I.1. Heat Equation

$Au_{xx} + Bu_{xy} + C\,u_{yy} + F(u_x, u_y, u, x, y) = 0$

- Parabolic if $B^2 - 4AC = 0$
- Elliptic if $B^2 - 4AC < 0$

The one dimensional heat equation is

$$\frac{\partial u}{\partial t}(x,t) = \alpha \frac{\partial^2 u}{\partial x^2}(x,t) \qquad 0 \leq x \leq L. \quad t \geq 0.$$

- Two independent variables

One Initial Condition:

$$u(x,0) = f(x) \qquad 0 \leq x \leq L.$$

Two boundary condition:

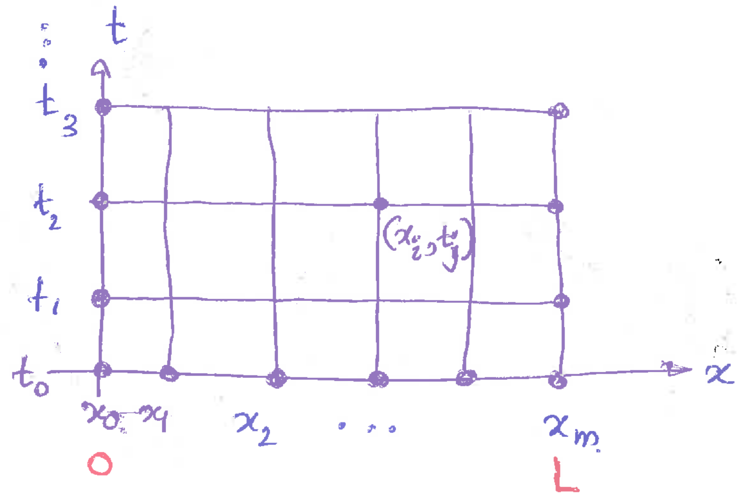
$$u(0,t) = 0$$

$$u(L,t) = 0 \qquad \text{for } t \geq 0.$$

First we make a grid of the domain of (x,t) with step size h and k, respectively.

$$x_i = i\,h. \qquad i = 0, 1, \ldots, m \qquad h = \frac{(L-0)}{m}.$$

$$t_j = j\,k. \qquad j = 0, 1, \ldots$$



Filled circles represent known initial and boundary condition

Difference Approximations for derivative terms:

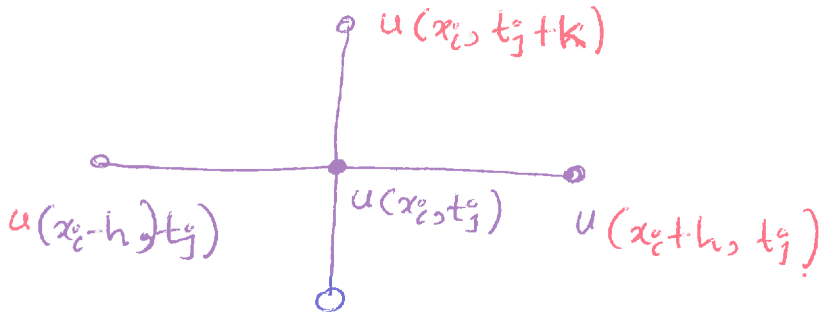
(*) $$\frac{\partial u}{\partial t}(x_i, t_j) \approx \frac{u(x_i, t_j + k) - u(x_i, t_j)}{k}.$$ $\Big\}$ Forward difference in time ~~approach~~

$$\frac{u_{i,j+1} - u_{i,j}}{k} \qquad O(k^1)$$

(*) Centred-difference formula for the second derivative

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) \approx \frac{1}{h^2}\left[u(x_i + h, t_j) - 2u(x_i, t_j) + u(x_i - h, t_j)\right]$$

$$\frac{1}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right] \qquad O(h^2)$$



Substituting into the heat eq^n at the point (x_i, t_j), we get

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}.$$

$$\frac{\left[u_{i,j+1} - u_{i,j}\right]}{k} = \frac{\alpha}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right]$$

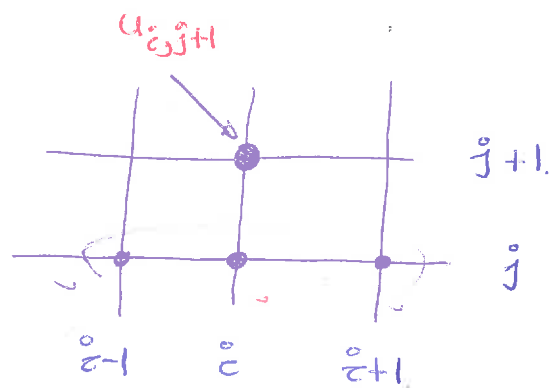
$$u_{i,j+1} = \frac{k\alpha^2}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right] + u_{i,j}.$$

let $\lambda = \frac{k\alpha^2}{h^2}$

$$\boxed{u_{i,j+1} = \lambda u_{i+1,j} + \lambda u_{i-1,j} + (1-2\lambda)u_{i,j}.}$$

$i = 1, \cdots, m-1$

$j = 0, 1, 2, \ldots$



By the IC $u(x,0) = f(x)$, we have

$$\left.\begin{aligned} u_{0,0} &= f(x_0) \\ u_{1,0} &= f(x_1) \\ &\vdots \\ u_{m,0} &= f(x_m) \end{aligned}\right\}$$

we can find $u_{0,1}, u_{1,1}, u_{2,0}, \cdots u_{m,1}$ and so on at different values of t_j.

