Chapter 5.

1. (a) $\cancel{Var(M_n) = \frac{Var(X_i)}{n} = 0.01}$,

$\cancel{n=100}$. $Var(M_n) = \frac{Var(X_i)}{n} = 0.0001$,

$Var(X_i) = 1$, $n = 10000$. $n \geq 10^4$.

(b) $P(|M_n - h| \leq 0.05) \geq 0.99$, $n = ?$

$$P(|M_n - h| \leq 0.05) = 1 - P(|M_n - h|^2 \geq 0.05^2)$$
$$\geq 1 - \frac{Var(M_n)}{0.05^2}$$
$$= 1 - \frac{1/n}{0.05^2} \geq 0.99$$

$n \geq 40000$.

(c) (revise 修改)

~~(revise revi~~

(revise 修改)

binomial$(n,p) \to$ Poisson$(\lambda)$

$\lambda = np$ is fixed, $(n \to \infty)$

2. (a) $\forall a$, $\forall s \geq 0$, $s$ is in a small finite interval, containing 0

if continuous.

$$\cancel{P(X \geq a) = \int_a^{+\infty} f_X(x)\,dx}$$

$$M(s) \cdot e^{-sa} = \int_{-\infty}^{+\infty} e^{-sa} \cdot e^{sx} \cdot f_X(x)\,dx$$
$$\geq \int_a^{+\infty} e^{-sa} \cdot e^{sx} \cdot f_X(x)\,dx$$
$$\geq \int_a^{+\infty} f_X(x)\,dx = P(X \geq a)$$

if discrete, $M(s) \cdot e^{-sa} \geq \sum_{x \geq a} e^{-sa} \cdot e^{sx} \cdot p_X(x) \geq \sum_{x \geq a} p_X(x) = P(X \geq a)$.

$\square$

(b) $e^{-sa} \cdot M(s) = e^{-sa} \cdot \int_{-\infty}^{+\infty} e^{xs} \cdot f_X(x)\, dx$ , for $s \leq 0$.

$$\geq e^{-sa} \cdot \int_{-\infty}^{a} e^{xs} \cdot f_X(x)\, dx$$

$$\geq e^{-sa} \cdot e^{as} \cdot \int_{-\infty}^{a} f_X(x) dx$$

$$= P(X \leq a).$$

(c) $s \geq 0$, we have $P(X \geq a) \leq e^{-sa} \cdot M(s) = e^{-(sa - \ln M(s))}$

for $\forall a$,

so $P(X \geq a) \leq \min_{s \geq 0} e^{-(sa - \ln M(s))}$

$$= e^{-(\max_{s \geq 0} sa - \ln M(s))}$$

(d) $\phi(a) = \max_{s \geq 0} (sa - \ln M(s)) > 0$

$\Leftrightarrow \min_{s \geq 0} e^{-sa + \ln M(s)} = \min_{s \geq 0} e^{-sa} \cdot M(s) < 1$

$a - E(X) > 0 \Leftrightarrow \int_{-\infty}^{+\infty} (a - x) f_X(x)\, dx > 0$

$e^{-sa} \cdot M(s) = \int_{-\infty}^{+\infty} e^{-s(a-x)} \cdot f_X(x)\, dx$

$\left.\dfrac{d\, e^{-sa} \cdot M(s)}{ds}\right|_{s=0} = \cancel{-(a-x)} + \int_{-\infty}^{+\infty} -(a-x) f_X(x) dx < 0.$

$\left. e^{-sa} \cdot M(s) \right|_{s=0} = 1$

so there must $\exists\, s_0 > 0$, s.t. $e^{-s_0 \cdot a} \cdot M(s_0) < e^{0a} \cdot M(0) = 1$.

$\min_{s \geq 0} e^{-sa} \cdot M(s) < 0 \Rightarrow \phi(a) > 0$.

(e) $M(s) = e^{s^2/2}$.

$P(X \geq a) \leq e^{-\phi(a)}$, $\quad sa - \ln M(s) = sa - \frac{s^2}{2} = -\frac{1}{2}(s-a)^2 + \frac{a^2}{2} \leq \frac{a^2}{2}$.

