Chapter 6.

1. (a) $P_R(r) = \binom{n}{r} p^r (1-p)^{n-r}$

$$E(R) = \sum_{r=0}^{n} r \cdot P_R(r) = E(\sum R_i) = np$$

$$Var(R) = np(1-p)$$

(b) $P(B) = P(\text{first item on Red car, the others on Green car})$

$+P(\text{first Green, others Red})$

$= p\cdot(1-p)^{n-1} + (1-p)\cdot p^{n-1}.$

(c) $P(C) = P(1 \text{ Red}, n-1 \text{ Green}) + P(1 \text{ Green}, n-1 \text{ Red})$

$= \binom{n}{1} p\cdot(1-p)^{n-1} + \binom{n}{1} p^{n-1}\cdot(1-p)$ (if $n \neq 2$)

if $n=2$, $P(C) = \binom{n}{1} p\cdot(1-p)^{n-1} = 2p(1-p)$.

(d) $R-G = R-(n-R) = 2R-n.$

$E(R-G) = 2E(R) - n = 2np-n,$

$Var(R-G) = 4Var(R) = 4np(1-p).$

(e) assume that $R = X_1 + X_2 + \cdots + X_n$,

$X_1, \cdots, X_n$ are independent.

$P(X_3 + \cdots + X_n \mid X_1, X_2 = 1) = P(X_3 + \cdots + X_n)$

$E(R \mid X_1, X_2 = 1) = E(2 + X_3 + \cdots + X_n) = 2 + (n-2)p.$

$Var(R \mid X_1, X_2 = 1) = (n-2)p\cdot(1-p).$

2. (a) $p = \binom{6}{2} \cdot \left(\frac{3}{4}\right)^4 \cdot \left(\frac{1}{4}\right)^2$

(b) Y_3 ~~X~~ denotes the number of all quizzes that ~~contains up failing up to 3 times.~~ the ~~next~~ last situation is 3rd failing

~~X_i denotes~~

$X_i \triangleq Y_i - Y_{i-1}$, $Y_0 = 0$, $p \triangleq \frac{1}{4}$.

so $Y \sim$ Pascal.

$$P_{Y_3}(y) = \binom{y-1}{2} p^2 \cdot (1-p)^{y-3} \cdot p$$

$$= \binom{y-1}{2} p^3 \cdot (1-p)^{y-3}.$$

$$E(Y_3) = E(X_1 + X_2 + X_3) = \frac{3}{p} = 12.$$

E(number of quizzes that he will pass) $= 12 - 3 = 9$.

(c) ~~$P(A) = P($he failed in first 7 quizzes 1 times$)$ quizzes $= \binom{7}{1} p \cdot (1-p$~~

~~$P(B) = P($he fails in first 7 quizzes 2 times$) = \binom{7}{2} p^2 \cdot (1-p)^5$~~

~~$P($2nd times$) = P($2nd failing in 8th quiz$) + P($2nd failing in 9th quiz~~

~~$= P(A) \cdot p + P(A) \cdot (1-p) \cdot p + (1-P(A)) \cdot p^2$~~

~~$= p^2 + 2(p-p^2) \cdot P(A)$~~

$P(A) \triangleq P($he faild 1 times in the first 7 quizzes$)$

$$P(C \text{ problems}) = P(A) \cdot p^2 = \binom{7}{1} p^3 \cdot (1-p)^6.$$

$$= \binom{7}{1} \left(\frac{1}{4}\right)^3 \cdot \left(\frac{3}{4}\right)^6.$$

(d)

① P(he pass 2 times in a row)

= 1 − P(he never pass 2 times in a row)

set H_k be the probability of never passing 2 times in a row.

S: Pass F: Fail. we let (SF) must together

$P(H_1) = P(F) = \frac{1}{4}$

$P(H_2) = P(H_1) \cdot P(F) + P(SF) = \frac{1}{4}$

$P(H_n) = P(H_{n-1}) \cdot P(F) + P(H_{n-2}) \cdot P(SF)$

$= \frac{1}{4} P(H_{n-1}) + \frac{3}{16} P(H_{n-2})$

since H_n 为压缩映射 (compressed mapping)

$P(H_n) = \frac{16}{9}$ or 0, as $n \to +\infty$.

⇒ written as $P(H) = \lim_{n\to\infty} P(H_n) = 0$

we can devided H_∞ into infinite parts of H_K, K is a constant

$P(H_\infty) \le (P(H_K))^n \quad (n \to \infty)$.
$\to 0$

we have P(pass 2 times in a row) = 1.

② P(2 ~~failing~~ fails in 2 row before 2 ~~quizzes~~ passing in a row)

= P(FF first time, and pass 2 times in a row)

$= \sum_{n=2}^{+\infty} P($ SFSF⋯FF or FSFS⋯FF | in n quizzes)

$= (\frac{1}{4})^2 \left(1 + \frac{3}{4} + \frac{3}{4} \cdot \frac{1}{4} + (\frac{3}{4})^2 \cdot \frac{1}{4} + (\frac{3}{4})^2 \cdot (\frac{1}{4})^2 + \cdots\right) = (\frac{1}{4})^2 \cdot \frac{\frac{7}{4} \cdot (1)}{1 - \frac{3}{16}} = \frac{7}{52}$. □

3\. (a) $p(1,B) = p_B \cdot p_{1|B} = \frac{5}{6} \cdot \frac{2}{5} = \frac{1}{3}$.

$p(A) = (1-p(1,B))^{3} \cdot p(1,B)$

$= (\frac{2}{3})^3 \cdot \frac{1}{3}$

(b) $p(B \text{ prob.}) = \frac{5}{6} \cdot \frac{1}{6}$

(c) ~~P(1 prob.)~~ $p(1\text{'s task} \mid n\text{th task}) = \frac{1}{3}$.

$p \triangleq \frac{1}{3}$,

T_k: interarrival kth

$Y_k \triangleq T_1 + \cdots + T_k$.

$E(T_1 + \cdots + T_5) = 5 \cdot E(T_5) = 15$. | if $X \sim \text{Geometric}(p)$, $E(X) = \frac{1}{p}$.

(d) find ~~E(busy slots | user 1 ≤ 5)~~

$E(\text{user } 1 \leq 5)$ about Busy slots.

