6.2. Gaussian Elimination

Key ideas

1. A linear system becomes upper triangular after elimination.
2. The upper triangular system is solved by back substitution (starting at the bottom).
3. If $a_{ii} = 0$ for some i and $a_{ji} \neq 0$ for some $j > i$, then we interchange E_i and E_j.

ex. Solve the linear system:

$$x_1 + x_2 + x_3 = 6$$
$$x_1 + 2x_2 + 2x_3 = 9$$
$$x_1 + 2x_2 + 3x_3 = 10$$

(ans. $x = 3, 2, 1$)

Note

Operation count for the elimination step of Gaussian Elimination

The elimination step for a system of n eq^{ns} in n-variables can be completed in $\frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{7}{6}n$ operations

6.3. Iterative methods

Jacobi and Gauss-Seidel methods

6.3.1. Jacobi method (JM)

This method is a form of fixed-point iteration for a system of equations.

Here, the 1st step is to solve the ith eq^n for the ith (variable) unknown. Then, iterate as in Fixed-point Iteration, starting with an initial guess.

ex. Apply the Jacobi Method to the system

$$3x_1 + x_2 = 5$$
$$x_1 + 2x_2 = 5.$$

Step 1

$$x_1 = \frac{1}{3}(5 - x_2)$$
$$x_2 = \frac{1}{2}(5 - x_1)$$

$$\begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(5 - x_2^k) \\ \frac{1}{2}(5 - x_1^k) \end{bmatrix}$$

Step 2 Initial vector $\begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 5/2 \end{bmatrix}$$

$$\begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\left(5 - \frac{5}{2}\right) \\ \frac{1}{2}\left(5 - \frac{5}{3}\right) \end{bmatrix} = \begin{bmatrix} 5/6 \\ 5/3 \end{bmatrix}$$

$$\begin{bmatrix} x_1^3 \\ x_2^3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\left(5 - \frac{5}{3}\right) \\ \frac{1}{2}\left(5 - \frac{5}{6}\right) \end{bmatrix} = \begin{bmatrix} 10/9 \\ 25/12 \end{bmatrix}$$

Further steps of Jacobi show convergence ~~to~~ towards the solu^n $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

<u>Note</u> This m/d makes two assumptions:

(i) The system given by

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1$$
$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2$$
$$\vdots$$
$$a_{n1} x_1 + a_{n2} x_2 + \ldots + a_{nn} x_n = b_n$$

has a unique solu^n.

(ii) The coefficient matrix A has no zeros on its main diagonal ex. Do ~~3~~ 2-iteration of the Jacobi m/d with $\vec{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ to approximate the solu^n of the system:

$$5x_1 - 2x_2 + 3x_3 = -1$$

$$-3x_1 + 9x_2 + x_3 = 2$$

$$2x_1 + x_2 - 7x_3 = 3.$$

$$\Rightarrow \quad x_1^{(k+1)} = \frac{1}{5}\left(-1 + 2x_2^{(k)} - 3x_3^{(k)}\right)$$

$$x_2^{(k+1)} = \frac{1}{9}\left(2 + 3x_1^{(k)} - x_3^{(k)}\right)$$

$$x_3^{(k+1)} = -\frac{1}{7}\left(3 - 2x_1^{(k)} + x_2^{(k)}\right)$$

$$\begin{bmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 2/9 \\ -3/7 \end{bmatrix}$$

$$\vec{x}^{(k+1)} = M\,\vec{x}^{(k)} + \vec{b}_2$$

$$\vec{x}^{(k+1)} = \begin{bmatrix} 0 & \frac{2}{5} & -\frac{3}{5} \\ \frac{3}{9} & 0 & -\frac{1}{9} \\ -\frac{2}{7} & \frac{1}{7} & 0 \end{bmatrix} \vec{x}^{(k)} + \begin{bmatrix} -\frac{1}{5} \\ \frac{2}{9} \\ -\frac{3}{7} \end{bmatrix}$$

$$\begin{bmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \end{bmatrix} = \begin{bmatrix} 0.146 \\ 0.203 \\ -0.517 \end{bmatrix}$$

Note: let D be the main diagonal of A.

