P119. ~~本~~ Chapter 2.

1. ~~$P(win) = P(A \cap B) =$~~

~~A denotes not losing the first games~~

~~B denotes not losing the second game.~~

~~$P(lose) = 1 - P(A \cap B) =$~~

2、 ~~$P = 1 - \frac{500!}{500}$~~

~~$P = 1$~~ ~~364~~

$P = 1 - (\frac{364}{365})^{500} = 0.746 \sim$

3、 draw 平局， tie 打平

(a) $P(F \text{ wins}) = \sum_{i=1}^{10} 0.4 \cdot 0.3^{i-1}$

(b) $P(\text{duration} = i) = \begin{cases} 0.7 \cdot 0.3^{i-1} & (i = 1, \text{—}, 10). \\ 0 & ~~(i = \text{otherwise})~~ \text{ otherwise.} \end{cases}$

4. (a) ~~$P(modem = i) = \{$~~

(lowercase) ~~$p(modem = i) = \begin{cases} \binom{50}{i} 0.01^i \cdot 0.99^{(50-i)} & i = 0, 1 \text{—} 50 \\ 0 & \text{otherwise} \end{cases}$~~

(lowercase)

(lowercase↓) (uppercase↓)

由 ~~p(mo~~ $p_M(i) = P(modem = i) = \begin{cases} \binom{1000}{i} 0.01^i \cdot 0.99^{(1000-i)} & i = 0, \text{—}, 49 \; \cancel{50} \\ \sum_{i=50}^{1000} 0.01^i \cdot 0.99^{(1000-i)} & i = 50 \end{cases}$

(b) according to 1000 is a high number.

$$p_M(i) = \binom{1000}{i} 0.01^{i}\, 0.99^{1000-i} \qquad (i=0,1,\cdots,49)$$

$$\approx e^{-\lambda}\cdot\frac{\lambda^i}{i!}, \qquad \lambda = 1000\cdot 0.01 = 10$$

$$\begin{cases} e^{-\lambda}\cdot\frac{\lambda^i}{i!} \approx \binom{n}{i}\cdot p^i\cdot(1-p)^{n-i}, & \lambda = np \\ \binom{n}{i}\cdot p^i \approx (np)^i/i! & \\ (1-p)^{n-i} \approx (1-p)^n \approx e^{-np} = e^{-\lambda}, & (1-p)\approx e^{-p} \end{cases}$$

$$p_M(i) \approx e^{-10}\cdot\frac{10^i}{i!} \qquad (i=0,1,\cdots,49)$$

$$p_M(50) \approx \sum_{i=50}^{\infty} e^{-10}\cdot\frac{10^i}{i!} \qquad (i=50)$$

(c) $P = \sum_{i=50}^{\infty} e^{-10}\cdot\frac{10^i}{i!} \approx ?$? Gw, G2.

5.6 (a)

(b) $\cancel{P(n=2k+1) = P(win \geq k+1)}$

$\cancel{= \sum_{i=k+1}^{n} \binom{2k+1}{i} p^i\cdot(1-p)^{n-i} = \sum_{i=k+1}^{2k+1} \binom{2k+1}{i} p^i (1-p)^{2k+1-i}}$

$\cancel{P(n=2k-1) = \sum_{i=k}^{2k-1} \binom{2k-1}{i} p^i (1-p)^{2k-1-i}}$

$\cancel{令\ P(n=2k+1) - P(n=2k-1)}$

令 assume that N is the numbers of winning games in the first (2k-1) games. A is (2k+1) games, B (2k-1) ga

$$P(A) = P(N \geq k+1) + P(N=k)\cdot(1-(1-p)^2) + P(N=k-1)\cdot p^2$$

$$P(B) = P(N \geq k) = P(N \geq k+1) + P(N=k)$$

$$P(A)-P(B) = P(N=k)\cdot(-(1-p)^2) + P(N=k-1)\cdot p^2$$

$$= P(N=k)\cdot(2p-1-p^2) + P(N=k-1)\cdot p^2$$

$$= \binom{2k-1}{k}\cdot p^k\cdot(1-p)^{k-1}\cdot(2p-1-p^2) + \binom{2k-1}{k-1} p^{k-1}\cdot(1-p)^k\cdot p^2$$

$$= \binom{2k-1}{k} p^{k-1}\cdot(1-p)^{k-1}\left(p(2p-1-p^2) - p^2(1-p)\right)$$

$$\cancel{= \binom{2k-1}{k} p^{k-1}\cdot(1-p)^{k-1}\cdot(2p^2-p^3-1)}$$

$$\cancel{\text{if } 2p^2-p^3-1>0,\ (p^2-p-1)(1-p)}$$

$$= \binom{2k-1}{k} p^{k-1}\cdot(1-p)^{k-1}\underline{\left(-p(1-p)^2+p^2(1-p)\right)}$$

$$= (1-p)p\cdot(2p-1)$$

if $p > \frac{1}{2}$, $P(A) > P(B)$. □ ✓

~~7. (a) P(A) =~~

8. $p_X(k) = \binom{n}{k}\cdot p^k\cdot(1-p)^{n-k}$

$$\frac{p_X(k+1)}{p_X(k)} = \frac{\binom{n}{k+1}\cdot p^{k+1}\cdot(1-p)^{n-k-1}}{\binom{n}{k}\cdot p^k\cdot(1-p)^{n-k}}$$

$$= \frac{(n-k-1)\cdot p}{(k+1)\cdot(1-p)}$$

9. $k^* \le (n+1)p$, $k^* = [(n+1)p]$, 且 k^* ~~都~~ $\le np$, and k^* 且 $\ge n$

$-1+np \le k^* \le np$

$p_X(k) = \binom{n}{k}\cdot p^k\cdot(1-p)^{n-k}$

if $\frac{p_X(k+1)}{p_X(k)} = \frac{(n-k-1)\cdot p}{\ldots} \ge 1$, $np-pk-p \ge -pk-p+k+1 \Rightarrow np \ge k+1$?