Y_i denotes number of busy slots when ith 1's task starts

$T_i = Y_i - Y_{i-1}$, $Y_0 = 0$.

$E(\text{user } 1 \leq 5) = E(Y_5) = 5 \cdot E(T) = 5 \cdot \frac{5}{2} = \frac{25}{2}$.

(e) ~~T_i denotes~~

Y_i denotes busy number of tasks until ith task from user 1.

Z denotes nuber of tasks from user 2 until 5th 1's task.

$Z = Y_5 - 5$. $p_{Y_5}(y) = \binom{y-1}{4} \cdot (\frac{2}{5})^4 \cdot (\frac{3}{5})^{y-5} \cdot (\frac{2}{5})$

$p_Z(z) = \binom{z+4}{4} \cdot (\frac{2}{5})^{5} \cdot (\frac{3}{5})^{z}$

$E(Y_5) = 5E(T)$, $Var(Y_5) = 5E(T)$.

$$E(Z) = 5E(T) - 5 = \frac{25}{2} - 5 = \frac{15}{2}.$$

$$Var(Z) = 5\,Var(T) = 5 \cdot \frac{1-(2/5)}{(2/5)^2}$$

$$= \frac{75}{4}. \quad \square$$

4. (a) let Y_k be the number of trials up to kth success.

$T_k = Y_k - Y_{k-1}$, $Y_0 = 0$

let X_k be the number of ~~failures~~ failures ---

$X_k = Y_k - k.$

~~$P_{Y_k}(y) = \binom{y-1}{k-1}(1-p)^k p^{y-k}$~~

$$P_{Y_k}(y) = \binom{y-1}{k-1} p^k \cdot (1-p)^{y-k}.$$

$$P_{X_k}(x) = P_{Y_k}(x+k) = \binom{x+k-1}{k-1} p^k \cdot (1-p)^x$$

(b) $E(X_k) = k \cdot E(T) - k$

$$= k \cdot \frac{1}{p} - k.$$

$$Var(X_k) = k \cdot \frac{1-p}{p^2}.$$

(c) P (ith failure before rth success)

$$= \sum_{x=i}^{+\infty} P_{X_k}(x) = 1 - \sum_{x=0}^{i-1} P_{X_k}(x)$$

$$= 1 - \sum_{x=0}^{i-1} \binom{x+k-1}{k-1} \cdot p^k \cdot (1-p)^x. \quad \square$$

5. Let t be the time I am in the room.

B be the latest time he won the game.

F be the first future time he win the game

since time $\leq t$ is independent with time $> t$.

$F - t \sim$ Geometric (p).

~~$E[B-t]$~~ $P_{B-t}(y) = (1-p)^{y-1} \cdot p$

symmetrically, ~~$P_{t-B}(x)$~~ $P_{t-B}(x) = (1-p)^{x-1} \cdot p$

~~$P_{F-B}(z) = P((F-t)+(t-B) = z)$~~

$Z = F - B = (F-t) + (t-B)$.

$Z \sim$ Pascal $(p, 2)$. since Bernoulli(p) + Bernoulli$(p) \sim$ Pascal$(p, 2)$

$P_Z(z) = \binom{z}{1} \cdot p^2 \cdot (1-p)^{z-1}$

$= z \cdot p^2 \cdot (1-p)^{z-1}$. $\square$

6. $Y = X_1 + X_2 + X_3 + \cdots + X_N$.

$X_i \sim$ Geometric(p), $N \sim$ Geometric (q).

$X_1, \cdots X_N, N$ are independent. independent.

independent.

① $P_Y(y) = \sum_{N=1}^{+\infty} P_{Y,N}(y, n) = \sum_{N=1}^{+\infty} P_{Y|N}(y|n) \cdot P_N(n)$

$= \sum_{n=1}^{y} \binom{y-1}{n-1} \cdot p^n \cdot (1-p)^{y-n} \cdot (1-q)^{n-1} \cdot q$

$$p_Y(y) = \sum_{n=1}^{y} \binom{y-1}{n-1} \cdot \left(\frac{p(1-q)}{1-p}\right)^{n} \cdot (1-p)^{y} \cdot \frac{q}{1-q}.$$

$$= \sum_{n=0}^{y-1} \binom{y-1}{n} \cdot \left(\frac{p(1-q)}{1-p}\right)^{n+1} \cdot (1-p)^{y} \cdot \frac{q}{1-q}$$

$$= \left(\frac{p(1-q)}{1-p}+1\right)^{y-1} \cdot \left(\frac{p(1-q)}{1-p}\right) \cdot (1-p)^{y} \cdot \frac{q}{1-q}$$

$$= (1-pq)^{y-1} \cdot pq$$

$\Rightarrow$ $Y \sim \text{Geometric}(pq)$. $\square$

③ Another solution:

(Another solution by hints.)

(Another solution by hints).

~~we can split each arrival time~~

we can assume that X_i denotes the ith interarrival times, and it fails $(X_i - 1)$ times with probability $(1-p)$, and success with p.

N denotes the nth success is accepted with probability q, and each arrival can be accept with pq. prob..

Hence we get Y is geometric distribution with parameter pq. $\square$

$$P_Y(y) = \sum_{n=1}^{y} \binom{y-1}{n-1} \cdot \left(\frac{p(1-q)}{1-p}\right)^{n} \cdot (1-p)^{y} \cdot \frac{q}{1-q}.$$

$$= \sum_{n=0}^{y-1} \binom{y-1}{n} \cdot \left(\frac{p(1-q)}{1-p}\right)^{n+1} \cdot (1-p)^{y} \cdot \frac{q}{1-q}$$

$$= \left(\frac{p(1-q)}{1-p} + 1\right)^{y-1} \cdot \left(\frac{p(1-q)}{1-p}\right) \cdot (1-p)^{y} \cdot \frac{q}{1-q}$$

$$= (1-pq)^{y-1} \cdot pq$$

$Y \sim \text{Geometric}(pq)$. $\square$

$\Rightarrow$

② Another solution:

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Hence we get Y is geometric distribution with parameter pq. $\square$

7. (a) ~~If we can proove~~

$$P(0 \le Y < \tfrac{1}{2}) = P(\tfrac{1}{2} \le Y < 1) = \tfrac{1}{2}$$

we have

since $P(0 \le Y < \frac{1}{2}) = P(\{Y = 0.0X_2 \cdots\})$

$P(\frac{1}{2} \le Y < 1) = P(\{Y \mid Y = 0.1X_2 \cdots\})$

and we can also get $P(0 \le Y < \frac{1}{4}) = P(\frac{1}{4} \le Y < \frac{1}{2}) = \cdots = \frac{1}{4}$.

