Chapter 7. p380

1. (facility 设备)

one of states should correspond to the times when an arrival occurs. ?

2. ~~因~~ $p_{i,i+1} = \begin{cases} 1 & i=1 \\ \frac{1}{2} & 1<i<2m \\ 0 & i=2m, \end{cases}$

$p_{i,i-1} = \begin{cases} \frac{1}{2}, & 2m>i>1 \\ 0 & i=1 \\ 1 & i=2m. \end{cases}$

$X_n = L$ means ~~在~~ $i \le m$,

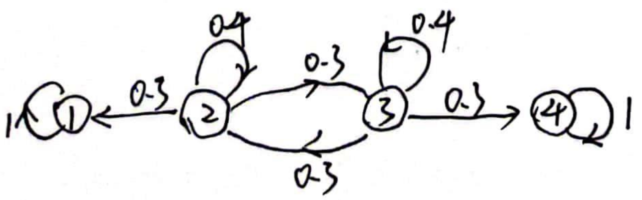
$X_n = R$ means $i > m$.

~~故~~ $P(X_{n+1}=L \mid X_n=R, X_{n-1}=L) = \frac{1}{2}$,

$P(X_{n+1}=L \mid X_n=R, X_{n-1}=R, X_{n-2}=L) = 0$,

it doesn't obey the property that X_{n+1} is independent of the history under condition of X_n.

3. no. $P(Y_2=2 \mid Y_1=1, Y_0=2) = 0.3$

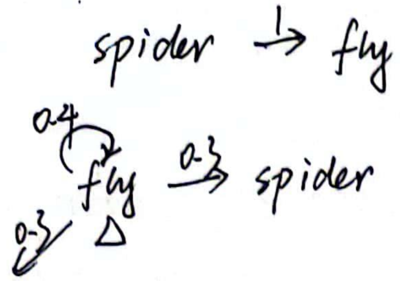


$\triangle 1$ $\triangle 2$

$P(Y_2=2 \mid Y_1=1, \cancel{\quad}) = P(Y_2=2, \cancel{\quad} X_1=1) + P(Y_2=2, X_1=2)$

$= P(Y_2=2, X_1=1) + P(Y_2=2 \mid \cancel{\quad} Y_1=1, Y_0=2) \ge 0$

4.



(a) since the distance between fly and spider are independent (irrelevant) of the history.

Assume that X_n denotes the distance at time n.

(b) ~~$P(X_n = i \mid X_{n-1} = i) = 1, \quad i \geq 0$~~

~~$P(X_n = i+1 \mid X_n =$~~

$P(X_n = 0 \mid X_{n-1} = 0) = p_{00} = 1 \qquad n \geq 0.$

$P_{10} = P(X_n = 0 \mid X_{n-1} = 1) = \cancel{P(0.4)}\ 0.4,$

$P_{11} = 0.3,$

$P_{i,(i-2)} = 0.3, \quad P_{i,(i-1)} = 0.4, \quad P_{i,(i)} = 0.3 \qquad (i \geq 2).$

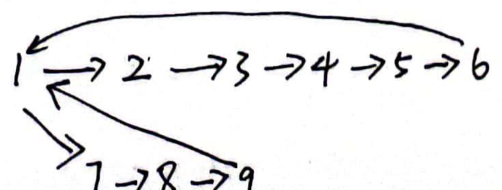
~~all state are recurrent~~

~~all state are recurrent state, since $p_{ii} > 0$ for all $i \geq 0$.~~

$\forall i > 0$, $A(i) = \{i, i-1, i-2\}$, i are transient state.

$A(0) = \{0\}$, 0 is recurrent state. □

5. Go! $p_{12} = p_{17} = 1/2$, $p_{i(i+1)} = 1$, for $i \neq 1, 6, 9$. $p_{61} = p_{91} = 1$.



$\{1, 2, \cdots, 9\}$ is reccurrent class.

it's periodic.

it can be seperated into $\{1, 3, 5, 8\}$, $\{2, 4, 6, 7, 9\}$

6. Go! 冲锋陷阵.

~~assume that~~ for any i in transient states.

assume that i is not in any $UA(j)$: j is recurrent state.

each transient states can be reached for 1 time.

it's contradictory.

7. (a) $\exists\, c > 0,\ 0 < \gamma < 1.$

show $P(X_n \text{ is transient} \mid X_0 = i) \le c \cdot \gamma^n,\ \forall\ i,\ n \ge 1.$

only need to concern if i is transient.

and path from i to j (j is transient), the states in the path are all transient.

~~set $\gamma = 1 - \min\{P_{ij} \mid i \text{ is transient}, j \text{ is recurrent}\}$.~~

we set $c > \frac{1}{\min(\gamma^n)} = \frac{1}{\gamma^N}$ $(n \in N)$

we have $c \cdot \gamma^n > 1.$

(b) $\sum_i P(X_n \text{ is first time recurrent} \mid X_0 = i) \cdot P(X_0 = i)$

$= \sum_i P(X_n \text{ is recurrent first tim}, X_{n-1} \text{ is transient}, X_0 = i)$

$\le \sum_i P(X_{n-1} \text{ is transient}, X_0 = i)$

$\le c \cdot \gamma^{n-1}$. , we can rewrite it to $c \cdot \gamma^n$ for simplicity

$E(T) = E(c \cdot \gamma^n) = \sum_{n=1}^{+\infty} n \cdot c \cdot \gamma^n = \frac{c \cdot \gamma}{(1-\gamma)^2} < +\infty.$

8. (Recurrent State)

definition of recurrent state is that

for $\forall$ $j \in A(i)$, $i \in A(j)$.

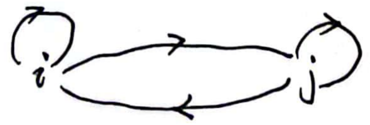
~~so assume that $i \to j$ i can reach j in m_1 steps, and j can reach i in m_2 steps.~~

assume that there exists a path that it can't return to i in any steps, starting from i.

~~set~~ ~~the~~ And there must exist a path $i \to i$, we denote it as T_i, $P(T_i) = c > 0$

$P(T_i) > 0 \Rightarrow P($ path can't return to $i) < 1 - c < 1$

In the recurrent class, we group the state except i into state j. p_{ij} transition graph shows as following.



$$P(X_n = i, X_{n-1} = X_{n-2} = \cdots = X_1 = j \mid X_0 = i)$$

$$= p_{ij} \cdot (p_{jj})^{n-2} \cdot p_{ji} \qquad (n \geq 2)$$

$$P(X_n = j = X_{n-1} = \cdots = X_1 \mid X_0 = i)$$

$$= p_{ij} \cdot (p_{jj})^{n-1} \qquad (n \geq 1)$$

$p_{jj} < 1$, we have $\lim\limits_{n \to \infty} p_{ij} \cdot (p_{jj})^{n-1} = 0$

$\Rightarrow P($ it will return to $i \mid X_0 = i) = 1$.

