A Class of Linear Complementarity Problems Solvable in Polynomial Time

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ABSTRACT

We describe a "condition" number for the linear complementarity problem (LCP), which characterizes the degree of difficulty for its solution when a potential reduction algorithm is used. Consequently, we develop a class of LCPs solvable in polynomial time. The result suggests that the convexity (or positive semidefiniteness) of the LCP may not be the basic issue that separates LCPs solvable and not solvable in polynomial time.

INTRODUCTION

In this paper, we are concerned with the linear complementarity problem (LCP), that is, to find a pair $x, y \in R^n$ such that

$$x^T y = 0, \qquad y = Mx + q, \quad \text{and} \quad x, y \geqslant 0,$$

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where we assume that all components of M and q are integers, and $\Omega^+ = \{(x, y): y = Mx + q, \ x > 0, \text{ and } y > 0\}$ is nonempty. Kojima, Megiddo, and Ye [7] have discussed how to transform an LCP with empty Ω^+ to an equivalent LCP with nonempty Ω^+. Thus, the last assumption is merely added for simplicity. We also use Ω to denote the feasible region, i.e., $\Omega = \{(x, y): y = Mx + q, \ x \geqslant 0, \text{ and } y \geqslant 0\}$.

If M is a Z matrix, Cottle and Veinott [2] and Mangasarian [10] showed that the LCP can be solved as a linear program; therefore, it can be solved in polynomial time. Pang and Chandrasekaran [15] also showed that some special LCPs can be solved in n pivots using several pivoting methods. If M is a positive semidefinite matrix, the LCP is simply a convex quadratic programming problem, and it can be solved in polynomial time by the ellipsoid method (Khachiyan [6]), the projective method (e.g., Kapoor and Vaidya [5] and Ye and Tse [24]), the path-following method (e.g., Kojima, Mizuno, and Yoshise [9] and Monteiro and Adler [12]), and the potential-reduction method (e.g., Kojima, Megiddo, and Ye [7], Kojima, Mizuno, and Yoshise [8], and Pardalos, Ye, and Han [17]). The best complexity result for a convex LCP is $O(\sqrt{n}\, L)$ iterations and $O(n^3 L)$ total arithmetic operations, where L is the size of the input data of the problem.

If M is a P matrix, Ye [22] showed that the potential-reduction algorithm of Kojima et al. [7] solves the LCP in $O(n^2 L \max(|\lambda|/n\theta, 1))$ iterations and each iteration solves a system of linear equations in at most $O(n^3)$ arithmetic operations, where λ is the least eigenvalue of $(M + M^T)/2$, and $\theta > 0$ is the so-called P-matrix number for M^T, that is,

$$\theta = \min_{x \neq 0} \left\{ \max_j \frac{x_j (M^T x)_j}{\|x\|^2} \right\}.$$

This indicates that $|\lambda|/n\theta$ can be used to measure the degree of difficulty for solving the P-matrix LCP (see also a related discussion of Mathias and Pang [11]). The algorithm is a polynomial-time algorithm if $|\lambda|/n\theta$ is bounded above by a polynomial in L and n.

In this paper, we describe a condition number for the general LCP, which characterizes the degree of difficulty for finding its solution when Kojima et al.'s potential-reduction algorithm is used. Consequently, we develop a new class of LCPs solvable in polynomial time. We show how the condition number varies as the data (M, q) changes. We also show that several existing classes and some examples of LCPs belong to our class. The result again suggests that the convexity (or positive semidefiniteness) of the LCP may not be the basic issue that separates LCPs that are solvable in polynomial time from ones that are not.

1. POTENTIAL FUNCTIONS AND POTENTIAL REDUCTION ALGORITHMS

We use the potential function

$$\phi(x,y) = \rho \ln(x^T y) - \sum_{j=1}^{n} \ln(x_j y_j)$$

to associate with an interior feasible solution (x, y) with $\rho > n$. This function first appeared in Todd and Ye [21], and was first used to develop an $O(n^3 L)$ potential-reduction algorithm for linear programming by Ye [23]. Soon after, this function was used for solving the convex LCP [7, 8].

