Chapter 1

P53. Section 1.1

P1: A: even B: 4, 5, 6

$(A\cup B)^c = \{1,3\}$,

$A^c \cap B^c = \{1,3,5\} \cap \{1,2,3\} = \{1,3\}$

$(A\cap B)^c = A^c \cup B^c = \{1,2,3,5\}$.

3. if $x \in A \cup (\bigcap_{n=1}^{\infty} B_n)$,

so $x \in A$ or $\bigcap_{n=1}^{\infty} B_n$.

① if $x \in A$, $\Rightarrow x \in \bigcap_{n=1}^{\infty}(A\cup B_n)$

② if $x \in \bigcap_{n=1}^{\infty} B_n \Rightarrow \forall n,\ x \in B_n, \Rightarrow x \in \bigcap_{n=1}^{\infty}(A\cup B_n)$

if $x \in \bigcap_{n=1}^{\infty}(A\cup B_n)$,

if $x \in A$, can prove $x \in A\cup(\bigcap_{n=1}^{\infty} B_n)$

assume that $x \notin A$, so $x \in B_n$, for $\forall n \geq 1$.

$\Rightarrow x \in \bigcap_{n=1}^{\infty} B_n$.

4. assume that all numbers in $[0,1]$, can be list as $\{c_i\}_{i=1}^{\infty}$

$c_i \triangleq 0.a_1^{(i)}a_2^{(i)}\cdots$ ~~express~~ $0 \leq c_i \leq 1$.

we make $c_0 = 0.(a_1^{(1)}+1)\ (a_2^{(2)}+1)\ (a_3^{(3)}+1)\ \cdots\cdots$

if $a_i^{(i)} = 9$, set $a_i^{(i)}+1 = 0$.

so c_0 is different ~~for~~ from $\forall\ c_i$, $c_0 \notin \{c_i\}_{i=1}^{\infty}$.

contradiction, get proved.

5. $P(G) = 60\%$, $P(C) = 70\%$, $P(G \cap C) = 40\%$

$P(G \cup C) = P(G) + P(C) - P(GC) = 90\%$

$P(G^c \cap C^c) = 10\%$

7. sample space $= \{ (a_1, a_2 \cdots, a_n)$ $a_1 \cdots a_{n-1} \in \{1,3\}$ and $a_n \in \{2\}$.

if we can get $\{2,4\}$ in finite dices,

$(a_1, a_2 \cdots)$ $a_i \in \{1,3\}$. $\}$

8. you need to win 2 games continuously,

$P = (p_1 + p_3 - p_1 \cdot p_3) \cdot p_2$

$\overset{!}{=} (1 - (1-p_1)(1-p_3))\, p_2$

compared $A = (p_1 + p_3 - p_1 p_3) p_2$ with $B = (p_1 + p_2 - p_1 \cdot p_2) p_3$

$A - B = p_1 p_2 + p_3 p_2 - p_1 p_3 + p_2 p_3$

$= p_1 (p_2 - p_3)$ if $p_2 > p_3$, $A > B$.

so if $p_2 = \max(p_1, p_2, p_3)$, it has the maximal prop. to win the tournament.

9. $P(A) = P(A \cap \Omega) = P(A \cap (\bigcup_{i=1}^{n} S_i))$

(a)

$= \sum_{i=1}^{n} P(A \cap S_i)$ since $\{A \cap S_i\}_{i=1}^{n}$ are disjount.

disjoint.

(b) $P(A) = P(A \cap \Omega)$.

$\Omega = B + C - B \cap C + (B \cup C)^c$

$= B + C - B \cap C + B^c \cap C^c$.

$P(A) = P(A \cap ((B \cup C) \cup (B \cup C)^c)$

$= P(A \cap (B \cup C)) + P(A \cap B^c \cap C^c)$

$P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C))$

$= P((A \cap B) \cup (A \cap (C \cap B^c)$

$= P((A \cap B) \cup (A \cap (C \cap B^c))$

$= P(A \cap B) + P(A \cap C \cap B^c)$

$P(A \cap C \cap B^c) + P(A \cap B \cap C) = P(A \cap C)$

$\Rightarrow P(A) = P(A \cap B) + P(A \cap C) + P(A \cap B^c \cap C^c) - P(A \cap B \cap C)$.

