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Chapter 5

Statistical Hypothesis

Contents

5.1 Introduction

The two principal areas of statistical inference are the areas of estimation of parameters and of statistics hypotheses. The problem of estimation of parameters, both point and interval estimation, has been treated. In this chapter some aspects of statistical hypothesis and tests of statistical hypotheses will be considered. The subject will be introduced by way of examples.

Example 5.1.1. *An individual claims that he can predict whether a coin would give a head or a tail, before flipping it. A conservative/natural GUESS is that his claim is false. We refer to this guess the "hypothesis", meaning a claim that is to be verified.*

To verify the guess, we repeat one experiment 10 times: record his prediction, and then let him flip the coin to see whether the prediction is correct. Suppose among 10 experiments, he managed to make X correct guesses.

Now there is a problem: these X correct guesses may really because he has the magic power, and may also because of randomness, since even an average people can make about 50% correct guesses.

If X is no greater than 50%, or SLIGHTLY greater than 50%, then people tend to draw the conclusion that this individual's claim is false. If X is much higher than 50%, it is fair to draw the conclusion that the individual's claim is true.

Note that both of the conclusions are made based on a random test, and are therefore under the risk of a mistake. For example, if the correct guesses are really from magic power yet they only take about 50% of the 10 experiments, then we make a mistake to ACCEPT the guess that "the claim is false"; if it is just because of randomness that the individual made 9 correct guesses (possible, right?), then we are making another kind of mistake by REJECTING the guess that "the claim is false".

We have now illustrated the following concepts :

(a) A statistical hypothesis.

(b) A test of a hypothesis against an alternative hypothesis and the associated concept of the critical region of the test.

(c) The power of a test.

These concepts will now be formally defined.

Definition 5.1.2. *A **statistical hypothesis** is an assertion about the distribution of one or more random variables. If the statistical hypothesis completely specifies the distribution, it is called a **simple statistical hypothesis**; if it does not, it is called a **composite statistical hypothesis**.*

Example 5.1.3. *Let it be known that the outcome X of a random experiment is $N(\theta, 100)$. For instance, X may denote a score on a test, which score we assume to be normally distributed with mean θ and variance 100. Let us say that past experience with this random experiment indicates that $\theta = 75$. Suppose, owing possibly to some research in the area pertaining to this experiment, some changes are made in the method of performing this random experiment. It is then suspected that no longer does $\theta = 75$ but that now $\theta > 75$. There is as yet no formal experiment evidence that $\theta > 75$; hence the statement $\theta > 75$ is a conjecture or a statistical hypothesis.*

In this example, we see that both $H_0 : \theta \leq 75$ and $H_1 : \theta > 75$ are composite statistical hypotheses, since neither of them completely specifies the distribution. If there, instead of $H_0 : \theta \leq 75$, we had $H_0 : \theta = 75$, then H_0 would have been a simple statistical hypothesis. We label these hypotheses as

$$H_0 : \theta \in \omega_0 \quad \text{versus} \quad H_1 : \theta \in \omega_1$$

H_0 is referred to as the null hypothesis; H_1 is referred to as the alternative hypothesis. Often H_0 represents no change or no difference from the past while H_1 represents change or difference.

Definition 5.1.4. *A **test** of statistical hypothesis is a decision rule which, when the experimental sample values have been obtained, leads to a decision to accept or to reject the hypothesis under consideration.*

Four possibilities

Decision Table for a Test of Hypothesis

	True State of Nature	
Decision	H_0 is true	H_1 is true
Reject H_0	Type I Error	Correct Decision
Accept H_0	Correct Decision	Type II Error

Definition 5.1.5. *Let C be that subset of the sample space which, in accordance with a prescribed test, leads to the rejection of the hypothesis under consideration. Then C is called the **critical region** of the test.*

A test is based on the set C.

Reject H_0, if $(X_1, \cdots, X_n) \in C$;

Accept H_0, if $(X_1, \cdots, X_n) \in C^c$.

