Chapter 4.

1. $f_X(x) = \frac{1}{2} \quad x \in [-1, 1].$

$$F_{\sqrt{|X|}}(x) = P(\sqrt{|X|} \le x)$$
$$= P(|X| \le x^2)$$
$$= x^2 \qquad (x \in [0,1].$$

$$f_{\sqrt{|X|}}(x) = 2x \qquad , x \in [0,1]$$

$$F_{-\ln|X|}(x) = P(-\ln|X| \le x)$$
$$= 1 - e^{-x} \qquad x \in [0, +\infty)$$

$$f_{-\ln|X|}(x) = e^{-x} \quad , \quad x \in [0, +\infty)$$

2. set $Y = e^X$,

$$F_Y(y) = P(e^X \le y) = P(X \le \ln y) \qquad y \in [1, e] \ \cancel{[0,+\infty)}.$$
$$= F_X(\ln y).$$

$\cancel{f_X(\ln y) =}$

$$\frac{d}{dy} F_X(\ln y) = f_X(\ln y) \cdot \frac{1}{y} = \frac{1}{y}$$

$$\frac{d}{dy} F_Y(y) = f_Y(y) = \frac{1}{y} \quad , \ y \in [1, e].$$

3. $F_{Y_1}(y) = P(|X|^{1/3} \le y) \qquad (y \in [0, +\infty))$

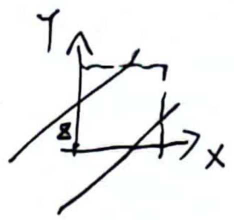
$$= P(|X| \le y^3)$$
$$= P(-y^3 \le X \le y^3)$$
$$= F_X(y^3) - F_X(-y^3)$$

$$\frac{d}{dy} F_{Y_1}(y) = f_{Y_1}(y) = f_X(y^3) \cdot 3y^2 + f_X(-y^3) \cdot 3y^2 \qquad y \in [0, +\infty)$$

$$f_{Y_2}(y) = 4y^3 \cdot f_X(y^4) + 4y^3 f_X(-y^4) \qquad y \in [0, +\infty)$$

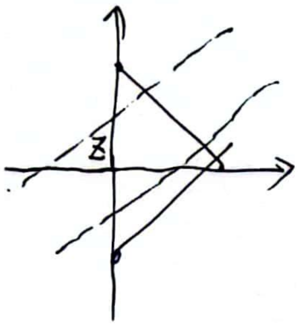
5. ~~P(|X-Y|~~ let $Z = |X-Y|$ $\quad z \in [0,1]$

$$F_Z(z) = P(|X-Y| \le z)$$
$$= P(-z \le X-Y \le z)$$
$$= P(-z \le X-Y \le z|$$
$$= 1-(1-z)^2, \quad z \in [0,1]$$



$$f_Z(z) = 2(-z+1) \quad z \in [0,1]$$

6.



$$F_Z(z) = P(|X-Y| \le z) \quad z \in [0,1]$$
$$= P(-z \le |X-Y| \le z)$$
$$= \left(\frac{1+z}{2}\right)^2 - \left(\frac{1-z}{2}\right)^2$$
$$= z$$

$$f_Z(z) = 1, \quad z \in [0,1].$$

7. X: denotes the distance between 0 and first point

Y: ——— 0 and second point

$Z = |X-Y|$, ~~$F_Z(z) = P$~~ $\quad F_Z(z) = P(Z \le z) \quad z \in [0,1].$

$$= P(|X-Y| \le z)$$
$$= 1-(1-z)^2$$

$$f_Z(z) = 2(1-z).$$

$$E(Z) = \int_0^1 2(1-z)\cdot z\, dz = \frac{1}{3}. \quad \square$$

8. $X, Y \sim$ exponential $(\lambda)$.

$$F_Z(z) = P(Z \le z) = P(X+Y \le z)$$

$$= \int_{-\infty}^{+\infty} P(X+Y \le z \mid X=x) \cdot f_X(x)\, dx$$

$$= \int P(Y \le z-x \mid X=x) \cdot f_X(x)\, dx$$

$$= \int \cancel{F_Y(z-x)}\ F_{Y|X}(z-x \mid x) \cdot f_X(x)\, dx$$

$$\frac{dF_Z(z)}{dz} = f_Z(z) = \int f_{Y|X}(z-x \mid x) \cdot f_X(x)\, dx$$ , since X and Y are independent.

$$= \int_{-\infty}^{+\infty} f_Y(z-x) \cdot f_X(x)\, dx$$

$$\cancel{= \int_0^{+\infty} \lambda e^{-\lambda(z-x)} \cdot \lambda e^{-\lambda x}\, dx}$$

$$\cancel{= \int_0^{+\infty} \lambda^2 \cdot e^{-\lambda z}\, dx}$$

since $(z-x) \ge 0$, $x \ge 0 \Rightarrow x \in [0, z]$.

$$f_Z(z) = \int_0^z \lambda \cdot e^{-\lambda(z-x)} \cdot \lambda e^{-\lambda x}\, dx$$

$$= \int_0^z \lambda^2 \cdot e^{-\lambda z}\, dx$$

$$= \lambda^2 \cdot z \cdot e^{-\lambda z}, \qquad z \in [0, z].$$

9. $F_Z(z) = P(X - Y \le z)$, we have $f_{X+Y}(z) = \int_{-\infty}^{+\infty} f_Y(z-x) \cdot f_X(x)\, dx$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{-Y}(z-x) \cdot f_X(x)\, dx$$

$$= \int_{-\infty}^{+\infty} f_Y(x-z) \cdot f_X(x)\, dx \qquad \begin{cases} x - z \ge 0 \\ x \ge 0 \end{cases} \Rightarrow x \in \max(0, z).$$

$$f_Z(z) = \begin{cases} \int_0^{+\infty} f_Y(x-z)\cdot f_X(x)\,dx, & \text{if } z \le 0 \\ \int_z^{+\infty} f_Y(x-z)\cdot f_X(x)\,dx, & \text{if } z > 0 \end{cases}$$

$$= \begin{cases} \int_0^{+\infty} \lambda\cdot e^{-\lambda(x-z)}\cdot\mu\cdot e^{-\mu x}\,dx, & \text{if } z \le 0 \\ \int_z^{+\infty} \lambda\cdot e^{-\lambda(x-z)}\cdot\mu\cdot e^{-\mu x}\,dx, & \text{if } z > 0 \end{cases}$$

