THURSDAY JULY 25, 2019

CH. 5 DIFFERENTIAL EQUATIONS

5.2 Euler's Method

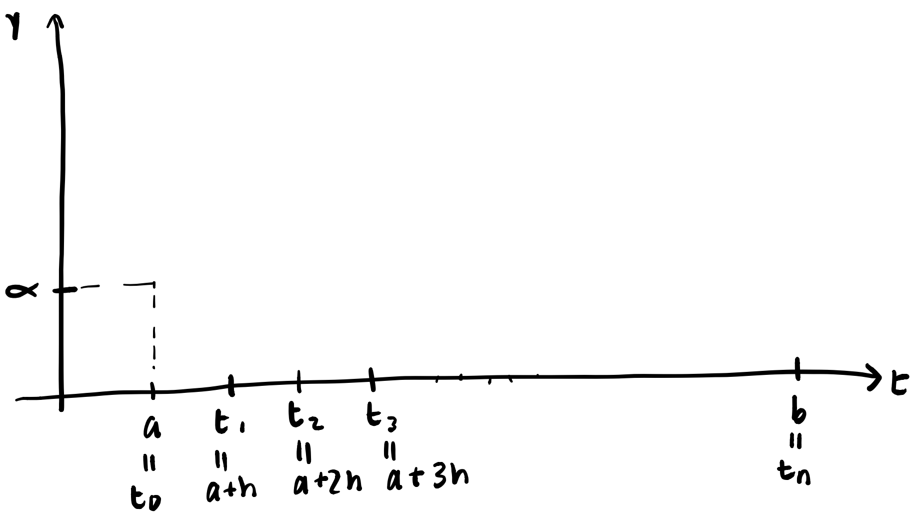
Consider the IVP:

$$\frac{dy}{dt} = f(t,y) \qquad \begin{matrix} a \le t \le b \\ y(a) = \alpha \end{matrix}$$

We begin with a grid of $(n+1)$ points $t_0 = a < t_1 < t_2 < t_3 < \dots < t_n = b$ along the t-axis with equal step size h.

$$t_i = a + ih \qquad i = 0, 1, 2, \dots, n$$

$$\uparrow h = \frac{b-a}{n}$$



The Euler's method finds $y_0, y_1, y_2, y_3, \dots, y_n$ such that

$$\underset{\text{approximate solution}}{y_i} \approx \underbrace{y(t_i)}_{\text{true solution}} \qquad \begin{aligned} y_0 &= \alpha \\ y_{i+1} &= y_i + hf(t_i, y_i) \\ i &= 0, 1, 2, \dots, n-1 \end{aligned}$$

