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1. Discuss whether the following functions are convex.

- (a) (5 points) $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f(x) = \cos^2 x_1 + \sin^2 x_2$.
- (b) (5 points) $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} |x| - \pi/2 & \text{if } |x| > \pi/2, \\ -\cos x & \text{if } -\pi/2 \leq x \leq \pi/2. \end{cases}$$

- (c) (10 points) $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by

$$f(x) = \sum_{i=1}^3 \ln(4 + (x_i)^2) + \frac{1}{4} \|x\|^2 + |8x_1 + 3x_2 - 6x_3 + 1| + |4x_1 - x_2 + 5x_3 - 12|$$

1. (a) $\nabla^2 f(x) = \begin{pmatrix} -2\cos 2x_1 & 0 \\ 0 & 2\cos 2x_2 \end{pmatrix}$

$\nabla^2 f(x)$ is not positive definite on \mathbb{R}^2 , so $f(x)$ is not convex.

(b) on $(-\frac{\pi}{2}, \frac{\pi}{2})$, $f'(x) = \cos x \geq 0$.

on $|x| > \frac{\pi}{2}$, $f'(x) = 0$.

when $x = \frac{\pi}{2}$, we have $f'(x) = 1$, $\lim_{x \rightarrow \frac{\pi}{2}^+} \frac{f(x) - f(\frac{\pi}{2})}{x - \frac{\pi}{2}} = \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{f(x) - f(\frac{\pi}{2})}{x - \frac{\pi}{2}}$.

$$\text{and } \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{f'(x) - f'(\frac{\pi}{2})}{x - \frac{\pi}{2}} = \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{f'(x) - f'(\frac{\pi}{2})}{x - \frac{\pi}{2}} = 0.$$

same as $x = -\frac{\pi}{2}$, so we have $f(x) \in C^2(\mathbb{R})$

and $f''(x) \geq 0$ for $x \in \mathbb{R}$, so $f(x)$ is convex.

(c) set $f_1(x) := \sum_{i=1}^3 \ln(4 + (x_i)^2)$, $f_2(x) := \frac{1}{4} \|x\|^2$

$$f_1(x) := |8x_1 + 3x_2 - 6x_3 + 1|, \quad f_2(x) := |4x_1 - x_2 + 5x_3 - 12|$$

$$\text{set } J_x^2 f_1(x) = \text{diag}\left(\frac{8-2x_1^2}{(x_1^2+4)^2}, \frac{8-2x_2^2}{(x_2^2+4)^2}, \frac{8-2x_3^2}{(x_3^2+4)^2}\right)$$

$$\nabla_x^2 f_2(x) = \frac{1}{2} I,$$

$$\text{since } \frac{8-2x_1^2}{(x_1^2+4)^2} + \frac{1}{2} = \frac{8-2x_1^2 + (x_1^2+4)^2 \cdot \frac{1}{2}}{(x_1^2+4)^2} \geq \frac{8+6x_1^2}{(x_1^2+4)^2} > 0$$

hence $\nabla_x^2 f_1(x) + \nabla_x^2 f_2(x) \succ 0 \Rightarrow f_1 + f_2$ is convex

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and assume $y := 8x_1 + 3x_2 - 6x_3 + 1$. y is linear transform of x .

we have $| \lambda y_1 | + | (1-\lambda)y_2 | \geq | \lambda y_1 + (1-\lambda)y_2 |$, triangular inequality.

so f_3 and f_4 are convex. (that is, 1-order norm of affine projection is convex)

$\Rightarrow f$ is convex. \square

2. (10 points) For the following function, find all the stationary points and determine their nature based on their Hessian, if possible.

$$f(x) = \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} 3 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - 6x_1 + 4x_2 - 3x_3^3 + 2024.$$

$$\nabla f(x) = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} -6 \\ 4 \\ -9x_3^2 \end{pmatrix} = 0$$

$$\Rightarrow x = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 1/9 \end{pmatrix}, \text{ 2 stationary points}$$

$$\nabla^2 f(x) = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -18x_3 \end{pmatrix}$$

if $x = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$, eigen values of $\nabla^2 f(x)$ is $1, 2\pm\sqrt{2}$. $\nabla^2 f(x) > 0$,

$\begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$ is local minimizer.

if $x = \begin{pmatrix} 1 \\ -3 \\ 1/9 \end{pmatrix}$, $\nabla^2 f(x)$ is not semi-definite matrix.

