Numerical Analysis MAT 362

Problem 1

Consider the Heat eq:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial t^2} \quad 0 \le x \le 2, \;\; t \ge 0.$$

$u(0,t) = u(2,t) = 0, \;\; t \ge 0$ and $u(x,0) = \cos\left(\frac{\pi(x-1)}{2}\right)$ for $0 \le x \le 2$. (a) Write the iteration-matrix formula to approximate the solution of the heat equation using $h = 0.4$ and $k = 0.1$. (b) Approximate the solution $u(x,t)$ at $t = 0.3$. (c) Compare the result at $t = 0.3$ using the exact solution $u(x,t) = e^{-\pi^2 t/8}\cos\left(\frac{\pi(x-1)}{2}\right)$.

Problem 2

Let $f(x) = \frac{1}{2^x}$ and $x_0 = 0, \;\; x_1 = 1$, and $x_2 = 2$. Determine the Lagrange polynomial of degree at most 2, which agrees with $f(x)$ at x_0, x_1, x_2.

Problem 3

Show that the sequence $\{\frac{1}{4^n}\}$ converges linearly to 0.

Problem 4

Consider the linear system:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 3x_3 & = & 0 \\ & & 2x_2 & + & x_3 & = & -8 \\ -x_1 & + & x_2 & + & 2x_3 & = & 3 \end{array}$$

(a) Use Gaussian Elimination to solve the system. (b) Apply three-iterations of the Gauss-Seidel method to solve the system with $\vec{x}_0 = [1\ 1\ 1]^T$.

Problem 5

Consider the cubic spline:

$$S(x) = \begin{cases} S_0(x) = 6 - 2x + \frac{1}{2}x^3 & \text{on } [0, 2] \\ S_1(x) = 6 + 4(x-2) + c(x-2)^2 + d(x-2)^3 & \text{on } [2, 3] \end{cases}$$

(a) Find the value of c. (b) Does there exist a number d such that the spline is *natural*? If so, find d.