2023 - 2024

1. compute gradient in terms of x

$$\nabla f_\mu(x) = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix} - \mu \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \frac{1}{8-x_1+3x_3} = 0$$

and $8 - x_1 + 3x_3 > 0$

assume $k := \frac{\mu}{2} \cdot \frac{1}{8-x_1+3x_3}$, we get $x_1 = -k, x_2 = 0, x_3 = 3k$, $k > 0$

$$k = \frac{\mu}{2} \cdot \frac{1}{8-x_1+3x_3} = \frac{\mu}{2} \cdot \frac{1}{8+10k} \Rightarrow 10k^2 + 8k - \frac{\mu}{2} = 0, \quad k > 0$$

we get solution $-\frac{2}{5} \pm \frac{\sqrt{\frac{16}{25} + \frac{\mu}{5}}}{2}$, since $k > 0$,

we have $k = -\frac{2}{5} + \frac{\sqrt{\frac{16}{25}+\frac{\mu}{5}}}{2}$

global minimizer is $\begin{pmatrix} \frac{2}{5} - \frac{\sqrt{\frac{16}{25}+\frac{\mu}{5}}}{2} \\ 0 \\ -\frac{6}{5} + \frac{3\sqrt{\frac{16}{25}+\frac{\mu}{5}}}{2} \end{pmatrix}$

$A := (-1, 0, 3)$,

and check $\nabla^2 f_\mu(x) = 2I + \mu\left(A^T \cdot \frac{1}{8-x_1+3x_3} \cdot A\right)$

it's convex. $\square$

$\exists\, c > 0$, s.t.

2. (a) $f(x) \geq \sum_{i=1}^{3} \sqrt{8+x_i^2} \geq \sum_{i=1}^{3} |x_i| \geq \|x\|_2 \cdot c$

$$c \cdot \|x^k\|_2 \leq f(x^k) \leq f(x^{k-1}) - \alpha_k \cdot \sigma \cdot \nabla f(x^k)^T \cdot \nabla f(x^k) \leq f(x^{k-1}) \cdots \leq f(x^0)$$

so $\|x^k\|_2$ is bounded, $\{x^k\}$ is bounded.

(b) $f \in C^2(\mathbb{R}^3)$, set $A := \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & -3 \end{pmatrix}$, $g(y) := \ln(5+y_1^2) + \sqrt{8+y_2^2}$

$$f(x) = g(Ax - b) + \sum_{i=1}^{3} \sqrt{8+x_i^2}$$

$b = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$

日期: /

$$\nabla^2 g(Ax-b) = A^T \nabla^2 g(y) A, \quad \nabla^2 g(y) = \begin{pmatrix} \frac{10-2y_1^2}{(5+y_1^2)^2} & 0 \\ 0 & \frac{8}{(8+y_2^2)^{3/2}} \end{pmatrix}$$

set $t = 5-y^2$, $t \le 5$

$$\frac{10-2y^2}{(y^2+5)^2} = \frac{2t}{t^2-20t+100} = \frac{2}{t-20+\frac{100}{t}} \le \frac{2}{5}, \quad \frac{8}{(8+y^2)^{3/2}} \le \frac{\sqrt{2}}{4}$$

set $h(x) := \sum_{i=1}^{3} \sqrt{8+x_i^2}$.

$$\nabla^2 h(x) = \begin{pmatrix} \frac{8}{(8+x_1^2)^{3/2}} & & 0 \\ & \frac{8}{(8+x_2^2)^{3/2}} & \\ 0 & & \frac{8}{(8+x_3^2)^{3/2}} \end{pmatrix}$$

$$\|\nabla^2 g(Ax-b)\|_2 \le \|A\|_2^2 \cdot \|\nabla^2 g(y)\|_2 \le \lambda_{max}\begin{pmatrix} 6 & -4 \\ -4 & 10 \end{pmatrix} \cdot \frac{2}{5} = \frac{16+4\sqrt{5}}{5}$$

$$\|\nabla^2 h(x)\|_2 \le \frac{\sqrt{2}}{4}.$$

$$\|\nabla^2 f(x)\|_2 \le \|\nabla^2 g(Ax-b)\|_2 + \|\nabla^2 h(x)\|_2 \le \frac{\sqrt{2}}{4} + \frac{16+4\sqrt{5}}{5} \approx 5.4 := L$$

now we know $\frac{\alpha}{2} \le \frac{2}{L} \Rightarrow \alpha \le \frac{2\alpha}{L} \approx 1.176$

hence if $\alpha = 1.176$, any accumulation point of $\{x^k\}$ is stationary. $\square$

