Problem 5

5 $g(x) = 2^{-x}$ on $[\frac{1}{3}, 1]$

$g(x)$ is a decreasing fun?

$g(\frac{1}{3}) \cong 0.7937$

$g(1) = 0.5.$

$\therefore$ $g(x)$ takes values on the interval $[\frac{1}{3}, 1]$

$g'(x) = -2^{-x} \ln 2$

$|g'(x)| = |\ln 2 \; 2^{-x}|$

$|g'(x)| = \ln 2. \; 2^{-x}$

$$\max_{x \in [\frac{1}{3}, 1]} |g'(x)| = \ln 2 (2^{-\frac{1}{3}})$$

$$\approx 0.55 < 1.$$

$$\therefore \max_{x \in [\frac{1}{3}, 1]} |g'(x)| < 1.$$

Hence $g(x)$ has a unique fixed point on the interval $[\frac{1}{3}, 1]$

Problem 6

$$g_1(x) = \frac{x^2-3}{2} \qquad g_2(x) = \sqrt{2x+3} \qquad g_3(x) = \frac{3}{(x-2)}$$

(a) Fixed points of each $g_i(x)$

$$x = g_1(x)$$
$$x = \frac{x^2-3}{2}$$
$$2x = x^2-3$$
$$0 = x^2-2x-3$$
$$0 = (x-3)(x+1)$$
$$\boxed{3, -1. = x}$$

$$x = g_2(x)$$
$$x = \sqrt{2x+3}$$
$$x^2 = 2x+3$$
$$x^2-2x-3=0$$
$$\boxed{x = 3, -1}$$

$$x = g_3(x)$$
$$x = \frac{3}{(x-2)}$$
$$x^2-2x = 3$$
$$x^2-2x-3=0$$
$$\boxed{x = 3, -1.}$$

Fixed-Points

Let's calculate the roots of $f(x)$

$$x^2-2x-3=0$$
$$(x-3)(x+1)=0$$
$$\boxed{x = 3, -1.}$$ ← roots of $f(x)$

Yes! the fixed-points of $x = g_i(x)$ are the roots of $f(x)=0$

(b). $g_1(x) = \frac{x^2-3}{2}$ $g_2(x) = \sqrt{2x+3}$ $g_3(x) = \frac{3}{x-2}$

Fixed-point Th^{m}.

(1) If $x \in [a,b]$ then $a \leq g(x) \leq b$. (Existence)

+

(2) $\max_{x\in(a,b)} |g'(x)| < 1$ (conditions 1 and 2 $\rightarrow$ uniqueness)

Consider

$g_1(x)$.

$g_1(x)$ is an increasing fun^{n}.

$g_1(1) = -1$ $g_1(4) = \frac{13}{2} = 6.5$.

So, $g_1(x) \notin [1,4]$ Condition 1 is not satisfied!

not in

Now, let's consider $g_2(x) = \sqrt{2x+3}$

$g_2(x)$ is also an increasing fun^{n}.

$g_2'(x) = \frac{1}{\cancel{2}\sqrt{2x+3}} \times \cancel{2} = \frac{1}{\sqrt{2x+3}}$

$g_2(1) = \sqrt{5} = 2.2361$

$g_2(4) = \sqrt{11} = 3.3166$.

$g_2'(x)$ is a decreasing fun^{n}.

$\therefore \left[g_2'(x)\right]_{max} = g_2'(1) = \frac{1}{\sqrt{5}} < 1$

$\Rightarrow$ $g_2(x) \in [1,4]$.

Condition 1 ✓

condition 2. ✓

unique fixed point!

2.3

Finally, $g_3(x) = \dfrac{3}{(x-2)}$

$g_3(x)$ is a decreasing fun^n.

$g_3(1) = -3 \qquad g_3(4) = \dfrac{3}{2}$.

$\Rightarrow \quad g(x) \notin [1,4]$

Problem 7

$f(x) = e^x - 3x = 0$ on $[1.1, 2]$ $x_0 = 1.5$

$e^x = 3x$

$\ln(e^x) = \ln(3x)$

$x = \underbrace{\ln(3x)}_{g(x)}$

Our candidate for $g(x) = \ln(3x)$

$g(1.1) = \ln(3.3) \approx 1.1939$

$g(2) = \ln(6) \simeq 1.7918.$

$g(x) = \ln(3x)$ is an increasing fun^n.

$\therefore$ $g(x) \in [1.1, 2]$ condition 1.

$g'(x) = \frac{1}{\cancel{3}x} \times \cancel{3} = \frac{1}{x}.$

$g'(x)$ is a decreasing fun^n and $\max\limits_{x \in [1.1, 2]} [g'(x)] = \underbrace{|g'(1.1)|}_{0.9090.} < 1$

$\Rightarrow$ unique fixed point!

Fixed-point iteration

$$x_{n+1} = g(x_n)$$

$$x_{n+1} = \ln(3x_n)$$

Iteration	x_n	e_n
1	1.5040	
2	1.5068	0.0027
3	1.5086	0.0018
4	1.5098	0.0011
5	1.5106	0.0007
6	1.5111	
7	1.5114	