Definition 5.1.6. *The **power function** of a test of statistical hypothesis H_0 against an alternative hypothesis H_1 is that function, defined for all distributions under consideration, which yields the probability that the sample point falls in the critical region*

*C of the test, that is, a function that yields the probability of rejecting the hypothesis under consideration. The value of the power function at a parameter point is called the **power** of the test at that point.*

$$\gamma_C(\theta) = P_\theta\{(X_1, \cdots, X_n) \in C\}, \quad \theta \in \omega_1$$

Definition 5.1.7. *Let H_0 denote a hypothesis that is to be tested against an alternative hypothesis H_1 in accordance with a prescribed test. The **significance level** of the test (or the size of the critical region C) is the maximum value (actually supremum) of the power function of the test when H_0 is true.*

$$\alpha = \max_{\theta \in \omega_0} P_\theta\{(X_1, \cdots, X_n) \in C\}$$

Example 5.1.8. *It is known that the random variable X has pdf of the form*

$$f(x;\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \qquad 0 < \theta < \infty$$

It is desired to test the simple hypothesis $H_0 : \theta = 2$ against the alternative simple hypothesis $H_1 : \theta = 4$. Thus $\omega_0 = \{2\}$, $\omega_1 = \{4\}, \Omega = \omega_0 \cup \omega_1 = \{\theta; \theta = 2, 4\}$. A random sample X_1, X_2 of size $n = 2$ will be used. The test to be used is defined by taking the critical region to be $C = \{(x_1, x_2)\,; 9.5 \le x_1 + x_2 < \infty\}$. The power function of the test and the significance level of the test will be determined.

Solution. $f(x_1, x_2; \theta = 2) = f(x_1; \theta = 2) f(x_2; \theta = 2) = \frac{1}{2^2} e^{-(x_1+x_2)/2}$, $x_1 > 0, x_2 > 0$

$f(x_1, x_2; \theta = 4) = f(x_1; \theta = 4) f(x_2; \theta = 4) = \frac{1}{4^2} e^{-(x_1+x_2)/4}$, $x_1 > 0, x_2 > 0$

$\alpha = \max_{\theta \in \omega_0} P_\theta\{(X_1, X_2) \in C\} = P(9.5 \le X_1 + X_2 < \infty; \theta = 2) = 0.05$

$\gamma(\theta) = P_\theta\{(X_1, X_2) \in C\}, \quad \theta \in \omega_1$

$\gamma(4) = P\{9.5 \le X_1 + X_2 < \infty; \theta = 4\} = 0.31$

Example 5.1.9 (for the tutorial). *Let* X_1, X_2 *be a random sample of size* $n = 2$ *from the distribution having pdf* $f(x;\theta) = (1/\theta)e^{-x/\theta}$, $0 < x < \infty$, *0 elsewhere. We reject* $H_0 : \theta = 2$ *and accept* $H_1 : \theta = 1$ *if the observed values of* X_1, X_2, *say* x_1, x_2, *are such that*

$$\frac{f(x_1;2)f(x_2;2)}{f(x_1;1)f(x_2;1)} \leq \frac{1}{2}.$$

Here $\Omega = \{\theta : \theta = 1, 2\}$. *Find the significance level of the test and the power the test when* H_0 *is false.*

Example 5.1.10 (for the tutorial)**.** *Let* Y *have a binomial distribution with parameters* n *and* p. *We reject* $H_0 : p = \frac{1}{2}$ *and accept* $H_1 : p > \frac{1}{2}$ *if* $Y \geq c$. *Find* n *and* c *to give a power function* $\gamma(p)$ *which is* $\gamma(\frac{1}{2}) = 0.10$ *and* $\gamma(\frac{2}{3}) = 0.95$, *approximately.*

Solution: $Y = X_1 + \cdots, X_n$ where the X_i's are i.i.d. $Ber(p)$.
$\sqrt{n}(Y/n - p) \to N(0, p(1-p))$,
$\frac{\sqrt{n}(Y/n-p)}{\sqrt{p(1-p)}} \to N(0,1)$.