we have $P(\frac{i}{2^k} \le Y < \frac{i+1}{2^k}) = \frac{1}{2^k}$, for $\forall k \ge 1, 0 \le i < 2^k$.

and we can conclude $P(Y < \frac{i}{2^k}) = \frac{i}{2^k}$.

for $\forall y \in (0,1)$, we have a series $\{a_n\}_{n=1}^{\infty}$,

$a_n = \frac{b_n}{2^n}$, $b_n \in \mathbb{N}^+$, and $a_n \le y < a_n + \frac{1}{2^n}$,

we have $P(Y \le a_n) \le P(Y \le y) < P(Y \le a_n + \frac{1}{2^n})$

$\Rightarrow a_n \le P(Y \le y) < a_n + \frac{1}{2^n}$

$\Rightarrow \lim_{n\to\infty} a_n \le P(Y \le y) \le \lim_{n\to\infty} a_n + \frac{1}{2^n} = y$

$\Rightarrow P(Y \le y) = y$, $Y \sim$ uniform distribution in $(0,1)$.

(b) $P(Y < \frac{1}{2}) = P(Y = 0.1 \cdots) = P(X_1 = 0) = \frac{1}{2}$.

$P(Y \ge \frac{1}{2}) = P(X_1 = 1) = \frac{1}{2}$.

$\Rightarrow X_1 \sim \text{Uniform}\{0,1\}$.

for any $n \ge 2$, we have ~~$P(Y \le \sum_{k=1}^{n-1} 2^{-k} X_k$~~

$\cancel{P(\sum_{k=1}^{n-1} 2^{-k} \cdot X_k \le Y < \sum_{k=1}^{n-1}}$

$\cancel{P(\sum_{k=1}^{n-1} 2^{-k} \cdot X_k + 2^{-n} \cdot X_n \le Y < \sum_{k=1}^{n-1} 2^{-k} \cdot X_k + 2^{-n} \cdot (X_n + 1))}$

$\cancel{\sum_{k=1}^{n-1} 2^{-k} \cdot X_k \le Y < \sum_{k=1}^{n-1} 2^{-k} \cdot X_k + 2^{-(n-1)})}$

$\cancel{= \frac{1}{2} = P(0.X_1 \cdots X_{n-1} X_n \cdots \mid 0.X_1, \cdots, X_{n-1} \cdots)}$

$\cancel{= P(X_1 = x_1, X_2 = x_2, \cdots, X_{n-1} = x_{n-1}, X_n = x_n \mid X_1 = x_1, \cdots X_n = x_n,}$

$\cancel{= P(X_n = x_n) = P(X_n = 0) = P(X_n = 1)}$

~~since~~ $\cancel{X_1, X_2, \cdots, X_n, \cdots}$ ~~are independent.~~

~~are independent.~~

$\cancel{\Rightarrow P(X_n = 0) = P(X_n = 1) = \frac{1}{2}}$

~~since~~ $\cancel{P(\sum_{k=1}^{n-1}}$ assume that we have

$P(X_1 = x_1, \cdots, X_i = x_i) = \frac{1}{2^i}$, $\forall i \le n-1$

$P(X_i = x_i) = \frac{1}{2}$, $\forall i \le n-1$.

And $P(\sum_{k=1}^{n} 2^{-k} X_k \le Y < \sum_{k=1}^{n} 2^{-k} \cdot X_k + 2^{-n}) = \frac{1}{2^n}$.

$= P(X_1 = x_1, \cdots X_n = x_n)$

$P(X_n = x_n) = \sum_{x_1} \sum_{x_2} \cdots \sum_{x_{n-1}} P(X_1 = x_1, \cdots, X_n = x_n)$

$= \frac{1}{2}$,

$\Rightarrow P(X_n = 0) = P(X_n = 1) = \frac{1}{2}$.

through inductive method, we have $P(X_n = 0) = P(X_n = 1) = \frac{1}{2}$, $\forall n \ge 1$.

$X_n \sim \text{Bernoulli}\overset{(\frac{1}{2})}{\{0, 1\}}$, for $\forall n \in \mathbb{N}^+$. □

8. During rush hour, from 8. a.m. to 9. a.m. traffic accidents occur according to a Poisson process with a rate of 5 accidents per hour.

8. - 9. a.m. : Poisson(5)

9 - 10 a.m. : Poisson(3)

10 - 11 a.m. : Poisson(3)

~~Mx(t)~~ $M_X(s) = E(e^{sx}) = \sum_{x=0}^{+\infty} e^{sx} \cdot e^{-\lambda} \cdot \frac{\lambda^x}{x!}$

$= \sum_{x=0}^{+\infty} e^{-\lambda} \cdot \frac{(e^s \cdot \lambda)^x}{x!}$

$= e^{-\lambda} \cdot e^{e^s \cdot \lambda} = e^{(e^s - 1)\lambda}$

$M_{X_1}(s) \cdot M_{X_2}(s) = e^{(e^s-1)\lambda_1} \cdot e^{(e^s-1)\lambda_2} = e^{(e^s-1)(\lambda_1+\lambda_2)}$

$\Rightarrow$ Poisson(λ_1) + Poisson(λ_2) = Poisson($\lambda_1 + \lambda_2$)

Poisson(5) + Poisson(3) + Poisson(3) = Poisson(11). □

9. time ~ Exponential

5 tennis courts ?

(4 + 8k)?

~~且~~ we can assume that each table of 5 tables are busy all the time,

so we get E(1 pairs, 1 table) = ~~E~~ $\frac{\;}{5}$E(1 pairs, / 5 t.

. □

?