Starting from an interior point (x^0, y^0) with

$$\phi(x^0, y^0) \leqslant O(\rho L),$$

the potential-reduction algorithm generates a sequence of interior feasible solutions $\{x^k, y^k\}$ terminating at a point such that

$$\phi(x^k, y^k) \leqslant -(\rho - n) L.$$

From the arithmetic–geometric-mean inequality,

$$n \ln\left[(x^k)^T y^k \right] - \sum_{j=1}^{n} \ln\left(x_j^k y_j^k \right) \geqslant n \ln n \geqslant 0.$$

Thus,

$$\left(x^k \right)^T y^k \leqslant 2^{-L},$$

and an exact solution to LCP can be obtained in $O(n^3)$ additional operations [9].

To achieve a potential reduction, Kojima et al. used the scaled gradient projection method. The gradient vector of the potential function with respect to x is

$$\nabla \phi_x = \frac{\rho}{\Delta} y - X^{-1} e,$$

and the one with respect to y is

$$\nabla\phi_y = \frac{\rho}{\Delta}x - Y^{-1}e,$$

where $\Delta = x^T y$, $X = \text{diag}(x)$, $Y = \text{diag}(y)$, and e is the vector of all ones. Now, let us solve the following linear program subject to an ellipsoid constraint at the kth iteration:

minimize $\nabla\phi_{x^k}\,\delta x + \nabla\phi_{y^k}\,\delta y$

subject to $\delta y = M\,\delta x,$

$$\left\|(X^k)^{-1}\delta x\right\|^2 + \left\|(Y^k)^{-1}\delta y\right\|^2 \leqslant \beta^2 < 1,$$

and denote by $\delta\bar{x}$ and $\delta\bar{y}$ its minimal solutions. Then, we have

$$\begin{pmatrix} (X^k)^{-1}\delta\bar{x} \\ (Y^k)^{-1}\delta\bar{y} \end{pmatrix} = -\beta\frac{p^k}{\|p^k\|},$$

where

$$p^k = \begin{pmatrix} p_x^k \\ p_y^k \end{pmatrix} = \begin{pmatrix} \dfrac{\rho}{\Delta^k}X^k\big(y^k + M^T\pi^k\big) - e \\[2mm] \dfrac{\rho}{\Delta^k}Y^k\big(x^k - \pi^k\big) - e \end{pmatrix},$$

$$\pi^k = \Big[(Y^k)^2 + M(X^k)^2 M^T\Big]^{-1}(Y^k - MX^k)\left(X^k y^k - \frac{\Delta^k}{\rho}e\right),$$

$\Delta^k = (x^k)^T y^k$, and X^k (Y^k) designates the diagonal matrix of x^k (y^k). Let $x^{k+1} = x^k + \delta\bar{x}$ and $y^{k+1} = y^k + \delta\bar{y}$. Then it can be verified that

$$\phi\big(x^{k+1}, y^{k+1}\big) - \phi\big(x^k, y^k\big) \leqslant -\beta\|p^k\| + \frac{\beta^2}{2}\left(\rho + \frac{1}{1-\beta}\right).$$

Therefore, choosing

$$\beta = \min\left(\frac{\|p^k\|}{\rho + 2}, \frac{1}{2}\right) \leqslant \frac{1}{2},$$

we have

$$\phi(x^{k+1}, y^{k+1}) - \phi(x^k, y^k) \leqslant -\alpha(\|p^k\|^2), \tag{1}$$

where

$$\alpha(\|p^k\|^2) = \begin{cases} \dfrac{\|p^k\|^2}{2(\rho + 2)} & \text{if} \quad \|p^k\|^2 \leqslant (\rho + 2)^2/4, \\ \dfrac{\rho + 2}{8} & \text{otherwise.} \end{cases}$$

2. A CONDITION NUMBER FOR THE LCP

Naturally, $\|p^k\|^2$ can be used to measure the potential reduction at the kth iteration of the potential-reduction algorithm. Let

$$g(x, y) = \frac{\rho}{\Delta} Xy - e$$

and

$$H(x, y) = 2I - (XM^T - Y)(Y^2 + MX^2M^T)^{-1}(MX - Y).$$

It can be verified that $H(x, y)$ is positive semidefinite (PSD). Note that

$$\|p^k\|^2 = g^T(x^k, y^k)H(x^k, y^k)g(x^k, y^k). \tag{2}$$

Let us use $\|g(x, y)\|_H^2$ to denote $g^T(x, y)H(x, y)g(x, y)$. Then, we define the condition number for the LCP (M, q) as