9. (i) R can be grouped into $\{S_1, \cdots, S_d\}$, disjoint, $d>1$.

$$\begin{tikzcd} S_1 \arrow[r] & S_2 \arrow[r] & \cdots \arrow[r] & S_d \arrow[lll, bend left] \end{tikzcd}$$

(ii) $\exists\, n$, s.t. $\forall\, i,j \in R$, $r_{ij}(n) > 0$.

① (i) $\Rightarrow \neg$(ii), $\quad \neg$(ii) $\Leftrightarrow \forall\, n$, $\exists\, i_0, j_0 \in R$, $r_{i_0 j_0}(n) = 0$.

since $p_{ij} > 0$ if and only if $i \in S_k$, $j \in S_{k+1}$,

assume that ~~S_{k+1}~~ $S_n = S_{n \% d}$.

we have $r_{ij}(n) > 0$ ~~only~~ ~~if~~ ~~and~~ only if $i \in S_t$ and $j \in S_{t+n}$

so if $i_0 \in S_t$, and $j_0 \notin S_{t+n}$, we have $r_{i_0 j_0}(n) = 0$.

~~$\neg$(ii) $\Rightarrow$ (i)~~,

② $\neg$(i) $\Rightarrow$ (ii), $\quad \neg$(i) $\Leftrightarrow \forall$ group $\{S_1, \cdots, S_d\}$, $d > 1$,

there must $\exists\, S_{t_1} \cap S_{t_2} \neq \emptyset$,

(ii) $\Leftrightarrow \exists\, n$, $\forall\, i \in R$, we set $S_1 = \{i\}$, we have $S_n = R$.

we need to find such the n parameter.

we need to proove there exists such a n parameter.

first, we consider $S_1 = \{i_1\}$, $\exists\, S_{p_1}$, s.t. $S_1 \cap S_{p_1} \neq \emptyset$,

that is, $i_1 \in S_{p_1}$, ~~and~~

second, we consider $i_2 \in S_{p_1} \setminus S_1$, $\exists\, S_{p_2}$, if $S_1 = \{i_2\}$,

s.t. $S_2 \cap S_{p_2} \neq \emptyset$, and $S_{p_1+p_2} \supset S_{p_1} \cup S_{p_2}$,

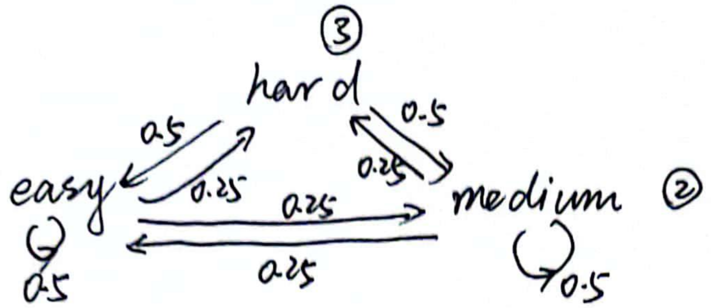
Similarly, we get $\{i_k\}_{k=1}^{m}$, i_k are all different, and

$S_{p_1+\cdots+p_m} \supset S_{p_1} \cup \cdots \cup S_{p_m} \supset R$.

for $\forall\, \{i_i\} \subset R$, it's only changing the order of sequence $\{i_k\}_{k=1}^{m}$,

~~题~~

11.



steady-state probabilities.

Markov chain: set the number of tests 为 N.

X_n ~~...~~ · denotes the n^{th} test ~~...~~

$$X_n = \begin{cases} 1 & \text{easy} \\ 2 & \text{medium} \\ 3 & \text{hard} \end{cases}.$$

$$\pi_i = \sum_{k=1}^{3} \pi_k \cdot p_{ki} \quad (i = 1, 2, 3)$$

$$\begin{cases} \pi_1 = 0.5 \cdot \pi_1 + 0.25\,\pi_2 + 0.5\,\pi_3 \\ \pi_2 = 0.25\,\pi_1 + 0.5\,\pi_2 + 0.5\,\pi_3 \\ \pi_3 = 0.25\,\pi_1 + 0.25\,\pi_2 \end{cases}$$

normalization equation: $\pi_1 + \pi_2 + \pi_3 = 1$.

$$\begin{cases} \pi_1 = 0.4 \\ \pi_2 = 0.4 \\ \pi_3 = 0.2 \end{cases}. \quad \square$$

12. Go! 冲冲冲冲冲冲.

13. when we start to observe the chain, it is already in the steady-state.

(a) balance equation: $0.6 \cdot \pi_1 = 0.3 \cdot \pi_2$, $0.2 \cdot \pi_2 = 0.2 \cdot \pi_3$

normalization equation: $\pi_1 + \pi_2 + \pi_3 = 1$

$$\Rightarrow \begin{cases} \pi_1 = 0.2 \\ \pi_2 = 0.4 \end{cases}$$

(b) $\cancel{P(X_1=}$

$p(X_1=2, X_0=1) + p(X_1=3, X_0=2)$

$= P_{12}\cdot\lambda_1 + P_{23}\cdot\lambda_2$

$= 0.6\times0.2 + 0.2\times0.4 = 0.2$

(c) $\cancel{P(X_1=2,}$ $P(X_1=2, X_0=1 \mid X_1\neq 1)\cdot\lambda_1 + P(X_1=3, X_0=2 \mid X_1\neq X_0)\cdot\lambda_2$

$= 0.2 + 0.4\times0.4$

$= 0.36.$

(d) $P(\text{state was } 2 \mid \text{first transition is birth})$

$$= \frac{P(\text{state } 2, \text{ first transition is birth})}{P(\text{first transition is birth})}$$

$$= \frac{\lambda_2\cdot0.2}{0.2} = 0.4.$$

(e) $$P = \frac{P(\text{state was } 2, \text{ first change is birth})}{P(\text{first change is birth})}$$

$$= \frac{\lambda_2\cdot0.4}{0.36} = \frac{4}{9}$$

(f) $P(\text{first transition is birth} \mid \text{change of state})$

$$= \frac{0.2\times0.6 + 0.4\times0.2}{0.2\times0.6 + 0.4\times0.5 + 0.4\times0.2} = 0.5$$

(g) $P(\text{lead to state } 2 \mid \text{change of state})$

$$= \frac{0.2\times0.6 + 0.4\times0.2}{0.2\times0.6 + 0.4\times0.5 + 0.4\times0.2} = 0.5. \quad \square$$

14. $n \geq 500$,

(a) $P(X_{1000}=j, X_{1001}=k, X_{2000}=l \mid X_0=i)$

$= P(X_{2000}=l \mid X_{1001}=k) \cdot P(X_{1001}=k \mid X_{1000}=j) \cdot P(X_{1000}=j \mid X_0=i)$

$\approx \pi_l \cdot p_{jk} \cdot \pi_j$.

