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1. ① $E(X)=\theta$,

$\hat{\theta}_{MME}=\bar{X}$

② $\log L(\theta, x_1 \cdots x_n)=\sum_{i=1}^{n} x_i \log\theta+(1-x_i)\log(1-\theta)$

$\frac{\partial \log L(\theta)}{\partial\theta}=\sum_{i=1}^{n}\frac{x_i}{\theta}+\frac{1-x_i}{\theta-1}=\sum_{i=1}^{n}\frac{x_i-\theta}{\theta(1-\theta)}=0$

$\frac{\partial^2 \log L(\theta)}{\partial\theta^2}=\sum-\frac{x_i}{\theta^2}-\frac{1-x_i}{(\theta-1)^2}<0$

so $\hat{\theta}_{MLE}=\bar{X}$

2. $f_X(x)=\frac{1}{\theta}$, $x\in[0,\theta]$, $\theta>0$

(a) $E(X)=\int_0^\theta \frac{1}{\theta}x\,dx=\frac{1}{2}\theta$,

$\hat{\theta}_{MME}=2\bar{X}$

(b) Find $\hat{\theta}_{MLE}$ first.

$L(\theta, x_1\cdots x_n)=\begin{cases}(\frac{1}{\theta})^n, & \text{if } 0\le x_i\le\theta \\ 0 & \text{otherwise}\end{cases}$

montonically decrease.

$\hat{\theta}_{MLE}=\max\{x_1 \cdots x_n\}$

(c) ① $Var(2\bar{X})=Var(\hat{\theta}_{MME})=4Var(\bar{X})=\frac{4}{n}Var(X)$

$=\frac{4}{n}\cdot\frac{\theta^2}{12}=\frac{\theta^2}{3n}$

② $E(2\bar{X})=\theta$, since $MSE(\hat{\theta}_{MME})=Var(\hat{\theta}_{MME})+bias^2(\hat{\theta})$

$=\frac{\theta^2}{3n}\to 0$ as $n\to\infty$.

then we have $\hat{\theta}_{MME} \xrightarrow{P} \theta$. consistent. ✓

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(d) $F_{X_{(n)}}(x) = P(X_{(n)} \le x) = P(X_1 \le x, X_2 \le x \cdots X_n \le x)$

$= P(X \le x)^n = (\frac{x}{\theta})^n$, if $x \in [0, \theta]$, $\theta > 0$

$\Rightarrow f_{X_{(n)}}(x) = n \cdot (\frac{x}{\theta})^{n-1} \cdot \frac{1}{\theta}$, if $x \in [0, \theta]$, 0 otherwise.

(e) $$\frac{f(x_1;\theta) \cdots f(x_n;\theta)}{f(x_{(n)};\theta)} = \frac{(\frac{1}{\theta})^n}{n(\frac{x}{\theta})^{n-1} \cdot \frac{1}{\theta}} = \frac{1}{n \cdot x^{n-1}}$$

independent with θ.

it is sufficient for θ

exponential family need the domain is independent with θ

uniform distribution is not exponential family.

3. Gamma distribution : $f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot x^{\alpha-1} \cdot e^{-x/\beta}$, $x > 0$

$f_X(x) = \frac{1}{\Gamma(5)\theta^5} \cdot x^4 \cdot e^{-\frac{x}{\theta}}$, $x > 0$

(a) $E(X) = \alpha \cdot \beta = 5\theta$,

$\hat{\theta}_{MME} = \frac{\bar{X}}{5}$

(b) $\log L(\theta; x_1 \cdots x_n) = \sum_{i=1}^{n} \log \frac{1}{\Gamma(5)} - 5\log\theta + 4\log x_i - \frac{x_i}{\theta}$

$\frac{\partial \log L(\theta)}{\partial\theta} = \sum_{i=1}^{n} -\frac{5}{\theta} + \frac{x_i}{\theta^2}$, $\hat{\theta}_{MLE} = \frac{1}{5}\bar{X}$

we find $\sum_{i=1}^{n} -5\theta + x_i$ maximizer is $\frac{1}{5}\bar{X}$.

so $\hat{\theta}_{MLE} = \frac{1}{5}\bar{X}$.

