HW2

1. $g(x) = 2^{-x}$ on $\left[\frac{1}{3}, 1\right]$

$g(x)$ is a decreasing fun^{n}.

$g\left(\frac{1}{3}\right) \cong 0.7937$ $\qquad g(1) = 0.5$.

$\therefore$ $g(x)$ takes values on the interval $\left[\frac{1}{3}, 1\right]$

$$g'(x) = -2^{-x} \ln 2$$

$$|g'(x)| = |\ln 2 \; 2^{-x}|$$

$$|g'(x)| = \ln 2 . \; 2^{-x}$$

$$\max_{x \in \left[\frac{1}{3}, 1\right]} |g'(x)| = \ln 2 \left(2^{-\frac{1}{3}}\right)$$

$$\approx 0.55 < 1.$$

$$\therefore \max_{x \in \left[\frac{1}{3}, 1\right]} |g'(x)| < 1.$$

Hence $g(x)$ has a unique fixed point on the interval $\left[\frac{1}{3}, 1\right]$.

Problem 2.

$g_1(x) = \dfrac{x^2-3}{2}$ $\qquad g_2(x) = \sqrt{2x+3}$ $\qquad g_3(x) = \dfrac{3}{(x-2)}$

(a) Fixed points of each $g_i(x)$

$$x = g_1(x)$$
$$x = \frac{x^2-3}{2}$$
$$2x = x^2-3$$
$$0 = x^2-2x-3$$
$$0 = (x-3)(x+1)$$
$$\boxed{3, -1 = x}$$

$$x = g_2(x)$$
$$x = \sqrt{2x+3}$$
$$x^2 = 2x+3$$
$$x^2-2x-3=0$$
$$\boxed{x = 3, -1}$$

$$x = g_3(x)$$
$$x = \frac{3}{(x-2)}$$
$$x^2-2x = 3$$
$$x^2-2x-3=0$$
$$\boxed{x = 3, -1}$$

Fixed-Points

Let's calculate the roots of $f(x)$

$$x^2-2x-3=0$$
$$(x-3)(x+1)=0$$
$$\boxed{x = 3, -1} \leftarrow \text{roots of } f(x)$$

Yes! the fixed-points of $x = g_i(x)$ are the roots of $f(x)=0$

(b). $g_1(x) = \frac{x^2-3}{2}$ $\quad g_2(x) = \sqrt{2x+3}$ $\quad g_3(x) = \frac{3}{x-2}$

Fixed-point Th^{m}.

(1) If $x \in [a,b]$ then $a \leq g(x) \leq b$. (Existence)

+

(2) $\max\limits_{x\in(a,b)} |g'(x)| < 1$ (conditions 1 and 2 $\rightarrow$ uniqueness)

Consider

$g_1(x)$.

$g_1(x)$ is an increasing fun^{n}.

$g_1(1) = -1 \qquad g_1(4) = \frac{13}{2} = 6.5.$

So, $g_1(x) \notin [1,4]$ Condition 1 is not satisfied!

not in

Now, let's consider $g_2(x) = \sqrt{2x+3}$

$g_2(x)$ is also an increasing fun^{n}.

$g_2(1) = \sqrt{5} = 2.2361$

$g_2(4) = \sqrt{11} = 3.3166.$

$\Rightarrow g_2(x) \in [1,4].$

Condition 1 ✓

$g_2'(x) = \frac{1}{\cancel{2}\sqrt{2x+3}} \times \cancel{2} = \frac{1}{\sqrt{2x+3}}$

$g_2'(x)$ is a decreasing fun^{n}.

$\therefore [g_2'(x)]_{max} = g_2'(1) = \frac{1}{\sqrt{5}} < 1$

condition 2. ✓

unique fixed point!

Finally, $g_3(x) = \dfrac{3}{(x-2)}$

$g_3(x)$ is a decreasing funⁿ.

$g_3(1) = -3$ $\quad$ $g_3(4) = \frac{3}{2}$.

$\Rightarrow g(x) \notin [1,4]$

Problem 3

$$f(x) = e^x - 3x = 0 \quad \text{on} \quad [1.1, 2] \quad x_0 = 1.5$$

$$e^x = 3x$$

$$\ln(e^x) = \ln(3x)$$

$$x = \underbrace{\ln(3x)}_{g(x)}$$

Our candidate for $g(x) = \ln(3x)$

$$g(1.1) = \ln(3.3) \approx 1.1939$$

$$g(2) = \ln(6) \simeq 1.7918.$$

$g(x) = \ln(3x)$ is an increasing fun^n.

$\therefore \; g(x) \in [1.1, 2]$ condition 1.

$$g'(x) = \frac{1}{\cancel{3}x} \times \cancel{3} = \frac{1}{x}.$$

$g'(x)$ is a decreasing fun^n and $\max\limits_{x\in[1.1,2]}[g'(x)] = \underbrace{|g'(1.1)|}_{0.9090.} < 1$

$\Rightarrow$ unique fixed point!

Fixed-point iteration

$$x_{n+1} = g(x_n)$$

$$x_{n+1} = \ln(3x_n)$$

Iteration	x_n	e_n
1	1.5040	
2	1.5068	0.0027
3	1.5086	0.0018
4	1.5098	0.0019
5	1.5106	0.0007
6	1.5111	
7	1.5114	

Problem 4

$f(x) = x - \cos x$ on $[0, \frac{\pi}{3}]$ accuracy 10^{-8} $x_0 = \frac{\pi}{4}$

$x = g(x)$

$x = \cos x$

$g(x) = \cos x$.

$\cos x \in [0, \frac{\pi}{3}]$ on $[0, \frac{\pi}{3}]$

$g'(x) = \sin x$

$\sin x$ is an increasing fun^n on $[0, \frac{\pi}{3}]$

$$\max_{x \in [0, \frac{\pi}{3}]} |g'(x)| = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} < 1$$

Error analysis.

$$|P - x_n| \leq K^n \max\{x_0 - a, b - x_0\}$$

we need $|P - x_n| < 10^{-8}$.

If we find n such that

$$K^n \max\{x_0 - a, b - x_0\} < 10^{-8},$$

the $|P - x_n| < 10^{-8}$.

$$K = \max_{x\in[0,\frac{\pi}{3}]} |g'(x)| = \frac{\sqrt{3}}{2}$$

$$\left(\frac{\sqrt{3}}{2}\right)^n \max\left\{ \frac{\pi}{4} - 0, \frac{\pi}{3} - \frac{\pi}{4} \right\} < 10^{-8}$$

$$\left(\frac{\sqrt{3}}{2}\right)^n \frac{\pi}{4} < 10^{-8}.$$

$$\left(\frac{\sqrt{3}}{2}\right)^n < \frac{4}{\pi} \times 10^{-8}.$$

$$\log_{10}\left[\frac{\sqrt{3}}{2}\right]^n < \log_{10}\left[\frac{4}{\pi} \times 10^{-8}\right]$$

$$n \log_{10}\left[\frac{\sqrt{3}}{2}\right] < \log_{10}\left[\frac{4}{\pi} \times 10^{-8}\right]$$

$$n \overset{*}{>} \frac{\log_{10}\left[\frac{4}{\pi} \times 10^{-8}\right]}{\log_{10}\left[\frac{\sqrt{3}}{2}\right]}$$

Note* $\log_{10}\left(\frac{\sqrt{3}}{2}\right)$ is a negative number.

$$n > 126.38$$

$$\boxed{n = 127}$$