Chapter 2 - Random Variables

Q1: show $P(X=x) = F(x^+) - F(x^-)$

by lemma 2.15 $P(x) = F(x) - F(x^-)$

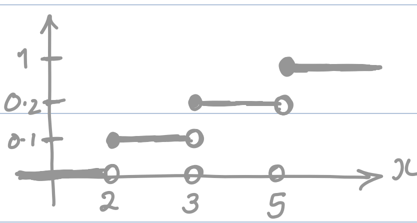
Since F is right cont. $F(x) = F(x^+) \Rightarrow P(x) = F(x^+) - F(x^-)$

Q2: $P(X=2) = P(X=3) = 1/10$ $P(X=5) = 8/10$

plot CDF F, $P(2 < X \leqslant 4.8) = ?$ $P(2 \leqslant X \leqslant 4.8) = ?$

$P(2 < x \leqslant 4.8) = F(4.8) - F(2) = 0.1$

$P(2 \leqslant x \leqslant 4.8) = F(4.8) - F(2^-) = 0.2$



Q3: prove lemma 2.15

a) $P(X=x) = F(x) - F(x^-)$ $F(x^-) = \lim_{y \uparrow x} F(y)$

let $y_1 < y_2 < \cdots$, $A_1 = (-\infty, y_1)$, $A_2 = (-\infty, y_2)$, $\ldots$, $A_\infty = (-\infty, x) = A$

$\Rightarrow \lim_{i \to \infty} y_i = x$, $\bigcup_{i=1}^{\infty} A_i = A$, $P(A_i) = F(y_i) - P(Y = y_i)$

$F(x^-) = \lim_{i \to \infty} P(A_i) = \lim_{i \to \infty} \left(F(y_i) - P(y_i) \right) = F(x) - P(x)$

$$\Rightarrow P(x) = F(x) - F(x^-)$$

b) $P(x < X \leq y) = F(y) - F(x)$

$$P(x < X \leq y) = P(X \leq y) - P(X \leq x)$$
$$= F(y) - F(x)$$

c) $P(X > x) = 1 - F(x)$

$$P(X > x) = 1 - P(X \leq x) = 1 - F(x)$$

d) if X is cont. $\Rightarrow$ $F(b) - F(a) = P(a < X < b) = P(a \leq x < b)$

$$= P(a < X \leq b) = P(a \leq x \leq b)$$

if X is cont $\Rightarrow$ $P(X = x) = F(x) - F(x^-) = 0$ for all x

$$\Rightarrow P(a < x \leq b) = P(X \leq b) - P(X \leq a)$$
$$= P(X \leq b) - P(X < a)$$
$$= P(X < b) - P(X \leq a)$$
$$= P(X < b) - P(X < a)$$

Q4: $f_X(x) = \begin{cases} 1/4 & 0 < x < 1 \\ 3/8 & 3 < x < 5 \\ 0 & \text{o.w.} \end{cases}$

a) find CDF $F(x)$

$$F(x) = P(X \leqslant x) = \begin{cases} \frac{x}{4} & 0 \leqslant x \leqslant 1 \\ \frac{1}{4} & 1 \leqslant x \leqslant 3 \\ \frac{3}{8}x - \frac{7}{8} & 3 \leqslant x \leqslant 5 \\ 1 & x \geqslant 5 \end{cases}$$

b) $Y = \frac{1}{X}$ $\quad f_Y(y) = ?$ $\quad$ hint: consider $\begin{cases} \frac{1}{5} \leqslant y \leqslant \frac{1}{3} \\ \frac{1}{3} \leqslant y \leqslant 1 \\ y \geqslant 1 \end{cases}$

$$f_Y(y) = F_Y'(y)$$

$$F_Y(y) = P(Y < y) = P\left(X > \frac{1}{y}\right) = 1 - F_X\left(\frac{1}{y}\right)$$

$$= \begin{cases} 1 - \frac{1}{4y} & 0 < \frac{1}{y} \leqslant 1 \iff y \geqslant 1 \\ 1 - \frac{1}{4} & 1 \leqslant \frac{1}{y} \leqslant 3 \iff \frac{1}{3} \leqslant y \leqslant 1 \\ 1 + \frac{7}{8} - \frac{3}{8y} & 3 \leqslant \frac{1}{y} \leqslant 5 \iff \frac{1}{5} \leqslant y \leqslant \frac{1}{3} \\ 0 & \frac{1}{y} \geqslant 5 \iff y \leqslant \frac{1}{5} \end{cases}$$

$$\Rightarrow f_Y = F_Y' = \begin{cases} 1/4y^2 & y > 1 \\ 3/8y^2 & 1/5 < y < 1/3 \\ 0 & \text{o.w.} \end{cases}$$