$\phi(a) = \max_{s \geq 0} (sa - \ln M(s)) = \frac{a^2}{2}$, $\quad (a>0)$

$P(X \geq a) \leq e^{-\frac{a^2}{2}} < 1$. $\square$

(f) if $a > E(X)$, then $\phi(a) > 0$, ~~$P(X \geq a) \leq e^{-\phi(a)}$~~.

assume that $Y = \sum_{i=1}^{n} X_i$,

~~$E(Y) = E(X)$, $Var(Y) = \frac{1}{n} Var(X)$.~~

$M_Y(s) = M_X(s)^n$

$\phi_Y(a) = \max_{s \geq 0} (sa - n \ln M_X(s))$

~~$P(Y \geq na) \leq e^{-\phi_Y(a)} =$~~

$P(Y \geq na) = P(\frac{1}{n} \sum_{i=1}^{n} X_i \geq a) \leq e^{-\phi_Y(na)}$

$= e^{-n\phi(a)}$. $\square$

3.

convex $\quad$ concave

(a) $f''(x) = a^2 e^{ax} > 0$, $\quad f''(x) = \frac{1}{x^2} > 0$, $\quad f''(x) = 12x^2 > 0$

(b) ~~$f(x) = f(a) + (x-a)\frac{df}{dx}(a) + \frac{(x-a)^2}{2} \cdot \frac{d^2f}{dx^2}(a) + \frac{(x-a)^3}{3!} \cdot \frac{d^3f}{dx^3}(\xi)$~~

$f(x) = f(a) + (x-a)\frac{df}{dx}(a) + \frac{(x-a)^2}{2} \cdot \frac{d^2f}{dx^2}(\xi)$. $\quad \xi \in (a, x)$.

$f(x) - f(a) - (x-a) \cdot \frac{df}{dx}(a) \geq 0$. $\square$

Taylor expansion: $f(x) = f(a) + \cdots + \frac{f^{(n)}(x)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$.

(c) $f(X) \approx f(E(X)) + (X - E(X)) \cdot \frac{df}{dx} \cancel{f}(E(X))$

$\Rightarrow E(f(X)) \approx E(f(E(X))) + 0$

$\Rightarrow E(f(X)) \approx f(E(X))$. $\square$

4. Go!

$M_n = \frac{1}{n} \cdot S_n$, $S_n = \sum_{i=1}^{n} X_i$,

chebyshev inequality: $P(|M_n - f| \geq \varepsilon) \leq \frac{Var(X)}{n\varepsilon^2} = \delta$

(a) as $\varepsilon_1 = \frac{1}{2}\varepsilon$,

$P(|M_n - f| \geq \frac{\varepsilon}{2}) \leq \frac{Var(X)}{n \cdot \varepsilon^2/4} = 4\delta$,

$\Rightarrow n_1 = 4n$, $P(|M_n - f| \geq \frac{\varepsilon}{2}) \leq \delta$.

(b) if $\delta_1 = \frac{1}{2}\delta$, $n_1 = 2n$, $\frac{Var(X)}{n_1 \cdot \varepsilon} = \delta_1 = \frac{1}{2}\delta$.

$n_1 = 2n$.

5. (a) $\lim_{n\to\infty} P(|Y_n| \geq \varepsilon) \leq \lim_{n\to\infty} \frac{Var(Y_n)}{\varepsilon^2}$

$= \lim_{n\to\infty} \frac{1}{3n\varepsilon^2} = 0$.

$\Rightarrow \lim_{n\to\infty} P(|Y_n - 0| \geq \varepsilon) = 0$,

$Y_n \xrightarrow{P} 0$.

(b) $\lim_{n\to\infty} P(|Y_n| \geq \varepsilon) = \lim_{n\to\infty} (2 - 2\sqrt[n]{\varepsilon}) = 0$, for $\forall\ \varepsilon < 1$.

$Y_n \xrightarrow{P} 0$.

$Y_n \xrightarrow{P} 0$. $Y_n \xrightarrow{P} 0$.

