Fixed point theorem

1.30.19
lecture

Theorem

sufficient conditions - for existence & uniqueness of a fixed-point.

existence: Let $g(x)$ be a continuous function on $[a,b]$ and $a \le g(x) \le b$ for $x \in [a,b]$, then $g(x)$ has at least one fixed point.

uniqueness: Moreover if $|g'(x)| < 1$ for all $x \in [a,b]$ then $g(x)$ has a unique fixed point in $[a,b]$

→ Then for any initial point $x_0 \in [a,b]$ the sequence

$$x_n = g(x_{n-1})$$

converges to the unique fixed-point p.

Example) Consider the function $f(x) = e^x - x - 2$ on the interval $[0,2]$. Find a function $g(x)$ that has a unique fixed point on the interval $[0,2]$.

Step 1) initial guess

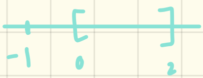
$$x = \underbrace{e^x - 2}_{g(x)} \qquad [0,2]$$

* $f(x) = 0$, solve for x

Step 2) check for existence

$0 \le g(x) \le 2$

$g(0) = -1$ X



Step 1) initial guess

$e^x = x + 2$

$\ln e^x = \ln(x+2)$

$x = \ln(x+2) = g(x)$

step 2) check for existence

$g(0) = \ln(2) \approx 0.693$

$g(2) = \ln(4) \approx 1.386$

$0 \le \ln(x+2) \le 2$ ✓

step 3) check for uniqueness

$g'(x) = \frac{1}{x+2}$

$g'(0) = 1/2 \quad g'(2) = 1/4$

$\Rightarrow |g'(x)| < 1$ ✓

* this is a trial & error method, there is no method for coming up with guesses for g(x).

Maximum error

Let ε_n be the absolute error for the n^{th} iteration x_n. Then, $\varepsilon_n = |p - x_n| \le K^n \cdot \max\{x_0 - a, b - x_0\}$ where $K = \max |g'(x)|$ $x \in [a,b]$

example) The FPI

$$X = \frac{x^2+3}{5} \text{ on } [0,1]$$

estimate how many iterations, n, are required to obtain an absolute error less than 10^{-4} when $x_0 = 1$.

$g(x) = \frac{x^2+3}{5} \quad g(0) = \frac{3}{5} \quad g(1) = \frac{4}{5}$

$0 < \frac{3}{5} \le g(x) \le \frac{4}{5} < 1$

$g'(x) = \frac{2x}{5} \quad [0,1] \quad \max|g'(x)| = \frac{2}{5} = K < 1$

$|p - x_n| \le K^n \max\{x_0 - a, b - x_0\}$

$\le (2/5)^n \max\{1, 0\} = (2/5)^n < 10^{-4}$

⋮

$n \ln(0.4) < \ln(10^{-4})$

$n > \frac{\ln(10^{-4})}{\ln(0.4)} = 10.05 \Rightarrow n = 11$