$$\begin{aligned} 0.1 = \gamma(1/2) &= P(Y \geq c; p = 1/2) \\ &= P(\sqrt{n}(Y/n - 0.5)/0.5 \geq \sqrt{n}(c/n - 0.5)/0.5; p = 0.5) \\ &= 1 - \Phi(\sqrt{n}(c/n - 0.5)/0.5) \\ &= \Phi(-\sqrt{n}(c/n - 0.5)/0.5). \end{aligned}$$

$-\sqrt{n}(c/n - 0.5)/0.5) = \Phi^{-1}(0.1)$.

$$0.95 = \gamma(2/3) \quad = \quad P(Y \geq c; p = 2/3)$$

Example 5.1.11 (for the tutorial)**.** *Let* $X_1, X_2, ..., X_8$ *be a random sample of size* $n = 8$ *from a Poisson distribution with mean* μ*. Reject the simple null hypothesis* $H_0 : \mu = 0.5$ *and accept* $H_1 : \mu > 0.5$ *if the observed sum* $\sum_{i=1}^{8} x_i \geq 8$*.*

(a) Compute the significance level α *of the test.*

(b) Find the power function $\gamma(\mu)$ *of the test as a sum of Poisson probabilities.*

(c) Determine $\gamma(0.75)$ *.*

5.2 Certain Best Tests

Definition 5.2.1. *Let C denote a subset of the sample space. Then C is called a best critical region of size α for the testing the simple hypothesis $H_0 : \theta = \theta'$ against the alternative simple hypothesis $H_1 : \theta = \theta''$ if, for every subset A of the sample space for which $P[(X_1, \cdots, X_n) \in A; H_0] = \alpha$:*

(a) $P[(X_1, \cdots, X_n) \in C; H_0] = \alpha.$

(b) $P[(X_1, \cdots, X_n) \in C; H_1] \geq P[(X_1, \cdots, X_n) \in A; H_1]$

This definition states, in effect, the following : First assume that H_0 to be true. In general, there will be a multiplicity of subsets A of the sample space such that

$$P[(X_1, \cdots, X_n) \in A] = \alpha.$$

Suppose that there is one of these subsets, say C, such that when H_1 is true, the power of the test associated with C is at least as great as the power of the test associated with each other A. Then C is defined as a best critical region of size α for testing H_0 against H_1.

Theorem 5.2.2 (Neyman-Pearson Theorem)**.** *Let $X_1, X_2, ..., X_n$ where n is a fixed positive integer, denote a random sample from a distribution that has pdf or pmf $f(x;\theta)$. Therefore the likelihood of $X_1, X_2, ..., X_n$ is*

$$L(\theta; x_1, x_2, ..., x_n) = f(x_1;\theta) f(x_2;\theta) \cdots f(x_n;\theta).$$

Let $\theta' \neq \theta''$. Write $\Omega = \{\theta', \theta''\}$, and let k be a positive number. Let C be a subset of the sample space such that

(a) $\frac{L(\theta';x_1,x_2,...,x_n)}{L(\theta'';x_1,x_2,...,x_n)} \leq k,$ *for each point* $(x_1, x_2, ..., x_n) \in C$

(b) $\frac{L(\theta';x_1,x_2,...,x_n)}{L(\theta'';x_1,x_2,...,x_n)} > k,$ *for each point* $(x_1, x_2, ..., x_n) \in C^c$

(c) $\alpha = P[(X_1, X_2, ..., X_n) \in C; H_0]$

Then C is a best critical region of size α for testing the simple hypothesis $H_0 : \theta = \theta'$ against the alternative simple hypothesis $H_1 : \theta = \theta''$

Proof. We shall give the proof when the random variables are of the continuous type. If C is the only critical region of size α, the theorem is proved. If there is another critical region of size α, denote it by A. For convenience, we shall let $\int \cdots \int_{\Re} L(\theta; x_1, \cdots, x_n)\, dx_1 \cdots dx_n$ be denoted by $\int_R L(\theta)$. In this notation we wish to show that $\int_C L(\theta'') - \int_A L(\theta'') \geq 0$. Since C is the union of the disjoint sets $C \cap A$ and $C \cap A^c$ and A is the union of the disjoint sets $A \cap C$ and $A \cap C^c$, we have

$$\begin{aligned} \int_C L(\theta'') - \int_A L(\theta'') &= \int_{C\cap A} L(\theta'') + \int_{C\cap A^c} L(\theta'') - \int_{A\cap C} L(\theta'') - \int_{A\cap C^c} L(\theta'') \\ &= \int_{C\cap A^c} L(\theta'') - \int_{A\cap C^c} L(\theta'') \qquad (5.1) \end{aligned}$$