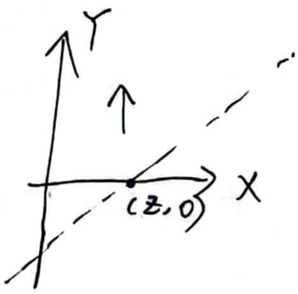
$$= \begin{cases} \lambda\cdot\mu\cdot\frac{1}{\lambda+\mu}\cdot e^{\mu z}, & \text{if } z \le 0 \\ \frac{\lambda\mu}{\lambda+\mu}\cdot e^{\lambda z}\cdot e^{-(\lambda+\mu)z}, & \text{if } z > 0. \end{cases}$$

$$= \frac{\lambda\mu}{\lambda+\mu}\cdot e^{-\lambda z}$$

another solution:

$F_Z(z) = P(X - Y \le z)$

and $(X, Y)$ in $[0,+\infty)\times[0,+\infty)$




we need to seperate into 2 situation, $z \ge 0$ and $z < 0$.

if $z \ge 0$, $F_Z(z) = \int_0^z dx \int_0^{+\infty} f_{X,Y}(x,y)\,dy$

$$+ \int_z^{+\infty} dx \int_{x-z}^{+\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_0^z dx \int_0^{+\infty} e^{-\lambda x-\mu y}\cdot\lambda\mu\,dy + \int_z^{+\infty} dx \int_{x-z}^{+\infty} e^{-\lambda x-\mu y}\lambda\mu\,dy$$

$$= 1 - e^{-\lambda z} + \frac{\lambda}{\mu+\lambda}\cdot e^{-\lambda z}, \quad \frac{dF_Z(z)}{dz} = f_Z(z) = \frac{\mu\lambda}{\mu+\lambda}\cdot e^{-\lambda z}$$

10. $f_Z(z) = \int_{-\infty}^{+\infty} f_Y(z-x) \cdot f_X(x)\, dx$

$p_Z(z) = \sum_x p_Y(z-x) \cdot p_X(x)$

$z \in \{1, 2, 3, 4, 5\}$

$p_Z(1) = p_Y(1) \cdot p_X(0) + p_Y(0) \cdot p_X(1) = \frac{1}{6}$.

— —

11. ~~$f_Z(x) =$~~ $X \sim Poisson(\lambda)$, $Y \sim Poisson(\mu)$

$p_X(x) = e^{-\lambda} \cdot \frac{\lambda^x}{x!}$ $x \geq 0$.

$$p_Z(z) = \sum_x p_X(z-x) \cdot p_Y(x)$$

$$= \sum_{x=0}^{z} e^{-\lambda} \cdot \frac{\lambda^{(z-x)}}{(z-x)!} \cdot e^{-\mu} \cdot \frac{\mu^x}{x!}$$

$$= \sum_{x=0}^{z} \cdot e^{-\lambda-\mu} \cdot \lambda^z \cdot \cancel{\frac{(\lambda\mu)^z}{x!}} \frac{(\frac{\mu}{\lambda})^x}{x!(z-x)!}$$

$$= \sum_{x=0}^{z} \frac{\lambda^z}{e^{\lambda+\mu}} \cdot \frac{1}{z!} \cdot \binom{z}{x} \cdot (\frac{\mu}{\lambda})^x$$

$$= \frac{\lambda^z}{e^{\lambda+\mu}} \cdot \frac{1}{z!} (1 + \frac{\mu}{\lambda})^z \quad , z \geq 0.$$

$\Rightarrow$ $Z \sim Poisson(\lambda+\mu)$.

12. $f_Z(z) = \int_{-\infty}^{+\infty} f_Y(z-x) \cdot f_X(x)\, dx$ $z \in [0, 2]$.

$1 \geq (z-x) \geq 0$, $1 \geq x \geq 0 \Rightarrow$

$\min(1, z) \geq x \geq \max(0, z-1)$

$$f_Z(z) = \begin{cases} \int_{z-1}^{1} dx = 2-z & , \text{if } z \in [1, 2] \\ \int_{0}^{z} dx = z & , \text{if } z \in [0, 1). \end{cases}$$

$\blacksquare$

~~法~~. 12. and then we calculate $V+Z$, $V=X+Y$.

13. $X+Y=(b-a)$

$Y=(b-a)-X$, since $X$ is distributed symmetricly with $\frac{a+b}{2}$.

$(b-a)-X$, $X$, $Y$ are identically distribution

$X-Y = X+\big((b-a)-Y\big)-(b-a)$ , $Z \triangleq (b-a)-Y$.

$= X+Z-(b-a)$

$\sim X+Y-(b-a)$

$X-Y \sim (X+Y)-(b-a)$, $X-Y$ distributes as $X+Y$ shifting by left of $(b-a)$.

14. $F_Z(z) = P(\min(X,Y) \le z)$

$= 1-P(\min(X,Y) > z)$

$= 1-P(X>z, Y>z)$

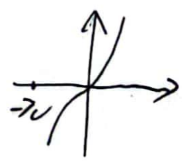
$= 1-(1-F_X(z))(1-F_Y(z))$

$\cancel{= 1-e^{-\lambda x}\cdot e^{-\mu y}}$

$= 1-e^{-\lambda z}\cdot e^{-\mu z}$ $\quad (z \geq 0)$

$\Rightarrow Z \sim \text{exponential}(\lambda+\mu)$. $\square$

15. (a) $X$: $f_X(x)=1$, $x\in[-\frac{1}{2},\frac{1}{2}]$



$Y$ is monototically increasing.

