Math. Program., Ser. A (2016) 160:477–493
DOI 10.1007/s10107-016-0996-4

FULL LENGTH PAPER

Some LCPs solvable in strongly polynomial time with Lemke's algorithm

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Received: 17 October 2014 / Accepted: 19 February 2016 / Published online: 28 March 2016

Abstract We identify a class of Linear Complementarity Problems (LCPs) that are solvable in strongly polynomial time by Lemke's Algorithm (Scheme 1) or by the Parametric Principal Pivoting Method (PPPM). This algorithmic feature for the class of problems under consideration here is attributable to the proper selection of the covering vector in Scheme 1 or the parametric direction vector in the PPPM which leads to solutions of limited and monotonically increasing support size; such solutions are sparse. These and other LCPs may very well have multiple solutions, many of which are unattainable by either algorithm and thus are said to be elusive. The initial conditions imposed on the new matrix class identified in Sect. 2 are subsequently relaxed in later sections.

Mathematics Subject Classification 90C33

J.-S. Pang: The work of this author is based on research supported by the National Science Foundation under grants CMMI 1402052 and 0969600.
Part of this work was done while the authors were visiting the Institute for Mathematical Sciences at the National University of Singapore.

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1 Introduction

It has long been known that the standard Linear Complementarity Problem (LCP)

$$0 \le z \perp q + Mz \ge 0 \tag{1}$$

is NP-complete [4]. Papers such as [10,16], and [11] exhibit examples of problem classes on which familiar complementary pivot methods [14] and [6] execute an exponential number of steps in solving certain cleverly conceived LCPs. Relations of these problems to that of Klee and Minty [13] for the Simplex Algorithm of linear programming are elaborated in [5]. At the opposite end of the complexity spectrum are results [1,2], and [18] pertaining to a specific class of meaningful LCPs which can be solved in a strongly polynomial way by Lemke's Scheme 1 (abbreviated LS1) and by (a variant of) the Principal Pivoting Method [7]. The latter approach was extended in a study [17] that introduced the notion of an n-step vector and a strongly polynomial algorithm of the Parametric Principal Pivoting type. The works cited above constitute the primary background for the present investigation; more recent studies include [12,15], and [3]. For a comprehensive summary of the LCP, we refer the reader to the monograph [7], which also contains extensive notes and comments on results of this kind.

Terminology and notation

Here we recall some critical notations and terminology; in addition we introduce some new ones. It is easy to infer from (1) that the matrix M must be square, say $n \times n$ for some positive integer n. Accordingly, we write $q \in \mathbb{R}^n$, $M \in \mathbb{R}^{n\times n}$; we use the pair (q, M) to denote the corresponding LCP and refer to it as an LCP of *order* n. We write $\mathbb{R}^n_+$ for $\{x \in \mathbb{R}^n : x \ge 0\}$ and $\mathbb{R}^n_{++}$ for $\{x \in \mathbb{R}^n : x > 0\}$.

In the conventional statement of the LCP (q, M), the goal is to find a solution—*any solution*—of the system (1). We say that an algorithm *resolves* the LCP (q, M) if it always either finds a solution or produces a certificate (evidence) that the system (1) has no solution. In most cases, the latter outcome is due to the *infeasibility* of the problem's linear inequality constraints, but for an arbitrary LCP this is not always the case.

A common formulation of the LCP (q, M) expressed in (1) is the system

$$Iw - Mz = q \tag{2a}$$

$$w \ge 0, \quad z \ge 0 \tag{2b}$$

$$z^{\mathrm{T}} w = 0. \tag{2c}$$

We refer to (2a) as the *system equation* for the LCP (q, M). In this representation of the system equation, the w-variables are dependent on the z-variables which are independent. Again, relative to this representation and the explicit basis I, the w-variables are said to be *basic* whereas the z-variables are called *nonbasic*. This terminology agrees with that of linear programming.

In reference to a solution (w, z) of the system equation (2a), variables w_i and z_i, for $i \in \mathbf{N} \triangleq \{1, \ldots, n\}$, are called *complements* of each other. When the vectors w and z satisfy (2b) and (2c), the pairs of complementary variables w_i and z_i satisfy $z_i w_i = 0$ for every $i \in \mathbf{N}$.

The present work makes rather heavy use of certain index sets and families of index sets. We adopt the system of notation for rows, columns, and submatrices of a given matrix $A \in \mathbb{R}^{m \times n}$ used in [7]. A basis B in the matrix $[I \ \ -M]$ is said to be *complementary* if for all $i \in \mathbf{N}$, $B_{\bullet i} \in \{I_{\bullet,i}, \ -M_{\bullet i}\}$. If B is a complementary matrix relative to M, there is an associated index set $\alpha \subseteq \mathbf{N}$ such that $B_{\bullet i} = -M_{\bullet i}$ if and only if $i \in \alpha$. In this event, one can write $B = C_M(\alpha)$. The convention $I = C_M(\emptyset)$ applies here. For every $\alpha \subseteq \mathbf{N}$, the set $\mathrm{pos}(C_M(\alpha)) \triangleq \left\{\sum_{i \in \mathbf{N}} B_{\bullet i} v_i : v_i \geq 0 \ \text{ for all } i \in \mathbf{N}\right\}$ is called a *complementary cone*. For a given LCP (q, M),

$$\mathrm{FEA}\,(q, M) \triangleq \{z \in \mathbb{R}^n_+ : q + Mz \geq 0\} \quad \text{and}$$
$$\mathrm{SOL}\,(q, M) \triangleq \{z \in \mathrm{FEA}\,(q, M) : z^{\mathrm{T}}(q + Mz) = 0\}.$$

For every $M \in \mathbb{R}^{n \times n}$, the set $K(M)$ consists of all $q \in \mathbb{R}^n$ such that $\mathrm{SOL}\,(q, M) \neq \emptyset$. It is well known that $K(M)$ is a closed cone containing $\mathbb{R}^n_+$; indeed, it is the union of all complementary cones relative to M. For an arbitrary $M \in \mathbb{R}^{n \times n}$ the cone $K(M)$ need not be convex.[1] The cone $K(M)$ is convex, if and only if $K(M) = \mathrm{pos}([I \ -M])$ and when this is the case, we say that M belongs to the matrix class $\mathbf{Q_0}$. (See [9] and [7, **3.2.1**].) When $K(M) = \mathbb{R}^n$, we say that M belongs to the matrix class $\mathbf{Q}$.

In the sequel, we write $\mathcal{N}$ for the set of all *nonempty* index sets α whose elements belong to $\mathbf{N}$. We take particular interest in the subset of $\mathcal{N}$ consisting of index sets of cardinality s. With $|\alpha|$ representing the cardinality of α, we define

$$\mathcal{N}_{=s} \triangleq \{\alpha \in \mathcal{N} : |\alpha| = s\}.$$

Going a step further, we write

$$\mathcal{N}_{\leq s} \triangleq \{\alpha \in \mathcal{N} : |\alpha| \leq s\}.$$

For every $x \in \mathbb{R}^n$, the set

$$\sigma(x) \triangleq \{i \in \mathbf{N} : x_i \neq 0\}$$

is called the *support* of x. Thus, if x is a nonzero n-vector, its support is an element of $\mathcal{N}_{\leq s}$ for some s.

