Chapter 4 - Inequalities

Q1: $X \sim Exp(\beta)$ $\quad p(|X-\mu| > k\sigma)$ for $k>1$

Compare it to the bound from Chebyshev

$f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}} \quad x>0 \qquad E(X) = \mu = \beta \qquad V(X) = \sigma^2 = \beta^2$

$\Rightarrow p(|X-\mu| > k\sigma) = p(|X-\beta| > k\beta)$

$= p(X > \beta + k\beta) + \underbrace{\cancel{p(X < \beta - k\beta)}}_{\to 0}$

note: $k>1 \to \beta(1-k) < 0 \to p(X < \beta - k\beta) = 0$

$\Rightarrow p(|X-\mu| > k\sigma) = p(X > \beta(1+k))$

$= \int_{\beta(1+k)}^{\infty} \frac{1}{\beta} e^{-\frac{x}{\beta}} dx = -e^{-x/\beta}\Big)_{\beta(1+k)}^{\infty} = e^{-\frac{1}{\beta}\beta(1+k)} = e^{-(k+1)}$

Using Chebyshev: $p(|X-\mu| > k\sigma) \leqslant \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2}$

$e^{-(k+1)} < \frac{1}{k^2}$ or $\boxed{e^{k+1} > k^2}$ $k>1$ always true.

Q2: $X \sim poisson(\lambda)$

use chebyshev to show $p(X \geqslant 2\lambda) \leqslant \frac{1}{\lambda}$

$f(x) = e^{-\lambda} \frac{\lambda^x}{x!} \quad x \geqslant 0 \qquad \mu = \lambda \quad \sigma^2 = \lambda$

$P(|X-\lambda|>\lambda) = P(x>2\lambda) + \cancel{P(x<0)}^{0}$

from chebyshev: $P(|X-\mu|>t) \leq \frac{\sigma^2}{t^2}$

$\Rightarrow P(x>2\lambda) = P(|X-\lambda|>\lambda) \leq \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$

Q3: $X_1, \dots, X_n \sim \text{Bernoulli}(p)$ $\quad \bar{X} = n^{-1}\sum X_i$

- find $P(|\bar{X}-p|>\varepsilon)$ using chebyshev & Hoeffding
- Show for large n, Hoeff. bound is smaller.

$\mu = E(\bar{X}) = p \qquad \sigma^2 = V(\bar{X}) = \frac{p(1-p)}{n}$

chebyshev: $P(|\bar{X}-\mu|>\varepsilon) \leq \frac{\sigma^2}{\varepsilon^2} = \frac{p(1-p)}{n\varepsilon^2}$

Hoeffding: $P(|\bar{X}-\mu|>\varepsilon) \leq 2e^{-2n\varepsilon^2}$

for large n, exponential decays faster than $\frac{1}{n}$.

Q4: $X_1, \dots, X_n \sim \text{Bernoulli}(p)$

$\varepsilon = \sqrt{\frac{1}{2n}\log\left(\frac{2}{\alpha}\right)} \qquad \alpha>0$

$\hat{p} = n^{-1}\sum X_i \qquad C = (\hat{p}-\varepsilon, \hat{p}+\varepsilon)$

Use Hoeffding to show $P(C \text{ contains } p) \geq 1-\alpha$

prob C does not contain p

$$= P(|\hat{p}-p| > \varepsilon) \leq 2e^{-2n\varepsilon^2} = \alpha$$

$$\Rightarrow P(C \text{ contain } p) \geq 1-\alpha$$

Q5: prove Mill's inequality

Hint: $P(|Z|>t) = 2P(Z>t)$

$$P(Z>t) = \frac{1}{2\pi}\int_t^\infty e^{-x^2/2}\,dx$$

$\Rightarrow$

$$t\,P(Z>t) = \frac{t}{2\pi}\int_t^\infty e^{-x^2/2}\,dx \leq \frac{1}{2\pi}\int_t^\infty x e^{-x^2/2}\,dx$$

note $\displaystyle \frac{1}{2\pi}\int_t^\infty x e^{-x^2/2} = \frac{-1}{\sqrt{2\pi}} e^{-x^2/2}\Big)_t^\infty = \frac{e^{-t^2/2}}{\sqrt{2\pi}}$

$$\Rightarrow \quad P(Z>t) \leq \frac{e^{-t^2/2}}{t\sqrt{2\pi}}$$

$\Rightarrow$

$$P(|Z|>t) \leq \sqrt{\frac{2}{\pi}}\,\frac{e^{-t^2/2}}{t}$$

Q6 computer experiment.

Q7 $X_1, \dots, X_n \sim N(0,1)$

Bound $P(|\bar{X}| \geq t)$ using Mill & Chebyshev

$\bar{X} = \frac{1}{n} \sum x_i$

$E(\bar{X}) = 0 \qquad V(\bar{X}) = \frac{1}{n} \rightarrow \sigma = \frac{1}{\sqrt{n}}$

$Z = \frac{\bar{X}}{\sigma} = \sqrt{n}\, \bar{X}$

$$\Rightarrow P(|\bar{X}| \geq t) = P(|Z| > t\sqrt{n}) \leq \begin{cases} \sqrt{\frac{2}{\pi}} \frac{e^{-t^2 n}}{t\sqrt{n}} & \text{Mill} \\ \frac{1}{t^2 n} & \text{cheb.} \end{cases}$$

Mill's bound is smaller.

Comparison plots in Jupyter notes.