10. ~~[illegible]~~ $p_X(k) = e^{-\lambda}\cdot\frac{\lambda^k}{k!}$.

if $\frac{p_X(k)}{p_X(k+1)} = \frac{\lambda^k/k!}{\lambda^{k+1}/(k+1)!}$

$= \frac{k+1}{\lambda} \geq 1$

$\lambda \leq (k+1)$, $k \geq \lambda - 1$.

$k^* \geq \lambda - 1$, and $k^* \leq \lambda$

$\lambda \geq k^* \geq \lambda - 1$.

$k^* = [\lambda]$ satisfies.

11. X denotes the number of remaining matches.
(a)

$p_X(k) = \cancel{P_L(0)}\ P_L(k) + P_R(k) = 2P_L(k)$, L denotes the number of left pockets.

$\cancel{p_L(k) = \binom{2n}{k}\cdot(\tfrac{1}{2})^k\cdot(\tfrac{1}{2})^{2n-k}}$

$\cancel{= \binom{2n}{k}\cdot(\tfrac{1}{2})^{2n}}$.

$\cancel{p_X(k) = 2\cdot\binom{2n}{k}\cdot(\tfrac{1}{2})^{2n} \quad (k=0,1,\cdots,n)}$

$\cancel{p_L(k) = \binom{2n+k}{k}\cdot(\tfrac{1}{2})^k\cdot(\tfrac{1}{2})^n}$

$p_L(k) = \binom{2n-k}{n}\cdot(\tfrac{1}{2})^n\cdot(\tfrac{1}{2})^{n-k}$

$= \frac{1}{2}\binom{2n-k}{n}\cdot(\tfrac{1}{2})^{2n-k}$, $k = 0, 1, \cdots, n$.

$p_X(k) = 2p_L(k) = \binom{2n-k}{n}\cdot(\tfrac{1}{2})^{2n-k}$

since if we choose A pocket as the remaining k matches pocket, the last match must from B pocket. That's why multiply 1/2.

(b) $P_X(k) = p\cdot\binom{2n-k}{n}\cdot p^n\cdot(1-p)^{n-k} + (1-p)\binom{2n-k}{n}\cdot p^{n-k}\cdot(1-p)^n$

$= \binom{2n-k}{n}\cdot\left(p^{n+1}\cdot(1-p)^{n-k} + p^{n-k}\cdot(1-p)^{n+1}\right)$

$k = 0, 1, \ldots, n.$

When A pocket remains 0 match and

B pocket ~~reains~~ remains (k) matches,

the next ~~match~~ pocket mathematician chooses must be A pocket with probability p. ~~then~~

12. ~~$P_B(x)$~~

$p_B(k) = \binom{n}{k}\cdot p^k\cdot(1-p)^{n-k}$

$P_P(k) = e^{-\lambda}\cdot\frac{\lambda^k}{k!}$.

for $\forall$ fixed k,

$p_B(k) = \binom{n}{k}\cdot p^k\cdot(1-p)^{n-k}$

$= \frac{n!}{k!\cdot(n-k)!}\cdot p^k\cdot(1-p)^{n-k}$.

~~$\binom{n}{k} \to n^k$ as $(n\to+\infty)$.~~

$\binom{n}{k}/n^k \to 1$ as $n\to+\infty$.

$\frac{(1-p)^{n-k}}{(1-p)^n} \to 1$ as $n\to+\infty$

so $\frac{P_B(k)}{P_P(k)} \to 1$ as $n\to+\infty$, $p\to0^+$, fix λ. $\square$

since fix $\lambda = np$, $n\to+\infty$, $p\to0^+$,

$(1-p)^n = (1-\frac{\lambda}{n})^n \to e^{-\lambda}$ as $n\to+\infty$

13. $P_G(x) = \begin{cases} \binom{5}{x-2} \cdot \left(\frac{1}{2}\right)^5 & \text{if } x = 2, 3, \cdots, 7 \\ 0 & \text{otherwise.} \end{cases}$

14. (a) $P_X(x) = \frac{1}{10}$ $x = 0, \cdots, 9$

$Y = X \bmod (3)$

$P_Y(y) = P(Y = y) = P(X (\bmod 3) = y)$

$$= \begin{cases} P(X=0) + P(X=3) + P(X=6) + P(X=9) & \text{if } y = 0 \\ P(X=1) + P(X=4) + P(X=7) & \text{if } y = 1 \\ P(X=2) + P(X=5) + P(X=8) & \text{if } y = 2 \\ 0 & \text{otherwise} \end{cases}$$

(b) $P_Y(y) = P(Y = y) = P(5 \bmod (X+1) = y)$

$$= \begin{cases} P(X=0) + P(X=4) & \text{if } y = 0 \\ P(X=1) + P(X=3) & \text{if } y = 1 \\ P(X=2) & \text{if } y = 2 \\ P(X=5) + P(X=6) + \cdots + P(X=9) & \text{if } y = 5 \\ 0 & \text{otherwise.} \end{cases}$$

$X \to$	Y
0	0
1	1
2	2
3	1
4	0
5-9	5

15. $P_K(x) = \frac{1}{2n+1}$, $x \in [-n, n]$, $x \in \mathbb{Z}$

$P_X(x) = P(a^{|K|} = x)$

$P_Y(y) = P(\ln X = y) = P(\ln(a^{|K|}) = y) = P(|K| \cdot \ln(a) = y)$

$= P\left(|K| = \frac{y}{\ln(a)}\right)$

if $|K| = 0$, $y = 0$. $P_Y(0) = \frac{1}{2n+1}$, otherwise ~~otherwise~~, $P_Y(y) = 0$.

if $|K| \neq 0$, $0 < |K| \leq n$, $y = \ln(a) \cdot |K|$, $P_Y(\ln(a) \cdot |k|) = \frac{2}{\ldots}$ $|k| = 1, \cdots, n$

16\.

