Chapter 3. P184.

1. $f_X(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$

① $Y = g(x)$, $E(Y) = E(g(X)) = 1 \cdot P(g(X)=1) + 2 \cdot P(g(X)=2)$

$= 1 \cdot \int_0^{1/3} f_X(x)\,dx + 2 \cdot \int_{1/3}^{1} f_X(x)\,dx$

$= \frac{5}{3}.$

② $E(Y) = \int_0^1 g(x) \cdot f_X(x)\,dx$

$= \int_0^{1/3} g(x) \cdot f_X(x)\,dx + \int_{1/3}^{1} g(x) \cdot f_X(x)\,dx$

$= \frac{5}{3}$

③ $P_Y(y) = \begin{cases} P(g(X)=1) & y=1 \\ P(g(X)=2) & y=2 \\ 0 & \text{otherwise} \end{cases}$

$P(g(x)=1) = P(0 < X \leq 1/3)$,
$P(g(x)=2) = P(1/3 < X \leq 1)$.

2. $\int_{-\infty}^{+\infty} f_X(x)\,dx = (\int_0^{+\infty} + \int_{-\infty}^{0}) f_X(x)\,dx$

$= -\frac{1}{2}e^{-\lambda x}\Big|_0^{+\infty} + \frac{1}{2}e^{-\lambda x}\Big|_{-\infty}^{0}$

$= 1$ so $f_X(x)$ satisfies the normalization condition.

$E(X) = \int_{-\infty}^{+\infty} x \cdot e^{-\lambda|x|} \cdot \frac{\lambda}{2} dx = 0$, since $x \cdot e^{-\lambda|x|}$ is odd function.

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 e^{-\lambda|x|} \cdot \frac{\lambda}{2} dx$$

$$= \int_0^{+\infty} x^2 e^{-\lambda x} \cdot \lambda \, dx$$

$$= \int_0^{+\infty} -x^2 \, d e^{-\lambda x}$$

$$= -x^2 \cdot e^{-\lambda x} \Big|_0^{+\infty} + \int_0^{+\infty} e^{-\lambda x} \cdot (2x) \, dx$$

$$= \int_0^{+\infty} 2x \left(-\frac{1}{\lambda}\right) d e^{-\lambda x}$$

$$= 2x\left(-\frac{1}{\lambda}\right) e^{-\lambda x} \Big|_0^{+\infty} + \int_0^{+\infty} \frac{2}{\lambda} \cdot e^{-\lambda x} dx$$

$$= \frac{2}{\lambda^2},$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{2}{\lambda^2}.$$

If X continuous RV,

3. (a) $\int_0^{+\infty} P(X > x) dx = \int_0^{+\infty} \int_x^{+\infty} f_X(t) \, dt \, dx$

$$= \int_0^{+\infty} \int_0^{t} f_X(t) \, dx \, dt$$

$$= \int_0^{+\infty} t \cdot f_X(t) \, dt$$

$$\int_0^{+\infty} P(X < -x) dx = \int_0^{+\infty} \int_{-\infty}^{-x} f_X(t) \, dt \, dx$$

$$= \int_{-\infty}^{0} \int_0^{-t} f_X(t) \, dx \, dt$$

$$= \int_{-\infty}^{0} -t \cdot f_X(t) \, dt$$

$$\Rightarrow \int_0^{+\infty} P(X > x) dx - \int_0^{+\infty} P(X < -x) dx = E(X).$$

(b) if X is discrete RV,

$$\int_0^{+\infty} P(X>x)\,dx = \int_0^{+\infty} \sum_{t>x} P_X(t)\,dx \qquad (P_X(t)>0 \text{ for countable } t.)$$

$$\cancel{= \sum_{t>0} \int_0^t P_X(t)\,dx}$$

$$= \sum_{t>0} \int_0^t P_X(t)\,dx$$

$$= \sum_{t>0} t\cdot P_X(t)$$

$$\int_0^{+\infty} P(X<-x)\,dx = \int_0^{+\infty} \sum_{t<-x} P_X(t)\,dx$$

$$= \sum_{t\le 0} \int_0^{-t} P_X(t)\,dx$$

$$= \sum_{t\le 0} (-t)\cdot P_X(t)$$

$$\Rightarrow E(X) = \sum_t t\cdot P_X(t) = \int \sim - \int \sim . \quad \square$$

4. (a) from 3 conclusion, $E(g(X)) = \int_0^{\infty} P(g(X)>x)\,dx - \int_0^{+\infty} P(g(X)<-x)\,dx$

we use the proposition in (3): that is

$$P(X>x) = \int_x^{+\infty} f_X(t)\,dt.$$

(b) $$\int_0^{+\infty} P(g(X)>t)\,dt = \int_0^{+\infty} \int_{\{x \mid g(x)>t\}} f_X(x)\,dx\,dt$$

$$= \int_{-\infty}^{+\infty} \int_0^{\max(g(x),0)} f_X(x)\,dt\,dx$$

$$= \int_{-\infty}^{+\infty} \cancel{g(x)\cdot f_X(x)\,dx}\ \max(g(x),0)\, f_X(x)\,dx.$$

$$\int_0^{+\infty} P(g(X) < -t)\, dt = \int_0^{+\infty} \int_{\{x \mid g(x) < -t\}} f_X(x)\, dx\, dt$$

$$= \int_{-\infty}^{+\infty} \int_0^{\max(-g(x),0)} f_X(x)\, dt\, dx$$

$$= \int_{-\infty}^{+\infty} \max(-g(x),0)\, f_X(x)\, dx$$

$$= \int_{-\infty}^{+\infty} -\min(g(x),0)\, f_X(x)\, dx$$

$$\max(g(x),0) + \min(g(x),0) = g(x)$$

hence, $E(g(x)) = \int_{-\infty}^{+\infty} (\max(g(x),0) + \min(g(x),0)) \cdot f_X(x)\, dx$

$$= \int_{-\infty}^{+\infty} g(x) \cdot f_X(x)\, dx.$$

5. Let b be the length of bases, h be the height of triangle.

$S = \frac{1}{2} bh.$

$P(X \geq x) = \left(\frac{h-x}{h}\right)^2$

CDF: $F_X(x) = P(X \leq x) = 1 - \left(\frac{h-x}{h}\right)^2 \quad (0 \leq x \leq h)$