However, by the hypothesis of the theorem, $L(\theta'') \geq (1/k)\, L(\theta')$ at each point of C, and hence at each point of $C \cap A^c$; thus

$$\int_{C\cap A^c} L(\theta'') \geq \frac{1}{k} \int_{C\cap A^c} L(\theta')$$

But $L(\theta'') \leq (1/k)\, L(\theta')$ at each point of C^c, and hence at each point of $A \cap C^c$; accordingly,

$$\int_{A\cap C^c} L(\theta'') \leq \frac{1}{k} \int_{A\cap C^c} L(\theta')$$

These inequalities imply that

$$\int_{C\cap A^c} L(\theta'') - \int_{A\cap C^c} L(\theta'') \geq \frac{1}{k} \int_{C\cap A^c} L(\theta') - \frac{1}{k} \int_{A\cap C^c} L(\theta');$$

and, from Equation (1), we obtain

$$\int_C L(\theta'') - \int_A L(\theta'') \geq \frac{1}{k} \left[\int_{C\cap A^c} L(\theta') - \int_{A\cap C^c} L(\theta') \right]. \qquad (5.2)$$

However,

$$\begin{gathered} \int_{C\cap A^c} L(\theta') - \int_{A\cap C^c} L(\theta') \\ = \int_{C\cap A^c} L(\theta') + \int_{C\cap A} L(\theta') - \int_{A\cap C} L(\theta') - \int_{A\cap C^c} L(\theta') = \alpha - \alpha = 0 \end{gathered}$$

If this result is substituted in inequality (2), we obtain the desired result,

$$\int_C L\left(\theta''\right) - \int_A L\left(\theta''\right) \geq 0.$$

If the random variables are of the discrete type, the proof is the same, with integration replaced by summation. $\square$

Example 5.2.3. *Let $X_1, X_2, ..., X_n$ denote a random sample from the distribution that has the p.d.f.*

$$f\left(x;\theta\right) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\left(x-\theta\right)^2}{2}\right\}, \qquad -\infty < x < \infty$$

It is desired to test the simple hypothesis $H_0 : \theta = \theta' = 0$ against the alternative simple hypothesis $H_1 : \theta = \theta'' = 1$.

Solution.

$$\frac{L(\theta')}{L(\theta'')} = \frac{(1/\sqrt{2\pi})^n \exp\{-\sum_{i=1}^n x_i^2/2\}}{(1/\sqrt{2\pi})^n \exp\{-\sum_{i=1}^n (x_i-1)^2/2\}} = e^{-\sum_{i=1}^n x_i + \frac{n}{2}}$$

$$\frac{L(\theta')}{L(\theta'')} \leq k \quad \Longleftrightarrow \quad e^{-\sum_{i=1}^n x_i + \frac{n}{2}} \leq k \quad \Longleftrightarrow \quad \sum_{i=1}^n x_i \geq c.$$

By Neyman-Pearson Theorem, a best critical region is

$$C = \{(x_1, \cdots, x_n) : \sum_{i=1}^n x_i \geq c\}$$

or

$$C = \{(x_1, \cdots, x_n) : \bar{x} \geq c_1\}.$$

Note that $\bar{X} \sim N(\theta, 1/n)$. So,

$$\alpha = P(\bar{X} \geq c_1; \theta = 0) = \int_{c_1}^{\infty} \frac{1}{\sqrt{2\pi/n}} e^{-\frac{y^2}{2/n}} dy = \int_{c_1\sqrt{n}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 1 - \Phi(c_1\sqrt{n}),$$

and the power

$$\gamma(1) = P(\bar{X} \geq c_1; \theta = 1) = \int_{c_1}^{\infty} \frac{1}{\sqrt{2\pi/n}} e^{-\frac{(y-1)^2}{2/n}} dy$$
$$= \int_{(c_1-1)\sqrt{n}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 1 - \Phi((c_1 - 1)\sqrt{n}).$$