$$u_{1,j+1} = (1-2\lambda)\, u_{1,j} + \lambda\, u_{2,j} + \lambda\, \underbrace{u_{0,j}}_{\text{zero}}$$

$$u_{2,j+1} = (1-2\lambda)\, u_{2,j} + \lambda\, u_{3,j} + \lambda\, u_{1,j}$$

$$\vdots$$

$$u_{m-1,j} = (1-2\lambda)\, u_{m-1,j} + \lambda\, \underbrace{u_{m,j}}_{\text{zero}} + \lambda\, u_{m-2,j}$$

BCs

Nok

$$\underset{(m-1,1)}{\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \vdots \\ u_{m-1,j+1} \end{bmatrix}} = \underset{(m-1,m-1)}{\begin{bmatrix} (1-2\lambda) & \lambda & 0 & \cdots & 0 \\ \lambda & (1-2\lambda) & \lambda & 0 \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ 0 & & & & 0 \\ & & & & \lambda \\ & & 0 & \lambda & (1-2\lambda) \end{bmatrix}} \underset{m-1,1}{\begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{m-1,j} \end{bmatrix}}$$

let $\vec{u}_{(j)} = [u_{1,j}, u_{2,j}, \ldots, u_{m-1,j}]^T$, then

$$\vec{u}_{(j+1)} = A\, \vec{u}_{(j)} \qquad j = 0, 1, 2, \ldots$$

where $\vec{u}_0 = [f(x_0), f(x_1), \ldots, f(x_{m-1})]^T$

Note

- The order of error is $O(k) + O(h^2)$.

- Difference method is conditionally stable if $\rho(A) < 1$, which implies $\lambda = \frac{k\alpha^2}{h^2} \le \frac{1}{2}$.

ex. Approximate the solu^n of the following heat eq^n at $t = 0.4$ by the difference m/d using $h = 0.25$ and $k = 0.2$

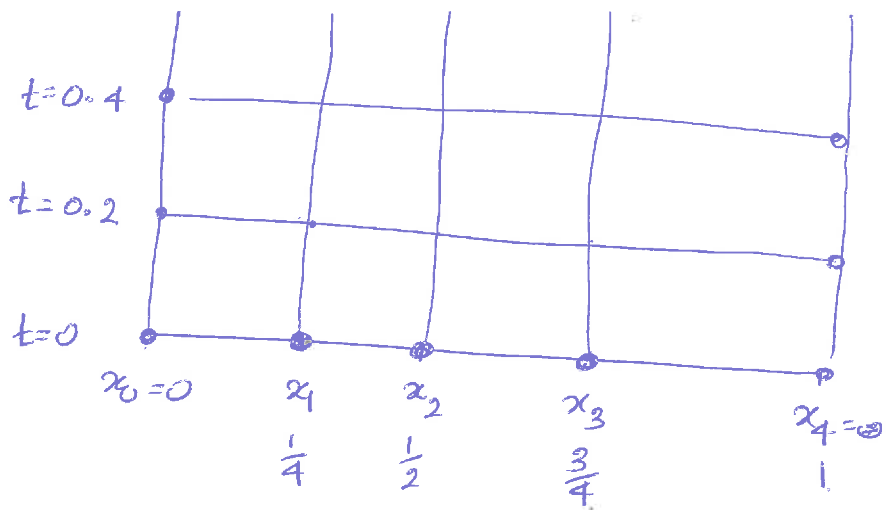
$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{16}\frac{\partial^2 u}{\partial x^2}(x,t) \qquad 0 \le x \le 1, \quad t \ge 0$$

$$u(0,t) = u(1,t) = 0 \quad t > 0 \quad \text{and} \quad u(x,0) = \sin(2\pi x) \quad 0 \le x \le 1.$$

Compare the result at $t = 0.4$ using the exact solu^n

$$u(x,t) = e^{-\pi^2 \frac{t}{4}} \sin(2\pi x)$$

$\lambda = \frac{k\alpha^2}{h^2} = \frac{0.2}{(0.25)^2}\frac{1}{16} = 0.2 = \frac{1}{5}.$ $\qquad 1-2\lambda = 0.6$



$m = \frac{\ell - 0}{h} = \frac{1-0}{0.25}$

$m = 4$

$\uparrow$

4-intervals

$u_{0,0} = 0$

$u_{1,0} = \sin\left(2\pi \cdot \frac{1}{4}\right) = 1$

$u_{2,0} = \sin\left(2\pi \cdot \frac{1}{2}\right) = 0$

$u_{3,0} = \sin\left(2\pi \cdot \frac{3}{4}\right) = -1$

$u_{4,0} = \sin\left(2\pi \cdot 1\right) = 0$

$$\vec{u}_0 = \begin{bmatrix} u_{1,0} \\ u_{2,0} \\ u_{3,0} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{u}_{(j+1)} = A\,\vec{u}_{(j)}$$

$$\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.6 & 0.2 \\ 0 & 0.2 & 0.6 \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{2,j} \end{bmatrix}$$

$$\begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.6 & 0.2 \\ 0 & 0.2 & 0.6 \end{bmatrix} \begin{bmatrix} u_{1,0} & 1 \\ u_{2,0} & 0 \\ u_{3,0} & -1 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.0 \\ -0.6 \end{bmatrix}$$

$$\begin{bmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.6 & 0.2 \\ 0 & 0.2 & 0.6 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.0 \\ -0.6 \end{bmatrix} = \begin{bmatrix} 0.36 \\ 0 \\ -0.36 \end{bmatrix}$$

i	x_i	$u_{i,0}$	$u_{i,1}$	$u_{i,2}$	True $u(x_i, 0.4)$
0	0.00	0	0.00	0.00	0.00
1	0.25	1	0.60	0.36	0.3727
2	0.50	0	0.00	0.00	0.00
3	0.75	-1	-0.60	-0.36	-0.3727
4	1.00	0	0.00	0.00	0.00