PDF: $\frac{dF_X(x)}{dx} = f_X(x) = \frac{2(h-x)}{h^2}. \quad (0 \leq x \leq h)$

$$F_X(x) = \begin{cases} 1 - \left(\frac{h-x}{h}\right)^2 & \text{if } 0 \leq x \leq h \\ 0 & \text{if } x < 0 \\ 0 & \text{if } x > h. \end{cases}$$

6. exponential distribution: $f(x) = \lambda \cdot e^{-\lambda x} \quad x \geq 0$

$F(x) = 1 - e^{-\lambda x} \quad x \geq 0.$

6. $P(X \le x \mid \text{one gustomer}) = 1 - e^{-\lambda x} \quad (x \ge 0)$

$P(X \le x \mid 0 \text{ customer}) = 1.$

$P(X \le x) = P(1 \text{ customer}) \cdot (1 - e^{-\lambda x}) + P(0 \text{ customer}) \cdot 1$

$= \frac{1}{2}(2 - e^{-\lambda x}) \quad (\text{if } x \ge 0.)$

7. (a) $P(X \le x) = \frac{2x^2}{2r^2} = \frac{x^2}{r^2}.$

$f_X(x) = \frac{2x}{r^2}. \quad (0 \le x \le r) \quad 0, \text{ otherwise}.$

$E(X) = \int_0^r x f_X(x)\,dx = \frac{2}{3} r$

$E(X^2) = \int_0^r x^2 \cdot \frac{2x}{r^2}\,dx = \frac{1}{2} r^2$

$Var(X) = \frac{1}{2} r^2 - \frac{4}{9} r^2 = \frac{1}{18} r^2.$

(b). $S = \begin{cases} 1/X & \text{if } X \le t \\ 0 & \text{otherwise} \end{cases}$

$\cancel{P(S \le s) = P(\frac{1}{X} \le s) = P(X \ge \frac{1}{s}) = P(r \ge X \ge \frac{1}{s})}$

$\cancel{= \begin{cases} \frac{t^2}{r^2} - \frac{1}{s^2 r^2} & (s \ge \frac{1}{t}) \\ 0 & \text{if } s < \frac{1}{t}. \end{cases}}$

$P(S \le s) = P(\frac{1}{X} \le s \mid X \le t) \cdot P(X \le t) + P(0 \le s \mid X > t) \cdot P(X > t)$

集导 $P(S \le s) = P(X \ge \frac{1}{s} \mid X \le t) \cdot \frac{t^2}{r^2} + P(0 \le s \mid X > t) \cdot (1 - \frac{t^2}{r^2})$

if $0 \le s < \frac{1}{t}$, $P(S \le s) = 0 + (1 - \frac{t^2}{r^2})$

if $s \ge \frac{1}{t}$, $P(S \le s) = \frac{\frac{t^2}{r^2} - \frac{1}{s^2 r^2}}{\frac{t^2}{r^2}} \cdot \frac{t^2}{r^2} + (1 - \frac{t^2}{r^2}) = 1 - \frac{1}{s^2 r^2}.$

8.

(a) $P(X \le x) = p \cdot P(Y \le x) + (1-p) P(Z \le x)$

$\Rightarrow F_X(x) = p \cdot F_Y(x) + (1-p) F_Z(x)$

$f_X(x) = p\, f_Y(x) + (1-p) F_Z(x)$.

(b) $f_Y(x) = \begin{cases} \lambda \cdot e^{\lambda x} & x < 0 \\ 0 & x \geq 0 \end{cases}$

$f_Z(z) = \begin{cases} 0 & x < 0 \\ \lambda \cdot e^{-\lambda x} & x \geq 0 \end{cases}$.

$F_Y(x) = \begin{cases} e^{\lambda x} & x < 0 \\ 1 & x \geq 0 \end{cases}$ $\lambda > 0$.

$F_Z(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$.

$F_X(x) = p \cdot F_Y(x) + (1-p) F_Z(x) = \begin{cases} p \cdot e^{\lambda x} & x < 0 \\ 1 - (1-p) e^{-\lambda x} & x \geq 0. \end{cases}$ √

9. Let X be the waiting time of both ways,

Y be the waiting time of taxi

Z — — bus.

$P(X \le x \mid \text{taxi is waiting}) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{otherwise}. \end{cases}$

$P(Y \le y \mid \text{taxi is not waiting}) = \begin{cases} y/10 & 0 \le y \le 10 \\ 0 & y < 0 \\ 1 & y > 10. \end{cases}$

$P(Z \le z \mid \text{taxi is not waiting}) = 1$, if $z \geq 5$

$P(z \mid \text{taxi no}(z)) = 1/2$ if [illegible]

$P(X \le x \mid \text{taxi no waiting}) = P(\min(Y, Z) \le x \mid \text{taxi not waiting})$.

$$= \begin{cases} x/10 & \text{if } 0 \le x < 5 \\ 1 & \text{if } x \ge 5 \\ 0 & \text{if } x < 0 \end{cases}.$$

$$F_X(x) = \begin{cases} \cancel{2/3} & \cancel{\text{if } x = 0} \\ x/30 + 2/3 & \text{if } 0 \le x < 5 \\ 1 & \text{if } x \ge 5 \\ 0 & \text{otherwise}. \end{cases}$$

$\cancel{F_X(X \le x \mid no) =} \left\{ \cancel{F_{Y|no}}(Y/no) \right.$

$E(X) = E(X \mid \text{taxi no waiting}) \cdot P(no) + E(X \mid yes) \cdot P(yes)$.

i) $P(X \le x \mid Y, no) = x/5$ if $0 \le x < 5$, $\quad P(Y \mid no) = 5/10 = 1/2$

$P(X \le x \mid Z, no) = 1$ if $x \ge 5$, $\quad P(Z/no) = 1/2$

$E(X \mid no) = E(X \mid Y, no) \cdot P(Y \mid no) + E(X \mid Z, no) \cdot P(Z \mid no)$

$= \frac{5}{2} \cdot \frac{1}{2} + 5 \cdot \frac{1}{2} = \frac{15}{4}$.

(or)

ii) $E(X \mid no) = \int_0^{10} f_{X|no}(x) \cdot x \, dx = \int_0^5 y \cdot f_{Y|no}(y) \, dy + \int_5^{10} z \cdot f_{Z|no}(z) dz$

$= \int_0^5 y \cdot \frac{1}{10} dy + 5 \cdot P_{Z|no}(5)$

$= \frac{15}{4}$

$E(X) = E(X \mid no) \cdot P(no) + E(X \mid yes) \cdot P(yes)$

$= \frac{15}{4} \cdot \frac{1}{3} = \frac{5}{4}$. (reference solution is wrong).

