Review:

2.20.19

Forward Difference Method:

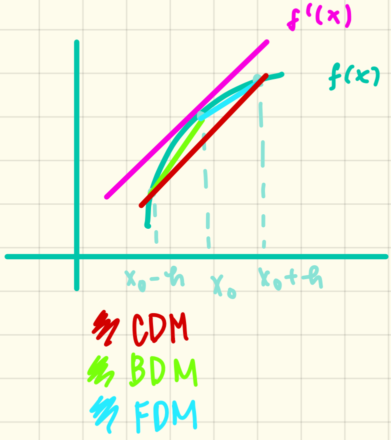
$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h}$$

Backward Difference Method:

$$f'(x_0) = \frac{f(x_0) - f(x_0-h)}{h}$$

Center Difference Method:

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h}$$



Example) Let $f(x) = x^2e^x$. Approximate $f'(1)$ using FDF, BDF, & CDF with $h = 0.2$

FDF: $f'(1) \approx \frac{f(1.2) - f(1)}{0.2}$

≈ 10.313

BDF: $f'(1) \approx \frac{f(1) - f(0.8)}{0.2}$

≈ 6.47

CDF: $f'(1) \approx \frac{f(1.2) - f(0.8)}{2(0.2)}$

≈ 8.392

$f'(1) = 8.15$ which proves CDF is the most accurate approx.

Errors in finite difference formulas

By the Taylor's theorem on $f(x)$ about x_0 we get:

$$f(x) = f(x_0) + \frac{f'(x_0)(x-x_0)}{1!} + \frac{f''(\xi)(x-x_0)^2}{2!}$$

Set $x = x_0 + h$

$$f(x_0+h) = f(x_0) + \frac{f'(x_0)h}{1} + \frac{f''(\xi)h^2}{2}$$

$$f'(x_0) = \frac{f(x_0+h) - f(x_0) - \frac{f''(\xi)h^2}{2}}{h}$$

$$= \underbrace{\frac{f(x_0+h) - f(x_0)}{h}}_{\text{FDF}} - \frac{f''(\xi)h}{2}$$

$$\left| f'(x_0) - \frac{f(x_0+h) - f(x_0)}{h} \right| = \left| -\frac{f''(\xi)h}{2} \right|$$

$$= \frac{f''(\xi)h}{2}$$

Therefore, the maximum error in FDF is:

$$\frac{h}{2} \cdot \max_{x \in (x_0, x_0+h)} |f''(\xi)|$$

- Error is proportional to h'
- FDF/BDF are order 1 approximations ($O(h')$)
- We can make the error small by making h small.

What does it mean for the formula to be 1st order?

- cutting h in half cuts the error approximately in half.

For CDF:

$$f(x_0+h) = f(x_0) + \frac{f'(x_0)h}{1!} + \frac{f''(x_0)h^2}{2!} + \frac{f'''(z)h^3}{3!} \quad (1)$$

$$f(x_0-h) = f(x_0) - \frac{f'(x_0)h}{1!} + \frac{f''(x_0)h^2}{2!} - \frac{f'''(z)h^3}{3!} \quad (2)$$

(1) - (2)

$$f(x_0+h) - f(x_0-h) = 2f'(x_0)h + (f'''(z_1) + f'''(z_2))\frac{h^3}{6}$$

$$\frac{f(x_0+h) - f(x_0-h)}{2h} = f'(x_0) + (f'''(z_1) + f'''(z_2))\frac{h^2}{12}$$

by the IVT one can show that

$$f'''(z) = \frac{f'''(z_1) + f'''(z_2)}{2} \quad z \in (x_0-h, x_0+h)$$

Finally we get

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} - \frac{f'''(z)}{6} \cdot h^2$$

$$\text{maximum error} = \frac{h^2}{6} \max_{x \in (x_0-h,\, x_0+h)} |f'''(z)|$$

CDF is 2nd order accurate $(O(h^2))$