For example, if $n = 25$ and $\alpha = 0.05$, then $c_1 = 1.645/\sqrt{25} = 0.329$, and the power

$$\gamma(1) = 1 - \Phi((0.329 - 1)\sqrt{25}) = 1 - \Phi(-3.355) = 0.9996.$$

Example 5.2.4 (for the tutorial). *If $X_1, ..., X_n$ is a random sample from a beta distribution with parameters $\alpha = \beta = \theta > 0$, find a best critical region for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$.*

5.3 Uniformly Most Powerful Tests

Definition 5.3.1. *The critical region C is a uniformly most powerful critical region of size α for testing the simple hypothesis H_0 against an alternative composite hypothesis H_1 if the set C is a best critical region of size α for testing H_0 against each simple hypothesis in H_1. A test defined by this critical region C is called a uniformly most powerful (UMP) test, with significance level α, for testing the simple hypothesis H_0 against the alternative composite hypothesis H_1.*

Example 5.3.2. *Consider the pdf,*

$$f(x|\theta) = \begin{cases} \frac{1}{\theta}e^{-x/\theta}, & 0 < x < \infty, \\ 0, & \textit{elsewhere,} \end{cases}$$

of Example 5.1.8. It is desired to test the simple hypothesis $H_0 : \theta = 2$ against the alternative composite hypothesis $H_1 : \theta > 2$. Thus $\Omega = \{\theta; \theta \geq 2\}$. A random sample, X_1, X_2, of size $n = 2$ will be used, and the critical region is $C = \{(x_1, x_2)\,; 9.5 \leq x_1 + x_2 < \infty\}$. It was shown in the example cited that the significance level of the test is approximately 0.05 and that the power of the test when $\theta = 4$ is approximately 0.31. This test is a UMP test.

Solution.

$$H_0 : \theta = \theta' = 2 \qquad \text{versus} \qquad H_1 : \theta = \theta'' > 2$$

$$\begin{aligned} \frac{L(\theta')}{L(\theta'')} &= \frac{\prod_{i=1}^n \frac{1}{\theta'}\mathrm{e}^{-\frac{x_i}{\theta'}}}{\prod_{i=1}^n \frac{1}{\theta''}\mathrm{e}^{-\frac{x_i}{\theta''}}} \\ &= \left(\frac{\theta''}{\theta'}\right)^n \mathrm{e}^{-\left(\frac{1}{\theta'}-\frac{1}{\theta''}\right)\sum_{i=1}^n x_i} \\ &\leq k \end{aligned}$$

$$\Longleftrightarrow \qquad \sum_{i=1}^n x_i \geq c, \quad n = 2$$

Take $c = 9.5$. Thus, $C = \{(x_1, x_2) : x_1 + x_2 \geq 9.5\}$ is a uniformly most powerful region for testing $H_0 : \theta = 2$ versus $H_1 : \theta > 2$.

Example 5.3.3. *Let $X_1, \ldots, X_n$ denote a random sample from the distribution that is $N(0, \theta)$, where the variance θ is an unknown positive number. It will be shown that there exists a uniformly most powerful test with significance level α for testing the simple hypothesis $H_0 : \theta = \theta'$, where θ' is a fixed positive number, against the alternative composite hypothesis $H_1 : \theta > \theta'$. Thus $\Omega = \{\theta; \theta \geq \theta'\}$.*

Proof.

$$H_0 : \theta = \theta' \qquad \text{versus} \qquad H_1 : \theta = \theta'' > \theta'$$