10. Co.

(a) $F_U(u) = u$ for u in $(0,1)$.

$F(x) = u$.

$F_X(x) = P(X \le x) = P(F(X) \le F(x))$ since F is strictly increasing

$= F_U(u)$

$= u$

$= F(x)$

(b) let $X \sim$ exponential (λ), $\lambda > 0$.

$F_X(x) = 1 - e^{-\lambda x}$ $(x \geq 0)$. $F_U(u) = u$

$\cancel{F_X(x) = P(X \le x) = P(}$

$F(x) \triangleq 1 - e^{-\lambda x}$, $F(x)$ is strictly increasing,

so let $F(x) = u$, $U \sim$ uniform$(0,1)$,

$x = \dfrac{\ln(1-u)}{-\lambda}$, $X \sim F =$ exponential (λ).

(c) $\cancel{F(k)}$ X is discrete RV, defined in $\mathbb{Z}$,

$\cancel{F(k) =}$ so $F_X(x)$ is strictly increasing. $X \sim F$.

if $F(k) < u \le F(k)$, let k maps u.

$\cancel{\text{so } F_X(k) = P(X \le k) = P(F(X) \le F(k))}$

$\cancel{= P(U \le F(k))}$

$\cancel{= F_U(F(k))}$

$\cancel{= F(k)}$

function $F: X \to U$

$\cancel{(k-1,k] \mapsto u}$

$k \mapsto \cancel{(k-1,k]}$ $(F(k-1), F(k)]$

$(k-1, k] \mapsto (F(k-1), F(k)]$

(discrete $\to$ interval)

so $\cancel{F_X(k) =}$ $p_X(k) = P(X = k) = P(k-1 < X \le k)$

$= P(F(k-1) < F(X) \le F(k))$ (discrete)

$= F_X(k) - F_X(k-$

[illegible]

11. (a) $X \sim Normal(0,1)$, $Y \sim Normal(1,4)$, ϕ ~ standard Gaussian

$P(X \le 1.5) = \phi(1.5)$

$P(X \le -1) = 1 - \phi(1)$

(b) $(Y-1)/2$ is still Gaussian distribution.

$E(\frac{Y-1}{2}) = 0$, $Var(\frac{Y-1}{2}) = 1$,

$\frac{Y-1}{2}$ ~ standard normal

(c) $P(-1 \le Y \le 1) = P\left(\frac{-1-1}{2} \le \frac{Y-1}{2} \le \frac{1-1}{2}\right)$

$= P(-1 \le \frac{Y-1}{2} \le 0)$

$= \phi(1) - \frac{1}{2}$.

12. $P(X \ge k6) = P(\frac{X}{6} \ge k)$

$= 1 - \phi(k)$

$P(|X| \le k6) = P(\frac{|X|}{6} \le k)$

$= P(-k \le \frac{X}{6} \le k)$

$= 2\phi(k) - 1$.

$d x(r,\theta) = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta$.

$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta$

$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta$

$\Rightarrow \begin{pmatrix} dx \\ dy \end{pmatrix} = \frac{\partial(x,y)}{\partial(r,\theta)} \begin{pmatrix} dr \\ d\theta \end{pmatrix}$

14. $f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$, show $\int_{-\infty}^{+\infty} f(x)\,dx = 1$.

$\left(\int_{-\infty}^{+\infty} f(x)\,dx\right)^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{2\pi}}\right)^2 \cdot e^{-\frac{x^2+y^2}{2}}\,dx\,dy$ set $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$= \int_0^{2\pi}\int_0^{+\infty} \frac{1}{2\pi} \cdot e^{-\frac{r^2}{2}} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}\,dr\,d\theta$.

$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{+\infty} r \cdot e^{-\frac{r^2}{2}}\,dr\,d\theta = 1$

15. (coordinate 坐标)

(a) $P_{X,Y}(x,y) = \begin{cases} \frac{2}{\pi r^2} & \text{if } (x,y)\in\{(x,y) \mid x^2+y^2\le r^2, y\ge 0\} \\ 0 & \text{otherwise.} \end{cases}$

(b) $P_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx$

$= \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{2}{\pi r^2}\,dx$

$= \frac{4\sqrt{r^2-y^2}}{\pi r^2}$ if $y\in[0,r]$.

$E(Y) = \int_0^r P_Y(y)\,y\,dy = \int_0^r \frac{4\sqrt{r^2-y^2}}{\pi r^2}\,y\,dy = \frac{4r^3}{3\pi r^2} = \frac{4}{3\pi}r$

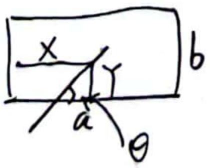
$= \frac{4r}{3\pi}$

(c) $E(Y) = \iint_{(x,y)\in D} y\cdot f_{X,Y}(x,y)\,dx\,dy$

$= \int_0^{\pi}\int_0^r s\cdot s\cdot \sin\theta\cdot\frac{2}{\pi r^2}\,ds\,d\theta$

$\begin{cases} x = s\cdot\cos\theta \\ y = s\cdot\sin\theta \end{cases}$

$= \frac{4r}{3\pi}.$

16. 

the middle point of needle is in the rectangle.

assume that X is the distance between middle of needle and b length sides, Y ~ .

let A be the event that needle crosses a length side.

B ~

~~$P(A) = P(\frac{l}{2}\sin\theta \ge Y)$, $P(B) = P(\frac{l}{2}\cos\theta \ge X)$~~

16. $$f_{X,Y,\Theta}(x,y,\theta) = \begin{cases} \frac{4}{ab\pi} & x \le \frac{a}{2},\ y \le \frac{b}{2},\ \theta \in (0,\pi) \\ 0 & \text{otherwise} \end{cases}$$

$$P(A) = P\left(\tfrac{l}{2}\sin\theta \ge Y \text{ and } X \ge \tfrac{l}{2}\cos\theta\right)$$

$$P(B) = P\left(\tfrac{l}{2}\cos\theta \ge X \text{ and } Y \ge \tfrac{l}{2}\sin\theta\right)$$

$$P(A) = \int_0^{\pi}\int_{\frac{l}{2}\cos\theta}^{a/2}\int_{b/2}^{\frac{l}{2}\sin\theta} \frac{4}{ab\pi}\, dy\, dx\, d\theta$$

$$= 1 - \frac{2l}{b\pi}$$

$$P(B) = 1 - \frac{2l}{a\pi}.$$

$$P(A+B) = 2 - \frac{2l}{\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$$

$$P(A\cup B) = 1 - P(A^c \cap B^c)$$

$$= P(A) + P(B) - P(A\cap B) \quad ?$$

17. $\cancel{E(Z) = \int_{-\infty}^{+\infty} \frac{1}{n}\sum_{i=1}^{n} Y_i}$

$$E(Z) = \frac{1}{n}\sum_{i=1}^{n} E\left(Y_i \cdot \frac{f_X(Y_i)}{f_Y(Y_i)}\right)$$