$\begin{pmatrix} 1 \\ -3 \\ 1/9 \end{pmatrix}$ is not minimizer or maximizer locally. \square

3. (15 points) Consider the function

$$f(x) = \frac{1}{2} x^T \begin{bmatrix} 6 & -3 \\ -3 & 8 \end{bmatrix} x + 2x_2^2 + x_1^4 - 2x_1^3.$$

Perform two iterations of the BFGS method with $\alpha_k \equiv 1.5$, $x^0 = (1, 0)$ and $H_0 = I$. Write down x^1 and x^2 . You may correct your answers to 4 d.p.

$$3. H_{k+1} = \left(I - \frac{s^k y^k s^k}{y^k T s^k} \right) H_k \left(I - \frac{y^k s^k T}{y^k T s^k} \right) + \frac{s^k s^k T}{y^k T s^k}$$

$$\begin{cases} 1. d^k = -H^k \cdot \nabla f(x^k) \\ 2. x^{k+1} = x^k + 2^k \cdot d^k \\ 3. y^k = \nabla f(x^{k+1}) - \nabla f(x^k), s^k = x^{k+1} - x^k \\ 4. H^{k+1}. \end{cases}$$

↑
It's the procedure of BFGS.

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$$H_0 = I, \quad x^0 = (1, 0), \quad 2x_1 \leq 1.5$$

$$\nabla f(x) = \begin{pmatrix} 6 & -3 \\ -3 & 8 \end{pmatrix} x + \begin{pmatrix} 4x_1^3 - 6x_1^2 \\ 4x_2 \end{pmatrix}$$

$$(1) \quad d^0 = -H^0 \cdot \nabla f(x^0) = \begin{pmatrix} -4 \\ 3 \end{pmatrix}, \quad \nabla f(x^0) = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

$$x^1 = x^0 + 2 \cdot d^0 = \begin{pmatrix} -5 \\ 4.5 \end{pmatrix}, \quad \nabla f(x) = \begin{pmatrix} -10 - 13.5 \\ 15 + 36 \end{pmatrix} + \begin{pmatrix} -500 - 150 \\ 18 \end{pmatrix} = \begin{pmatrix} -6935 \\ 69 \end{pmatrix}$$

$$y^0 = \begin{pmatrix} -697.5 \\ 72 \end{pmatrix}, \quad s^0 = x^1 - x^0 = \begin{pmatrix} -6 \\ 4.5 \end{pmatrix}$$

$$H^{k+1} = \left(I - \frac{s^k y^k T}{y^k T s^k} \right) H_k \left(I - \frac{y^k s^k T}{y^k T s^k} \right) + \frac{s^k y^k T}{y^k T s^k} \quad H^1 = \left(1 - \frac{s^0 y^0 T}{y^0 T s^0} \right) \left(1 - \frac{y^0 s^0 T}{y^0 T s^0} \right) + \frac{s^0 y^0 T}{y^0 T s^0}$$

$$= \begin{pmatrix} 0.0223 & 0.1329 \\ 0.1329 & 1.3505 \end{pmatrix}$$

$$(2) \quad d^1 = -H^1 \cdot \nabla f(x) = \begin{pmatrix} 6.3095 \\ -0.9802 \end{pmatrix}$$

$$x^2 = x^1 + 2 \cdot d^1 = \begin{pmatrix} 4.4643 \\ 3.0295 \end{pmatrix}$$

$$x_1 = \begin{pmatrix} -5 \\ 4.5 \end{pmatrix}. \square$$

4. (15 points) Let $a \in \mathbb{R}$ and consider the following linear programming problem.

$$\begin{array}{ll} \text{Maximize } & ax_1 - 3x_2 - 2x_3 + 3x_4 \\ \text{subject to } & 3x_1 - x_2 + x_4 \geq 6, \\ & 3x_2 + 2x_3 - x_4 = 8, \\ & x_1 + x_2 + x_3 + x_4 \leq 1, \\ & x_1, x_2, x_3, x_4 \text{ unrestricted.} \end{array}$$

By considering its dual or otherwise, find the optimal value of the above problem for all values of a .

4.

	Maximize $c^T x$ subjected to $Ax \leq b$	Minimize $b^T y$ subject to $A^T y \leq c$
◆	i/jh constraint \leq i/jh constraint \geq i/jh constraint $=$ j/h variable ≤ 0 j/h variable unrestricted	i/jh variable ≥ 0 i/jh variable ≤ 0 i/jh variable unrestricted j/h constraint \geq j/h constraint \leq j/h constraint $=$
◇	j/h variable unrestricted	j/h variable unrestricted

$$c = (a \ -3 \ -2 \ 3)^T, \quad A := \begin{pmatrix} 5 & -1 & 0 & 1 \\ 0 & 3 & 2 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$b := (-6 \ -8 \ 1)^T$$

$$\text{dual:} \quad \text{Min } b^T y$$

$$\text{s.t. } A^T y = c$$

$$y_1 \leq 0, \quad y_3 \geq 0.$$

$$\text{solve the constraints } A^T y = c, \text{ we get } y = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}, \text{ if } a = -3.$$

if $a \neq -3$, feasible set is \emptyset .

① if $a = -3$, optimal value is -20 .

② if $a \neq -3$, $\exists x = (2, 0, 7/3, -10/3)^T \in$ primal feasible set,
so optimal value $\rightarrow +\infty$. \square

5. Consider the function $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by

$$f(x) = \ln(4 + (x_1 - x_2 - 3)^2) + \ln(4 + (x_1 + 2x_3 + 1)^2) + \sum_{i=1}^3 \sqrt{7 + x_i^2}.$$

Define

$$B := \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad \text{and} \quad h(y) = \ln(4 + (y_1 - 3)^2) + \ln(4 + (y_2 + 1)^2).$$

(a) (5 points) Write down the first and second-order derivatives of the function $g(t) = \ln(4 + t^2)$.

