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1. (a) Use 3 steps of the Bisection method to find an approximating root of $f(x) = x^3 - 20$ on $[2, 3]$.

5

i	a_i	$f(a_i)$	c_i	$f(c_i)$	b_i	$f(b_i)$
1	2	$-$	2.5	$-$	3	$+$
2	2.5	$-$	2.75	$+$	3	$+$
3	2.5	$-$	2.6250	$-$	2.75	$+$

(b) How many iterations of the Bisection method are necessary to approximate the root of $f(x) = x^3 - 20$ on $[2, 3]$ with accuracy within 5×10^{-6}?

$$|r^* - c_n| \leq \frac{b_1 - a_1}{2^n}$$

4

$$\frac{3-2}{2^n} \leq 5 \times 10^{-6}$$

$$\frac{1}{2^n} \leq 5 \times 10^{-6}$$

$$2^{-n} \leq 5 \times 10^{-6}$$

$$-n \log_{10} 2 \leq \log_{10}(5 \times 10^{-6})$$

$$n \geq \frac{\log_{10}(5 \times 10^{-6})}{-\log_{10} 2}$$

$$n \geq 17.6096$$

$$\boxed{n = 18.}$$

2. (a) State the Intermediate Value Theorem.

$\boxed{3}$

(b) Let $f(x) = e^x - 3x$. Show that $f(x)$ has a root in the interval $[0, 1]$.

$\boxed{2}$

$f(0) = 1.$

$f(1) = e - 3 < 0$

From the IVT, there is a point c^* in $[0,1]$ s.t $f(c^*) = 0$

3. Perform 2 iterations of the Newton's method to approximate a solution to $x^2 - 6 = 0$ with initial value $x_0 = 1$.

$\boxed{5}$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $\qquad f'(x) = 2x$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(-5)}{2} = 1 + \frac{5}{2} = \frac{7}{2} = \boxed{3.5}$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= \frac{7}{2} - \frac{\left(\frac{49}{4} - 6\right)}{7}$

$= \frac{7}{2} - \frac{25}{28} = \boxed{2.6071}$

$=$

4. Let $f(x) = \frac{1}{x}$.

(a) Find the third order Taylor polynomial $P_3(x)$ approximation for $f(x)$ about $x_0 = 1$.

$\boxed{4}$

$$f(x) = f(x_0) + f'(x_0)\frac{(x-x_0)}{1!} + f''(x_0)\frac{(x-x_0)^2}{2!} + f'''(x_0)\frac{(x-x_0)^3}{3!} + \frac{f^{(iv)}(\xi)(x-x_0)^4}{4!}$$

$f(x) = x^{-1}$

$f'(x) = -x^{-2}$ $\quad f(x) = 1 + \frac{-1}{1!}(x-1) + \frac{2(x-1)^2}{2!} + (-6)\frac{(x-1)^3}{3!}$

$f''(x) = 2x^{-3}$

$f'''(x) = -6x^{-4}$ $\quad f^{(iv)}(x) = 24x^{-5}$

(b) Approximate $f(0.8)$ using $P_3(x)$.

$\boxed{2}$

$f(x) = 1 - \frac{1}{1}(x-1) + (x-1)^2 - (x-1)^3$

$f(0.8) = 1 - \frac{1}{1}(0.8-1) + (0.8-1)^2 - (0.8-1)^3$

$= 1.2480$

(c) Give an expression for the Taylor remainder.

$\boxed{2}$

$R_n(x) = \frac{24}{\xi^5} \frac{(x-1)^4}{4!}$

5. (a) Consider the Fixed-Point Iteration method $x_{n+1} = g(x_n)$ on $[a, b]$. Discuss sufficient conditions on $g(x)$ for convergence to the unique fixed-point.

$g(x)$ is a continuous fun^n.

(1) $a \leq g(x) \leq b$

2 (2) $\max_{x \in (a,b)} |g'(x)| < 1$

(b) Consider the function $g(x) = \sqrt{2x+3}$. Show that fixed points of $g(x)$ are roots of $f(x) = x^2 - 2x - 3$.