U denotes the upper triangle of A.

L " " lower " of A.

Then $A = L + D + U$.

$$Ax = Lx + Dx + Ux$$

Since, $Ax = b$

$$Lx + Dx + Ux = b$$

$$D\vec{x} = \vec{b} - (L\vec{x} + D\vec{x})$$

$$D\vec{x} = \vec{b} - (L + D)\vec{x}$$

$$\vec{x}^{(k+1)} = D^{-1}\left[\vec{b} - (L + D)\vec{x}_k^{(k)}\right]$$

6.3.2. The Gauss-Seidel m/d

The only difference between Gauss-Seidel and Jacobi is that in the former, the most recently updated values of the unknowns are used ~~at~~ in the current step.

~~That is, new values of each x_i as~~

That is, with the GS-method, we use the new values of each x_i as soon as they are known.

ex. Once you have determined x_1 from the 1st eq^n, its value is then used in the ~~second~~ 2nd eq^n to obtain the new x_2.
Similarly, the new x_1 and x_2 are used in the 3rd eq^n to obtain the new x_3, and so on.

ex. Apply the GS-method to solve the system:

$$\begin{cases} 3x_1 + x_2 - x_3 = 4 \\ 2x_1 + 4x_2 + x_3 = 1 \\ -x_1 + 2x_2 + 5x_3 = 1. \end{cases}$$

with $\vec{x_0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$x_1^{k+1} = \frac{1}{3}\left(4 - x_2^{(k)} + x_3^{(k)}\right)$$

$$x_2^{k+1} = \frac{1}{4}\left(1 - 2x_1^{(k+1)} - x_3^{(k)}\right)$$

$$x_3^{k+1} = \frac{1}{5}\left(1 + x_1^{(k+1)} - 2x_2^{(k+1)}\right)$$

$$\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} = \begin{bmatrix} 4/3 \\ -5/12 \\ 19/30 \end{bmatrix}$$

$$\begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{bmatrix} = \begin{bmatrix} \frac{101}{60} \\ -\frac{3}{4} \\ \frac{251}{300} \end{bmatrix}$$

Algorithm

$\vec{x_0}$

$$\vec{x}_{k+1} = D^{-1}\left(b - U\vec{x}_k - L\vec{x}_{k+1}\right)$$

Convergence of the Jacobi and Gauss-Seidel Methods

Def^n. The matrix $A_{n\times n}$ is strictly diagonally dominant if, for each $1 \le i \le n$,

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$$

← Sum of the absolute values of the other entries in the row.

That is $|a_{11}| > |a_{12}| + |a_{13}| + \dots + |a_{1n}|$

$$|a_{22}| > |a_{21}| + |a_{23}| + \dots + |a_{2n}|$$

$$\vdots$$

$$|a_{nn}| > |a_{n1}| + |a_{n2}| + \dots + |a_{n(n-1)}|$$

Determine whether the matrices

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -5 & 2 \\ 1 & 6 & 8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 8 & 1 \\ 9 & 2 & -1 \end{bmatrix}$$

✓ No!

are strictly diagonally dominant.

Th^m. Sufficient condition for the convergence of either the Jacobi or Gauss-Seidel m/d^s.

If $A_{n\times n}$ matrix is strictly diagonally dominant, then (1) A^{-1} exists and (2) for every vector $\vec{b}$ and every $\vec{x}^{(0)}$, the Jacobi (or GS) method applied to $A\vec{x}=\vec{b}$ converges to ~~the~~ a unique solu^n.

ex. Interchanging Rows to obtain convergence.

$$x_1 - 5x_2 = -4$$
$$7x_1 - x_2 = 6.$$

$$A^1 = \begin{bmatrix} 1 & -5 \\ 7 & -1 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 7 & -1 \\ 1 & -5 \end{bmatrix}$$

HW Find the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$