日期: /2021 - 2022

1. (a) (i) $P_{Y_1, Y_2}(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2)$

$$= P(X_1 X_2 = y_1, X_2 = y_2)$$

$$= P(X_1 = \frac{y_1}{y_2}, X_2 = y_2) \quad (y_2 \neq 0)$$

$$= p(\frac{y_1}{y_2}, y_2) = y_1/18.$$

(ii) $P_{Y_1}(y_1) = P(Y_1 = y_1) = P(X_1 \cdot X_2 = y_1)$

$$= \sum_{x_1 \cdot x_2 = y_1} y_1/18$$

$$= \begin{cases} 1/18 & y_1 = 1 \\ 4/18 & y_1 = 2 \\ 3/18 & y_1 = 3 \\ 4/18 & y_1 = 4 \\ 6/18 & y_1 = 6 \\ 0 & \text{otherwise} \end{cases}$$

(iii) $E(Y_1) = \sum y_1 \cdot P_{Y_1}(y_1) = \frac{1}{18} + \frac{8}{18} + \frac{9}{18} + \frac{16}{18} + \frac{36}{18} = \frac{35}{9}$

$Var(Y_1) = E(Y_1^2) - E(Y_1)^2 = \frac{1}{18} + \frac{16}{18} + \frac{27}{18} + \frac{64}{18} + \frac{216}{18} - (\frac{35}{9})^2 = \frac{1175}{324}$

MGF: $M_{Y_1}(t) = E(e^{ty_1}) = e^t \cdot \frac{1}{18} + e^{2t} \cdot \frac{4}{18} + e^{3t} \cdot \frac{3}{18} + e^{4t} \cdot \frac{4}{18} + e^{6t} \cdot \frac{6}{18}$

(b) Beta distribution ?

$$f(x; \alpha = \theta, \beta = 1) = \frac{\Gamma(\theta+1)}{\Gamma(\theta) \cdot \Gamma(1)} \cdot x^{\theta-1}, \quad x \in (0,1)$$

$$= \theta \cdot x^{\theta - 1}$$

$$\frac{\partial \log L(\theta, x_1 - x_n)}{\partial \theta} = \frac{\partial \sum_{i=1}^{n} (\log \theta + (\theta-1)\log x_i)}{\partial \theta} = 0 \Rightarrow \hat{\theta} = -\frac{n}{\sum_{i=1}^{n} \log x_i}$$

$$\frac{\partial^2 \log L(\theta, x_1 - x_n)}{\partial \theta^2} = -\frac{n}{\theta^2} < 0$$

日期: /

so $\hat{\theta} = -\dfrac{n}{\sum_{i=1}^{n} \log x_i}$ is maximizer, that is MLE of θ.

$\tau(\hat{\theta}) = \dfrac{\hat{\theta}}{1+\hat{\theta}} = \dfrac{n}{n - \sum_{i=1}^{n} \log x_i}$ is MLE of $\tau(\theta)$.

(ii) $f(x; \alpha=\theta, \beta=1) = \theta \cdot x^{\theta-1} = \exp\{\log\theta + (\theta-1)\log x\}$

is exponential class,

so $\sum_{i=1}^{n} \log(x_i)$ is sufficient statistics

since $L(\theta; x_1 - x_n) = \exp\{\sum \log\theta + (\theta-1)\log x_i\} = g(\sum\log x_i, \theta)$

is also function of $\hat{\theta}_{MLE}$.

so $\hat{\theta}_{MLE} = -\dfrac{n}{\sum \log x_i}$ is sufficient statistics.

2、 (a) MLE: $\dfrac{\partial \log L(\theta; x_1 \cdots x_n)}{\partial \sigma} = \dfrac{\partial \sum_{i=1}^{n} \log 2 - \log(\sqrt{2\pi}\sigma) - \frac{(x_i-\mu)^2}{2\sigma^2}}{\partial \sigma}$

$$= \sum_{i=1}^{n} -\frac{1}{\sigma} + \frac{2(x_i-\mu)^2}{2\sigma^3} = 0$$

$$\Rightarrow \hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}$$

$$\left.\frac{\partial^2 \log L(\theta; x_1 - x_n)}{\partial \sigma^2}\right|_{\sigma=\hat{\sigma}} = \left.\sum_{i=1}^{n} \frac{1}{\sigma^2} - 3 \cdot \frac{(x_i-\mu)^2}{\sigma^4}\right|_{\sigma=\hat{\sigma}} < 0$$

so $\hat{\sigma}^2 = \dfrac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}$.

