WEDNESDAY JULY 24, 2019

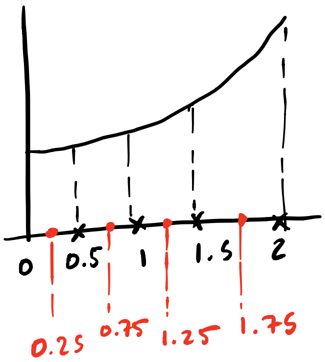
CH. 4 NUMERICAL DIFFERENTIATION AND INTEGRATION

* Lecture 10 – MATLAB INTRO DOCUMENT *

4.4 COMPOSITE NUMERICAL INTEGRATION (CONT)

ex. Use CMR to approximate $\int_0^2 e^x\,dx$ with $n=4$

$h = \frac{2-0}{4} = 0.5$



$$\int_0^2 e^x\,dx \approx 0.5\left[f(0.25) + f(0.75) + f(1.25) + f(1.75)\right]$$

$$\approx 0.5\left[e^{0.25} + e^{0.75} + e^{1.25} + e^{1.75}\right]$$

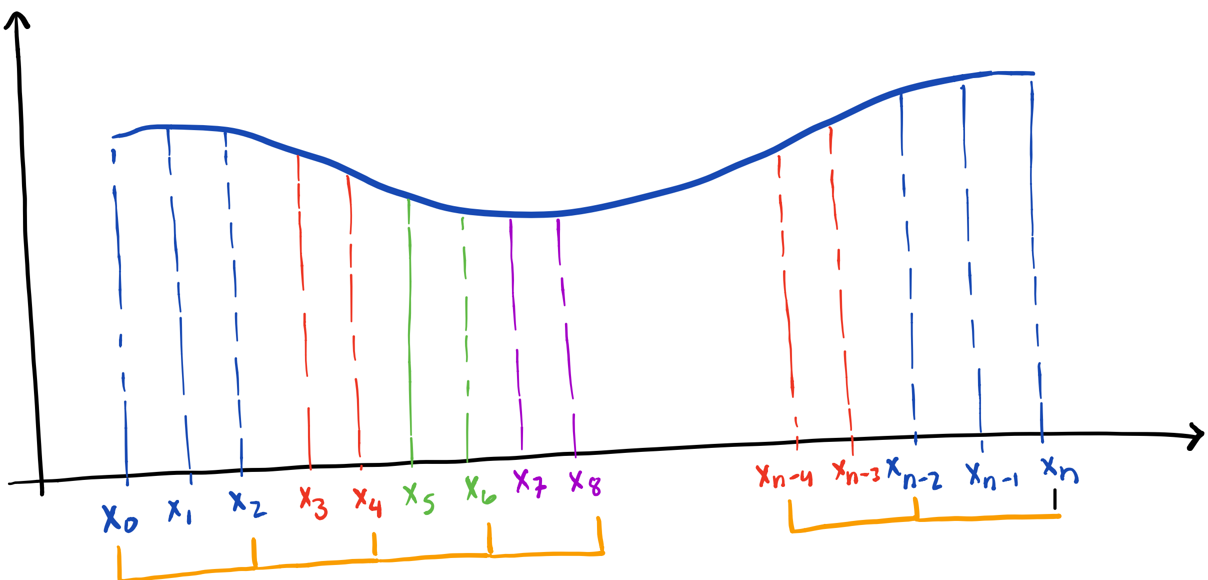
$$= 6.3229$$

- Composite Simpson's Rule

  consider an evenly spaced grid (n-even)

  subintervals:

  $[x_0, x_2], [x_2, x_4], [x_4, x_6], \ldots, [x_{n-2}, x_n]$



$[x_0, x_2]$: $\frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$

$[x_2, x_4]$: $\frac{h}{3}\left[f(x_2) + 4f(x_3) + f(x_4)\right]$

$[x_4, x_6]$: $\frac{h}{3}\left[f(x_4) + 4f(x_5) + f(x_6)\right]$

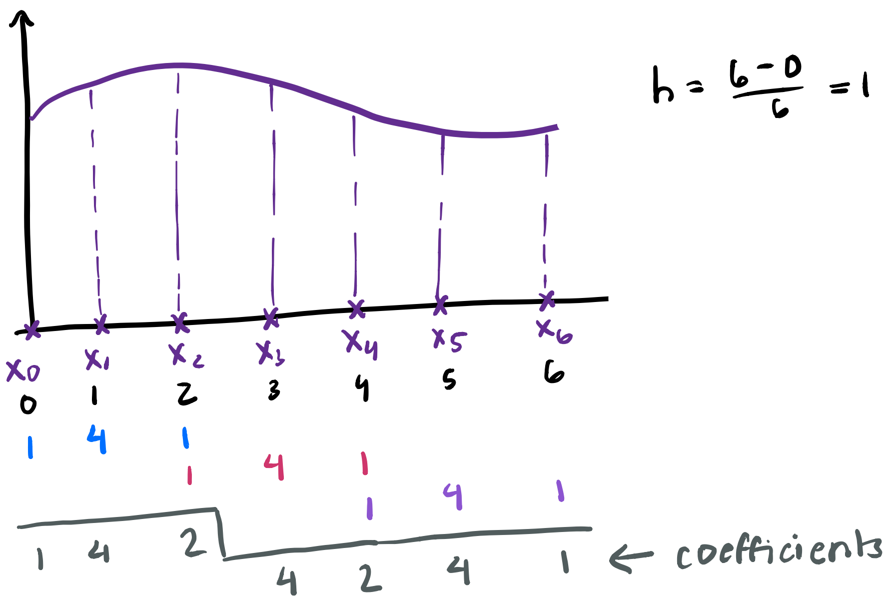
$\vdots$

$[x_{n-2}, x_n]$: $\frac{h}{3}\left[f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$

$$\int_a^b f(x) \approx \frac{h}{3}\left[f(x_0) + 4(f(x_1) + f(x_3) + f(x_5) + \dots + f(x_{n-1})) + 2(f(x_2) + f(x_4) + f(x_6) + \dots + f(x_{n-2})) + f(x_n)\right]$$

$$\approx \frac{h}{3}\left[f(x_0) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{\frac{n}{2}-1} f(x_{2i}) + f(x_n)\right]$$

ex. use CSR to approximate $\int_0^6 e^x dx$ using 6-subintervals.



$h = \frac{6-0}{6} = 1$

$$\int_0^6 e^x dx \approx \frac{1}{3}\left[f(0) + 4(f(1) + f(3) + f(5)) + 2(f(2) + f(4)) + f(6)\right]$$

$$\approx \frac{1}{3}\left[e^0 + 4(e^1 + e^3 + e^5) + 2(e^2 + e^4) + e^6\right]$$

$$\approx \frac{1}{3}\left[e^0 + 4e^1 + 2e^2 + 4e^3 + 2e^4 + 4e^5 + e^6\right] \Leftarrow$$ leave answer like this!

- Errors in Composite Rules

1. Composite Trapezoid Rule

$$E_{CTR} = -\frac{h^2}{12}(b-a) f''(\xi) \quad \xi \in [a,b]$$

$h = \frac{b-a}{n}$

2. Composite Mid-point Rule

$$E_{CMR} = \frac{(b-a)}{6} h^2 f''(\xi) \quad \xi \in [a,b]$$

3. Composite Simpson's Rule

$$E_{CSR} = \frac{-(b-a)}{180} h^4 f^{4}(\xi) \quad \xi \in [a,b]$$

ex. Find the number of subintervals (n) required to approximate $\int_0^2 e^x\,dx$ correct to within 10^{-2} using all three methods: CMR, CSR, CTR

<u>CTR</u>: $E_{CTR} = -\frac{h^2}{12}(b-a)f''(z)$

$|E_{CTR}| = \frac{-h^2}{12}(b-a)\,|f''(z)|$

$|E_{CTR}| = \frac{(b-a)^2}{n^2\,12}\cdot(b-a)\,|f''(z)|$

$|E_{CTR}| \le \frac{(b-a)^3}{n^2 12} \max_{z\in[0,2]} |f''(z)| \le 10^{-2}$

$f(x) = e^x$
$f'(x) = e^x$
$f''(x) = e^x$

$\frac{(2-0)^3}{n^2 12}\left(e^x\Big|_{max}\right) \le 10^{-2}$

$\frac{2^3}{12n^2}\cdot e^2 \le 10^{-2}$

$\sqrt{\frac{\left(\frac{2^3\cdot e^2}{12}\right)}{10^{-2}}} \le n$

$22.19 \le n$

$\boxed{n = 23}$

<u>CMR</u>: $E_{CMR} = \frac{(b-a)}{6}h^2 f''(z)$

$|E_{CMR}| = \frac{(b-a)}{6}\frac{(b-a)^2}{n^2}|f''(z)|$

$\le \frac{(b-a)(b-a)^2}{6n^2}|f''(z)|_{max} \le 10^{-2}$

$\le \frac{(2-0)(2-0)^2}{6n^2}e^2 \le 10^{-2}$

$$\sqrt{\frac{\left(\frac{8e^2}{6}\right)}{10^{-2}}} \leq n$$

$$31.38 \leq n$$

$$\boxed{n = 32}$$

CSR: $E_{CSR} = \frac{-(b-a)}{180} h^4 f^4(\xi)$

$$|E_{CSR}| \leq -\frac{(b-a)}{180} \cdot \frac{(b-a)^4}{n^4} \, |f^4(\xi)|_{max} \leq 10^{-2}$$