(c) $I(\theta) = -E\left(\frac{\partial^2 \log f(x|\theta)}{\partial\theta^2}\right)$

$= -E\left(\frac{5}{\theta^2} - \frac{2x}{\theta^3}\right)$ $\qquad \frac{1}{\Gamma(5)\cdot\theta^5}\cdot x^4 \cdot e^{-x/\theta}$

$= -\frac{5}{\theta^2} + \frac{2}{\theta^3}\cdot E(X)$ $\qquad \frac{1}{\Gamma(\alpha)\cdot\beta^\alpha}\cdot x^{\alpha-1}\cdot e^{-x/\beta}$

$= \frac{5}{\theta^2}$

$\alpha\beta^2 \quad 5\cdot\theta^2$

(d) $Var(\hat{\theta}_{MLE}) = Var(\frac{1}{5}\bar{X}) = \frac{1}{25}\cdot\frac{1}{n}Var(X) = \frac{1}{25}\cdot\frac{1}{n}\cdot 5\cdot\theta^2 = \frac{1}{5}\cdot\frac{1}{n}\cdot\theta^2$

R-C lower bound $= \frac{1}{nI(\theta)} = \frac{1}{n\cdot\frac{5}{\theta^2}} = \frac{1}{5n}\cdot\theta^2 = Var(\hat{\theta}_{MLE})$

efficient.

$\left(\frac{1}{x}\right)\theta^x$

4. (a) $b(1,\theta)$ $\qquad p_X(x) = \theta^x(1-\theta)^{1-x}$, $x = 0,1$

$= \begin{cases} \theta & \text{if } x=1 \\ 1-\theta & \text{if } x=0 \end{cases}$

$p_X(x) = \exp\{x\cdot\log\theta + (1-x)\log(1-\theta)\}$ $\qquad \theta^2$ lower

$= \exp\{\log(1-\theta) + x\cdot\log\frac{\theta}{1-\theta}\}$

hence $\sum_{i=1}^n x_i$ is complete sufficient.

(b) MVUE of θ,

$E(\bar{X}) = E(X) = \theta$,

MVUE is $E(\bar{X} | \sum_{i=1}^n x_i) = \bar{X}$

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(c) $E(X^2)=\theta$, $E(\bar{X}^2)=\frac{1}{n^2}\left((n^2-n)E(X)^2+nE(X^2)\right)=\frac{n-1}{n}\theta^2+\frac{1}{n}\theta$

$E\left(\frac{n}{n-1}\left(\bar{X}^2-\frac{1}{n}\bar{X}\right)\right)=\theta^2.$

$\frac{n}{n-1}\left(\bar{X}^2-\frac{1}{n}\bar{X}\right)$ is unbiased estimator for θ^2.

$\bar{X}$ is sufficient complete statistic for θ and for θ^2

hence UMVUE of $e^2(\theta-\theta^2)$ is $E\left(e^2\left[\bar{X}-\frac{n}{n-1}\left(\bar{X}^2-\frac{1}{n}\bar{X}\right)\right]\middle|\bar{X}\right)$

$=\left(\frac{n}{n-1}\bar{X}-\frac{n}{n-1}\bar{X}^2\right)e^2.$

5. (a) reject H_0 if $Y=\sum x_i\le c$

$\forall\ \theta'=\frac{1}{3},\ \theta''<\frac{1}{3}$

$$\frac{L(\theta')}{L(\theta'')}=\frac{\prod_{i=1}^{n}\theta'^{x_i}(1-\theta')^{1-x_i}}{\prod\theta''^{x_i}(1-\theta'')^{1-x_i}}=\left(\frac{\theta'}{\theta''}\right)^{\sum x_i}\cdot\left(\frac{1-\theta'}{1-\theta''}\right)^{4-\sum x_i}$$

$$=\left(\frac{1-\theta'}{1-\theta''}\right)^4\cdot\left(\frac{\theta'}{\theta''}\cdot\frac{1-\theta''}{1-\theta'}\right)^{\sum x_i}\le k\qquad\frac{\theta'}{\theta''}\cdot\frac{1-\theta''}{1-\theta'}>1$$

$\Longleftrightarrow\ \sum_{i=1}^{4}x_i\le k_1$

we find $\sum_{i=1}^{4}x_i\le c$ is the critical region in UMP test.

$\alpha=P\left(\sum_{i=1}^{4}x_i\le c\,\middle|\,\theta=\frac{1}{3}\right)$

(b) $\sum_{i=1}^{4}x_i\sim\text{Binomial}(4,\theta)$

$P_{\sum x_i}(x)=\binom{4}{x}\theta^x(1-\theta)^{4-x}$

$\alpha=P\left(\sum_{i=1}^{4}x_i\le 0\,\middle|\,\theta=\frac{1}{3}\right)=\left(\frac{2}{3}\right)^4$

(c) $\alpha=P\left(\sum_{i=1}^{4}x_i\le 2\,\middle|\,\theta=\frac{1}{3}\right)=\left(\frac{2}{3}\right)^4+4\cdot\frac{1}{3}\cdot\left(\frac{2}{3}\right)^3+6\cdot\left(\frac{1}{3}\right)^2\cdot\left(\frac{2}{3}\right)^2=\frac{8}{9}$

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