(b) $P(X_{1000}=i \mid X_{1001}=j) = \dfrac{P(X_{1001}=j \mid X_{1000}=i)\cdot P(X_{1000}=i)}{P(X_{1001}=j)}$

$\approx \dfrac{p_{ij}\cdot \pi_i}{\pi_j}$. □

15. 加油！没事的，肯定能行。

(Ehrenfest model of diffusion)

n balls, black and white

nothing, ε, $0<\varepsilon<1$, choose ball $\frac{1-\varepsilon}{n}$.

(vice versa 反之亦然) vice versa.

(white → black, black → white).

solve steady-state distribution of number of white balls. ?

Build the Markov chain: $X_n=i$ denotes the nth turn, there is i white balls among n balls.

$P(X_{n+1}=i+1 \mid X_n=i) = \frac{1-\varepsilon}{n}\cdot(n-i)$,

$P(X_{n+1}=i-1 \mid X_n=i) = \frac{1-\varepsilon}{n}\cdot(i)$

$P(X_{n+1}=i \mid X_n=i) = \varepsilon$.

$\begin{cases} p_{ij}\cdot\pi_i = p_{ji}\cdot\pi_j & (j=i\pm1) \end{cases}$

$\Rightarrow$

$$p_{i,i+1} \cdot x_i = p_{i+1,i} \cdot x_{i+1},$$

$$\frac{x_{i+1}}{x_i} = \frac{p_{i,i+1}}{p_{i+1,i}} = \frac{\frac{1-k}{n} \cdot (n-i)}{\frac{1-k}{n} \cdot (i+1)} = \frac{n-i}{i+1} \qquad (i \geq 1)$$

$$\cancel{x_i = \left(\frac{n-1}{i}\right)^i \cdot x_0,}$$

~~assume that~~ $\cancel{\frac{n-1}{i} = p_i > 0,}$

$$\cancel{\sum_{i=0}^{n} x_i = \sum_{i=0}^{n} \left(\frac{n-1}{i}\right)^i \cdot x_0 = \frac{1-p^{(n+1)}}{1-p} \cdot x_0 = 1,}$$

$$\cancel{x_0 = \frac{1-p}{1-p^{(n+1)}}, \quad x_i = p^i \cdot \frac{(1-p)}{1-p^{(n+1)}}, \quad (i \geq 0).}$$

$$\cancel{x_i = \left(\frac{n-i}{i}\right)^i}$$

$$\cancel{\sum_{i=0}^{n} x_i = \sum_{i=0}^{n} \left(\frac{n-1}{i}\right)^i \cdot x_0 = x_0.}$$

$$\frac{x_{i+1}}{x_i} = \frac{n-i}{i+1}, \qquad x_i = \prod_{k=0}^{i-1} \left(\frac{n-k}{1+k}\right) \cdot x_0 = \binom{n}{i} \cdot x_0.$$

$$\cancel{x_1 \cdot p_{10} = x_0 \cdot p_{01}, \quad \frac{x_1}{x_0} = \frac{p_{01}}{p_{10}}}$$

$$\sum_{i=0}^{n} x_i = \sum_{i=0}^{n} \binom{n}{i} \cdot x_0 = 1.$$

$$x_0 = \frac{1}{2^n}, \quad x_i = \binom{n}{i} \cdot \frac{1}{2^n} \qquad (i \geq 0).$$

16. (Bernoulli-Laplace model of diffusion)

(out of the total of 占总数的)

assume that $X_n = i$ denotes in the n th times, the urn ~~with m white ball initially~~ have i ~~bal~~ white balls.

$$P(X_n = i+1 \mid X_{n-1} = i) = \frac{m-i}{m} \cdot \frac{(m-i)}{m} = \left(\frac{m-i}{m}\right)^2$$

$$P(X_n = i-1 \mid X_{n-1} = i) = \left(\frac{i}{m}\right)^2$$

$$P(X_n = i \mid X_{n-1} = i) = 1 - \left(\frac{m-i}{m}\right)^2 - \left(\frac{i}{m}\right)^2$$

$$\begin{cases} \cancel{p_{ij} \cdot \pi_i = p} \quad p_{i,i+1} \cdot \pi_i = p_{i+1,i}\, \pi_{i+1} & \forall\ 0 \le i < m. \\ \sum_{i=0}^{m} \pi_i = 1 \end{cases},$$

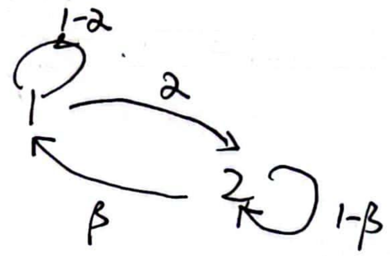
$$\frac{\pi_{i+1}}{\pi_i} = \frac{p_{i,i+1}}{p_{i+1,i}} = \left(\frac{m-i}{1+i}\right)^2$$

$$\pi_{i+1} = \binom{m}{i+1}^2 \pi_0, \quad \pi_i = \binom{m}{i}^2 \pi_0 \ (0 \le i \le m).$$

$$\sum_{i=0}^{m} \binom{m}{i}^2 \cdot \pi_0 = 1 \Rightarrow \binom{2m}{m} \cdot \pi_0 = 1$$

$$\Rightarrow \pi_i = \frac{\binom{m}{i}^2}{\binom{2m}{m}} \ (0 \le i \le m).$$

17. 冲!



(a) obviously, it is aperiodic and recurrent class.

(b) $r_{11}(1) = p_{11} = 1-\alpha$

$r_{11}(2) = r_{11}(1) \cdot p_{11} + r_{12}(1) \cdot p_{21} = (1-\alpha)^2 + \alpha(1-\alpha).$

use inductive method, we can verify it.

(c) $\pi_1 = \frac{\beta}{\alpha+\beta}$, $\pi_2 = \frac{\alpha}{\alpha+\beta}$, as $n \to +\infty$.

19. (steady-state convergence).

(a) pass. assume that $c \cdot \gamma^n \to 1$.