Fixed points of $g(x)$ $\rightarrow$ $x = \sqrt{2x+3}$

$x^2 = 2x+3$

$x^2 - 2x - 3 = 0 \Rightarrow f(x)$

$x = \sqrt{2x+3}$

$x^2 = 2x+3$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3 \quad x = -1$

2 $\Rightarrow$ Fixed points of $g(x)$ are roots of $f(x)$.

(c) Show that $g(x)$ has a unique fixed point in $[1, 4]$.

$g(x) = \sqrt{2x+3}$

2 $g(x)$ is an increasing function

$g(1) = \sqrt{5} \approx 2.2361$

$g(4) = \sqrt{11} \approx 3.3166$

$1 \leq g(x) \leq 4$

$g(x)$ has atleast one fixed point.

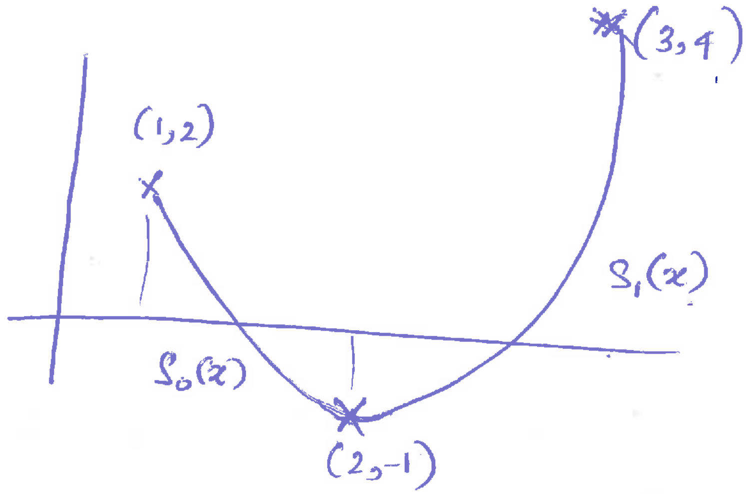
$g'(x) = \dfrac{\cancel{2}}{\cancel{2}\sqrt{2x+3}}$ ← decreasing fun^n.

2 $\max_{x \in (1,4)} |g'(x)| = \dfrac{1}{\sqrt{5}} < 1$

$\Rightarrow g(x)$ has a unique fixedpoint.

6. Obtain the system of equations that is necessary to find the **natural** cubic spline through $(1, 2)$, $(2, -1)$, and $(3, 4)$.

8pts



$$S_0(x) = a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3$$
$$S_1(x) = a_1 + b_1(x-2) + c_1(x-2)^2 + d_1(x-2)^3$$

at $(1,2)$ $\boxed{a_0 = 2}$ — (1)

$\boxed{a_1 = -1}$ — (2)

at $(2,-1)$ $\boxed{a_0 + b_0 + c_0 + d_0 = -1}$ — (3)

$\boxed{a_1 + b_1 + c_1 + d_1 = 4}$ — (4)

Continuity of the derivatives at $(2,-1)$ $S_0'(2) = S_1'(2)$ and $S_0''(2) = S_1''(2)$

$S_0'(x) = b_0 + 2c_0(x-1) + 3d_0(x-1)^2$ $S_0''(x) = 2c_0 + 6d_0(x-1)$

$S_1'(x) = b_1 + 2c_1(x-2) + 3d_1(x-2)^2$ $S_1''(x) = 2c_1 + 6d_1(x-2)$

$\boxed{b_0 + 2c_0 + 3d_0 = b_1}$ — (5) $\boxed{2c_0 + 6d_0 = 2c_1}$ — (6)

Natural Cubic Splines. $S_0''(1) = 0$

$\boxed{2c_0 = 0}$ — (7) $\boxed{2c_1 + 6d_1 = 0}$ — (8)

7. Find the Lagrange Interpolating polynomial of degree 2 that passes through the points $(1,2)$, $(2,4)$, and $(5,1)$. You do NOT need to simplify the polynomial completely.

x_i	1	2	5
y_i	2	4	1

5

$$P_2(x) = \frac{(x-2)(x-5)}{(1-2)(1-5)}\,2 + \frac{(x-1)(x-5)}{(2-1)(2-5)}\,4 + \frac{(x-1)(x-2)}{(5-1)(5-2)}\,1.$$

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1. Consider the data in the following table.

x	1.0	1.1	1.2	1.3
$f(x)$	0.0000	0.1269	0.3151	0.5764

$\boxed{3}$ (a) Use the Two-point Forward-Difference Formula to approximate $f'(1.1)$.