2 A new matrix class $\mathcal{SP}$

The concept of an n-step vector or its generalization to an extended n-step vector defines a class of $n \times n$ matrices for which such a vector exists and for which the LCP

[1] Obviously, this property is of importance for (algorithms to solve) parametric linear complementarity problems.

can be solved in $O(n^3)$ time. For any matrix in such a class, both LS1 and the Parametric Principal Pivoting Algorithm (PPPA)[2] will compute a solution of the LCP (q, M) for an arbitrary (constant) vector q in no more than $n+1$ (for LS1) or n pivots (for PPPA). The proof of Theorem 3 in [17] reveals that results about LS1 can be readily translated to statements about the PPPA, under a matrix-theoretic nondegeneracy assumption if necessary. Thus, from this point on, we will state and prove our results with respect to LS1 only. In dealing with pivoting algorithms for the LCP—and LS1 in particular—it is customary to invoke a nondegeneracy assumption on the pivots (as opposed to matrix nondegeneracy) or include a mechanism to eliminate "ties" in minimum ratio tests so as to prevent cycling. It will be seen that when the hypotheses of Theorem 1 hold, such a pivot nondegeneracy assumption is *not* actually needed.

Throughout the paper, we fix an integer $s \in \mathbf{N}$. We say that an $n \times n$ matrix $\mathcal{M}$ is *nondegenerate of order s* if

(a) $\det(M_{\alpha\alpha}) \neq 0$ for all $\alpha \in \mathcal{N}_{\leq s}$.

For $s = n$, this matrix nondegeneracy condition was employed in [17] to define an n-step vector and was removed by Chu [3] when she defined an *extended n-step vector*. Generalizing Chu's definition, we say that a vector $d \in \mathbb{R}^n_{++}$ is an *extended s-step vector* for $M \in \mathbb{R}^{n \times n}$ if it satisfies the following two conditions:

(b1) for every $\alpha \in \mathcal{N}_{\leq s}$, there exists $y_\alpha \geq 0$ such that $M_{\alpha\alpha} y_\alpha = d_\alpha$, and
(b2) for every $\alpha \in \mathcal{N}_{=s}$, $-d \in \text{pos}(C_M(\alpha))$.

The same vector $d \in \mathbb{R}^n_{++}$ is an *s-step* vector for $M \in \mathbb{R}^{n \times n}$ if condition (a) holds in addition to (b1) and (b2). Clearly, under condition (a), (b1) and (b2) can be equivalently stated as:

(b1′) $(M_{\alpha\alpha})^{-1} d_\alpha \geq 0$, for all $\alpha \in \mathcal{N}_{\leq s}$,
(b2′) $d_{\bar{\alpha}} - M_{\bar{\alpha}\alpha}(M_{\alpha\alpha})^{-1} d_\alpha \leq 0$, for all $\alpha \in \mathcal{N}_{=s}$ (where $\bar{\alpha}$ is the complement of α with respect to $\mathbf{N}$)

Further, whenever $s = n$, the above definitions coincide, respectively, with Chu's definition of an extended n-step vector (without (a)) and that of n-step definition (with (a)). In general, conditions (b1′) and (b2′) are easier to verify than (b1) and (b2)

Our first result extends Theorem 3 in [17] for an n-step vector and Chu's generalization [3] for an extended n-step vector. The result requires neither the matrix or pivot nondegeneracy assumption.

Theorem 1 *Let $M \in \mathbb{R}^{n \times n}$ and let d be an extended s-step vector for M for some $s \in \mathbf{N}$. Then, for every vector $q \in \mathbb{R}^n$, LS1, with covering vector d, will compute a solution to the LCP (q, M) in no more than $s + 1$ pivots.*

[2] This acronym is due to Chu [3]. Under the present conditions, it is the same as the PPPM [7, Algorithm **4.5.2**].

Proof Consider the application of LS1 to the auxiliary system (q, d, M) given by

$$w = q + dz_0 + Mz \tag{3a}$$

$$w \geq 0, \quad z_0 \geq 0, \quad z \geq 0 \tag{3b}$$

$$z^{\mathrm{T}} w = 0. \tag{3c}$$

It can be assumed that $q \notin \mathbb{R}^n_+$. The first pivot will make z_0 basic replacing some w-variable. At a subsequent current iteration wherein z_0 is positive (and thus is still a basic variable), we have available an index set α, of cardinality $|\alpha| \in \{0, \ldots, s-1\}$, corresponding to the basic z-variables and an index $t \notin \alpha$ such that w_t has just become nonbasic so that z_t is the incoming variable and $\{w_t, z_t\}$ is the nonbasic pair. For each basic variable, the effect of increasing z_t can be expressed as a partial derivative, and that quantity is just the coefficient of z_t in the updated system Eq. (3a). No other terms are needed because the other nonbasic variables (including w_t) remain fixed at their current value of zero. After a principal rearrangement, if necessary, we may write (noting that the current basic z-variables are z_α and z_0)

$$\frac{\partial}{\partial z_t}\begin{pmatrix} z_\alpha \\ z_0 \end{pmatrix} = -\begin{bmatrix} M_{\alpha\alpha} & d_\alpha \\ M_{t\alpha} & d_t \end{bmatrix}^{-1}\begin{bmatrix} M_{\alpha t} \\ M_{tt} \end{bmatrix} \tag{4}$$

$$\frac{\partial w_\gamma}{\partial z_t} = M_{\gamma t} - [\, M_{\gamma\alpha}\; d_\gamma \,]\begin{bmatrix} M_{\alpha\alpha} & d_\alpha \\ M_{t\alpha} & d_t \end{bmatrix}^{-1}\begin{bmatrix} M_{\alpha t} \\ M_{tt} \end{bmatrix} \tag{5}$$

where γ is the complement of $\{t\} \cup \alpha$ in $\mathbf{N}$. We claim that the next pivot will not occur in a z_α-row. By condition (b1), there exists $(y_\alpha, y_t) \geq 0$ such that $\begin{bmatrix} M_{\alpha\alpha} & M_{\alpha t} \\ M_{t\alpha} & M_{tt} \end{bmatrix}\begin{pmatrix} y_\alpha \\ y_t \end{pmatrix} = \begin{pmatrix} d_\alpha \\ d_t \end{pmatrix}$. Since the matrix $\begin{bmatrix} M_{\alpha\alpha} & d_\alpha \\ M_{t\alpha} & d_t \end{bmatrix}$ is nonsingular, by the nature of the pivots, the vector $\begin{pmatrix} d_\alpha \\ d_t \end{pmatrix}$ is independent of $\begin{bmatrix} M_{\alpha\alpha} \\ M_{t\alpha} \end{bmatrix}$; hence, it follows that $y_t \neq 0$. It is not difficult to show that

$$\begin{bmatrix} M_{\alpha\alpha} & d_\alpha \\ M_{t\alpha} & d_t \end{bmatrix}^{-1}\begin{bmatrix} M_{\alpha t} \\ M_{tt} \end{bmatrix} = \frac{1}{y_t}\begin{pmatrix} -y_\alpha \\ 1 \end{pmatrix}.$$

Since $y_\alpha \geq 0$ and $y_t > 0$, it follows that every component of z_α is a nondecreasing function of z_t and z_0 is a strictly decreasing function of z_t. This is enough to establish that the next pivot will not occur in a z_α-row.