(b) $\hat{\sigma}_{MLE} = \sqrt{\hat{\sigma}^2_{MLE}} = \sqrt{\dfrac{\sum(x_i-\mu)^2}{n}}$

(c) assume that $\sigma_1^2 = 1$, $\sigma_2^2 = 4$.

$$\frac{L(\sigma_1^2)}{L(\sigma_2^2)} = \frac{\prod \frac{2}{\sqrt{2\pi}} \exp\left(\frac{-(x_i-\mu)^2}{2}\right)}{\prod \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x_i-\mu)^2}{8}\right)}$$

$$= 2^n \cdot \exp\left(-\frac{3}{8} \cdot \sum_{i=1}^{n} (x_i-\mu)^2\right) \le k$$

$\Rightarrow \sum_{i=1}^{n} (x_i-\mu)^2 \ge c$ is critical region.

significance level $\alpha = P\left(\sum_{i=1}^{n} (x_i-\mu)^2 \ge c \,\middle|\, \sigma^2 = 1\right)$.

(d) $E(X) = \int_0^{+\infty} \frac{2}{\sqrt{2\pi}\cdot\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx$

$= \int_0^{+\infty} \frac{2}{\sqrt{2\pi}\cdot\sigma} \exp(-w)\, d\sqrt{2w}\cdot\sigma$?

$\left(\int_0^{+\infty} \exp(-x^2)\, dx\right)^2 = \int_0^{+\infty}\int_0^{+\infty} \exp(-x^2-y^2)\, dx\, dy$

substitute $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $= \int_0^{+\infty} dr \int_0^{\pi/2} \exp(-r^2)\, r\, d\theta = \frac{\pi}{4}$

$\Rightarrow \int_0^{+\infty} \exp(-x^2)\, dx = \frac{\sqrt{\pi}}{2}$

so $E(X) = \int_0^{+\infty} \frac{2}{\sqrt{2\pi}\cdot\sigma} \exp(-t^2)\, d\sqrt{2}\sigma t = 1$?

3. (a) $\frac{\partial \log L(\theta; x_1 \cdots x_n)}{\partial\theta} = \frac{\partial \sum_{i=1}^{n} \log(2x_i) - \log\theta - \frac{x_i^2}{\theta}}{\partial\theta}$

$= -\frac{n}{\theta} + \frac{\sum x_i^2}{\theta^2} = 0$

$\Rightarrow \hat{\theta} = \frac{\sum x_i^2}{n}$.

$\left.\frac{\partial^2 \log L(\theta)}{\partial\theta^2}\right|_{\theta=\hat{\theta}} = \left.\frac{n}{\theta^2} - \frac{2\sum x_i^2}{\theta^3}\right|_{\theta=\hat{\theta}} < 0$, so $\hat{\theta}_{MLE}$ is local maximizer

日期: /

$\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} x_i^2}{n}.$

(b) $E(X_1^2) = E(X^2) = \int_0^{+\infty} \frac{2x^3}{\theta} \cdot e^{-x^2/\theta}\, dx$

$$= \int_0^{+\infty} \frac{x^2}{\theta} e^{-x^2/\theta}\, dx^2$$

$$= \int_0^{+\infty} \frac{t}{\theta} e^{-t/\theta}\, dt$$

$$= \int_0^{+\infty} -t \; de^{-t/\theta}$$

$$= -t\, e^{-t/\theta}\Big|_0^{+\infty} + \int_0^{+\infty} e^{-t/\theta}\, dt$$

$$= \theta$$

so X_1^2 is unbiased estimator of θ.

(c) $f(x|\theta) = \exp\left(\log 2x - \log\theta - \frac{x^2}{\theta}\right) \qquad x > 0$

so X is exponential class

(d) R-C lower bound $= \frac{1}{nI(\theta)}$

$I(\theta) = -E\left(\frac{\partial^2 \log f(x|\theta)}{\partial\theta^2}\right)$

$= -E\left(\frac{\partial^2 \log 2x - \log\theta - \frac{x^2}{\theta}}{\partial\theta^2}\right)$

$= -E\left(\frac{1}{\theta^2} - 2\frac{x^2}{\theta^3}\right)$

$= -\frac{1}{\theta^2} + 2\frac{1}{\theta^3}\cdot\theta = \frac{1}{\theta^2}. \qquad$ R-C lower bound $= \frac{\theta^2}{n}.$

(e) uniformly? <u>MVUE</u> 求 MVUE. unique

$f(x|\theta) = \exp\left(\log 2x - \log\theta - \frac{x^2}{\theta}\right),$

we know $\sum_{i=1}^{n} X_i^2$ is sufficient statistics

日期:　　/

and $E(X_i^2)=\theta$, X_i^2 is unbiased.

