Chapter 3.1 Lagrange Interpolation

Suppose that we are given three points (x_0, y_0), (x_1, y_1), and (x_2, y_2)

Then, the polynomial:

$$P_2(x) = \underbrace{\boxed{\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}}}_{(x_0,y_0)} y_0 + \underbrace{\boxed{\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}}}_{(x_1,y_1)} y_1 + \underbrace{\boxed{\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}}}_{(x_2,y_2)} y_2$$

* Lagrange Interpolation for 3-points

Given $(x_0, x_1, x_2, \dots, x_n)$, define the "Cardinal" fun$^{\underline{ns}}$ $\ell_0, \ell_1, \dots, \ell_n$ such that

$$\ell_i(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)(x_i-x_1)\dots(x_i-x_1)(x_i-x_{i+1})\dots(x_i-x_n)}$$

$\ell_i(x_i) = 1$

$\ell_i(x_k) = 0$ for $i \neq K$

Lagrange Interpolation Polynomial

$$P_n(x) = \sum_{i=0}^{n} \ell_i(x)\, y_i$$

Example: Find the Lagrange Polynomial for:

x_i	0	2/3	1
y_i	1	1/2	0

$x_0 = 0$
$x_1 = 2/3$
$x_2 = 1$

$$P_2(x) = \frac{(x-2/3)(x-1)}{(0-2/3)(0-1)}\underbrace{[1]}_{y_0} + \frac{(x-0)(x-1)}{(2/3-0)(2/3-1)}\underbrace{[1/2]}_{y_1} + \frac{(x-0)(x-2/3)}{(1-0)(1-2/3)}\underbrace{[0]}_{y_2}$$

$$\boxed{P_2(x) = \left(\frac{-3}{4}\right)x^2 + \left(\frac{-1}{4}\right)x + (1)}$$

Example: Find the Lagrange Polynomial for

x_i	0	1	3
y_i	1/2	0	1

$x_0 = 0$
$x_1 = 1$
$x_2 = 3$

$$P_2(x) = \frac{(x-1)(x-3)}{(0-1)(0-3)}[1/2] + \frac{(x-0)(x-3)}{(1-0)(1-3)}[0] + \frac{(x-0)(x-1)}{(3-0)(3-1)}[1]$$

$$= \frac{(x-1)(x-3)}{3}[1/2] + 0 + \frac{(x-0)(x-1)}{6}$$

$$=$$

Main Theorem

Let $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ be $(n+1)$ points in the xy-plane with distinct x_i.

Then, there exists "one and only one" polynomial $p_n(x)$ of

Let $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ be $(n+1)$ points in the xy-plane with distinct x_i.

Then, there exists "<u>one and only one</u>" polynomial $p_n(x)$ of "<u>degree n or less</u>" that satisfies $P_n(x_i) = y_i$ for $i = 0, 1, 2, \dots, n$

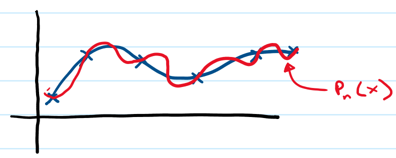
Maximum Error

Suppose $x_0, x_1, \dots, x_n$ are distinct numbers in the interval $[a,b]$ and $f \in C^{n+1}[a,b]$ (in)

Then, for each x in $[a,b]$ a number (zeta) $\xi(x)$ (unknown) in (a,b) exists with

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n),$$

Where $P_n(x)$ is the Lagrange interpolating polynomial



$$\left| f(x) - P_n(x) \right| = \left| \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n) \right|$$