Q5: X & Y indep $\Leftrightarrow f_{X,Y}(x,y) = f_X(x)\, f_Y(y) \;\; \forall x,y$

assume X & Y to be discrete

part 1: if X & Y indepn $\overset{?}{\Rightarrow} f_{X,Y} = f_X\, f_Y$

$$f_{X,Y}(x,y) = P(X\in\{x\}, Y\in\{y\}) = P(X\in\{x\})\, P(Y\in\{y\})$$
$$= f_X(x)\, f_Y(y)$$

part 2: if $\forall x,y \; f_{X,Y} = f_X\, f_Y \overset{?}{\Rightarrow} X$ & Y indep

$$P(X\in\{x\}, Y\in\{y\}) = \sum_{(X,Y)\in x\times y} f_{X,Y}(x,y) = \sum_{(X,Y)\in x\times y} f_X(x)\, f_Y(y)$$

$$= \sum_{X\in x} f_X(X=x) \sum_{Y\in y} f_Y(Y=y)$$

$$= P(X\in\{x\})\; P(Y\in\{y\})$$

Q6: let A subset of real line, $I_A(x) = \begin{cases} 1 & x\in A \\ 0 & x\notin A \end{cases}$

$Y = I_A(X)$, $X \sim F_X, f_X$

CDF $F_Y = ?$

hint: first find prob. mass function for Y

$$f_Y(y) = P(I_A(x) = y) \Rightarrow f_Y(y) = \begin{cases} P(x \in A) & y=1 \\ P(x \notin A) & y=0 \\ 0 & o.w. \end{cases}$$

$$\Rightarrow F_Y(y) = \begin{cases} 0 & y<0 \\ P(x \notin A) & 0 \leq y < 1 \\ 1 & y \geq 1 \end{cases}$$

Q_7: X & Y indep. $\sim$ Unif$(0,1)$

$Z = \min\{X, Y\}$ find $f_Z(z) = ?$

$$P(Z>z) = P(X>z, Y>z) = P(X>z)\,P(Y>z) = (1-z)^2$$

$$\Rightarrow F_Z = 1-(1-z)^2 \Rightarrow f_Z = 2(1-z) \quad \forall z \in [0,1]$$

Q_8: $X \sim F_X$ find CDF of $X^+ = \max\{0, X\}$

$$Y = X^+ \Rightarrow P(Y<y) = \begin{cases} 0 & y<0 \\ f_X(x) & y \geq 0 \end{cases} \Rightarrow F_Y(y) = \begin{cases} 0 & y<0 \\ F_X(y) & y \geq 0 \end{cases}$$

Q9: $X \sim Exp(\beta)$ $\quad F(x) = ?$ $\quad F^{-1}(q) = ?$

$x > 0 \quad , \quad \beta > 0$

$f_{(x)} = \beta^{-1} e^{-x\beta^{-1}}$

$F_X(x) = \beta^{-1} \int_0^x e^{-x\beta^{-1}} dx = 1 - e^{-x\beta^{-1}}$

$F^{-1}(q) = ?$ $\quad F(x) = q = 1 - e^{-x\beta^{-1}} \Rightarrow e^{-x\beta^{-1}} = 1 - q$

$\Rightarrow x = -\beta \ln(1-q)$

$\Rightarrow F^{-1}(q) = -\beta \ln(1-q)$

Q10: let X & Y indep, show $g(X)$ & $h(Y)$ indep

let $A' = \{x : g(x) \in A\}$, $B' = \{y : h(y) \in B\}$

$g(x) \in A \iff x \in A'$, $h(y) \in B \iff y \in B'$

$\Rightarrow P(g(x) \in A, h(y) \in B) = P(x \in A', y \in B')$

$\overset{X \perp\!\!\!\perp Y}{=\!=} P(x \in A') \, P(y \in B')$

$= P(g(x) \in A) \, P(h(y) \in B)$

Q_{11}: X, Y number of heads/tails with prob. p

a) toss once, prove X & Y are dependent

b) toss coin N times, $N \sim \text{Poisson}(\lambda)$

show X & Y are independent!

