A Dynamics Perspective of Pursuit-Evasion: Capturing and Escaping When the Pursuer Runs Faster Than the Agile Evader

Wei Li, *Senior Member, IEEE*

***Abstract*—Pursuit-evasion is fascinating in both nature and artificial world. Typically, a pursuer runs faster than its targeted evader while with less agile maneuverability. Naturally, there is a wonder that how an evader escapes from a faster pursuer or how faster a pursuer is able to capture an agile evader? This is not yet answered from the dynamics (i.e., Lagrangian or Newtonian) perspective. In this paper, we first provide a concise dynamics formulation from a bio-inspired perspective, in which the evader's escape strategy consists of two simplest possible yet efficient ingredients integrated as an organic whole, i.e., the suddenly turning-left or turning-right propelling maneuver, together with the early alert condition for starting and maintaining this maneuver. Then, we characterize the dynamic properties of the system at two different levels: 1) the maneuvers and non-trivial escape of the evader, at the level of individual runs of the system; and further 2) the non-trivial escape zones, the sharp phase-transitions and the phase–transition lines of the gaming outcome, at the level of the running results with respect to different ranges of the system parameters. The results are consistent with natural observations and may disclose some clues of natural laws, as well as imply applications in competition of autonomous mobile robots.**

***Index Terms*—Behavioral modes, biological-inspired systems, capture, escape, escape strategy, hybrid systems, maneuverability, motion pattern, non-trivial escape, phase transition, predator-prey, pursuit-evasion.**

I. Introduction

Pursuit-evasion is fascinating and ubiquitous with a long history of research [1], [7]–[13], [28], [29] from many disciplines, e.g., physics, mathematics, and engineering, and has recently received much renewed attention from different perspectives, e.g., the confinement-escape problem [2], [3], the *Homicidal Chauffeur* problem [5], [6], relay pursuit [4], harbor defense [19], [20], and motion camouflage [22]–[24]. Pursuer and evader in nature usually correspond to predator and prey, respectively; in an artificial world, they may be robotic vehicles. Evaders survive in several ways, e.g., in aggregation to counteract a single pursuer or group pursuers [13], [14], [27]; more often, the survival of an individual evader by itself is vitally important. Generally, a pursuer runs faster than its targeted evader, e.g., the cheetah is the ultimate cursorial pursuer [25]; however, a faster pursuer generally has less agile maneuvers, i.e., it has a larger turning radius as it runs at a higher speed, while an evader may have more agile (quick-turning) maneuvers to possibly avoid capture [25]. An escape strategy of an evader that utilizes such advantage is ultimately important for survival, and the investigations on such maneuverability are definitely not new (e.g., animal pursuit-evasion [9]–[12], the *Homicidal Chauffeur* problem [1], [5], [6]), ranging from straight-line escape to complex and random behaviors investigated in different disciplines [1], [9]–[12], [15], [16], [26].

However, there are some aspects to reconsider such problem: 1) This is not yet satisfyingly answered from a perspective of the *dynamics* (i.e., Lagrangian or Newtonian) mechanism, as compared with many existing results from, e.g., *kinematics, motion-geometry or statistics* [1], [4]–[13], [16], in different disciplines, without consideration of any possibly "driving forces" to achieving "how to move." 2) Moreover, the *constant-speed assumption* of agents in the previous work may be not realistic or at least restricted for many cases (as seen universally in nature, e.g., the cheetah [25] and almost all hunting animals in chasing a prey, especially when they make turns). For example, for a typical *Homicidal Chauffeur* problem, the evader has a quick-turning maneuver as many natural preys do and the pursuer has a bounded minimal turning radius; but they are generally modeled both in kinematics and move with the constant-speed assumption. 3) Additionally, for the evader, the maneuverability alone is not enough, how to demonstrate it is also important. 4) Also, it is not clear that how fast (or what characteristics) a pursuer is able to capture an agile evader if the pursuer (e.g., as the cheetah in nature does, which may also imply possible applications in artificial world) simply relies on its velocity advantage.

Certainly, inferring possible dynamics rules at the individual-level from macroscopic phenomena is much challenging, it is an inverse problem that also arises universally, e.g., from many-body systems in many disciples which is hard to solve, if not impossible.

This paper does not intend to investigate a particular chasing-escape of a specific type of pursuer(s) and evader(s), but tries to provides an effective escape strategy that approximates general pursuit-evasion phenomena *from a bio-inspired perspective* (e.g., as in [18]), instead of formulating the problem as a differential game [1], which may disclose some clues of natural laws, as well as imply bio-inspired intelligence in robotics competition or combats, in which a dynamics formulation with force control is vitally important for agents with inertia in physical implementations.

In this paper, we provide a dynamics framework using the *self-propelled accelerations* (or called self-driving forces per unitary mass) of the agents to model the struggling of the live-or-death pursuit-evasion (here the self-propelled accelerations are expressed as scalars instead of vectors), and provide an effective escape strategy of the evader that consists of two simplest possible yet efficient ingredients as an organic whole: 1) the suddenly turning-left or turning-right propelling maneuver (or simply called the *agile maneuver*), together with 2) the *early alert* condition for starting and maintaining the maneuver (or called the *scale* of the agile maneuver). From the control perspective, the evader has two discrete control modes, with two control transitions between the maneuvers and non-maneuvers. As expected that, any mathematical modeling of such phenomenon is definitely *hybrid* [21] and nonlinear. Although such non-linearity

Manuscript received December 23, 2015; revised April 6, 2016; accepted April 11, 2016. Date of publication June 1, 2016; date of current version December 26, 2016. Recommended by Associate Editor X. Yu.