10. (a) $P(\text{0 fish} \mid \text{an hour}) = e^{-\lambda}$

$P(\text{stay} \geq \text{2 hours}) = P(\text{0 fish} \mid \text{1 hour})^2 = e^{-2\lambda} = e^{-1.2}$

(b) $P(\text{2-5 hour}) = 1 - P(\cancel{\cdot}\, T \geq 5 \text{ hour}) - P(T < \overset{2}{\cancel{5}} \text{ hour})$

$= P(T \geq 2 \text{ hours}) - P(T \geq 5 \text{ hours})$

$= e^{-1.2} - e^{-3}.$

(c) he catches ~~2~~ at least 2 fish in the first 2 hours

$\sum_{k=2}^{+\infty} P(F=k) = 1 - P(F=0) - P(F=1)$

$= 1 - e^{-2\lambda} - e^{-2\lambda}(2\lambda)$

(d) ~~$E(X) = E(\cancel{x}\, E(X|F))$. F denotes the~~

$\cancel{=}$

$E(X) = E(X \mid \text{0 fish in the first 2 hours}) \cdot P(\overset{0/}{2} \text{ hour})$

$+ \cancel{\cdot}\, E(X \mid \geq 1 \text{ fish, 2 hours}) \cdot P(\geq 1 \text{ fish})$

$= 1 \cdot P(\text{0 fish}) + P(\geq 1 \text{ fish}) \cdot E(\text{Poisson}(\lambda))$

$= e^{-2\lambda} + (1 - e^{-2\lambda})(2\lambda)$

$= 2\lambda + (1 - 2\lambda) \cdot e^{-2\lambda}$

$= 1.2\cancel{\cdot} - 0.2 \cdot e^{-1.2}$

Time ~ Poisson(k, t)

(e) Due to the memoryless property of Poisson distribution

$P(\text{Time} \leq t) = 1 - P(0, t) = 1 - e^{-\lambda t}$, $f_T(t) = \lambda \cdot e^{-\lambda t}$, $E(T) = \frac{1}{\lambda}$,

$E(T+4) = 4 + \frac{1}{\lambda}$ [illegible]

1. ~~(depart 出发)~~ (depart 离开).

~~(depart 出发)~~.

when a customer departs from the bookstore, he will buy a book with probability p.

$P(k, y)$ denotes k customers depart in y time.

~~$P(kp, y)$~~ ~~去~~ the probability of selling books in y time obeys Poisson(λp)

(a) $P(T \le t \mid \text{first customer}) = 1 - e^{-\lambda t}$.

$P(T \le t \mid \text{first book}) = 1 - e^{-\lambda p t}$.

b) $e^{-\lambda p}$ $\quad 1 - P(T \le 1)$

(c) $E(\text{Poisson}(\lambda p)) = E(\text{Poisson}(\lambda p)) = \lambda p$. □

Poisson(λp).

~~12. Go! Go! Poisson(λp)~~

12. 冲 Go! A pizza serves n different types of pizza, and is visited by a number k of customers in a given period of time, where k is a Poisson random variables with mean λ.

k customers : $K \sim \text{Poisson}(\lambda)$.

n types of pizza. : uniform distribution.

~~we assume that X_k 表 denotes the number of customers choose the kth different pizza.~~

~~表~~ $T_k = X_k - X_{k-1}$, $T_1 = 1$, T_k ~~表~~ $\sim$ Geometric($\frac{n-k+1}{n}$)

~~$P_{T_k}(t) = \frac{n-k+1}{n} \cdot \left(\frac{k-1}{n}\right)^{t-1}$,~~

~~$Y_k = T_1 + T_2 + \cdots + T_k$~~ ~~$P(K=k) = P(Y_k \le k < Y_{k+1})$~~

~~$E(K) = \sum_{k=1}^{n} P$~~

assume that X_k $(1 \le k \le n)$ denotes the number of customers who choose kth pizza.

$X_k \sim \text{Poisson}(\frac{\lambda}{n})$

assume that $Y_k = \begin{cases} 1 & \text{if } X_k \geq 1 \\ 0 & \text{if } X_k = 0. \end{cases}$

$P(Y_k = 1) = 1 - P(X_k = 0) = 1 - e^{-\frac{\lambda}{n}}$.

$E(Y_k) = 1 - e^{-\frac{\lambda}{n}}$,

$E(Y_1 + Y_2 + \cdots + Y_n) = n\left(1 - e^{-\frac{\lambda}{n}}\right) = n \cdot (1 - e^{-\lambda/n})$

13. Go!

A messages $\sim \text{Poisson}(\lambda_A)$

B ---- $\sim \text{Poisson}(\lambda_B)$

words in a message $\sim P_W(w)$.

(a) $C \triangleq A + B \sim \text{Poisson}(\lambda_A + \lambda_B)$

$P(A \text{ problem}) = P_C(9; (\lambda_A + \lambda_B) \cdot t)$

$= e^{-(\lambda_A + \lambda_B)t} \cdot \frac{[(\lambda_A + \lambda_B) \cdot t]^9}{9!}$.

(b) W_i denotes the word length of i th message.

令 $N = W_1 + W_2 + — + W_c$

$$E(N) = E(W_1 + — + W_c) = E(E(W_1 + \cdots + W_c \mid c))$$

$$= E(c \cdot E(W))$$

$$= E(c) \cdot E(W).$$

$c \sim \text{Poisson}(\lambda_A + \lambda_B)t$, $W \sim$ ~ (in time duration t).

$E(c) = t(\lambda_A + \lambda_B)$, $E(W) = \frac{11}{6}$.

$E(N) = \frac{11}{6}(\lambda_A + \lambda_B) \cdot t$,

(c) Determine the PDF of the time from $t=0$ until receiver has received exactly eight three-word messages from A.

~~$P_W(w)$~~ assume that C_3 denotes the number of 3-word message.

each message can be calculated with prob. $p_W(3)$,

and $c \sim \text{Poisson}((\lambda_A + \lambda_B)t)$.

C denotes the number of A or B message.