3. (a) (1) set primal problem is:

$$\text{Min} \quad f(x) := 3x_1 - (x_2)^3$$
$$\text{s.t.} \quad g_1(x) := (x_1)^2 + 2(x_2)^2 - 5 \le 0$$
$$g_2(x) := -2x_1 - x_2 + 3 \le 0$$

$\nabla^2 g_1(x) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \succ 0$, so $g_1(x)$ is convex, $g_2(x)$ is convex

we find slater point $\bar{x} = \begin{pmatrix} 1 \\ 1.1 \end{pmatrix}$ satisfy $g_1(\bar{x}) < 0$, $g_2(\bar{x}) < 0$

hence, MFCQ holds on feasible region.

(2) KKT conditions: ① $g_1(x) = x_1^2 + 2x_2^2 - 5 \le 0$, $g_2(x) = -2x_1 - x_2 + 3 \le 0$

② $\nabla f(x) + \lambda_1 \nabla g_1(x) + \lambda_2 \nabla g_2(x)$

$$= \begin{pmatrix} 3 \\ -3x_2^2 \end{pmatrix} + \lambda_1 \cdot \begin{pmatrix} 2x_1 \\ 4x_2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ -1 \end{pmatrix} = 0$$

日期: /

③ $\lambda_1 \geq 0, \lambda_2 \geq 0$, $\lambda_1 \cdot g_1(x) = \lambda_1 (x_1^2 + 2x_2^2 - 5) = 0$

$\lambda_2 \cdot g_2(x) = \lambda_2 (-2x_1 - x_2 + 3) = 0$

(3) if $g_1(x) \neq 0, g_2(x) \neq 0$, we have $\lambda_1 = \lambda_2 = 0$

$\nabla f(x) + \lambda_1 \nabla g_1(x) + \lambda_2 g_2(x) = \begin{pmatrix} 3 \\ -3x_2^2 \end{pmatrix} = 0$, it's contradictory

so there's no stationary point satisfied $g_1(x), g_2(x) \neq 0$. □

(b) if $g(x) \in C^1(\mathbb{R}^1)$,

$g(x) = (x+1)^2 - 1$ is the counterexample

if $g(x) \in C^1(\mathbb{R}^n)$, $n \geq 2023$,

$g(x) := (x_1+1)^2 - 1 + \sum_{i=2}^{n} x_i^2$, slater point $\tilde{x} = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, satisfy $g(\tilde{x}) < 0$.

and $g(x)$ is convex, since $(x+1)^2$, x^2 is convex, $\frac{\partial^2 (x+1)^2}{\partial x^2} = 2 > 0$.

so MFCQ holds on Ω,

consider S, only feasible point is $\hat{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$,

$\lambda_0 \cdot \nabla g(x) - I(\lambda_1 \cdots \lambda_n)^T = \lambda_0 \cdot \begin{pmatrix} 2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = 0$,

$\lambda_0, \lambda_1, \cdots, \lambda_n \geq 0$

$\Rightarrow \lambda_1, \cdots \lambda_n = 0$, $2\lambda_0 - \lambda_1 = 0$, it's not positive independent.

so $\hat{x}$ does not satisfy MFCQ. □

(D) consider MFCQ, if $\tilde{x} \in \Omega$ satisfies $g(\tilde{x}) = 0$.

and MFCQ fails, $\Rightarrow$ $\lambda \cdot \nabla g(\tilde{x}) = 0$, $\lambda \geq 0$, doesnot have unique solution. $\lambda = 0$.

if $\nabla g(\tilde{x}) \neq 0$, it has unique solution $\lambda = 0$.

hence $\nabla g(\bar{x}) = 0$.