11. (a) $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$\geq P(A) + P(B) - 1$.

(b) $P(A_1 \cap A_2 \cap A_3) = P(A_1) + P(A_2 \cap A_3) - P(A_1 \cup (A_2 \cap A_3))$

$\geq P(A_1) + P(A_2 \cap A_3) - 1$ use (a) solution.

$\geq P(A_1) + P(A_2) + P(A_3) - 2$.

another solution:

$$1 - P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1^c \cup A_2^c \cup \cdots \cup A_n^c)$$
$$\leq P(A_1^c) + P(A_2^c) + \cdots + P(A_n^c)$$
$$= 1 - P(A_1) + 1 - P(A_2) + \cdots + 1 - P(A_n)$$

$$\Rightarrow P(A_1 \cap A_2 \cap \cdots \cap A_n) \geq P(A_1) + P(A_2) + \cdots + P(A_n) - (n-1).$$

$\square$

12. (a) $P(A \cup B \cup C) = P(A \cup (B \cup C))$

$$= P(A) + P(B \cup C) - P((A \cap B) \cup (A \cap C))$$

$$= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$$

(b) ① use induction to ~~prove~~ prove.

② ~~for $\forall$ $A_{j_1}, A_{j_2} \cdots A_{j_k}$ in $\{A_i\}_{i=1}^{n}$~~

~~appears in set S~~

~~appears in S_k 1 times.~~

~~S_{k+1} $\binom{n-k}{1}$ times~~

~~S_{k+2} $\binom{n-k}{2}$ times~~

assume that E_k means the set appears only in k of $\{A_i\}_{i=1}^{n}$.

(1) prove $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(E_i)$, it's obviously.

(2) $\forall x \in$ ~~prove~~ E_k appears k times in S_1,

appears $\binom{k}{2}$ times in S_2,

$\cdots\cdots$ ~~sum is $k + \binom{k}{2}$~~

$k - \binom{k}{2} + \binom{k}{3} \pm \cdots \pm (-1)^{k+1} \cdot \binom{k}{k} = 1$

since

$(1-1)^k = 1 - \binom{k}{1} + \binom{k}{2} \pm \cdots \pm (-1)^k \cdot \binom{k}{k}$.

To get the conclusion, ~~P(A)~~ $\cancel{\sum_{i=1}^{n} S_i = \sum_{i=1}}$

$$P(S_1) - P(S_2) + \cdots + (-1)^{n-1} \cdot P(S_n) = \sum_{i=1}^{n} P(E_i) = P(\bigcup_{i=1}^{n} A_i)$$

13. (a) $A_n \subset A_{n+1}$, for $\forall n$. $A \triangleq \bigcup_{n=1}^{\infty} A_n$.

prove $P(A) = \lim_{n\to\infty} P(A_n)$.

we have axiom that if $\{B_i\}_{i=1}^{\infty}$ is disjoint, $P(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} B_i$

assume that $B_1 = A_1$, $B_2 = A_2 / A_1$, $B_3 = A_3 / (A_1 \cup A_2)$ ——.

so $\{B_i\}_{i=1}^{\infty}$ is disjoint.

$\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i = A_n$.

$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$, $P(A) = P(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} P(B_i) = \lim_{n\to\infty} P(A_n)$.

(b) from (a)'s conclusion, we get $P(A^c) = \lim_{n\to\infty} P(A_n^c)$

$1 - P(A^c) = \lim_{n\to\infty} 1 - P(A_n^c)$

$\Rightarrow P(A) = \lim_{n\to\infty} P(A_n)$

(c) $\because \lim_{n\to\infty} [0, n] = [0, \infty)$, $\Rightarrow P([0, \infty)) = \lim_{n\to\infty} P([0, n])$.

$1 - \lim_{n\to\infty} P([n, \infty)) = \lim_{n\to\infty} P((-\infty, n)) = P((-\infty, +\infty)) = 1$,

$\Rightarrow \lim_{n\to\infty} P(n, \infty) = 0$.

