

MONDAY JULY 29, 2019

CH. 5 DIFFERENTIAL EQUATIONS

S.3 Higher-Order Taylor Methods

NOTE: Euler's m/d was derived by dropping the $\Theta(h^2)$ term in the Taylor series expansion of $y(t)$ about $t = t_i$. One can derive methods with higher-order by retaining more terms in the Taylor series expansion.

$$y(t) = y(t_i) + \frac{y'(t_i)(t - t_i)}{1!} + \frac{y''(t_i)(t - t_i)^2}{2!} + \frac{y'''(t_i)(t - t_i)^3}{3!} + \dots \\ + \frac{y^n(t_i)(t - t_i)^n}{n!} + \frac{y^{(n+1)}(t_i)(t - t_i)^{n+1}}{(n+1)!}$$

$$t = t_i + h$$

$$y(t_i + h) = y(t_i) + \frac{y'(t_i)h}{1!} + \frac{y''(t_i)h^2}{2!} + \dots + \frac{y^{(n)}(t_i)h^n}{n!} + \frac{y^{(n+1)}(t_i)h^{n+1}}{(n+1)!} \quad \text{drop}$$

$$y(t_i + h) \approx y(t_i) + \frac{y'(t_i)h}{1!} + \frac{y''(t_i)h^2}{2!} + \dots + \frac{y^n(t_i)h^n}{n!}$$

$$\text{But } y' = f(t, y)$$

nth-order Taylor method:

$$\therefore \left[y(t_i + h) \approx y(t_i) + \frac{h}{1!} f(t_i, y_i) + \frac{h^2}{2!} f'(t_i, y_i) + \frac{h^3}{3!} f''(t_i, y_i) + \dots + \frac{h^n}{n!} f^{(n-1)}(t_i, y_i) \right]$$

$\xrightarrow{\text{Euler's m/d}}$
 $\xrightarrow{\text{2nd order Taylor's m/d}}$
 $\xrightarrow{\text{3rd order Taylor's m/d}}$

ex 1) use Taylor's m/d of order 2 with step size $h=0.5$ to approx. the soln of the IVR

$$\frac{dy}{dt} = te^{3t} - 2y \quad 0 \leq t \leq 1 \quad y(0) = 0$$

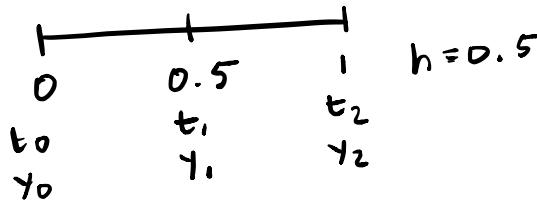
$$f(t, y) = te^{3t} - 2y$$

$$f'(t, y) = t \cdot 3e^{3t} + e^{3t} - 2y'$$

$$= 3te^{3t} + e^{3t} - 2(te^{3t} - 2y)$$

$$y_{i+1} = y_i + h f(t_i, y_i) + \frac{h^2}{2!} f'(t_i, y_i) \quad f'(t, y) = e^{3t} + te^{3t} + 4y$$

$$y_{i+1} = y_i + h [te^{3t_i} - 2y] + \frac{h^2}{2!} [e^{3t_i} + t_i e^{3t_i} + 4y_i]$$



$$(t_0, y_0) \Rightarrow (0, 0)$$

$$y_1 = 0 + (0.5) [0 \cdot e^{3 \cdot 0} - 2 \cdot 0] + \frac{(0.5)^2}{2!} [e^{3 \cdot 0} + 0 \cdot e^{3 \cdot 0} + 4 \cdot 0]$$

$$y_1 = 0.125$$

$$(t_1, y_1) \Rightarrow (0.5, 0.125)$$

$$y_2 = 0.125 + (0.5) (0.5 \cdot e^{3 \cdot 0.5} - 2 \cdot 0.125) + \frac{(0.5)^2}{2!} [e^{3 \cdot 0.5} + 0.5 \cdot e^{3 \cdot 0.5} + 4 \cdot 0.125]$$

$$y_2 = 2.02324$$

ex 2) Using 3rd-order Taylor's m/d w/ $h=0.5$. "set up" an iteration formula for $\{y_i\}$ to approx. the solution of the IVR

$$\frac{dy}{dt} = t^2 - y \quad 0 \leq t \leq 2 \quad y(0) = 1$$

$$f(t, y) = t^2 - y$$

$$f'(t, y) = 2t - (t^2 - y)$$

$$f''(t, y) = 2 - 2t + y'$$

$$= 2 - 2t + (t^2 - y)$$

$$y_{i+1} = y_i + 0.5[t_i^2 - y_i] + \frac{0.5^2}{2!}[2t_i - t_i^2 + y_i] + \frac{0.5^3}{3!}[2 - 2t_i + t_i^2 - y_i]$$

S.4 Runge - kutta m/ds (RK-m/d²)

In the Euler's m/d, we use the information on the slope of y at a given time-step to predict the soluⁿ to the next time step.