$C_3 \sim \text{Poisson}((\lambda_A + \lambda_B)t \cdot p_W(3))$.

$$\cancel{P(C_3 = 8) = e^{-(\lambda_A + \lambda_B)t \cdot p_W(3)} \cdot \frac{[(\lambda_A + \lambda_B)t \cdot p_W(3)]^8}{8!} = P_T(t)}$$

~~in duration t.~~

$$\cancel{P(C_3 = 8) = e^{-(\lambda_A + \lambda_B)t/6} \cdot \frac{((\lambda_A + \lambda_B)t \cdot \frac{1}{6})^8}{8!}}$$

$$P(C_3 = 8)(t) = e^{-\lambda_A/6} \cdot \frac{(\lambda_A t \cdot \frac{1}{6})^8}{8!}.$$

(d) ① we know that each messages ~~have~~ from ~~prob.~~ A ~~of~~ with prob. $\frac{\lambda_A}{\lambda_A+\lambda_B}$, B with prob. $\frac{\lambda_B}{\lambda_A+\lambda_B}$.

$$p(8\text{ A and }4\text{ B} \mid 12\text{ messages}) = \binom{12}{8}\left(\frac{\lambda_A}{\lambda_A+\lambda_B}\right)^8 \cdot \left(\frac{\lambda_B}{\lambda_A+\lambda_B}\right)^4.$$

②
$$p(8\text{A and }4\text{B} \mid 12\text{ messages}) = \frac{p(8A)\cdot p(4B)}{p(12C)}$$

$$= \frac{p(A=8)\cdot p(B=4)}{p(C=12)}$$

$$= \frac{e^{-\lambda_A t}\cdot \frac{\lambda_A^8}{8!}\cdot e^{-\lambda_B t}\cdot \frac{\lambda_B^4}{4!}}{e^{-(\lambda_A+\lambda_B)t}\cdot \frac{(\lambda_A+\lambda_B)^{12}}{12!}}$$

$$= \binom{12}{8}\left(\frac{\lambda_A}{\lambda_A+\lambda_B}\right)^8 \cdot \left(\frac{\lambda_B}{\lambda_A+\lambda_B}\right)^4. \quad \square$$

14. (illuminate 阐明，照亮)

(illuminate 阐明，照亮)

(a)
$$E(X) = E(X|A)\cdot P(A) + E(X|B)\cdot P(B)$$
$$= E(X|A) + E(X|B)\cdot P(B)$$
$$= 1\cdot\tfrac{1}{2} + \tfrac{1}{3}\cdot\tfrac{1}{2} = \tfrac{2}{3}$$

(b)
$$P(\text{No failures } \le t) \cancel{= P(\min(A, B) > t)}$$
$$= P(A>t|A)\cdot P(A) + P(B>t)\cdot P(B) \cancel{= P(A>t)\cdot P(B>t)}$$
$$= \tfrac{1}{2}(e^{-t}+e^{-3t}). \qquad \cancel{= (e^{-x})\cdot(e^{-3x})}$$
$$\cancel{= e^{-4x}}$$

(c) N: No failures until time t.

$$p(A|N=t) = \frac{P(N=t|A)\cdot P(A)}{P(N=t)} = \frac{f_{X|A}(t)\cdot\frac{1}{2}}{f_N(t)} = \frac{\frac{1}{2}e^{-t}}{\frac{1}{2}(e^{-t}+e^{-3t})} = \frac{e^{-t}}{e^{-t}+e^{-3t}}$$

(d) $Var(N) = \cancel{Var(N)}\ E(N^2) - E(N)^2$

$$= E(N^2|A)\cdot P(A) + E(N^2|B)\cdot P(B) - (E(N|A)\cdot P(A) + E(N|B)\cdot P(B)$$

$$E(N^2|A) = (Var(N|A) + E(N|A)^2)\cdot P(A) + (Var(N|B) + E(N|B)^2)\cdot P(B)$$

$$-E(N)$$

$$= (1+1)\cdot\frac{1}{2} + (\frac{2}{9})\cdot\frac{1}{2} - \frac{4}{9}.$$

$$= \frac{2}{3}$$

(e) $P = \binom{11}{3}(\frac{1}{2})^{12}$

(f) $P = \binom{12}{4}(\frac{1}{2})^{12}$

(g) for each bulb, life it is independent ~~between~~ among different bulbs.

$$P(N_i \le t) = P(N_i \le t|A)\cdot P(A) + P(N_i \le t|B)\cdot P(B)$$

$$= \frac{1}{2}(1 - e^{-t} + 1 - e^{-3t})$$

let $T_i \triangleq N_1 + N_2 + \cdots + N_i$.

~~$P(T_i \le t) =$~~

$$M_{N_i}(s) = \frac{1}{2}\left(\frac{1}{1-s} + \frac{3}{3-s}\right),$$

$$M_{T_i}(s) = \left[\frac{1}{2}\left(\frac{1}{1-s} + \frac{3}{3-s}\right)\right]^i$$, when $i=12$, $M_{T_{12}}(s) = (\text{---})^{12}$.

(h) Q_1: ~~first~~ total period of first 2 B bulbs

Q_2: total period of A bulb.

~~$P(Q_1 \le t)$~~ $Q_1 \sim$ Erlang ($\lambda = 3$, order = 2).

$Q_2 \sim$ Exponential ($\lambda = 1$).

$Q_1 - Q_2 \sim ?$

$P(Q_1 > Q_2) = ?$

~~$f_{Q_1}(x) = \frac{\lambda^k y^{k-1} e^{-\lambda x}}{}$~~ Erlang $(\lambda, k) \sim f = \frac{\lambda^k \cdot y^{k-1} \cdot e^{-\lambda y}}{(k-1)!}$

$f_{Q_1}(x) = 9x \cdot e^{-3x}$. $(x \geq 0)$

$f_{Q_2}(x) = e^{-x}$ $(x \geq 0)$

~~$\sum_x P(Q_1 > x > Q_2 | X = x) P(X = x) = P(Q_1 > Q_2)$~~

~~$= \sum_x P(Q_1 > x > Q_2) \cdot P(X = x)$~~

$$P(Q_1 > Q_2) = \sum_{q_2} P(Q_1 > Q_2 | Q_2 = q_2) P(Q_2 = q_2)$$

$$= \int_{-\infty}^{+\infty} P(Q_1 > q) \cdot f_{Q_2}(q)\, dq$$

$$= \int_{-\infty}^{+\infty} P(Q_2 < q) \cdot f_{Q_1}(q)\, dq$$

$$= \int_0^{+\infty} (1 - e^{-q}) \cdot 9q \cdot e^{-3q}\, dq$$

$$= \frac{7}{16}.$$

(i) assume that Y_i denotes the period of ith B bulbs ~~tumi~~ illumination. N is the number of B bulbs in 12 bulbs

$Y = Y_1 + \cdots + Y_N$.

$E(Y) = E(Y) \cdot E(N) = 6 \cdot \frac{1}{3} = 2$.

