TUESDAY JULY 30, 2019

CH. 5 DIFFERENTIAL EQUATIONS

5.4 Runge - Kutta m/ds (cont.)

ex. use RK4 with step size $h = 0.5$ to approx. the solution of the IVP.

$$\frac{dy}{dt} = t^2 - y \qquad 0 \le t \le 2$$
$$y(0) = 1$$

Find y_1

$$y_{i+1} = y_i + \frac{h}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

where $K_1 = f(t_i, y_i)$

$K_2 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}K_1\right)$

$K_3 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}K_2\right)$

$K_4 = f(t_i + h, y_i + hK_3)$

K	$f(t, y)$	$f(t_i, y_i)$	$t_i^2 - y_i$	value
1	$f(t_0, y_0)$	$f(0, 1)$	$0^2 - 1$	-1
2	$f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}K_1)$	$f(0.25, 0.75)$	$0.25^2 - 0.75$	-0.6875
3	$f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}K_2)$	$f(0.25, 0.8281)$	$0.25^2 - 0.8281$	-0.7656
4	$f(t_0 + h, y_0 + hK_3)$	$f(0.5, 0.6172)$	$0.5^2 - 0.6172$	-0.3672

$$y_1 = 1 + \frac{0.5}{6}\left[-1 + 2(-0.6875) + 2(-0.7656) + -0.3672\right] = \boxed{0.6439}$$

* MATLAB COMMAND — "ode45"

CH. 6 LINEAR ALGEBRA

* Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & & & a_{mn} \end{bmatrix}_{m \times n}$$

a_{ij} i: rows
j: columns

ex. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$ $A(1,3) = 3$
$A(2,3) = 6$

$A(i,j) =$ _

Column vector: $\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix}_{3\times 1}$ row vector: $\vec{u} = \begin{bmatrix} 1 & 2 & 5 & 9 \end{bmatrix}_{1\times 4}$

* Transpose

The transpose of an $m \times n$ matrix A, denoted A^T, is an $n \times m$ matrix whose columns are corresponding rows of A

ex. $A = \begin{bmatrix} 1 & -2 & 5 \\ 7 & 0 & -3 \end{bmatrix}_{2\times 3}$ $A^T = \begin{bmatrix} 1 & 7 \\ -2 & 0 \\ 5 & -3 \end{bmatrix}_{3\times 2}$

* sum of two matrices

ex. $A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 5 & 6 \end{bmatrix}$ $B = \begin{bmatrix} -3 & 4 & 1 \\ 0 & -2 & 7 \end{bmatrix}$

Note: matrices must be the same size

$$A + B = \begin{bmatrix} -2 & 4 & 3 \\ -3 & 3 & 13 \end{bmatrix}$$

* Scalar multiplication

ex. $A = \begin{bmatrix} 1 & 0 & 2 & 5 \\ -3 & 1 & 4 & 0 \end{bmatrix}$

$$2A = \begin{bmatrix} 2 & 0 & 4 & 10 \\ -6 & 2 & 8 & 0 \end{bmatrix}$$

* Matrix multiplication

If $A_{m \times n}$ and $B_{n \times p}$, then their product $(AB)_{m \times p}$, whose (i,j)-entry is the "dot" product: the row i of A and the column j of B

ex. $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 6 & 8 \end{bmatrix}_{2\times 3}$ $B = \begin{bmatrix} 1 & -1 \\ 4 & -3 \\ 0 & -2 \end{bmatrix}_{3\times 2}$

$$AB = \begin{bmatrix} (-7)_{11} & (-1)_{12} \\ (24)_{21} & (-34)_{22} \end{bmatrix}_{2\times 2} \Rightarrow \begin{bmatrix} -7 & -1 \\ 24 & -34 \end{bmatrix}_{2\times 2}$$

$$BA = \underbrace{\begin{bmatrix} 1 & -1 \\ 4 & -3 \\ 0 & -2 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 0 & 6 & 8 \end{bmatrix}}_{2\times 3} \Rightarrow \begin{bmatrix} 1 & -8 & -5 \\ 4 & -26 & -12 \\ 0 & -12 & -16 \end{bmatrix}_{3\times 3}$$

