THURSDAY JULY 18, 2019

CH. 4 NUMERICAL DIFFERENTIATION AND INTEGRATION

4.1 NUMERICAL DIFFERENTIATION (CONT.)

· Error in CDF

$$f(x_0+h) = f(x_0) + \frac{f'(x_0)h}{1!} + \frac{f''(x_0)h^2}{2!} + \frac{f'''(\xi_1)h^3}{3!} \quad — ①$$

$$f(x_0-h) = f(x_0) - \frac{f'(x_0)h}{1!} + \frac{f''(x_0)h^2}{2!} - \frac{f'''(\xi_2)h^3}{3!} \quad — ②$$

$$\text{CDF} - \quad f'(x_0) \approx \frac{f(x_0+h) - f(x_0-h)}{2h}$$

① − ②

$$f(x_0+h) - f(x_0-h) = 2f'(x_0)h + \frac{[f'''(\xi_1) + f'''(\xi_2)]h^3}{3!}$$

$$\frac{f(x_0+h) - f(x_0-h)}{2h} = f'(x_0) + \frac{[f'''(\xi_1) + f'''(\xi_2)]}{2} \cdot \frac{1}{3!} \cdot \frac{h^2}{1}$$

$$-\frac{[f'''(\xi_1) + f'''(\xi_2)]}{2} \frac{h^2}{3!} = f'(x_0) - \frac{f(x_0+h) - f(x_0-h)}{2h}$$

using the IVT, $f'''(\xi) = \dfrac{f'''(\xi_1) + f'''(\xi_2)}{2}$

Finally, we have

$$-\frac{f'''(\xi)h^2}{6} = f'(x_0) - \left[\frac{f(x_0+h) - f(x_0-h)}{2h}\right]$$

Then, the max^m error in CDF is

$$\boxed{\max_{\xi \in [x_0-h,\, x_0+h]} \left|\frac{f'''(\xi)h^2}{6}\right|}$$

EX. Consider $f(x) = xe^x$. Find the maxi$^{\underline{m}}$ error in approximating $f'(1)$ by FDF, BDF, and CDF with $h=0.2$.

$f(x) = xe^x$

$f'(x) = e^x + xe^x$

$f''(x) = e^x + e^x + xe^x$

$= e^x(x+2)$

$f'''(x) = e^x + e^x(x+2)$

$= e^x(x+3)$

FDF:

$$\max_{\xi\in[x_0, x_0+h]} \left| \frac{f''(\xi)h}{2!} \right|$$

$$= \max_{\xi\in[1,1.2]} \frac{[e^{\xi}(\xi+2)](0.2)}{2!}$$

$$= \frac{e^{1.2}(1.2+2)(0.2)}{2!} = \boxed{1.0624}$$

BDF:

$$\max_{\xi\in[x_0-h, x_0]} \left| \frac{f''(\xi)h}{2!} \right| = \max_{x\in[0.8,1]} \left| \frac{e^{\xi}(\xi+2)(0.2)}{2!} \right| = \frac{e^{1}(1+2)(0.2)}{2!}$$

$$= \boxed{0.8155}$$

CDF:

$$\max_{\xi\in[x_0-h, x_0+h]} \left| \frac{f'''(\xi)h^2}{6} \right| = \max_{\xi\in[0.8,1.2]} \left| \frac{e^{\xi}(\xi+3)h^2}{6} \right| = \frac{e^{1.2}(1.2+3)(0.2)^2}{6}$$

$$= \boxed{0.09296}$$

- Three-point formulas

  - 3-point FDF

$$f'(x_0) \approx \frac{[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)]}{2h}$$

  - 3-point BDF

$$f'(x_0) \approx \frac{[3f(x_0) - 4f(x_0-h) + f(x_0-2h)]}{2h}$$

ex. From the following table approximate $f'(1)$ by the 3-point FDF and BDF.

$\uparrow$ x_0

x	0.6	0.8	$\overset{x_0}{\boxed{1.0}}$	1.2	1.4
$f(x)$	0.65	1.42	2.71	4.78	7.94

$h = 0.2$

$(0.8 - 0.6) = 0.2$

FDF $\Rightarrow$ $f'(1) \approx \dfrac{[-3f(1) + 4f(1.2) - f(1.4)]}{2(0.2)}$

$$\approx \frac{[-3(2.71) + 4(4.78) - 7.94]}{2(0.2)} = \boxed{7.625}$$

BDF $\Rightarrow$ $f'(1) \approx \dfrac{[3f(1) - 4f(0.8) + f(0.6)]}{2(0.2)}$

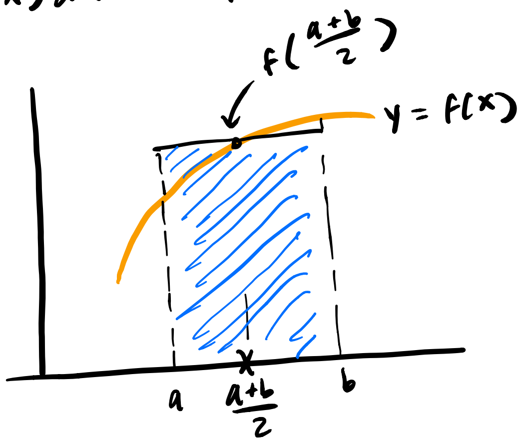
$$\approx \frac{3(2.71) - 4(1.42) + 0.65}{2(0.2)} = \boxed{7.75}$$

4.3 ELEMENTS OF NUMERICAL INTEGRATION

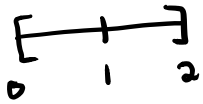
The basic method involved in approximating $\int_a^b f(x)\,dx$ is called numerical quadrature. Sometimes, it is hard to calculate a definite integral analytically. ex. $\int_0^1 e^{x^2}dx$

- Mid-point rule

$$\int_a^b f(x)\,dx \approx f\left(\frac{a+b}{2}\right)(b-a)$$



ex. Use the midpoint rule to approximate



$$\int_0^2 \underset{\uparrow}{x^2}\,dx \approx f(1)(2-0) = (1)^2(2) = 2$$

$f(x)$