$$\gamma(M, q) = \inf\left\{\|g(x, y)\|_H^2 : x^T y \geqslant 2^{-L}, \phi(x, y) \leqslant O(\rho L), \text{ and } (x, y) \in \Omega^+\right\}.$$

$$\tag{3}$$

The quantity $\|g(x,y)\|_H^2$ first appeared in Pardalos and Ye [16] for the row-sufficient matrix M of Cottle, Pang, and Venkateswaran [1]. Our condition number is different from the one defined in Kojima et al. [7], where

$$\gamma(M) = \frac{(\rho - n)^2}{n} \inf\{\lambda(M, X, Y) : (x, y) \in \Omega^+\}$$

and $\lambda(M, X, Y)$ is the smallest eigenvalue of the matrix

$$(I + Y^{-1}MX)(I + XM^T Y^{-2}MX)^{-1}(I + XM^T Y^{-1}).$$

Of course, the two capture similar aspects of positivity related to the matrix $H(x, y)$.

We present a sequence of propositions for $\gamma(M, q)$.

PROPOSITION 1.   *Let $\rho \geqslant 2n$. Then, for M a diagonal and PSD matrix, and any $q \in R^n$,*

$$\gamma(M, q) \geqslant n.$$

Proof. If M is diagonal and PSD, then $I - (XM^T - Y)(Y^2 + MX^2M^T)^{-1}(MX - Y)$ is diagonal. It is also PSD, since the jth diagonal component is

$$1 - \frac{(M_{jj}x_j - y_j)^2}{y_j^2 + M_{jj}^2 x_j^2} = \frac{2M_{jj}x_j y_j}{y_j^2 + M_{jj}^2 x_j^2} \geqslant 0.$$

Therefore, for all $(x, y) \in \Omega^+$ and $\rho \geqslant 2n$,

$$\gamma(M, q) \geqslant \|g(x, y)\|^2 \geqslant \frac{(\rho - n)^2}{n} \geqslant n. \qquad \blacksquare$$

PROPOSITION 2 (Kojima, Megiddo, and Ye [7]).   *Let $\rho \geqslant 2n + \sqrt{2n}$. Then, for M a PSD matrix and any $q \in R^n$,*

$$\gamma(M, q) \geqslant 1.$$

PROPOSITION 3 (Ye [22]). *Let $\rho \geqslant 3n + \sqrt{2n}$. Then, for M a P matrix and any $q \in R^n$,*

$$\gamma(M, q) \geqslant \min(n\theta / |\lambda|, 1),$$

where λ is the least eigenvalue of $(M + M^T)/2$, and θ is the P-matrix number of M^T, i.e.,

$$\theta = \min_{x \neq 0} \left\{ \max_j \frac{x_j(M^T x)_j}{\|x\|^2} \right\}. \qquad \blacksquare$$

PROPOSITION 4. *Let $\rho > n$ and be fixed. Then, for M a row-sufficient matrix and $\{(x, y) \in \Omega^+ : \phi(x, y) \leqslant O(\rho L)\}$ bounded,*

$$\gamma(M, q) > 0.$$

Proof. Both Pang [14] and Pardalos and Ye [16] showed that for any $(x, y) \in \Omega^+$

$$\| g(x, y) \|_H^2 > 0.$$

Moreover, for all $(x, y) \in \Omega^+$, $x^T y \geqslant 2^{-L}$, and $\phi(x, y) \leqslant O(\rho L)$ we have

$$O(\rho L) \geqslant \phi(x, y)$$

$$= \rho \ln(x^T y) - \sum_{j=1}^n \ln(x_j y_j)$$

$$= (\rho - n + 1)\ln(x^T y) + (n - 1)\ln(x^T y) - \sum_{j \neq i} \ln(x_j y_j) - \ln(x_i y_i)$$

$$\geqslant (\rho - n + 1)\ln(x^T y) + (n - 1)\ln(x^T y - x_i y_i)$$

$$\quad - \sum_{j \neq i} \ln(x_j y_j) - \ln(x_i y_i)$$

$$\geqslant (\rho - n + 1)\ln(x^T y) + (n - 1)\ln(n - 1) - \ln(x_i y_i)$$

$$\geqslant -(\rho - n + 1)L + (n - 1)\ln(n - 1) - \ln(x_i y_i),$$

where $i \in \{1, 2, \ldots, n\}$. Thus,

$$\ln(x_i y_i) \geqslant -(\rho - n + 1)L + (n - 1)\ln(n - 1) - O(\rho L) = -O(\rho L),$$