6.2 Solving system of Linear equations

$$x_1 + 2x_2 - x_3 = 5 \quad - E1$$
$$3x_1 + x_2 - 2x_3 = 9 \quad - E2$$
$$-x_1 + 4x_2 + 2x_3 = 0 \quad - E3$$

Augmented Matrix

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -2 \\ -1 & 4 & 2 \end{bmatrix}}_{\text{coefficient matrix}} \left| \begin{matrix} 5 \\ 9 \\ 0 \end{matrix} \right]$$

↑ constant vector

Elementary - row operations

(1) multiply a row by a non-zero constant $E_i \leftarrow cE_i$

(2) Interchange two rows $E_i \leftrightarrow E_j$

(3) Add a scalar multiple of one row to another $E_i \leftarrow E_i + cE_j$

$$\begin{array}{c|c} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} E \\ ① \\ ② \\ ③ \end{array} & \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{array}\right] \end{array}$$

· step 1:

$E_2 \leftarrow E_2 - 3E_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 1 & -6 \\ -1 & 4 & 2 & 0 \end{array}\right]$$

· step 2:

$E_3 \leftarrow E_3 + E_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 1 & -6 \\ 0 & 6 & 1 & 5 \end{array}\right]$$

· step 3:

$E_2 \leftarrow \frac{1}{5} E_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -1 & \frac{1}{5} & -\frac{6}{5} \\ 0 & 6 & 1 & 5 \end{array}\right]$$

· step 4:

$E_3 \leftarrow \frac{1}{6} E_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -1 & -\frac{1}{5} & -\frac{6}{5} \\ 0 & 1 & \frac{1}{6} & \frac{5}{6} \end{array}\right]$$

· step 5:

$E_3 \leftarrow E_3 + E_2$

upper-triangle matrix

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -1 & -\frac{1}{5} & -\frac{6}{5} \\ 0 & 0 & \frac{11}{30} & -\frac{11}{30} \end{array}\right] \end{array}$$

① get zeroes top to bottom

② get zeroes left to right

③ solve from bottom to top

$\rightarrow \frac{11}{30} x_3 = -\frac{11}{30} \quad \therefore \boxed{x_3 = -1}$

$\rightarrow -x_2 - \frac{1}{5} x_3 = -\frac{6}{5}$

$-x_2 - \frac{1}{5} = -\frac{6}{5}$

$-x_2 = -\frac{6}{5} + \frac{1}{5} = -1$

$\boxed{x_2 = 1}$

$\rightarrow x_1 + 2x_2 - x_3 = 5$

$x_1 + 2(1) - (-1) = 5$

$\boxed{x_1 = 2}$

solution vector:

$$\boxed{\vec{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}}$$

$x_1 + x_2 + x_3 = 6$

$x_1 + 2x_2 + 2x_3 = 9$

$x_1 + 2x_2 + 3x_3 = 10$

① Augmented Matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 2 & 9 \\ 1 & 2 & 3 & 10 \end{array}\right]$$

② $E_2 \leftarrow E_2 - E_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 3 \\ 1 & 2 & 3 & 10 \end{array}\right]$$

③ $E_3 \leftarrow E_3 - E_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 2 & 4 \end{array}\right]$$

④ $E_3 \leftarrow E_3 - E_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

upper triangle matrix

back substitution

$[x_3 = 1]$

$x_2 + x_3 = 3$

$[x_2 = 2]$

$x_1 + x_2 + x_3 = 6$

$x_1 + 2 + 1 = 6$

$[x_1 = 3]$

$$\vec{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

• Key ideas: Gaussian Elimination

(1) The linear system becomes upper triangle after elimination

(2) The upper triangular system is solved by back-substitution

(3) If $a_{ij} = 0$ for some i and $a_{ji} \neq 0$ for some $j > i$, then we interchange E_i and E_j

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 0 & -1 & -1 \\ 1 & 2 & 3 & 10 \end{array}\right]$$

we can switch these two rows