Numerical Analysis MAT 362: HW Problem set

Please read the Instructions

- **Show all the steps that you go between the question and the answer. Show how you derived the answer. For your work to be complete, you need to explain your reasoning and make your computations clear.**
- **You will be graded on the readability of your work.**
- **The correct answer with no or incorrect work will earn you NO marks**
- **Show ALL your work**
- Use only **four decimal places** for all numbers.
- **If possible, use** 8.5" $\times$ 11" **white paper (not torn from spiral binders) and staple sheets together**.
- Print your name legibly in the upper corner of the page.
- Write your solutions as though you're trying to convince someone that you know what you're talking about.
- **Failure to follow these instructions will result in loss points (up to the full amount of the homework total)**

Problem 1

Show that the following equations have at least one solution in the given interval.

(a) $x \cos x - 2x^2 + 3x - 1 = 0, \quad [0.2, 0.3]$

(b) $x - (\ln x)^x = 0, \quad [4, 5]$

Problem 2

Find c satisfying the Mean Value Theorem for $f(x)$ on the interval $[0, 1]$.

(a) $f(x) = e^x$

(b) $f(x) = x^2$

Problem 3

Find the fifth iteration (c_5) of the Bisection Method to approximate the root of $f(x) = \sqrt{x} - \cos x = 0$ on $[0, 1]$.

Problem 4

Find n for which the nth iteration by the Bisection Method guarantees to approximate the root of $f(x) = x^4 - x^3 - 10$ on $[2, 3]$ with accuracy within 10^{-8}.

Problem 5

Use the fixed-point iteration theorem to show that $g(x) = 2^{-x}$ has a unique fixed point on $\left[\frac{1}{3}, 1\right]$. Hint: $\dfrac{d}{dx}a^{-x} = -a^{-x}\ln a$.

Problem 6

Consider three functions:

$$g_1(x) = \frac{x^2 - 3}{2}, \quad g_2(x) = \sqrt{2x + 3}, \quad \text{and} \quad g_3(x) = \frac{3}{x - 2}.$$

(a) Show that fixed points of each $g_i(x)$ are roots of $f(x) = x^2 - 2x - 3$.

(b) Which $g_i(x)$ has a unique fixed point in $[1, 4]$ guaranteed by the Fixed Point Theorem?

Problem 7

Use the fixed point iteration to find the solution of $e^x = 3x$ on $[1.1, 2]$ with $x_0 = 1.5$ correct to roughly within 10^{-3}. Hint: Use the correct $g(x)$ for $f(x) = e^x - 3x$.

Problem 8

Find n for which the nth iteration by the Fixed-point method guarantees to approximate the root of $f(x) = x - \cos x$ on $[0, \frac{\pi}{3}]$ with accuracy within 10^{-8} using $x_0 = \frac{\pi}{4}$.

Problem 9

Apply **two steps** of Newton's Method with initial guess $x_0 = 0$.

(a) $x^3 + x - 2 = 0$

(b) $x^4 - x^2 + x - 1 = 0$

Problem 10

Apply **two steps** of Newton's Method with initial guess $x_0 = 1$.

(a) $x^3 + x^2 - 1 = 0$

(b) $5x - 10 = 0$

Problem 11

Apply **two steps** of the Secant Method to the following equations with initial guesses $x_0 = 1$ and $x_1 = 2$.

(a) $e^x + x = 7$

(b) $x^3 = 2x + 2$

Problem 12

(a) State the Taylor's theorem with remainder.

(b) Construct the $3^{rd}-$order Taylor polynomial about $x_0 = 1$ approximating $\ln(x)$.

(c) Use this polynomial to approximate $\ln(1.1)$.

(d) Give an expression for the Taylor remainder.

Problem 13

Use Lagrange interpolation to find a polynomial that passes through the points:

(a) $(0, 1)$, $(2, 3)$, $(3, 0)$

(b) $(0, -2)$, $(2, 1)$, $(4, 4)$

(c) $(-1, 0)$, $(2, 1)$, $(3, 1)$, $(5, 2)$

Problem 14

The estimated mean atmospheric concentration of carbon dioxide in earth's atmosphere is given in the table that follows, in parts per million (ppm) by volume.

Year	1800	1850	1900	2000
CO_2 (ppm)	280	283	291	370

(a) Find the degree 3 Lagrange interpolating polynomial of the data set.

(b) Then, use it to estimate the CO_2 concentration 2050.

Problem 15

Find the **maximum error** in approximating $f(x) = e^x$ on $[-1, 1]$ by the Lagrange polynomial $P_2(x)$ using points $x = \{-1, 0, 1\}$.

Problem 16

Decide whether the equations form a cubic spline.

(a) $S(x) = \begin{cases} S_0(x) = x^3 + x - 1 & \text{on } [0,1] \\ S_1(x) = 1 + 3(x-1) + 3(x-1)^2 - (x-1)^3 & \text{on } [1,2] \end{cases}$

(b) $S(x) = \begin{cases} S_0(x) = 2x^3 + x^2 + 4x + 5 & \text{on } [0,1] \\ S_1(x) = 12 + 12(x-1) + 7(x-1)^2 + (x-1)^3 & \text{on } [1,2] \end{cases}$

Problem 17

Find the **natural** cubic spline through $(0,3)$, $(1,2)$, and $(2,1)$.