(a) $\sum_{x=-3}^{3} p_X(x) = 1 \Rightarrow a = 28$

$$E(X) = \sum_{x=-3}^{3} x \cdot p_X(x) = 0$$

(b) $Z = (X-0)^2$

$$p_Z(z) = P(Z=z) = P(X^2 = z) \quad (\text{PMF})$$

$$= \begin{cases} \frac{9}{14} & z=9 \\ \frac{2}{7} & z=4 \\ \frac{1}{14} & z=1 \\ 0 & \text{otherwise} \end{cases}$$

(c) $Var(X) = E((X-E(X))^2)$

$= E(Z)$

$= \frac{9}{14} \cdot 9 + \frac{2}{7} \cdot 4 + \frac{1}{14} \cdot 1$

$= 7$

Thm: if $Z = f(X)$, $E(Z) = E(f(X))$

~~Prof~~ Proof: $E(Z) = \sum_z z \cdot p_Z(z)$

$\cancel{= \sum_z f(x)}$

$p_Z(z) = P(Z=z) = P(f(X)=z)$

$= p_X(f^{-1}(z)) = p_X(x)$

$E(Z) = \sum_x f(x) \cdot p_X(f^{-1}(z))$

$= \sum_x f(x) \cdot p_X(x)$

$= E(x)$.

$f(X) = Z$

$f(x) = z$.

$P(Z=z) = P(f(X)=x)$

(d) $Var(X) = (-3)^2 \cdot$ ~~~

Thm: $E(Z) = E(f(X))$, if

18. $p_X(x) = \begin{cases} \frac{1}{b-a+1} & \text{if } x = 2^k, \text{ for } a \le k \le b. \\ 0 & \text{otherwise} \end{cases}$

$$E(X) = \sum_{k=a}^{b} p_X(x) \cdot 2^k$$

$$= \frac{2^a \cdot (1 - 2^{b-a+1})}{1-2} \cdot \frac{1}{b-a+1}$$

$$= \frac{2^{b+1} - 2^a}{b-a+1}$$

$$E(X^2) = \frac{4^{b+1} - 4^a}{3(b-a+1)}$$

$$\text{Var}(X) = E(X)^2 - E(X^2)$$

$$= \left(\frac{2^{b+1} - 2^a}{b-a+1}\right)^2 - \frac{4^{b+1} - 4^a}{3(b-a+1)}$$

19. (a) set X ~~set~~ denotes the number of questions.

$p_X(k) = \frac{1}{10}$ for $\forall\, k = 1, 2, \cdots, 10.$

$E(X) = 5.5$

(b) if prize is in 1st boxes, we need 4 questions.

① $k=5$, ② $k=3$ ③ $k=2$ ④ $k=1$

as following, we ~~need to ask~~ have the function:

number of questions	→ index of prize
4	1, 2, 6, 7
3	3, 4, 5, 8, 9, 10

$E(X) = 4 \cdot \frac{4}{10} + 3 \cdot \frac{6}{10} = 3.4.$

20.

21. X denotes the number of tosses.

$p_X(k) = (\frac{1}{2})^k$

$E(X) = \sum_{k=1}^{\infty} (\frac{1}{2})^k \cdot 2^k = +\infty$

a man can accept any price to play the game, since the gain expectation is infinity.

22. (a) X denotes the times of tossing.

~~$p_X(k) =$~~ $p_X(1) = p(1-q) + (1-p)\cdot q$

$p_X(k) = (1 - p_X(1))^{k-1} \cdot p_X(1) \qquad k = 1, 2 \cdots$

$E(X) = \sum_{k=1}^{\infty} k \cdot (1-p_X(1))^{k-1} \cdot p_X(1)$

$(1-p_X(1)) \cdot E(X) = \sum_{k=1}^{\infty} k \cdot (1-p_X(1))^{k} \cdot p_X(1)$

$= \sum_{k=2}^{\infty} (k-1) \cdot (1-p_X(1))^{k-1} \cdot p_X(1)$

$E(X) - (1-p_X(1)) E(X) = p_X(1) \cdot E(X) = (1-p_X(1))^0 \cdot p_X(1) + \sum_{k=2}^{\infty} (1-p_X(1))^{k-1} p_X$

$\Rightarrow E(X) = \frac{1}{p_X(1)} = \frac{1}{p(1-q)+q(1-p)}$

$E(X^2) = \sum_{k=1}^{\infty} k^2 (1-p_X(1))^{k-1} \cdot p_X(1)$

$E(X^2) - (1-p_X(1)) E(X^2) = p_X(1) E(X^2) = p_X(1) \left(1 + \sum_{k=2}^{\infty} (2k-1) \cdot (1-p_X(1))^{k-1}\right)$

$E(X^2) = 1 + \sum_{k=1}^{\infty} (2k+1)(1-p_X(1))^k$

$= \frac{1}{p_X(1)} + \frac{2}{p_X(1)^2}$.

$Var(X) = E(X^2) - E(X)^2 = \frac{1}{p_X(1)} + \frac{1}{p_X(1)^2}$. ?

