5.3 Higher-order Taylor Methods

Note. Euler's method was derived by dropping the $O(h^2)$ term in the Taylor series expansion of y about $t=t_i$. One can derive methods with higher-order by retaining more terms in the Taylor series.

Consider the IVP:

$$y' = f(t,y) \qquad a \le t \le b$$

$$y(a) = b.$$

Suppose the Sol^n $y(t)$ has $(n+1)$ continuous derivatives.

Now, use Taylor's th^m on y about $t=t_i$.

$$y(t) = y(t_i) + \frac{y'(t_i)}{1!}(t-t_i) + \frac{(t-t_i)^2}{2!} y''(t_i) + \dots + \frac{(t-t_i)^n}{n!} y^{(n)}(t_i) + \frac{(t-t_i)^{n+1}}{(n+1)!} y^{(n+1)}(z_i),$$

where $z_i \in (t_i, t)$

Let $t = t_i + h = t_{i+1}$

$$y(t_{i+1}) = y(t_i) + \frac{h}{1!} f(t_i, y(t_i)) + \frac{h^2}{2!} f'(t_i, y(t_i)) + \dots + \frac{h^n}{n!} f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(z_i, y(z_i))$$

$$y(t_{i+1}) \approx y_i + h\left[f(t_i, y(t_i)) + \frac{h}{2!} f'(t_i, y(t_i)) + \dots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, y(t_i))\right]$$

$$+ \, O(h^{n+1})$$

Taylor Method of order (n)

$$y(t_{i+1}) \approx y_i + h \, T_n(t_i, y_i) = y_{i+1},$$

where $F_n(t_i, y_i) = f(t_i, y(t_i)) + \frac{h}{2!} f'(t_i, y(t_i)) + \frac{h^2 f''(t_i, y(t_i))}{3!} + \dots$

$$+ \frac{h^{n-1} f^{(n-1)}(t_i, y(t_i))}{n!}$$

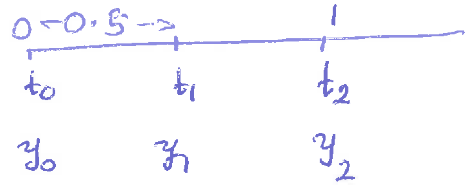
Taylor's Method of order . n

ex Use Taylor's method of order 2 with step size $h = 0.5$ to approximate the solu^n of the IVP

$$y' = te^{3t} - 2y \qquad 0 \le t \le 1$$

$$y(0) = 0.$$

We have



$$f(t,y) = te^{3t} - 2y.$$

$t_0 = 0 \qquad y_0 = 0.$

Then $$y_{i+1} = y_i + h\left[f(t_i,y_i) + \frac{h}{2!} f'(t_i,y_i)\right].$$

$$f'(t,y) = e^{3t} + 3te^{3t} - 2y'$$

$$= e^{3t} + 3te^{3t} - 2\left[te^{3t} - 2y\right]$$

$$= e^{3t} + te^{3t} + 4y$$

$$f'(t,y) = e^{3t}(1+t) + 4y$$

$$y_{i+1} = y_i + h\left[te^{3t} - 2y + \frac{h\left(e^{3t}(1+t) + 4y\right)}{2!}\right]\Bigg|_{(t_i,y_i)}$$

$$y_1 = y_0 + 0.5\left[t_0 e^{3t_0} - 2y_0 + 0.5\frac{\left(e^{3t_0}(1+t_0) + 4y_0\right)}{2!}\right]$$

$$y_1 = 0.1250.$$

$t_1 = 0.5 \quad y_1 = 0.1250$

$$y_2 = y_1 + h\left[t_1 e^{3t_1} - 2y_1 + h \frac{\left(e^{3t_1}(1+t_1) + 4y_1 \right)}{2} \right]$$

$$y_2 = 2.0232$$

True Solu^n

$$y' = te^{3t} - 2y$$

$$y' + 2y = t e^{3t}$$

$$IF = e^{\int 2\,dt} = e^{2t}$$

$$\frac{d}{dt}\left[y e^{2t}\right] = t e^{5t}$$

$$y e^{2t} = \int \underset{u}{t}\, \underset{dv.}{e^{5t}}\, dt$$

$$= \frac{t e^{5t}}{5} - \int \frac{e^{5t}}{5} \cdot 1\, dt$$

$y(0.5) = 0.2836$

$y(100) = 3.2191$

$$y e^{2t} = \frac{t e^{5t}}{5} - \frac{e^{5t}}{25} + C$$

$$y = \frac{t e^{3t}}{5} - \frac{e^{3t}}{25} + c\, e^{-2t}$$

$$0 = \frac{-1}{25} + c \qquad c = \frac{1}{25}$$

$$y = \frac{(5t-1)e^{3t} + e^{-2t}}{25}.$$

ex. Using Taylor's method of order 3 with step size $h = 0.5$, set up an iteration formula for $\{y_i\}$ to approximate the solu^n of the IVP:

$$\frac{dy}{dt} = t^2 - y \qquad 0 \leq t \leq 2$$

$$y(0) = 1.$$

$$y_{i+1} = y_i + h\left[f(t,y) + \frac{h\, f'(t,y)}{2!} + \frac{h^2 f''(t,y)}{3!} \right]\Bigg|_{(t_i,\, y_i)}$$

$$f(t,y) = t^2 - y.$$

$$f'(t,y) = 2t - y' = 2t - t^2 + y$$

$$f''(t,y) = 2 - 2t + y' = 2 - 2t + t^2 - y.$$

$$y_{i+1} = y_i + 0.5\left[t_i^2 - y_i + \frac{0.5\,(2t_i - t_i^2 + y_i)}{2!} + \frac{(0.5)^2 (2 - 2t_i + t_i^2 - y_i)}{3!} \right]$$

$$y_{i+1} = \frac{19t_i^2 + 10t_i + 2 + 29y_i}{48}$$

$$y_0 = 1$$

$$t_i = t_0 + hi$$

$$t_i = 0 + 0.5i$$

Ⓜ Definition   Local Truncation Error.

$$e_{i+1}(h) = \frac{y(t_{i+1}) - (y_i + h\,\varphi(t_i, y_i))}{h}$$

where $y_{i+1} = y_i + h\varphi(t_i, y_i)$

ex. LTE for Euler's method

$$e_{i+1}(h) = \frac{y(i+1) - y_i}{h} - f(t_i, y_i) = \frac{h}{2}\, y''(c_i)$$

$$|e_{i+1}(h)| \leq \frac{h}{2} M \qquad \mathcal{O}(h^1)$$

Thm. If Taylor's method of order n is used to approximate the Solu^{n} of $\frac{dy}{dt} = f(t, y)$ with step size h, and if $y \in C^{n+1}[a,b]$, then the LTE is $\mathcal{O}(h^n)$.

Proof. $y(i+1) - y_i - hf(t_i, y_i) - \frac{h^2}{2} f'(t_i, y_i) - \cdots - \frac{h^n}{n!} f^{(n-1)}(t_i, y_i)$

$$= \frac{h^{n+1}}{(n+1)!} f^{(n)}(c_i, y(c_i))$$

Then LTE $e_{i+1}(h) = \frac{h^n}{(n+1)!} f^{(n)}(c_i, y(c_i)) \leq M_0 h^n = \mathcal{O}(h^n)$