Problem 18

(Elements of Numerical Integration)
Apply the Trapezoid and Simpson's Rule to approximate

$$\int_1^2 \ln x \, dx$$

and find an upper bound for the error in the your approximation.

Problem 19

(Composite numerical integration)

Approximate $\int_0^1 e^{x^2}\,dx$ using 4 subintervals in (a) Composite Trapezoidal Rule, (b) Composite Midpoint Rule, (c) Composite Simpson's Rule.

Problem 20

(Composite numerical integration)

Approximate $\int_1^4 \sin x\,dx$ using 6 subintervals in (a) Composite Trapezoidal Rule, (b) Composite Midpoint Rule, (c) Composite Simpson's Rule.

Problem 21

(Composite numerical integration)

Find the step size h and the number of subintervals n required to approximate $\int_1^3 x^2 \ln x\,dx$ correct within 10^{-4} using (a) Composite Trapezoidal Rule, (b) Composite Midpoint Rule, (c) Composite Simpson's Rule.

Problem 22

Use Euler's method with step size $h = 0.5$ to approximate the solution of the IVP:

$$\frac{dy}{dt} = \frac{e^{y^2}}{t}, \quad 1 \leq t \leq 2, \quad y(1) = 0.$$

Problem 23

Consider the IVP:

$$\frac{dy}{dt} = 2ty + t, \quad 0 \leq t \leq 1, \quad y(0) = 1.$$

(a) Use Euler's method with step size $h = 0.25$ to approximate $y(0.5)$.
(b) Find the exact solution of the IVP.
(c) Find the maximum error in approximating $y(0.5)$ by y_2.
(d) Calculate the actual absolute error in approximating $y(0.5)$ by y_2.

Problem 24

Derive the $n^{th}-$order Taylor's method for the IVP:

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha.$$

Problem 25

Use Taylor's method of order 2 with step size $h = 0.5$ to approximate the solution of the following IVP:

$$\frac{dy}{dt} = t - y^2, \quad 1 \le t \le 3, \quad y(1) = 0.$$

Problem 26

Use RK2 with step size $h = 0.5$ to approximate the solution of the following IVP:

$$\frac{dy}{dt} = \frac{e^y}{2t}, \quad 1 \le t \le 3, \quad y(1) = 0.$$

Problem 27

Consider the IVP:

$$\frac{dy}{dt} = (1 - y)\cos t, \quad 0 \le t \le 3, \quad y(0) = 3.$$

(a) Use RK4 with step size $h = 1$ to set up an iteration formula for y_{i+1} to approximate the solution of the above IVP. (Hint. write k_1, k_2, k_3, k_4 and y_{i+1} in terms of them).
(b) Approximate the solution of the IVP.
(c) Find the absolute error in approximating $y(1)$ by y_1 using the actual solution
$y = 1 + 2^{-\sin t}$.

Problem 28

(a) Use RK4 with step size $h = 1$ to approximate the solution of the following IVP:

$$\frac{dy}{dt} = (1 - y)\cos t, \quad 0 \le t \le 3, \quad y(0) = 3.$$

at $y(1)$ (ONLY 1 step). (b) Find the absolute error in approximating $y(1)$ by y_1 using the actual solution
$y = 1 + 2e^{-\sin t}$.

Problem 29

(Introduction to Linear Algebra)
2.1 If A and B are 5×7 matrices, and C is a 2×5 matrix, which of the following are defined?
(a) AB (b) A^T (c) $A + B$ (d) $B^T C^T$ (e) CB (f) $B + C$

2.2 Evaluate the matrix product BA^T where, $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$.

Problem 30

Use Gaussian Elimination to solve the system:

$$\begin{array}{rcrcrcl} 2x_1 & - & 6x_2 & + & 2x_3 & = & 4 \\ x_1 & & & + & 2x_3 & = & 5 \\ 3x_1 & - & 5x_2 & + & 3x_3 & = & 6 \end{array}$$

Problem 31

Do 2-iterations of the **Jacobi and Gauss-Seidel methods** with $\vec{x}^{(0)} = [0, 0, 0]^T$ to approximate the solution of the following system:

$$\begin{array}{rcrcrcl} 3x_1 & + & x_2 & + & x_3 & = & 6 \\ x_1 & + & 3x_2 & + & x_3 & = & 3 \\ x_1 & + & x_2 & + & 3x_3 & = & 5 \end{array}$$

Problem 32

For the following coefficient matrices A, determine if the Jacobi and Gauss-Seidel iterations converge.

(a) $A = \begin{bmatrix} 2 & 1 & 0 \\ -3 & -5 & 1 \\ 0 & -2 & -3 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & 1 & 0 \\ -3 & -4 & 1 \\ 0 & -2 & 3 \end{bmatrix}$

Problem 33

(a) Do three iterations by the power method with $\vec{x}^{(0)} = [-1, 1]^T$ to approximate the largest eigenvalue of $A = \begin{bmatrix} 5 & -4 \\ -1 & 2 \end{bmatrix}$ with its eigenvector.
(b) Calculate the *actual* dominant eigenvalue and the corresponding eigenvector.