Assignment 3

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1. $$P(X_n \le t \mid x_1 < x_2 < \cdots < x_n)$$

$$= \frac{P(X_n \le t,\ x_1 < x_2 < \cdots < x_n)}{P(x_1 < x_2 < \cdots < x_n)}$$

$$= \frac{\int_0^t f(x_n)\,dx_n \int_0^{x_n} f(x_{n-1})\,dx_{n-1} \cdots \int_0^{x_2} f(x_1)\,dx_1}{\int_0^\theta f(x_n)\,dx_n \int_0^{x_n} f(x_{n-1})\,dx_{n-1} \cdots \int_0^{x_2} f(x_1)\,dx_1}$$

$$= \frac{\left.\frac{x_n^{3n}}{n!\cdot\theta^{3n}}\right|_0^t}{\left.\frac{x_n^{3n}}{n!\cdot\theta^{3n}}\right|_0^\theta} = \frac{t^{3n}}{\theta^{3n}}$$

hence, $$P\left(c < \frac{X_n}{\theta} < 1 \,\middle|\, x_1 < x_2 < \cdots < x_n\right)$$

$$= 1 - P(X_n \le c\cdot\theta \mid x_1 < x_2 < \cdots < x_n)$$

$$= 1 - \frac{(c\theta)^{3n}}{\theta^{3n}} = 1 - c^{3n}. \quad \square$$

2. (a) $H_0: \theta = 1/2 \quad H_1: \theta < 1/2$

set $\theta' = 1/2$, $\theta'' \in H_1$

$$\frac{L(\theta', x_1 \cdots x_5)}{L(\theta'', x_1 \cdots x_5)} = \frac{\prod_{i=1}^5 \theta'^{x_i}(1-\theta')^{1-x_i}}{\prod_{i=1}^5 \theta''^{x_i}(1-\theta'')^{1-x_i}}$$

$$= \left(\frac{\theta'(1-\theta'')}{\theta''(1-\theta')}\right)^{\sum_{i=1}^5 x_i} \cdot \left(\frac{1-\theta'}{1-\theta''}\right)^5 \le k$$

$$\Leftrightarrow \sum_{i=1}^5 x_i \cdot \ln\left(\frac{\theta'(1-\theta'')}{\theta''(1-\theta')}\right) \le \ln k$$

$$\Leftrightarrow \sum_{i=1}^5 x_i \le c, \quad \text{set } c \triangleq \frac{\ln k}{\ln\left(\frac{\theta'(1-\theta'')}{\theta''(1-\theta')}\right)}$$

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so $C \triangleq \{ \sum_{i=1}^{5} x_i \le c \}$ is critical region.

$Y \le c$ is uniformly most powerful test.

(b) significance level $\alpha = P(\sum_{i=1}^{5} x_i \le 1 \mid H_0)$

$= P(Y \le 1 \mid \theta = 1/2)$

$Y \sim \text{Binomial}(n, \theta)$

$\alpha = \binom{5}{0} \cdot (\frac{1}{2})^5 + \binom{5}{1} (\frac{1}{2})^5 = \frac{3}{16}$

(c) $\alpha = P(Y = 0 \mid \theta = 1/2) = \binom{5}{0} \cdot (\frac{1}{2})^5 = \frac{1}{32}$. □

3. (a) $E(\bar{X}) = E(k\bar{X}_1 + (1-k)\bar{X}_2)$

$= k \cdot \mu + (1-k)\mu$

$= \mu,$

so it's unbiased for μ.

(b) $\lim_{n \to \infty} P(|\bar{X} - \mu| \ge \varepsilon) = 0$ is what we need to proove.

First, prove $\bar{X}_1 \xrightarrow{P} \mu$,

$MSE(\bar{X}_1) = Var(\bar{X}_1) + bias(\bar{X}_1)^2 = \frac{1}{n} Var(X_1) + 0 \to 0$ as $n \to +\infty$

so we have $\bar{X}_1 \xrightarrow{P} \mu$.

Similarly, $\bar{X}_2 \xrightarrow{P} \mu$.

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$\bar{X}_1 \xrightarrow{P} \mu$, $\bar{X}_2 \xrightarrow{P} \mu$ $\Rightarrow$ $\bar{X} = k\bar{X}_1 + (1-k)\bar{X}_2 \xrightarrow{P} \mu$.

(c) $Var(\bar{X}) = k^2 Var(\bar{X}_1) + (1-k)^2 Var(\bar{X}_2)$

$$= k^2 \cdot \frac{\sigma_1^2}{n} + (1-k)^2 \frac{\sigma_2^2}{n}$$

$$\frac{\partial Var(\bar{X})}{\partial k} = 2k \cdot \frac{\sigma_1^2}{n} + 2(k-1)\frac{\sigma_2^2}{n} = 0$$

$$\Rightarrow \quad k = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

$\frac{\partial^2 Var(\bar{X})}{\partial k^2} = 2\left(\frac{\sigma_1^2 + \sigma_2^2}{n}\right) > 0$, so $k = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$ is minimizer. $\square$