Therefore, as LS1 proceeds, after the first pivot, once a z-variable becomes basic it will remain so, as long as $|\alpha| \leq s - 1$. Hence the algorithm will either terminate with a solution of the LCP having fewer than s basic z-variables, or it will reach an iteration with exactly $s - 1$ basic z-variables and the next entering variable will be a z-variable. In terms of the system (4), the cardinality of $\widehat{\alpha} \triangleq \{t\} \cup \alpha$ will then equal s. When this happens, we claim that condition (b2) guarantees that z_0 must be the outgoing variable, and therefore the algorithm must terminate with a solution of the LCP (q, M) having exactly s basic z-variables. Indeed, in this case, since $-d \in \operatorname{pos}(C_M(\alpha))$, we may let $\widehat{z}$ be a solution of the LCP$(-d, M)$ such that

$$-\begin{pmatrix} d_\alpha \\ d_t \end{pmatrix} + \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha t} \\ M_{t\alpha} & M_{tt} \end{bmatrix} \begin{pmatrix} \widehat{z}_\alpha \\ \widehat{z}_t \end{pmatrix} = 0 \le \begin{pmatrix} \widehat{z}_\alpha \\ \widehat{z}_t \end{pmatrix}$$

and $-d_\gamma + M_{\gamma\alpha}\widehat{z}_\alpha + M_{\gamma t}\widehat{z}_t \ge 0$. With $\begin{pmatrix} \widehat{z}_\alpha \\ \widehat{z}_t \end{pmatrix}$ playing the role of $\begin{pmatrix} y_\alpha \\ y_t \end{pmatrix}$, we deduce that $\widehat{z}_t > 0$ and

$$M_{\gamma t} - [\, M_{\gamma\alpha} \; d_\gamma \,] \begin{bmatrix} M_{\alpha\alpha} & d_\alpha \\ M_{t\alpha} & d_t \end{bmatrix}^{-1} \begin{bmatrix} M_{\alpha t} \\ M_{tt} \end{bmatrix} = \frac{1}{\widehat{z}_t} \left[-d_\gamma + M_{\gamma\alpha}\widehat{z}_\alpha + M_{\gamma t}\widehat{z}_t \right] \ge 0.$$

Consequently, when $|\alpha| = s - 1$ and z_t with $t \notin \alpha$ is the entering variable, the only candidate to become nonbasic is z_0. □

Remark 2 It can be seen from the argument above that—under the hypotheses of the theorem—no degeneracy resolution apparatus is required with LS1. For cycling to occur, a basic z-variable would have to become nonbasic, and this can not happen in such a case.

For a fixed integer $s \in \mathbf{N}$, let $\mathcal{SP}_s$ denote the set of all $n \times n$ matrices for which an extended s-step vector exists; and let $\mathcal{SP} \triangleq \bigcup_{s=1}^{n} \mathcal{SP}_s$.

Theorem 1 has several important consequences, the first and foremost being the strongly polynomial solvability of *every* LCP (q, M) for $M \in \mathcal{SP}$. Most interestingly, the $(s + 1)$-step termination of LS1 is based on the property that once a z-variable becomes basic, it stays basic; when $s - 1$ of the z-variables have become basic, then the next entering variable, which must be a z-variable, will force the auxiliary variable z_0 to become nonbasic, thereby returning a solution of the LCP (q, M). A second consequence is that the solution obtained at termination of LS1 has at most s positive z-variables; such a solution is considered to be *sparse* if s/n is small. A third consequence is that any $z \in \mathrm{SOL}\,(q, M)$ having $|\sigma(z)| > s$, i.e., more than s positive components, will be *elusive* in the sense that it cannot be computed by LS1 applied to (q, d, M) where d is an extended s-step vector. (The reader is referred to a recent paper [19] where it is demonstrated that LS1 can only compute solutions with a particular property of the LCP formulation of a *generalized Nash equilibrium problem*; such problems provide a class of realistic LCPs where many solutions are elusive when they are solved by LS1.) A fourth consequence of Theorem 1 is that all matrices $M \in \mathcal{SP}$ must belong to the class $\mathbf{Q}$, which is to say that for *every* q, the LCP (q, M) has at least one solution.

It is also of interest to consider the set $\mathcal{D}(s, M)$ of all extended s-step vectors for M. When $\mathcal{D}(s, M) \neq \emptyset$, i.e., an extended s-step vector exists for M, the main diagonal elements of M must be positive. It is easily seen that when $D \in \mathbb{R}^{n\times n}$ is any diagonal matrix with positive diagonal elements, then $\mathcal{D}(s, M) = \mathcal{D}(s, MD)$. Combining these last two observations, it follows that when $\mathcal{D}(s, M) \neq \emptyset$, the columns of M can be scaled so as to make its diagonal entries equal 1. It is clear that every positive multiple of an element of $\mathcal{D}(s, M)$ is another element of this set. In that sense, it is a so-called *blunt cone*, but it is not a true cone as it does not contain the zero vector nor is it necessarily convex because of the unusual condition (b2). If condition (a) is in place, then the set of nonnegative vectors d satisfying conditions (b1′) and (b2′) is indeed a polyhedral cone.

As shown by the discussion above, $\mathcal{SP}$ is a very interesting class of matrices, but it is not easy to identify members of this class except when s is either very small (see next section) or equals n (see [17]). Condition (b1) on the vector d is reasonable as checking whether it is satisfied is not difficult; it involves solving a linear program. On the other hand, condition (b2) is indeed highly restrictive. It basically demands that for every index set of cardinality exactly s, a solution of the LCP$(-d, M)$ exists with the given index set as support. Nevertheless, when these two conditions are in place, then for all $q \in \mathbb{R}^n$, using the single covering vector d, the LS1 must terminate after no more than $(s + 1)$ iterations. As one can understand, this is a rather tall order in general. Thus we will devote some effort in a later section to relaxing this requirement whereby strong polynomiality of the algorithm can be maintained. We first discuss some further consequences of the theorem and then present its generalizations.

3 Discussion with examples

In this section, we consider the cases of $s = 1$ and $s = 2$ and give examples of matrices M for which an extended s-step vector exists. We also discuss the case where $s \geq 3$.

