5.1. Differential Equations

Elementary theory of Initial-value Problems

Def^n. A fun^n $f(t,y)$ is said to satisfy a Lipschitz Condition in the variable y on a set $D \subset \mathbb{R}^2$ if

$$|f(t,y_1) - f(t,y_2)| \leq L\,|y_1 - y_2|$$

for all (t,y_1) and (t,y_2) in D with $L > 0$.

L - Lipschitz constant.

ex. Find the Lipschitz constant for

$f(t,y) = ty + t^3$ on the ~~interval~~ set $D = \{(t,y) \mid 0 \leq t \leq 1 \text{ and } -\infty < y < \infty\}$.

$$|f(t,y_1) - f(t,y_2)| = |ty_1 + t^3 - (ty_2 + t^3)|$$

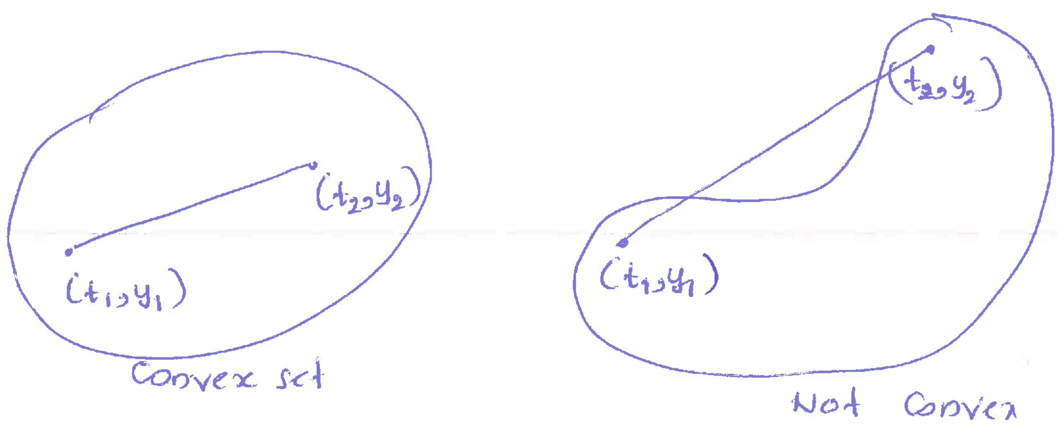
$$= |t(y_1 - y_2)|$$

$$\leq |t|\,|y_1 - y_2|$$

$$\leq 1 \cdot |y_1 - y_2|$$

Defⁿ Convex set.

A set $D \subset \mathbb{R}^2$ is said to be convex, if whenever (t_1, y_1) and (t_2, y_2) belong to D, then $((1-\lambda)t_1 + \lambda t_2, (1-\lambda)y_1 + \lambda y_2)$ also belongs to D for every λ in $[0,1]$



Thm. Suppose $f(t,y)$ is defined on a convex set $D \subset \mathbb{R}^2$. If a constant $L > 0$ exists with

$$\left| \frac{\partial f}{\partial y}(t,y) \right| \leq L \quad \forall \ (t,y) \in D,$$

then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L.

In this chapter, we numerically solve the IVP

$$\frac{dy}{dt} = f(t, y) \qquad a \leq t \leq b \qquad \text{—} \; (*)$$
$$y(a) = y_0.$$

Before, approximating a solu^n $y = y(t)$, we must ask if $(*)$ has a unique solu^n. The answer is given by the existence and uniqueness Th^m.

Th^m The IVP $(*)$ has a unique solu^n $y(t)$ on $[a, b]$, if

1. f is continuous on $D = \{ (t, y) \mid a \leq t \leq b, \; -\infty < y < \infty \}$,

and

2. f satisfies a Lipschitz condition on D with constant L:

$$| f(t, y_1) - f(t, y_2) | \leq L | y_1 - y_2 | \quad \text{for all } (t, y_1), (t, y_2) \in D$$

ex. Use the EUT to show that there is a unique solu^n to the IVP.

$$y' = 1 + t \sin(t y) \qquad 0 \leq t \leq 2, \; y(0) = 0$$

$$\frac{\partial f}{\partial y} = t^2 \cos(t y) \qquad \left| \frac{\partial f}{\partial y} \right| = t^2 |\cos(t y)| \leq 2^2 = L.$$

f satisfies a Lipschitz condition in the variable y with $L = 4$. Th^m implies a unique solu^n

5.2. Euler's Method

Consider the IVP

$$\frac{dy}{dt} = f(t,y), \qquad a \le t \le b, \quad y(a) = \alpha.$$

We begin with a grid of $(n+1)$ points

$$a = t_0 < t_1 < t_2 < \dots < t_n = b$$

along the t-axis with equal step size h.

$$t_i = a + ih \qquad \text{and} \qquad h = \frac{b-a}{n}.$$

The Euler's method finds $y_0, y_1, y_2, \dots, y_n$ such that

approximate Soln at time t_i. → $y_i \approx y(t_i)$ ← True solution at time t_i.

$$\boxed{\begin{aligned} y_0 &= \alpha. \\ y_{i+1} &= y_i + h\, f(t_i, y_i) \end{aligned}} \qquad i = 0, 1, \dots, n-1.$$

new value

old value

step size

slope!