$F_Y(y) = P(Y\le y) = P(g(X)\le y)$ $\qquad X=\frac{1}{\pi}\arctan(Y)$. $Y\in(-\infty,+$

$f_Y(y) = f_X(g^{-1}(y))\cdot\left|\frac{dx}{dy}\right| = \frac{1}{\pi}\cdot\frac{1}{y^2+1}$, $y\in(-\infty,+\infty)$.

(b) $\tan(X) = Y$, $X \in (-\pi/2, \pi/2)$.

$$F_X(x) = P(X \le x) = P(\tan X \le \tan(x))$$
$$= F_Y(\tan x)$$

$$f_X(x) = f_Y(\tan x)\cdot(\sec^2 x)$$
$$= \frac{1}{\pi(1+\tan^2 x)}\cdot \sec^2 x$$
$$= \frac{\sec^2 x}{\pi\cdot\sec^2 x}$$
$$= \frac{1}{\pi}, \quad \forall x \in (-\pi/2, \pi/2)$$

16. (a) $X, Y \sim \text{Normal}(0,1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$$

$$f_{X,Y}(x,y) = \frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right).$$

$$F_{R,\Theta}(r,\theta) = P(R \le r, \Theta \le \theta).$$
$$= P((X,Y)\in A)$$
$$= \iint_A f_{X,Y}(x,y)\,dx\,dy$$
$$= \int_0^r\int_0^\theta f_{X,Y}(r_1\cos\theta_1, r_1\sin\theta_1)\,\frac{\partial(x,y)}{\partial(r,\theta)}\,d\theta_1 dr_1$$
$$= \int_0^r\int_0^\theta \frac{1}{2\pi}\exp\left(-\frac{r_1^2}{2}\right) r_1\,d\theta_1 dr_1$$
$$= \frac{\theta}{2\pi}\left(1-e^{-\frac{r^2}{2}}\right). \qquad (r \ge 0, \theta \in [0, 2\pi])$$

$$f_R(r) = d\,\frac{2\pi}{2\pi}\left(1-e^{-\frac{r^2}{2}}\right)/dr = r\cdot e^{-r^2/2}$$

$$f_\Theta(\theta) = \frac{d\left(\frac{\theta}{2\pi}\left(1-e^{-\frac{\infty^2}{2}}\right)\right)}{d\theta} = \frac{1}{2\pi}.$$

since $f_{R,\Theta}(r,\theta) = g(r)\cdot h(\theta)$, $R, \Theta$ are independent.

(b) $P(R^2 \le t) = P(R \le \sqrt{t}) \quad t \in [0, +\infty)$.

$$= \int_0^{\sqrt{t}} r \cdot e^{-r^2/2}\, dr$$

$$= 1 - e^{-t/2} \quad t \in [0, +\infty)$$

so $R^2 \sim$ exponetial with parameter $\lambda = \frac{1}{2}$.

17. $Cov(X-Y, X+Y) = E((X-Y)(X+Y)) - E(X-Y)E(X+Y)$

$$= E(X^2) - E(X) - (E(Y^2) - E(Y))$$

$$= 0,$$

so $X-Y$ and $X+Y$ are uncorrelated.

18. $$\rho(R,S) = \frac{Cov(W+X, X+Y)}{\sqrt{Var(W+X) \cdot Var(X+Y)}}$$

$$= \frac{E((W+X)(X+Y)) - E(W+X) \cdot E(X+Y)}{\sqrt{(E((W+X)^2) - E(W+X)^2) \cdot (E((X+Y)^2) - E(X+Y)^2}}$$

$$= \frac{Var(X)}{\sqrt{(Var(W) + Var(X))(Var(X) + Var(Y))}}$$

$$= \frac{1}{2}$$

$\rho(R,T) = 0$

19. $$\rho(X,Y) = \frac{E(XY) - E(X) \cdot E(Y)}{\sqrt{Var(X) \cdot Var(Y)}}$$

$$= \frac{b - 0}{\sqrt{b^2 + 2c^2}}$$

$$= \frac{b}{\sqrt{b^2 + 2c^2}}. \quad \square$$

20. $E(XY)^2 = (\iint_{-\infty}^{+\infty} x^2 y\, f_{X,Y}(x,y)\,dx\,dy)^2$

$E(X^2)\cdot E(Y^2) = (\iint_{-\infty}^{+\infty} x^2 \cdot f_{XY}(x,y)\,dx\,dy)(\iint_{-\infty}^{+\infty} y^2 \cdot f_{X,Y}(x,y)\,dx\,dy)$

$\geq (\iint_{-\infty}^{+\infty} xy \cdot f_{XY}(x,y)\,dx\,dy)^2$. ( xxx inequality)
Cauchy?

another solution:

$$0 \leq E\left((X - \frac{E(XY)}{E(Y^2)}\cdot Y)^2\right) = E\left(X^2 - 2XY\cdot\frac{E(XY)}{E(Y^2)} + \frac{E(XY)^2}{E(Y^2)^2}\cdot Y^2\right)$$

$$= E(X^2) - 2\frac{E(XY)^2}{E(Y^2)} + \frac{E(XY)^2}{E(Y^2)}$$

$$= E(X^2) - \frac{E(XY)^2}{E(Y^2)}.$$

if $E(Y^2) \neq 0$, inequality holds.

if $E(Y^2) = 0$, $\forall y \neq 0$, $P_Y(y) = 0$, or $f_Y(y) = 0$

$\Rightarrow E(XY) = 0$. inequality holds too.

21. (a) Schwarz inequality: ~~Cov~~ $E(XY)^2 \leq E(X^2)\cdot E(Y^2)$

$\Rightarrow E((X-E(X))(Y-E(Y)))^2 \leq E((X-E(X))^2)\cdot E((Y-E(Y))^2)$

$\Rightarrow |\rho(X,Y)| \leq 1$.

(b) If $Y - E(Y) = c(X - E(X))$, assume that $c > 0$.

故 $Cov(X,Y) = c\cdot Var(X)$, $\rho(X,Y) = 1$

if $c < 0 \Rightarrow \rho(X,Y) = -1$.