The joint p.d.f. of $X_1, \ldots, X_n$ is

$$L(\theta; x_1, x_2, \cdots, x_n) = \left(\frac{1}{2\pi\theta}\right)^{n/2} \exp\left(-\frac{\sum_1^n x_i^2}{2\theta}\right).$$

$$\frac{L(\theta')}{L(\theta'')} = \left(\frac{\theta''}{\theta'}\right)^{\frac{n}{2}} \mathrm{e}^{-\left(\frac{\theta''-\theta'}{2\theta'\theta''}\right)\sum\limits_{i=1}^{n} x_i^2} \leq k$$

$$\Longleftrightarrow \qquad \sum_{i=1}^{n} x_i^2 \geq \frac{2\theta'\theta''}{\theta''-\theta'}\left\{\frac{n}{2}\log(\frac{\theta''}{\theta'}) - \log k\right\} = c$$

So, $C = \{(x_1, x_2, \cdots, x_n) : \sum_{i=1}^n x_i^2 \geq c\}$ is a uniformly most powerful critical region for testing $H_0 : \theta = \theta'$ versus $H_1 : \theta > \theta'$.

We want to determine c. If H_0 is true, $\frac{\sum_{i=1}^n X_i^2}{\theta'} \sim \chi^2(n)$.

$\alpha = P(\sum_{i=1}^n X_i^2 \geq c; H_0) = P(\frac{\sum_{i=1}^n X_i^2}{\theta'} \geq \frac{c}{\theta'})$.

$\frac{c}{\theta'} = \chi^2_{\alpha,n}, \Longrightarrow c = \theta'\chi^2_{\alpha,n}$.

If we take $n = 15, \alpha = 0.05$, and $\theta' = 3$, then $\frac{c}{3} = 25$. $\Rightarrow c = 75$.

Example 5.3.4. *Let $X_1, \ldots, X_n$ denote a random sample from the distribution that is $N(\theta, 1)$, where the mean θ is unknown. It will be shown that there is no uniformly most powerful test of the simple hypothesis $H_0 : \theta = \theta'$, where θ' is a fixed number,* *against the alternative composite hypothesis $H_1 : \theta \neq \theta'$. Thus $\Omega = \{\theta; -\infty < \theta < \infty\}$. Let θ'' be a number not equal to θ'.*

Proof.

$$H_0 : \theta = \theta' \qquad \text{versus} \qquad H_1 : \theta = \theta'' \neq \theta'$$

$$L(\theta) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \mathrm{e}^{-\frac{\sum_{i=1}^{n}(x_i-\theta)^2}{2}}$$

$$\begin{aligned} \frac{L(\theta')}{L(\theta'')} &= \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta')^2+\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta'')^2} \\ &= \exp\left\{-(\theta''-\theta')\sum_{i=1}^{n} x_i + \frac{n}{2}(\theta''^2 - \theta'^2)\right\} \\ &\leq k \end{aligned}$$

$$\Longleftrightarrow \qquad (\theta''-\theta')\sum_{i=1}^{n} x_i \geq \frac{n}{2}(\theta''^2 - \theta'^2) - \log k$$

Two cases:

(i) If $\theta'' > \theta'$, $\frac{L(\theta')}{L(\theta'')} \leq k \Longleftrightarrow \sum_{i=1}^{n} x_i \geq \frac{n}{2}(\theta' + \theta'') - \frac{\log k}{\theta''-\theta'}$.

(ii) If $\theta'' < \theta'$, $\frac{L(\theta')}{L(\theta'')} \leq k \Longleftrightarrow \sum_{i=1}^{n} x_i \leq \frac{n}{2}(\theta' + \theta'') - \frac{\log k}{\theta''-\theta'}$.

Therefore, there is no uniformly most powerful test for testing $H_0 : \theta = \theta'$ versus $H_1 : \theta \neq \theta'$.

5.4 Likelihood Ratio Tests

The notion of using the magnitude of the ratio of two probability density function as the basis of a test or of a uniformly most powerful test can be modified, and made intuitively appealing, to provide a method of constructing a test of a composite hypothesis against an alternative composite hypothesis or of constructing a test of a simple hypothesis against an alternative composite hypothesis when a uniformly most powerful test does not exist. This method leads to tests called *likelihood ratio tests.* A likelihood ratio test, as just remarked, is not necessarily a uniformly most powerful test, but it has been proved in the literature that such a test often has desirable properties.