$$\leq \frac{(2-0)}{180} \cdot \frac{(2-0)^4}{n^4} \cdot e^2 \leq 10^{-2}$$

$$\left[\frac{\left(\frac{2^5 e^2}{180}\right)}{10^{-2}}\right]^{1/4} \leq n$$

$$3.385 \leq n$$

$$\boxed{n = 4} \qquad n \rightarrow h \Rightarrow \frac{b-a}{n}$$

CH. 5 DIFFERENTIAL EQUATIONS

5.1 Elementary theory of Initial Value Problems

In this chapter, we numerically solve the IVP (initial value problem)

$$\left[\begin{array}{ll} \dfrac{dy}{dt} = f(t,y) & a \le t \le b \\ & y(a) = y_0 \end{array}\right] \text{ — } \circledast$$

Before approximating a solu^{n} $y(t)$, we must ask if eq^{n} $\circledast$ has a unique solu^{n}. The answer is given by the existence and uniqueness th^{m}.

- EUT Th^{m} (exam # 3 – state theorem) ← don't forget to mention what $\circledast$ is!

  The IVP $\circledast$ has a unique solu^{n} $y(t)$ on $[a,b]$, if

  (1) $f(t,y)$ is continuous on the domain $D = \left\{ (t,y) \;\middle|\; \begin{array}{l} a \le t \le b \\ -\infty \le y \le \infty \end{array} \right\}$

  and

  (2) $f(t,y)$ satisfies the Lipschitz condition on D with constant L.

  $$|f(t,y_1) - f(t,y_2)| \le \underset{\uparrow \text{ Lipschitz constant}}{L}\, |y_1 - y_2| \quad \text{for all } (t,y) \underset{\uparrow \text{ in}}{\in} D$$

$$\left| f(t,y_1) - f(t,y_2) \right| = \left| (ty_1 + t^3) - (ty_2 + t^3) \right| = \left| t(y_1 - y_2) \right|$$

$$= \underset{\substack{\uparrow \\ \text{choose max} \\ t\text{-value}}}{t}\, |y_1 - y_2| \le \underset{\substack{\uparrow \\ L}}{1} \cdot |y_1 - y_2|$$

Th^{m}: suppose $f(t,y)$ is defined on a convex set $D \subseteq \mathbb{R}^2$. If a constant $L > 0$ exists with $\left| \dfrac{\partial f(t,y)}{\partial y} \right| \le L$ for all (t,y) in D,

then $f(t,y)$ satisfies the Lipschitz condition on D with Lipschitz constant L.

ex 1) Find the Lipschitz constant for $f(t,y) = \underbrace{ty}_{f} + t^3$ on the set

$D = \left\{ (t,y) \text{ s.t. } \begin{matrix} 0 \le t \le 1 \\ -\infty \le y \le \infty \end{matrix} \right\}$

such that

$$\frac{\partial f}{\partial y} = t \qquad t \in [0,1] \qquad \left| \frac{\partial f}{\partial y} \right| \le 1 \leftarrow L$$

ex 2) use the EUT to show that there is a unique solu^n to the IVP:

$$\frac{dy}{dt} = 1 + t\sin(ty) \qquad \begin{matrix} 0 \le t \le 2 \\ y(0) = 0 \end{matrix}$$

(1) $f(t,y) = 1 + t\sin(ty)$ is a continuous function $\therefore$ the diff. eq^n has a solution

(2) $\frac{\partial f}{\partial y} = t^2 \cos(ty) \Rightarrow \left| \frac{\partial f}{\partial y} \right| = t^2 \left| \cos(ty) \right| \le 4(1)$

max t^2 max $\cos(ty)$

$\left| \frac{\partial f}{\partial y} \right| \le 4 \leftarrow$ Lipschitz constant (non-zero constant)

$f(t,y)$ satisfies the Lipschitz condition

$\Rightarrow$ Therefore, the given DE has a unique solution