~~$Var(X) = \frac{1}{p_X(1)^2} - \left(\frac{1}{p_X(1)} + \frac{2}{p_X(1)^2}\right) = -\frac{1}{p_X(1)^2} - \frac{1}{p_X(1)}$~~

(b) P(first coin is head | last tose))

$$= \frac{(p(1-q))\cdot(1-P_X(1))^{k-1}}{(p(1-q)+q(1-p))\cdot(1-P_X(1))^{k-1}}$$

$$= \frac{P(\text{first coin is head, last tose})}{P(\text{last tose})}$$

23. (a) assume that X is the times of tosing.

$p_X(k) = (\frac{1}{2})^k \cdot 2 = (\frac{1}{2})^{k-1} \quad k \geq 2$

$$p_X(k) = \begin{cases} (\frac{1}{2})^{k-1} & k \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

$E(X) = $ — , $Var(X) = $ -

~~E~~ set Y, $P_Y(k) = (\frac{1}{2})^k \quad k \geq 1.$

$\Rightarrow Y = X - 1, \quad X = Y+1$

~~E~~) $E(X) = E(Y) + 1 = 3$

$Var(X) = Var(Y) = 2.$

(b) ~~若~~ we will tose all head, then all tail, the next tail, will be the situation we want.

$p_X(k) = \frac{k-1}{2^k} \quad k \geq 2,$

$E(X) = \sum_{k=2}^{\infty} k \cdot \frac{k-1}{2^k} = \sum_{k=1}^{\infty} (k^2+k)(\frac{1}{2})^{k+1}$ ~~...~~ $= \sum_{k=1}^{\infty} (k^2-k)(\frac{1}{2})^k$

$= Var(Y) + E(Y)^2 - E(Y) = 2+4-2 = 4.$

24. easy

25. ~~easy~~

~~26~~ (a) $P_{I,J}(i,j) = \begin{cases} \dfrac{1}{\sum_{k=1}^{n} m_k} & \text{for } i \le n \text{ and } j \le m_i \\ 0 & \text{otherwise} \end{cases}$

marginal PMF: $P_I(i) = \sum_{j=1}^{m} P_{I,J}(i,j) = \dfrac{m_i}{\sum_{k=1}^{n} m_k}$

$P_J(j) = \sum_{i=1}^{n} P_{I,J}(i,j) = \dfrac{\text{card}\{i \mid m_i \le j\}}{\sum_{k=1}^{n} m_k}$

(b) $P_{J|I}(j|i) = \dfrac{P_{I,J}(i,j)}{P_I(i)} = \dfrac{1}{m_i}$ for $i \le n$ and $j \le m_i$.

$\cancel{E(c \mid I=i) = \sum_{j=1}^{m} P_{J|I}(j|i) \cdot E \cdot j}$

(c is the points including a and b)

~~(since c is~~ $\cancel{= \sum_{j=1}^{m} P}$

~~set A that the ~~correctness~~ points of the (i,j) question~~

$\cancel{P_{A|I,J}(a|i,j) = p_{i,j}}$

a denotes the correctness.

set $f(i,j,a)$ that the scores of the (i,j) question

$P_{J|I}(j|i) = \dfrac{P_{I,J}(i,j)}{P_I(i)} = \dfrac{1}{m_i}$ for $i \le n$ and $j \le m_i$

$E(f(i,j,a) \mid I=i) = \cancel{\sum_{j=1}^{m} f(i,j,a) \cdot P_{J|I}(j|i) =}$

$= \sum_{j=1}^{m} \sum_{a} f(i,j,a)\, P_{J|I}(j,a|i)$

$= \sum_{j=1}^{m_i} \frac{1}{m_i} (p_{ij} \cdot a + (1-p_{ij}) \cdot b).$

26.

(a) $X = min(X_1, X_2, X_3)$

$$P_X(X=x) = P(\min(X_1, X_2, X_3) = x)$$

$$= P(X_1, X_2, X_3 \geq x) - P(X_1, X_2, X_3 \geq x+1)$$

(for x in $[101, 110]$)

$$= P(X_1 \geq x) \cdot P(X_2 \geq x) \cdot P(X_3 \geq x) - P(X_1 \geq x+1) \cdot P(X_2 \geq x+1) \cdot P(X_3 \geq x+1);$$

$$= \left(\frac{111-x}{10}\right)^3 - \left(\frac{110-x}{10}\right)^3$$

$$\cancel{P_X(X=} \; P_X(x) = \begin{cases} \left(\frac{111-x}{10}\right)^3 - \left(\frac{110-x}{10}\right)^3 & \text{for } x \text{ in } [101, 110],\ x \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

(b) $E(X) = \sum_{x=101}^{110} P_X(x) \cdot x = \sim\sim .$ $\square$

27. (a) $P_{X_1 X_2 \cdots X_r}(k_1, \cdots, k_r) = \binom{n}{k_1\ k_2 \cdots k_r} p_1^{k_1} \cdots p_r^{k_r}$.

otherwise, $P_{X_1 \cdots X_r}(k_1 \cdots k_r) = 0$.

(b) $P_{X_i}(k_i) = \binom{n}{k_i} p_i^{k_i} \cdot (1-p_i)^{1-k_i}$.

$$E(X_i) = \sum_{k_i=1}^{n} \binom{n}{k_i} \cdot k_i \cdot p_i^{k_i} (1-p_i)^{1-k_i}.$$

$$= np_i$$

$$Var(X_i) = np_i(1-p_i)$$

(c) $E(X_i, X_j) = E(X_i \cdot X_j) = \sum_{k_i} \sum_{k_j} k_i \cdot k_j \cdot P_{X_i X_j}(k_i, k_j)$

Let $Y_{i,k}$ be the RV that takes 1 if the dice roll the i face on

$\sum_{k=1}^{n} Y_{i,k} = E_i$, and $Y_{i,k}$ is Bernoulli distribution.