a) $P(X=1, Y=1) = 0 \neq P(X=1) \cdot P(Y=1)$

$\Rightarrow X, Y$ dependent

b) $P(X=x, Y=y) = P(X=x \mid n=x+y)\; P(n=x+y)$

$= \text{Binomial}(n, p) \cdot \text{Poisson}(\lambda)$

$= \binom{n}{x} p^x (1-p)^{n-x} \, e^{-\lambda} \frac{\lambda^n}{n!}$

$\boxed{n-x=y}$

$$= \frac{\cancel{n!}\; p^x (1-p)^y \, e^{-\lambda} \lambda^{x+y}}{x!\, y! \;\; \cancel{n!}}$$

lets calculate $P(X=x) = ?$

N is random variable where $N \geq X$

$$\Rightarrow P(X=x) = \sum_{k=0}^{\infty} P(X=x \mid N=x+k)\, P(N=x+k)$$

$$P(X \mid N = x+k) = \binom{k+x}{x} p^x (1-p)^k$$

$$P(N = x+k) = e^{-\lambda} \lambda^{x+k} / (x+k)!$$

$$\Rightarrow P(X=x) = \sum_{k=0}^{\infty} \frac{\cancel{(k+x)!}}{x!\,k!} p^x (1-p)^k \frac{e^{-\lambda} \lambda^{x+k}}{\cancel{(x+k)!}}$$

$$= \frac{p^x e^{-\lambda} \lambda^x}{x!} \sum_{k=0}^{\infty} \frac{(\lambda - p\lambda)^k}{k!}$$

note: $\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a$

$$\Rightarrow P(X=x) = p^x \cancel{e^{-\lambda}} \lambda^x \cancel{e^{\lambda}} e^{-p\lambda} / x! = p^x \lambda^x e^{-p\lambda} / x!$$

$$\Rightarrow P(Y=y) = (1-p)^y \lambda^y e^{-(1-p)\lambda} / y!$$

$$\Rightarrow P(X=x)\,P(Y=y) = \frac{p^x (1-p)^y \lambda^{x+y} e^{-\lambda}}{x!\,y!}$$

$$= P(X=x, Y=y) \Rightarrow \text{independent.}$$

Q_{12}: Suppose range of X, Y possibly infinite rectangle

show if $f(x,y) = g(x)\,h(y) \Rightarrow X, Y$ indep.

g & h are some functions, not necessairly density

$$f(x,y) = g(x)\,h(y)$$

$$\Rightarrow \begin{cases} f(x) = \int f(x,y)\,dy = g(x) \int h(y)\,dy \\ f(y) = \int f(x,y)\,dx = h(y) \int g(x)\,dx \end{cases}$$

$$\Rightarrow f(x)\,f(y) = g(x)\,h(y) \iint g(x)\,h(y)\,dx\,dy$$

$$= f(x,y) \underbrace{\iint f(x,y)\,dx\,dy}_{=1}$$

$\Rightarrow f(x)\,f(y) = f(x,y) \Rightarrow$ independent.

Q_{13}: $X \sim N(0,1)$, $Y = e^X$

a) find PDF for Y and plot it

Since $r(x) = e^x$ is strictly monotone increasing

$$\Rightarrow f_Y(y) = f_X(s(y)) \left| \frac{ds(y)}{dy} \right| \quad , \quad s = r^{-1}$$