$\cancel{f_Y(y) > 0 \Leftarrow f_X(y) \ne 0}$

$\cancel{E(Y_i) = \int_{-\infty}^{+\infty} y \cdot f_Y(y)\, dy}$ $$E\left(Y_i \cdot \frac{f_X(Y_i)}{f_Y(Y_i)}\right) = \int_{-\infty}^{+\infty} y \cdot \frac{f_X(y)}{f_Y(y)} \cdot f_Y(y)\, dy$$

$$= \int_S y \cdot f_X(y)\, dy$$

$$= \int_{-\infty}^{+\infty} y \cdot f_X(y)\, dy$$

$$= E(X)$$

$$\Rightarrow E(Z) = E(X). \quad \square$$

18.

(a) $E(X) = \int_1^3 x \cdot f_X(x)\, dx = \frac{13}{6}.$

$P(A) = P(X \geq 2) = \int_2^3 f_X(x) dx = \frac{5}{8}.$

$f_{X|A}(x) = \frac{f_{X,A}(x)}{P(A)} = \frac{f_X(x) \text{ in } A}{P(A)}$

$= \frac{2x}{5}$ if $x \in [2,3]$.

$E(X|A) = \int_2^3 \frac{2x}{5} \cdot x\, dx = \frac{38}{15},$ $E(X|A) \cdot P(A) \triangleq E(X,A).$

(b) $E(Y) = E(X^2) = \int_1^3 x^2 f_X(x)\, dx = 5$

$E(X^4) = E(Y^2) = \int_1^3 x^4 \cdot f_X(x)\, dx = \frac{91}{3}$

$Var(Y) = \frac{91}{3} - 5^2 = \frac{16}{3}.$

19. Go. (a) $\int_1^2 f_X(x)\, dx = 1 \Rightarrow c=2.$

(b) $P(A) = \int_{1.5}^2 f_X(x) dx = \frac{1}{3}.$

$f_{X|A}(x) = 6x^{-2}$ if $x \in (1.5, 2]$.

(c) $E(Y|A) = \int_{1.5}^2 x^2 \cdot 6x^{-2}\, dx = 3$

$E(Y^2|A) = \int_{1.5}^2 x^4 \cdot 6x^{-2}\, dx = \frac{37}{4}$

$Var(Y^2|A) = \frac{37}{4} - 9 = \frac{1}{4}.$

20. let X be the ~~1st~~ time 2nd departure - 1st arrival.

?

21. $f_Y(y) = 1/\ell \quad 0 \le y \le \ell.$

(a)

$P(X \le x \mid Y = y) = \frac{x}{y}$, if $0 \le x \le y$

$f_{X|Y}(x|y) = \frac{1}{y}, \quad 0 \le x \le y.$

$f_{X,Y}(x, y) = f_{X|Y}(x|y) \cdot f_Y(y) = \begin{cases} \frac{1}{\ell y} & \text{if } 0 \le x \le y \le \ell \\ 0 & \text{otherwise.} \end{cases}$

(b) $f_X(x) = \int_x^{\ell} f_{X,Y}(x, y)\, dy = \frac{1}{\ell}(\ln \ell - \ln x).$

(c) $E(X) = \int_0^{\ell} x \cdot \frac{1}{\ell} \cdot \ln \frac{\ell}{x}\, dx$

$= \frac{1}{4}\ell.$

d) $E(X) = E(E(X|Y))$,

$E(X) = E(Y) \cdot E(\frac{X}{Y})$? why ?

22. (i) let X_1 be the first break point, X_2 be the second.

$f_{X_1}(x_1) = 1, \quad f_{X_2}(x_2) = 1.$

We get three length,

if $X_1 > X_2$, get $X_2, X_1 - X_2, 1 - X_1$

if $X_2 > X_1$, get $X_1, X_2 - X_1, 1 - X_2$.

$P(X_2 + (X_1 - X_2) > 1 - X_1,\ X_2 + (1 - X_1) > X_1 - X_2,\ (X_1 - X_2) + (1 - X_1) > X_2 \mid X_1 > X_2) \cdot P(X_1 > X_2)$

$+ P(X_1 + (X_2 - X_1) > 1 - X_2,\ X_1 + (1 - X_2) > X_2 - X_1,\ (X_2 - X_1) + (1 - X_2) > X_1 \mid X_1 < X_2) \cdot P(X_1 < X_2)$

$= P(X_1 > \frac{1}{2},\ \cancel{X_2 \ldots}\ X_1 - X_2 < \frac{1}{2},\ X_2 < \frac{1}{2} \mid X_1 > X_2) \cdot P(X_1 > X_2) + \text{——}$

$$P = 2\int_{\frac{1}{2}}^{1}\int_{x_1-\frac{1}{2}}^{\frac{1}{2}} f_{X_1}(x_1)\cdot f_{X_2}(x_2)\,dx_2\,dx_1$$

$$= \frac{1}{4}.$$

(ii) $f_{X_1}(x_1) = 1, \quad 0 \le x_1 \le 1$

if $0 \le x_2 \le 1 - x_1$, $\quad f_{X_2|X_1}(x_2|x_1) = \frac{1}{1-x_1}$.

we get 3 sticks, with length of $x_1, x_2, 1-x_1-x_2$.

$$P(X_1+X_2 > 1-X_1-X_2,\ X_1+(1-X_1-X_2) > X_2,\ X_2+(1-X_1-X_2) > X_1)$$

$$= P(X_1+X_2 > \tfrac{1}{2},\ X_2 < \tfrac{1}{2},\ X_1 < \tfrac{1}{2})$$

$$= \int_0^{\frac{1}{2}}\int_{\frac{1}{2}-x_1}^{\frac{1}{2}} f_{X_1}(x_1)\cdot f_{X_2|X_1}(x_2|x_1)\,dx_2\,dx_1$$

$$= \ln 2 - \frac{1}{2}.$$

(iii) $f_{X_1}(x_1) = 1, \quad 0 \le x_1 \le 1.$

$$f_{X_2|X_1}(x_2|x_1) = \begin{cases} \frac{1}{1-x_1} & \text{if } x_1 < \frac{1}{2} \\ \frac{1}{x_1} & \text{if } x_1 > \frac{1}{2}. \end{cases}$$