$E(X_i X_j) = E((Y_{i,1} + \cdots + Y_{i,n})(Y_{j,1} + \cdots + Y_{j,n}))$

$\cancel{= (n^2 - n) E(}$ since $E(Y_{i,t} \cdot Y_{j,t}) = 0$.

$= (n^2 - n) E(Y_{i,k_i} \cdot Y_{j,k_j})$ $k_i \neq k_j$, $i \neq j$.

$= (n^2 - n) E(Y_{i,k}) E(Y_{j,k})$

$= (n^2 - n) \cdot p_i \cdot p_j$.

28. assume that L is the order of quiz problems.

set $L_1 = (i_1, i_2, \cdots, i_n)$

$L_2 = (i_1, i_2, \cdots i_{k+1}, i_k, \cdots i_n)$ change the position of i_k, i_{k+1} with L_1.

assume that i_1 takes value 1 if it's correct.

$P(L_1 = (1, 1, \cdots, 1))$ means the prob. that all problems are correct.

$\cancel{E(L_1) =}$ $\cancel{E(L_1 \cdot V) =}$ $V(L_1)$ means the reward of L_1 order.

$\cancel{E(L_1 \cdot V_1) =}$ $E(V(L_1)) = p_{i_1} \cdot V_{i_1} + p_{i_1} \cdot p_{i_2} \cdot V_{i_2} + \cdots + \prod_{t=1}^{n} p_{i_t} \cdot V_{i_n}$.

$E(V(L_2)) = p_{i_1} \cdot V_{i_1} + \cdots + \prod_{t=1}^{k+1} p_{i_t} \cdot V_{i_{k+1}} + \prod_{t=1}^{k} p_{i_t} \cdot V_{i_k} + \cdots + \cdots$

if $E(V(L_1)) \geq E(V(L_2))$,

$\Leftrightarrow p_{i,k+1} \cdot V_{i,k+1} + p_{i,k} \cdot p_{i,k+1} \cdot V_{i,k} \leq p_{i,k} \cdot V_{i,k} + p_{i,k} \cdot p_{i,k+1} \cdot V_{i,k+1}$

$\Leftrightarrow p_{i_{k+1}} \cdot V_{i_{k+1}} - p_{i_{k+1}} \cdot V_{i_{k+1}} \cdot p_{i,k} \leq p_{i,k} \cdot V_{i_k} (1 - p_{i_{k+1}})$

$$\Leftrightarrow \quad \frac{p_{ik+1} \cdot v_{ik+1}}{1-p_{ik+1}} \le \frac{p_{ik} \cdot v_{ik}}{1-p_{ik}}$$

$$\Leftrightarrow \quad E(V(L_2)) \le E(V(L_1))$$

we can get information that the problem with larger $\frac{p_i \cdot v_i}{1-p_i}$ should place on the prior position

so the maximum $E(V(L))$ should choose the problem on ~~the order of~~ a nonincreasing order of $\frac{p_i v_i}{1-p_i}$.

29.

$$P((1-A_1)(1-A_2)\cdots(1-A_n))$$

$$= 1 - P(\cup A_k)$$

$$= 1 - \sum P(A_i) + \sum P(A_{i_1} \cap A_{i_2}) + \cdots + (-1)^n P(\cap A_i)$$

$$\Rightarrow P(\cup A_k) = \sum P(A_i) - \sum P(A_{i_1} A_{i_2}) + \cdots + (-1)^{n+1} P(\cap A_i)$$

30. Let A_i be the event that ith friend accept the correct card

$$A_i = \begin{cases} 1 & \text{accept} \\ 0 & \text{no} \end{cases}$$

we should calculate $P(\bigcup_i A_i)$

$$P(\bigcup_i A_i) = \sum P(A_i) - \sum P(A_{i_1} A_{i_2}) + \cdots + (-1)^{n-1} P(A_{i_1} \cdots A_{i_n})$$

$$= \binom{n}{1} \cdot \frac{1}{n} - \binom{n}{2} \cdot \frac{1}{n} \cdot \frac{1}{n-1} + \cdots + (-1)^{n-1} \cdot \binom{n}{n} \frac{1}{n} \cdots \frac{1}{1}$$

$$= \frac{1}{1!} - \frac{1}{2!} + \cdots + (-1)^{n-1} \cdot \frac{1}{n!} \to (1-e^{-1}) \text{ as } (n \to +\infty).$$

31.

$$p_{X,Y}(x,y) = \begin{cases} \binom{4}{x\ \ y\ \ 4-x-y} \left(\frac{1}{6}\right)^{x+y} \cdot \left(\frac{4}{6}\right)^{4-x-y} & \text{if } 4-x-y \geq 0,\ x,y \geq 0. \\ 0 & \text{otherwise} \end{cases}$$

, otherwise

33. H denotes head, T denotes tails.

so the situation of tossing is like

HTHT---THH or THTH---HTT

or HTHT---HTT or THTH---THH

if the last toss is H or T, the other tosses are confirmed

$p_X(x) = \cancel{2 \cdot (\frac{1}{2})^{x} = (\frac{1}{2})^{x-1}}$.

$p_{X|H}(x) \cdot p(\text{last coin is } H) + p_{X|T}(x) \cdot p(T)$

$E(X) = P(H) \cdot E(X|H) + P(T) \cdot E(X|T)$

$= p \cdot E(X|H) + (1-p)E(X|T)$

$E(X|H) = \sum_{k=1}^{\infty} \overset{(2k)}{\cancel{(k+1)}} \cdot \cancel{4}\, p^{k+1} \cdot (1-p)^{k-1} + \cancel{(k+1)}\, p^{k} \cdot (1-p)^{k-1}$

since $(1-p)^{k-1} \cdot (k+1) \to 0$ as $k \to \infty$, and monototically decrease.

and $\sum_{k=1}^{\infty} p^{k+1}$ is converge,

we get $E(X|H)$ is converge.