(c) $E(Y_n) = \frac{1}{n} E(X_1) \cdots E(X_n) = 0.$

$Var(Y_n) = E(X_1^2) \cdots E(X_n^2) = \frac{1}{3n} \to 0$ as $n \to \infty$

$$\lim_{n\to\infty} P(|Y_n| \geq \varepsilon) \leq \lim_{n\to\infty} \frac{Var(Y_n)}{\varepsilon^2} = 0$$

(d) $F_{Y_n}(y) = P(X_1, \cdots X_n \leq y) = (F_X(y))^n$

$= (\frac{1}{2} + \frac{1}{2}y)^n \qquad y \in [-1, 1].$

$$= \begin{cases} 0 & y \in [-1, 1) \\ 1 & y \in \{1\}. \end{cases}$$

since $F_n \xrightarrow{P} F \Rightarrow F_n \xrightarrow{W} F.$

$Y_n \xrightarrow{W} 1.$

we now proove $Y_n \xrightarrow{P} 1.$

$$\lim_{n\to\infty} P(|Y_n - 1| \geq \varepsilon) = \lim_{n\to\infty} P(Y_n \geq 1+\varepsilon \cup Y_n \leq 1-\varepsilon)$$

$$= \lim_{n\to\infty} P(X_1 \leq 1-\varepsilon, \cdots, X_n \leq 1-\varepsilon)$$

$$= \lim_{n\to\infty} (1 - \tfrac{1}{2}\varepsilon)^n \qquad \varepsilon \in (0, 2]$$

$= 0.$

Hence, $Y_n \xrightarrow{P} 1.$

6. we know: $\lim_{n\to\infty} P(|X_n - x| \geq \varepsilon) = 0$, $\lim_{n\to\infty} P(|Y_n - y| \geq \varepsilon) = 0$

$X_n \xrightarrow{P} x$, $Y_n \xrightarrow{P} y.$

① Proove $c \cdot X_n \xrightarrow{P} cx.$ $\lim_{n\to\infty} P(|cX_n - cx| \geq \varepsilon) = 0$

② $\lim_{n\to\infty} P(|X_n + Y_n - x - y| \geq \varepsilon) \leq \lim_{n\to\infty} P(|X_n - x| + |Y_n - y| \geq \varepsilon)$

$\cancel{\leq \lim_{n\to\infty} P(}$

$|X_n - x| + |Y_n - y| \geq \varepsilon$,

since $|X_n - x|$ and $|Y_n - y|$, one of which is at least $\varepsilon/2$.

we have $\{|X_n - x| + |Y_n - y| \geq \varepsilon\} \subset \{|X_n - x| \geq \varepsilon/2\} \cup \{|Y_n - y| \geq \varepsilon\}$.

$P\{|X_n - x| + |Y_n - y| \geq \varepsilon\} \leq P\{|X_n - x| \geq \varepsilon/2\} + P\{|Y_n - y| \geq \varepsilon/2\}$

$\to 0$ as $n \to \infty$.

$\Rightarrow X_n + Y_n \xrightarrow{P} x + y$.

③ ~~P(|max{0, Xn}~~ since $F_n \xrightarrow{P} F \Rightarrow F_n \xrightarrow{W} F$.

$P(\max(X_n, 0) \geq y) = P(X_n \geq x \cup 0 \geq y)$

$= P(0 \geq y) + P(0 < y) \cdot P(X_n \geq y)$.

$= P(0 \geq y) + P(0 < y) \cdot P(x \geq y)$

$= P(0 \geq y \text{ or } x \geq y)$

$= P(\max(0, x) \geq y)$

$\max(X_n, 0) \xrightarrow{W} \max(0, x)$.

$\lim_{n \to \infty} P(|\max(X_n, 0) - \max(0, x)| \geq \varepsilon)$

since $\max(x, y) = \frac{1}{2}(|x - y| + (x + y))$.

from ④ we have $|X_n| \xrightarrow{P} |x|$.

~~max(Xn, 0)~~

$|X_n - 0| \xrightarrow{P} |X|$, $\Rightarrow |X_n| + (X_n) \xrightarrow{P} |x| + x = \max(x, 0)$.

$\Rightarrow \max\{X_n, 0\} \xrightarrow{P} \max\{x, 0\}$.

④ $\lim_{n \to \infty} P(||X_n| - |x|| \geq \varepsilon) \leq \lim_{n \to \infty} P(|X_n - x| \geq \varepsilon) = 0$.

