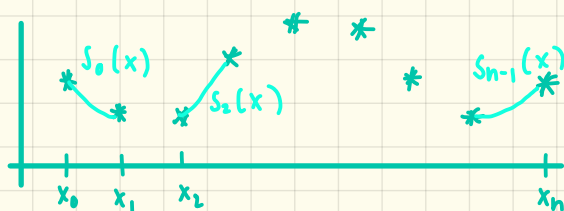


Cubic Splines

02.15.19

Suppose there are $(n+1)$ data points:

$(x_0, y_0) (x_1, y_1) \dots (x_n, y_n)$



$(n+1)$ data points \Rightarrow n intervals

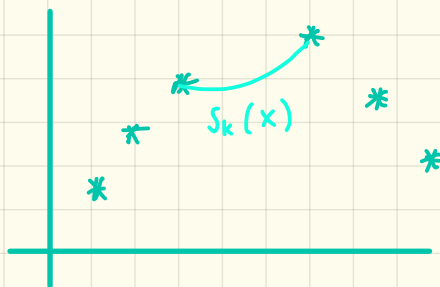
Each cubic polynomial is expressed in the form:

$$S_k(x) = a_k + b_k(x - x_k) + c_k(x - x_k)^2 + d_k(x - x_k)^3$$

in the interval $[x_k, x_{k+1}] \quad k = 0, 1, 2, \dots, n-1$

- There are n such polynomials ($S_k(x)$)
 - Each has 4 unknown constants
- \rightarrow System of 4 equations
- } 4n total unknowns

Properties of Cubic Splines



$$1. \begin{aligned} S_k(x_k) &= y_k & [n \text{ equations}] \\ S_k(x_{k+1}) &= y_{k+1} & [n \text{ equations}] \end{aligned}$$

$$2. S'_k(x_{k+1}) = S'_{k+1}(x_{k+1}) \quad [n-1 \text{ equations}]$$

$$3. S''_k(x_{k+1}) = S''_{k+1}(x_{k+1}) \quad [n-1 \text{ equations}]$$

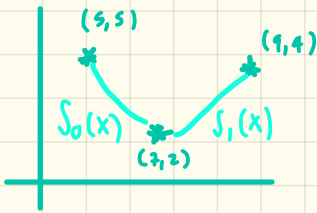
4. (a) Natural splines

$$S''_0(x_0) = 0 \quad S''_{n-1}(x_n) = 0$$

(b) Clamped cubic splines

total $4n - 2$ equations

Example) Construct a piecewise cubic spline interpolant for the curve passing through $(5, 5)$, $(7, 2)$, $(9, 4)$ with natural boundary conditions. Approximate $f(8)$.



$$[S_k(x) = a_k + b_k(x - x_k) + c_k(x - x_k)^2 + d_k(x - x_k)^3]$$

$$S_0(x) = a_0 + b_0(x - 5) + c_0(x - 5)^2 + d_0(x - 5)^3$$

$$S_1(x) = a_1 + b_1(x - 7) + c_1(x - 7)^2 + d_1(x - 7)^3$$

1. Checking Property 1

$$S_k(x_k) = y_k$$

$$S_0(5) = a_0 = 5 \quad (1)$$

$$S_1(7) = a_1 = 2 \quad (2)$$

$$S_k(x_{k+1}) = y_{k+1}$$

$$S_0(7) = a_0 + 2b_0 + 4c_0 + 8d_0 = 2 \quad (3)$$

$$S_1(9) = a_1 + 2b_1 + 4c_1 + 8d_1 = 4 \quad (4)$$

2. Checking Property 2 & 3

$$S'_k(x_{k+1}) = S'_{k+1}(x_{k+1})$$

$$S'_0(x) = b_0 + 2c_0(x - 5) + 3d_0(x - 5)^2$$

$$S'_1(x) = b_1 + 2c_1(x - 7) + 3d_1(x - 7)^2$$

$$S'_0(7) = S'_1(7) \Rightarrow S'_0(x) = b_0 + 4c_0 + 12d_0 = b_1 = S'_1(7) \quad (5)$$

$$S''_k(x_{k+1}) = S''_{k+1}(x_{k+1})$$

$$S''_0(x) = 2c_0 + 6d_0(x - 5)$$

$$S''_1(x) = 2c_1 + 6d_1(x - 7)$$

$$S''_0(7) = 2c_0 + 12d_0 = 2c_1 \quad (6)$$

$$S''_0(5) = 2c_0 = 0 \quad (7)$$

$$S''_1(9) = 2c_1 + 12d_1 = 0 \quad (8)$$