$Var(Y) = \cancel{Var}\ E(Y^2) - E(Y)^2$

$= E(E(Y^2 | N)) - E(E(Y|N))^2$

$= E(Var(Y|N) + E(Y|N)^2) - E(E(Y|N))^2$

$= E(Var(Y|N)) + Var(E(Y|N))$

$= E(N \cdot Var(Y_i)) + Var(N \cdot E(Y_i)) = 6 \cdot \frac{1}{9} + \frac{1}{9} \cdot Var(N) = 1$

j) $P(T-t \le x \mid T \ge t)$

$E(T-t \mid T\ge t) = E(T-t, A \mid T\ge t) + E(T-t, B \mid T \ge t)$

$= E(T-t \mid T\ge t, A)\cdot P(A\mid T\ge t) + E(T-t \mid T\ge t, B)\cdot P(B\mid T\ge$

$E(T-t \mid T\ge t, A) = E(T_A) = 1, \frac{1}{\lambda_A}$

$P(A\mid T\ge t) = \frac{P(T\ge t\mid A)\cdot P(A)}{P(T\ge t)} = \frac{P(T\ge t\mid A)}{P(T\ge t\mid A)+P(T\ge t\mid B)} = \frac{e^{-t}}{e^{-t}+e^{-3t}}$

$E(T-t \mid T\ge t) = \frac{e^{-t}}{e^{-t}+e^{-3t}} + \frac{1}{3}\cdot\frac{e^{-3t}}{e^{-t}+e^{-3t}}$. □

15、Go!

16、Go! 冲冲冲！相信自己！

(a) ~~$P(N\le n)=$~~ assume that X_i means interarrival time
~~$P(N=n)=$~~ between two police car.

$P(X_i \le x) = 1 - P(X_i > x) = 1 - P(0, x)) = 1 - e^{-\lambda x}$.

$P(0, x)$ means in time x, 0 cars are noticed.

if $X_i < \tau$, U-turn won't happen. if $X_i \ge \tau$, U-turn happens

$P(N=n) = P(X_i < \tau)^n \cdot P(X_i \ge \tau)$

$= (1-e^{-\lambda\tau})^n \cdot e^{-\lambda\tau}$

$E(N) = \sum_{n=1}^{+\infty} n\cdot(1-e^{-\lambda\tau})^n\cdot e^{-\lambda\tau} = e^{\lambda\tau}-1$.

(b)

(b) ~~find~~ $X_1, X_2 \cdots, X_n$

find $E(X_n \mid N \geq n)$

$$E(X_n \mid N \geq n) = E(X_n \mid X_1 < \tau, X_2 < \tau \cdots, X_n < \tau)$$

$$= E(X_n \mid X_n < \tau)$$

$$= \frac{\int_0^\tau x \cdot f_{X_n}(x)\, dx}{P(X_n < \tau)}$$

$$P(X_n < x) = 1 - P(0, x) = 1 - e^{-\lambda x}.$$

$$E(X_n \mid N \geq n) = \frac{\int_0^\tau x \cdot (\lambda e^{-\lambda x})\, dx}{1 - e^{-\lambda \tau}}$$

$$= \left(-(\tau + \tfrac{1}{\lambda}) \cdot e^{-\lambda \tau} + \tfrac{1}{\lambda}\right) / (1 - e^{-\lambda \tau}). \quad \square$$

(c) $\cancel{E(X_1 + \cdots + X_N) = E(E(X_1 + \cdots + X_N \mid N))}$

$\cancel{= E(X_1 + \cdots + X_N \mid N = n)}$

$\cancel{= E(X_i) \cdot E(N)}$

$\cancel{E(N) = e^{\lambda\tau} - 1, \quad E(X_i) = E(\text{Poisson}(\lambda)) = \lambda}$

$$E(X_1 + \cdots + X_N + \tau) = E(E(X_1 + \cdots + X_N \mid N)) + \tau$$

$$= \sum_{n=1}^{+\infty} [P(N = n) \cdot E(X_1 + \cdots + X_N \mid N = n)] + \tau$$

$$= \tau + \sum_{n=1}^{+\infty} P(N = n) \cdot E(X_i \mid X_i \leq \tau) \cdot n$$

$$= \tau + E(N) \cdot E(X_i \mid N \geq i)$$

$$= \tau + (e^{\lambda\tau} - 1) \cdot \left(\frac{\frac{1}{\lambda} - (\tau + \frac{1}{\lambda}) \cdot e^{-\lambda\tau}}{1 - e^{-\lambda\tau}}\right)$$

$$= \frac{1}{\lambda}(e^{\lambda\tau} - 1). \quad \square$$

完成

17.

burrow $\xrightarrow{20s}$ tray → eating $\xrightarrow{20s}$ burrow. → rest

exponential(30) exponential(30)

(1/3 stand) (momentary 片刻的) (1/3 stand still for 0 time)

assume that X_i denotes the time of finishing ith cycle.

$X_i = T_{i,1} + T_{i,2} + T_{i,3} + T_{i,4}$

$T_{i,1} = T_{i,3} = 20$,

$T_{i,2}, T_{i,4} \sim \text{Exponential}(30)$.

assume that photographer arrive at c.

$P(c \text{ in } T_{i,1}) = P(c \text{ in } T_{i,3}) = \frac{2}{5}$,

we can discard $T_{i,3}$ and $T_{i,4}$ without loss of generality.

$X_i = T_{i,1} + T_{i,2}$.

assume that photographer arrive at 0 time unit.

waiting time is Y, A means snapping the bear.

$\cancel{E(Y) = E(Y|A)\cdot P(A) + E(Y|A^c)\cdot P(A^c)}$

$\cancel{= 10\cdot\frac{1}{3} + (50 + E(Y))\cdot\frac{2}{3}},$

$\cancel{E(Y) = 110.}$ Z, is the time he waits totally.

~~assume that~~ C is the time photographer arrives.