(c) if $\rho(X,Y) = 1$, $\Rightarrow$ 故 $Cov(X,Y) \leq \sqrt{Var(X)\cdot Var(Y)}$ takes equality

$Cov(X,Y)$ $Cov(X,Y)$

22. ~~对~~ Assume that at the begining of the $n$ th gambling, gambler's fortune is $X_n$,

If gambler wins, he will get $\cancel{p}\cdot(2p-1)\cdot X_n$ fortune, otherwise he will lose $(2p-1)\cdot X_n$.

$$E(X_{n+1}|X_n) = X_n + \left(p\cdot(2p-1)X_n - (1-p)(2p-1)X_n\right)$$

$$= \left(1+(2p-1)^2\right)X_n.$$

$$E(X_{n+1}) = E(E(X_{n+1}|X_n)) = \left(1+(2p-1)^2\right) \cancel{X_0}\ E(X_n)$$

$$E(X_1) = \left(1+(2p-1)^2\right) x_1.$$

$$E(X_n) = \left(1+(2p-1)^2\right)^n x_1,$$

23.

P: arrived in $[8,10]$. $X_1$ time

if $X<1$, 3 hours

if $X>1$. $[0, 3-X]$ ~~uniformly~~ uniform distribution,

N: 9:00

(a) ~~E~~ $E(W) = E(W|X<1)\cdot P(X<1) + E(W|X>1)\cdot P(X>1)$

$$= 0 + \int_0^1 w\,dw \cdot \frac{1}{2}$$

$$= \frac{1}{4}.$$

W denotes waiting time by N.

(b) $E(D) = E(D|X<1)\cdot P(X<1) + E(D|X>1)\cdot P(X>1)$

$$= 3\cdot\frac{1}{2} + \int_0^1 \frac{(3-x)}{2}\,dx \cdot \frac{1}{2}$$

$$= \frac{15}{8}$$

$E(D|X>1) = E\left(\frac{3-X}{2}\middle| X>1\right) = \frac{3}{2} - E(X|X$

(c) ~~$P(B_n) = \frac{7}{8}$~~.

~~$B_n$ denotes the $n^{th}$ times breaking~~ obeying the time ~~(dating on time)~~.

An denotes the first ~~the~~ n times ~~all~~ that all dates on times

Bn denotes in the first n times. P didn't date on time for 1 times

C denotes 2 times not on time.

$P(A_1) = p$, $\quad P(B_1) = (1-p)$, $\quad p = \frac{7}{8}$.

$P(A_n \mid A_{n-1}) = p$. $\quad P(B_n \mid A_{n-1}) = (1-p)$

$P(A_n \mid B_{n-1}) = 0$, $\quad P(B_n \mid B_{n-1}) = p$.

$E(C) = E(C \mid A_n) \cdot P(A_n) + E(C \mid B_n) \cdot P(B_n) + E(C \mid (B_n \cup A_n)^c) \cdot (1 - P(A_n) + P(B_n)$

$= P(A_n) \cdot (E(C) + n) + E(C \mid B_n) \cdot P(B_n) +$

$E(C) = \sum_{n=1}^{\infty} n \cdot P($2 times not on time in first $(n+1)$ times$)$

$= \cancel{\sum_{n=1}^{\infty} n \cdot \frac{\binom{n+1}{2} \cdot p^{n-1} \cdot (1-p)^2}{1}}$

$= \sum_{n=1}^{\infty} \binom{n}{1} \cdot p^{n-1} \cdot (1-p)^2 \cdot n$

$= \sum_{n=1}^{\infty} n^2 \cdot p^{n-1} \cdot (1-p)^2$.

set $f(p) = \sum_{n=1}^{\infty} p^n = \frac{p}{1-p}$.

$f'(p) = \sum_{n=1}^{\infty} n \cdot p^{n-1} = \frac{1}{(1-p)^2}$

since $f(p)$ and $f'(p)$ are absolute convergence.

$= \sum_{n=1}^{\infty} (n(n+1) - n) p^{n-1} \cdot (1-p)^2$

$= \left[\left(\frac{p^2}{1-p}\right)'' - \left(\frac{p}{1-p}\right)'\right](1-p)^2$

another solution: $X_i = \begin{cases} 1 & \text{denotes } n\text{-th dating is not on time} \\ 0 & \text{on time} \end{cases}$

$$E(N \mid X_1 + \cdots + X_N = 1,\ X_{N+1} = 1) = ?$$

25. $E(X \cdot g(Y) \mid Y \overset{=y}{\downarrow}) = \int_{-\infty}^{+\infty} x \cdot g(y) \cdot f_{X|Y}(x|y)\, dx$

$$= g(y) \cdot \int_{-\infty}^{+\infty} x \cdot f_{X|Y}(x|y)\, dx$$

$$= g(y) \cdot E(X \mid Y = y)$$

$\Rightarrow E(X \cdot g(Y) \mid Y) = g(Y) \cdot E(X \mid Y)$.

$E(X \cdot g(Y) \mid Y = y) = g(y) \cdot E(X \mid Y = y)$, holds the equality for all $y$,

so $E(X \cdot g(Y) \mid Y) = g(Y) \cdot E(X \mid Y)$.

26. Law of Total Variance:

$$\mathrm{var}(X) = E(\mathrm{var}(X|Y)) + \mathrm{var}(E(X|Y)).$$

$$\mathrm{Var}(XY) = E(\mathrm{Var}(XY|Y)) + \mathrm{Var}(E(XY|Y))$$

$$E(\mathrm{Var}(XY|Y)) = E(E((XY - E(XY|Y))^2 \mid Y))$$

$$= E(Y^2 \cdot E(X^2|Y) - E(X|Y)^2 \cdot Y^2)$$

$$= E(Y^2 \cdot \mathrm{Var}(X|Y))$$

$\mathrm{Var}(X|Y) = E((X - E(X|Y))^2|Y) = E(X^2|Y) - E(X|Y)^2$

$$= E(Y^2 \cdot E(X^2|Y)) - E(E(X|Y)^2 \cdot Y^2)$$

$$= E(E(X^2 Y^2 | Y)) - E(E(X)^2 \cdot Y^2) \quad \text{since } X, Y \text{ are independent}$$

$$= E(X^2) \cdot E(Y^2) - E(X)^2 \cdot E(Y^2)$$

$$= E(Y^2) \cdot \mathrm{Var}(X).$$

$\cancel{var(E(X|Y)) = E(}$

$$var(E(XY|Y)) = E(E(XY|Y)^2) - E(E(XY|Y))^2$$

$$= E(Y^2 \cdot E(X)^2) - E(XY)^2$$

$$= E(X)^2 \cdot Var(Y)$$.