⑤ $\lim_{n \to \infty} P(|X_n Y_n - xy| \geq \varepsilon) = \lim_{n \to \infty} P(|(X_n - x)(Y_n - y) + xY_n + yX_n - 2xy| \geq \varepsilon)$

$= \lim_{n \to \infty} P(|(X_n - x)(Y_n - y) + x(Y_n - y) + y(X_n - x)| \geq \varepsilon)$

~~since~~ since $x \cdot (Y_n - y) \xrightarrow{P} 0$ and $y \cdot (X_n - x) \xrightarrow{P} 0$

ⓒ $x\cdot(Y_n-y)+y\cdot(X_n-x) \xrightarrow{P} 0.$

$$\lim_{n\to\infty} P(|x\cdot(Y_n-y)+y\cdot(X_n-x)|\geq\varepsilon)=0, \quad \forall\varepsilon>0.$$

$$\lim_{n\to\infty} P(|(X_n-x)(Y_n-y)|\geq\varepsilon) \leq \lim_{n\to\infty} P(|X_n-x|\geq\sqrt{\varepsilon})+P(|Y_n-y|)\geq\sqrt{\varepsilon})=0.$$

$\Rightarrow (X_n-x)\cdot(Y_n-y) \xrightarrow{P} 0.$

$\Rightarrow X_n\cdot Y_n \xrightarrow{P} xy.$ □

7. ao! $\lim_{n\to\infty} E((X_n-c)^2)=0$

$X_n$ converge to $c$ in mean ~~等~~ square. 均方收敛.

(a) $\lim_{n\to\infty} E((X_n-c)^2)=0 \Rightarrow \lim_{n\to\infty} P(|X_n-c|^{\circ}\geq\varepsilon)=0 \quad \forall\varepsilon>0.$

~~$E(|X_n-c|^2)$~~ set $Y_n=(X_n-c)^2$.

$$E(Y_n)=\int_0^{+\infty} y_n\cdot f_Y(y)\,dy \geq \varepsilon\cdot P(Y_n\geq\varepsilon) \to 0 \quad \text{as } n\to\infty.$$

$\Rightarrow P(Y_n\geq\varepsilon)\to 0$ as $n\to\infty$.

another solution:

$$P(|X_n-c|\geq\varepsilon) \leq \frac{E(|X_n-c|^2)}{\varepsilon^2} \to 0 \quad \text{as } n\to\infty.$$

hence $X_n \xrightarrow{P} c$.

(b) assume that

$$P_{X_n}(x)=\begin{cases} 1-\frac{1}{n} & \text{if } x=0 \\ \frac{1}{n} & \text{if } x=n, \end{cases}$$

$P(|X_n|\geq\varepsilon)=\frac{1}{n}\to 0$ as $n\to\infty$

$E(|X_n|^2)=n\to+\infty$ as $n\to\infty$. □

$X_n \xrightarrow{P} c \not\Rightarrow X_n \to c$ (mean square).

8. P( choose an odd number in one round) = $\frac{1}{2}$.

P( wrong decision) = P( odd $\geq 55$ | $N=100$)

assume that $X_n$ means the result of $n$ th test.

$P(\sum_{i=1}^{100} X_i \geq 55)$ let $X = \sum_{i=1}^{100} X_i$,

$E(X) = 50$, $Var(X) = 100\, Var(X_i) = 25$.

$P(X \geq 55) = P\left(\frac{X - E(X)}{\sqrt{Var(X)}} \geq 1\right) \approx 1 - \Phi(1) \approx 0.1587$. □

$P(X \geq 55) = \frac{1}{2} P(X \geq 55 \text{ or } X \leq 45)$

$= \frac{1}{2} P(|X - 50| \geq 5)$

$\leq \frac{1}{2} \cdot \frac{Var(X)}{25} = 0.5$.

9. ( crash 撞，死机)

( crash 电脑死机).

(a) $X_i$ means $i$th day situation.

$$X_i = \begin{cases} 1 & \text{crash-free} \\ 0 & \text{crash} \end{cases} \quad (n \leq 50)$$

$P(\sum_{i=1}^{50} X_i \geq 45) = P\left(\frac{X - E(X)}{\sqrt{Var(X)}} \geq \frac{45 - 47.5}{\sqrt{2.375}} = \frac{-2.5}{1.54}\right) \approx \Phi(1.62)$.