$P(X_1 < \frac{1}{2},\ x_1, x_2, 1-x_1-x_2$ is the triangle$)$

$+\,P(X_1 > \frac{1}{2},\ 1-x_1, x_2, x_1-x_2$ is the triangle$)$

$$= \ln 2 - \frac{1}{2} + P(X_1 > \tfrac{1}{2},\ X_1 - X_2 < \tfrac{1}{2},\ X_2 < \tfrac{1}{2},\ X_1 > \tfrac{1}{2})$$

$$= \int_{\frac{1}{2}}^{1}\int_{x_1-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{x_1}\cdot 1\,dx_2\,dx_1 + \ln 2 - \frac{1}{2}$$

$$= \ln 2 - \frac{1}{2} + \ln 2 - \frac{1}{2} = 2\ln 2 - 1.$$

23. (a) $f_{X,Y}(x,y) = \frac{1}{2}$ if $(x,y) \in$ triangle.

(b) $f_Y(y) = \int_0^{1-y} \frac{1}{2}\, dx = \frac{1}{2}(1-y)$, $y \in [0,1]$

(c) $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{1}{1-y} & \text{if } x \in (0,1-y), \text{ and } y \in [0,1) \\ 0 & \text{otherwise,} \end{cases}$

(d) $E(X|Y=y) = \int_0^{1-y} x \cdot (1-y)\, dx = \frac{1}{2}(1-y)$ $y \in [0,1)$.

$E(X) = \cancel{\xi}\, E(E(X|Y=y)) = \int_0^1 \frac{1}{2}(1-y) \cdot f_Y(y)\, dy$

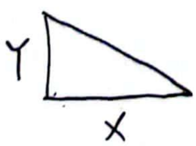
$= \int_0^1 \frac{1}{2} f_Y(y) - \frac{1}{2} y \cdot f_Y(y)\, dy$

$= \frac{1}{2} - \frac{1}{2} E(Y)$.

e) ~~xo~~ we can notice that the situation of X, Y are symmetric, so $E(X) = E(Y)$,

and $E(X) = \frac{1}{2} - \frac{1}{2}E(Y)$, $E(Y) = \frac{1}{2} - \frac{1}{2}E(X)$

$\Rightarrow E(X) = E(Y) = \frac{1}{3}$.

24. 

$f_{X,Y}(x,y) = 1$ if (x,y) in triangle.

~~$f_X(x)$~~ $f_X(x) = \int_0^{\frac{2-x}{2}} f_{X,Y}(x,y)\, dy = \frac{2-x}{2}$ $x \in [0,2]$

~~f~~ $E(X) = \int_0^2 x \cdot f_X(x)\, dx = +\frac{2}{3}$ ~~$f_{X|Y}$~~

$$\cancel{f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2}{2-x} \qquad 0 \le y \le \frac{2-x}{2}.$$

$$E(Y) = \iint_S f_{Y|X}(y|x) \cdot f_X(x) \cdot y \, dx\, dy$$

$$E(Y|X) = \int_0^{(2-x)/2} y \cdot \frac{2}{2-x}\, dy = \frac{2-x}{4} \qquad x \in [0,2]. \quad \text{Since } E(Y) = E(E(Y|X))$$

$$E(Y) = \int_0^2 E(Y|X)\, f_X(x)\, dx$$

$$= \int_0^2 \frac{2-x}{4} f_X(x)\, dx$$

$$= \frac{1}{2} - \frac{1}{4} E(X),$$

$$\Rightarrow E(Y) = \frac{1}{2} - \frac{1}{4} \cdot \frac{2}{3} = \frac{1}{3}.$$

~~25.~~ (coordinate 坐标)

(coordinate 坐标)

~~$X, Y \sim N(0, \sigma^2)$~~

~~$P(X \le x, Y \le y \mid \sqrt{X^2+Y^2} \ge c)$~~

~~$P(X \le x, Y \le y, X^2+Y^2 \ge c^2)$~~

~~$= P(X \le x, Y \le y, X \ge \sqrt{c^2-Y^2} \text{ or } X \le -\sqrt{c^2-Y^2}, c^2-Y^2 \ge 0)$~~

~~$=$~~

~~No, it's hard. can't solve.~~

26. $X_1, \cdots X_n$ are independent.

$$\operatorname{Var}\left(\prod_{i=1}^{n} X_i\right) = E\left(\prod X_i^2\right) - E\left(\prod X_i\right)^2$$

$$= \prod E(X_i^2) - \prod E(X_i)^2$$

$$\frac{\operatorname{Var}(\prod X_i)}{\prod E(X_i^2)} = \frac{\prod E(X_i^2) - \prod E(X_i)^2}{\prod E(X_i^2)}$$

$$\cancel{\prod E(X_i^2) \cdot \left(\prod\left(\frac{\operatorname{Var}(X_i)}{E(X_i^2)} + 1\right)\right) - \prod E(X_i^2)}$$

$$\cancel{= \prod\left(\operatorname{Var}(X_i) + E(X_i^2)\right)} - \prod E(X_i^2)$$

$=$

The formula should be $\dfrac{\operatorname{Var}(\prod X_i)}{\prod E(X_i)^2} = \prod\left(\dfrac{\operatorname{Var}(X_i)}{E(X_i)^2} + 1\right) - 1$

Since $\prod E(X_i)^2 \cdot \prod\left(\dfrac{\operatorname{Var}(X_i)}{E(X_i)^2} + 1\right) - \prod E(X_i)^2$

$$= \prod\left(\operatorname{Var}(X_i) + E(X_i)^2\right) - \prod E(X_i)^2$$

$$= \prod E(X_i^2) - \prod E(X_i)^2$$

$$= \operatorname{Var}\left(\prod X_i\right). \quad \square$$

$\cancel{27}$. 25. $f_{X,Y|A}(x,y) \cdot P(A) = f_{X,Y}(x,y)$ on $(x,y) \in A$.

for $(x,y) \in A$, $A \triangleq \{(x,y) \mid \sqrt{x^2+y^2} \geq c\}$.

