


In mathematical [optimization theory](#), the **linear complementarity problem (LCP)** arises frequently in [computational mechanics](#) and encompasses the well-known [quadratic programming](#) as a special case. It was proposed by Cottle and [Dantzig](#)  1968.^{[1][2][3]}

Formulation

Given a real matrix M and vector q , the linear complementarity problem $\text{LCP}(M, q)$ seeks vectors z and w which satisfy the following constraints:

- $w, z \geq 0$, (that is, each component of these two vectors is non-negative)
- $z^T w = 0$ or equivalently $\sum_i w_i z_i = 0$. This is the [complementarity](#) condition, since it implies that, for all i , at most one of w_i and z_i can be positive.
- $w = Mz + q$

A sufficient condition for existence and uniqueness of a solution to this problem is that M be [symmetric positive-definite](#). If M is such that $\text{LCP}(M, q)$ have a solution for every q , then M is a [Q-matrix](#). If M is such that $\text{LCP}(M, q)$ have a unique solution for every q , then M is a [P-matrix](#). Both of these characterizations are sufficient and necessary.^[4]

The vector w is a [slack variable](#),^[5] and so is generally discarded after z is found. As such, the problem can also be formulated as:

- $Mz + q \geq 0$
- $z \geq 0$
- $z^T (Mz + q) = 0$ (the complementarity condition)

Convex quadratic-minimization: Minimum conditions

Finding a solution to the linear complementarity problem is associated with minimizing the quadratic function

$$f(z) = z^T (Mz + q)$$

subject to the constraints

$$\begin{aligned} Mz + q &\geq 0 \\ z &\geq 0 \end{aligned}$$

These constraints ensure that f is always non-negative. The minimum of f is 0 at z if and only if z solves the linear complementarity problem.

If M is **positive definite**, any algorithm for solving (strictly) convex QPs can solve the LCP. Specially designed basis-exchange pivoting algorithms, such as **Lemke's algorithm** and **a variant of the simplex algorithm of Dantzig** have been used for decades. Besides having polynomial time complexity, **interior-point methods** are also effective in practice.

Also, a quadratic-programming problem stated as minimize $f(x) = c^T x + \frac{1}{2} x^T Q x$ subject to $Ax \geq b$ as well as $x \geq 0$ with Q symmetric

is the same as solving the LCP with

$$q = \begin{bmatrix} c \\ -b \end{bmatrix}, \quad M = \begin{bmatrix} Q & -A^T \\ A & 0 \end{bmatrix}$$

This is because the **Karush–Kuhn–Tucker** conditions of the QP problem can be written as:

$$\begin{cases} v = Qx - A^T \lambda + c \\ s = Ax - b \\ x, \lambda, v, s \geq 0 \\ x^T v + \lambda^T s = 0 \end{cases}$$

with v the Lagrange multipliers on the non-negativity constraints, λ the multipliers on the inequality constraints, and s the slack variables for the inequality constraints. The fourth condition derives from the complementarity of each group of variables (x, s) with its set of KKT vectors (optimal Lagrange multipliers) being (v, λ) . In that case,

$$z = \begin{bmatrix} x \\ \lambda \end{bmatrix}, \quad w = \begin{bmatrix} v \\ s \end{bmatrix}$$

If the non-negativity constraint on the x is relaxed, the dimensionality of the LCP problem can be reduced to the number of the inequalities, as long as Q is non-singular (which is guaranteed if it is **positive definite**). The multipliers v are no longer present, and the first KKT conditions can be rewritten as:

$$Qx = A^T \lambda - c$$

or:

$$x = Q^{-1}(A^T \lambda - c)$$

pre-multiplying the two sides by A and subtracting b we obtain:

$$Ax - b = AQ^{-1}(A^T \lambda - c) - b$$

The left side, due to the second KKT condition, is s . Substituting and reordering:

$$s = (AQ^{-1}A^T)\lambda + (-AQ^{-1}c - b)$$

Calling now

$$\begin{aligned} M &:= (AQ^{-1}A^T) \\ Q &:= (-AQ^{-1}c - b) \end{aligned}$$

we have an LCP, due to the relation of complementarity between the slack variables s and their Lagrange multipliers λ . Once we solve it, we may obtain the value of x from λ through the first KKT condition.

