from the pmf $f(x;\theta) = \frac{(\log\theta)^x}{\theta x!}$, $x = 0, 1, 2, \ldots$, $\theta > 1$, zero elsewhere. Find the method of moments estimator (MME) of θ.

4. $E(x) = \sum_{x=0}^{+\infty} \frac{(\log\theta)^x}{\theta \cdot x!} \cdot x$

$= \sum_{x=0}^{+\infty} \frac{\log\theta}{\theta} \cdot \frac{(\log\theta)^x}{x!} = \theta \cdot \frac{\log\theta}{\theta} = \log\theta$

MME: $\log\hat{\theta} = \bar{x} \Rightarrow \hat{\theta} = e^{\bar{x}} > 0$. □

5. (6.1.1) Let $X_1, \ldots, X_n$ be a random sample from a Gamma$(\alpha = 3, \beta = \theta)$ distribution, $0 < \theta < \infty$. Determine the MLE of θ.

5. MLE: $\hat{\theta} = \arg\min_{\theta} \sum_{i=1}^{n} -\log f(x_i ; \alpha=3, \beta=\theta)$

$$= \arg\min_{\theta} -\sum_{i=1}^{n} (\alpha-1)\log x_i - \frac{x_i}{\beta} - \log(\beta^{\alpha} \cdot \Gamma(\alpha))$$

$$= \arg\min_{\theta} \sum_{i=1}^{n} \frac{x_i}{\theta} + 3\log\theta$$

$$\frac{\partial L(\theta)}{\partial\theta} = \left(\sum_{i=1}^{n} x_i\right)\cdot\left(-\frac{1}{\theta^2}\right) + 3n\cdot\frac{1}{\theta} = 0 \overset{(\theta>0)}{\Longrightarrow} \hat{\theta} = \frac{\bar{X}}{3}.$$

$$\frac{\partial^2 L(\theta)}{\partial\theta^2} = \left(\sum_{i=1}^{n} x_i\right)\cdot\frac{2}{\theta^3} - \frac{3n}{\theta^2}$$

$$\left.\frac{\partial^2 L(\theta)}{\partial\theta^2}\right|_{\theta=\hat{\theta}} = \frac{27n}{\bar{X}^2} > 0. \Rightarrow \hat{\theta} = \frac{\bar{X}}{3} \text{ is minimizer.}$$

6. (6.1.4) Suppose $X_1, \ldots, X_n$ are iid with pdf $f(x;\theta) = 2x/\theta^2$, $0 < x \le \theta$, zero elsewhere. Note this is a nonregular case. Find the MLE $\hat{\theta}$ for θ. Find a constant c so that $E(c\hat{\theta}) = \theta$.

6. MLE: $\hat{\theta} = \arg\min_{\theta} -\sum_{i=1}^{n} \log f(x_i;\theta)$ $\quad \theta \ge \max\{x_i\}_{i=1}^{n}$

$$= \arg\min_{\theta} -\sum_{i=1}^{n} \log(2x_i) - 2\log\theta \qquad \theta \ge \max\{x_i\}_{i=1}^{n}$$

$\frac{\partial L(\theta)}{\partial\theta} = 2n\cdot\frac{1}{\theta}$, $L(\theta)$ is stricty increasing on $[\{x_i\}_{i=1}^{n}, +\infty)$.

$\Rightarrow \hat{\theta} = \max\{x_i\}_{i=1}^{n}$.

$E(\hat{\theta}) = ?$, $P(\max\{X_i\} \le y) = P(X_1 \le y)\cdots P(X_n \le y)$

$$= P(X \le y)^n = \left(\frac{y^2}{\theta^2}\right)^n, \quad y \in (0, \theta]$$

pdf of $\max\{x_i\}$ is $f_{\max\{x\}}(y) = n\cdot\left(\frac{y^2}{\theta^2}\right)^{n-1}\cdot\frac{2y}{\theta^2}$

$$E(\hat{\theta}) = \int_0^{\theta} y\cdot n\left(\frac{y^2}{\theta^2}\right)^{n-1}\cdot\frac{2y}{\theta^2}\,dy = \frac{2n}{\theta^{2n}}\,\frac{1}{2n+1}\,\theta^{2n+1}. = \frac{2n}{2n+1}\cdot\theta$$

$E(c\hat{\theta}) = \theta \Rightarrow c = \frac{2n+1}{2n}$. □

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7. (6.1.6) Let the table

x	0	1	2	3	4	5
Frequency	6	10	14	13	6	1

represent a summary fo a sample of size 50 from a binomial distribution having $n = 5$. Find the MLE of $P(X \geq 3)$.

7. $f(x) = \binom{5}{x} p^{x} (1-p)^{5-x}$

$$\hat{p}_{MLE} = \arg\min_{p} -\sum_{i=1}^{50} \log\binom{5}{x_i} + x \cdot \log p + (5-x)\log(1-p)$$

$$= \arg\min_{p} -\sum_{i=1}^{50} x \cdot \log p + (5-x)\log(1-p)$$

$$\frac{\partial L(p)}{\partial p} = -\sum_{i=1}^{50}\left(\frac{x_i}{p} + \frac{5-x_i}{p-1}\right) = 0$$

$$\Rightarrow \sum_{i=1}^{50} x_i(p-1) + (5-x_i)p = 0 \Rightarrow \hat{p} = \frac{\bar{X}}{5} \quad // \quad \frac{\partial^2 L(p)}{\partial p^2} = +\sum\left[\frac{x_i}{p^2} + \frac{(5-x_i)}{(p-1)^2}\right] > 0$$

$$P(X \geq 3) = \binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p) + p^5$$

$$= \left(\frac{\bar{X}}{5}\right)^3 \left(10\left(1-\frac{\bar{X}}{5}\right)^2 + 5\frac{\bar{X}}{5}\cdot\left(1-\frac{\bar{X}}{5}\right) + \left(\frac{\bar{X}}{5}\right)^2\right)$$

$$= \left(\frac{\bar{X}}{5}\right)^3 \cdot \left(\frac{6\bar{X}^2}{25} - 3\bar{X} + 10\right). \qquad \bar{X} = 2.12.$$

$= ?$ ~

8. (6.1.8) Let the table

x	0	1	2	3	4	5
Frequency	7	14	12	13	6	3

represent a summary of a random sample of size 55 from a Poisson distribution. Find the MLE of $P(X = 2)$.

