2-6-19

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Chapter 2.4: Order of convergence

Question:

How fast does an alogorithm converge?

One way to measure the speed of convergence is to use the ratio of the errors between successive iterations

x_n $\quad p \leftarrow$ root

Error $e_n = |x_n - p|$

$e_{n+1} = |x_{n+1} - p|$

$$\frac{e_{n+1}}{e_n} = \frac{|x_{n+1} - p|}{|x_n - p|} < 1$$

$$e_n = |x_n - x_{n-1}|$$

$$\frac{e_{n+1}}{e_n} \approx \frac{|x_{n+1} - x_n|}{|x_n - x_{n-1}|}$$

clearly we would like the error ratio to be less than 1 as n goes to infinity

- To measure the speed of convergence, we use a concept called the "order of convergence"

Def^{n} Suppose the seq. $\{x_n\}_{n=0}^{\infty}$ converges to value p with $x_n \neq p$ for all n.

If there exists positive constants α and λ such that

$$\lim_{n\to\infty} \frac{|x_{n+1} - p|}{|x_n - p|^{\alpha}} = \lambda$$

then $\{x_n\}$ is said to converge to p of order α with asymptotic error constant α

- If $\alpha = 1$ and $\lambda < 1$, the convergence is linear
- If $\alpha = 2$, the convergence is quadratic
- Larger α means "faster convergence"

$$f(x) = \cos x - \sin x = 0 \longrightarrow x = \underbrace{x + \cos x - \sin x}_{g(x)}$$

$$x_{n+1} = x_n + \cos(x_n) - \sin(x_n)$$

x_1 $\quad |x_1 - \pi/4| = e_1$

x_2 $\quad |x_2 - \pi/4| = e_2$

x_3 $\quad |x_3 - \pi/4| = e_3$

x_4

x_5

Th^{m}

Assume that $g(x)$ is continuously differentiable, $g(p) = p$ and $S = |g'(x)| < 1$

Then, the FPI converges linearly with rate S to the fixed point p for initial guess sufficiently close to p

Example:

$x_{n+1} = g(x_n)$ $\qquad$ Goal $\rightarrow e_{n+1} = S\, e_n$

$$e_{n+1} = |x_{n+1} - p| = |g(x_n) - g(p)|$$

$$= |g'(c_n)(x_n - p)| \quad \text{for some } x_n \leq c_n \leq p$$

$$e_{n+1} = |g'(c_n)|\,|x_n - p|$$

$$e_{n+1} = |g'(c_n)|\; e_n$$

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n} = \lim_{n\to\infty} |g'(c_n)| = |g'(p)|$$

$$\lim_{n\to\infty} \frac{e_{n+1}}{e^{n}} = S, \text{ where } S = |g'(p)|$$

Converges linearly!

Note:

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(1) The order of convergence of the Bisection method is linear

(2) The Newton's method is quadratically convergent