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}$$

$$f'(1.1) \approx \frac{f(1.2) - f(1.1)}{0.1} = \frac{0.3151 - 0.1269}{0.1} = 1.882$$

$\boxed{3}$ (b) Use the Two-point Centered-Difference Formula to approximate $f'(1.2)$.

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0-h)}{2h} = \frac{f(1.3) - f(1.1)}{2(0.1)} = \frac{0.5764 - 0.1269}{0.2}$$

$$= 2.2475$$

$\boxed{3}$ (c) Use the Three-point Forward-Difference Formula to approximate $f'(1.0)$.

12.50

$$f'(x_0) \approx \frac{-3f(x_0) + 4f(x_0+h) - f(x_0+2h)}{2h}$$

$$f'(1.0) \approx \frac{-3f(1.0) + 4f(1.1) - f(1.2)}{2(0.1)} = \frac{-3[0] + 4[0.1269] - 0.3151}{0.2}$$

$$= 0.9625.$$

4 (d) If $f(x) = x^3 \ln x$, find the maximum error in approximating $f'(1.1)$ by the Two-point Backward-Difference Formula.

$f(x) = x^3 \ln x$

$f'(x) = 3x^2 \ln x + x^2$

$f''(x) = 6x \ln x + 3x + 2x$

$f''(x) = x(5 + 6\ln x)$

$$E_{BDF} \le \frac{h}{2} \max_{x \in (x_0 - h, x_0)} |f''(x)|$$

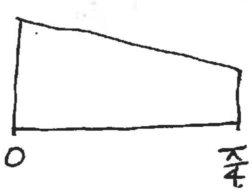
$$\frac{0.1}{2} \max_{x \in (1.0,\, 1.1)} |x(5 + 6\ln x)|$$

$$\frac{0.1}{2}\left[1.1(5 + 6\ln 1.1)\right] = \boxed{0.3065}$$

2. Consider the definite integral $\int_0^{\pi/4} \cos(x^2)\, dx$, where x is in **radians**. Approximate the definite integral using the

3 (a) Non-composite Trapezoidal rule.

$$\frac{[f(a) + f(b)]}{2}(b - a)$$



$$\approx \frac{\left[\cos(0) + \cos\left(\left(\frac{\pi}{4}\right)^2\right)\right]}{2} \frac{\pi}{4} = \boxed{0.7130}$$

3 (b) Non-composite Midpoint rule.

$$= \left[f\left(\frac{a+b}{2}\right)\right](b - a)$$

$$\approx \frac{\pi}{4}\left[\cos\left(\frac{\pi^2}{8^2}\right)\right] = \boxed{0.7760}$$

5 (c) Non-composite Simpson's rule.

$$\frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$\frac{(b-a)/2}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \frac{\pi}{8(3)}\left[\cos(0^2) + 4\cos\left(\frac{\pi^2}{8^2}\right) + \cos\left(\frac{\pi^2}{4^2}\right)\right] \boxed{= 0.7551}$$

3.

$\boxed{4}$ (a) Determine the step size h required to approximate $\int_2^6 x^2 \ln x \, dx$ correct within 10^{-5} using the Composite Midpoint rule.

$$E_{CMR} = \frac{(b-a)h^2}{24} f''(c)$$

$$f(x) = x^2 \ln x.$$
$$f' = 2x \ln x + x$$
$$f'' = 2 \ln x + 3.$$

$$|E_{CMR}| \le \frac{(b-a)}{24} h^2 \left|f''(x)\right|_{max} = \frac{4}{24} h^2 (2 \ln x + 3)\Big|_{x=6} \leqslant 10^{-5}.$$

$h \leqslant 0.0030$

$\boxed{h = 0.003}$

$\boxed{5}$ (b) Use the Composite Trapezoidal Rule with $n = 4$ to approximate the integral $\int_2^6 x^2 \ln x \, dx$.