8. $\bar{X} = \frac{116}{55}$.

$f(x;\lambda) = \frac{\lambda^x}{x!} \cdot e^{-\lambda} \quad (x \geq 0).$

MLE: $L(\lambda) := -\sum_{i=1}^{55} \log f(x_i;\lambda)$

$$= -\sum_i \left(x_i \log\lambda - \log x! - \lambda\right)$$

$$\frac{\partial L(\lambda)}{\partial \lambda} = -\sum_i\left(\frac{x_i}{\lambda} - 1\right) = 0 \Rightarrow \hat{\lambda} = \bar{x}.$$

$$\frac{\partial^2 L(\lambda)}{\partial \lambda^2} \geq 0. \Rightarrow \hat{\lambda} = \arg\min_{\lambda} L(\lambda), \quad \overset{(MLE)}{P(X=2)} = \frac{\hat{\lambda}^2}{2} \cdot e^{-\hat{\lambda}} = \frac{\bar{X}^2}{2} \cdot e^{-\bar{X}}.$$

9. (6.1.9)* Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli distribution with parameter p. If p is restricted so that we know that $\frac{1}{2} \le p \le 1$, find the MLE of this parameter.

9. $f(x) = p^{x} \cdot (1-p)^{1-x} \quad , x \in \{0, 1\}.$

$$\hat{p}_{MLE} = \arg\min_{p} -\sum_{i=1}^{n} x_i \log p + (1-x_i) \log(1-p).$$

$$\left.\frac{\partial L(p)}{\partial p}\right|_{p=\hat{p}} = -\sum_{i=1}^{n} \frac{x_i}{\hat{p}} + \frac{1-x_i}{\hat{p}-1} = 0$$

$$\Rightarrow \hat{p} = \bar{x}. \quad \triangle$$

$$\left.\frac{\partial^2 L(p)}{\partial p^2}\right|_{p=\hat{p}} = \sum_{i=1}^{n} \frac{x_i}{\hat{p}^2} + \frac{1-x_i}{(\hat{p}-1)^2} > 0 \quad , \text{ so } \hat{p} \text{ is minimizer.}$$

10. (6.1.10)* Let $X_1, \ldots, X_n$ be a random sample from a $N(\theta, \sigma^2)$ distribution, where σ^2 is fixed but $-\infty < \theta < \infty$.

a. Show that the MLE of θ is $\overline{X}$.

b. If θ is restricted by $0 \le \theta < \infty$, show that the MLE of θ is $\hat{\theta} = \max\{0, \overline{X}\}$.

10. (a) $$\hat{\theta}_{MLE} = \arg\min_{\theta} -\sum_{i=1}^{n} \log \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left(-\frac{(x_i-\theta)^2}{2\sigma^2}\right)$$

$$= \arg\min_{\theta} \sum_{i=1}^{n} \log \sigma + \frac{(x_i-\theta)^2}{2\sigma^2}$$

$$\left.\frac{\partial L(\theta)}{\partial \theta}\right|_{\theta=\hat{\theta}} = \sum_{i=1}^{n} \frac{2(\hat{\theta}-x_i)}{2\sigma^2} = 0 \Rightarrow \hat{\theta} = \bar{x}.$$

since $\left.\frac{\partial^2 L(\theta)}{\partial \theta}\right|_{\theta=\hat{\theta}} > 0$. $\hat{\theta}$ is minimizer of $L(\theta)$.

(b) $\frac{\partial^2 L(\theta)}{\partial \theta} = \frac{1}{\sigma^2} > 0$, so $L(\theta)$ is convex.

if $\bar{x} < 0$, $L(\theta)$ is strictly increasing on $[0, +\infty)$

hence $\arg\min_{\theta} L(\theta) = 0$

if $\bar{x} \ge 0$, as above, $\hat{\theta} = \bar{x}$.

$\hat{\theta} = \max\{0, \bar{x}\}$.

11. (6.1.12)* Let $X_1, \ldots, X_n$ be a random sample from a distribution with one of two pdfs. If $\theta = 1$ then $f(x; \theta = 1) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $-\infty < x < \infty$. If $\theta = 2$, then $f(x; \theta = 2) = \frac{1}{\pi(1+x^2)}$, $-\infty < x < \infty$. Find the MLE of θ.

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11. $\hat{\theta} = \underset{\theta}{\arg\min} \sum_{i=1}^{n} -\log f(x_i;\theta)$.

$$\sum_{i=1}^{n} -\log f(x_i;\theta=1) = \sum_{i=1}^{n} \log\sqrt{2\pi} + \frac{x_i^2}{2},$$

$$\sum_{i=1}^{n} -\log f(x_i;\theta=2) = \sum_{i=1}^{n} \log\pi + \log(1+x_i^2)$$

so if $\sum_{i=1}^{n} \log\frac{\pi}{\sqrt{2\pi}} + \log(1+x_i^2) - \frac{x_i^2}{2} > 0$, $\hat{\theta}=2$,

otherwise $\hat{\theta}=1$.