$Var(X) = 50\, Var(X_i) = 50 \cdot (0.05 \times 0.95^2 + 0.95 \times 0.05^2) = 2.375$

(b) fixed $np$. $Binomial(n, p) \to Poisson(n(1-p)) = Poisson(2.5)$

$P(\sum_{i=1}^{50} X_i \geq 45) =$

10. (a) $P(\sum_{i=1}^{100} X_i \le 440)$

$$= P\left(\frac{\sum_{i=1}^{100} X_i - E(X)}{\sqrt{Var(X)}} \le \frac{440-500}{\sqrt{9\times 100}}\right)$$

$= \phi(-2)$.

$= 1-\phi(2)$.

(b) $P(X_1 + \cdots + X_n - 5n \geq 200)$

$$= P\left(\frac{X_1+\cdots+X_n-5n}{\sqrt{9n}} \geq \frac{200}{\sqrt{9n}}\right)$$

$\le 0.05$

$\phi^{-1}(0.05) = -1.644$

$\frac{200}{\sqrt{9n}} \geq 1.644 \Rightarrow n \le 1644.$

(c) $\cancel{P(N \geq 220)}$

$\cancel{P(X_1+\cdots+X_n \geq 1000)}$

$\cancel{P(X_1+\cdots+X_{220} \geq 1000,\ X_1+\cdots+X_{219} < 1000)}$

$P(N \geq 220) = P(X_1+\cdots+X_{219} < 1000)$

$$= P\left(\frac{X_1+\cdots+X_{219}-219\cdot 5}{\sqrt{219\cdot 9}} < \frac{1000-219\cdot 5}{\sqrt{219\cdot 9}}\right)$$

$\approx \phi(-2.14) = 1-\phi(2.14)$

$\cancel{0.96}$. $\approx 0.016.$

0.016

11. $X_1, Y_1, X_2, Y_2$ are independent RV,

$$P(|W-E(W)|<0.001) = P\left(\frac{|W-E(W)|}{\sqrt{Var(W)}} < \frac{0.001}{\sqrt{1/96}}\right) \approx 2\phi(0.0098)-1.$$
$$\approx 0.008.$$

$\sqrt{Var(W)} = \sqrt{1/96}$

12. ( central limit theorem)

(a) ~~$X_i \sim$~~ $E(X_i)=0$, $Var(X_i)=6^2$.

$M_X(s)$, $s\in(-d,d)$, $d>0$.

$$Z_n = \frac{X_1+\cdots+X_n}{6\cdot\sqrt{n}}.$$

(a) $M_{Z_n}(s) = E\left(e^{s\cdot\frac{X_1+\cdots+X_n}{6\cdot\sqrt{n}}}\right)$

$= E\left(e^{s\cdot\frac{X_1}{6\sqrt{n}}}\cdot e^{s\frac{X_2}{6\sqrt{n}}}\cdots\cdot e^{s\cdot\frac{X_n}{6\sqrt{n}}}\right)$

$= \left(M_X\left(\frac{s}{6\sqrt{n}}\right)\right)^n.$

(b) (order 阶, second order 二阶).

$M_X(s) = E(e^{sX})$

$a = M_X(0) = 1$

$b = \frac{dM_X(s)}{ds}\Big|_{s=0} = 0$

$2c = \frac{d^2M_X(s)}{ds^2}\Big|_{s=0} = 6^2,$

$a=1$, $b=0$, $c=\frac{6^2}{2}$

$M_X(s) = 1+\frac{6^2}{2}s^2+o(s^2).$

$\lim_{\frac{s}{\sqrt{n}}\to 0}\frac{o(\frac{s^2}{n6^2})}{\frac{s^2}{n6^2}} = 0 \Rightarrow$

fixed s, $\lim_{n\to\infty}\frac{o(\frac{s^2}{n})}{\frac{s^2}{n}} = 0.$

(c) ~~$M_{Z_n}(s) = (1+\frac{6^2}{2}s^2+o(s^2))^n$~~

$M_{Z_n}(s) = \left(1+\frac{6^2}{2}\cdot\left(\frac{s}{6\sqrt{n}}\right)^2+o\left(\frac{s^2}{n6^2}\right)\right)^n = \left(1+\frac{s^2}{2n}+o\left(\frac{s^2}{n}\right)\right)^n = e^{s^2/2}$

$\Rightarrow Z_n \to \text{Normal}(0,1)$ as $n \to +\infty$.

since ~~[illegible]~~ $M_{Z_n}(s) \to e^{s^2/2}$ as $n \to \infty$. $\square$

13. $X_n \xrightarrow{a.s.} a$, $Y_n \xrightarrow{a.s.} b$.