(b) (occupancy 占用率)

$T = \min\{n \mid X_n = T_n\}$,

$P(X_n = j \mid T \le n) = \sum_{t=1}^{n} P(X_n = j, T = t \mid T \le n)$

$= \sum_{t=1}^{n} P(X_n = j \mid T \le n, T = t) \cdot P(T = t \mid T \le n)$

$$P(X_n=j \mid T\le n) = \sum_{t=1}^{n} P(X_n=j \mid T\cancel{\le n}=t)\cdot P(T=t \mid T\le n)$$

$$= \sum_{t=1}^{n} \sum_{k\in R} P(X_n=j,\ X_t=k \mid T=t)\cdot P(T=t \mid T\le n)$$

$$= \sum_{t=1}^{n} \sum_{k\in R} P(X_n=j,\ \cancel{Y_t=k}\ X_t=k)\cdot P(T=t \mid T\le n)$$

$$= \sum_{t=1}^{n} \sum_{k\in R} P(X_n=j \mid X_t=k, T=t, T\le n)\cdot P(X_t=k \mid T=t, T\le n)$$
$$\cdot P(T=t \mid T\le n)$$

$$= \sum_{t=1}^{n} \sum_{k\in R} P(X_n=j \mid X_t=k)\cdot P(X_t=k,\ T=t \mid T\le n)$$

$$= \sum_{t=1}^{n} \sum_{k\in R} r_{ij}(n-t)\cdot P(Y_t=k,\ T=t \mid T\le n)$$

$$= P(Y_n=j \mid T\le n).$$

(c) $r_{ij}(n)\ \cancel{-r_{kj}(n)} = P(X_n=j \mid X_0=i)\cdot P(X_0=i)$

$$= P(X_n=j)$$

$$= P(X_n=j \mid T\le n)\cdot P(T\le n) + P(X_n=j \mid T>n)\cdot P(T>n)$$

$$r_{kj}(n) = P(Y_n=j \mid T\le n)\cdot P(T\le n) + P(Y_n=j \mid T>n)\cdot P(T>n).$$

$$|r_{ij}(n) - r_{kj}(n)| = |P(T>n)|\cdot\left|P(Y_n=j \mid T>n) - P(X_n=j \mid T>n)\right|$$

$$\le |P(T>n)| \le c\cdot\gamma^n$$ (from (a) solution)

(from (a) solution).

(d) $q_j^+(n) \triangleq \max_i r_{ij}(n)$, $\quad q_j^-(n) \triangleq \min_i r_{ij}(n)$.

$$r_{ij}(n+1) = \sum_{k\in R} r_{ik}^{(n)} p_{kj} = \sum_{k\in R} p_{ik}\cdot r_{kj}(n)$$

$$\le \sum_{k\in R} p_{ik}\cdot q_j^+(n) = q_j^+(n). \quad \forall i$$

$\Rightarrow q_j^+(n+1) \le q_j^+(n)$. similarly $q_j^-(n) \le q_j^-(n+1)$.

(e) we have $|r_{ij}(n) - r_{kj}(n)| \le c \cdot \gamma^n$ for $\forall\, i, j, k$

from (c).

$\Rightarrow \lim_{n\to\infty} r_{ij}(n) - r_{kj}(n) = 0 \quad \forall\, i, j, k$

since $q_j^-(n) \le q_j^-(n+1)$, $q_j^-(n)$ is montotic and bounded

same as $q_j^+(n)$,

since $q_j^+(n) - q_j^-(n) = \max_{i,k} (r_{ij}(n) - r_{kj}(n)) \le c \cdot \gamma^n \to 0$ as $n \to +\infty$

$\Rightarrow \lim_{n\to\infty} q_j^-(n) = \lim_{n\to\infty} q_j^+(n)$.

and $q_j^-(n) \le r_{ij}(n) \le q_j^+(n) \quad \forall\, i, j$,

$\Rightarrow$ $r_{ij}(n)$ have the limitation as $n \to +\infty$, $\forall\, i, j$.

$r_{ij}(n)$ uniformly converge to the limitation. $\square$

20. (a) show uniqueness,

assume that $r_{ij}(n)$ $\forall i$ has two limitation, π_j, $\bar{\pi}_j$.

$P(X_n = j \mid X_0 = i) = r_{ij}(n)$

$\bar{\pi}_j = \lim_{n\to\infty} P(X_n = j)$

$\pi_j = \lim_{n\to\infty} P(X_n = j)$

$\Rightarrow \bar{\pi}_j = \pi_j$.

(b) assume that a new Markov chain, has the transition probabilities $\bar{p}_{ij}$.

$\bar{p}_{ii} \triangleq (1-\alpha) \cdot p_{ii} + \alpha$, $\quad \bar{p}_{ij} = (1-\alpha) \cdot p_{ij} \quad (i \ne j)$.

the balance equations are

$$\pi_j = \pi_j \cdot \bar{p}_{jj} + \sum_{k\neq j} \pi_k \cdot \bar{p}_{kj}$$

$$= \pi_j \cdot \left((1-\alpha) p_{jj} + \alpha \right) + \sum_{k\neq j} \pi_k \cdot (1-\alpha) p_{kj}$$

$$\Leftrightarrow \quad (1-\alpha)\cdot \pi_j = \sum_{k\in R} \pi_k \cdot (1-\alpha) \cdot p_{kj} \qquad (0<\alpha<1)$$

$\Leftrightarrow$ the balance equations with transition probabilities p_{ij}.

hence, $\bar{\pi}_j = \pi_j$, and $\bar{p}_{jj} > 0, \forall j$. new Markov chain is aperiodic obviously.

the new Markov chain have the same unique π_j.

19. (a) ~~set β = min~~

$$P(T \geq n) = 1 - P(T<n)$$

$$= 1 - P(\min\{n \mid X_n = Y_n\} < n)$$

$$< 1 - P(X_n = Y_n)$$

$$< 1 - P(X_n = Y_n = \ell)$$

$$= 1 - P(X_n = \ell, Y_n = \ell)$$

assume that ℓ is the recurrent-state,

since X, Y are aperiodic with one recurrent class,

$\exists\, \bar{n} > 0$, s.t. $r_{i\ell}(\bar{n}) > 0$, $\forall\, i, \ell$

set $\beta = \min_i r_{i\ell}(\bar{n}) > 0$,

$$\Rightarrow P(T \geq \bar{n}) \leq 1 - \sum_{k\in R} P(X_0 = k) \sum P(Y_0 = k) \cdot \beta^2$$

$$= 1 - \beta^2.$$

$P(T \geq \bar{n}) \leq 1-\beta^2.$

set $a = [\frac{n}{\bar{n}}]$, denotes the max(integer $| \ a \leq \frac{n}{\bar{n}}$).