The author is with the Department of Control and Systems Engineering, Nanjing University, Nanjing 210093, China (email: wei.utdallas@live.com).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2016.2575008

TABLE I
NOTATIONS OF THE MODEL AND CHARACTERIZATION

Notations of the Model	**Meaning**
$x_1(t), \dot{x}_1(t), \ddot{x}_1(t) \in \mathbb{R}^2$	the position, velocity, and acceleration, of the pursuer, respectively
$x_e(t), \dot{x}_e(t), \ddot{x}_e(t) \in \mathbb{R}^2$	the position, velocity, and acceleration, of the evader, respectively
$\ell > 0$	the initial distance between the agents
$\mu > 0$	the unit-mass velocity damping coefficient
$a_1 > 0$	the unit-mass self-propelled acceleration of the pursuer
$a_e > 0$	the unit-mass self-propelled acceleration of the evader
$d(t) \in \mathbb{R}^2$, $\|\|d(t)\|\| \equiv 1$	the unitary relative-positional vector from the pursuer to the evader
$f_e(t) \in \mathbb{R}^2$, $\|\|f_e(t)\|\| \equiv 1$	the escape strategy of the evader
$R_l, R_r \in \mathbb{R}^{2\times 2}$	the matrices parameters to model the turning-left and turning-right propelling maneuvers of the evader, respectively
$c > 0$	the critical distance for the evader to take and maintain the agile maneuvers
$\epsilon > 0$	the capture radius of the pursuer
The Characterization Variables	**Meaning**
$e(t) \in \{-1, 0, 1\}$	the maneuver-state indicator of the evader
$\theta(t)$	the angle between the vectors $d(t)$ and $\dot{x}_1(t)$

exists, our work provides a concise and effective formulation; but it may be not easy to derive analytical solutions for the system.

Then, from the proposed framework, we characterize the dynamic properties of the system at two different levels. 1) First, at the level of individual runs of the system, we illustrate the maneuvers of the evader and the response of the pursuer, which are reminiscent of different motion patterns of natural observations. 2) At the higher level of the running results with respect to the parameters' ranges of the system, i) we characterize the sharp *phase-transitions* of the gaming outcome, and then provide the notion of the *phase-transition lines* (PTLs); ii) we characterize quantitatively the *non-trivial-escape* zones about the self-propelled accelerations of the agents, in which the pursuer's self-propelled acceleration is (much) greater than the evader's self-propelled acceleration and thus the pursuer is (much) faster than the evader; and iii) we indicate that, if the pursuer wants to utilize merely its velocity advantage to successfully capture the agile evader, then the pursuer should have a very larger self-propelled acceleration than the evader has; while for the evader, a larger scale of the early alert for critical danger improves dramatically the escape, which are rather consistent with natural observations.

The remainder of the paper is arranged as follows: Section II presents the problem. Section III provides the dynamics framework and the strategies of the agents. Section IV characterizes the properties and motion patterns at the level of individual runs of the system. Section V analyzes the phase-transitions, the PTLs, and the winning sets of the agents, with respect to different ranges of the system parameters. Section VI is the conclusion. Table I lists the main notations for convenience of reference.

II. PROBLEM DESCRIPTION

In this paper, we are interested in modeling and investigating pursuit-evasion of a fast pursuer and an agile evader from the *dynamics* (i.e., Lagrangian or Newtonian) formulation. It is frequently observed that, in a predation chase scenario of terrestrial animals (as compared with the sudden predator-prey encounters of, e.g., aquatic animals), the pursuer first approaches very slowly to its targeted evader, while the evader is also in high alert, once the distance of the pursuer and the evader shrinks to a certain threshold (denoted as $\ell > 0$), the pursuer suddenly runs toward the evader as fast as possible, and almost simultaneously the evader begins running and struggling for its possible escape.

Here we model the framework and strategies of the agents from a bio-inspired perspective, instead of formulating the problem as a differential game [1]; the perspective is itself meaningful for disclosure of natural laws, and is also a good start point for considering complicated behaviors of either natural or artificial agents [2].

Then, there is naturally a wonder that: from the dynamics perspective, how an evader escapes from a faster pursuer? or how faster a pursuer is able to capture an agile evader if the pursuer simply relies on its velocity advantage (as some animal predators do in nature)?

III. FORMULATION AND DESIGN OF CONTROL LAWS

Denote the positions of the pursuer and the evader in the 2D plane as $x_1(t), x_e(t) \in \mathbb{R}^2$, respectively. It is reasonable (refer to the first paragraph in Section II) to assume that, the two agents have zero initial velocities; then, without loss of generality, assume the initial positions as: $x_1(0) = (0, -\ell)^T$, $x_e(0) = (0, 0)^T$. The evader is said *captured* once it is within the capture radius ϵ of the pursuer, i.e., $\|x_e(t) - x_1(t)\| \le \epsilon$, where $\|\cdot\|$ is the Euclidean norm and $\epsilon > 0$.

A. Dynamics and Control of the Pursuer

For convenience, denote

$$d(t) := \frac{x_e(t) - x_1(t)}{\|x_e(t) - x_1(t)\|} \in \mathbb{R}^2$$

as the *unitary relative-positional vector* from the pursuer to the evader with the unitary magnitude (i.e., $\|d(t)\| \equiv 1$).

The pursuer runs as fast as possible toward the evader. Within the dynamics framework, we describe the pursuer as

$$\ddot{x}_1(t) = -\mu \dot{x}_1(t) + a_1 d(t) \tag{1}$$

where $\mu > 0$ is the velocity damping coefficient that models friction, $a_1 > 0$ is the unit-mass *self-propelled acceleration* of the pursuer.

Remark 1: Here, without loss of generality, the mass of the pursuer is not expressed explicitly, since μ and a_1 can be viewed as *the unit-mass parameters* or the parameters per unitary mass (the same is for the evader in the following).

As the direction of $d(t)$ changes, the term $a_1 d(t)$ also models the responding maneuver of the pursuer.

B. Dynamics and Control of the Evader

The dynamics of the evader is described as

$$\ddot{x}_e(t) = -\mu \dot{x}_e(t) + a_e f_e(t) \tag{2}$$

where $a_e > 0$ is the unit-mass (refer to Remark 1) *self-propelled acceleration* of the evader, and $f_e(t) \in \mathbb{R}^2$ represents an escape control strategy with the unitary constraint $\|f_e(t)\| \equiv 1$ that will be provided in Section III-C.