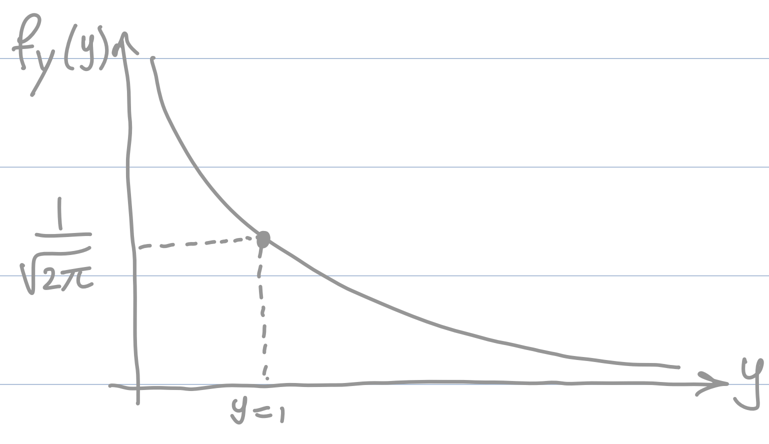
$$y = e^x \Rightarrow x = \ln(y) \Rightarrow \begin{cases} s(y) = \ln(y) \\ ds/dy = 1/y \end{cases}$$

$$\Rightarrow f_Y(y) = f_X(\ln y) \left| \frac{1}{y} \right| = \frac{e^{-\frac{(\ln y)^2}{2}}}{|y|\sqrt{2\pi}} \qquad 0 < y < \infty$$

$$f_Y(y=\infty) = 0/\infty = 0$$

$$f_Y(y=1) = 1/\sqrt{2\pi}$$

$$\lim_{y \to 0} f_Y(y) = +\infty$$ using Hopital rule 2 times



b) computer experiment

Find Solution in Jupyter notebook.

Q_{14}: (X, Y) uniformly dist. on unit disk $\{x^2+y^2 \leqslant 1\}$

$R = \sqrt{X^2+Y^2}$ find CDF & PDF of R

$F_R = P(R<r) = P(\sqrt{x^2+y^2} < r)$

$f(x,y) = 1/\pi \quad \forall\, x^2+y^2<1$ (disc area is π)

$\Rightarrow F_R = P(\sqrt{x^2+y^2} < r) = \frac{\pi r^2}{\pi} \quad 0<r<1$

$\Rightarrow f_R = F_R' = 2r \quad 0<r<1$

Q_{15}: Universal Random Number Generator

F_X strictly inc. CDF of Cont. var. X

$Y = F(X)$

a) find density of Y

b) $U \sim \text{Unif}(0,1)$, $X = F^{-1}(U)$, Show $X \sim F$

c) write code takes Unif$(0,1)$ and generates random variable from Exp(β) distribution

a) $F_Y = P(F(X) \leqslant y) = P(X \leqslant F^{-1}(y)) = F(F^{-1}(y)) = y$

$\Rightarrow f_Y = 1 \quad 0 \leqslant y \leqslant 1$

b) note: prev section $\begin{cases} X \sim F_X \\ Y = F_X \end{cases} \xrightarrow{\text{show}} y$ is uniform

this section $\begin{cases} U \text{ uniform} \\ X = F_X^{-1}(U) \end{cases} \xrightarrow{\text{show}} X \sim F$

$$P(X < x) = P(F_X^{-1}(U) \leq x) = P(U \leq F_X(x))$$
$$= F_X(x) \Rightarrow X \sim F_X$$

c) $U \sim unif(0,1)$, $X = F^{-1}(U)$

exp. dist. : $f_y = \frac{1}{\beta} e^{-y/\beta}$ $\quad y > 0$

$$\Rightarrow F_y(y) = 1 - e^{-\frac{y}{\beta}}$$

$$\Rightarrow F_y^{-1}(c) = -\beta \ln(1-c)$$

$$\Rightarrow X = -\beta \ln(1-u) \qquad u \sim Unif(0,1)$$

verify that $X \sim Exp(\beta)$ dist

$\Rightarrow$ find in Jupyter notebook.