that is, $x_i y_i$ is bounded away from zero by $e^{-O(\rho L)}$ for all i. Since $\{(x, y) \in \Omega^+ : \phi(x, y) \leqslant O(\rho L)\}$ is bounded, there must exist a positive number $\bar{\epsilon}$, independent of (x, y), such that

$$x_i \geqslant \bar{\epsilon} \quad \text{and} \quad y_i \geqslant \bar{\epsilon}, \qquad i = 1, 2, \ldots, n,$$

for all (x, y) such that $x^T y \geqslant 2^{-L}$, $\phi(x, y) \leqslant O(\rho L)$, and $(x, y) \in \Omega^+$. Therefore,

$$\gamma(M, q) = \inf\{\|g(x, y)\|_H^2 : x^T y \geqslant 2^{-L}, \phi(x, y) \leqslant O(\rho L), \text{ and } (x, y) \in \Omega^+\}$$

$$\geqslant \inf\{\|g(x, y)\|_H^2 : x \geqslant \bar{\epsilon} e, y \geqslant \bar{\epsilon} e, \phi(x, y) \leqslant O(\rho L), \text{ and } (x, y) \in \Omega\}$$

$$> 0.$$

The last inequality holds because the inf is taken in a compact set where $\|g(x, y)\|_H^2$ is always positive.   ∎

Note that $\phi(x, y) \leqslant O(\rho L)$ implies that $x^T y \leqslant O(\rho L / (\rho - n))$. Hence, the boundedness of $\{(x, y) \in \Omega : x^T y \leqslant O(\rho L / (\rho - n))\}$ guarantees the boundedness of $\{(x, y) \in \Omega^+ : \phi(x, y) \leqslant O(\rho L)\}$.

We now derive

THEOREM 1. *The potential-reduction algorithm with $\rho = \theta(n) > n$ solves the LCP for which $\gamma(M, q) > 0$ in $O(nL / \alpha(\gamma(M, q)))$ iterations and each iteration solves a system of linear equations in at most $O(n^3)$ operations, where $\alpha(\cdot)$ is defined in (1).*

Proof. Since Ω^+ is nonempty, by solving a linear program in polynomial time, we can find an interior feasible point (x^0, y^0), each component of which is greater than 2^{-L} and less than 2^L. The resulting point has an initial potential value less than $O(nL)$. Due to (1), (2), and (3), the potential function is reduced by $O(\alpha(\gamma(M, q)))$ at each iteration. Hence, in the total of $O(nL / \alpha(\gamma(M, q)))$ iterations we have $\phi(x^k, y^k) < -(\rho - n)L$ and $(x^k)^T y^k < 2^{-L}$.   ∎

COROLLARY 1. *An instance (M, q) of the LCP is solvable in polynomial time if $\gamma(M, q) > 0$ and if $1/\gamma(M, q)$ is bounded above by a polynomial in L and n.*

The condition number $\gamma(M, q)$ represents the degree of difficulty for the potential-reduction algorithm in solving the LCP (M, q). The larger the condition number, the easier the LCP. We know that some LCPs are very hard, and some are easy. Here, the condition number builds a connection from easy LCPs to hard LCPs. In other words, the corresponding degree of difficulty shifts continuously from easy to hard LCPs. We feel that such a condition number will be an important criterion in analyzing the complexity of algorithms for optimization problems.

3. A CLASS OF LCP'S SOLVABLE IN POLYNOMIAL TIME

We now further study $\|p^k\|$ by first introducing the following lemma.