Remark 2: Here accelerations a_1, a_e are scalars instead of vectors.

C. The Escape Strategy and Design of Control Law of the Evader

Note that the accelerations $a_1 > a_e$ in the agents' dynamics, which reflects the fact that the pursuer runs faster than the evader; otherwise, the pursuer can never capture the evader. Thus for the evader, it is not a good strategy to move straight away from the pursuer, which will result in eventual capture sooner or later. It is often observed in nature that, the evader first moves straight away from the pursuer when the pursuer is far away, and then turns its moving direction suddenly once the pursuer approaches near the evader.

Denote the situation

$$\|x_e(t) - x_1(t)\| \leq c$$

as the *emergency situation*, where $c > 0$ is called the *critical distance* or the *scale* of the *early alert* of the critical danger for the evader to *begin to take and maintain the agile maneuver* (the term *early alert* is often heard from, but not exclude to, military scenarios); for convenience, c is simply called the *scale* of the agile maneuver. A larger the value of c is, a larger scale of the agile maneuver the evader endeavors. Naturally, $c > \epsilon$, i.e., the evader is able to take possible maneuvers before being captured; otherwise, it is too late for the evader to take any response for survival.

This escape strategy is described as the control law

$$f_e(t) := \begin{cases} d(t), & \|x_e(t) - x_1(t)\| > c \\ Rd(t), & \text{otherwise} \end{cases} \tag{3}$$

where the matrix R takes one of the constant values of R_l and R_r

$$R_l := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; R_r := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where the physical meaning of $R_l d(t)$ represents the suddenly turning-left propelling maneuver in the emergency; similarly, $R_r d(t)$ represents the turning-right propelling maneuver. The unitary constraint $\|f_e(t)\| \equiv 1$ satisfies, i.e., $\|Rd(t)\| \equiv 1$, which renders that a_e is the self-propelled acceleration of the evader. Note that $f_e(t)$ represents inherently discontinuous at condition $\|x_e(t) - x_1(t)\| = c$.

In nature, an evader may take the turning-left/right propelling maneuver randomly or according to some different situations (e.g., the terrain of the landscape, the obstacles or other possible pursuers in the surrounding); ideally (or without these considerations), the criteria of the maneuvers' selection is reasonably provided as:

- if $d(t)$ and $\dot{x}_1(t)$ are parallel, the results of the turning-left/right maneuvers are thus symmetric for the microscopical motion patterns, then, *without loss of generality*, assume that evader always takes the turning-left maneuver;
- if $\dot{x}_1(t)$ points to the right (left) side of vector $d(t)$, then the evader reasonably takes the turning-left (right) maneuver.

From the control perspective, the evader has two discrete control modes: $d(t)$ as the non-maneuver control and $Rd(t)$ as the maneuver control, with a control transition at condition $\|x_e(t) - x_1(t)\| = c$ between the two modes (or maneuvers and non-maneuvers). Thus the system is *hybrid* [21]. Note that the dynamics of the ordinary differential equations (1)–(3) is nonlinear and inherently discontinuous, possibly without analytical solutions for a general case.

IV. Characterization at the Level of Individual Runs

A. Definitions

Definition 1: To characterize the state of the escape strategy, define the maneuver-state $e(t) \in \{-1, 0, 1\}$ to describe when and what kind of the maneuvers the evader currently takes:

- $e(t) = 1$ denotes the currently turning-left maneuver of the evader,
- $e(t) = -1$ denotes the currently turning-right maneuver of the evader, and
- $e(t) = 0$ denotes no maneuver (i.e., $\|x_e(t) - x_1(t)\| > c$) of the evader.

Define $\theta(t)$ as the angle between the two vectors $d(t)$ and $\dot{x}_1(t)$, i.e.,

$$\theta(t) := \arccos \frac{\langle d(t), \dot{x}_1(t) \rangle}{\|\dot{x}_1(t)\|} \in [0, \pi]$$

where $\arccos(\cdot)$ is the inverse cosine (arccosine) function, $\langle \cdot, \cdot \rangle$ represents the inner product. Here $\theta(t)$ is the difference between the actual heading direction $(\dot{x}_1(t)/\|\dot{x}_1(t)\|)$ and the desired heading direction $d(t)$ of the pursuer, which, to a certain degree, reflects the agility of the responding-maneuverability of the pursuer, i.e., the higher the value of $\theta(t)$ is, the less agile responding-maneuverability the pursuer has.

Denote t_e as the time when the evader is captured (denote $t_e = \infty$ if the pursuer can never capture the evader). t_e is a function of a_1, a_e, μ, c, ϵ, ℓ, i.e.,

$$t_e = t_e(a_1, a_e, \mu, c, \epsilon, \ell)$$

here we use the abbreviation t_e. Naturally, a pursuer will lose patience if it cannot capture the targeted evader after a limited period of the exhausted chasing, and then will give up and wait for a next opportunity in order to save energy or reduce the risk of injury [25], this phenomenon is modeled by the terminal time $t_f > 0$ for the evolution of the system.

Another characteristic is the ratio

$$\gamma := \frac{\bar{v}_1}{\bar{v}_e} \tag{4}$$

where

$$\bar{v}_1 := \frac{1}{\min\{t_e, t_f\}} \int_{t=0}^{\min\{t_e, t_f\}} \|\dot{x}_1\| dt$$

$$\bar{v}_e := \frac{1}{\min\{t_e, t_f\}} \int_{t=0}^{\min\{t_e, t_f\}} \|\dot{x}_e\| dt$$

are the average speeds of the pursuer and the evader, respectively. Actually, γ is a *functional*, $\gamma = \gamma(a_1, a_e, \mu, c, \epsilon, \ell, t_e, t_f)$, here we use the abbreviation γ. Obviously, $\gamma = 1$ for $a_e = a_1$. The non-trivial result is that, the evader can survive with $\gamma > 1$ or even $\gamma \gg 1$, due to the effectiveness of the escape strategy. γ increases as a_1 increases and a_e decreases.