$\cancel{[-\infty, n]}$ $\bigcap_{n=1}^{\infty} [n, \infty) = \emptyset$, ~~[illegible]~~.

$\Rightarrow \lim_{n\to\infty} P([n, +\infty)) = P(\emptyset) = 0$.

15. Let A be the first tose is head.

B be the second tose is head.

$$P(AB \mid A) = \frac{P(AB)}{P(A)} = \frac{1}{2},$$

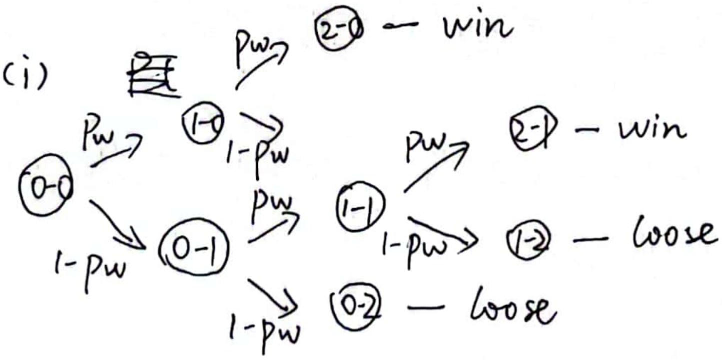
$$P(AB \mid A \cup B) = \frac{\frac{1}{4}}{P(A \cup B)} = \frac{P(AB)}{1 - P(A^c \cap B^c)} = \frac{1}{3}.$$

$$P(AB \mid A) > P(AB \mid A \cup B).$$

17. a batch of 一批.

$$\frac{\binom{95}{4}}{\binom{100}{4}} = \frac{95 \cdot 94 \cdot 93 \cdot 92}{100 \cdot 99 \cdot 98 \cdot 97}$$

20. (a) (i)



$$P = Pw \cdot Pw + 2Pw \cdot (1-Pw) \cdot Pw = Pw^2 + 2Pw^2 - 2Pw^3$$

$$= 3Pw - 2Pw^3.$$

(ii) $P = Pd^2 \cdot Pw$

(iii) $P = Pw \cdot Pd + Pw \cdot (1-Pd) \cdot Pw + (1-Pw) Pw^2$

(b) $P(3^{rd} \text{ strategy}) = Pw \cdot (Pd + Pw - Pwd + Pw - Pw^2)$

22. $P(\cancel{\text{fir}} \text{ white} \mid \text{first jar}) = \frac{m}{m+n}$.

$P(\text{white} \mid \text{second jar}) = P(\text{white} \mid \text{first white}) \cdot P(\text{first white})$

$P(\text{white} \mid \text{first black}) \cdot P(\text{first black})$

$$= \frac{m+1}{m+n+1} \cdot \frac{m}{m+n} + \frac{m}{m+n+1} \cdot \frac{n}{m+n}$$

$$= \frac{m}{m+n},$$

by induction, $P(\text{white} \mid k^{th} \text{ jar}) = \frac{m}{m+n}$.

by induction method,

~~24.~~

26. (a) $P(A|B) = P(A)$. since A, B are independent

(b) $\cancel{P(C)=}$

$$P(A|C) = \frac{P(AC)}{P(C)} = \frac{P(AC)}{P(C|A) \cdot P(A) + P(C|A^c) P(A^c)}$$

$$= \frac{pq}{pq + (1-p) \cdot q \cdot \frac{1}{2}} = \frac{p}{\frac{1}{2}p + \frac{1}{2}} = \frac{2p}{1+p}$$

since $P(C|A) = q$, $P(C|A^c) = q \cdot \frac{1}{2}$

28. (Conditional version of the total probability theorem)

$P(A) = \sum_{i=1}^{n} P(A|C_i) \cdot P(C_i)$.

$P(A|B) = P(AB) / P(B)$.