$\cancel{P(T\geq\bar{n})}$ $P(T \geq 2\bar{n}) = P(T \geq 2\bar{n} \mid T \geq \bar{n}) \cdot P(T \geq \bar{n})$

$P(T \geq 2\bar{n} \mid T \geq \bar{n}) = P(T \geq 2\bar{n} \mid T \geq \bar{n},\ X_{\bar{n}-1} \neq Y_{\bar{n}-1})$

$= P(T \geq \bar{n}) < 1-\beta^2.$

$\Rightarrow P(T \geq k\bar{n}) \leq (1-\beta^2)^k \qquad \forall\ k \in \mathbb{N}^+.$

$P(T \geq n) \leq P(T \geq [\frac{n}{\bar{n}}] \cdot \bar{n}) \leq (1-\beta^2)^a$, $\quad a \triangleq [\frac{n}{\bar{n}}]$

$(1-\beta^2)^a = (1-\beta^2)^{\frac{1}{\bar{n}}(a \cdot \bar{n})}$,

since $a \cdot \bar{n} \leq n$, $\ a \cdot \bar{n} \geq n - \bar{n}$,

$P(T \geq n) \leq (1-\beta^2)^{\frac{1}{\bar{n}}(n-\bar{n})} = (1-\beta^2)^{\frac{n}{\bar{n}}-1}$,

set $\gamma = (1-\beta^2)^{\frac{1}{\bar{n}}} < 1$, $\quad c = (1-\beta^2)^{-1}$, we have the inequality

$P(T \geq n) \leq c \cdot \gamma^n.$ $\square$

21. Go! 冲!

show $r_{ij} = \lim_{n\to\infty} \frac{V_{ij}(n)}{n}$

since we have ~~property~~ proposition which can be proved:

for series $\{a_n\}_{n=1}^{\infty}$, we have $\lim_{n\to\infty} a_n = a$,

$\Rightarrow \lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_i}{n} = a.$

$V_{ij}(n) = \sum_{k=1}^{n} r_{ij}(k)$, $\quad \lim_{n\to\infty} \frac{V_{ij}(n)}{n} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} r_{ij}(k)}{n} = \lim_{n\to+\infty} r_{ij}(n).$

22. doubly stochastic: $\sum_{i=1}^{m} p_{ij} = 1, \ \forall j$

$\iff \sum_{i=1}^{m} P(X_{n+1}=j \mid X_n = i) = 1, \ \forall j$

(a)

(b) balance equation:

$\pi_j = \sum_{k=1}^{m} \pi_k \cdot p_{kj}, \ \forall j$

$\sum_{k=1}^{m} p_{kj} = 1, \ \forall j,$

$\sum_{j=1}^{m} \pi_j = 1,$

we know that the equations above have ~~one~~ unique solution.

~~the~~ Through verification, $\pi_1 = \pi_2 = \cdots = \pi_m = \frac{1}{m}$ confirms.

(c) In 20 questions, (b) we know,

When the periodic situation holds, equations above have the same unique solutions, that is $\pi_1 = \pi_2 = \cdots = \pi_m = \frac{1}{m}$.

23. (Queueing).

24. (dependence of the balance equations).

$\sum_{j=1}^{m-1} \pi_j = \sum_{k=1}^{m} \pi_k \left(\sum_{j=1}^{m-1} p_{kj} \right)$

$\iff \pi_m = \sum_{k=1}^{m} \pi_k \cdot (p_{km})$

so with equation $\sum_{j=1}^{m} \pi_j = 1$, we have a system of inhomogeneous equations, ~~we~~ these equations must

have one or more solutions (homogeneous: ...

25. (a) balance equations:

$$\pi_j = \sum_{k=1}^{m} \cancel{\text{}}\ \pi_k \cdot p_{kj}, \quad \forall j.$$

$$\cancel{\pi_j \cdot p_{ji} = \sum_{k=1}^{m} \pi_k \cdot p}$$

proove $\pi_i p_{ij} = \pi_j \cdot p_{ji} \quad \forall j \Rightarrow \pi_j = \sum_{k=1}^{m} \pi_k \cdot p_{kj}, \forall j.$

$$\sum_{k=1}^{m} \pi_k \cdot p_{kj} = \sum_{k=1}^{m} \pi_k \cdot p_{kj} = \sum_{k=1}^{m} \pi_j \cdot p_{jk} = \pi_j . \ \square$$

(b) $\pi_i p_{ij} = \lim_{n\to\infty} r_{si}(n) \cdot p_{ij}$

$$= \lim_{n\to\infty} P(X_n = i \mid X_0 = s) \cdot P(X_{n+1} = j \mid X_n = i)$$

$$= \lim_{n\to\infty} P(X_{n+1} = j, X_n = i \mid X_0 = s)$$

$$\lim_{n\to\infty} P(X_{n+1} = j, X_n = i) = \lim_{n\to\infty} P(X_{n+1} = i, X_n = j).$$

$\pi_i p_{ij}$ can be explained that the frequency from i to j.

$$i \rightleftarrows j$$

~~to cool~~

(c) $\pi_1 \cdot p_{12} = \pi_2 \cdot p_{21}$ local balance equation.

$$\cancel{\pi_2 \cdot p_{21} = \pi_2 \cdot p}$$

balance equation:

$$\pi_1 = \pi_1 \cdot p_{11} + \pi_2 \cdot p_{21}$$

Can't.

$\pi_1 \cdot p_{12} = \pi_2 \cdot p_{21}$, $\pi_1 \cdot p_{13} = \pi_3 \cdot p_{31}$, $\pi_2 \cdot p_{23} = \pi_3 \cdot p_{32}$.
(local balance equation ↗)

balance equation:

$\pi_1 = \cancel{p}\ \pi_1 \cdot p_{11} + \pi_2 \cdot p_{21} + \pi_3 \cdot p_{31}$, $\Rightarrow$ $(p_{23} \cdot p_{12} + p_{13} \cdot p_{21} + p_{23} \cdot p_{13}) \cdot \pi_1 =$
$(p_{31} \cdot p_{21} + p_{31} \cdot p_{23} + p_{32} \cdot p_{21})\ \pi_1$

we conclude from local balance equation $\Rightarrow$

$$p_{13}\cdot\lambda_1 = p_{31}\cdot\lambda_3.$$

if we wan't to make a specific λ_1, λ_3 not obeying the equation above, we have

$$\frac{p_{31}}{p_{13}} \neq \frac{p_{31}\cdot p_{21}+p_{31}\cdot p_{23}+p_{32}\cdot p_{21}}{p_{23}\cdot p_{12}+p_{13}\cdot p_{21}+p_{23}\cdot p_{13}}$$

$$\Leftrightarrow p_{31}\cdot p_{23}\cdot p_{12} \neq p_{13}\cdot p_{32}\cdot p_{21}.$$

we have the contradictory example.

$$(p_{ij})_{m\times m} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

26. (a) $r_{ij}(n+l) = P(X_{n+l}=j \mid X_0=i)$

$$= \sum_{k=1}^{m} P(X_{n+l}=j, X_l=k \mid X_0=i)$$

$$= \sum_{k=1}^{m} P(X_{n+l}=j \mid X_l=k)\cdot P(X_l=k \mid X_0=i)$$

$$= \sum_{k=1}^{m} r_{kj}(n)\cdot r_{ik}(l).$$

(b) ~~we assume the transition of Y is $r_{ij}(l) = P(Y_l=j \mid Y_0$~~

first we proove Y is a Markov chain:

$$P(Y_{n+1}=j \mid Y_n=i) = P(X_{l(n+1)} \mid X_{nl}=i) = r_{ij}(l).$$

transition probabilities are $r_{ij}(l)$, for $\forall i,j$.