A certain terminology and notation will be introduced by means of an example.

Example 5.4.1. *Let the random variable X be $N(\theta_1, \theta_2)$ and let the parameter space be*

$$\Omega = \{(\theta_1, \theta_2)\,; -\infty < \theta_1 < \infty, 0 < \theta_2 < \infty\}\,.$$

Let the composite hypothesis be $H_0 : \theta_1 = 0, \theta_2 > 0$, and let the alternative composite hypothesis be $H_1 : \theta_1 \neq 0, \theta_2 > 0$. The set $\omega = \{(\theta_1, \theta_2)\,; \theta_1 = 0, 0 < \theta_2 < \infty\}$ is a subset of Ω and will be called the subspace specified by the hypothesis H_0. Then for instance, the hypothesis H_0 may be described as $H_0 : (\theta_1, \theta_2) \in \omega$. It is proposed that we test H_0 against all alternative in H_1.

Let

$$L(\hat{\omega}) = \max_{(\theta_1,\theta_2)\in\omega} L(\theta_1, \theta_2) \quad \text{and} \quad L(\hat{\Omega}) = \max_{(\theta_1,\theta_2)\in\Omega} L(\theta_1, \theta_2).$$

The likelihood ratio is given by

$$\Lambda(x_1, \cdots, x_n) = \frac{L(\hat{\omega})}{L(\hat{\Omega})}.$$

Note that $0 \leq \Lambda(x_1, \ldots, x_n) \leq 1$. The likelihood ratio test is determined by

$$C = \{(x_1, \cdots, x_n) : \Lambda(x_1, \cdots, x_n) \leq \lambda_0\}, \quad 0 < \lambda_0 < 1$$

The size or significance level of the test is

$$\alpha = P_{H_0}\{(X_1, \cdots, X_n) \in C\} = P_{H_0}\{\Lambda(X_1, \cdots, X_n) \leq \lambda_0\}$$

$$L(\theta_1, \theta_2) = \left(\frac{1}{2\pi\theta_2}\right)^{n/2} e^{-\sum_{i=1}^n (x_i-\theta_1)^2/(2\theta_2)}$$

When $(\theta_1, \theta_2) \in \omega$,

$$L(\theta_1 = 0, \theta_2) = \left(\frac{1}{2\pi\theta_2}\right)^{n/2} e^{-\sum_{i=1}^n x_i^2/(2\theta_2)}$$

$$\log L(\theta_1 = 0, \theta_2) = (-n/2)\log(2\pi) - (n/2)\log\theta_2 - \sum_{i=1}^n x_i^2/(2\theta_2)$$

$$\frac{\partial \log L(\theta_1 = 0, \theta_2)}{\partial\theta_2} = -\frac{n}{2\theta_2} + \frac{\sum_{i=1}^n x_i^2}{2\theta_2^2} = 0 \quad \Longrightarrow \theta_2 = \frac{\sum_{i=1}^n x_i^2}{n}$$

$$\Longrightarrow L(\hat{\omega}) = \left(\frac{1}{2\pi(\sum_{i=1}^n x_i^2/n)}\right)^{n/2} e^{-\sum_{i=1}^n x_i^2/(2\sum_{i=1}^n x_i^2/n)} = \left(\frac{ne^{-1}}{2\pi\sum_{i=1}^n x_i^2}\right)^{n/2}$$

We know that $L(\hat{\Omega}) = L(\hat{\theta}_{1,mle}, \hat{\theta}_{2,mle})$, where

$$\hat{\theta}_{1,mle} = \bar{x}, \quad \hat{\theta}_{2,mle} = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}.$$