$\cancel{P(A|B)}$ $P(AB) = \sum_{i=1}^{n} P(AC_iB) \cdot \cancel{P(C_iB)}$

$$= \sum_{i=1}^{n} P(A|C_iB) \cdot P(C_iB)$$

$$\sum_{i=1}^{n} P(A|C_iB) \cdot P(C_i|B) \cdot P(B) \quad \square$$

30. $P(\text{they agree, and choose the right way}) = p^2$

$P(\text{they agree, but the wrong path}) = (1-p)^2$.

$P(\text{they disagree}) = 2p(1-p)$

$P(\text{first strategy works}) = p^2 + 2p(1-p)\cdot\frac{1}{2} = p$.

$P(\text{second strategy works}) = p$.

They are the same.

34. $p_1 = p$, $p_2 = 1-(1-p)\cdot(1-p\cdot(1-(1-p)^3))$

$p_3 = 1-(1-p)^2$.

35. $\cancel{p = \frac{\binom{n}{k}+\binom{n}{k+1}+\cdots+\binom{n}{n}}{2^n}}$

$p = \binom{n}{k}\cdot p^k\cdot(1-p)^{n-k} + \binom{n}{k+1}\cdot p^{k+1}\cdot(1-p)^{n-k-1} + \cdots + \binom{n}{n}\cdot p^n$.

$= \sum_{i=k}^{n}\binom{n}{i}\cdot p^i\cdot(1-p)^{n-i}$.

~~38.~~ 40. $q_1 = 1-p$, $q_2 = p^2 + q_1\cdot(1-p) = p^2+(1-p)^2$

$\cancel{q_n = q_{n-2}\cdot p^2 + q_{n-1}\cdot(1-p)}$

$\cancel{x^2 = (1-p)x + p^2 \Rightarrow x_1 =}$

$q_n = q_{n-1}\cdot(1-p) + q_{n-2}\cdot p^2$

$x^2 + (p-1)x - p^2 = 0 \Rightarrow x_1 = \frac{1-p\pm\sqrt{5p^2-2p+1}}{2}$

$q_n = A x_1^n + B x_2^n$, $q_n = x_1^n + x_2^n = \frac{(1-p+\sqrt{4p^2+(1-p)^2})^n + (\ -\ )}{4}$

40. $q_1 = 1-p$, $q_2 = p^2+(1-p)^2$

assume t_n is odd toses is head, in n times toses.

$t_1 = p$. $t_2 = (1-p)\cdot p + p\cdot(1-p) = -2p^2+2p$.

$q_n = q_{n-1}\cdot(1-p) + t_{n-1}\cdot p$ ①

$t_n = t_{n-1}\cdot(1-p) + q_{n-1}\cdot p = 1-q_n$ ②

$t_{n-1} = (1-q_n - p\, q_{n-1})/(1-p)$ ③ substitute this formula into ①,

$$q_n = (1-p)\,q_{n-1} + \frac{p}{1-p}(1-q_n - p\cdot q_{n-1})$$

$$= \frac{p}{1-p} - \frac{p}{1-p}\,q_n + \left(1-p-\frac{p^2}{1-p}\right) q_{n-1}$$

$$\frac{1}{1-p}\,q_n = \frac{p}{1-p} + \frac{-2p+1}{1-p}\,q_{n-1}.$$

$q_n = p + (1-2p)\,q_{n-1}$. $\cancel{q_n - p = 2}$

$\cancel{x^2+(2p-1)x-p=0}$

$q_n + A = (1-2p)(q_{n-1}+A)$

$\Rightarrow A = -\frac{1}{2}$, $q_n - \frac{1}{2} = (1-2p)(q_{n-1}-\frac{1}{2})$

$= (1-2p)^{n-1}\cdot(1-p-\frac{1}{2})$

$= (1-2p)^n\cdot\frac{1}{2}$.

$q_n = \frac{1}{2} + \frac{1}{2}\cdot(1-2p)^n$. □

42. assume that $w_{k,t}$ is the probability of having k \$. and it is t^{th} times gamble.