B. Motion Patterns and Characterization

Fig. 1 illustrates one instance of the capture patterns, which pattern is often seen in nature when a pursuer is chasing a vulnerable juvenile evader (which is modeled with a smaller value of a_e, as compared with the case of Fig. 2). At the beginning, the evader moves straight away from the pursuer; while the pursuer moves much faster, and thus the velocity difference $\|\dot{x}_1(t)\| - \|\dot{x}_e(t)\|$ increases monotonically; although the evader takes the turning-left maneuver later at about 1.4(s) and after that, it is still captured at $t_e \approx 2.1$ (s). The distance $\|x_e(t) - x_1(t)\|$ decreases monotonically. The agile maneuver does not take the expected effect in this case.

Fig. 2 illustrates one instance of the escape patterns, which evolves to be periodic. At the beginning, the evader moves straight away from the pursuer; as the distance between the agents less than c, the evader takes the maneuver to increase the distance. As the system evolves, *the evader takes and maintains the maneuvers periodically to avoid*

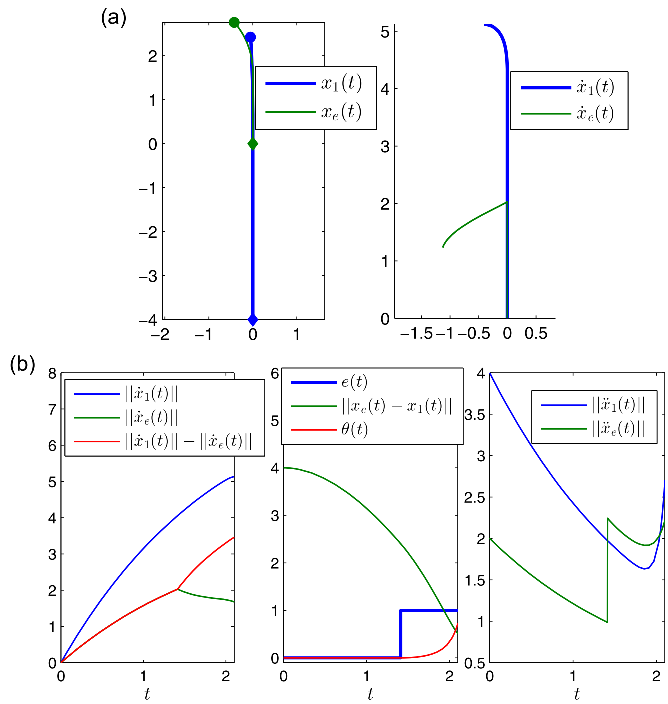

Fig. 1. Illustration of the dynamics and characteristic variables as the system evolves. $\ell = 4$. $\mu = 0.5$, $a_1 = 4$, $a_e = 2$, $c = 2.4$, $\epsilon = 0.5$. The plot in (a) is the trajectories of the positions and velocities of the agents in the x–y plane respectively, notations $\diamond$ and $\circ$ represent the initial and end positions of the agents, respectively. (b) The variables as the system evolves.

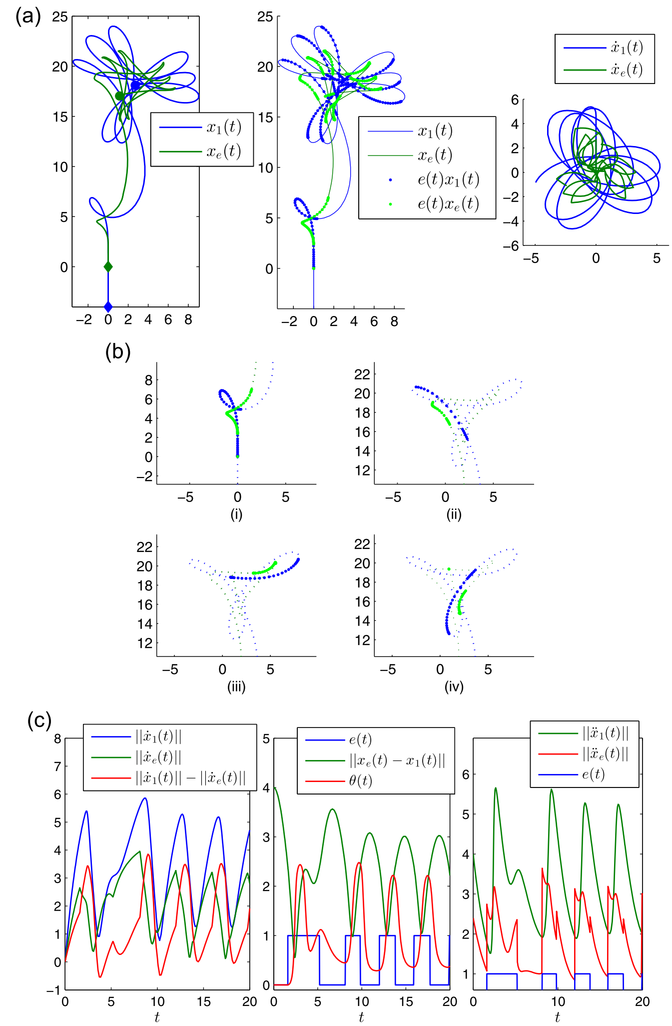

Fig. 2. Illustration of one motion pattern. $\ell = 4$. $\mu = 0.5$, $a_1 = 4$, $c = 2.4$, $\epsilon = 0.5$. $a_e = 2.4$. $t_f = 40$ (s). (a) Trajectories of the agents in the x–y plane, notations $\diamond$ and $\circ$ represent the initial and end positions of agents, respectively; notations $e(t)x_e(t)$ and $e(t)x_1(t)$ represent the trajectories of the evader and the pursuer, respectively, during the evader's maneuvers. (b) The four plots are the first four durations of the evader's maneuver trajectories (the green dotted lines) and the responding trajectories of the pursuer (the blue dotted lines). (c) Characteristics of the system. Note that $\ddot{x}_1(t)$ is a vector, its magnitude $\|\ddot{x}_1(t)\|$ is not the derivative of $\|\dot{x}_1(t)\|$, the same is for $\|\ddot{x}_e(t)\|$.