$$\Rightarrow Var(XY) = E(Y^2) \cdot Var(X) + E(X)^2 \cdot var(Y)$$

$$= (var(Y) + E(Y)^2) \cdot var(X) + E(X)^2 \cdot Var(Y)$$

$$= var(X) \cdot var(Y) + var(X) \cdot E(Y)^2 + var(Y) \cdot E(X)^2. \quad \square$$

27. (a) $E(X_i) = E(E(X_i|Q))$

$$= E(Q) = \mu.$$

$$E(X) = E(X_1 + \text{—} + X_n) = n \cdot E(Q) = n\mu$$

(b) $cov(X_i, X_j) = E(X_i \cdot X_j) - E(X_i) \cdot E(X_j)$.

$$E(X_i \cdot X_j) = E(E(X_i \cdot X_j | Q))$$

$$= E(E(X_i|Q) \cdot E(X_j|Q))$$ since $X_i, X_j$ are conditional independent, given $Q$.

$$= E(Q^2)$$

$$= Var(Q) + E(Q)^2$$

$$= \sigma^2 + \mu^2.$$

$cov(X_i, X_j) = \sigma^2 > 0$, so $X_i, X_j$ are not independent.

(c) $Var(X) = E(Var(X|Q)) + var(E(X|Q))$

$$= E(n \cdot var(X_i|Q)) + var(nQ)$$ since conditional independent

$$= E(nQ(1-Q)) + var(nQ)$$

$$= n(\mu - \sigma^2 - \mu^2) + n^2(\sigma^2).$$

$$\mathrm{Var}(X) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + \sum_{i\neq j} \mathrm{cov}(X_i, X_j)$$

$$= (\mu-\mu^2)n + (n^2-n)\cdot\sigma^2.$$

two methods get the same results. $\square$

28. Go. 冲.

(a) $$q(x,y) = \left[\left(\frac{x}{\sigma_x} - \rho\frac{y}{\sigma_y}\right)^2 + (1-\rho^2)\cdot\frac{y^2}{\sigma_y^2}\right] / 2(1-\rho^2)$$

$$= \frac{1}{1-\rho^2}\cdot\left(\frac{x}{\sigma_x} - \rho\cdot\frac{y}{\sigma_y}\right)^2 + \frac{1}{2}\cdot\frac{y^2}{\sigma_y^2}.$$

(b) $$E(X) = \int_{-\infty}^{+\infty} x\cdot\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy\,dx$$ Go!

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x\cdot c\cdot \exp\{-(\alpha x-\beta y)^2 - \gamma y^2\}\,dx\,dy$$

$$E(Y) = \int_{-\infty}^{+\infty} y\cdot c\cdot\exp\{-\gamma y^2\}\int_{-\infty}^{+\infty} \exp\{-(\alpha x-\beta y)^2\}\,dx\,dy$$

$$\int_{-\infty}^{+\infty} \exp\{-(\alpha x-\beta y)^2\}\,dx = \int_{-\infty}^{+\infty}\frac{1}{\alpha}\cdot\exp\{-u^2\}\,du$$

$$= \frac{\sqrt{\pi}}{\alpha},$$

$$\alpha = \frac{1}{\sqrt{1-\rho^2}}\cdot\frac{1}{\sigma_x},\quad \beta = \frac{1}{\sqrt{1-\rho^2}}\cdot\frac{\rho}{\sigma_y},\quad \gamma = \frac{1}{2\sigma_y^2}.$$

$$\int_{-\infty}^{+\infty} y\cdot\exp\{-\gamma y^2\}\,dy = \int_{-\infty}^{+\infty}\frac{1}{2}\exp\{-\gamma y^2\}\,dy^2 = 0.$$

$\Rightarrow E(Y)=0$, and we find that $q(x,y)$ is symmetric between $x$ and $y$

so $E(X)=0$.

$$E(Y^2) = c\cdot\frac{\pi}{2}\cdot\int_{-\infty}^{+\infty} y^2\cdot\exp\{-ry^2\}\,dy$$

$$= c\cdot\frac{\pi}{2}\cdot\left(-\frac{2}{r}\right)\cdot\int_{-\infty}^{+\infty} y\, d\exp\{-ry^2\}$$

$$= c\cdot\frac{\pi}{2}\cdot\left(-\frac{2}{r}\right)\cdot\left(y\cdot e^{-ry^2}\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} e^{-ry^2}\,dy\right)$$

$$f_Y(y) = \int_{-\infty}^{+\infty} c\cdot\exp\{-ry^2\}\cdot\exp\{-(2x-\beta y)^2\}\,dx$$

$$= c\cdot\exp\{-ry^2\}\cdot\frac{\pi}{2}.$$

$$\Rightarrow Var(Y) = \sigma_y^2.$$

(c) since $\int_{-\infty}^{+\infty} c\frac{\pi}{2}\exp\left\{-\frac{y^2}{2\sigma_y^2}\right\}dy = 1$

$$\Rightarrow \frac{\sqrt{2\pi}}{2}c = \frac{1}{\sqrt{2\pi}\,\sigma_y}$$

$$c = \frac{1}{\sqrt{2\pi}\,\sigma_x\cdot\sigma_y}\cdot\frac{1}{\sqrt{1-\rho^2}}.$$

(d) $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \dfrac{c\cdot e^{-q(x,y)}}{c\cdot\frac{\pi}{2}\cdot\exp\{-ry^2\}}$ $\quad y\in\mathbb{R}.$

$$= \frac{2}{\sqrt{2\pi}}\cdot\exp\{-(2x-\beta y)^2 - ry^2 + ry^2\}$$

$$= \frac{2}{\sqrt{2\pi}}\cdot\exp\{-(2x-\beta y)^2\}.$$

$$E(X|Y) = \int_{-\infty}^{+\infty} x\cdot f_{X|Y}(x|y)\,\cancel{dy}\,dx$$

---

It's a lot of computation, I'm tired.