- RK-m/d² are a class of techniques that use the information on the "slope" at "more than one point" to predict the solution to the future time step (y_{i+1})

- Recall IVP:

$$\frac{dy}{dt} = f(t, y) \quad a \leq t \leq b \\ y(a) = y_0$$

The general form of the approximate soluⁿ:

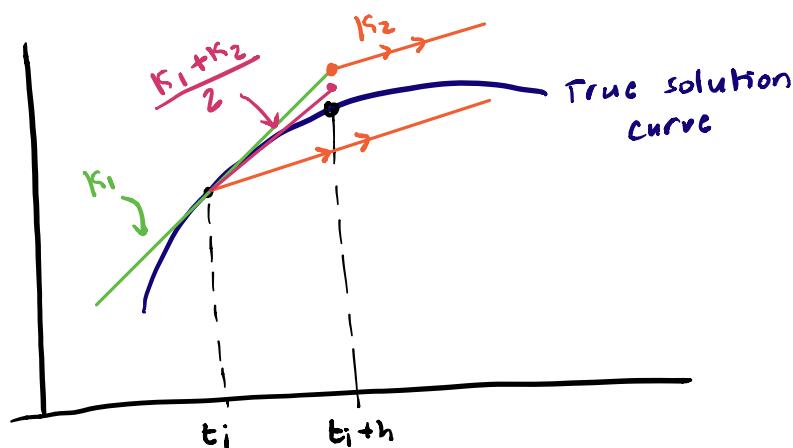
$$y_{i+1} = y_i + h \underbrace{\psi(t_i, y_i)}_{\text{Increment fun}^n}$$

* $\psi(t_i, y_i)$ is essentially a suitable slope over the interval $[t_i, t_{i+1}]$

• second-order RK-m/d² (RK-2)

(1) Modified-Euler method

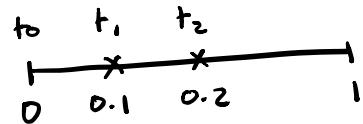
$$y_{i+1} = y_i + h \underbrace{\left[\frac{k_1 + k_2}{2} \right]}_{\text{average of 2 slopes}} \quad \text{where } k_1 = f(t_i, y_i) \\ k_2 = f(t_i + h, y_i + k_1 h) \\ = f(t_i + h, y_i + h \underbrace{f(t_i, y_i)}_{k_1})$$



ex 1) Consider the IVP:

$$\frac{dy}{dt} = 2t^2 + t^2 y \\ = t^2(2+y)$$

$$0 \leq t \leq 1 \\ y(0) = 1 \\ h = 0.1$$



Compute the estimated value of y_1 using the Modified-Euler m/d.

$$t_0 = 0, y_0 = 1$$

$$y_{i+1} = y_i + h \left[\frac{k_1 + k_2}{2} \right]$$

$$y_1 = y_0 + 0.1 \left[\frac{0 + 0.03}{2} \right]$$

$$= 1 + 0.1 \left(\frac{0.03}{2} \right) = 1.0015$$

$$y_2 = y_1 + h \left[\frac{k_1 + k_2}{2} \right]$$

$$= 1.0015 + 0.1 \left[\frac{0.030015 + 0.12018}{2} \right] K_2$$

$$= 1.009$$

$$K_1 = f(t_0, y_0) \Rightarrow t^2(2+y)$$

$$f(0, 1) \quad K_1 = 0^2(2+1) = 0$$

$$K_2 = f(t_1 + h, y_1 + K_1 h)$$

$$f(t_0 + h, y_0 + K_1 h)$$

$$f(0.1, 1) = 0.1^2(2+1) = 0.03$$

.....

$$K_1 = f(t_1, y_1)$$

$$f(0.1, 1.0015) = 0.1^2(2+1.0015) = 0.030015$$

$$K_2 = f(0.2, 1.0015 + 0.030015(0.1))$$

$$f(0.2, 1.0045)$$

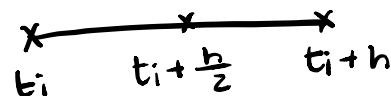
$$= 0.2^2(2+1.0045) = 0.12018$$

* calculate new K_1, K_2
for each iteration

(2) Mid-point m/d

$$y_{i+1} = y_i + h k_2 \quad \text{where } K_1 = f(t_i, y_i)$$

$$k_2 = f(t_i + \frac{h}{2}, y_i + \frac{h}{2} K_1)$$



. Runge-Kutta m/d of order four (RK4)

$$y_{i+1} = y_i + \frac{h}{2} (K_1 + 2K_2 + 2K_3 + K_4)$$

$$K_3 = f(t_i + \frac{h}{2}, y_i + \frac{h}{2} K_2)$$

$$\text{where } K_1 = f(t_i, y_i)$$

$$K_2 = f(t_i + \frac{h}{2}, y_i + \frac{h}{2} K_1)$$

$$K_4 = f(t_i + h, y_i + h K_3)$$