capture, refer to the trajectories of the agents and the value of $e(t)$ during the evader's maneuvers in Fig. 2. The evader has a sharp change of its acceleration when it maneuvers; while the pursuer has a higher speed (compare $\|\dot{x}_1(t)\|$ and $\|\dot{x}_e(t)\|$), a higher acceleration, and a relatively less sharper change of its acceleration but with a certain delay (compare $\|\ddot{x}_1(t)\|$ and $\|\ddot{x}_e(t)\|$). The pursuer has a sharp decrease in the speed just before turning which would enable much tighter turns for maximizing possible capture, as observed in [25]; the vibration of the velocity $\|\dot{x}_1(t)\|$ and the acceleration $\|\ddot{x}_1(t)\|$ of the pursuer are rather roughly similar to the observations of the hunting behaviors of cheetahs, and the top speed is usually only sustained for a few seconds [25]. This shows the merit of the modeling, instead of the constant-speed motion of agents mentioned in the literature of the Introduction section. The evader survives due to its strategy in this case.

Fig. 3 illustrates other periodic motion patterns of the system, although the speed of the pursuer is approximately twice of the speed of the evader, the evader still survives owning to the strategy, the evader takes the maneuvers periodically to avoid capture, refer to the values of $e(t)$. Fig. 3(d) is one stable motion pattern where the agents eventually evolve to a synchronized motion on two concentric circles. Fig. 3(e) illustrates a more complex periodic motion pattern. Fig. 3(f) is a *traveling* motion pattern with an increased value of a_e, as compared with other cases.

The evader adopts the turning-left maneuver at the first symmetric case it encounters; and from the values of $e(t) \geq 0$ and $\theta(t) > 0$, there is no more symmetric cases after that during the evolution, i.e., $\dot{x}_1(t)$ always points to the right side of $d(t)$ after the only symmetric case; as a result, the evader tends to take the turning-left maneuver (if the evader takes the turning-right maneuver at the first symmetric case, it tends to take the turning-right maneuver after that), which is reasonable and rather consistent with natural observations (note that in nature, many random factors, e.g., the terrain of the landscape, the obstacles or other possible pursuers in the surrounding, may also influence the decision of the evader).

V. Characterization at the Level of the Running Results With Respect to the Parameters' Ranges

It is straightforward that, for $a_e \geq a_1$ in the dynamics of the agents, the evader runs in a straight line and easily escapes, which is trivial. In the following, we show non-trivial escape, and particularly the characterizations of: 1) the phase-transitions, 2) the phase-transition lines, and 3) non-trivial escape zones, when $a_1 > a_e$.

A. Definitions of Trivial- and Non-Trivial Escape Zones

Definition 2: Note that $t_e = t_e(a_1, a_e, \mu, c, \epsilon, \ell)$. Define the level sets as

$$\mathcal{L}_{t_f}(t_e) := \{(a_1, a_e, \mu, c, \epsilon, \ell) \mid t_e = t_f\}$$
$$\mathcal{L}^n_{t_f}(t_e) := \{(a_1, a_e, \mu, c, \epsilon, \ell) \mid a_1 > a_e,\ t_e = t_f\}.$$