$$h = \frac{b-a}{n} = \frac{4}{4} = 1.$$

$$CTR = \frac{h}{2}\left[f(x_0) + 2[f(x_1) + f(x_2) + f(x_3)] + f(x_4)\right]$$

2	3	4	5	6.
x_0	x_1	x_2	x_3	x_4

$$= \frac{1}{2}\left[f(2) + 2[f(3) + f(4) + f(5)] + f(6)\right]$$

$$= \boxed{105.9421}$$

$\boxed{6}$ (c) Use the Composite Simpson's Rule with $n = 4$ to approximate the integral $\int_2^6 x^2 \ln x \, dx$.

$$CSR = \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

2	3	4	5	6
x_0	x_1	x_2	x_3	x_4
1	4	1		
		1	4	1.

$$= \frac{1}{3}\left[f(2) + 4[f(x_1) + f(x_3)] + 2f(x_2) + f(x_4)\right]$$

$$= \boxed{104.0437}$$

4\. (a) For each of the two following functions, show that it satisfies a Lipschitz condition, wit.. respect to y, on the corresponding domain, and find the Lipschitz constant L.

[2] i. $f(t, y) = \dfrac{3y}{\sqrt{t}}$ for $1 \le t < \infty$

$$\frac{\partial f}{\partial y} = \frac{3}{\sqrt{t}}$$

$$\left|\frac{\partial f}{\partial y}\right| \leq \left.\frac{3}{\sqrt{t}}\right|_{1} = 3 = L.$$

[3] ii. $f(t, y) = 2 + t\cos(t^2 y)$ for $0 \le t \le 2$

$$\left|\frac{\partial f}{\partial y}\right| = \left|t^3 \sin(t^2 y)\right| \leq t^3 = 8.$$

$$\left|\frac{\partial f}{\partial y}\right| \leq 8.$$

[2] (b) State the existence and uniqueness theorem for initial value problems (IVPs).

Consider the IVP $\frac{dy}{dt} = f(t, y)$; $y(a) = \alpha$ and $\underbrace{a \leq t \leq b \text{ and } -\infty \leq y \leq \infty}_{\text{Domain } D}$.

1. If $f(t,y)$ is continuous on domain D and
2. $f(t,y)$ satisfies a Lipschitz condition, then there exists a ! Solu^n to the IVP.

[2] (c) Prove that the initial value problem:

$$\frac{dy}{dt} = 2 + t\cos(t^2 y), \quad y(0) = 1$$

has a unique solution for t in $[0, 2]$.

1. $f(t,y) = 2 + t\cos(t^2 y)$ is continuous and
2. It satisfies the Lipschitz condition ((a)(ii))

$\therefore$ the IVP has a unique solu^n.

5.

[2] (a) State Euler's method for computing an approximate solution to the IVP:

$$\frac{dy}{dt} = f(t, y) \quad \text{for } a \leq t \leq b \text{ with } y(a) = \alpha.$$

$$y_{i+1} = y_i + f(t_i, y_i)$$

$$y(a) = \alpha.$$

[4] (b) Derive the Euler's method from a truncated Taylor series expansion.

Consider the Taylor Series expansion for $y(t)$ about $t = t_i$.

$$y(t) = y(t_i) + \frac{y'(t_i)(t - t_i)}{1!} + \frac{y''(\xi_i)(t - t_i)^2}{2!}$$

Then,

$$y(t_i + h) = y(t_i) + \frac{f(t_i, y_i)\,h}{1!} + \frac{y''(c_i)\,h^2}{2!}$$

$$\Rightarrow y(t_i + h) \approx y(t_i) + f(t_i, y_i)h = y_{i+1}.$$

[6] (c) Use Euler's method with step size $h = 0.5$ to compute an approximate solution to the IVP:

$$\frac{dy}{dt} = 1 - t^2 + y \quad \text{for } 0 \leq t \leq 1.5 \text{ with } y(0) = 1.$$

Be sure to label your approximations for $y(0.5)$, $y(1.0)$, and $y(1.5)$.