Then,

$$L(\hat{\Omega}) = \left(\frac{1}{2\pi\sum_{i=1}^n (x_i - \bar{x})^2/n}\right)^{n/2} e^{-\sum_{i=1}^n (x_i-\bar{x})^2/(2\sum_{i=1}^n (x_i-\bar{x})^2/n)}$$
$$= \left(\frac{ne^{-1}}{2\pi\sum_{i=1}^n (x_i - \bar{x})^2}\right)^{n/2}$$

$$\Lambda(x_1, \ldots, x_n) = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \left[\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{\sum_{i=1}^n x_i^2}\right]^{n/2} = \left[\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{\sum_{i=1}^n (x_i - \bar{x})^2 + n\bar{x}^2}\right]^{n/2}$$

$$= \left[\frac{1}{1 + \frac{n\bar{x}^2}{\sum_{i=1}^n (x_i-\bar{x})^2}}\right]^{n/2} \leq \lambda_0$$

$$\Longrightarrow \frac{n\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \geq c$$

$$C = \{(x_1, \ldots, x_n) : \ \Lambda(x_1, \ldots, x_n) \leq \lambda_0\} = \left\{(x_1, \ldots, x_n) : \ \frac{n\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \geq c\right\}$$

If H_0 is true, $\frac{\sqrt{n}\bar{X}}{\sqrt{\sum_{i=1}^n (X_i-\bar{X})^2/(n-1)}} \sim t(n-1)$.

The preceding example should make the following generalization easier to read. Let $X_1, \ldots, X_n$ denote n mutually stochastically independent random variables having, respectively, the probability density function

$$f_i\left(x_i; \theta_1, \cdots, \theta_m\right), i = 1, 2, ..., n.$$

The set that consists of all parameter points $(\theta_1, \cdots, \theta_m)$ is denoted by Ω, which we have called the parameter space. Let ω be a subset of the parameter space Ω. We wish to test the (simple or composite) hypothesis $H_0 : (\theta_1, \cdots, \theta_m) \in \omega$ against all alternative hypothesis.

Let

$$L(\hat{\omega}) = \max_{(\theta_1, \ldots, \theta_m) \in \omega} L(\theta_1, \cdots, \theta_m) \quad \text{and} \quad L(\hat{\Omega}) = \max_{(\theta_1, \ldots, \theta_m) \in \Omega} L(\theta_1, \cdots, \theta_m).$$

The likelihood ratio is given by

$$\Lambda(x_1, \cdots, x_n) = \frac{L(\hat{\omega})}{L(\hat{\Omega})}.$$

Note that $0 \leq \Lambda(x_1, \cdots, x_n) \leq 1$. The likelihood ratio test is determined by

$$C = \{(x_1, \cdots, x_n) : \Lambda(x_1, \cdots, x_n) \leq \lambda_0\}, \quad 0 < \lambda_0 < 1$$

The size or significance level of the test is

$$\alpha = P_{H_0}\{(X_1, \cdots, X_n) \in C\} = P_{H_0}\{\Lambda(X_1, \cdots, X_n) \leq \lambda_0\}.$$

Example 5.4.2 (for the tutorial). *Consider a distribution having a pmf of the form* $f(x;\theta) = \theta^x(1-\theta)^{1-x}, x = 0, 1,$ *zero elsewhere. Let* $H_0 : \theta = \frac{1}{20}$ *and* $H_1 : \theta > \frac{1}{20}$. *Use the central limit theorem to determine the sample size* n *of a random sample so that a uniformly most powerful test of* H_0 *against* H_1 *has a power function* $\gamma(\theta)$, *with approximately* $\gamma(\frac{1}{20}) = 0.05$ *and* $\gamma(\frac{1}{10}) = 0.90$.

Example 5.4.3. *Show that the likelihood ratio principle leads to the same test when testing a simple hypothesis H_0 against an alternative simple hypothesis H_1, as that given by the Neyman-Pearson theorem. Note that there are only two points in Ω.*

Example 5.4.4 (for the tutorial)**.** *Let X have the pdf $f(x;\theta) = \theta^x(1-\theta)^{1-x}, x = 0, 1,$ zero elsewhere. We test $H_0 : \theta = \frac{1}{2}$ and $H_1 : \theta < \frac{1}{2}$ by taking a random sample $X_1, ..., X_5$ of size $n = 5$ and rejecting H_0 if $Y = \sum_{i=1}^{n} X_i$ is observed to be less than or equal to a constant c.*

(a) Show that this is a uniformly most powerful test.

(b) Find the significance level when $c = 1$.

(c) Find the significance level when $c = 0$.