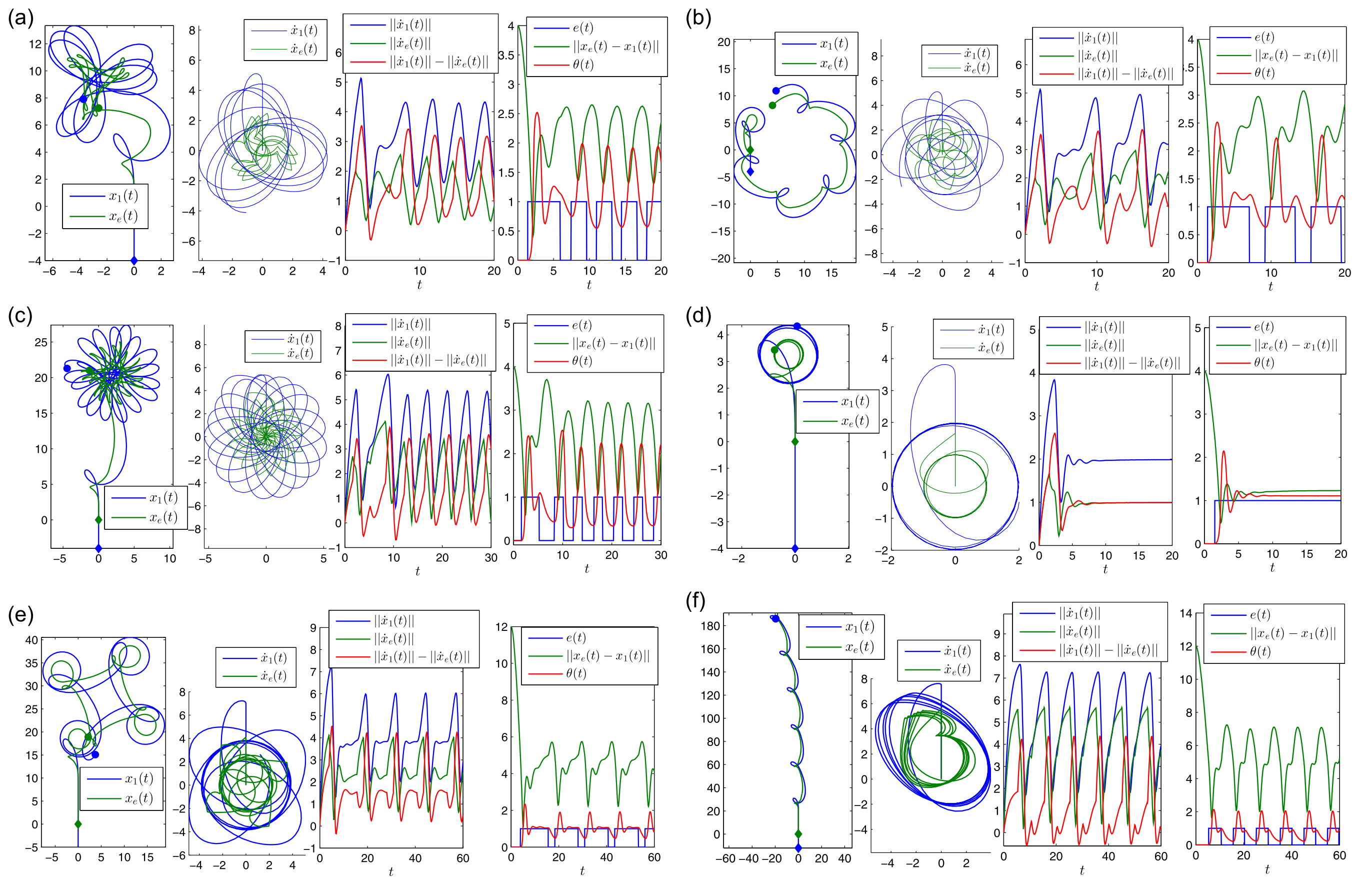

Fig. 3. Dynamics of agents. $a_1 = 4$, $\epsilon = 0.5$, $\ell = 4$, $\mu = 0.5$, $a_e = 2$, except values specified. (a) $c = 2.44$. (b) $c = 2.45$. (c) $c = 2.45$, $a_e = 2.45$. (d) $c = 2.45$, $\mu = 0.9$. (e) $a_e = 2.4$, $c = 5$, $\ell = 12$. (f) $a_e = 3$, $c = 5$, $\ell = 12$.

Mathematically, the escape zone of the system is the level set $\mathcal{L}_\infty(t_e)$, the non-trivial escape zone of the system is the level set $\mathcal{L}^n_\infty(t_e)$, and the trivial escape zone is $\mathcal{L}_\infty(t_e) - \mathcal{L}^n_\infty(t_e)$.

However, $\mathcal{L}_\infty(t_e)$ and $\mathcal{L}^n_\infty(t_e)$ are not computable. Fortunately, there are *the sharp phase-transitions* of $\mathcal{L}_{t_f}(t_e)$ as follows.

B. Phase-Transitions and Non-Trivial Escape Zones

Consider the gaming outcome, if the pursuer cannot capture the evader until the end of the designated deadline t_f (refer to the physical meaning of t_f in Section IV), then for easy illustrations of the simulations, assign the value of t_e as t_f; thus $t_e = t_f$ in the illustrations means that: the pursuer can never capture the evader or cannot capture the evader before t_f.

Fig. 4(a)–(c) illustrates t_e as a function of a_1 and a_e, with $c = 4$, $c = 5$, and $c = 6$, respectively. Note that there is *a sharp phase-transition* between the survival and the capture for every case. From the sharp phase-transition, it is almost sure that the evader cannot be captured for the values of a_1 and a_e belonging to the *magenta-color-zone* in Fig. 4, even for a still larger value of t_f (the more illustrations omitted for the limited space) for the system evolution.

The *magenta-color-zone* is a level set of t_e, which has two parts:

1) the zone of $a_e \geq a_1$, which is the zone of trivial escape of the evader, and
2) the zone of $a_1 > a_e$, which is the *non-trivial escape zone* of the evader that embodies the very effect of its escape strategy.

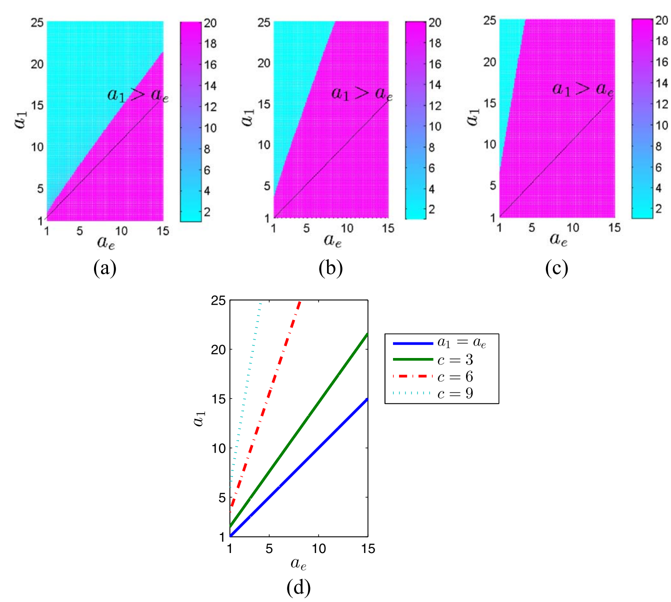

Fig. 4. Illustration of t_e as a function of a_1 and a_e in the 2D x–y plot. $\ell = 12$. $\mu = 0.5$. $\epsilon = 0.5$. $t_f = 20$. The evader escapes for the values of a_1 and a_e that are located in the magenta-color-zone, and is captured for the values of a_1 and a_e that are located in the cyan-color-zone; the magenta-color-zone with $a_1 > a_e$ is the non-trivial escape zone. The black line in (a)–(c) is the line $a_1 = a_e$. (a) $c = 3$. The PTL is approximately $a_1 = (7/5)a_e + (3/5)$. (b) $c = 6$. The PTL is approximately $a_1 = 3a_e + (1/2)$. (c) $c = 9$. The PTL is approximately $a_1 = 6a_e$. (d) The PTLs with different values of c.

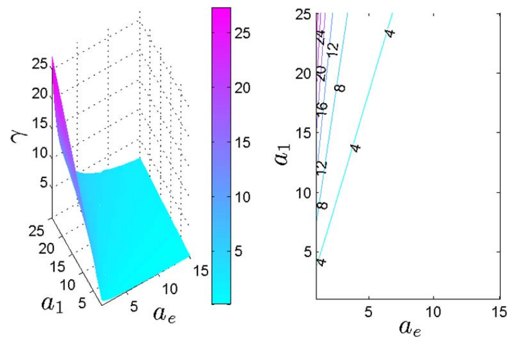

Fig. 5. Illustration of γ as a function of a_1 and a_e in the 3D x–y–z plot and the contour lines in the 2D x–y plot, respectively. The parameters are same as in Fig. 4(c), with $c = 6$.

