Chapter 6 - Models

Q_1: $X_1, \dots, X_n \sim \text{Poisson}(\lambda)$ $\quad \hat{\lambda} = n^{-1} \sum_{i=1}^{n} X_i$

find bias, se, MSE of the estimator

$E(\hat{\lambda}) = n^{-1} \sum E(X_i) = \lambda \Rightarrow \text{bias}(\hat{\lambda}) = \lambda - \lambda = 0$

$V(\hat{\lambda}) = n^{-2} \sum V(X_i) = \frac{\lambda}{n} \Rightarrow se(\hat{\lambda}) = \sqrt{\frac{\lambda}{n}}$

$MSE = \text{bias}^2 + se^2 = \lambda / n$

Q_2: $X_1, \dots, X_n \sim \text{Unif}(0, \theta)$

$\hat{\theta} = \max\{X_1, \dots, X_n\} \Rightarrow$ bias, se, MSE ?

$y = \hat{\theta} \Rightarrow$

$F_y = P(y < y) = P(X_1 < y, \dots, X_n < y) = (y/\theta)^n \qquad 0 < y < \theta$

$\Rightarrow f_y = F_y' = \frac{n}{\theta^n} y^{n-1} \Rightarrow E(y) = \int_0^\theta \frac{n}{\theta^n} y^n \, dy = \frac{n\theta}{n+1}$

$\Rightarrow \text{bias}(\hat{\theta}) = \frac{-\theta}{n+1}$

$E(y^2) = \int_0^\theta \frac{n}{\theta^n} y^{n+1} \, dy = \frac{n}{n+2} \theta^2$

$\Rightarrow V(y) = E(y^2) - E^2(y) = \frac{n\theta^2}{n+2} - \frac{n^2\theta^2}{(n+1)^2}$

$\Rightarrow se^2(\hat{\theta}) = \theta^2 \left(\frac{n}{n+2} - \frac{n^2}{(n+1)^2} \right)$

$$MSE = bias^2 + se^2 = \frac{\theta^2 n}{n+2} + \frac{\theta^2(1-n^2)}{(n+1)^2}$$

Q_3: $X_1, \dots, X_n \sim Unif(0, \theta)$

$\hat{\theta} = 2\bar{X}_n \Rightarrow$ bias, se, MSE ?

note: uniform dist. $E(X_i) = \frac{\theta}{2}$, $V(X_i) = \frac{\theta^2}{12}$

$$E(\hat{\theta}) = \frac{2}{n}\sum E(X_i) = \frac{2}{n} \times \frac{n\theta}{2} = \theta \Rightarrow bias(\hat{\theta}) = 0$$

$$V(\hat{\theta}) = \frac{4}{n^2}\sum V(X_i) = \frac{4}{n^2} \times n \times \frac{\theta^2}{12} = \frac{\theta^2}{3n} \Rightarrow se^2(\hat{\theta}) = \frac{\theta^2}{3n}$$

$$\Rightarrow MSE = bias^2 + se^2 = \frac{\theta^2}{3n}$$