$E(Z) = E(Z|C \text{ in } T_{i,1})P(T_{i,1}) + \cancel{E(Z|C \text{ in } T_{i,2})P(T_{i,2})}$

$\cancel{E(Z|C \text{ in } T_{i,2}) = 30 + E(Y) = 140}$

$E(Z|C \text{ in } T_{i,1}) = \cancel{E(T_{i,1}) = 10}$

$E(Z|T_{i,1}, B)\cdot P(B|T_{i,1}) + E(Z|T_{i,1}, B^c)\cdot P(B^c|T_{i,1})$

B denotes bear stand still.

$$\cancel{E(Z \mid T_{i,1}) = 10 \cdot \frac{1}{3} + \frac{2}{3} \cdot (20 + 30 + E(T)) = 110.}$$

$$\cancel{E(Z) = \frac{2}{5} \cdot 110 + \frac{3}{5} \cdot 140}$$

$$\cancel{= 128.}$$

$$E(T) = E(T \mid A) \cdot P(A) + E(T \mid A^c) \cdot P(A^c)$$

$$= 10 \cdot \frac{1}{3} + (50 + E(T)) \cdot \frac{2}{3}$$

$$\Rightarrow E(T) = 110.$$

$$E(Z \mid T_{i,2}) = 30 + E(T) = 140$$

$$E(Z) = E(Z \mid T_{i,1}) \cdot P(T_{i,1}) + E(Z \mid T_{i,2}) \cdot P(T_{i,2})$$

$$E(Z \mid T_{i,1}) = \int_0^{20} E(Z \mid T_{i,1}, c) \cdot f_{c \mid T_{i,1}}(c)\, dc$$

$$= \int_0^{20} \Big(E(Z \mid T_{i,1}, c, B) \cdot P(B \mid T_{i,1}, c) + E(Z \mid T_{i,1}, c, B^c) \cdot P(B^c \mid T_{i,1}, c) \Big) \cdot f_{c \mid T_{i,1}}(c)\, dc$$

$$= \int_0^{20} \left(\frac{(20-c)}{2} \cdot \frac{1}{3} \cdot \frac{20-c}{20} + (20 - c + 30 + E(T)) \left(1 - \frac{1}{3} \cdot \frac{20-c}{20}\right) \right) \cdot \frac{1}{20}\, dc$$

$$= 140 - \frac{40}{3} - \frac{10}{9}$$

$$\cancel{E(Z) = \left(10 - \frac{10}{9}\right) \cdot \frac{2}{5} + 140 \cdot \frac{3}{5}}$$

$$E(Z) = \left(140 - \frac{40}{3} - \frac{10}{9}\right) \cdot \frac{2}{5} + (30 + E(T)) \cdot \frac{3}{5}$$

$$= 140 - \frac{16}{3} - \frac{4}{9} \approx 134.22. \quad \square$$

18. Go! 冲，春来我不先开口，哪个虫儿敢作声。

$P(k, t)$ means in t time interval $[0, t]$, exactly exists k arrivals.

$\cancel{P(T_1 \le x \mid T_1 \le t) =}$

$\cancel{P(T_1 \le x)} =$

$$P(T \le x \mid 1 \text{ arrival in } [0,t]) = \frac{P(1,x)\cdot P(0,t-x)}{P(1,t)}$$

$$= \frac{e^{-\lambda x}\cdot(\lambda x)\cdot e^{-\lambda(t-x)}}{e^{-t\lambda}\cdot(t\lambda)}$$

$$= \frac{x}{t}.$$

$T \mid 1$ arrival in $[0,t] \sim \text{Uniform}([0,t])$. $\square$

19. $X_1 \sim \text{Exponential}(\lambda_1)$, $X_2 \sim \text{Exponential}(\lambda_2)$.

(a) $P(\max\{X_1, X_2\} \le x) = P(X_1 \le x)\cdot P(X_2 \le x)$

$$E(\{\max\{X_1,X_2\}\}) = \int_0^{+\infty} x\cdot[(1-e^{-\lambda_1 x})\cdot\lambda_2 e^{-\lambda_2 x} + (1-e^{-\lambda_2 x})\cdot\lambda_1 e^{-\lambda_1 x}]\cdot$$

$$f_{\{X_1,X_2\}}(x) = \frac{dP(\{\max\{X_1,X_2\}\le x)}{dx} = F_{X_1}(x)\cdot f_{X_2}(x) + F_{X_2}(x)\cdot f_{X_1}(x)$$

$$E(\max\{X_1,X_2\}) = E(X_1) + E(X_2) - E(\text{exponential}(\lambda_1+\lambda_2))$$

$$= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2}$$

$$= \frac{\lambda_1^2+\lambda_2^2+\lambda_1\cdot\lambda_2}{\lambda_1\cdot\lambda_2(\lambda_1+\lambda_2)}.$$

(b) $f_{\{T,Z\}}(x) = f_T(x)\cdot F_Z(x) + F_T(x)\cdot f_Z(x)$

$$= (\lambda_1\cdot e^{-\lambda_1 x})\cdot(1-e^{-\lambda_2 x} - e^{-\lambda_2 x}\cdot\lambda_2 x) + (1-e^{-\lambda_1 x})\cdot\lambda_2^2 x\cdot$$

$$= \lambda_1\cdot e^{-\lambda_1 x} + \lambda_2\cdot e^{-\lambda_2 x} + e^{-(\lambda_1+\lambda_2)x}\cdot(-\lambda_1 - \lambda_1\lambda_2 x - \lambda_2^2 x)$$

$$E(\max\{T,Z\}) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{\lambda_1}{(\lambda_1+\lambda_2)^2} - \frac{3\lambda_2}{(\lambda_1+\lambda_2)^2}.\quad \square$$

20. $Y_k \sim \text{Erlang}(k)$,

$$f_{Y_k}(y) = \lambda^2 y^{(k-1)} \cdot e^{-\lambda y} \cdot / (k-1)!$$

$$\sum_{k=1}^{\infty} f_{Y_k}(y) = \sum_{k=1}^{\infty} \lambda^k y^{(k-1)} \cdot e^{-\lambda y} / (k-1)!$$

$$= \sum_{k=1}^{\infty} \frac{\lambda \cdot (\lambda y)^{(k-1)} \cdot e^{-\lambda y}}{(k-1)!}$$

$$= \lambda \cdot e^{-\lambda y} \cdot \sum_{k=0}^{\infty} \frac{(\lambda y)^k}{k!}$$

$$= \lambda \cdot e^{-\lambda y} \cdot e^{\lambda y} = \lambda. \quad \square$$

21. $X_1(k)$: time of k th arrival in Poisson (λ_1) Process.

$X_2(k)$: time of kth arrival in Poisson (λ_2) Process.