$P(\lim_{n\to\infty} X_n = a) = P(\lim_{n\to\infty} Y_n = b) = 1$.

show $X_n + Y_n \xrightarrow{a.s.} a+b$, $X_n / Y_n \xrightarrow{a.s.} a/b$.

① solution 1: $P(\lim_{n\to\infty} X_n = a) = 1 \Leftrightarrow \forall \varepsilon > 0, \lim_{n\to\infty} P(\bigcap_{m=n}^{\infty} |X_n - a| < \varepsilon) = 1$.

$\cancel{P(\bigcap_{m=n}^{\infty} |X_n - a| < \varepsilon)}$

$P(\bigcap_{m=n}^{\infty} |X_m + Y_m - a - b| < \varepsilon) \geq P(\bigcap_{m=n}^{\infty} |X_m - a| + |Y_m - b| < \varepsilon)$

$\geq P(\bigcap_{m=n}^{\infty} |X_m - a| < \varepsilon/2 \cap \bigcap_{m=n}^{\infty} |Y_m - b| < \varepsilon/2)$

assume that $A_n = \bigcap_{m=n}^{\infty} \{|X_m - a| < \varepsilon/2\}$, $B_n = \bigcap_{m=n}^{\infty} \{|Y_m - b| < \varepsilon/2\}$

$\forall \varepsilon > 0$, $\exists N$, s.t. $P(A_n) > 1 - \varepsilon/2$, $P(B_n) > 1 - \varepsilon/2$.

$P(A_n \cap B_n) = P(A_n) + P(B_n) - P(A_n \cup B_n) > 1 - \varepsilon$.

$\Rightarrow \lim_{n\to\infty} P(A_n \cap B_n) = 1$,

$\Rightarrow \lim_{n\to\infty} P(\bigcap_{m=n}^{\infty} |X_m + Y_m - a - b| < \varepsilon) = 1$, $\forall \varepsilon > 0$.

$\Rightarrow X_n + Y_n \xrightarrow{a.s.} a+b$. $X_n + Y_n \xrightarrow{a.s.} a+b$.

solution 2: $P(\lim_{n\to\infty} X_n = a) = P(\forall \varepsilon > 0, \exists N > 0, \text{s.t.} |X_n - a| < \varepsilon) = 1$.

since ~~$|X_n - a|$~~ $|X_n + Y_n - a - b| < \varepsilon \Leftarrow |X_n - a| + |Y_n - b| < \varepsilon$

$\Leftarrow |X_n - a| < \varepsilon/2 \cap |Y_n - b| < \varepsilon/2$.

let $A = \{X_n : \lim_{n\to\infty} X_n = a\}$, $B = \{Y_n | \lim_{n\to\infty} Y_n = b\}$,

$P(\forall \varepsilon > 0, \exists N > 0, \text{s.t. } |X_n - a| < \varepsilon/2 \text{ and } |Y_n - b| < \varepsilon/2) = P(A \cap B)$

$\leq P(\lim_{n\to\infty} X_n + Y_n = a+b)$

$P(A \cap B) \geq P(A) + P(B) - 1 = 1.$

$\Rightarrow P(\lim_{n\to\infty} X_n + Y_n = a+b) = 1$

$\Rightarrow X_n + Y_n \xrightarrow{a.s.} a+b. \quad \square$

② proove $X_n / Y_n \xrightarrow{a.s.} a/b \qquad b \neq 0, Y_n \neq 0$

$$|X_n/Y_n - a/b| = \left| \frac{X_n b - a Y_n}{b Y_n} \right|$$

$$= \left| \frac{X_n(b-a) + a(X_n - Y_n)}{b Y_n} \right|$$

since $X_n(b-a) \xrightarrow{a.s.} a(b-a)$, $a(X_n - Y_n) \xrightarrow{a.s.} a(a-b)$.

$X_n(b-a) + a(X_n - Y_n) \xrightarrow{a.s.} 0.$

if $b \neq 0$, we assume $b > 0$ with loss of generality.