$\nabla f(\bar{x}) \neq 0$, so $\nabla f(\bar{x}) + \lambda \nabla g(\bar{x}) = \nabla f(\bar{x}) \neq 0$, contradiction.

hence, cannot find $\lambda \geq 0$, s.t. $\nabla f(\bar{x}) + \lambda \nabla g(\bar{x}) = 0$. □

4. (a) Primal Problem $\Leftrightarrow$

$$\begin{aligned} \text{Min} \quad & x_1 + x_2 + x_3 + y \\ \text{s.t.} \quad & y^2 \geq x_1^2 + (x_2 + x_3)^6 + 2023 \\ & \frac{(x_3)^2}{x_2+7} + (x_1 \ \ x_2) \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq 2 \\ & x_2 \geq 0, \ x_3 \geq 0 \end{aligned}$$

consider $y^2 \geq x_1^2 + (x_2 + x_3)^6 + 2023$

$\Leftrightarrow$ $y^2 \geq x_1^2 + t_1^2 + 2023$, $t_1^2 \geq (x_2 + x_3)^6$, $t_1 \geq 0$.

$\Leftrightarrow$ $y^2 \geq x_1^2 + t_1^2 + 2023$, $t_1 \geq |x_2 + x_3|^3$, $t_1 \geq 0$

$\Leftrightarrow$ $y^2 \geq x_1^2 + t_1^2 + 2023$, $t_2 t_1 \geq t_2^3 \cdot t_2$, $t_2 \geq |x_2 + x_3|$, $t_1, t_2 \geq 0$

$\Leftrightarrow$ $y^2 \geq x_1^2 + t_1^2 + 2023$, $t_1 t_2 \geq t_3^2$, $t_3 \geq t_2^2$, $t_2 \geq |x_2 + x_3|$, $t_1 \geq 0$, $t_2 \geq 0$, $t_3 \geq 0$

$\Leftrightarrow$ $\begin{pmatrix} y \cdot I_3 & (x_1 \ \ t_1 \ \ \sqrt{2023})^T \\ (x_1 \ \ t_1 \ \ \sqrt{2023}) & y \end{pmatrix} \succeq 0$, $\begin{pmatrix} t_1 & t_3 \\ t_3 & t_2 \end{pmatrix} \succeq 0$, $\begin{pmatrix} t_3 & t_2 \\ t_2 & 1 \end{pmatrix} \succeq 0$,

$t_2 \geq x_2 + x_3 \geq -t_2$

consider $\frac{(x_3)^2}{x_2+7} + (x_1 \ \ x_2) A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq 2$, $x_2 \geq 0$, set $A := \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix} \succ 0$

$\Leftrightarrow$ $x_2 \geq 0$, $\frac{(x_3)^2}{x_2+7} + z \leq 2$, $z \geq (x_1 \ \ x_2) A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$A^{-1} = \frac{1}{4} \begin{pmatrix} 5 & -1 \\ -1 & 1 \end{pmatrix}$.

$\Leftrightarrow$ $x_2 \geq 0$, $(2 - z)(x_2 + 7) \geq (x_3)^2$, $z \geq (x_1 \ \ x_2) A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\Leftrightarrow$ $x_2 \geq 0$, $\begin{pmatrix} 2-z & x_3 \\ x_3 & x_2+7 \end{pmatrix} \succeq 0$, $\begin{pmatrix} z & x_1 & x_2 \\ x_1 & 5/4 & -1/4 \\ x_2 & -1/4 & 1 \end{pmatrix} = \begin{pmatrix} z & x_1 & x_2 \\ x_1 & \multicolumn{2}{c}{A^{-1}} \\ x_2 & & \end{pmatrix} \succeq 0$.

日期: /

so we reformulate it to SDP.

(b) primal problem:

$$\begin{aligned} \text{Min} \quad & tr(CX) \\ s.t. \quad & tr(A_1 X) = b_1 \\ & tr(A_2 X) = b_2. \\ & X \succeq 0. \end{aligned}$$

$$C \succ 0, \quad A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1/2 & 1 \\ 1/2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

dual problem:

$$\begin{aligned} \text{Max} \quad & b^T y \\ s.t. \quad & C - y_1 A_1 - y_2 A_2 \succeq 0 \end{aligned}$$

(2) $\bar{X} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \succ 0$, satisfies $tr(A_1 \bar{X}) = b_1$, $tr(A_2 \bar{X}) = b_2$.

so $V_p = V_d$.