~~E(X|H) = assume that~~

~~E(X|H)~~ assume that Y is like HTHT--- or THTH---, each adjacent two dice is different. Y is number of ~~?~~. tosses

~~H E(Y|H) = E(X|H), ? E(Y|T) = E(X|T)~~

~~E(Y|H) =~~

$1+E(Y|H) = E(X|H)$, $1+E(Y|T) = E(X|T)$

$$E(Y|T) = \sum_{y=1}^{\infty} P_{Y|T}(y|T)\cdot y$$

$$= \sum_{y=1}^{\infty} P_{X|H}(y-1|H)\cdot(1-p)\cdot y$$

$$= 1+\cancel{p}\ (1-p)\ E(Y|H)$$

$$E(Y|H) = 1+p\,E(Y|T)$$

$$E(X|H) = \sum_{k=1}^{\infty}(2k)\,p^{k+1}(1-p)^{k-1} + \sum_{k=1}^{\infty}(2k-1)\,p^{k}(1-p)^{k-1}$$

$$S \triangleq \sum_{k=1}^{\infty} k\cdot p^{k}\cdot(1-p)^{k} = \frac{p(1-p)}{(1-p(1-p))^2} = \frac{pq}{(1-pq)^2}$$

$$E(X|H) = \frac{2p}{1-p}\cdot S + \frac{2}{1-p}\cdot S - \frac{p}{1-p(1-p)}$$

$$= \frac{p(-p^2+3p+1)}{(1-p(1-p))^2} = \frac{p(2p+1+pq)}{(1-pq)^2}$$

$$E(X|T) = \frac{q(2q+1+pq)}{(1-pq)^2}$$

$$E(X) = p\,E(X|H) + q(E(X|T) = \frac{(1+pq)(p^2+q^2)+2(p^3+q^3)}{(1-pq)^2}$$

做错了,

wrong?

34. Let T be the time. sipder lands on the fly.
Ad: initially spider are d units apart from fly.
Bd: after 1 second, spider are d units apart from fly.

compute $E(T|Ad)$

$$Ad = (Ad\cap Bd)\cup(Ad\cap B_{d-1})\cup(Ad\cap B_{d-2}) \quad (d\geq 2)$$

~~$P(Ad\cap Bd)=p$, $P(Ad\cap Bd-1)$~~ $A_1 = (A_1\cap B_1)\cup(A_1\cap B_0)$

~~$E(T)=E(T$~~ $E(T|A_1) = E(T|A_1B_1)\cdot P(B_1|A_1) + E(T|A_1B_0)\cdot P(B_0|A_1)$

$$= 2p\cdot E(T|A_1B_1) + (1-2p)E(T|A_1B_0)$$

34. $E(T|A_1 B_1) = 1 + E(T|A_1)$,

$E(T|A_1 B_0) = 1$

$\Rightarrow E(T|A_1) = 2p(1 + E(T|A_1)) + (1-2p)$,

$E(T|A_1) = \frac{1}{1-2p}$.

if $d \geq 2$,

$$E(T|A_d) = E(T|A_d \cap B_d) \cdot P(B_d|A_d) + E(T|A_d \cap B_{d-1}) \cdot P(B_{d-1}|A_d)$$
$$+ E(T|A_d \cap B_{d-2}) \cdot P(B_{d-2}|A_d)$$

$E(T|A_d \cap B_d) = E(T|A_d) + 1$

$E(T|A_d \cap B_{d-1}) = E(T|\cancel{B} A_{d-1}) + 1$

$E(T|A_d \cap B_{d-2}) = E(T|A_{d-2}) + 1$

$E(T|A_d \cap B_0) = 1$, $\cancel{\Rightarrow E(T|A_d) =}$

$$E(T|A_d) = p(E(T|A_d) + 1) + (1-2p)(1 + E(T|A_{d-1})) + p(1 + E(T|A_{d-2})),$$

$$= 1 + p\,E(T|A_d) + (1-2p)\,E(T|A_{d-1}) + p\,E(T|A_{d-2})$$

$$\begin{cases} E(T|A_1) = \frac{1}{1-2p}. \\ E(T|A_0) = 0 \end{cases}$$

$$\Rightarrow (1-p)\,E(T|A_d) = 1 + (1-2p)\,E(T|A_{d-1}) + p\,E(T|A_{d-2})$$

$$\Rightarrow E(T|A_d) - E(T|A_{d-1}) = 1 + (-1)^d \cdot \left(\frac{p}{1-p}\right)^{d-2} \cdot \left(\frac{2p}{1-2p}\right)$$

$(d \geq 2)$

$$\Rightarrow E(T|A_d) = d - 1 + E(T|A_1) - \frac{2p(1-p)}{1-2p} \cdot \left(\frac{p}{p-1}\right)^{d-1}$$

$$E(T) = \sum_d E(T|A_d) \cdot P(A_d)$$

$= \sim\sim\sim$. □

35. $E(g(X,Y)) = \cancel{\sum_y g(x,y)} \sum_g P_a(g) \cdot g(X,Y)$

$= \sum_y E(g(X,Y) \mid Y=y) \cdot p_Y(y)$

$= \sum_y E(g(X,y)) \cdot p_Y(y)$

$\cancel{= \sum_y p_Y(y) \cdot \sum_x E(g(x,y) \mid X=x) \cdot p_X(x)}$

$\cancel{= \sum_y}$

$= \sum_y \cdot p_Y(y) \cdot E(g(X, \cancel{y}\,Y) \mid Y=y)$

$= \sum_y p_Y(y) \cdot \sum_x E(g(x,y) \mid X=x, Y=y) \cdot p_{X|Y}(x|y)$

$= \sum_y \sum_x p_{X,Y}(x,y) \cdot E(g(x,y))$

$= \sum_x \sum_y p_{X,Y}(x,y) \cdot g(x,y).$

36. (a) $p_{X,Y,Z}(x,y,z) = p_X(x) \cdot p_{Y,Z|X}(y,z|x)$

$= p_X(x) \cdot p_{Z|X,Y}(z|x,y) \cdot p_{Y|X}(y|x)$

(b) the formula (PMF)