• $s = 1$. We call $\bar{z} \in \text{SOL}\,(q, M)$ with $|\sigma(\bar{z})| = 1$ a *super-sparse* solution of the LCP. Suppose the diagonal elements of M are all positive. In that case, it is not restrictive to assume that they are all equal to 1. Such a matrix is nondegenerate of order 1. If, in addition, the off-diagonal elements are all greater than or equal to 1, then M and d, taken to be the vector of all ones, denoted e, easily satisfy conditions (b1) and (b2). This implies that for every $q \in \mathbb{R}^n \backslash \mathbb{R}^n_+$, the LCP will have a super-sparse solution, found in two pivot steps by LS1. Obviously, matrices such as these are positive, hence they belong to a matrix class $\mathbf{L}$ for which Eaves [9, p. 621] proved that every nondegenerate LCP (q, M) has an *odd* number of solutions.

It follows directly from the definitions that for a given $M \in \mathbb{R}^{n\times n}$ with positive diagonal entries, the conditions for extended 1-step vector are equivalent to the existence of $d \in \mathbb{R}^n$ such that

$$d > 0 \quad \text{and} \quad -d \in \text{pos}\,(C_M(i)) \;\; \text{for all } i \in \mathbf{N}. \tag{6}$$

This means that the condition above implies the conclusion of Theorem 1. Rephrasing condition (6) more explicitly, we obtain the following corollary.

Corollary 3 *Let $d \in \mathbb{R}^n_{++}$ be given. Then LS1 using d as covering vector will solve (q, M) in exactly two pivot steps for all $q \not\geq 0$ if and only if M has positive diagonal elements and*

$$\frac{m_{ji}}{m_{ii}} \geq \frac{d_j}{d_i} \quad \textit{for all} \;\; j \neq i. \tag{7}$$

Proof It is clear that with $m_{ii} > 0$ and $d_i > 0$ for all $i \in \mathbf{N}$, the condition (7) is equivalent to (6) which is exactly condition (b2) for $s = 1$. Hence the assertion on LS1 follows readily from Theorem 1.

Conversely, suppose that LS1 with d as covering vector solves every LCP (q, M) with $q \ngeq 0$ in exactly two pivot steps. For every index i, there must exist a vector $q \ngeq 0$ such that the first pivot step will be $\langle w_i, z_0 \rangle$. Obtaining a solution with just one more pivot step requires it to be $\langle z_0, z_i \rangle$. But the minimum ratio test used in LS1 leads to this pivot only if $m_{ii} > 0$ and

$$\frac{\partial w_j}{\partial z_i} = \frac{m_{ji} d_i - m_{ii} d_j}{m_{ii}} \geq 0 \quad \text{for all } j \neq i. \tag{8}$$

Clearly this implies (7). □

Given M with $m_{ii} = 1$ for all $i \in \mathbf{N}$ and all off-diagonal elements $m_{ij} \geq 1$, the inequalities (7) can be formulated as a linear inequality system in the vector variable d allowing us to search for elements of $\mathcal{D}(1, M)$ other than those of the form $(\theta, \ldots, \theta)$ for $\theta > 0$. The inequalities in (8) being homogeneous, we can require d to satisfy $d \geq e$. Thus we arrive at a system of linear inequalities of the form

$$Ad \geq 0, \quad d \geq e, \tag{9}$$

where A is an $n(n-1) \times n$ matrix whose rows are labeled by ij for $i \neq j$ in $\mathbf{N}$ and whose columns are labeled by $k \in \mathbf{N}$ with the (ij, k)-entry given by

$$A_{ij,k} \triangleq \begin{cases} m_{ji} & \text{if } k = i \\ -m_{ii} & \text{if } k = j \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Given M, the question of whether there exists a covering vector d that satisfies (9) can be answered by the Phase One approach of linear programming. The question of uniqueness is another matter.

For the matrix

$$M = \begin{bmatrix} 1.0000 & 1.5578 & 1.9879 & 1.0740 \\ 1.0773 & 1.0000 & 1.1704 & 1.6841 \\ 1.9138 & 1.1662 & 1.0000 & 1.4024 \\ 1.7067 & 1.6225 & 1.3968 & 1.0000 \end{bmatrix},$$

a vector d found in this way is:

$$d = (0.9474, 0.8111, 0.7105, 0.9702).$$

As noted above, when $M > 0$, every nondegenerate LCP (q, M) will have an odd number of solutions, possibly as many as $2^n - 1$. For instance, if M is the matrix in the previous paragraph and q is the negative of the vector of all ones, then the LCP (q, M) has 9 solutions; the supports of the z vectors in these solutions are: {1}, {2}, {3}, {4}, {1, 2}, {1, 4}, {2, 3}, {3, 4}, and {1, 2, 3, 4}.

Finally, we note that any matrix M with a unit diagonal and a constant bigger than 1 elsewhere and a vector q of minus ones will produce all $2^n - 1$ possible complementary solutions. In addition, as noted above, the vector of all ones is a 1-step vector for such an M.

• $s = 2$. Let M be an $n \times n$ matrix all of whose diagonal elements are positive and all of whose principal submatrices of order 2 are nonsingular. It is not difficult to see that if a positive n-vector d exists such that every $\alpha \in \mathcal{N}_{=2}$ yields a solution of the LCP $(-d, M)$ with support contained in α, then for every $q \in \mathbb{R}^n$, the LCP (q, M) has a solution whose support is contained in α and can be obtained by LS1 in at most 3 pivots. The following example confirms that such a case can occur. Let

$$M = \begin{bmatrix} 4 & 115 & 40 & 90 \\ 40 & 25 & 60 & 70 \\ 25 & 80 & 1 & 80 \\ -10 & 45 & 40 & 5 \end{bmatrix} \quad \text{and} \quad d = M_{\bullet 4}.$$

The diagonal entries of M are positive and all 6 of its 2×2 principal submatrices are nonsingular. The LCP (q, M) having $q = (-60, 200, -10, 10)$ and the M displayed above has 3 solutions; they correspond to the index sets $\{3\}$, $\{1, 3\}$, and $\{1, 4\}$. The latter solution is found by LS1 with the chosen covering vector d.

• $s \geq 3$. A matrix $M \in \mathcal{SP}$ has many hereditary properties. In what follows, we establish one such property for a matrix in $M \in \mathcal{SP}_s$ which satisfies condition (a) (that is, a matrix for which an s-step vector exists). Let $d \in \mathcal{D}(s, M)$. For each diagonal entry m_{ii} of M, which must be nonzero, let $\widehat{M}_i$ denote the Schur complement of m_{ii} in M. Such $\widehat{M}_i$ will be of order $n-1$. Define the $(n-1)$-vector $\widehat{d}$ where

$$\widehat{d}_j = d_j - \frac{m_{ji}}{m_{ii}} d_i, \quad \text{for all } j \neq i.$$

Theorem 4 *If* $\det(M_{\alpha\alpha}) > 0$ *and* $(M_{\alpha\alpha})^{-1} d_\alpha > 0$ *for all index sets* $\alpha \in \mathcal{N}_{=2}$*, i.e., if for all* $i \neq j$*,* $m_{ii}m_{jj} - m_{ij}m_{ji} > 0$ *and*

$$\frac{m_{ji}}{m_{ii}} < \frac{d_j}{d_i}, \tag{11}$$

then $\widehat{M}_i \in \mathcal{SP}_{s-1}$ *for all* $i \in \mathbf{N}$ *and* $\widehat{d} \in \mathcal{D}(s-1, \widehat{M}_i)$.