And $P(Y_n=j \mid Y_{n-1}=i, Y_t=k)$ $t<n-1$,

$=P(Y_n=j \mid Y_{n-1}=i)$

. Y_n is independent of the history under condition of Y_{n-1}, thus Y has Markov property.

Since X is single aperiodic, from problem 9,

we have $\exists\, n$, s.t. $r_{ij}(n)>0$ for all $i,j \in$ recurrent class.

$\Rightarrow r_{ij}(kn)>0$, $\forall\, i,j,k$.

$\Rightarrow$ recurrent class in X is also held in Y.

and it is aperiodic.

$\forall$ the ~~point~~ state i $\in$ transient states.

$\forall$ $j \in A(i)$, $A(i)$ is the set that there exists a path from i to $A(i)$.

if i is transient, $\forall\, j \in A(i)$, $i \notin A(j)$

it holds in Y too.

so Y is single aperiodic recurrent class.

(c) $\lim_{n\to\infty} r_{ij}(n) = \lim_{n\to\infty} r_{ij}(ln) = \pi_j$.

$\star$ $P(Y_n=j \mid Y_0=i) = P(X_{ln}=j \mid X_0=i) = r_{ij}(ln)$

X and Y have the same steady-states probability π_j.

27. (a) $P((X_{n-1}, X_n) = (i, j)) = P(X_{n-1} = i, X_n = j)$

$$= P(X_n = j \mid X_{n-1} = i) \cdot P(X_{n-1} = i)$$

$$= p_{ij} \cdot P(X_{n-1} = i) \cancel{= \pi_i \cdot p_{ij}}.$$

since X_n chain is a single recurrent aperiodic chain.

it has ~~stead~~ steady-state, $P(X_n = i) = \pi_i$ as $n \to +\infty$.

$P((X_{n-1}, X_n) = (i, j)) \to \pi_i \cdot p_{ij}$ as $n \to \infty$.

(b) $P((X_{n-k}, \cdots, X_n) = (i_0, \cdots, i_k))$

$= P((X_{n-k} = i_0, \cdots, X_n = i_k)$

$= P(X_{n-k} = i_0) \cdot p_{i_0, i_1} \cdots p_{i_{(k-1)}, i_k}$

$\to \pi_{i_0} \cdot p_{i_0, i_1} \cdots p_{i_{(k-1)}, i_k}$ as $n \to +\infty$.

29.

(a) $X_0 = 1$,

$r_{11}(6) = P(X_0 = 1, X_1 = 2, X_2 = 2, X_3 = 2, X_4 = 3, X_5 = 4, X_6 = 1)$

$+ P(1\ 2\ 2\ 3\ 4\ 4\ 1) + P(1\ 2\ 3\ 4\ 4\ 4\ 1)$

$= (\frac{1}{3})^2 \cdot \frac{2}{3} \cdot \frac{2}{5} \cdot \frac{3}{5} + \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{2}{5} \cdot \frac{3}{5} + \frac{2}{3} \cdot (\frac{2}{5})^2 \cdot \frac{3}{5}$

$= \frac{182}{1125}$.

(b) A: $X_{1000} \neq X_{999}$, $X_{1000} \neq X_{1001}$

$P(A) = \cancel{P(X_{1001} = 2, X_{1000} = 1, X_{999} = 4) +}$

$P(X_{999} = 1) \cdot p_{12} \cdot p_{23} + P(X_{999} = 2) \cdot p_{23} \cdot p_{34} + P(X_{999} = 3) \cdot$

$+ \cdots$

31. $\begin{cases} a_s = 1 \\ a_i = 0 & \text{, for all recurrent } i \neq s \\ a_i = \sum_{k=1}^{m} p_{ik} \cdot a_k & \text{, for all transient } i. \end{cases}$

there is m equations and m variables, we know that the solution must exist.

Now proove the solution is unique.

assume that $\bar{a}_i$ is another solution.

$\delta_i \triangleq \bar{a}_i - a_i$,

$$\Rightarrow \begin{cases} \delta_s = 0 \\ \delta_i = 0 & \text{for all recurrent } i \\ \delta_i = \sum_{k=1}^{m} p_{ik} \cdot \delta_k & \text{for } \forall \text{ transient } i. \end{cases}$$

$\delta_i = \sum_{k=1}^{h} p_{ik} \cdot \delta_k$, assume that $1-h$ states are transient.

since all path of transient states will reach recurrent state.

we assume that $H = \max\{h:$ ~~$i \to \cdots \to j \to i$~~ transient state i reaches recurrent state for first time, within h transitions

$$\delta_i = \sum_{k=1}^{h} p_{ik} \cdot \delta_k$$

$$= \sum_{k_1=1}^{h} p_{ik_1} \cdot \sum_{k_2=1}^{h} p_{k_1k_2} \cdot \delta_{k_2}$$

$$= \sum_{k_1=1}^{h} p_{ik_1} \cdots \sum_{k_H=1}^{h} p_{k_{H-1}k_H} \cdot \delta_H.$$

H must be recurrent, $\Rightarrow \delta_H = 0$,

$\Rightarrow \delta_i = 0$, $\forall$ $i \in$ transient state.

32. 7| recurrent aperiodic class.

(a) $a_k(k) = 1$

$a_t(k) = 0$, $\forall$ $\overset{t6}{\text{recurrent}}$ class 'k' $\neq k$.

$\forall$ $i \in$ trasient states,

$$a_i(k) = \sum_{j=1}^{m} p_{ij} \cdot a_j(k)$$ 显

(b) ~~since~~ ~~each~~

if i in kth recurrent aperiodic class, j is too.

$r_{ij}(n) = 0$

违 $r_{ij}(n) = \begin{cases} 0 & \text{if } j \text{ is not in kth class} \\ 1 & \text{, if } j \text{ is in kth class.} \end{cases}$

if i is trasient, j is in kth class.

$$\lim_{n\to+\infty} r_{ij}(n) = \lim_{n\to+\infty} \sum_{h=1}^{m} r_{ih}(c) \cdot r_{hj}(n-c) \quad (c \text{ is large enoug}$$

* if $c \geq m$, ~~h must~~ if $r_{ih}(c) > 0$, h must be recurren state,

$$= \lim_{n\to+\infty} \sum_{k\in K} a_i(k) \cdot \pi_j$$

$$= a_i(k) \cdot \pi_j \quad (\text{, if } j \in \text{kth class})$$

π_j is the steady-state probability ~~时~~ when starting from the state in the same reurrent state. $\square$

33. (passage 通道)

$t_i = E(\min\{n \geq 0 \mid X_n = s\} \mid X_0 = i\}$

$t_i = E(\min\{ i \text{ first reach } s\})$

assume that there is another solution ~~为~~. $\bar{t}_i$, $\forall i$.