$p_{X,Y,Z}(x,y,z) = p_X(x) \cdot p_{Y|X}(y|x) \cdot p_{Z|X,Y}(z|x,y)$

can be written as

$P(X=x, Y=y, Z=z) = P(X=x) \cdot P(Y=y \mid X=x) \cdot P(Z=z \mid X=x, Y=y)$

(c) $p_{X_1, \ldots X_n}(x_1 \ldots x_n) = p_{X_1}(x_1) \cdot p_{X_2|X_1}(x_2|x_1) \ldots p_{X_k|X_1, \ldots X_{k-1}}(\ldots)$

37. assume that X denotes the number of 1s,

Y denotes the number of transmission.
transmissions

$$P_Y(y) = \frac{\lambda^y}{y!} \cdot e^{-\lambda} \quad (y \geq 0).$$

$$P_{X|Y}(x|y) = \binom{y}{x} p^x \cdot (1-p)^{y-x}.$$

$$P_{X,Y}(x,y) = \binom{y}{x} p^x (1-p)^{y-x} \cdot \frac{\lambda^y}{y!} \cdot e^{-\lambda}$$

$$P_X(x) = \sum_{y=0}^{\infty} \binom{y}{x} p^x \cdot (1-p)^{y-x} \cdot \frac{\lambda^y}{y!} e^{-\lambda}$$

$$= \sum_{y=x}^{\infty} \binom{y}{x} p^x \cdot (1-p)^{y-x} \cdot \frac{\lambda^y}{y!} e^{-\lambda}$$

$$= \sum_{y=x}^{\infty} \frac{1}{x!(y-x)!} p^x \cdot (1-p)^{y-x} \cdot \lambda^y \cdot e^{-\lambda}.$$

$$= e^{\lambda(1-p)} \cdot e^{-\lambda} \cdot \frac{p^x \cdot \lambda^x}{x!}$$

$$= e^{-\lambda p} \cdot \frac{p^x \cdot \lambda^x}{x!}$$

$\Rightarrow X \sim \text{Poisson}(\lambda p)$.

39. let X_i be the ~~egg~~ RV of eggs each day.

$P_{X_i}(x) = \frac{1}{6}$, $x = 1, 2, \cdots, 6$, $i = 1, 2, \cdots 10$.

~~$P_X(x) =$~~ $X = \sum_{i=1}^{10} X_i$

$E(X) = 10 E(X_i) = 35$

$Var(X) = 10\, Var(X_i) =$ ~~~

40. assume that X_i denotes RV of ~~tossing~~ papers ~~times~~ to get the ith different grades

$$E(X_i) = E(X_i \mid \text{first paper get different grade}) \cdot \frac{6-(i-1)}{6}$$

$$+ E(X_i \mid \text{first paper } \cancel{\text{get}} \text{ didn't get different grade}) \cdot \frac{i-1}{6}$$

$$= \frac{6-(i-1)}{6} \cdot (1) + \frac{i-1}{6}(1 + E(X_i))$$

$$\Rightarrow E(X_i) = \frac{6}{7-i}, \quad (i \geq 1)$$

$$E(X) = \sum_{i=1}^{6} E(X_i) = \frac{6}{6} + \frac{6}{5} + \cdots + \frac{6}{1} \approx 14.7$$

41. (a) 250 days to tally.

$$E(X) = 250 \cdot p = 5.$$

$$P(X=5) = \binom{250}{5} \cdot p^5 \cdot (1-p)^{245}$$

$$= 0.1773.$$

(b) $p(x=5) = \binom{n}{5} p^5 (1-p)^n \rightarrow \frac{(np)^5}{5!} \cdot e^{-np}$ as $n \to \infty$

$$\approx \frac{5^5}{5!} \cdot e^{-5} \qquad \lambda = np = 5$$

$$\approx 0.1755.$$

(c) set Y_i be the tickets of ~~a~~ ith day. $1 \leq i \leq 250$

$$P_{Y_i}(y) = \begin{cases} 0.98 & y=0 \\ 0.02 \times 0.5 & y=10 \\ 0.02 \times 0.3 & y=20 \\ 0.02 \times 0.2 & y=50 \end{cases}$$

$Var(Y) = 250 \cdot Var(Y_i)$

$E(Y_i) = 0.42$, $E(Y) = 250 \cdot E(Y_i) = \sim$, $Var(Y_i) = \sim$

(d)

5 times the standard deviation. $= 5 \cdot \sqrt{250p(1-p)}$

? $|(p-\hat{p})| \leq 5\sqrt{250p(1-p)}$.

42. (a) $P(X_i=1) = Area(S)$

$E(X_i) = Area(S)$

$Var(X_i) = Area(S) - Area(S)^2$

$E(S_n) = E(X_i) = Area(S)$

$Var(S_n) = \frac{Var(X_i)}{n} \to 0$ as $n \to \infty$.

(b) $S_n = \frac{n-1}{n} \cdot S_{n-1} + \frac{1}{n} X_n$

(c) ① Generate $X_1, X_2, \cdots, X_{10000}$. as random Variable in $[0,2] \times [0,2]$

② Calculate $S_1 - S_{10000}$, through formula (b).

③ Compute S_n as approximate value of π.