$$f_{X,Y|A}(x,y) = \frac{f_{X,Y}(x,y)}{P(A)}, \quad X, Y \sim N(0, \sigma^2).$$

$$P(A) = \iint_A \frac{1}{\sigma^2 \cdot 2\pi} \cdot e^{-\frac{x^2}{2\sigma^2}} \cdot e^{-\frac{y^2}{2\sigma^2}}\, dx\, dy$$

$$= \int_0^{2\pi}\int^{+\infty} \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{r^2}{2\sigma^2}}\, r\, dr\, d\theta = e^{-\frac{c^2}{2\sigma^2}}$$

$$f_{X,Y|A} = \begin{cases} \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{x^2+y^2-c^2}{2\sigma^2}} & \text{if } (x,y) \in A, \text{ that is } \sqrt{x^2+y^2} \geq c. \\ 0 & \text{otherwise.} \end{cases}$$

27. (a) $\iint_A f_{X,Y|C}(x,y)\,dx\,dy$

$= P(C)/P(C) = 1$

(b) $P_{X,Y}(x \in S_1, y \in S_2) = \sum_{i=1}^{n} P(x \in S_1, y \in S_2 | C_i) \cdot P(C_i)$

$\frac{\partial^2}{\partial x \partial y} P_{X,Y}(X \leq x, Y \leq y) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x \partial y} P(X \leq x, Y \leq y | C_i) \cdot P(C_i)$

$\Rightarrow f_{X,Y}(x,y) = \sum_{i=1}^{n} f_{X,Y|C_i}(x,y) \cdot P(C_i)$.

28. (a) $E(X) = \int_{-\infty}^{+\infty} x \cdot f_X(x)\,dx$

$= \int_0^{+\infty} x \cdot p \cdot \lambda \cdot e^{-\lambda x}\,dx + \int_{-\infty}^{0} x \cdot (1-p)\lambda e^{\lambda x}\,dx$

$= \frac{p}{\lambda} + \frac{p-1}{\lambda} = \frac{2p-1}{\lambda}$.

由 $E(X^2) = \frac{2}{\lambda^2}$, $Var(X^2) = \frac{2}{\lambda^2} - (\frac{2p-1}{\lambda})^2$.

(b) $f_{X|A}(x) = \lambda \cdot e^{-\lambda x}$, ~~let A be~~

$A = \{x | x \geq 0\}$, $A^c = \{x | x < 0\}$.

$f_{X|A^c}(x) = \lambda \cdot e^{\lambda x}$.

$E(X) = E(X|A) + E(X|A^c)$

$= \int_0^{+\infty} x \cdot f_{X|A}(x)\,dx \cdot P(A) + \int_{-\infty}^{0} x f_{X|A^c}(x)\,dx \cdot P(A^c) =$

$\frac{1}{\lambda} \cdot p + (-\frac{1}{\lambda}) \cdot (1-p)$

$= \frac{2p-1}{\lambda}$.

$$E(X^2) = P(A)\cdot E(X^2|A) + P(A^c)\cdot E(X^2|A^c)$$
$$= \frac{2}{\lambda^2}\cdot p + \frac{2}{\lambda^2}(1-p)$$
$$= \frac{2}{\lambda^2}.$$
$$Var(X^2) = \frac{2}{\lambda^2} - \left(\frac{2p-1}{\lambda}\right)^2.$$

keypoint $E(X)$ $\quad E(X) = E(X|A)\cdot P(A) + E(X|A^c)\cdot P(A^c)$

keypoint: if A, A^c is partition of Ω, of course.

29. Go! $\quad f_{X,Y,Z}(x,y,z) = f_{X|Y,Z}(x|y,z)\cdot f_{Y,Z}(y,z).$

$$f_{Y,Z}(y,z) = f_{Y|Z}(y|z)\cdot f_Z(z).$$

$$\Rightarrow f_{X,Y,Z}(x,y,z) = f_{X|Y,Z}(x|y,z)\cdot f_{Y|Z}(y|z)\cdot f_Z(z).$$

30. Go! 冲冲冲 冲 冲. 冲

PDF. $\alpha > 0$, $\beta > 0$.

$$f_X(x) = \begin{cases} \frac{1}{B(\alpha,\beta)}\cdot x^{\alpha-1}\cdot(1-x)^{\beta-1}, & \text{if } 0\le x\le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) $\cancel{E(X^m) = \left(\int_0^1 x^m\cdot x^{\alpha-1}(1-x)^{\beta-1}dx\right)^{-1}}$.

$$E(X^m) = \int_0^1 \frac{1}{B(\alpha,\beta)}\cdot x^m\cdot x^{\alpha-1}\cdot(1-x)^{\beta-1}dx$$
$$= \frac{B(\alpha+m,\beta)}{B(\alpha,\beta)}.$$

(b)

$$B(\alpha,\beta)=\int_0^1 x^{\alpha-1}\cdot(1-x)^{\beta-1}\,dx$$

$$=\int_0^1 (1-x)^{\beta-1}\cdot\frac{1}{(\alpha-1)+1}\,d\,x^{\alpha}$$

$$=x^{\alpha}\cdot\frac{1}{\alpha}\cdot(1-x)^{\beta-1}\Big|_0^1-\int_0^1\frac{1}{\alpha}x^{\alpha}\,d(1-x)^{\beta-1}$$

$$=\int_0^1(\beta-1)\frac{1}{\alpha}\cdot x^{\alpha}\cdot(1-x)^{\beta-2}\,dx$$

$$=\cdots$$

$$=(\beta-1)!\cdot\int_0^1 x^{\alpha+\beta-2}\cdot\frac{(\alpha-1)!}{(\alpha+\beta-2)!}\,dx.$$

$$=\frac{(\beta-1)!\,(\alpha-1)!}{(\alpha+\beta-1)!}.$$

$$\Rightarrow\quad E(X^m)=\frac{(\beta-1)!\,(\alpha+m-1)!\,/\,(\alpha+m+\beta-1)!}{(\beta-1)!\,(\alpha-1)!\,/\,(\alpha+\beta-1)!}$$

$$=\frac{(\alpha+m-1)\cdot\cdots(\alpha)}{(\alpha+m+\beta-1)\cdot\cdots(\alpha+\beta)}.$$

Solution 2: assume that $Y, Y_1, \cdots Y_{\alpha+\beta}$ are independent uniform distribution in $(0,1)$

$A=\{Y_1\le\cdots\le Y_\alpha\le Y\le Y_{\alpha+1}\le\cdots Y_{\alpha+\beta}\}$.