$\bar{y} = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}$, $C - y_1 A_1 - y_2 A_2 = \begin{pmatrix} 3/2 & 1/4 & 0 \\ 1/4 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \succ 0$, since $3/2 > 0$, $\det\begin{pmatrix} 3/2 & 1/4 \\ 1/4 & 1 \end{pmatrix} > 0$.

so $V_p = V_d$ and $tr(C\bar{X}) \geq V_p = V_d \geq b^T \bar{y}$,

so V_p, V_d is finite $\Rightarrow$ we get V_p, V_d is attainable. □

(3) assume $x_{12} = 0$, $a := x_{11}$

we have $x_{13} = \frac{2-a}{2}$, $C = \begin{pmatrix} a & 0 & \frac{2-a}{2} \\ 0 & x_{22} & 0 \\ \frac{2-a}{2} & 0 & x_{33} \end{pmatrix}$

assume $\underline{x_{22} > 0}$, $a \cdot x_{33} = (\frac{2-a}{2})^2 \Rightarrow x_{33} = \frac{1}{a} \cdot (\frac{2-a}{2})^2$

$x_{22} = \frac{4 - x_{11} - x_{33}}{2}$, $tr(CX) = x_{11} + x_{22} + 3x_{33} - x_{13} = a + \frac{4 - a - x_{33}}{2} + 3x_{33} - \frac{2-a}{2}$

assume $tr(CX) = a + 1 + \frac{5}{2} x_{33} = \frac{5}{2} - \varepsilon$, $\varepsilon > 0$.

$$\Rightarrow a^2 + \frac{5}{2}\left(\frac{2-a}{2}\right)^2 = \left(\frac{3}{2} - \varepsilon\right)a,$$

$$\Rightarrow \frac{13}{8} a^2 + \left(\varepsilon - \frac{13}{2}\right)a + \frac{5}{2} = 0$$

$$\Rightarrow a = \frac{-\varepsilon + \frac{13}{2} \pm \sqrt{(\varepsilon - \frac{13}{2})^2 - 4 \cdot \frac{13}{8} \cdot \frac{5}{2}}}{2 \cdot \frac{13}{8}}$$

we consider $a = \dfrac{(-\varepsilon+\frac{13}{2}) - \sqrt{(\varepsilon-\frac{13}{2})^2 - 4\cdot\frac{13}{8}\cdot\frac{5}{2}}}{2\cdot\frac{13}{8}}$

if $\varepsilon=0$, $a=\dfrac{26-2\sqrt{104}}{13}$, it satisfy $x_{22}>0$, so that $X\succeq 0$.

and $tr(CX)=2.5$

if $\varepsilon=0.001$, $tr(CX)=2.5-\varepsilon<2.5$. and $X\succ 0$ can be verified.

hence $v_p<2.5$. $\square$

5.(a) $A=\begin{pmatrix} 5 & & & & & & -2 \\ & 3 & & & & -1 & \\ & & 2 & & 0 & & \\ & & & \ddots & & & \\ & & 0 & & 2 & & \\ & -1 & & & & 3 & \\ -2 & & & & & & 5 \end{pmatrix}$

$3-\frac{1}{3} \quad 0 \quad \frac{1}{3}$

$0 \quad 3-\frac{1}{3}$

$P_1 := \begin{cases} 1 & \text{if } i=j,\ i\neq 2023, 2 \\ 1 & \text{if } (i=2023, j=2) \text{ or } (i=2, j=2023) \\ 0 & \text{otherwise} \end{cases}$

$P_1^T A P_1 = \begin{pmatrix} \begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix} & & & & \\ & 2 & & & \\ & & \ddots & & \\ & & & 2 & \\ & & & & \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \end{pmatrix}$, it means we switch row 2023, row 2 and column 2023, column 2.

Solve $\begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}-\lambda I=0$, $\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}-\lambda I=0$.

eigenvalues are 7, 3, 4, 2 (2 is of 2020 power)

(b) Apply Luenberger theorem.

$$(x^4-x^*)^T A (x^4-x^*) \le \left(\frac{\lambda_{n-3}-\lambda_1}{\lambda_3+\lambda_1}\right)^2 (x^0-x^*)^T A (x^0-x^*)$$

$n=2023$, $\lambda_{2020}=\lambda_1=0$.

$$(x^4-x^*)^T A (x^4-x^*) \ge \lambda_{min}(A)\,\|x^4-x^*\|_2^2$$

$\Rightarrow \quad x^4=x^*$,

日期:　　　/

$x^* = A^{-1} \cdot b$,　　$Ax^* = b$,

$\Rightarrow$ we have $\begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_{2023} \end{pmatrix} = \begin{pmatrix} 1 \\ 2023 \end{pmatrix}$

$\Rightarrow (x^*)^1 = (x^4)^1 = \frac{4051}{21}$. $\square$