Proof To establish this claim we first show that every principal submatrix $(\widehat{M}_i)_{\widehat{\alpha}\widehat{\alpha}}$ of $\widehat{M}_i$ of order $|\widehat{\alpha}| \leq s-1$ is nonsingular, where $\widehat{\alpha} \subset \mathbf{N} \backslash \{i\}$. Indeed, any such principal submatrix is the Schur complement of m_{ii} in the principal submatrix $M_{\alpha\alpha}$ of M, where $\alpha = \widehat{\alpha} \cup \{i\}$. The nonsingularity of $(\widehat{M}_i)_{\widehat{\alpha}\widehat{\alpha}}$ therefore follows from the positivity of m_{ii} and the nonsingularity of $M_{\alpha\alpha}$. The inequalities in (11) imply that $\widehat{d}$ is a positive vector. To complete the proof, it suffices to show that conditions (b1) and (b2) hold for the matrix $\widehat{M}_i$ with the vector $\widehat{d}$. We first show (b1); for this we need to show $\widetilde{d} = \{(\widehat{M}_i)_{\widehat{\alpha}\widehat{\alpha}}\}^{-1} \widehat{d}_{\widehat{\alpha}} \geq 0$, where $\widehat{\alpha} \subset \mathbf{N} \backslash \{i\}$ is of cardinality not exceeding $s-1$. It is not difficult to verify that

$$\widetilde{d} = \left[M_{\widehat{\alpha}\widehat{\alpha}} - \frac{1}{m_{ii}} M_{\widehat{\alpha}i} M_{i\widehat{\alpha}} \right]^{-1} \left(d_{\widehat{\alpha}} - \frac{M_{\widehat{\alpha}i}}{m_{ii}} d_i \right).$$

By Schur's inversion formula we have

$$\begin{bmatrix} M_{\widehat{\alpha}\widehat{\alpha}} & M_{\widehat{\alpha}i} \\ M_{i\widehat{\alpha}} & m_{ii} \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} \left[M_{\widehat{\alpha}\widehat{\alpha}} - \frac{1}{m_{ii}} M_{\widehat{\alpha}i} M_{i\widehat{\alpha}} \right]^{-1} & -\left[M_{\widehat{\alpha}\widehat{\alpha}} - \frac{1}{m_{ii}} M_{\widehat{\alpha}i} M_{i\widehat{\alpha}} \right]^{-1} \frac{M_{\widehat{\alpha}i}}{m_{ii}} \\ -\frac{M_{i\widehat{\alpha}}}{m_{ii}} \left[M_{\widehat{\alpha}\widehat{\alpha}} - \frac{1}{m_{ii}} M_{\widehat{\alpha}i} M_{i\widehat{\alpha}} \right]^{-1} & \dfrac{1 + M_{i\widehat{\alpha}} \left[M_{\widehat{\alpha}\widehat{\alpha}} - \frac{1}{m_{ii}} M_{\widehat{\alpha}i} M_{i\widehat{\alpha}} \right]^{-1} \frac{M_{\widehat{\alpha}i}}{m_{ii}}}{m_{ii}} \end{bmatrix};$$

hence $\widetilde{d}$ is the $\widehat{\alpha}$-component of $\begin{bmatrix} M_{\widehat{\alpha}\widehat{\alpha}} & M_{\widehat{\alpha}i} \\ M_{i\widehat{\alpha}} & m_{ii} \end{bmatrix}^{-1} \begin{pmatrix} d_{\widehat{\alpha}} \\ d_i \end{pmatrix}$ and is thus nonnegative. Next, to show (b2), let $j \in \mathbf{N}\backslash\alpha$. We need to show that $\widehat{d}_j - (\widehat{M}_i)_{j\widehat{\alpha}} \widetilde{d}_{\widehat{\alpha}} \leq 0$. We have

$$\begin{aligned} \widehat{d}_j - (\widehat{M}_i)_{j\widehat{\alpha}} \widetilde{d}_{\widehat{\alpha}} &= d_j - \frac{m_{ji}}{m_{ii}} d_i - \left[M_{j\widehat{\alpha}} - \frac{m_{ji}}{m_{ii}} M_{i\widehat{\alpha}} \right] \\ &\quad \times \left[M_{\widehat{\alpha}\widehat{\alpha}} - \frac{1}{m_{ii}} M_{\widehat{\alpha}i} M_{i\widehat{\alpha}} \right]^{-1} \left(d_{\widehat{\alpha}} - \frac{M_{\widehat{\alpha}i}}{m_{ii}} d_i \right). \end{aligned}$$

It follows that

$$d_j - M_{j\alpha}(M_{\alpha\alpha})^{-1} d_\alpha = d_j - (M_{j\widehat{\alpha}}\; m_{ji}) \begin{bmatrix} M_{\widehat{\alpha}\widehat{\alpha}} & M_{\widehat{\alpha}i} \\ M_{i\widehat{\alpha}} & m_{ii} \end{bmatrix}^{-1} \begin{pmatrix} d_{\widehat{\alpha}} \\ d_i \end{pmatrix},$$

which can be seen to equal $\widehat{d}_j - (\widehat{M}_i)_{j\widehat{\alpha}} \widetilde{d}_{\widehat{\alpha}}$. □

4 Relaxing condition (b1)

The gist of Theorem 1 is that under the conditions (b1) and (b2) of the extended s-step vector, LS1, with an extended s-step covering vector, will compute a solution to the LCP (q, M) for any vector q in at most $s + 1$ pivots, without a z-variable becoming basic and later nonbasic and without a nondegeneracy assumption on the pivots. From this conclusion, the three solution properties: strong polynomiality, sparsity, elusiveness follow. While this is very desirable, the two conditions (b1) and (b2) are admittedly quite restrictive; for one thing, it seems difficult to identify matrices for which an extended s-step vector exists for $s \geq 3$. Starting in this section, we will relax the conditions in various ways. Such relaxations come with a price; they will lack at least one of the four properties: strong polynomiality, sparsity, elusiveness, and nondegeneracy assumption on the pivots. The loss of elusiveness is not yet thought to be undesirable.

We next present a variant of Theorem 1 wherein we relax condition (b1) retaining the property that LS1 terminates in a solution once the number of z-variables in the basis reaches s. However, we lose the strong polynomiality, though a strong polynomiality for *fixed* s is achieved.

Theorem 5 *Given $M \in \mathbb{R}^{n\times n}$, $d \in \mathbb{R}^n_{++}$, and $s \in \mathbf{N}$ suppose that:*

(i) for every $\alpha \in \mathcal{N}_{\leq s}$, no nonzero u exists such that $\sigma(u) \subseteq \alpha$ and $u \in \mathrm{SOL}\,(d\theta, M)$ for some $\theta \geq 0$;

(ii) for every $\alpha \in \mathcal{N}_{=s}$, $-d \in pos(C_M(\alpha))$.