① $P(A)=\cancel{P(Y_1)\cdot P(Y_1\le Y_2|Y_1)\cdot P(Y_2\le Y_3|Y_2)\cdots P(Y_{\alpha+\beta-1}\le Y_{\alpha+\beta})}$

$$\int_0^1 dy_1\int_{y_1}^1 dy_2\int_{y_2}^1\cdots\int_{y_{\alpha+\beta-1}}^1 f_{Y_1,Y_2,\cdots,Y_{\alpha+\beta}}(\cdots)\,dy_{\alpha+\beta}$$

$$=\int_0^1 dy_{\alpha+\beta}\int_0^{y_{\alpha+\beta}}dy_{\alpha+\beta-1}\cdots\int_0^{y_2}1\,dy_1$$

$$=\frac{1}{(\alpha+\beta+1)!}$$

$B=\{\max\{Y_1,\cdots Y_\alpha\}\le Y\}$, $C=\{Y\le\min\{Y_{\alpha+1},\cdots Y$

we also have $P(A \mid B\cap C) = P(Y_1 \le Y_2 \le \cdots \le Y_\alpha,\ Y_{\alpha+1} \cdots Y_{\alpha+\beta})$

$$= \frac{1}{\alpha!}\cdot\frac{1}{\beta!}$$ as the proof above.

$P(B\cap C)\cdot P(A \mid B\cap C) = P(A, B\cap C) = P(A)$.

$P(B\cap C) = P(\max\{Y_1 \cdots Y_\alpha\} \le T,\ \min\{Y_{\alpha+1}, \cdots Y_{(\alpha+\beta)}\} \ge T)$

$$= \int_0^1 P(\max\{\ \} \le y,\ \min\{\ \} \ge T \mid T=y)\cdot f_T(y)\,dy$$

$$= \int_0^1 P(Y_1 \le y)\cdot P(Y_2 \le y) \cdots P(Y_\alpha \le y)\cdot P(y \le Y_{\alpha+1}) \cdots P(y \le Y_{\alpha+\beta})\,dy$$

$$= \int_0^1 y^\alpha \cdot (1-\alpha)^\beta\,dy$$

since $\cancel{P(A)} = P(A \mid T=y)\cdot f_T(y) = \cancel{P(T=y)}\ f_{T|A}(y)\cdot P(A)$.

$P(A)\cdot\int_{-\infty}^{+\infty} f_{T|A}(y)\,dy$

$= P(A)\cdot P(T\in\Omega \mid A)$

$= P(A)$.

---

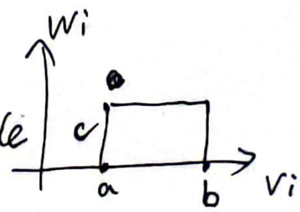
we have $P(B\cap C)\cdot P(A \mid B\cap C) = P(A)$

$$\Rightarrow \int_0^1 y^\alpha\cdot(1-\alpha)^\beta\,dy = \frac{\alpha!\,\beta!}{(\alpha+\beta+1)!}. \quad \square$$

31. Col 对! $f_X(x) = \begin{cases} 0 & \text{if } x\in[a,b]^c \end{cases}$

$\forall x$, $x f_X(x) \le c$.

$f_{V_i,W_i}(v_i, w_i) = \frac{1}{(b-a)c}$ if (V_i, W_i) in rectangle



$Y_i = \begin{cases} 1 & \text{if } W_i \le V_i \cdot f_X(V_i) \\ 0 & \text{otherwise} \end{cases}$

$\cancel{E(Y_i)} = P(w_i \le v_i \cdot f_X(v_i)) = \int_a^b \int_0^{v_i \cdot f_X(v_i)} f_{V_i, W_i}(v_i, w_i)\, dw_i dv_i$

$$= \int_a^b \frac{1}{(b-a)c} v_i \cdot f_X(v_i) dv_i$$

$$= \frac{1}{(b-a)c} \cdot E(X)$$

$$\Rightarrow P_{Y_i}(y_i) = \begin{cases} \frac{E(X)}{(b-a)c}, & \text{if } y_i = 1 \\ 1 - \frac{E(X)}{(b-a)c}, & \text{if } y_i = 0. \end{cases}$$

$$E(Y_i) = \frac{1}{(b-a)c} \cdot E(X),$$

$$E(Z) = E(Y_i) = \frac{E(X)}{c \cdot (b-a)}.$$

$$Var(Z) = \frac{1}{n^2} \sum_{i=1}^{n} Var(Y_i)$$

$$= \frac{1}{n} \cdot Var(Y_i).$$

$$\cancel{E(Y_i^2)} \; Var(Y_i) = \frac{E(X)}{(b-a)c} - \left(\frac{E(X)}{(b-a)c}\right)^2$$

$$= -\left(\frac{E(X)}{(b-a)c} - \frac{1}{2}\right)^2 + \frac{1}{4} \le \frac{1}{4}.$$

$$\Rightarrow Var(Z) = \frac{1}{n} Var(Y_i) \le \frac{1}{4n},$$

32. $F_{X,Y}(x,y) = P(X \le x, Y \le y)$

$$= P(X \le x) \cdot P(Y \le y)$$

$$= F_X(x) \cdot F_Y(y)$$

$$\frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} = f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y).$$

33. $E(T) = \sum_{n=1}^{\infty} E(T|N=n) \cdot p_N(n)$

$E(T|N=n) = E(X_1 + X_2 + \cdots + X_N | N=n)$

$= E(X_1|N=n) + \cdots + E(X_n|N=n)$ since, X_i, N are independen

$= n \cdot E(X)$

$E(T) = \sum_{n=1}^{\infty} n \cdot E(X) \cdot p_N(n)$

$= E(N \cdot E(X))$

$= E(N) \cdot E(X).$

~~Var(T) =~~ $E(T^2) = \sum_{n=1}^{\infty} E(T^2|N=n) \cdot p_N(n).$

$E(T^2|N=n) = E((X_1 + X_2 + \cdots + X_n)^2)$ since, $X_1, \cdots X_n$, N are independent.