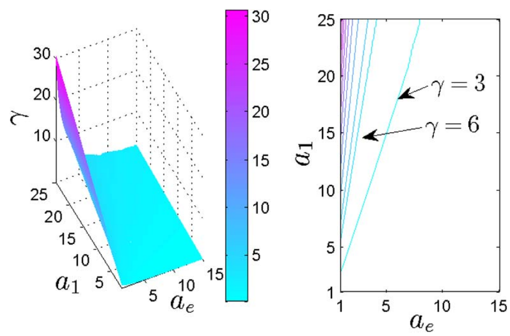

Fig. 6. Illustration of γ as a function of a_1 and a_e in the 3D x–y–z plot and the contour lines in the 2D x–y plot, respectively. The parameters are same as in Fig. 4(c), with $c = 9$.

C. Notion of the Phase-Transition Lines and the Expressions

Denote $a_{e\_\min}$ as the minimum value of a_e that ensures successful escape of the evader, which is a function of $a_1, \mu, c, \epsilon, \ell$, i.e., $a_{e\_\min} = a_{e\_\min}(a_1, \mu, c, \epsilon, \ell)$; certainly $a_{e\_\min} < a_1$.

Definition 3: The line $a_e = a_{e\_\min}$ is just called the *phase-transition line* (PTL), which separates the gaming outcome between the successful capture zone $(0 < a_e < a_{e\_\min})$ and the non-trivial escape zone $(a_{e\_\min} \le a_e < a_1)$.

For example, the PTLs for different values of c are illustrated in Fig. 4(d), which are roughly linear functions of a_1 and a_e; here the PTLs are appropriately

$$a_1 \approx \frac{7}{5}a_e + \frac{3}{5}, \qquad c = 3$$

$$a_1 \approx 3a_e + \frac{1}{2}, \qquad c = 6$$

$$a_1 \approx 6a_e, \qquad c = 9$$

for $\ell = 12$, $\mu = 0.5$, $\epsilon = 0.5$. As the critical distance c increases, the evader has more possibility for survival, e.g., for $c = 6$, the evader survives even $a_1 \approx 3a_e + (1/2)$; and such tendency is more distinct as c increases (the illustrations are omitted for the limited space).

Also, we can conclude from Figs. 5 and 6 that, *for non-trivial escape of the evader, the values of the parameters that maximize the functional* γ (4) *are just the values on the PTLs*; or equivalently, for successful escape, the values of the parameters that minimize the functional γ (4) are just the values on the PTLs.

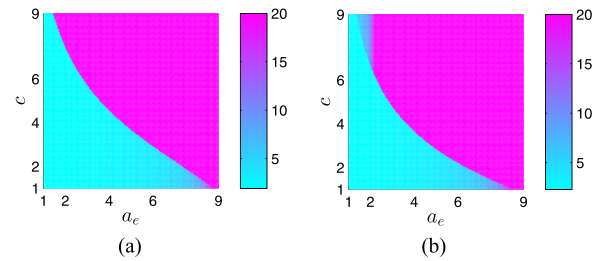

Fig. 7. Illustration of t_e as a function of c and a_e. $\ell = 12$, $\epsilon = 0.5$, $a_1 = 9$. (a) $\mu = 0.5$. (b) $\mu = 1$.

D. Effects of the Critical Distance

As c increases, the non-trivial escape zone will still expand, i.e., the evader has more capability for survival even the pursuer is much faster (also refer to Figs. 5 and 6).

For example, as illustrated in Fig. 6, here the PTL is approximately $a_1 \approx 6a_e$ [refer to Fig. 4(c)], the evader can successfully escape even when $\gamma \approx 6$. As c increases, the evader can escape even γ is more larger (refer to Figs. 5 and 6).

A large value of c favors the evader's survival, but there is a trade-off: a too larger value of c will make the evader harder to move to a faraway possible shelter (since the maneuver tends to make the evader moving around the pursuer). The larger difference of $a_1 - a_e$ also has this effect (compare Fig. 3(f) with other patterns in Fig. 3).

Fig. 7 illustrates t_e as a function of c and a_e, with a fixed a_1. The larger a_e and c favor successful survival of the evader. As a_e approaches to a_1, i.e., the mobility of the evader approaches to the mobility of the pursuer $(a_e \to a_1)$, the minimum of c that ensures the successful survival of the evader decreases dramatically.

As the value of the velocity damping μ increases, the zone of the parameters of c and a_e for survival of the evader expands, i.e., the physical meaning is that, the pursuer exhibits a reduced velocity advantage than the evader at a higher value of μ, as illustrated in Fig. 7 (here μ takes the values in the range as in the natural observations in [25]).

VI. Conclusion

In this paper, we propose a dynamics formulation of pursuit-evasion with two players, in which the evader's escape control strategy consists of two simplest possible yet efficient ingredients as an organic whole. Then, 1) we characterize the non-trivial escape of the evader at the level of individual runs of the system, and further 2) we characterize the non-trivial escape zones, the sharp phase-transitions and the PTLs of the outcome, at the level of the parameters' ranges of the system. The results may disclose some clues of natural laws, as well as suggest applications in competition of mobile robots.

The two-level results are provided mainly from numerical computations (the characterizations of the capture-escape zones and the PTLs are computationally intensive). The system has very rich dynamical properties, further characterization (e.g., full ranges of the parameters $a_1, a_e, \mu, c, \epsilon, \ell$), and a still higher level of the properties (e.g., more properties of the PTLs) will be addressed in future. The modeling of the fatigue effects of the agents, the randomness of their behaviors will also be considered in a future paper, as well as pursuit-evasion evolving on non-Euclidean space (e.g., the sphere manifold [17]).