$y' = f(t, y)$

$y_1 = y_0 + hf(t_0, y_0)$

$y_2 = y_1 + hf(t_1, y_1)$

$y_3 = y_2 + hf(t_2, y_2)$

$\vdots$

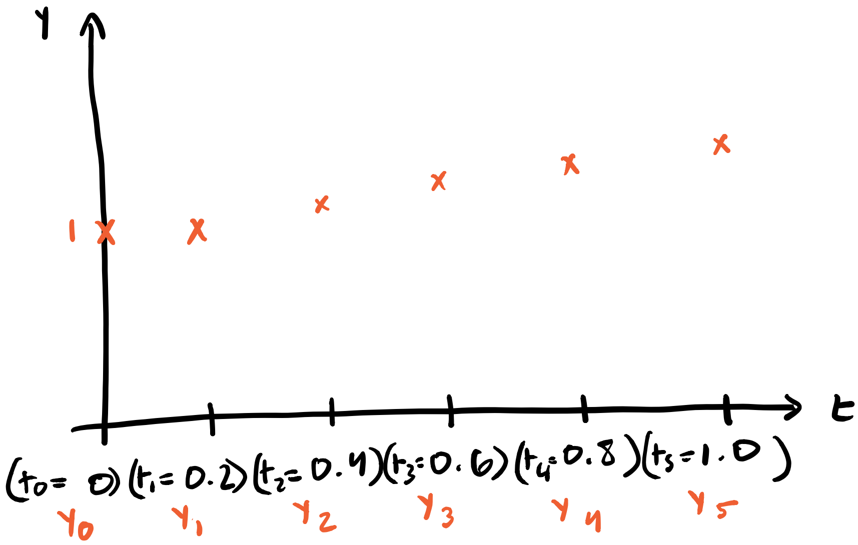
$y_n = y_{n-1} + hf(t_{n-1}, y_{n-1})$

* QUIZ * ↴

Ex 1) Apply Euler's method to IVP : $\frac{dy}{dt} = \underbrace{ty + t^3}_{f(t,y)}$ $y(0) = 1$

$0 \le t \le 1$

with step size $h = 0.2$



$y_{i+1} = y_i + hf(t_i, y_i)$

$y_1 = y_0 + hf(t_0, y_0)$

$= 1 + (0.2)[f(0,1)]$

$= 1 + (0.2)(0)$

$y_1 = 1$

$y_2 = y_1 + hf(t_1, y_1)$

$= 1 + (0.2)[f(0.2, 1)]$

$= 1 + (0.2)[0.2(1) + (0.2)^3]$

$y_2 = 1.0416$

$y_3 = y_2 + hf(t_2, y_2)$

$= 1.0416 + (0.2)[0.4(1.0416) + (0.4)^3]$

$y_3 = 1.13773$

$y_4 = y_3 + hf(t_3, y_3)$

$= 1.13773 + (0.2)(0.6(1.13773) + 0.6^3)$

$y_4 = 1.31746$

$y_5 = y_4 + hf(t_4, y_4)$

$= 1.31746 + 0.2(0.8 \cdot 1.31746 + 0.8^3)$

$y_5 = 1.6306$

$t_0 = 0$

$t_1 = 0.2$

$t_2 = 0.4$

$t_3 = 0.6$

$t_4 = 0.8$

$t_5 = 1.0$

ex 2) Use Euler's method with step size $h=0.5$ to approximate the soln^{ns} of the IVP: $\frac{dy}{dt} = t^2 - y$ $\quad y(0)=1$

$0 \leq t \leq 1.5$

$f(t,y) = t^2 - y$

$t_0 = 0$

$t_1 = 0.5$

$t_2 = 1.0$

$t_3 = 1.5$

$$y_1 = y_0 + hf(\overset{0,1}{t_0, y_0})$$

$$= 1 + (0.5)[0^2 - 1]$$

$y_1 = 0.5$

$$y_2 = y_1 + hf(\overset{0.5,0.5}{t_1, y_1})$$

$$= 0.5 + (0.5)(0.5^2 - 0.5)$$

$y_2 = 0.375$

$$y_3 = y_2 + hf(\overset{1.0,0.375}{t_2, y_2})$$

$$= 0.375 + (0.5)(1.0^2 - 0.375)$$

$y_3 = 0.6875$

☆ QUIZ ☆

Proof: Use Taylor's Th^{m} on y about $t = t_i$

$$y(t) = y(t_i) + \frac{y'(t_i)(t-t_i)'}{1!} + \frac{y''(\xi)(t-t_i)^2}{2!}$$

Now, set $t = t_0 + h$

$$y(t_i + h) = y(t_i) + \frac{y'(t_i)h}{1!} + \frac{y''(\xi)h^2}{2!}$$

Euler's mtd is obtained by dropping $O(h^2)$ term in the formula

$$y(t_i + h) \approx y(t_i) + \underset{\substack{\uparrow \\ \frac{dy}{dt} = f(t,y)}}{y'(t_i)}\, h$$

$$y(t_i + h) \approx y(t_i) + f(t_i, y_i)h \Rightarrow y_{i+1} = y_i + hf(t_i, y_i)$$