$E\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right)=\theta$, $\overline{X^2}$ is unbiased,

so $E\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 \,\middle|\, \sum_{i=1}^{n} X_i^2\right) = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ is unique MVUE. □

4. (a) (1) $L(\theta_0)=\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma}\cdot \exp\left\{\frac{-(x_i-\theta_0)^2}{2\sigma^2}\right\}$

$$\hat{\theta} \triangleq \arg\max_{\theta\in\mathbb{R}} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma}\cdot \exp\left\{\frac{-(x_i-\theta)^2}{2\sigma^2}\right\}$$

$$= \arg\max_{\theta} \sum_{i=1}^{n} -\frac{1}{2}\log(2\pi) - \log\sigma - \frac{(x_i-\theta)^2}{2\sigma^2}$$

$$= \arg\max_{\theta} -\sum_{i=1}^{n} \frac{(x_i-\theta)^2}{2\sigma^2}$$

set $g(\theta) \triangleq -\sum_{i=1}^{n} \frac{(x_i-\theta)^2}{2\sigma^2}$,

$g'(\theta) = -\sum_{i=1}^{n} \frac{(\theta-x_i)}{\sigma^2} = 0 \Rightarrow \hat{\theta} = \bar{x}$,

$g''(\theta) = -\frac{n}{\sigma^2} < 0$, so $\hat{\theta} = \arg\max_{\theta\in\mathbb{R}} L(\theta)$

$$\Lambda(x_1, \ldots, x_n) = \frac{L(\theta_0)}{L(\hat{\theta})} = \exp\left\{\sum_{i=1}^{n} \frac{-(x_i-\theta_0)^2}{2\sigma^2} + \frac{(x_i-\bar{x})^2}{2\sigma^2}\right\}$$

critical region is $C = \{(x_1, \ldots, x_n) \mid \Lambda \le k\}$

$$= \left\{(x_1 \ldots x_n) \,\middle|\, \sum_{i=1}^{n} (x_i-\bar{x})^2 - (x_i-\theta_0)^2 \le c_1\right\}$$

$$= \{(x_1 \ldots x_n) \mid -n\bar{x}^2 + 2n\theta_0\bar{x} \le c_2\}$$

$$= \{(x_1 \ldots x_n) \mid \bar{x}^2 - 2\theta_0\bar{x} \ge c_3\}$$

so $\alpha = P\{\bar{x}^2 - 2\theta_0\bar{x} \ge c \mid \theta=\theta_0\}$.

日期: /

4(b) (i) $\log L(p) = \sum_{i=1}^{n} \log [p^{x_i} (1-p)^{1-x_i}]$

$$= \sum_{i=1}^{n} x_i \log p + (1-x_i) \log (1-p)$$

$$\frac{\partial \log L(p)}{\partial p} = \frac{n\bar{x}}{p} + \frac{n-n\bar{x}}{p-1} = 0$$

$$\Rightarrow \hat{p} = \bar{X}$$

$$\frac{\partial^2 \log L(p)}{\partial p^2} = -\frac{n\bar{x}}{p^2} - \frac{n(1-\bar{x})}{(p-1)^2} < 0$$, $\hat{p}$ is maximizer.

hence $\hat{p} = \bar{X}$ is MLE estimator.

(ii) $E(\hat{p}) = E(\bar{X}) = E(X) = p$, so it's unbiased of p.

(iii) $I(p) = -E\left(\frac{\partial^2 \log f(x;p)}{\partial p^2}\right)$

$$= -E\left(-\frac{x}{p^2} - \frac{1-x}{(p-1)^2}\right) = E\left(\frac{x}{p^2}\right) + E\left(\frac{1-x}{(p-1)^2}\right)$$

$$= \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)}$$

(iv) $Var(\hat{p}) = \frac{1}{n} Var(X) = \frac{1}{n} p(1-p)$

$MSE(\hat{p}) = Var(\hat{p}) + bias(\hat{p})^2 = \frac{1}{n} p(1-p) \to 0$ as $n \to +\infty$.

so $\hat{p} \xrightarrow{P} p$, $\hat{p}$ is consistent of p.