Finally, it is also possible to handle additional equality constraints:

$$A_{eq}x = b_{eq}$$

This introduces a vector of Lagrange multipliers μ , with the same dimension as b_{eq} .

It is easy to verify that the M and Q for the LCP system $s = M\lambda + Q$ are now expressed as:

$$\begin{aligned} M &:= [A \quad 0] \begin{bmatrix} Q & A_{eq}^T \\ -A_{eq} & 0 \end{bmatrix}^{-1} \begin{bmatrix} A^T \\ 0 \end{bmatrix} \\ Q &:= -[A \quad 0] \begin{bmatrix} Q & A_{eq}^T \\ -A_{eq} & 0 \end{bmatrix}^{-1} \begin{bmatrix} c \\ b_{eq} \end{bmatrix} - b \end{aligned}$$

From λ we can now recover the values of both x and the Lagrange multiplier of equalities μ :

$$\begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} Q & A_{eq}^T \\ -A_{eq} & 0 \end{bmatrix}^{-1} \begin{bmatrix} A^T \lambda - c \\ -b_{eq} \end{bmatrix}$$

In fact, most QP solvers work on the LCP formulation, including the [interior point method](#), principal / complementarity pivoting, and [active set](#) methods.^{[1][2]} LCP problems can be solved also by the [criss-cross algorithm](#),^{[6][7][8][9]} conversely, for linear complementarity problems, the criss-cross algorithm terminates finitely only if the matrix is a sufficient matrix.^{[8][9]} A [sufficient matrix](#) is a generalization both of a [positive-definite matrix](#) and of a [P-matrix](#), whose [principal minors](#) are each positive.^{[8][9][10]} Such LCPs can be solved when they are formulated abstractly using [oriented-matroid theory](#).^{[11][12][13]}

See also

- [Complementarity theory](#)
- [Physics engine](#) Impulse/constraint type physics engines for games use this approach.
- [Contact dynamics](#) Contact dynamics with the nonsmooth approach.
- [Bimatrix games](#) can be reduced to a LCP.

Notes

1. [Murty \(1988\)](#)
2. [Cottle, Pang & Stone \(1992\)](#)
3. R. W. Cottle and [G. B. Dantzig](#). Complementary pivot theory of mathematical programming. *Linear Algebra and its Applications*, 1:103-125, 1968.
4. Murty, Katta G. (January 1972). "On the number of solutions to the complementarity problem and spanning properties of complementary cones". *Linear Algebra and its Applications*. **5** (1): 65–108. doi:10.1016/0024-3795(72)90019-5 .
5. Taylor, Joshua Adam (2015), *Convex Optimization of Power Systems* , Cambridge University Press, p. 172, ISBN 9781107076877 .
6. [Fukuda & Namiki \(1994\)](#)
7. [Fukuda & Terlaky \(1997\)](#)
8. den Hertog, D.; Roos, C.; Terlaky, T. (1 July 1993). "The linear complementarity problem, sufficient matrices, and the criss-cross method" (pdf). *Linear Algebra and its Applications*. **187**: 1–14. doi:10.1016/0024-3795(93)90124-7 .
9. Csizmadia, Zsolt; Illés, Tibor (2006). "New criss-cross type algorithms for linear complementarity problems with sufficient matrices" (pdf). *Optimization Methods and Software*. **21** (2): 247–266. doi:10.1080/10556780500095009 .
10. [Cottle, R. W.](#); Pang, J.-S.; Venkateswaran, V. (March–April 1989). "Sufficient matrices and the linear complementarity problem" . *Linear Algebra and its Applications*. 114–115: 231–249. doi:10.1016/0024-3795(89)90463-1 . MR 0986877 .
11. [Todd \(1985\)](#)
12. [Terlaky & Zhang \(1993\)](#): Terlaky, Tamás; Zhang, Shu Zhong (1993). "Pivot rules for linear programming: A Survey on recent theoretical developments". *Annals of Operations Research*. Degeneracy in optimization problems. Springer Netherlands. 46–47 (1): 203–233. CiteSeerX 10.1.1.36.7658 . doi:10.1007/BF02096264 . ISSN 0254-5330 . MR 1260019 .
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Further reading

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External links

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- [LCPSolve](#) — A simple procedure in GAUSS to solve a linear complementarity problem
 - [Siconos](#)/Numerics open-source GPL implementation in C of Lemke's algorithm and other methods to solve LCPs and MLCPs

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