Consider $X_1 + X_2$ obeys Poisson $(\lambda_1 + \lambda_2)$ distribution,

For each arrival, it is from X_1 with $\frac{\lambda_1}{\lambda_1+\lambda_2}$ probability.

$p(X_1(n) < X_2(m))$ means when X_1 has n arrivals, X_2 must less than or equal to $(m-1)$ arrivals.

It's the same as in first $(n+m-1)$ trials, $X_1 \geq n$.

if $X_1 + X_2 = n+m-1$, $X_1 \geq n \iff X_1(n) > X_2(m)$.

$$P(X_1(n) < X_2(m)) = P(X_1 \geq n \mid X_1 + X_2 = n+m-1)$$

$$= \sum_{k=n}^{n+m-1} \binom{n+m-1}{k} \cdot \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \cdot \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n+m-1-k}. \quad \square$$

22. $Y = X_1 + X_2 + \cdots + X_N$

$N \geq 0$, $X_i \sim \text{Bernoulli}(p)$

(a) $Y = X_1 + X_2 + \cdots + X_N$. $N \sim \text{Binomial}(m, \cancel{p}\, q)$.

N can be explaned that in m trials, N = the number of success with probability q.

and Y = the number of success with probabity pq in m trials

$Y \sim \text{Binomial}(m, pq)$.

(b) if $N \sim \text{Poisson}(\lambda)$

N can be seen as an interval $[0,1]$, splitted in to n intervals equally. $(n \to \infty)$. each interval is a Bernoulli distribution with prob. λ/n. N = the number of all ~~success~~ points being chosen.

and Y denotes that in each chosen points, it has prob. p ~~being~~ having value 1, that is ~~being~~ chosen with prob. $p \cdot \lambda/n$.

$Y \sim \text{Poisson}(\lambda p)$. $\square$

23. $X_i \sim \text{Exponential}(\lambda)$, $Y = X_1 + \cdots + X_N$.

$N \sim \text{Geometric}(p)$.

X_i is the time of the ith arrival in Poisson(λ).

$X_1 + \cdots + X_N$ denotes the first accepted arrival with probability p, that is, the first accepted arrival with probability λp.

$Y \sim \text{Exponential}(\lambda p)$. $\square$

24. $T \sim$ Exponential (v)

N = ~~[illegible]~~ number of Poisson process in $[0, T]$.

$x_i \triangleq$ the ith time of Poisson process. (λ)

$P(N=n) = P(x_1 + \cdots + x_n \leq T,\ x_1 + \cdots + x_n > T)$.

that is, when T first success, N is exactly ~~[illegible]~~ having N arrivals. assume that H is the number of Poisson(λ) + Poisson(v)

$\cancel{P(N=n)} = \cancel{P(H=n+1,\ N=n \mid H=n+1)}$ $Z \sim$ Erlang $(n+1, v+\lambda)$

$Z(n+1)$.

$= P(N=n) = \cancel{P(\text{exactly } n+1, v+\lambda)} \cdot \left(\frac{v}{v+\lambda}\right) \cdot \left(\frac{\lambda}{v+\lambda}\right)^n$

$\cancel{P(n+1, v+\lambda)}$ $Z(n+1) = 1 - \sum_{i=0}^{n} P(i, v+\lambda)$

$= 1 - \left(\sum_{i=0}^{n} e^{-(v+\lambda)} \cdot \frac{(v+\lambda)^i}{i!}\right)$

$P(N=n \mid Z(n+1)) = \left(\frac{v}{v+\lambda}\right) \cdot \left(\frac{\lambda}{v+\lambda}\right)^n$. ? ? ?

25. Go!!! 冲 ✓ An infinite ~~severe~~ server queue.

customer arrives $\sim$ Poisson(λ) rate.

x_i = ith customer stay

$x_i \in \{1, 2, \cdots, n\}$

? N_t = number of customers at time t. (assume that $t \geq n$)

$E(N_t) = E(E(N_t \mid N)) = \sum_{n=1}^{\infty} E(N_t \mid N) \cdot P(N=n)$

$E(N_t \mid N=n) = \cancel{P(\ldots n, \lambda)}$ $E(\text{Poisson}(n\lambda)) = n\lambda$.

$E(N_t) = E(N \cdot \lambda) = \lambda \cdot E(N) = \lambda \cdot E(X_i)$. ~~[illegible]~~ $(N \triangleq X_i)$. □

26. $P(X_1=x_1, X_2=x_2) = P(X_1+X_2=x_1+x_2, X_1=x_1)$

$X_1 \sim Poisson(\lambda p)$, $X_2 \sim Poisson(\lambda(1-p))$

$$P(X_1=x_1, X_2=x_2) = e^{-\lambda \cancel{(x_1+x_2)}} \cdot \frac{\lambda^{(x_1+x_2)}}{(x_1+x_2)!} \cdot \binom{x_1+x_2}{x_1} \cdot p^{x_1} \cdot (1-p)^{x_2}$$

$$= e^{-\lambda} \cdot p^{x_1} \cdot (1-p)^{x_2} \cdot \frac{\lambda^{x_1} \cdot \lambda^{x_2}}{x_1! \cdot x_2!} \cdot p^{x_1} \cdot (1-p)^{x_2}$$

$$= e^{-\lambda p} \cdot p^{x_1} \cdot \lambda^{x_1} \cdot \cancel{p} \frac{1}{x_1!} \cdot e^{-\lambda(1-p)} \cdot \frac{(\lambda(1-p))^{x_2}}{x_2!}$$

$$= P(X_1=x_1) \cdot P(X_2=x_2). \quad \square$$

27. $\cancel{F(E=e) = F(L=e+E(}$

$F(L_2=l) = \cancel{F(L_1=l-E(Poisson(\frac{2}{\lambda})))}$

$= F(L_2-L_1+L_1=l)$

$h_1(L_2-L_1)$ is the time from t to the first time ~~when~~ of arriv

since the memorylessness of Poisson process, $h_1(L_2-L_1)$ obeys Poisson($\frac{2}{\lambda}$) distribution, L_1 is a Poisson($\frac{2}{\lambda}$) distribution.

Symmetrically, $(L_2-L_1)-h_1(L_2-L_1)$ is also Poisson($\frac{2}{\lambda}$) distributi

$L_2 \sim Erlang(3, \frac{\lambda}{2})$. $\square$