5 (a) $f_T(y) = f_X\left(\frac{y}{\frac{1}{\theta} - t}\right) \cdot \left|\frac{dx}{dy}\right|$

$$= \frac{1}{\Gamma(2)\theta^2} y^{2-1} \cdot \left(\frac{1}{\frac{1}{\theta} - t}\right)^2 \cdot \exp\left\{\frac{y}{\theta t - 1}\right\}$$

MGF: $M_X(t) = E(e^{tX}) = \int_0^{+\infty} e^{tx} \cdot \frac{1}{\Gamma(2) \cdot \theta^2} x^{2-1} \cdot e^{-\frac{x}{\theta}} dx$

$$= \int_0^{+\infty} \frac{1}{\Gamma(2) \cdot \theta^2} \exp\left(\left(t - \frac{1}{\theta}\right)x\right) \cdot x^{2-1} dx$$

assume $a \triangleq \operatorname{sgn}(\frac{1}{\theta}-t)$,

$$= \int_0^{+\infty} \frac{1}{\Gamma(\alpha)\cdot\theta^{\alpha}} \exp(-ay) \cdot \left(\frac{ay}{\frac{1}{\theta}-t}\right)^{\alpha-1} d\, \frac{ay}{\frac{1}{\theta}-t}$$

$$= \int_0^{+\infty} \frac{1}{\Gamma(\alpha)\cdot\theta^{\alpha}} \cdot \left(\frac{a}{\frac{1}{\theta}-t}\right)^{\alpha} \cdot y^{\alpha-1} \cdot \exp(-ay)\, dy$$

we know that $\frac{1}{\theta}-t>0$ must be held.

$$= \frac{1}{\Gamma(\alpha)\cdot\theta^{\alpha}} \cdot \left(\frac{a}{\frac{1}{\theta}-t}\right)^{\alpha} \cdot (\alpha-1)! \cdot a^{\alpha-1} \cdot \int_0^{+\infty} \exp(-ay)\, dy$$

$$= \frac{1}{\Gamma(\alpha)\cdot\theta^{\alpha}} \cdot \left(\frac{1}{\frac{1}{\theta}-t}\right)^{\alpha} \cdot (\alpha-1)! \int_0^{+\infty} \exp(-y)\, dy$$

$$= \frac{1}{\Gamma(\alpha)} \cdot \left(\frac{1}{1-\theta t}\right)^{\alpha} \cdot (\alpha-1)!$$

$$= \left(\frac{1}{1-\theta t}\right)^{\alpha} \qquad \left(\frac{1}{\theta}-t>0\right)$$

(ii) set $Y \triangleq \sum_{i=1}^{n} X_i$

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t) = \left(\frac{1}{1-\theta t}\right)^{n\alpha}$$

so $Y \sim \Gamma(n\alpha, \theta)$

(5b) $$\frac{f(x_1;2)\cdot f(x_2;2)}{f(x_1;1)\cdot f(x_2;1)} = \frac{\frac{1}{4}\cdot \exp\{-\frac{x_1+x_2}{2}\}}{\exp\{-x_1-x_2\}} = \frac{1}{4}\exp\left\{\frac{x_1+x_2}{2}\right\} \le \frac{1}{2}$$

$$\Longleftrightarrow \quad x_1+x_2 \le 2\log 2$$

$$\alpha = P(x_1+x_2 \le 2\log 2 \mid \theta=2)$$

$$M_X(t) = \int_0^{+\infty} e^{tx}\cdot\frac{1}{\theta}\cdot e^{-x/\theta}\, dx = \frac{1}{1-\theta t}$$

$$M_{X_1+X_2}(t) = M_X(t)^2 = \left(\frac{1}{1-\theta t}\right)^2$$

so $X_1+X_2 \sim \Gamma(2,\theta)$

$$\alpha = P(X_1+X_2 \le 2\log 2 \mid \theta=2) = \int_0^{2\log 2} \frac{1}{4}\, x\cdot e^{-\frac{x}{2}}\, dx$$

$$= \frac{1}{2} - \frac{1}{2}\log 2$$

日期:　　　/

(ii) $P(x_1+x_2 \le 2\log 2 \mid \theta=1) = \int_0^{2\log 2} x \cdot \exp\{-x\}\, dx$

$= \frac{3}{4} - \frac{1}{2}\log 2.$ $\square$