Then, for every vector $q \in \mathbb{R}^n$, LS1 under a nondegeneracy assumption on the pivots will compute a solution $\bar{z}$ to the LCP (q, M) with $|\sigma(\bar{z})| \leq s$.

Proof Suppose that using the vector d, the algorithm terminates with s' basic z-variables, for some positive integer s' not exceeding $s - 1$. By (i), the algorithm cannot terminate on a secondary ray, so it must terminate with $\bar{z} \in \mathrm{SOL}\,(q, M)$ with $|\sigma(\bar{z})| < s$. Thus, under the nondegeneracy assumption on the pivots, the algorithm (if not terminating in a solution) must reach a point where there are exactly $(s - 1)$ basic z-variables and the next pivot entry is a z-variable, say z_t. Since condition (ii) is identical to condition (b2), and it also implies condition (b1) for $|\alpha| = s$, we can use the same argument as the one used in the proof of Theorem 1 to show that if LS1 reaches step s, it terminates with a solution at that step. □

In what follows, we make several remarks about the above theorem.

Remark 6 Condition (i) is precisely that of a d-regular matrix; see [7, **3.9.20]** for discussion of such a matrix. In particular, if M is strictly semimonotone, then this condition is readily satisfied for all positive vectors d.

Remark 7 Many matrices M have the property that for all $q \in \mathbb{R}^n$ and $d \in \mathbb{R}^n_{++}$, any secondary ray obtained during the application of LS1 certifies that $\mathrm{FEA}\,(q, M) = \emptyset$ (and thus $\mathrm{SOL}\,(q, M) = \emptyset$). Hence, if any such matrix satisfies (ii) in Theorem 5 for which $\mathrm{FEA}\,(q, M) \neq \emptyset$, then there exists a vector $\bar{z} \in \mathrm{SOL}\,(q, M)$ with $|\sigma(\bar{z})| \leq s$.

Remark 8 For $n > s$, condition (ii) implies that the LCP $(-d, M)$ has multiple solutions. Since $M \in \mathbf{P}$ if and only if (q, M) has a unique solution for every $q \in \mathbb{R}^n$, it follows that Theorem 5 excludes all matrices belonging to the class **P**.

Remark 9 Under the assumptions of Theorem 5 and with a non-cycling device if necessary, the LCP (q, M) is solvable by LS1 (with covering vector d) in strongly polynomial time for fixed s. The argument is as follows. Count the number of bases with z_0 plus exactly r more z-variables as basic variables. Any such specific basis has an additional $(n - r - 1)$ basic w-variables out of $n - r$ which are eligible to keep complementarity. So altogether there are $\binom{n}{r}(n - r)$ such bases. This number is polynomial in n with degree $r + 1$. Now sum up these terms for $r = 1$ to $s - 1$. This sum is an upper bound for the number of complementary bases made up from the basic z_0 plus up to $(s - 1)$ basic z-variables. Since when we arrive at the first basis with s basic z-variables the algorithm terminates, we have a polynomial in n of degree s as an upper bound on the number of steps.

5 Relaxing condition (b2)

As mentioned before, condition (b2) is rather restrictive. In this section, we consider the removal of this condition. This will yield classes of LCPs that remain strongly polynomially solvable and yet the computed solution z is not necessarily sparse.

Consider a class of matrices $M \in \mathbb{R}^{n\times n}$ satisfying conditions (b1) for $s = n-1$, but not condition (b2). Unlike the matrices discussed in the previous sections, a matrix M herein may be such that for some $q \in \mathbb{R}^n$ the LCP (q, M) has no solution. The class of matrices satisfying the assumptions of Theorem 10 below is reminiscent of the class of $\mathbf{P_1}$-matrices [8] where the defining properties are of the "all-but-one" type, although the properties are different in these two cases.

Before stating the theorem, we introduce some further notation. For $M \in \mathbb{R}^{n\times n}$ and each $i \in \mathbf{N}$, let $M^{(-i)}$ denote the principal submatrix of M obtained by deleting the ith row and column of M. Similarly, let $M_{-i,\bullet}$ ($M_{\bullet,-i}$) denote the submatrix of M obtained by deleting the ith row (column). To put these notations a little differently, let $\beta = \mathbf{N}\setminus\{i\}$. Then

$$M^{(-i)} = M_{\beta\beta}, \quad M_{-i,\bullet} = M_{\beta,\bullet}, \quad \text{and} \quad M_{\bullet,-i} = M_{\bullet,\beta}.$$

Likewise, put

$$x_{-i} = x_\beta.$$

We remark that a matrix M satisfying the first assumption in the theorem below must be nondegenerate of order $n-1$.

Theorem 10 *Suppose $M \in \mathbb{R}^{n\times n}$ is such that for every $i \in \mathbf{N}$ the principal submatrix $M^{(-i)}$ is of class $\mathbf{P}$ and there exists a vector $d^{-i} \in \mathbb{R}^{n-1}_{++}$ such that*

$$(M^{(-i)}_{\alpha\alpha})^{-1} d^{(-i)}_\alpha \geq 0, \quad \textit{for all } \alpha \subseteq \{1, \ldots, n\}\setminus\{i\}.$$

For any $q \in \mathbb{R}^n$, the following statements hold:

(i) the LCP (q, M) can be resolved in strongly polynomial time by LS1;

(ii) there are at most n solutions z with at least one zero component (called type-I solution*), each of which is characterized by an index i such that z_{-i} is the unique solution of the strongly polynomially solvable LCP $(q^{(-i)}, M^{(-i)})$, i.e.,*

$$0 \leq z_{-i} \perp q_{-i} + M^{(-i)} z_{-i} \geq 0 \tag{12}$$

and $z_i = 0 \leq q_i + M_{i,-i} z_{-i}$;

(iii) if a solution z has all components positive (called a type-II solution*), then $q + Mz = 0$; hence any such solution is characterized by an index $i \in \mathbf{N}$ such that*

$$\begin{aligned} z_{-i} &= -[M^{(-i)}]^{-1}(q_{-i} + M_{-i,i} z_i) > 0; \quad z_i > 0 \\ 0 &= \left(q_i - M_{i,-i}[M^{(-i)}]^{-1} q_{-i}\right) + \left(m_{ii} - M_{i,-i}[M^{(-i)}]^{-1} M_{-i,i}\right) z_i, \end{aligned} \tag{13}$$

which demonstrates that if a type-II solution exists, it can be computed in strongly polynomial time.