The control system (1), (2) has the three-fold nonlinearities: the unitary vector $d(t)$, the turning-left/right maneuver, and the control law switch (3). Although the system formulation is very concise, its analysis is not easy which will be addressed in future.

References

[1] R. Isaacs, *Differential Games*. New York, NY, USA: Wiley, 1965.

[2] W. Li, "The confinement-escape problem of a defender against an evader escaping from a circular region," *IEEE Trans. Cybern.*, vol. 46, no. 4, pp. 1028–1039, Apr. 2016.

[3] W. Li, "Escape analysis on the confinement-escape problem of a defender against escape of an evader with respect to a circular region," *IEEE Trans. Cybern.*, vol. 46, no. 9, pp. 2166–2172, Sep. 2016.

[4] E. Bakolas and P. Tsiotras, "Relay pursuit of a maneuvering target using dynamic Voronoi diagrams," *Automatica*, vol. 48, pp. 2213–2220, 2012.

[5] S. D. Bopardikar, F. Bullo, and J. P. Hespanha, "A cooperative homicidal chauffeur game," *Automatica*, vol. 45, pp. 1771–1777, 2009.

[6] S. D. Bopardikar, F. Bullo, and J. P. Hespanha, "On discrete-time pursuit-evasion games with sensing limitations," *IEEE Trans. Robot.*, vol. 24, no. 6, pp. 1429–1439, 2008.

[7] M. Pachter, "Simple-motion pursuit-evasion in the half plane," *Comput. Math. Appl.*, vol. 13, pp. 69–82, 1987.

[8] J. Sgall, "Solution of David Gale's lion and man problem," *Theor. Comput. Sci.*, vol. 259, pp. 663–670, 2001.

[9] H. C. Howland, "Optimal stratagies for predator avoidance: The relative importance of speed and manoeuvrability," *J. Theor. Biol.*, vol. 134, pp. 56–76, 1974.

[10] D. Weihs and P. W. Webb, "Optimal avoidance and evasion tactics in predator-prey interactions," *J. Theor. Biol.*, vol. 106, pp. 189–206, 1984.

[11] T. Caro, *Antipredators Defenses in Birds and Mammals*. Chicago, IL, USA: Univ. Chicago Press, 2005.

[12] A. Hedenström and M. Rosén, "Predator versus prey: On aerial hunting and escape strategies in birds," *Behav. Ecol.*, vol. 12, pp. 150–156, 2001.

[13] S. Kopparty and C. V. Ravishankar, "A framework for pursuit evasion games in $\mathbb{R}^n$," *Inf. Process. Lett.*, vol. 96, no. 3, pp. 114–122, 2005.

[14] M. Chen, Z. Zhou, and C. J. Tomlin, "A path defense approach to the multiplayer reach-avoid game," in *Proc. IEEE Conf. Decision Control*, 2014, pp. 2420–2426.

[15] J. M. Eklund, J. Sprinkle, and S. S. Sastry, "Switched and symmetric pursuit/evasion games using online model predictive control with application to autonomous aircraft," *IEEE Trans. Control Syst. Technol.*, vol. 20, no. 3, pp. 604–620, May 2012.

[16] P. J. Nahin, *Chases and Escapes: The Mathematics of Pursuit and Evasion*. Princeton, NJ, USA: Princeton Univ. Press, 2007.

[17] W. Li and M. W. Spong, "Unified cooperative control of multiple agents on a sphere for different spherical patterns," *IEEE Trans. Autom. Control*, vol. 59, no. 5, pp. 1283–1289, May 2014.

[18] M. Haque, A. Rahmani, M. Egerstedt, and A. Yezzi, "Efficient foraging strategies in multi-agent systems through curve evolutions," *IEEE Trans. Autom. Control*, vol. 59, no. 4, pp. 1036–1041, Apr. 2014.

[19] J. Walrand, E. Polak, and H. Chung, "Harbor attack: A pursuit-evasion game," in *Proc. 49th Annu. Allerton Conf.*, Monticello, IL, USA, 2011, pp. 1584–1591.

[20] S. Lee, E. Polak, and J. Walrand, "A receding horizon control law for harbor defense," in *Proc. 51st Annual Allerton Conf.*, Monticello, IL, USA, 2013, pp. 70–77.

[21] D. Liberzon, "Switching in systems and control," in *Systems & Control: Foundations & Applications*. Boston, MA, USA: Birkhaüser, 2003.

[22] P. Glendinning, "The mathematics of motion camouflage," *Proc. R. Soc. B*, vol. 271, pp. 477–481, 2004.

[23] E. W. Justh and P. S. Krishnaprasad, "Steering laws for motion camouflage," *Proc. R. Soc. A*, vol. 462, pp. 3629–3643, 2006.

[24] K. S. Galloway, E. W. Justh, and P. S. Krishnaprasad, "Symmetry and reduction in collectives: Cyclic pursuit strategies," *Proc. R. Soc. A*, vol. 469, 2013, Art. no. 20130264.

[25] A. M. Wilson, J. C. Lowe, K. Roskilly, P. E. Hudson, K. A. Golabek, and J. W. McNutt, "Locomotion dynamics of hunting in wild cheetahs," *Nature*, vol. 498, pp. 185–189, 2013.

[26] I. Exarchos, P. Tsiotras, and M. Pachter, "On the suicidal pedestrian differential game," *Dyn. Games Appl.*, vol. 5, no. 3, pp. 297–317, Sep. 2015.

[27] W. Ewert, R. J. Marks, B. B. Thompson, A. Yu, "Evolutionary inversion of swarm emergence using disjunctive combs control," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 43, no. 5, pp. 1063–1076, Sep. 2013.

[28] B. P. Gerkey, S. Thrun, and G. Gordon, "Visibility-based pursuit-evasion with limited field of view," *Int. J. Robot. Res.*, vol. 25, no. 4, pp. 299–315, 2006.

[29] M. Pachter and Y. Yavin, "A stochastic homicidal chauffeur pursuit-evasion differential game," *J. Optim. Theory and Appl.*, vol. 34, no. 3, pp. 405–424, 1981.