29. $\bigstar$ $M_X(s) = E(e^{sX}) = e^{s}\cdot\frac{1}{2} + e^{2s}\cdot\frac{1}{4} + e^{3s}\cdot\frac{1}{4}$.

$E(X) = \frac{dM_X}{ds}(0) = \frac{1}{2} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{4} = \frac{7}{4}$,

$E(X^2) = \frac{d^2M_X}{ds^2}(0)$

$E(X^3) = \frac{d^3M_X}{ds^3}(0)$. $\square$

$\bigstar$ 30. $M_X(s) = e^{s^2/2}$

$E(X) = 0$, $E(X^2) = 1$, $E(X^3) = 0$, $E(X^4) = 3$.

$f_X(x) = \frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x^2}{2}}$,

$$E(e^{sX}) = \int_{-\infty}^{+\infty} e^{sx}\cdot\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x^2}{2}}\,dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{(x-s)^2}{2}}\cdot e^{s^2/2}\,dx$$

$$= e^{s^2/2}.$$

31. $f_X(x) = \lambda\cdot e^{-\lambda x}$. $(x \geq 0)$.

~~Ete~~ $M_X(s) = E(e^{sX}) = \int_0^{+\infty} e^{sx}\cdot\lambda\cdot e^{-\lambda x}\,dx$

$$= \int_0^{+\infty} \lambda\cdot e^{(s-\lambda)x}\,dx$$

$$= \frac{\lambda}{\lambda - s} \quad (s < \lambda).$$

$E(X^3) = \frac{6}{\lambda^3}$, $E(X^4) = \frac{24}{\lambda^4}$, $E(X^5) = \frac{120}{\lambda^5}$. $\square$

32. Go!

(a) $M(0) \neq 1$, $M(s) = e^{2(e^{(e^{s}-1)}-1)}$

(b) since $x \geq 0$, $x \in \mathbb{Z}$. $\lim_{s\to-\infty} M_X(s) = P_X(0) = e^{2(e^{-1}-1)}$

33. we know that if $X \sim \text{Exponential}(\lambda)$, $M_X(s) = \frac{\lambda}{\lambda - s}$ $(s<\lambda)$.

$E(e^{s(aX_1+X_2)}) = E(e^{asX_1} \cdot e^{sX_2}) = E(e^{asX_1}) \cdot E(e^{sX_2})$, if $X_1, X_2$ are independent

if $f_X(x) = p_1 f_{X_1}(x) + \cdots + p_n f_{X_n}(x)$.

$E(e^{sX}) = p_1 \int e^{sx} \cdot f_{X_1}(x)\,dx + \cdots + p_n \int f_{X_n}(x) \cdot e^{sx}\,dx$

$= p_1 E(e^{sX_1}) + \cdots + p_n \cdot E(e^{sX_n})$.

---

$$f_X(x) = \begin{cases} \frac{1}{3} \cdot 2 \cdot e^{-2x} + 2 \cdot e^{-3x} & (x \geq 0). \\ 0 & (\text{otherwise}). \end{cases}$$

(otherwise)

34. Soccer team has 3 designated players. go.

$X_1, X_2, X_3$ denotes each playerss, $p_{X_1+X_2}(z) = \sum_{x_1+x_2=z} p_{X_1}(x_1) \cdot p_{X_2}(x_2)$

$$p_{X_1+X_2}(x) = \begin{cases} (1-p_1)(1-p_2) & , x=0 \\ (1-p_1)p_2 + p_1(1-p_2) & , x=1 \\ p_1 p_2 & x=2 \\ 0 & , \text{otherwise}. \end{cases}$$

otherwise.

$$p_{X_1+X_2+X_3}(z) = \begin{cases} (1-p_1)(1-p_2)(1-p_3) & , x=0 \\ \cdots \\ \sim \end{cases}$$

② $M_X(s) = M_{X_1}(s) \cdot M_{X_2}(s) \cdot M_{X_3}(s)$

$= (e^s \cdot p_1 + (1-p_1))(e^s \cdot p_2 + 1 - p_2)(e^s p_3 + 1 - p_3)$

$= e^{3s} \cdot p_1 p_2 p_3 + \cdots$

$p_X(z) = \begin{cases} p_1 p_2 p_3 & z=3 \end{cases}$

$M_X(s) = \sim$

$$E(X) = \frac{dM_X(s)}{ds}\Big|_{s=0} = c\frac{(8e^{2s}+6e^{3s})(3-e^s)-(3+4e^{2s}+2e^{3s})(-e^s)}{(3-e^s)^2}\Big|_{s=0}$$

$$= \frac{37}{4}c$$

$$M_X(0) = E(1) = 1 = \frac{9}{2}c \Rightarrow c = \frac{2}{9}.$$

$$E(X) = \frac{37}{18}.$$

since X takes nonnegative value,

$$p_X(0) = \lim_{s\to-\infty} M_X(s) = \frac{2}{9}.$$

$$p_X(1) = \lim_{s\to-\infty} \frac{M_X(s) - p_X(0)}{e^s} = \frac{2}{27}$$

$$M_X(s) = \frac{2}{9}(3+4\cdot e^{2s}+2\cdot e^{3s})\left(\frac{1}{3}+\frac{1}{9}e^s+\frac{1}{27}e^{2s}+ \cdots\right)$$

taylor expansion formula can also get the same answer.

$$E(X|X\neq 0) = \frac{\sum_{x\neq 0} p_X(x)\cdot x}{P(X\neq 0)} = \frac{E(X)-0}{P(X\neq 0)} = \frac{37/18}{1-\frac{2}{9}} = \frac{37}{14}. \ \square$$