$= Var(X_1 + X_2 + \cdots + X_n) + E(X_1 + \cdots + X_n)^2$

$= \sum_{i=1}^{n} Var(X_i) + (\sum_{i=1}^{n} E(X_i))^2$

$E(T^2) = \sum_{n=1}^{\infty} p_N(n) \cdot \left(\sum_{i=1}^{n} Var(X_i) + (\sum_{i=1}^{n} E(X_i))^2 \right)$

$= \sum_{n=1}^{\infty} p_N(n) \cdot (n Var(X) + n^2 E(X)^2)$

$= E(N) \cdot Var(X) + E(N^2) \cdot E(X)^2$

$Var(T) = E(N) \cdot Var(X) + E(N^2) \cdot E(X)^2 - E(N)^2 \cdot E(X)^2$

$= E(N) \cdot Var(X) + Var(N) \cdot E(X)^2.$

34. $f_P(p)$ is the probability of the coins ~~has~~ toss results in heads with prob. p.

let A: the coin is head.

$$P(A) = \int P(A \mid P=p)\, f_P(p)\, dp$$

$$= \int_0^1 p \cdot p e^p \, dp$$

$$= e-2.$$

b) $$f_{P|A}(p) = \frac{P(A \mid P=p) \cdot f_P(p)}{P(A)}$$

$$= \frac{p \cdot p \cdot e^p}{e-2} \quad \text{if } p \in [0,1]$$

$$= \begin{cases} \dfrac{p^2 \cdot e^p}{e-2} & \text{if } p \in [0,1] \\ 0 & \text{if otherwise.} \end{cases}$$

(c) assume that B: toss the coin repeatly, the probability of head.

$$\cancel{P(B \mid A) = \int P(B \mid A, P=p)}$$

$$\cancel{= \int P(B, P=p \mid A) \cdot f_P(p)\, dp}$$

$$\cancel{= \int P(B \mid A, P=p) \cdot P(P=p \mid A) \cdot f_P(p)\, dp}$$

$$\cancel{= \int P(B \mid A, P=p) \cdot f_{P|A}(p) \cdot f_P(p)\, dp}$$

$$\cancel{= \int_0^1 p \cdot p^2 \cdot e^p \cdot p \cdot e^p\, dp}$$

$$P(B|A) = \int P(B, P=p|A)\,dp$$

$$= \int P(B|A, P=p) \cdot P(P=p|A)\,dp$$

$$= \int P(B|P=p) \cdot f_{P|A}(p)\,dp$$

$$= \int_0^1 p \cdot \frac{p^2 \cdot e^p}{e-2}\,dp$$

$$= \frac{6-2e}{e-2}.$$

35. (a) ① $f_{Z|X}(z|x) = P(Z=z|X=x)$

$= P(X+Y=z|X=x)$

$= P(\cancel{X}\ Y = z-x|X=x)$

$= P_{Y|X}(z-x|x)$ since X, Y are independent.

$= P_Y(z-x)$

$= f_Y(z-x)$

② $F_{Z|X}(z|x) = P(Z \le z|X=x)$

$= P(X+Y \le z|X=x)$

$= P(Y \le z-x|X=x)$ since X, Y are independe

$= P(Y \le z-x)$

$= F_Y(z-x)$

$$\frac{dF_{Z|X}(z|x)}{dz} = f_{Z|X}(z|x) = \frac{dF_Y(z-x)}{d(z-x)} \cdot \frac{d(z-x)}{dz} = \frac{dF_Y(z-x}{dz}$$

$= \cancel{F_Y(z)} \cdot f_Y(z-x) \cdot 1.$ $\square$ easy

(b)

$\cancel{F_{Z|X}}$

$F_{X|Z}(x|z) = ?$

$$f_{X|Z}(x|z) = \frac{f_{Z|X}(z|x)\cdot f_X(x)}{f_Z(z)} = \frac{f_Y(z-x)\cdot f_X(x)}{f_Z(z)}$$ if $z-x,\ x \geq 0$, that is $x \in [0, z]$

$$f_Z(z) = \int_0^{+\infty} f_{Z|X}(z|x)\cdot f_X(x)\,dx$$

$$= \int_0^{+\infty} f_Y(z-x)\cdot f_X(x)\,dx$$

$$= \int_0^{z} \lambda^2\cdot e^{-\lambda(z-x)}\cdot e^{-\lambda x}\,dx \qquad (z-x \geq 0,\ x \geq 0)$$

$$= \int_0^{z} \lambda^2\cdot e^{-\lambda z}\,dx$$

$$= z\cdot\lambda^2\cdot e^{-\lambda z}.$$

$f_Y(z-x)\cdot f_X(x) = \lambda^2\cdot e^{-\lambda z}$ if $x \in [0, z]$.

$f_{X|Z}(x|z) = \frac{1}{z}$ if $x \in [0, z]$,

and since $f_{X|Z}(x|z) = \dfrac{\lambda^2\cdot e^{-\lambda z}}{f_Z(z)}$, $f_{X|Z}(x|z)$ is uniform distribution in $[0, z]$, $f_{X|Z}(x|z) = \frac{1}{z}$, $F_{X|Z}(x|z) = \frac{x}{z}$, if $x \in [0, z]$.

(c) $$f_{X|Z}(x|z) = \frac{f_Y(z-x)\cdot f_X(x)}{f_Z(z)} = \frac{1}{2\pi\sigma_x\cdot\sigma_y}\cdot\exp\left(-\frac{(z-x)^2}{2\sigma_y^2} - \frac{x^2}{2\sigma_x^2}\right)\cdot\frac{1}{f_Z(z)}$$

$$= \frac{1}{f_Z(z)\cdot 2\pi\sigma_x\cdot\sigma_y}\cdot\exp\left\{-\frac{x^2(\sigma_y^2+\sigma_x^2) - 2xz\sigma_x^2}{2\sigma_x^2\sigma_y^2} + \frac{z^2\sigma_x^2}{\sigma_x^2\cdot\sigma_y^2}\right\}$$

$$= h(z)\cdot\exp\left\{-\frac{(\sigma_y^2+\sigma_x^2)(x-c)^2}{\sigma_x^2\sigma_y^2}\right\}$$ $h(z)$ is not depend on x.

so $f_{X|Z}(x|z)$ is normal distribution.

$E(X|Z) = \dfrac{\sigma_x^2 z}{\sigma_x^2+\sigma_y^2}$ $\quad Var(X|Z) = \dfrac{\sigma_x^2\cdot\sigma_y^2}{\sigma_x^2+\sigma_y^2}$ $\quad X|Z \sim N\left(\dfrac{\sigma_x^2 z}{\sigma_x^2+\sigma_y^2}, \dfrac{\sigma_x^2\cdot\sigma_y^2}{\sigma_x^2+\sigma_y^2}\right)$