• Error Bounds for Euler's Method

Let $D = \{(t,y)$ such that $a \le t \le b \quad -\infty \le y \le \infty\}$ and $F(t,y)$ has a Lipschitz constant L. Suppose that $y(t)$ is the unique solu^{n} to the IVP: $\frac{dy}{dt} \approx F(t,y)$, where $|y''(t)| \le M$ for all t in $[a,b]$. Then for the approximation y_i of $y(t_i)$ by the Euler's method with step size h, we have

$$\underbrace{|y(t_i) - y_i|}_{\text{Error}} \le \underbrace{\frac{hM}{2L}\left[e^{L(t_i-a)} - 1\right]}_{\text{maximum error}}$$

max $|y''(t)|$

ex.) consider the IVP: $\frac{dy}{dt} = \frac{2}{t}y + t^2 e^t \qquad 1 \le t \le 2$, with exact

$y(1) = 0$

solu^{n} $y(t) = t^2(e^t - e^1)$

(a.) Use Euler's method with $h = 0.2$ to approximate the solu^{n} at $t = 1.4$

(b.) Compare it with the actual value of y

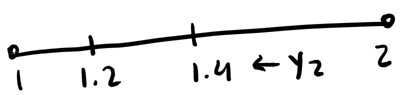
(c.) Find the maximum error in approximating $y(1.4)$ by y_2.

...

(a.) $t_0 = 1,\ t_1 = 1.2,\ t_2 = 1.4$

$$y_1 = y_0 + hF(\overset{1}{t_0}, \overset{0}{y_0}) = 0 + (0.2)\left[\tfrac{2}{1}(0) + (1)^2 e^1\right] = 0.5437$$

$$y_2 = y_1 + hF(\overset{1.2}{t_1}, \overset{0.5437}{y_1}) = 0.5437 + (0.2)\left[\tfrac{2}{1.2}(0.5437) + (1.2)^2(e^{1.2})\right] = \boxed{1.6811}$$



(b.) $y(1.4) = (1.4)^2(e^{1.4} - e^1) = \boxed{2.6204}$

True error $= |2.6204 - 1.6811| = 0.9393$

(c.) $|y(\underset{t_i}{1.4}) - y_2| \le \frac{hM}{2L}\left[e^{L(t_i-a)} - 1\right] = \frac{0.2M}{2L}\left[e^{L(1.4-1)} - 1\right]$

FIND L – $\frac{\partial f}{\partial y} = \frac{2}{t} \Rightarrow \max_{t\in[1,2]} \left|\frac{2}{t}\right| = \frac{2}{1} = 2 \Leftarrow L$

FIND M — $y''(t)$

$$y(t) = t^2 e^t - t^2 e^1$$

$$y'(t) = (t^2 e^t + e^t 2t) - (2te^1)$$

$$y''(t) = (t^2 e^t + e^t 2t) + (e^t 2 + 2te^t) - 2e^1$$

$$= t^2 e^t + 4te^t + 2e^t - 2e^1 = e^t(t^2 + 4t + 2) - 2e$$

max $\Rightarrow$ $y''(2) = e^2(2^2 + 4 \cdot 2 + 2) - 2e = 98.0102 = M$

$$|y(1.4) - y_2| \leq \frac{0.2(98.0102)}{2(2)}\left[e^{2(1.4-1)} - 1\right]$$

$$\boxed{|y(1.4) - y_2| \leq 6.005}$$

5.3 Higher-Order Taylor Methods

NOTE: Euler's m/d was derived by dropping the $O(h^2)$ term in the Taylor series expansion of $y(t)$ about $t = t_i$. One can derive methods with higher-order by retaining more terms in the Taylor series expansion.

$$y(t) = y(t_i) + \frac{y'(t_i)(t-t_i)}{1!} + \frac{y''(t_i)(t-t_i)^2}{2!} + \frac{y'''(t_i)(t-t_i)^3}{3!} + \ldots$$

$$+ \frac{y^n(t_i)(t-t_i)^n}{n!} + \frac{y^{(n+1)}(\xi)(t-t_i)^{n+1}}{(n+1)!}$$

$t = t_i + h$

$$y(t_i + h) \approx y(t_i) + \frac{y'(t_i)h}{1!} + \frac{y''(t_i)h^2}{2!} + \ldots + \frac{y^n h^n}{n!}$$