36. (a) $M_U(u) = E(e^{(XY+(1-X)Z)s})$

$$= E(e^{(XY+(1-X)Z)s}|X)$$

$$= p_X(0)E(e^{Zs}) + p_X(1)\cdot E(e^{Ys})$$

$$= \frac{2}{3}\cdot e^{3(e^s-1)} + \frac{1}{3}\cdot\frac{2}{2-s}.$$

$Z \sim \text{Poisson}(\lambda)$, $\lambda=3$, $P_Z(z) = e^{-\lambda}\cdot\frac{\lambda^z}{z!}$, $E(e^{sz}) = M_Z(z) = e^{\lambda(e^s-1)}$

$Y \sim \text{exponential}(\mu)$, $\mu=2$, $f_Y(y) = \mu\cdot e^{-\mu y}$, $E(e^{sy}) = M_Y(s) = \frac{\mu}{\mu-s}$

(b) $E(e^{(2Z+S)s}) = e^{3s} \cdot e^{3(e^{2s}-1)}$

$= e^{3(e^{2s}+s-1)}$

(c) $E(e^{Ys}) \cdot E(e^{Zs}) = \frac{2}{2-s} \cdot e^{3(e^{s}-1)}$. $\square$

37. $K$ customers, $M_K(s) = E(e^{sK})$.

$n$ types of pizza.

assume that $X_i$ denotes first $i$th customers different types of pizza in.

$E(X_1) = 1$, ~~$E(X_i | X_{i-1}) =$~~

$(X_i \le i)$, $E(X_i | X_{i-1} = k) = (1 - \frac{k}{n}) \cdot (k+1) + \frac{k}{n} \cdot k = k+1 - \frac{k}{n}$

$E(X_i) = E(E(X_i | X_{i-1})) = E(X_{i-1}) + 1 - \frac{1}{n} E(X_{i-1})$

$= (1 - \frac{1}{n}) E(X_{i-1}) + 1$.

$E(X_K) = E(E(X_K | K))$

$= E((1-\frac{1}{n})^{K-1} + (1-\frac{1}{n})^{K-2} + \cdots + 1)$

$= E(n - n(1-\frac{1}{n})^K)$

$= n - n E((1-\frac{1}{n})^K)$

$= n - n \cdot M_K(\log(1-\frac{1}{n}))$. $\square$

38 (a) $M(s) = \sum_{k=0}^{+\infty} P(X=k) \cdot e^{sk}$,

since $\sum_{k=0}^{+\infty} P_X(k) \cdot e^{sk}$ are convergent and $\sum_{k=0}^{+\infty} P_X(k) = 1$,

we have $\sum_{k=1}^{+\infty} P(X=k) e^{sk} \to 0$ as $s \to -\infty$.

$M(s) \to P_X(0)$, $s \to -\infty$.

(b)

$X \sim \text{Binomial}(n, p)$

$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$, $M_X(s) = (1-p+p\cdot e^s)^n$.

$X \sim \text{Poisson}(\lambda)$, $M_X(s) = e^{\lambda(e^s-1)}$.

(c) $X \geq \bar{k}$, $E(e^{sX}) = \sum_{k=\bar{k}}^{+\infty} e^{sk} \cdot p_X(k)$

$$p_X(\bar{k}) = \lim_{s\to-\infty} \frac{E(e^{sX})}{e^{s\bar{k}}}.$$

$$= \lim_{s\to-\infty} \frac{M_X(s)}{e^{s\bar{k}}}. \quad \square$$

39. (a) $p_X(k) = \begin{cases} \frac{1}{b-a+1} & \text{, if } k = a, \text{—}, b. \\ 0 & \text{otherwise.} \end{cases}$

$$M_X(s) = E(e^{sk}) = \sum_{k=a}^{b} \frac{1}{b-a+1} \cdot e^{sk}$$

$$= \frac{1}{b-a+1} \cdot \frac{e^{sa} - e^{s(b+1)}}{1-e^s}.$$

(b) $f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$

$$M_X(s) = \int_a^b \frac{1}{b-a} \cdot e^{sx}\, dx$$

$$= \frac{1}{(b-a)s} \cdot (e^{sb} - e^{sa}).$$

40. degree$(A(t))$ < degree$(B(t))$.

( common 共同的，普通的)

common roots 同根

(a) $\lim\limits_{s \to \infty} M(s) = p(x=0) = \sum_{i=1}^{m} a_i$

$$\lim_{s \to -\infty} M(s) - p(X=0) = \lim_{s \to -\infty} \sum_{i=1}^{m} a_i \left( \frac{1}{1 - r_i \cdot e^s} - 1 \right) / e^s$$

$$= \lim_{s \to -\infty} \sum_{i=1}^{m} a_i \left( \frac{r_i \cdot e^s}{1 - r_i \cdot e^s} \right) / e^s$$

$$= \sum_{i=1}^{m} a_i \cdot r_i$$

through induction method, assume that for $\forall k \le k$, the formula holds.

$$\lim_{s \to -\infty} (M(s) - p(X=0) - \cdots - p(X=k)) / e^{s(k+1)}$$

$$= \lim_{s \to -\infty} \sum_{i=1}^{m} a_i \left( \frac{1}{1 - r_i e^s} - 1 - e^{2s} - \cdots - e^{ks} \right) / e^{s(k+1)}$$

$$= \lim_{s \to -\infty} \sum_{i=1}^{m} a_i \left( 1 + r_i e^s + r_i^2 e^{2s} + \cdots - 1 - e^{2s} - \cdots - e^{ks} \right) / e^{s(k+1)}$$

$$= \sum_{i=1}^{m} a_i \cdot r_i^{(k+1)}.$$

(b) $M(s) = \sum_{i=1}^{n} a_i (1 + r_i e^s + r_i^2 e^{2s} + \cdots) \cdot e^{bs}$

$$p(x=k) = \sum_{i=1}^{m} a_i \cdot r_i^{(k-b)}, \quad k \ge b.$$

(c) $\frac{1}{r_1}, \cdots, \frac{1}{r_m}$ are $B(t)$'s roots.