(b) (5 points) Compute $\nabla h(y)$, $\nabla^2 h(y)$ and BB^T .

(c) (15 points) Consider an iterate of the following form:

$$x^{k+1} = x^k - 0.6602 \cdot \nabla f(x^k).$$

Argue that any accumulation point of $\{x^k\}$ is a stationary point of f .

$$5. (a) g'(t) = \frac{2t}{4+t^2}, \quad g''(t) = \frac{8-2t^2}{(4+t^2)^2}$$

$$(b) \nabla h(y) = \begin{pmatrix} \frac{8-2(y_1-3)^2}{(4+(y_1-3)^2)^2} & 0 \\ 0 & \frac{8-2(y_2+1)^2}{(4+(y_2+1)^2)^2} \end{pmatrix}$$

$$\nabla^2 h(y) = \left(\frac{2(y_1-3)}{4+(y_1-3)^2}, \frac{2(y_2+1)}{4+(y_2+1)^2} \right)$$

$$BB^T = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$$

$$(c) \text{ set } L := \|\nabla^2 f(x)\|_2$$

$$\text{set } y := Bx, \quad f_1(x) = \ln(4 + (x_1 - x_2 - 3)^2) + \ln(4 + (x_1 + 2x_3 + 1)^2),$$

$$f_2(x) := \sum_{i=1}^3 \sqrt{7 + x_i^2}.$$

$$\nabla_x^2 f_1(x) = B^T \nabla_y^2 h(y) B$$

$$\nabla_x^2 f_2(x) = \text{diag}\left(\frac{1}{(7+x_1^2)^{3/2}}, \frac{1}{(7+x_2^2)^{3/2}}, \frac{1}{(7+x_3^2)^{3/2}}\right) \quad \text{set } c := 4-t^2, \quad c \in (-\infty, 4].$$

$$\begin{aligned} \|\nabla_x^2 f(x)\|_2 &\leq \|\nabla_x^2 f_1(x)\|_2 + \|\nabla_x^2 f_2(x)\|_2 & \left| \frac{8-2t^2}{(4+t^2)^2} \right| = 2 \left| \frac{c}{(8-c)^2} \right| \leq \frac{1}{16} \\ &\leq \|B\|_2^2 \|\nabla_y^2 h(y)\|_2 + \|\nabla_x^2 f_2(x)\|_2 & \frac{1}{(7+x^2)^{3/2}} \leq \frac{1}{\sqrt{7}} \\ &= \left(\frac{7+\sqrt{13}}{2}\right) \cdot \frac{1}{16} + \frac{1}{\sqrt{7}} \end{aligned}$$

hence step size $< \frac{2}{L}$, accumulation points is stationary.

(d) Consider an iterate of the following form:

$$x^{k+1} = x^k + \alpha_k d^k.$$

Let $d^k = -\nabla f(x^k)$ and α_k be chosen to satisfy the Wolfe's condition. Suppose that x^k is nonstationary for all k .

i. (5 points) Show that the sequence $\{x^k\}$ is bounded.

ii. (10 points) Show that any accumulation point of $\{x^k\}$ is stationary. You should specify the ℓ that can be used in Zoutendijk's theorem.

$$(d) (i) \|x^{k+1}\|_2 \leq \sum_{i=1}^3 \sqrt{7 + (x_i^{(k+1)})^2} \leq f(x^{k+1}) < f(x^k) + \alpha_k \cdot d^k \cdot \nabla f(x^k) < f(x^k)$$

$$\leq \dots \leq f(x^0)$$

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hence, $\{x_k\}_{k=1}^{\infty}$ is bounded.

(ii) since $\Phi f \in C^1(\mathbb{R}^3)$, $f \neq 0$.

② $\forall x, y \in \mathbb{R}^3, \exists \|Jf(x) - Jf(y)\|_2 \leq \ell \cdot \|x - y\|_2$

where $\ell = \sup_{t \in \mathbb{R}} \|J^2 f(t)\|_2$, since $f \in C^2(\mathbb{R}^3)$, we can apply this theorem.
and we know $\|J^2 f(x)\|_2$ has upper bound from 5.(c).

\Rightarrow Applying Zoutendijk's theorem, we have

$$\sum_{k=0}^{\infty} \cos^2 \theta_k \cdot \|Jf(x_k)\|_2^2 < \infty.$$

where $\cos \theta_k = \frac{-Jf(x_k)^T d^k}{\|Jf(x_k)\| \cdot \|d^k\|} = 1$,

hence we have $\|Jf(x_k)\|_2 \rightarrow 0$ as $k \rightarrow +\infty$.

\Rightarrow any accumulation point is stationary. \square

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