LEMMA 1. $\|p^k\| < 1$ *implies*

$$y^k + M^T \pi^k > 0, \qquad x^k - \pi^k > 0,$$

and

$$\frac{2n - \sqrt{2n}}{\rho} \Delta^k < \overline{\Delta} < \frac{2n + \sqrt{2n}}{\rho} \Delta^k,$$

where $\overline{\Delta} = (x^k)^T(y^k + M^T \pi^k) + (y^k)^T(x^k - \pi^k).$

Proof. The proof is by contradiction. Let $\bar{y} = y^k + M^T \pi^k$ and $\bar{x} = x^k - \pi^k$. It is obvious that if $\bar{y} \not> 0$ or $\bar{x} \not> 0$, then

$$\|p^k\|^2 \geq 1.$$

On the other hand, as is developed in Ye [22], we have

$$\|p^k\|^2 = \left(\frac{\rho}{\Delta^k}\right)^2 \left\| \begin{pmatrix} X^k \bar{y} \\ Y^k \bar{x} \end{pmatrix} - \frac{\overline{\Delta}}{2n} e \right\|^2 + \left\| \frac{\rho \overline{\Delta}}{2n \Delta^k} e - e \right\|^2$$

$$\geq \left(\frac{\rho \overline{\Delta}}{2n \Delta^k} - 1 \right)^2 2n.$$

Hence, the following must be true:

$$\left(\frac{\rho \overline{\Delta}}{2n\Delta^k} - 1 \right)^2 2n < 1,$$

that is,

$$\frac{2n - \sqrt{2n}}{\rho} \Delta^k < \overline{\Delta} < \frac{2n + \sqrt{2n}}{\rho} \Delta^k. \qquad \blacksquare$$

$\overline{\Delta}$ can be further expressed as

$$\overline{\Delta} = 2\Delta^k - q^T \pi^k.$$

Thus, we can prove the following propositions.

PROPOSITION 5.   *Denote by* Σ^+ *the set*

$$\{\pi : x^T y - q^T \pi < 0 \text{ for some } (x, y) \in \Omega^+$$

$$\text{that also satisfy } x - \pi > 0 \text{ and } y + M^T \pi > 0\},$$

and let Σ^+ *be empty for an LCP* (M, q). *Then, for* $\rho \geqslant 2n + \sqrt{2n}$,

$$\gamma(M, q) \geqslant 1.$$

Proof.   The proof directly results from (2) and Lemma 1. $\blacksquare$

PROPOSITION 6.   *Let*

$$\{\pi : x^T y - q^T \pi > 0 \text{ for some } (x, y) \in \Omega^+$$

$$\text{that also satisfy } x - \pi > 0 \text{ and } y + M^T \pi > 0\}$$

be empty for an LCP (M, q). *Then, for* $n < \rho \leqslant 2n - \sqrt{2n}$,

$$\gamma(M, q) \geqslant 1.$$

Proof.   The proof again results from (2) and Lemma 1.   ∎

Now, let

$$\mathcal{G} = \{(M, q) : \Omega^+ \text{ is nonempty and } \Sigma^+ \text{ is empty}\}.$$

As pointed out by Stone [20], the description of $\mathcal{G}$ is technically similar to those of Eaves's class [3] and Garcia's class [4]. These two classes and some others have been extensively studied. It may not be possible in polynomial time to tell if an LCP (M, q) is an element of $\mathcal{G}$ (this is also true for some other LCP classes published so far). However, the coproblem, to tell that an LCP (M, q) is not in $\mathcal{G}$, can be solved in polynomial time. We can simply run the potential-reduction algorithm for the LCP. In polynomial time the algorithm either gives the solution or concludes that (M, q) is not in $\mathcal{G}$. The coproblem is actually more important in practice.

In the following, we present a dual interpretation using the Lagrangian multiplier. For $(M, q) \in \{(M, q) : |\text{sol } (M, q)| \geqslant 1\}$ (the class where at least one LCP solution exists), the LCP can be represented as an optimization problem with known zero optimal value:

(P)   minimize $x^T y$

subject to $(x, y) \in \Omega = \{(x, y) : y = Mx + q, \ x, y \geqslant 0\}.$

A dual to this problem can be written as

(D)   maximize $q^T \pi - x^T y$

subject to $(x, y, \pi) \in \{(x, y, \pi) : x - \pi \geqslant 0,$

$$y + M^T \pi \geqslant 0, (x, y) \in \Omega\}.$$

Σ^+ being empty means that for all (x, y, π) in the interior of the feasible region of (D), the objective value of (D) is less than or equal to zero. Since the objective value of (P) is always greater than or equal to zero, $\mathcal{G}$ is a class of LCPs satisfying the weak duality condition for (P) and (D).