~~$w_k = w_{k-1}\cdot p + w_{k+1}\cdot(1-p)$.~~

~~w_{k+1}~~ $w_{k,t+1} = w_{k-1,t}\cdot p + w_{k+1,t}\cdot(1-p)$.

assume that for $\forall k$, $w_{k,t} \to w_k$ as $t \to +\infty$.

so $w_k = w_{k-1}\cdot p + w_{k+1}\cdot(1-p)$

$$\cancel{w_{k+1} - w_k = \frac{p}{p-1}(w_k - w_{k-1})}$$

$$\cancel{= \left(\frac{p}{p-1}\right)^{k-1}(w_0 - w_1)}$$

42. assume that w_k denotes the probabilities to get \$n. when starting with k\$.

$$w_k = w_{k+1} \cdot (1-p) + w_{k-1} \cdot p$$

$$= P(w_{k+1} \mid \text{lose}) + P(w_{k-1} \mid \text{win the first game}).$$

and we have $w_n = 1$, $w_0 = 0$

$$\Rightarrow w_{k+1} - w_k = \frac{1-p}{p}(w_k - w_{k-1})$$

$$= \left(\frac{1-p}{p}\right)^k w_1$$

$$\Rightarrow w_{k+1} = \left(\left(\frac{1-p}{p}\right)^k + \cdots + 1\right) w_1$$

$$= \frac{1 - \left(\frac{1-p}{p}\right)^{k+1}}{1 - \frac{1-p}{p}}$$

$$= \frac{p}{2p-1}\left(1 - \left(\frac{1-p}{p}\right)^{k+1}\right) \cdot w_1$$

if $p = \frac{1}{2}$, $w_k = k \cdot w_1$,

if $p \neq \frac{1}{2}$, $w_k = \frac{p}{2p-1}\left(1 - \left(\frac{1-p}{p}\right)^k\right) w_1$.

if $p = \frac{1}{2}$, $W_n = n w_1$, $\Rightarrow w_1 = \frac{1}{n}$, $\Rightarrow w_k = \frac{1}{n} \cdot w_1 = \frac{k}{n}$.

if $p \neq \frac{1}{2}$, $w_n = \frac{p}{2p-1}\left(1 - \left(\frac{1-p}{p}\right)^n\right) w_1 = 1$. $\Rightarrow \cancel{w_1 = \left(\frac{1-p}{p}\right)^{1-n}}$,

$$\cancel{w_k = \frac{p}{2p-1}\left(\left(\frac{1-p}{p}\right)^{1-n} - \left(\frac{1-p}{p}\right)^{n+k+1}\right)} \quad k > 0.$$

$$\cancel{= \frac{p}{\cdots}\left(\left(\frac{1-p}{p}\right)^{1-n} - \left(\frac{1-p}{p}\right)^{k+1-n}\right)} \quad (k>0)$$

$$w_1 = \frac{2p-1}{p} \cdot \frac{1}{1-(\frac{1-p}{p})^n},$$

$$w_k = \begin{cases} \dfrac{1-(\frac{1-p}{p})^k}{1-(\frac{1-p}{p})^n} & \text{when } (k>0),\ k\le n.\ \ p\neq\frac{1}{2} \\[2ex] \dfrac{k}{n} & \text{when } p=\frac{1}{2},\ \ 0\le k\le n. \end{cases}$$

43.

(a) $P(AB^c) = P(A(\Omega - B)) = P(A) - P(AB)$

$= P(A) - P(A)\cdot P(B)$

$= P(A)\cdot P(B^c)$

(b) apply the conclusion of (a), we can get it easily

44. $P(A\wedge B|C) = \dfrac{P(ABC)}{P(C)}$

$= \dfrac{P(A)P(B)P(C)}{P(C)}$

$= P(A)\cdot P(B)$

$= P(A|C)\cdot P(B|C)$

if A, B is independent, A, B, C is not mutually independent

we can't get $P(AB|C) = P(A|C)P(B|C)$.

example: set $A: \{1,3\}$, $B: \{1,4,6\}$, $C: \{1,2,3\}$ roll the dice.