Proof   use Taylor's thm on y about $t = t_i$

$\cancel{y(t_{i+1}) = y(t_i + h)}$

$$y(t) = y(t_i) + y'(t_i)\frac{(t-t_i)^1}{1!} + y''(\xi_i)\frac{(t-t_i)^2}{2!}$$

Now let $t = t_i + h$.

$$y(t_i + h) = y(t_i) + y'(t_i)\,h + \frac{y''(c_i)\,h^2}{2!}$$

Euler method is obtained by dropping the $O(h^2)$ term in the formula

$$y(t_i + h) \approx y(t_i) + h\,y'(t_i)$$

$$\boxed{y_{i+1} = y_i + h\,f(t_i, y_i)}$$

ex. Apply Euler's method to IVP

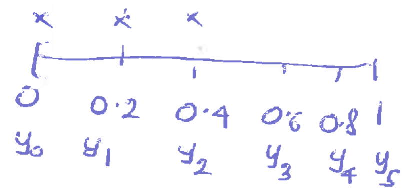
$$y' = ty + t^3 = f(t, y)$$

$$y(0) = 1$$

$$t \in [0, 1]$$

with step size $h = 0.2$

$$y_{i+1} = y_i + h\, f(t_i, y_i) \qquad i = 1:4$$



$$y_1 = y_0 + 0.2\, f(t_0, y_0)$$

$$\boxed{y_1 = 1.000}$$

$$y_2 = y_1 + h f(t_1, y_1)$$

$$y_2 = 1 + 0.2\, f(0.2, 1)$$

$$\boxed{y_2 = 1.0416}$$

$$y_3 = y_2 + h f(t_2, y_2)$$

$$y_3 = 1.0416 + 0.2\, f(0.4, 1.0416)$$

$$\boxed{y_3 = 1.1377}$$

$$\boxed{y_4 = 1.3175}$$

$$\boxed{y_5 = 1.6306}$$

ex. Use Euler's m/d with step size $h=0.5$ to approximate the solu^n of the following IVP.

$$\frac{dy}{dt} = t^2 - y \qquad 0 \le t \le 2 \qquad y(0)=1.$$

$$y_{i+1} = y_i + 0.5\,(t_i^2 - y_i)$$

$$y_{i+1} = 0.5\,(y_i + t_i^2)$$

$$y_1 = 0.5\,(y_0 + t_0^2) = 0.5$$

$$y_2 = 0.5\,(y_1 + t_1^2) = 0.5\,(0.5 + (0.5)^2) = 0.375.$$

$$y_3 = 0.6875.$$

$$y_4 = 1.46875$$

Error Bounds for Euler's method:

Let $D = \{ (t,y) \mid a \leq t \leq b, \ -\infty < y < \infty \}$ and $f(t,y)$ has a Lipschitz constant L. Suppose that $y(t)$ is the unique Solu^n to the IVP: $y' = f(t,y)$ where $|y''(t)| \leq M$ for all $t \in [a,b]$. Then for the approximation y_i of $y(t_i)$ by the Euler's m/d with step size h, we have

$$| y(t_i) - y_i | \leq \frac{hM}{2L} \left[e^{L(t_i - a)} - 1 \right] \qquad i = 0, 1, \ldots, n.$$

ex. Find the maximum error in approximating $y(1)$ by y_2 in the previous example.

$$0 \leq t \leq 2$$

$$| y(1) - y_2 | \leq \frac{0.5 \ M}{2 \ L} \left[e^{L(1-0)} - 1 \right]$$

$$f(t,y) = t^2 - y$$

$$\frac{\partial f}{\partial y} = -1 \qquad \left| \frac{\partial f}{\partial y} \right| \leq 1 = L$$

$$\frac{dy}{dt} = t^2 - y \qquad y(0) = 1$$

$$\frac{dy}{dt} + y = t^2$$

$$IF = e^{\int 1.dt} = e^t$$

$$\frac{d}{dt}\left[e^t y\right] = e^t t^2$$

$$e^t y = \int \underset{dv}{e^t}\,\underset{u}{t^2}\,dt$$

$$= t^2 e^t - \int \underset{u}{2t}\,\underset{dv}{e^t}\,dt$$

$$= t^2 e^t - \left[e^t 2t - \int 2e^t dt\right]$$

$$e^t y = t^2 e^t - e^t 2t + 2e^t + c$$

$$1 = 2 + c$$

$$-1 = c$$

$$\boxed{y = (t^2 - 2t + 2) - e^{-t}} \leftarrow \text{True solu}^n$$

$$y' = 2t - 2 + e^{-t}$$

$$y'' = 2 - e^{-t}$$

Since $y''' = e^{-t} > 0$ y'' is an increasing fun^n, and

$$\max|y''| = |2 - e^{-t}|_{max} = |2 - e^{-2}| = M = 1.8647$$

$$|y(1) - y_2| \leq \frac{hM}{2L}\left[e^{L(t_2-a)} - 1\right]$$

$$= \frac{0.5(1.8647)}{2 \cdot 1}\left[e^{1(1-0)} - 1\right]$$

$$= \underline{\underline{0.8010}}$$

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ex2. Given the IVP:

$$y' = \frac{2}{t}y + t^2 e^t \qquad 1 \leq t \leq 2 \qquad y(1) = 0$$

with exact solu^n $y(t) = t^2(e^t - e^1)$:

(a) Use Euler's m/d with $h = 0.2$ to approximate the solu^n and compare it with the actual values of y.

(b) Find the maxi^m error in approximating $y(1.6)$ by y_3.

(c) Compute the value of h necessary for $|y(t_i) - y_i| \leq 0.1$ using

$$|y(t_i) - y_i| \leq \frac{hM}{2L}\left[e^{L(t_i-a)} - 1\right]$$

(skip)