Q_{16}: $X \sim \text{Poisson}(\lambda)$ $Y \sim \text{Poisson}(\mu)$ X, Y indep

show if $X+Y=n \Rightarrow X \sim \text{Binomial}(n, \pi)$

$\pi = \lambda / \lambda+\mu$

hint 1: $X+Y \sim \text{Poisson}(\mu+\lambda)$

hint 2: $\{X=x, X+Y=n\} = \{X=x, Y=n-x\}$

$$P(X=x \mid X+Y=n) = \frac{P(X=x, X+Y=n)}{P(X+Y=n)} \quad \text{I}$$

hint 1 $$P(X+Y=n) = e^{-(\lambda+\mu)} \frac{(\lambda+\mu)^n}{n!} \quad \text{II}$$

hint 2 $$P(X=x) \cdot P(Y=n-x) = \frac{e^{-\lambda}\lambda^x}{x!} \frac{e^{-\mu}\mu^{(n-x)}}{(n-x)!} \quad \text{III}$$

I, II, III

$$P(X=x \mid X+Y=n) = \frac{\dfrac{\cancel{e^{-\lambda-\mu}} \lambda^x \mu^{(n-x)}}{x! \, (n-x)!}}{\cancel{e^{-(\lambda+\mu)}} \dfrac{(\lambda+\mu)^n}{n!}} = \frac{n!}{x!(n-x)!} \times \frac{\lambda^x \mu^{(n-x)}}{(\lambda+\mu)^n}$$

$\pi = \lambda / \lambda+\mu \Rightarrow$

$$P(X=x \mid X+Y=n) = \binom{n}{x} \frac{\lambda^{x} \mu^{(n-x)}}{(\lambda+\mu)^{x} (\lambda+\mu)^{n-x}} = \binom{n}{x} \pi^{x} (1-\pi)^{n-x}$$

$$\Longrightarrow P(X=x \mid X+Y=n) \sim \text{Binomial}(n, \pi)$$

Q17: $f_{X,Y} = \begin{cases} c(x+y^2) & 0 \leqslant x \leqslant 1 \;\&\; 0 \leqslant y \leqslant 1 \\ 0 & \text{o.w.} \end{cases}$

find $P(X < \frac{1}{2} \mid Y = \frac{1}{2})$

$$f_y = \int_0^1 c(x+y^2)\,dx = c(y^2 + \tfrac{1}{2})$$

$$f(x \mid y) = f(x,y)/f_y = \frac{x+y^2}{\frac{1}{2}+y^2}$$

$$\Rightarrow P(X < \tfrac{1}{2} \mid y = \tfrac{1}{2}) = \int_0^{\frac{1}{2}} \frac{x + \frac{1}{4}}{\frac{1}{2} + \frac{1}{4}}\,dx = \frac{1}{3}$$

Q_{18}: $X \sim N(3, 16)$

a) $P(X < 7)$

b) $P(X > -2)$

c) $P(X > x) = 0.05 \Rightarrow x = ?$

d) $P(0 \leq X < 4)$

e) $P(|X| > |x|) = 0.05 \Rightarrow x = ?$

Find simulations in Jupyter notebook.

a) $P(X < 7) = P\left(\frac{x-\mu}{\sigma} < \frac{7-3}{4}\right) = \Phi(1) = 0.84$

b) $P(X > -2) = 1 - \Phi\left(\frac{-2-3}{4}\right) = 1 - \Phi\left(\frac{-5}{4}\right) = 0.89$

c) $P(X > x) = 0.05 \Rightarrow P(X < x) = 0.95$

$\Rightarrow \Phi\left(\frac{x-3}{4}\right) = 0.95 \Rightarrow x = 4\Phi^{-1}(0.95) + 3 = 9.56$

d) $P(0 \leq X < 4) = \Phi\left(\frac{1}{4}\right) - \Phi\left(\frac{-3}{4}\right) = 0.37$

e) $P(|X| > |x|) = 0.05 \Rightarrow x = ?$

$\Rightarrow P(-a < x < a) = 0.95$

$\Rightarrow \Phi\left(\frac{x-3}{4}\right) - \Phi\left(\frac{-x-3}{4}\right) = 0.95$

$\Rightarrow x = 9.6$ numerical solution.

↳ veify in Jupyter notebook.

Q19: $y = r(x)$ r: strictly monotone inc./dec.