y_0	y_1	y_2	y_3
0	0.5	1.0	1.5

$$y_{i+1} = y_i + h\left[1 - t_i^2 + y_i\right] \qquad y_0 = 1.$$

$$y_1 = y_0 + 0.5\left[1 - 0 + 1\right] = \boxed{2}$$

$$y_2 = y_1 + 0.5\left[1 - (0.5)^2 + 2\right]$$

$$y_2 = 2 + 0.5\left[1 - (0.5)^2 + 2\right] = \boxed{3.375}$$

$$y_3 = y_2 + 0.5\left[1 - 1^2 + y_2\right]$$

$$= 3.375 + 0.5\left[1 - 1^2 + 3.375\right]$$

$$= \boxed{5.0625}$$

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1.

$\boxed{7}$ (a) Do 1-step of $RK4$ with step size $h = 1$ to approximate the solution to the IVP:

$$\frac{dy}{dt} = 4y \text{ with } y(0) = 1.$$

Show all your intermediate steps.

$y_{i+1} = y_i + \frac{h}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$t_0 = 0$ $y_0 = 1$

$f(t, y) = 4y$.

$k_1 = f(t_i, y_i)$

$k_2 = f(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_1)$

$k_3 = f(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_2)$

$k_4 = f(t_i + h, y_i + hk_3)$

$k_1 = f(t_0, y_0) = 4$

$k_2 = f(t_0 + \frac{1}{2}, 1 + \frac{1}{2}(4)) = 12$

$k_3 = f(t_0 + \frac{1}{2}, 1 + \frac{1}{2}(12)) = 28$.

$k_4 = f(t_0 + 1, 1 + 28) = 116$.

$y_1 = y_0 + \frac{1}{6}\left[4 + 2(12) + 2(28) + 116\right] = 34.3333$

$\boxed{2}$ (b) Find the absolute error in approximating $y(1)$ by y_1 using the actual solution $y = e^{4t}$.

$y(1) = e^4$.

absolute error $= |e^4 - 34.3333| = 20.26$.

2. Consider the linear system of equations:

$$\begin{aligned} 2x_1 \quad - \quad x_2 \quad &= 1 \\ -x_1 \quad + \quad 2x_2 \quad &= 1 \end{aligned} \tag{1}$$

$\boxed{3}$ (a) Formulate the Jacobi iteration to solve the system (1).

$\boxed{4}$ (b) Express the Jacobi iteration in the matrix form as $\mathbf{x}^{(k+1)} = M\,\mathbf{x}^{(k)} + \mathbf{b}$.

$\boxed{4}$ (c) Perform 2-iterations with starting vector $\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

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1.

[7] (a) Do 1-step of $RK4$ with step size $h = 1$ to approximate the solution to the IVP:

$$\frac{dy}{dt} = 4y \quad \text{with} \quad y(0) = 1.$$

Show all your intermediate steps.

[2] (b) Find the absolute error in approximating $y(1)$ by y_1 using the actual solution $y = e^{4t}$.

2. Consider the linear system of equations:

$$\begin{aligned} 2x_1 \; - \; x_2 &= 1 \\ -x_1 \; + \; 2x_2 &= 1 \end{aligned} \tag{1}$$

$\boxed{3}$ (a) Formulate the Jacobi iteration to solve the system (1).

$$x_1 = \frac{1}{2} + \frac{1}{2} x_2 \quad \Rightarrow \quad x_1^{(k+1)} = \frac{1}{2} + \frac{1}{2} x_2^{(k)}$$

$$x_2 = \frac{1}{2} + \frac{1}{2} x_1 \quad \Rightarrow \quad x_2^{(k+1)} = \frac{1}{2} + \frac{1}{2} x_1^{(k)}$$

$\boxed{4}$ (b) Express the Jacobi iteration in the matrix form as $\mathbf{x}^{(k+1)} = M\,\mathbf{x}^{(k)} + \mathbf{b}$.

$$\vec{x}^{(k+1)} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \vec{x}^{(k)} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$\boxed{4}$ (c) Perform 2-iterations with starting vector $\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

1st iteration

$x_1^{(1)} = \frac{1}{2}$

$x_2^{(1)} = \frac{1}{2}.$

2nd iteration

$$\begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$x_1^{(2)} = \frac{3}{4}$

$x_2^{(2)} = \frac{3}{4}$

3 (d) Formulate the Gauss-Seidel iteration to solve the system (2).