$E(S_n) = \pi$, $Var(S_n) \to 0$ as $n \to +\infty$.

$$X_i = \begin{cases} 1 & \text{if } |X_i - (1,1)| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(d) ~~cos x + sin xy~~

43. $P(X=i) = P(Y=i) = p \cdot (1-p)^{i-1}, \forall i \geq 1$.

$P(X=i \mid X+Y=n) = P(X=i, Y=n-i / X+Y=n)$

$P(X=i, Y=n-i, X+Y=n) = p^2 \cdot (1-p)^{i-1} \cdot (1-p)^{n-i-1}$

$= p^2 \cdot (1-p)^{n-2}$

Therefore $P(X=i, Y=n-i)$ are identical for $i=1, 2, \ldots, n-1$

$$P(X+Y=n) = \sum_{i=1}^{n-1} P(X=i, Y=n-i)$$

$$\Rightarrow P(X=i, Y=n-i) = \frac{1}{n-1}.$$

$$\Rightarrow P(X=i, Y=n-i / X+Y=n) = \frac{1}{n-1}.$$

44. X, Y are independent. set $U = f(x)$, $V = g(y)$

$$P_{X,Y}(x,y) = P_X(x) \cdot P_Y(y).$$

~~$P_{X,Y}(g(x), h(Y)) =$~~

$$P_{U,V}(u,v) = P(U=u, V=v) = \sum_{\substack{f(x)=u \\ g(y)=v}} P_{X,Y}(x,y)$$

$$= \sum_{\substack{f(x)=u \\ g(y)=v}} P_X(x) \cdot P_Y(y)$$

$$= \sum_{f(x)=u} P_X(x) \cdot \sum_{g(y)=v} P_Y(y)$$

$$= \cancel{P(f(x)=X=x)}$$

$$P(f(X)=u) \cdot P(g(Y)=v)$$

$$= P(U=u) \cdot P(V=v).$$

so U, V are independent.

45. (a) $P_{X_i}(x_i) = \begin{cases} p_i & x_i = 1 \\ 1-p_i & x_i = 0 \end{cases}$

$E(X_i) = p_i,$

$E(X) = \sum_{i=1}^{n} p_i / n = \mu.$

~~$Var(X) = \frac{1}{n^2} \cdot E((\sum_{i=1}^{n}(X_i - \mu))^2) \geq 0,$~~

~~if and only if~~ $Var(X) = \frac{1}{n^2} \sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2} \sum_{i=1}^{n} (p_i - p_i^2)$

$$= \frac{1}{n^2}\left(n\mu - \sum_{i=1}^{n} p_i^2\right)$$

$$Var(X) = \frac{1}{n^2}(n\mu - \sum_{i=1}^{n} p_i^2)$$

$$\cancel{\geq \frac{1}{n^2}(n\mu - \frac{\mu^2}{n})}$$

$$\leq \frac{1}{n^2}(n\mu - \frac{\mu^2}{n})$$

if and only if $p_1 = p_2 = \cdots = p_n$.

$$\Rightarrow p_1 = p_2 = \cdots = p_n = \frac{\mu}{n}.$$

(b) $\mu = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \frac{1}{p_i}$.

$$Var(X) = \sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n} \frac{1-p_i}{p_i^2} = \sum_{i=1}^{n} \frac{1}{p_i}(\frac{1}{p_i} - 1)$$

~~Var(X)~~ assume that $t_i = \frac{1}{p_i}$,

$$\sum_{i=1}^{n} t_i = \mu, \quad \sum_{i=1}^{n} \frac{1}{p_i}(\frac{1}{p_i} - 1) = \sum_{i=1}^{n} t_i(t_i - 1)$$

$$= \sum_{i=1}^{n} t_i^2 - t_i$$

$$\geq \frac{(\sum_{i=1}^{n} t_i)^2}{n} - \mu$$

if and only if $\frac{1}{p_1} = \cdots = \frac{1}{p_n} = \frac{\mu}{n}$,

it's the maximum,

46. (a) to prove

$$-\sum_{i=1}^{n} p_i \log p_i \leq -\sum_{i=1}^{n} p_i \log q_i$$

$$\Leftrightarrow \sum_{i=1}^{n} p_i \log(\frac{p_i}{q_i}) \geq 0.$$

$$\Leftrightarrow \sum_{i=1}^{n} p_i \log(\frac{q_i}{p_i}) \leq 0, \quad \text{since } \log(\frac{q_i}{p_i}) \leq \frac{q_i}{p_i} - 1.$$

$$\Rightarrow \sum_{i=1}^{n} p_i \log(\frac{q_i}{p_i}) \leq \sum_{i=1}^{n} p_i(\frac{q_i}{p_i} - 1) = 0. \quad \square$$

(b) from (a) conclusion,

$$-\sum_{x}\sum_{y} p_{X,Y}(x,y)\log p_{X,Y}(x,y) \le -\sum_{x}\sum_{y} p_{X,Y}(x,y)\cdot\log p_X(x)\cdot p_Y(y).$$

(c) easy

(d) $I(X,Y) = H(X) + H(Y) - H(X,Y)$

$$-(I(X,Y) - (H(X) - H(X|Y)))$$

$$= (H(Y) + H(X|Y) - H(X,Y))$$

$$= \sum_{x}\sum_{y}\left(p_{X,Y}(x,y)\cdot\log p_Y(y) - p_{X,Y}(x,y)\log p_{X,Y}(x,y)\right)$$

$$-H(X|Y)$$

$$= -\sum_{x}\sum_{y} p_{X,Y}(x,y)\cdot\log p_{X|Y}(x|y) - H(X|Y)$$

$$= -\sum_{x}\sum_{y} p_Y(y)\cdot p_{X|Y}(x|y)\cdot\log p_{X|Y}(x|y) - H(X|Y)$$

$$= 0. \quad \square$$