THURSDAY JULY 11, 2019

CH.2 SOLUTIONS OF EQ^{ns} OF ONE VARIABLE

2.2 FIXED-POINT ITERATION (CONT.)

NOTE: Let e_n be the absolute error for the n^{th}-iteration. Then,

$$e_n = |x_n - p| \le k^n \cdot \max\{x_0 - a, b - x_0\} \text{ where } |g'(x)| \le k \le 1$$

$$k = \max |g'(x)|$$

ex. Given the FPI $x_n = \frac{x_{n-1}^2 + 3}{5}$ on $[0,1]$, estimate how many iterations "n" are required to obtain the absolute error $|x_n - p| \le 10^{-4}$ when $x_0 = 0.6$.

$$\underbrace{|x_n - p|}_{\text{absolute error}} \le k^n \cdot \max\{x_0 - a, b - x_0\}$$

$$k^n \cdot \max\{0.6 - 0, 1 - 0.6\} \le 10^{-4}$$

$$k^n \cdot \max\{0.6, 0.4\} \le 10^{-4}$$

$$k^n \cdot 0.6 \le 10^{-4}$$

$$k = \max |g'(x)|$$

$$g(x) = \frac{x^2 + 3}{5} = \frac{x^2}{5} + \frac{3}{5}$$

$$g'(x) = \frac{2x}{5}$$

$$k = \max |g'(x)| = \frac{2(1)}{5} = \frac{2}{5}$$

$$\left(\frac{2}{5}\right)^n \cdot 0.6 \le 10^{-4}$$

$$\left(\frac{2}{5}\right)^n \le \frac{10^{-4}}{0.6}$$

$$n \log_{10}\left(\frac{2}{5}\right) \le \log_{10}\left(\frac{10^{-4}}{0.6}\right)$$

$$n \ge \frac{\log_{10}\left(\frac{10^{-4}}{0.6}\right)}{\log_{10}\left(\frac{2}{5}\right)}$$

$$n \ge 9.4943$$

$$\boxed{n = 10}$$

2.3 NEWTON-RAPHSON METHOD

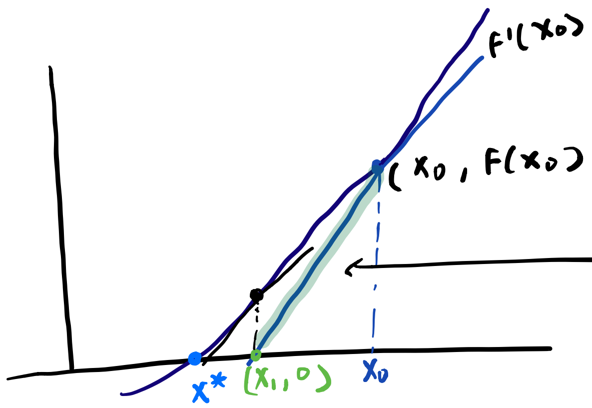
Suppose $f(x)$ is a funⁿ with a root x^* in $[a,b]$. Let x_0 be a "good" initial guess.

• NEWTON'S METHOD

x_0 - initial guess

$$\boxed{x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}} \quad n = 1, 2, 3, \ldots$$

* this method fails when $f'(x_{n-1}) = 0$



$$\frac{f(x_0) - 0}{x_0 - x_1} = f'(x_0)$$

solve for x_1

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

ex. Find the Newton's iteration formula for $x^3 + x - 1 = 0$

$$f(x) = x^3 + x - 1$$

$$f'(x) = 3x^2 + 1$$

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \Rightarrow \left[x_n = x_{n-1} - \frac{(x_{n-1}^3 + x_{n-1} - 1)}{(3x_{n-1}^2 + 1)} \right]$$

Iterate this formula from the initial guess $x_0 = -0.7$. Find x_1 and x_2.

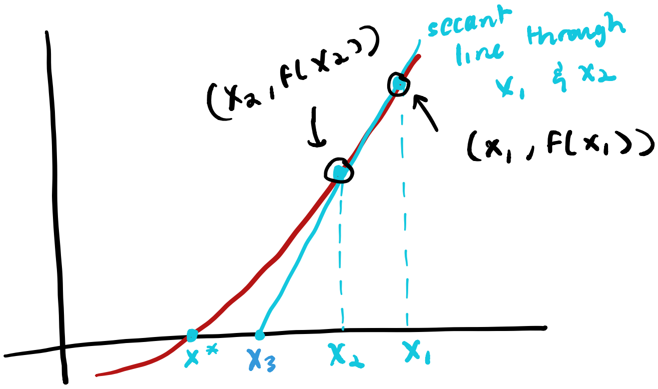
$$x_1 = x_0 - \frac{x_0^3 + x_0 - 1}{3x_0^2 + 1} = \boxed{0.1271}$$

$$x_2 = x_1 - \frac{x_1^3 + x_1 - 1}{3x_1^2 + 1} = \boxed{0.9577}$$

· SECANT METHOD

Main idea - avoid computing $f'(x)$

- need two initial conditions



$$f'(x_2) \approx \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

secant m/d

$$x_3 = x_2 - \frac{f(x_2)}{\left[\frac{f(x_1) - f(x_2)}{(x_1 - x_2)}\right]}$$

general formula $\Rightarrow$

$$\boxed{x_n = x_{n-1} - \frac{f(x_{n-1})(x_{n-2} - x_{n-1})}{[f(x_{n-2}) - f(x_{n-1})]}}$$

ex. Apply the secant method with starting guesses $x_0 = 0$, $x_1 = 1$ to find the root of $f(x) = x^3 + x - 1$. Find x_2 and x_3.

$$f(x) = x^3 + x - 1$$

$$x_n = x_{n-1} - \frac{(x_{n-1}^3 + x_{n-1} - 1)(x_{n-2} - x_{n-1})}{[(x_{n-2}^3 + x_{n-2} - 1) - (x_{n-1}^3 + x_{n-1} - 1)]}$$

$$x_2 = x_1 - \frac{(x_1^3 + x_1 - 1)(x_0 - x_1)}{[(x_0^3 + x_0 - 1) - (x_1^3 + x_1 - 1)]} = \boxed{0.5}$$

$$x_3 = x_2 - \frac{(x_2^3 + x_2 - 1)(x_1 - x_2)}{[(x_1^3 + x_1 - 1) - (x_2^3 + x_2 - 1)]} = \boxed{0.6364}$$

<u>NOTE</u>: stopping criteria

· Absolute error $|x_{n+1} - x_n| < \text{tolerance}$

MATLAB: use while loop

· Relative error $\frac{|x_{n+1} - x_n|}{x_{n+1}} < \text{tolerance}$