4.   SOME EXISTING CLASSES BELONG TO THE NEW CLASS

We see that the new class $\mathcal{G}$ has the same bound on the condition number as the PSD class, that is, $\gamma(M, q) \geqslant 1$. Various other classes of LCPs can be found in Eaves [3], Murty [13], Saigal [18], and Stone [19]. Here, we list several existing classes of LCPs that belong to $\mathcal{G}$.

(1) *M is positive semidefinite and q is arbitrary.* We have, if Σ^+ is not empty,

$$0 < (x - \pi)^T (y + M^T \pi) = x^T y - q^T \pi - \pi^T M^T \pi,$$

which implies

$$x^T y - q^T \pi > \pi^T M^T \pi \geq 0,$$

a contradiction.

(2) *M is copositive and $q \geq 0$.* We have

$$x^T y - q^T \pi = x^T M x + q^T (x - \pi).$$

Thus, $x > 0$ and $x - \pi > 0$ implies $x^T y - q^T \pi \geq 0$, that is, Σ^+ is empty.

(3) M^{-1} *is copositive and $M^{-1} q \leq 0$.* We have

$$x^T y - q^T \pi = y^T M^{-T} y - (M^{-1} q)^T (y + M^T \pi).$$

Thus, $y > 0$ and $y + M^T \pi > 0$ implies $x^T y - q^T \pi \geq 0$, that is, Σ^+ is empty.

Although a trivial solution may exist for the last two classes (e.g., $x = 0$ and $y = q$ for the second class), our computational experience indicates that the potential-reduction algorithm usually converges to a nontrivial solution if multiple solutions exist. For example, let

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Then the potential-reduction algorithm constantly generates the solution

$$x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

from virtually any interior starting point, avoiding the trivial solution $x = 0$ and $y = q$. Another example:

$$M = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Again, the algorithm constantly generates the solution

$$x = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

from virtually any interior starting point. Both problems converge in a few iterations, although the second is a nonconvex problem. This property certainly deserves further research.

Another nonconvex LCP also belongs to $\mathscr{S}$:

$$M = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

This is because Ω^+ is nonempty, since $x = (3 \ 1)^T$ is an interior feasible point; and Σ^+ is empty, since $x_1 - x_2 > 1$, $x_1 - \pi_1 > 0$, $x_2 - \pi_2 > 0$, $x_1 - x_2 - 1 + \pi_1 + 2\pi_2 > 0$, and $2x_1 - 1 - \pi_1 > 0$ imply

$$x^T y - q^T \pi = x^T(Mx + q) - q^T \pi$$

$$= x_1(x_1 - x_2) + 2x_1 x_2 - x_1 - x_2 + \pi_1 + \pi_2$$

$$= x_1^2 + x_1 x_2 - 2x_1 - x_2 + 1 + (x_1 - x_2 - 1 + \pi_1 + 2\pi_2) + (x_2 - \pi_2)$$

$$> x_1^2 + x_1 x_2 - 2x_1 - x_2 + 1$$

$$= (x_1 - 1)^2 + x_2(x_1 - 1) > 0,$$

which indicates that Σ^+ is empty.

5. CONCLUDING REMARKS

As a by-product, we have

$$\mathscr{S} \subset \{(M, q) : |\text{sol}(M, q)| \geqslant 1\}.$$

In fact, any LCP (M, q) with $\gamma(M, q) > 0$ belongs to $\{(M, q) : |\text{sol}(M, q)| \geqslant 1\}$. Furthermore, if $\gamma(M, q) > 0$ for all $q \in R^n$, then $M \in \mathscr{Q}$, the matrix class where the LCP (M, q) has at least one solution for all $q \in R^n$. How to

calculate $\gamma(M,q)$ or a lower bound for $\gamma(M,q)$ in polynomial time is a further research topic.

At this time, the condition number $\gamma(M,q)$ and the class $\mathscr{S}$ are only partially understood, and hence $\mathscr{S}$ has not been shown to be a particularly large class. Nevertheless, the above analysis illustrates that the condition number in the potential-reduction algorithm may lead researchers to the study of new and different classes of LCPs that are solvable in polynomial time. Furthermore, it is possible that many existing classes of LCPs will find a more meaningful and useful characterization through the analysis of various LCP potential-reduction algorithms.

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