$$a_i = \lim_{e^s \to \frac{1}{r_i}} (1 - r_i \cdot e^s) \cdot M(s).$$

Since $B(t)$ has distinct root, $\frac{A(t)}{B(t)} = \frac{a_1}{\cdots} + \frac{a_2}{\cdots} + \cdots + \frac{a_m}{\cdots}$ [illegible]

41. (a) $T = X_1 + X_2 + \cdots + X_N$,

$N \sim \text{Poisson}(\lambda)$,

$M_T(s) = E(e^{s(X_1 + \cdots + X_N)})$

$= E(E(e^{s(X_1 + \cdots + X_N)} | N))$

$= E([\frac{1}{s}(e^s - 1)]^N)$

$= E(M_X(s)^N)$

$M_N(s) = e^{\lambda(e^s - 1)}$

$M_T(s) = M_N(\log M_X(s)) = e^{\lambda(M_X(s) - 1)} = e^{\lambda(\frac{e^s - 1}{s} - 1)}$.

(b) $E(T) = \frac{d}{ds} M_T(s) \Big|_{s=0} = \frac{\lambda}{2}$.

(c) $E(T) = E(X_1 + \cdots + X_N)$

$E(X) \cdot E(N)$

$= E(E(X_1 + \cdots + X_N | N)) = E(E(X) \cdot N) = E(X) \cdot E(N)$

$= E(\frac{N}{2})$

$= \frac{1}{2}\lambda$.

42. $M_X(s) = e^{(\sigma^2 s^2/2) + \mu s}$. ?

43. ~~$X_i \sim \text{Normal}(1, \frac{1}{4})$~~

(a) $X = X_1 + X_2 + X_3 + X_4$.

~~$X_i =$~~ $\{$ ~~$P_{X_i}(0) = \frac{1}{2}$~~, $P_{X_i}(0) = \frac{1}{2}$, $X_i | \text{red} \sim \text{Normal}(1, \frac{1}{4})$.

~~$f_X(x) =$~~ $P(k \text{ lights are red}) = \binom{4}{k}(\frac{1}{2})^4$.

$f_{X|k \text{ red}} = \text{Normal}(k, \frac{k}{4})$, $k \geq 1$.

mixed distribution is not normal.

if $k=0$, ~~$f_{X|0 \text{ red}} =$~~ $P(X=0 | 0 \text{ red}) = 1$.

(b) let $K$ be the number of red lights.

$K \sim \text{Binomial}(4, \frac{1}{2})$ △ $X = X_1 + \cdots + X_K$, all red lights.

$M_K(s) = (\frac{1}{2} + \frac{1}{2} \cdot e^s)^4$,

$M_X(s) = E(E(e^{Xs} | K))$

$= E(M_{X_i}(s)^K)$

$= M_K(\log(M_{X_i}(s)))$

~~$M_{X_i}(s) = E(e^{Xs}) = \frac{1}{2} + \frac{1}{2}(e^{(\frac{1}{2})^2 \cdot s^2/2 + s})$~~

~~$M_X(s) = (\frac{1}{2} + \frac{1}{4} + \frac{1}{4}(e^{(\frac{1}{2})^2 \cdot s^2/2 + s}))^4$~~

$X_i$ means if ith lights are red, the time of the red light.

$M_{X_i}(s) = e^{(\frac{1}{2})^2 \cdot s^2/2 + s}$,

~~$\cancel{M_X}$~~ $M_X(s) = (\frac{1}{2} + \frac{1}{2}(e^{(\frac{1}{2})^2 \cdot s^2/2 + s}))^4$.

another solution: △ $X = X_1 + X_2 + X_3 + X_4$, all lights.

$X_i$ means the ith ~~lights's time~~ time of the ith light.

$M_{X_i}(s) = \frac{1}{2} + \frac{1}{2} \cdot e^{(\frac{1}{2})^2 \cdot s^2/2 + s}$

~~$M_K(s) =$~~

$M_X(s) = M_{X_i}(s)^4 = (\frac{1}{2} + \frac{1}{2} \cdot e^{(\frac{1}{2})^2 \cdot s^2/2 + s})^4$. □

44. ( derive 获得 ) derive from 获得, 起源.

(a) $E(N) = E(E(N|M))$

$= E(M \cdot E(K))$

$= E(M) \cdot E(K)$.

$Var(N) = E(Var(N|M)) + Var(E(N|M))$

$= E(M \cdot Var(K)) + Var(M \cdot E(K))$

$= E(M) \cdot Var(K) + E(K)^2 \cdot Var(M)$.

(b) $E(Y) = E(X) \cdot E(N) = E(X) \cdot E(M) \cdot E(K)$.

$var(Y) = E(N) \cdot var(X) + E(X)^2 \cdot Var(N)$

$= E(M) \cdot E(K) \cdot var(X) + E(X)^2 \cdot \left( E(M) \cdot var(K) + E(K)^2 \cdot var(M) \right)$.

(c) $N$: total widgets ~~weight~~ of a crate.

$N = k_1 + k_2 + \cdots + k_M$,

$Y$: total weight of a crate.

$Y = X_1 + \cdots + X_N$,

$X_i$: ~~ith~~ weight of the ith widget.

$Y$: $E(Y)$?, $Var(Y)$?

45. $Y = X_1 + \cdots + X_N$,

$M_{X_i}(s) = 1 - p + p \cdot e^s$

$X_i \sim$ Bernoulli $(p)$,

$M_N(s) = e^{\lambda(e^s - 1)}$

$N \sim$ Poisson $(\lambda)$.

$M_Y(s) = E(E(Y \mid N))$

$= M_N(\log(M_X(s)))$

$= e^{\lambda(p \cdot e^s - p)}$

$= e^{\lambda p(e^s - 1)}$

$Y \sim$ Poisson$(\lambda p)$, since $M: Y \mapsto M_Y(s)$ is injection. $\square$

$\begin{cases} \text{injection 单射} \\ \text{surjection 满射} \\ \text{bijection 双射.} \end{cases}$