$\delta_i \triangleq \bar{t}_i - t_i$ $\quad$ $\delta_s = 0$, $\delta_i = \sum_{j=1}^{m} p_{ij} \cdot \delta_j$.

~~[illegible]~~ 若

$$\delta_i = \sum_{j \neq s} p_{ij} \cdot \delta_j$$

$$= \sum_{j_1 \neq s} p_{ij_1} \sum_{j_2 \neq s} p_{j_1 j_2} \cdots \sum_{j_n \neq s} p_{j_{n-1} j_n} \cdot \delta_{j_n}$$

$$= \sum_{j_1 \neq s} \cdots \sum_{j_n \neq s} P(X_n = j_n, \cdots X_1 = j_1 | X_0 = i) \cdot \delta_{j_n}$$

$$= P(X_n \neq s, \cdots, X_1 \neq s | X_0 = i) \cdot \delta_{j_n}$$

(since i will reach s in the future)

$\to P(X_n \neq s, \cdots, X_1 \neq s | X_0 = i) < 1$,

we have $\delta_i \leq \delta_{j_n}$, $\forall j_n \neq s$, $\forall i \neq s$,

it is not satisfied,

unless $\delta_i = 0$, $\forall i \neq s$.

$\Rightarrow \bar{t}_i = t_i$, $\forall i$. unique solution has been prooved.

34. (Balance equations and mean recurrence times).

~~$p_i = E($ number of visits to $i |$ between $2 s)$~~

~~$= \sum_{x} x \cdot P($ between $2s$, x i's visits $|$ between $2s)$~~

~~$= \sum_{x} x \cdot P(x | X_0 = s, X_n = s,$ card$(k | X_k = i, k \leq n)$.~~

~~$= P(X_1 \overset{=i}{} |$ between $2s) + \cdots + P(X_n \overset{=i}{} |$ between $2s)$~~

~~(assume that $X_0 = s$, $X_n = s$)~~

~~$= P(X_1 = i | X_0 = s, X_1 \neq s \cdots, X_{n-1} \neq s, X_n = s) + \cdots$~~

$P(X_1 = k | X_0 = s, X_n = s, \cdots \neq s) \cdot p_{ki} =$

34.

if $i \neq s$,

(a) $p_i = E(X_0 = X_N = s,\ X_1, \dots X_{N-1} \neq s,\ \text{card}(X_h = i \mid h \leq N))$

$= E\left(\sum_{h=1}^{N} P(X_h = i,\ X_0 = X_n = s,\ X_1, \dots, X_{N-1} \neq s \mid N = n)\right)$

$= E\left(\sum_{h=1}^{N-1} P(X_h = i,\ X_0 = X_N = s,\ X_1 \dots, X_{N-1} \neq s)\right)$

$p_k \cdot p_{ki} = E\left(\sum_{h=1}^{N-1} P(X_h = k, \dots) \cdot p_{ki}\right)$.

$p_i = E\left(\sum_{h=1}^{N} P(X_h = i,\ X_0 = X_n = s,\ X_1 \dots, X_{N-1} \neq s \mid N = n)\right)$

$\cancel{p_i = \sum_{n=1}^{\infty} P(X_1 \neq s, \dots, X_{n-1} \neq s,}$

$p_i = E\left(\sum_{h=1}^{N-1} P(X_0 = s, X_1 \dots X_{h-1} \neq s, X_h = i, X_{h+1} \dots X_{N} \neq s, X_N = \right.$

assume that $I_h(N) = 1$, if $X_h = i$, and between 2 s states, there are N steps.

$p_i = E\left(E\left(\sum_{h=1}^{N} I_h(N) \mid N\right)\right)$

$= \sum_{N=1}^{+\infty} \sum_{h=1}^{N} P(X_0 = s, X_1 = \dots X_{h-1} \neq s, X_h = i, X_{h+1} \dots X_{N-1} \neq s, X_N = s)$

$= \sum_{h=1}^{+\infty} \sum_{N=h+1}^{+\infty} P(\dots)$ (since $P(X_{h+1} \neq s, \dots X_N \neq s) \to 0$ as $N \to +\infty$

$= \sum_{h=1}^{+\infty} P(X_0 = s, X_1, \dots X_{h-1} \neq s, X_h = i)$

$= \sum_{h=2}^{+\infty} \sum_{k \neq s} P(X_0 = s, X_1 \neq s, \dots X_h = i, X_{h-1} = k) + P(X_0 = s, X_1 = i)$

$= \sum_{h=2}^{+\infty} \sum_{k \neq s} P(X_0 = s, X_1 \dots, X_{h-2} \neq s, X_{h-1} = k) \cdot p_{k,i} + P(X_0 = s,$

$= \sum_{k \neq s} p_{ki} \cdot p_k + P(X_0 = s, X_1 = i)$

$= \sum^{m} p_{ki} \cdot p_k - p_{si} p_s + P(X_0 = s, X_1 = i)$

since $\rho_s = 1$, we have $P(X_0 = s) = 1$.

(b) $\sum_{i=1}^{m} \pi_i = \sum_{i=1}^{m} \frac{E(\text{number of } i \text{ between } 2\ s)}{E(\text{number between } 2s)} = 1$.

balance equation:

$\pi_i = \sum_{k=1}^{m} \pi_k \cdot p_{ki}, \iff \rho_i = \sum_{k=1}^{m} \rho_k \cdot p_{ki}$.

(c) if i is transient, $\rho_i = 0$ obviously. $s \not\to i$.

if i is recurrent,

proove $\pi_i = \frac{\rho_i}{t_s^*} = \frac{1}{t_i^*}$, that is, $\rho_i = \frac{t_s^*}{t_i^*}$.

$t_s^* = \sum_{i \in R} \rho_i$, $t_i^* = E(\text{number of states between } 2\ i)$

assume that t_j denotes the passage time from $j \neq s$ to s. (the first tim under condition $X_1 = j$.

$t_s^* = 1 + \sum_{j \neq s} P_{sj} \cdot t_j$.

?

(d) ~~obviously~~ (b) shows that $\pi_i = \frac{\rho_i}{t_s^*}$ satisfies the balance equation, and (c) shows that there is unique solution.

(c) we need to proove $\sum \pi_s \cdot t_s^* = 1$ if we fix s.

~~and~~ when $\{\pi_1 -- \pi_m\}$ satisfies the balanee equation.