Proof For each $i \in \mathbf{N}$, consider the application of the $(n-1)$-step scheme to the parametric LCP of order $n-1$ given by

$$0 \leq z_{-i} \perp w_{i-1} = q_{-i} + \tau d^{(-i)} + M^{(-i)} z_{-i} \geq 0.$$

By the choice of $d^{(-i)}$, the unique solution of the above LCP, denoted $\widehat{z}_{-i}$, can be obtained in no more than $n-1$ (diagonal) pivots. Augmenting $\widehat{z}_{-i}$ by zero in the omitted ith component, we deduce that the augmented vector, denoted $\widehat{z}^{(i)}$ is a solution of the LCP (q, M) if and only if $q_i + M_{i\bullet}\widehat{z}^{(i)} \geq 0$. In this manner, after we have carried out this procedure for all indices $i \in \mathbf{N}$, we will be able to declare that one of three situations must hold: (a) we have obtained a (type-I) solution to the LCP (q, M) that has at least one zero component (indexed by i); (b) the original LCP has no solution, or (c) its solutions must have the z-vector positive (type-II solution).

Every type-I solution is as given by (iii) for some i (cf. (12)). Thus there are at most n of them. Moreover, every type-II solution z must satisfy, by complementarity, the equation $q + Mz = 0$ which we rewrite in partitioned form as

$$\begin{pmatrix} q_{-i} \\ q_i \end{pmatrix} + \begin{bmatrix} M^{(-i)} & M_{-i,i} \\ M_{i,-i} & m_{ii} \end{bmatrix} \begin{pmatrix} z_{-i} \\ z_i \end{pmatrix} = 0.$$

Expression (13) follows easily from the nonsingularity of $M^{(-i)}$. Finally, the strongly polynomial time needed for obtaining (and verifying) a type-I or type-II solution is obvious. If these solutions do not exist, then the LCP (q, M) has no solution. □

It is worthwhile to make several remarks about the above theorem.

Remark 11 The vectors $d^{(-i)}$ need not be the same for different indices i. This relaxation significantly expands the class of strongly polynomially resolvable LCPs. A noteworthy fact is that an LCP of this class may not have a solution, and such insolvability can be determined in strongly polynomial time.

Remark 12 There is no stipulation about the Schur complement $m_{ii} - M_{i,-i}(M^{(-i)})^{-1} M_{-i,i}$ in statement (iii) of the theorem. In particular, under the **P**-property of the proper principal submatrices of M, the latter Schur complement is positive if and only if $M \in \mathbf{P}$; it is zero if and only if M is singular. A simple example of a singular matrix M satisfying the conditions of Theorem 10 is

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

We can further expand the class of strongly polynomially resolvable LCPs by adding rows and columns to matrices satisfying the conditions of Theorem 13 below. This is accomplished under the assumption that every submatrix of order $n-2$ has an $(n-2)$-step vector. Conditions are then imposed on the omitted rows and columns that allow us to resolve the LCP strongly polynomially. To this end, we introduce some notations. For each pair of distinct indices $i \neq j$ in $\mathbf{N}$, let $M^{(-ij)}$ be the principal submatrix obtained from M by deleting its ith and its jth rows and columns. For any vector x, we let x_{-ij} be the subvector of x whose i-component and j-component are deleted. Similarly, let $M_{-ij,\bullet}$ be the submatrix of M with the ith row and the jth row deleted.

The submatrix $M_{\bullet,-ij}$ would be defined analogously. The two special properties of the type-I and type-II solutions in the last two parts of Theorem 13 are key to its strongly polynomial time assertion.

Theorem 13 *Suppose $M \in \mathbb{R}^{n\times n}$ is such that for any distinct pair of indices $i \neq j$ in $\mathbf{N}$, the principal submatrix $M^{(-ij)}$ is of class $\mathbf{P}$ and a positive vector $d^{(-ij)} \in \mathbb{R}^{n-2}$ exists such that*

$$(M^{(-ij)}_{\alpha\alpha})^{-1} d^{(-ij)}_{\alpha} \geq 0 \quad \textit{for all } \alpha \subseteq \mathbf{N}\backslash\{i, j\}. \tag{14}$$

For any vector $q \in \mathbb{R}^n$, the following statements hold:

(i) the LCP (q, M) can be resolved in strongly polynomial time;
(ii) if $\mathrm{SOL}\,(q, M) \neq \emptyset$, the solutions z are of three types, depending on the size of their support, and any of them can be computed in strongly polynomial time: solutions of type I have support of size $n-2$; solutions of type II have support of size $n-1$; solutions of type III have support of size n;
(iii) there are at most $\binom{n}{2}$ type-I solutions, each of which is characterized by a pair of indices $i \neq j$ such that $z = (z_{-ij}, z_i, z_j)$ with z_{-ij} being the unique solution of the LCP $(q_{-ij}, M^{(-ij)})$ of order $n-2$:

$$0 \leq z_{-ij} \perp q_{-ij} + M^{(-ij)} z_{-ij} \geq 0, \tag{15}$$

and $z_i = z_j = 0 \leq \min\left(q_i + \sum_{k\neq i,j} m_{ik} z_k,\; q_j + \sum_{k\neq i,j} m_{jk} z_k\right)$;
(iv) each type-II solution is associated with an index $i \in \mathbf{N}$ such that

$$\begin{aligned} (q + Mz)_j &= 0 < z_j, \quad \textit{for all } j \neq i \\ (q + Mz)_i &\geq 0 = z_i; \end{aligned}$$

(v) a type-III solution z must satisfy $q + Mz = 0 < z$; any such solution can be characterized by a pair of indices $i \neq j$ such that

$$\begin{aligned} z_{-ij} &= (M^{(-ij)})^{-1}[q_{-ij} + M_{-ij,i} z_i + M_{-ij,j} z_j] > 0, \quad z_i > 0, \quad z_j > 0 \\ 0 &= q_\ell + M_{\ell,-ij} z_{-ij}, \quad \textit{for } \ell = i, j. \end{aligned}$$

Proof For any index $i \in \mathbf{N}$, the principal submatrix $M^{(-i)} \in \mathbb{R}^{(n-1)\times(n-1)}$ satisfies the conditions of Theorem 13. Thus, the LCP $(q_{-i}, M^{(-i)})$ is resolvable in strongly polynomial time. If this LCP of order $n-1$ has no solution, then any solution z of the LCP (q, M), if one exists, must have $z_i > 0$. If the LCP $(q_{-i}, M^{(-i)})$ has a type-I solution, then after augmenting this solution by zero in the ith component, the augmented vector can easily be tested as a candidate solution of the LCP (q, M). Assume that no type-I solution of the LCP $(q_{-i}, M^{(-i)})$ yields a solution of the LCP (q, M). Then all solutions of the LCP $(q_{-i}, M^{(-i)})$, if they exist, must be of type-II. Such vectors are solutions of the system of linear inequalities:

$$q_{-i} + M^{(-i)} z_{-i} = 0, \quad z_{-i} \geq 0, \tag{16}$$

which can be reduced to an equivalent system in just one variable by expressing z_{-i} in terms of a single component; cf. (13). To test if a solution of (13) yields a solution of the LCP (q, M), we consider the augmented system:

$$\begin{aligned} q_i + M_{-i,\bullet} z = 0, \quad z_{-i} \geq 0 \\ q_i + M_{i\bullet} z \geq 0, \quad z_i = 0 \end{aligned} \tag{17}$$

If the latter system has no solution, then no type-II solution of the LCP $(q_{-i}, M^{(-i)})$ yields a solution of the LCP (q, M). Incidentally, by using the **P**-property of the principal submatrices $M^{(-ij)}$, the system (17) can be reduced to an equivalent system in one single variable. Therefore, after processing the LCP $(q_{-i}, M^{(-i)})$ by the strongly polynomial procedure employed in the proof of Theorem 13, we deduce that either a solution to the LCP (q, M) is obtained readily, or every solution z of the LCP (q, M), if it exists, must be positive. To decide on the latter case, we consider the system of linear inequalities $q + Mz = 0 \leq z$ which, by using the **P**-property of the principal submatrices $M_{(-ij)}$ for any pair of indices $i \neq j$, can be reduced to an equivalent system in the two variables z_i and z_j. The feasibility of the reduced system can be easily checked in strongly polynomial time.