45. $P(A_1 \vee A_2 | A_3 \wedge A_4) = \dfrac{P((A_1\vee A_2)\wedge A_3 \wedge A_4)}{P(A_3\wedge A_4)} = P(A_1 \vee A_2)$

$= \dfrac{P(A_1A_3A_4 \vee A_2A_3A_4)}{P(A_3A_4)} = \dfrac{P(A_1\vee A_2)P(A_3)P(A_4)}{P(A_3)P(A_4)}$

46. R_n denotes draw red balls n times,

E ~~表~~ denotes $n+1$ th times red ball.

$$P(E|R_n) = \cancel{P}\ \frac{P(ER_n)}{P(R_n)} = \frac{P(R_{n+1})}{P(R_n)}$$

$$P(R_{n+1}) = \frac{1}{m+1}\sum_{k=0}^{m}\left(\frac{k}{m}\right)^{n+1}$$

$$P(R_n) = \frac{1}{m+1}\sum_{k=0}^{m}\left(\frac{k}{m}\right)^{n}$$

$$P(E|R_n) = \frac{\sum_{k=0}^{m}\left(\frac{k}{m}\right)^{n+1}}{\sum_{k=0}^{m}\left(\frac{k}{m}\right)^{n}}$$

$$P(R_n) = \sum_{k=0}^{m}\left(\frac{k}{m}\right)^{n}\cdot\left(\frac{1}{m+1}\right)$$

$$\cancel{故} \to \int_0^1 x^n\,dx \quad \text{as } m\to+\infty.$$

$$= \frac{1}{n+1}$$

$\Rightarrow$ as $m\to+\infty$, $P(E|R_n) = \frac{n+1}{n+2}$.

~~解~~ 48. (a) $P(N) = \prod_{i=1}^{\infty}(1-p_i) \leq \prod_{i=1}^{n}(1-p_i) \leq \left(\frac{\sum_{i=1}^{n}(1-p_i)}{n}\right)^n$

$$\ln\left(\prod_{i=1}^{n}(1-p_i)\right) = \sum_{i=1}^{n}\log(1-p_i) \leq \sum_{i=1}^{n}(-p_i) \to -\infty, \text{ as } n\to+\infty.$$

$$P(N) \leq e^{-\infty} = 0.$$

$P(I)$, I^c denotes the prob. of finite success.

assume that m is the last success.

$$P(I^c|\cancel{对}M) = \frac{P(N)\cdot P(M)}{P(M)} = 0.$$

$$\Rightarrow P(I^c) = \sum_{M} P(I^c|M)\cdot P(M) = 0.$$

why ~~do~~ the question is infinite problem? I think it's not

(b) assume $p_i + q_i = 1$,

$\sum_{i=1}^{\infty} q_i = \infty$, so $P(I^c) = 1$, $\Rightarrow P(I) = 0$.

50. easy

~~52~~.

56. (a) $\binom{8}{4} \cdot \binom{10}{3} = \sim$

(b) if we have k courses in $\{H_6, \cdots H_{10}\}$, L_2, L_3

$(3-k)$ courses in $\{H_1, \cdots, H_5\}$. L_1

① $k=0$, $\binom{5}{3} \cdot \binom{7}{4}$

$k=1$. $\binom{5}{2} \cdot \binom{5}{1} \cdot \binom{5}{1}$

$k=2$, $\binom{5}{2} \cdot \binom{5}{1} \cdot \binom{5}{1}$

$k=3$, $\binom{5}{3} \cdot \binom{6}{2}$

61. $$P = \frac{\binom{m}{i}\binom{n-m}{k-i}}{\binom{n}{k}}$$

62. ~~correct~~ (correct 修正)

(a) we ~~choose~~ permute n objects, have $n!$ sequences but each indistinguishable objects have $k!$ cequences so eventual sequences have $n!/k!$.

(b) $$\frac{n!}{k_1!\, k_2! \cdots k_r!}$$ ✓

assume $A_1 A_2 \cdots A_{k_1}$ are indistinguishable,

the order of $\{A_i\}_{i=1}^{k_1}$ is not matter,

it has $k_1!$ sequences. that is repeatable in n objects.

so we need to ~~divide~~ ~~div~~ divide it. $\square$