$\forall \varepsilon > 0, \exists N > 0$, s.t. $|Y_n - b| < \varepsilon \Rightarrow \exists N > 0$, s.t. $|Y_n| > |b/2|$.

$\Rightarrow P(\exists N > 0$, s.t. $|b| \cdot |Y_n| > |b|^2/2)$

$P(\lim_{n\to\infty} X_n / Y_n = a/b) \geq P(A, B, \forall \varepsilon > 0, \exists N > 0$, s.t. $|X_n/Y_n - a/b| < \frac{\varepsilon}{|b|^2/2})$

$= P(A \cap B) \geq 1$

$\Rightarrow P(\lim_{n\to\infty} X_n / Y_n = a/b)$

$\Rightarrow X_n / Y_n \xrightarrow{a.s.} a/b. \quad \square$

14. assume that

$$A_n = \sum_{i=1}^{n} X_i / n, \qquad B_n = \sum_{i=1}^{n} Y_i / n.$$

$$Z_n = A_n / B_n.$$

$$E(A_n) = E(X), \quad Var(A_n) = \frac{Var(X)}{n},$$

through strong law of large number, we have

$$A_n \xrightarrow{a.s.} E(X), \quad B_n \xrightarrow{a.s.} E(Y).$$

from 13 problem, we have $A_n / B_n \xrightarrow{a.s.} E(X)/E(Y)$.
(almost surely)

15. prove $Y_n \xrightarrow{a.s.} c \Rightarrow Y_n \xrightarrow{P} c$.

$$Y_n \xrightarrow{a.s.} c \iff P(\forall \varepsilon > 0, \exists N > 0, \text{ s.t. } \overset{\forall n > N}{|Y_n - c| < \varepsilon}) = 1$$

$$\cancel{\iff P(\bigcap_{\varepsilon>0} \bigcup_{n=1}^{+\infty} |Y_n - c| < \varepsilon) = 1 \iff \forall \varepsilon > 0, P(\bigcup_{n=1}^{+\infty} |Y_n - c| < \varepsilon) = 1}$$

$$\iff P(\bigcap_{\varepsilon>0} \bigcup_{n=1}^{+\infty} \bigcap_{m=n}^{+\infty} |Y_m - c| < \varepsilon) = 1 \iff \forall \varepsilon > 0, P(\bigcup_{n=1}^{+\infty} \bigcap_{m=n}^{+\infty} |Y_m - c| < \varepsilon) = 1$$

~~Since~~ let $B_n = \bigcap_{m=n}^{+\infty} |Y_m - c| < \varepsilon$, since $B_n \subset B_{n+1}$, for $\forall n > 0$.

$$P(\bigcup_{n=1}^{+\infty} \bigcap_{m=n}^{+\infty} |Y_m - c| < \varepsilon) = P(\lim_{n \to \infty} B_n), \quad P(B_n) \le P(B_{n+1})$$

$$\iff \forall \varepsilon > 0, \lim_{n \to +\infty} P(\bigcap_{m=n}^{+\infty} |Y_m - c| < \varepsilon) = 1.$$

$$Y_n \xrightarrow{P} c \iff \lim_{n \to +\infty} P(|Y_n - c| \geq \varepsilon) = 0,$$

$$\iff \lim_{n \to +\infty} P(|Y_n - c| < \varepsilon) = 1, \text{ for } \forall \varepsilon > 0.$$

since $P(\bigcap_{m=n}^{+\infty} |Y_m - c| < \varepsilon) = 1 \Rightarrow P(|Y_n - c| < \varepsilon) = 1$

$$Y_n \xrightarrow{a.s.} c \Rightarrow Y_n \xrightarrow{P} c. \quad \square$$

16. $E(\sum_{n=1}^{+\infty} Y_n) < +\infty$, $Y_n \geq 0$.

show $P(\lim_{n\to+\infty} Y_n = 0) = 1$.

$\Leftrightarrow \forall \varepsilon > 0$, $\lim_{n\to+\infty} P(\bigcap_{m=n}^{+\infty} |Y_n| < \varepsilon) = 1$.

since $\sum_{i=1}^{\infty} Y_i > 0$, if $P(\sum_{n=1}^{+\infty} Y_n = \infty) > 0$, $E(\sum_{n=1}^{\infty} Y_n) = +\infty$.

we have $P(\sum_{n=1}^{\infty} Y_n = \infty) = 0$,

$P(\sum_{n=1}^{\infty} Y_n < \infty) = 1$.

if we don't have $Y_n \xrightarrow{a.s.} 0$.