$t_s^* = 1 + \sum_{j\neq s} p_{sj} \cdot t_j$, ~~ts~~ $t_s = 0$ ①

$t_i = 1 + \sum_{j\neq s} p_{ij} t_j \quad (\forall i \neq s)$ ②

~~multiplying~~ $\pi_i = \sum_{k=1}^{m} \pi_k \cdot p_{ki}$, $\forall i$ ③

Multiplying ① with π_s, ② with π_i

we have $\pi_s \cdot t_s^* + \sum_{i\neq s} \pi_i \cdot t_i = \pi_s + \sum_{i\neq s} \pi_i + \sum_{j\neq s} \pi_s \cdot p_{sj} \cdot t_j$

$+ \sum_{i\neq s} \pi_i \cdot \sum_{j\neq s} p_{ij} t_j$

$RHS = \sum_{i=1}^{m} \pi_i + \sum_{j\neq s} \pi_s \cdot t_{sj} \cdot t_j + \sum_{j\neq s} t_j (\pi_j - \pi_s \cdot p_{sj})$

$= 1 + \sum_{j\neq s} t_j \cdot \pi_j$

$LHS = \pi_s \cdot t_s^* + \sum_{i\neq s} \pi_i \cdot t_i$

$LHS = RHS \iff \pi_s \cdot t_s^* = 1.$

we have $\pi_s = \frac{1}{t_s^*}$ is the unique solution of π_s,

$\forall s \in R$

R: Recurrent class.

35. (Strong laws of large numbers of Markov Chains)

Y_k: time of k th visit to s.

V_n: number of visits to s during first 10 transitions.

(a) $t_s^* = E($ between 2s. the number of transitions between 2s)

$\frac{Y_k}{k} = \frac{(Y_k - Y_{k-1}) + \cdots + (Y_1 - Y_0)}{k}$ $\to E(Y_{n+1} - Y_n) = t_s^*$ ($k \to \infty$

through strong law of

(b) Let v_n, y_k be the values of random variable V_n and Y_k respectively.

n trans, v_n – s

$y_n \to$ trans nth s

y_k k

we have $\lim_{k\to\infty} \frac{y_k}{k} = t_s^*$,

set we have $v_n = k$ times of s in first n transitions.

$y_k \le n < y_{k+1}$,

$$\Rightarrow \frac{y_k}{k} \le \frac{n}{k} = \frac{n}{v_n} < \frac{y_{k+1}}{k} \quad \text{as } k\to\infty$$

$$\lim_{n\to\infty} = \lim_{v_n\to\infty} \frac{n}{v_n} = t_s^*,$$

$$\Rightarrow \lim_{n\to\infty} \frac{v_n}{n} = \frac{1}{t_s^*}. \qquad P\left(\lim_{n\to\infty} \frac{v_n}{n} = \frac{1}{t_s^*}\right) = 1. \ \square$$

(c) from Problem 34, we have $\pi_s \cdot t_s^* = 1$.

$\lim_{n\to\infty} \frac{V_n}{n} = \pi_s$. $\frac{V_n}{n}$ and π_s ?

$$\frac{E(V_n)}{n} = \frac{E(X_1 + \cdots + X_n)}{n} = \frac{n\pi_s}{n} = \pi_s. \ \square$$

37. we consider a continuous Markov chain with state $\{0, 1, 2, 3, 4\}$, $X_n = i$ means the nth minutes, is with state i.

we are required to calculate $E(T_1+T_2+\cdots+T_N)=E(T)\cdot E(N)$

T_i denotes the time of ith passengers need to wait.

① $P(X_{n+1}=i+1 \mid X_n=i) = P(\delta,1) = e^{-\lambda\delta}(\lambda\delta) = (\lambda\delta)+o(\lambda\delta)$

$= \delta \quad (0\le i<4)$

$P(X_{n+1}=i \mid X_n=i+1) = P(\delta,1) = \lambda\delta+o(\lambda\delta) = 2\delta. \quad (0\le i<4)$

δ is a small enough number, approximate to 0.

we have stable-probability, $\pi_0 \cdots \pi_4$.

$\pi_i\cdot\delta = \pi_{i+1}\cdot 2\delta$, $\frac{\pi_{i+1}}{\pi_i}=\frac{1}{2}$, $\forall\ 0\le i<4$.

$\pi_4=(\frac{1}{2})^4\cdot\pi_0 \Rightarrow \pi_0=\frac{16}{31},\ \pi_1=\frac{8}{31},\ \pi_2=\frac{4}{31},\ \pi_3=\frac{2}{31},\ \pi_4=\frac{1}{31}.$

② The process is in steady state,

so $E(T_1+\cdots+T_N)=E(T)\cdot E(N)$.

$E(T)=\frac{1}{2}$, $E(N) = 0\pi_0+\pi_1+2\pi_2+\cdots+4\pi_4$

$=\frac{26}{31}$.

$\cancel{E(T_1+\cdots+T_N)=E(T)\cdot E(N)=\frac{13}{31}.}$ □

Given that he joins the queue,

$E(T_1+\cdots+T_N \mid N\le 3) = \left(\frac{\pi_1}{\pi_0+\pi_1+\cdots+\pi_3}+\frac{2\pi_2}{\pi_0+\cdots+\pi_3}+\frac{3\pi_3}{\pi_0+\cdots+\pi_3}\right)\cdot E(T)$

$=\frac{11}{30}$. □

~~39. Go! 冲冲冲.~~

$E(T_1+\cdots+T_N \mid N\le3) = \frac{E(T_1+\cdots+T_N,\ N\le3)}{P(N\le3)}$

$= \frac{\sum_{N=0}^{3} P(N=n)\cdot E(T_1+\cdots+T_N)}{P(N\le3)} = \frac{\sum_{N=0}^{3} P(N=n)\cdot NE(T)}{P(N\le3)} = \frac{E(E(T_1+\cdots+T_N,\ N\le3 \mid N))}{P(N\le3)} = \frac{E(T)\sum_{N=0}^{3} P(N=n)\cdot}{P(N\le3)}$

39. (a) Y_n : the time of the nth transition.

$Y_n(i)$ obeys exponential distribution with parameter v_i

assume that $v = v_i$.

$\{Y_n\}$, $P(Y_n \leq t) = 1 - P(0, t)$.

each arrival of Y_n obeys Poisson distribution,

so $\{Y_n\}$ obeys Poisson distribution. with rate v.

(b) balance equation of continuous chain:

$$\pi_j \sum_{k \neq j} q_{jk} = \sum_{k \neq j} \pi_k \cdot q_{kj}, \quad j = 1, 2 \cdots, m.$$

—— of discrete chain:

$$\bar{\pi}_j = \sum_{k=1}^{m} \bar{\pi}_k \cdot p_{kj}, \quad \forall j$$

$$\Leftrightarrow \bar{\pi}_j \cdot \sum_{k=1}^{+\infty} p_{jk} = \sum_{k=1}^{m} \bar{\pi}_k \cdot p_{kj}$$

$$\Leftrightarrow \bar{\pi}_j \cdot \sum_{k \neq j} p_{jk} = \sum_{k \neq j} \bar{\pi}_k \cdot p_{kj}, \quad \forall j$$

assume that $p_{jk} \cdot v_j = q_{jk}$, $\forall j \neq k$.

we have the same balance equation, so they have the same solutions. □.

□.