Further examples

An example of a large class of matrices satisfying the assumptions of Theorem 13 can be seen in the following instance. Suppose that a vector $h \in \mathbb{R}^n_{++}$ exists such that for every triple of pairwise distinct indices $\{i, j, k\} \subset \mathbf{N}$,

$$m_{ii} h_i > \sum_{\ell \neq i,j,k} |m_{i\ell}| h_\ell. \tag{18}$$

Such a matrix extends the concept of a "quasi-strictly diagonally dominant" matrix that stipulates:

$$m_{ii} h_i > \sum_{\ell \neq i} |m_{i\ell}| h_\ell, \quad \text{for all } i \in \mathbf{N}.$$

It follows that if (18) holds for all such triples, $\{i, j, k\}$, then for all vectors $q \in \mathbb{R}^n$, the LCP (q, M) is strongly polynomially resolvable.

In principle, we could continue extending Theorem 13 to the case of a few more omitted indices, i.e, to the class, $\mathbf{P}_{n-s}$, of matrices that are **P** of order $n - s$ for a fixed positive integer $s \in \mathbf{N}$ and for which positive vectors exist satisfying the analogs of (14). Since the argument would be the analogous, we shall not pursue this line any further. Instead, we present below a further class of matrices extending condition (18) to a fixed number of excluded indices. Specifically, let $s > 0$ be a given integer. For each row i, let the off-diagonal elements of row i, i.e., the $n - 1$ entries $\{m_{ij}\}_{j \neq i}$, be arranged in non-increasing order as follows:

$$|m_{i[1]}| \geq |m_{i[2]}| \geq \cdots |m_{i[n-s]}| \geq |m_{i[n-s+1]}| \geq \cdots \geq |m_{i[n-2]}| \geq |m_{i[n-1]}|$$

where each $[\ell]$ corresponds to an index $j \neq i$. Using such bracketed indices, let

$$\mathcal{R}_{i;s} = \{[1], [2], \ldots, [n-s]\} \subset \mathbf{N} \backslash \{i\}.$$

Define an $n \times n$ matrix $\widehat{M}^s$ as follows:

$$(\widehat{M}^s)_{ij} = \begin{cases} m_{ii} & \text{if } i = j \\ -|m_{ij}| & \text{if } i \neq j \text{ and } j \in \mathcal{R}_{i;s} \\ 0 & \text{if } i \neq j \text{ and } j \notin \mathcal{R}_{i;s}. \end{cases}$$

This matrix is obtained from the *comparison matrix* of M [7, **3.3.12]** by "zeroing out" in each row i those entries not in the columns indexed by the elements of $\mathcal{R}_{i;s}$. It follows that $\widehat{M}^s$ is a **Z**-matrix with at least $s-1$ zero off-diagonal elements. It can be shown that if $\widehat{M}^s$ is a **P**-matrix, then for any vector $q \in \mathbb{R}^n$, the LCP (q, M) is strongly polynomially resolvable in n with s fixed.

Acknowledgments The authors are grateful to two referees who have made some constructive comments that have helped to improve the presentation of the paper.

References

1. Chandrasekaran, R.: A special case of the complementary pivot problem. Opsearch **7**, 263–268 (1970)
2. Chandrasekaran, R., Kabadi, S.N.: Strongly polynomial algorithm for a class of combinatorial LCPs. Op. Res. Lett. **6**, 91–92 (1987)
3. Chu, T.H.: A class of polynomially solvable linear complementarity problems. Math. Progr. **107**, 461–470 (2006)
4. Chung, S.J.: NP-completeness of the linear complementarity problem. J. Optim. Theory Appl. **60**, 393–399 (1989)
5. Cottle, R.W.: Observations on a class of nasty linear complementarity problems. Discret. Appl. Math. **2**, 89–111 (1980)
6. Cottle, R.W., Dantzig, G.B.: Complementary pivot theory of mathematical programming. Linear Algebra Appl. **1**, 103–125 (1968)
7. Cottle, R.W., Pang, J.-S., Stone, R.E.: The Linear Complementarity Problem. Academic Press, Boston, 1992. [See the revised edition in the Classics in Applied Mathematics series, Philadelphia: SIAM, 2009.]
8. Cottle, R.W., Stone, R.E.: On the uniqueness of solutions to linear complementarity problems. Math. Progr. **27**, 191–213 (1983)
9. Eaves, B.C.: The linear complementarity problem. Manag. Sci. **17**, 612–634 (1971)
10. Fathi, Y.: Computational complexity of LCPs associated with positive definite symmetric matrices. Math. Progr. **17**, 335–344 (1979)
11. Gana, A.: Studies in the complementarity problem. Ph.D. thesis, Department of Industrial and Operations Engineering, University of Michigan, Ann Arbor
12. Kelly, L.M., Murty, K.G., Watson, L.T.: CP rays in simplicial cones. Math. Progr. **48**, 387–414 (1990)
13. Klee, V., Minty, G.J.: How good is the simplex algorithm? In: Shisha, O. (ed.) Inequalities—III, pp. 159–175. Academic Press, New York (1972)
14. Lemke, C.E.: Bimatrix equilibrium points and mathematical programming. Manag. Sci. **11**, 681–689 (1965)
15. Morris Jr., W.D., Lawrence, J.: Geometric properties of Hidden Minkowski matrices. SIAM J. Matrix Anal. Appl. **10**, 229–232 (1989)
16. Murty, K.G.: Computational complexity of complementary pivot methods. Math. Progr. Study **7**, 61–73 (1978)
17. Pang, J.S., Chandrasekaran, R.: Linear complementarity problems solvable by a polynomially bounded pivoting algorithm. Math. Progr. Study **25**, 13–27 (1985)

18. Saigal, R.: A note on a special linear complementarity problem. Opsearch **7**, 175–183 (1970)
19. Schiro, D.A., Pang, J.S., Shanbhag, U.V.: On the solution of affine generalized Nash equilibrium problems with shared constraints by Lemke's method. Math. Progr., Ser. A **146**, 1–46 (2013)