$P(\exists \varepsilon_0 > 0, \forall N > 0, \exists n_N > N$, s.t. $|Y_{n_N}| \geq \varepsilon_0) > 0$.

$\Rightarrow P(\exists \varepsilon_0 > 0, \exists \{n_k\}_{k=1}^{+\infty}$, s.t. $|Y_{n_k}| \geq \varepsilon_0) > 0$

$\Rightarrow P(\sum_{k=1}^{+\infty} Y_{n_k} = +\infty) > 0$, it's contradictory.

so $Y_n \xrightarrow{a.s.} 0$. □

17. $\sum_{n=1}^{\infty} p_n < \infty$, to proove $P(\sum_{n=1}^{\infty} X_n < \infty) = 1$.

since $E(\sum_{n=1}^{\infty} X_n) < \infty$, we have $P(\sum_{n=1}^{\infty} X_n < \infty) = 1$.

if not, $E(\sum_{n=1}^{\infty} X_n) > +\infty \cdot P(\sum_{n=1}^{\infty} X_n = +\infty) = +\infty$. □

18. $E(X_i^4) < \infty$, proove $S_n \xrightarrow{a.s.} E(X)$,

$S_n = \dfrac{X_1 + \cdots + X_n}{n}$, $\Leftrightarrow S_n - E(X) \xrightarrow{a.s.} 0$.

without loss of generality, we assume that $E(X_n) = 0$.

$E(S_n^4) = E(\dfrac{X_1^4 + \cdots + X_n^4}{n^4}) + \sum_{i\neq j} E(\dfrac{X_i^2 \cdot X_j^2}{n^4}) + \sum_{\text{others}} E(X_i \cdot X_j \cdot X_p \cdot X_q)$

there must ~~be~~ exist an item with order 1, in $\sum_{\text{others}} E(\cdots)$ for all, hence $\sum_{\text{other}} E(X_i \cdot X_j \cdot X_p \cdot X_q) = 0.$

$$E(X_i^2 \cdot X_j^2) \le E\left(\frac{X_i^4 + X_j^4}{2}\right)$$

$$\Rightarrow E(S_n^4) \le \frac{1}{n^4} \cdot \sum_{i=1}^{n} (E(X_i^4)) + 6(n-1) \sum_{i=1}^{n} E(X_i^4) \cdot \frac{1}{n^4}$$

$$= \frac{6(n-1)n + n}{n^4} \cdot E(X^4)$$

$$= \frac{6n^2 - 5n}{n^4} \cdot E(X^4) < \infty, \quad \text{for any } n, \ \forall n > 0.$$

$$E\left(\sum_{n=1}^{N} S_n^4\right) = \sum_{n=1}^{N} \frac{6n^2 - 5n}{n^4} \cdot E(X^4) < \infty, \quad \text{for } \forall N > 0.$$

$$\Rightarrow$$

$$E\left(\sum_{n=1}^{\infty} S_n^4\right) < \infty,$$

$\Rightarrow S_n^4 \xrightarrow{a.s.} 0$, through 16 problem results.

$\Leftrightarrow S_n \xrightarrow{a.s.} 0$, since $|S_n - 0|^4 < \varepsilon \Leftrightarrow |S_n| < \sqrt[4]{\varepsilon}, \ \forall \varepsilon > 0.$

□

$S_n \xrightarrow{a.s.} 0 \Rightarrow$ strong law of large number holds. □

holds. □

$$S_n \xrightarrow{a.s.} 0 \Leftrightarrow S_n \xrightarrow{a.s.} E(X)$$

$$\Leftrightarrow \frac{X_1 + \cdots + X_n - nE(X)}{n} \xrightarrow{a.s.} 0$$

$$\Leftrightarrow \frac{X_1 + \cdots + X_n}{n} \xrightarrow{a.s.} E(X). \ \square$$

$$\frac{X_1 + \cdots + X_n}{n} \xrightarrow{a.s.} E(X). \ \square$$

This chapter is ended. !!!

Move on, you need to speed up.

More classes you need to learn. □