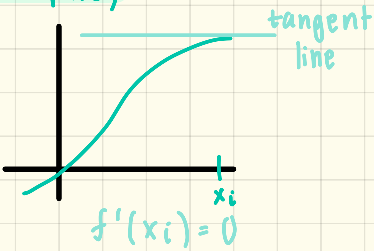
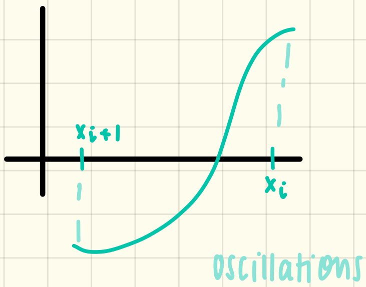
Newton's Method

02.04.19

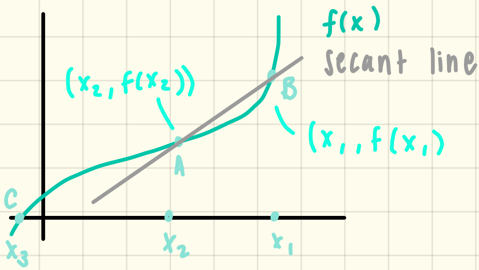
* Note: when an iterate is not sufficiently close to a solution, various types of "bad" behavior can occur with Newton's Method

examples)





2.3.2 Secant Method



Main idea:

* Avoid computing $f'(x)$
* we need two initial points

Slope of the secant line:

$$m = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

Slope of the line segment AC:

$$\text{slope} = \frac{f(x_2) - 0}{x_2 - x_3}$$

Then...

$$\frac{f(x_2)}{x_2 - x_3} = \frac{f(x_1) - f(x_2)}{x_1 - x_2} \Rightarrow \frac{f(x_2)}{\left[\frac{f(x_1) - f(x_2)}{x_1 - x_2}\right]} = x_2 - x_3$$

$$\Rightarrow x_3 = x_2 - f(x_2)\left[\frac{x_1 - x_2}{f(x_1) - f(x_2)}\right]$$

* In general

$$x_{n+1} = x_n - f(x_n)\left[\frac{x_{n-1} - x_n}{f(x_{n-1}) - f(x_n)}\right]$$

example) Apply the secant method with starting guesses $x_0 = 0$ $x_1 = 1$ to find the root of $f(x) = x^3 + x - 1$. Find x_2 and x_3.

$$x_2 = x_1 - f(x_1)\left[\frac{x_0 - x_1}{f(0) - f(1)}\right]$$

$$= 1 - f(1)\left[\frac{0 - 1}{f(0) - f(1)}\right]$$

$$= 1 - 1\left[\frac{0-1}{-1-1}\right] = 1 - 1\left[\frac{-1}{-2}\right] = 1 - \frac{1}{2} = \frac{1}{2}$$

$$x_2 = 1/2$$

$$x_3 = x_2 - f(x_2)\left[\frac{x_1 - x_2}{f(x_1) - f(x_2)}\right]$$

$$= 0.5 - f(0.5)\left[\frac{1 - 0.5}{f(1) - f(0.5)}\right]$$

$$= 0.5 + 0.375\left[\frac{1 - 0.5}{1 + 0.375}\right]$$

$$= 0.5 + 0.375\left[\frac{0.5}{1.375}\right]$$

$$= 0.636$$

QUIZ 03 FRIDAY 02/08 — Fixed Point theorem + iteration

Fixed Point Review:

given $x = g(x)$ $I = [a,b]$

existence: $g(x) \in [a,b]$

uniqueness: $\max |g'(x)| < 1$

Review: Taylor's Theorem with Remainders

Theorem

Let x and x_0 be real numbers, and let $f(x)$ be $(k+1)$ times continuously differentiable on the interval between x and x_0. Then there is a number c between x and x_0 such that

$$f(x) = \underbrace{\left[f(x_0) + \frac{f'(x_0)(x-x_0)}{1!} + \frac{f''(x_0)(x-x_0)^2}{2!} + \dots + \frac{f^{(k)}(x_0)(x-x_0)^k}{k!} \right]}_{\text{degree } k \text{ taylor polynomial for } f(x) \text{ centered at } x_0} + \underbrace{\frac{f^{(k+1)}(c)(x-x_0)^{k+1}}{(k+1)!}}_{\text{Taylor's remainder}}$$

example) Find the degree 4 polynomial, $P_4(x)$, for $f(x) = \sin x$ centered at the point $x_0 = 0$

$f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^4(x) = \sin x$

$$= \sin(0) + \frac{1 \cdot (x)}{1!} + 0 + \frac{(-1)x^3}{3!} + 0$$

$$\sin(x) \approx \frac{x}{1!} - \frac{x^3}{3!}$$