$s = r^{-1} \Rightarrow$ prove $f_Y = f_X(s(y)) \left| \frac{ds}{dy} \right|$

if $r(x)$ strictly inc. $\Rightarrow$

$F_Y = P(r(x) \leq y) = P(x \leq s(y)) = F_X(s(y))$

$\Rightarrow f_y = F_Y' = s'(y) f_X(s(y))$

if $r(x)$ strictly dec. $\Rightarrow$

$F_Y = P(r(x) \leq y) = P(x \geq s(y)) = 1 - F_X(s(y))$

$\Rightarrow f_y = F_Y' = -s'(y) f_X(s(y))$

$s'(y) > 0$ strictly inc

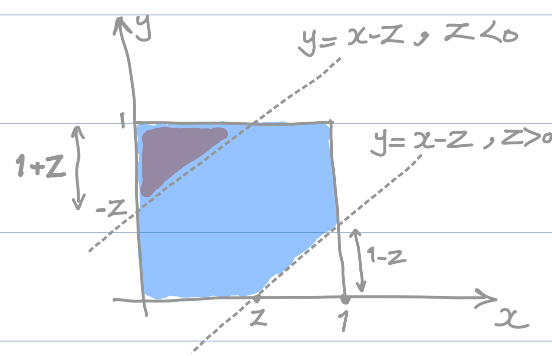
$s'(y) < 0$ strictly dec

$$\Rightarrow f_Y = \left| \frac{ds}{dy} \right| f_X(s(y))$$

Q20: $X, Y \sim \text{Unif}(0,1)$ and indep.

find PDF for $X-Y$ & X/Y

part 1:



$Z = X - Y$, $-1 < Z < 1$

$F_Z = P(x - y < z)$

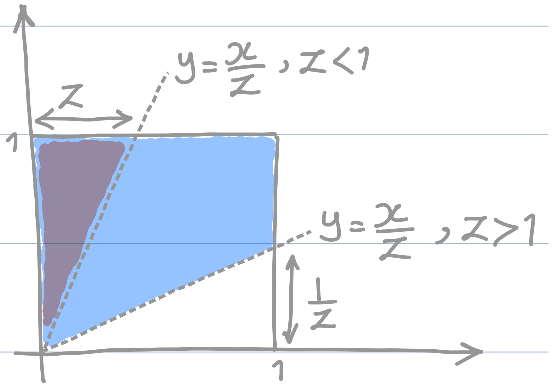
$$\Rightarrow F_Z = \begin{cases} \frac{1}{2}(1+Z)^2 & -1 < Z < 0 \text{ (red region)} \\ 1 - \frac{1}{2}(1-Z)^2 & 0 < Z < 1 \text{ (blue region)} \end{cases}$$

$$\Rightarrow f_Z = \begin{cases} Z+1 & -1 < Z < 0 \\ 1-Z & 0 < Z < 1 \\ 0 & \text{o.w} \end{cases}$$

part 2:

$Z = X/Y \,, \quad 0<Z<\infty$

$F_Z = P(X/Y < z) = P(Y > \frac{x}{z})$



$$\Rightarrow F_Z(z) = \begin{cases} z/2 & 0<z<1 \quad \text{(red region)} \\ 1-\frac{1}{2z} & z>1 \quad \text{(blue region)} \end{cases}$$

$$\Rightarrow f_Z(z) = \begin{cases} 0.5 & 0<z<1 \\ 0.5z^{-2} & z>1 \\ 0 & \text{o.w.} \end{cases}$$

Q_{21}: $X_1, \dots, X_n \sim Exp(\beta)$, IID

$Y = \max\{X_1, \dots, X_n\}$ find PDF of Y?

hint $Y \leq y \Leftrightarrow X_i \leq y \;\; \forall i$

$Exp(\beta) \rightarrow f(x) = \frac{1}{\beta} e^{-x/\beta} \qquad x>0$

$F_Y(y) = P(Y<y) = P(X_i < y) \;\; \forall i = 1, \dots, n$

$$\text{IID} \Rightarrow F_Y = \prod_{i=1}^{n} P(X_i < y)$$

$$P(X_i < y) = \int_0^y \frac{1}{\beta} e^{-\frac{x}{\beta}} = 1 - e^{-y/\beta}$$

$$\Rightarrow F_Y = \left(1 - e^{-y/\beta}\right)^n \Rightarrow f_Y = \frac{n e^{-y/\beta}}{\beta}\left(1 - e^{-y/\beta}\right)^{n-1}$$