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ -x_1 + 2x_2 &= 1 \end{aligned} \tag{2}$$

$x_1^{(k+1)} = \frac{1}{2} + x_2^{(k)}$

$x_2^{(k+1)} = \frac{1}{2} + \frac{1}{2} x_1^{(k+1)}$

4 (e) Now perform 1-iteration with starting vector $\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$x_1^{(1)} = \frac{1}{2}$

$x_2^{(1)} = \frac{3}{4}$

$\boxed{6}$ 3. Use Gaussian elimination with back-substitution to solve the system:

$$\begin{aligned} x_1 &+ x_2 &+ x_3 &= 6 \\ x_1 &+ 2x_2 &+ 2x_3 &= 9 \\ x_1 &+ 2x_2 &+ 3x_3 &= 10 \end{aligned}$$

Clearly indicate the elementary row operation used in each step.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 2 & 9 \\ 1 & 2 & 3 & 10 \end{array}\right]$$

$E_2 \leftarrow E_2 - E_1$ and then $E_3 \leftarrow E_3 - E_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 2 & 4 \end{array}\right]$$

$E_3 \leftarrow E_3 - E_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\boxed{x_3 = 1}$

$x_2 + x_3 = 3$

$\boxed{x_2 = 2}$

$x_1 + x_2 + x_3 = 6.$

$x_1 + 2 + 1 = 6$

$\boxed{x_1 = 3}$

4.

4 (a) Derive the n^{th}–order Taylor's method for the IVP: $\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha.$

See Notes!

5 (b) Consider the differential equation given by

$$\frac{dy}{dt} = 16\,t\,y \quad \text{with} \quad y(0) = 1.$$

$f(t,y) = 16ty.$

$f' = 16[ty' + y]$

Do 1-step of Taylor's method of order 2 with step size $h = 0.5$ to approximate the solution $y(0.5)$.

$y_{i+1} = y_i + h f(t_i, y_i) + \frac{h^2}{2} f'(t_i, y_i)$

$f' = 16[t(16ty) + y]$

$y_1 = y_0 + h f(t_0, y_0) + \frac{h^2}{2} 16[t_0(16t_0y_0) + y_0]$

$y_1 = 1 + \frac{1}{2}(0) + \frac{1}{\cancel{2}}\left(\frac{1}{\cancel{4}}\right) \times \overset{2}{\cancel{16}}[0 + 1]$

$\boxed{y_1 = 3}$

$\boxed{8}$ 5. Do **two** iterations by the power method with $\vec{x}^{(0)} = [0,1]^T$ to approximate the dominant eigenvector of $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ with the corresponding eigenvalue.

$$\vec{x}_1 = \frac{A\vec{x}_0}{\|A\vec{x}_0\|_\infty} = \frac{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\| \cdot \|_\infty} = \frac{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}{2} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\vec{x}_2 = \frac{A\vec{x}_1}{\|A\vec{x}_1\|_\infty} = \frac{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}}{\| \cdot \|_\infty} = \frac{\begin{bmatrix} -2 \\ 5/2 \end{bmatrix}}{5/2} = \begin{bmatrix} -\frac{4}{5} \\ 1 \end{bmatrix}$$

$$\lambda_1^{(1)} = \frac{\vec{x}_1^T A \vec{x}_1}{\vec{x}_1^T \vec{x}_1} = \frac{\begin{bmatrix} -\frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}}{\begin{bmatrix} -\frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}} = \frac{3.50}{1.25} = \boxed{2.80}$$

$$\lambda_1^{(2)} = \frac{\vec{x}_2^T A \vec{x}_2}{\vec{x}_2^T \vec{x}_2} = \frac{\begin{bmatrix} -\frac{4}{5} & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -\frac{4}{5} \\ 1 \end{bmatrix}}{\begin{bmatrix} -\frac{4}{5} & 1 \end{bmatrix}\begin{bmatrix} -\frac{4}{5} \\ 1 \end{bmatrix}} = \frac{4.88}{1.64} = \boxed{2.98}$$