MONDAY JULY 29, 2019

CH. 5 DIFFERENTIAL EQUATIONS

5.3 Higher-Order Taylor Methods

NOTE: Euler's m/d was derived by dropping the $\theta(h^2)$ term in the Taylor series expansion of $y(t)$ about $t = t_i$. One can derive methods with higher-order by retaining more terms in the Taylor series expansion.

$$y(t) = y(t_i) + \frac{y'(t_i)(t-t_i)}{1!} + \frac{y''(t_i)(t-t_i)^2}{2!} + \frac{y'''(t_i)(t-t_i)^3}{3!} + \dots + \frac{y^n(t_i)(t-t_i)^n}{n!} + \frac{y^{(n+1)}(\xi)(t-t_i)^{n+1}}{(n+1)!}$$

$t = t_i + h$

$$y(t_i+h) = y(t_i) + \frac{y'(t_i)h}{1!} + \frac{y''(t_i)h^2}{2!} + \dots + \frac{y^{(n)}(t_i)h^n}{n!} + \frac{y^{n+1}(\xi)h^{n+1}}{(n+1)!} \quad \text{drop}$$

$$y(t_i+h) \approx y(t_i) + \frac{y'(t_i)h}{1!} + \frac{y''(t_i)h^2}{2!} + \dots + \frac{y^n(t_i)h^n}{n!}$$

But $y' = f(t,y)$

n^{th}-order Taylor method:

$$\therefore \boxed{y(t_i+h) \approx y(t_i) + \underbrace{\frac{h}{1!}f(t_i,y_i)}_{\text{Euler's m/d}} + \underbrace{\frac{h^2}{2!}f'(t_i,y_i)}_{\text{2nd order Taylor's m/d}} + \underbrace{\frac{h^3}{3!}f''(t_i,y_i)}_{\text{3rd order Taylor's m/d}} + \dots + \frac{h^n}{n!}f^{n-1}(t_i,y_i)}$$

ex 1) use Taylor's m/d of order 2 with step size $h = 0.5$ to approx. the solu$^{\underline{n}}$ of the ivp

$$\frac{dy}{dt} = te^{3t} - 2y \qquad 0 \le t \le 1 \qquad y(0) = 0$$

$$f(t, y) = te^{3t} - 2y$$

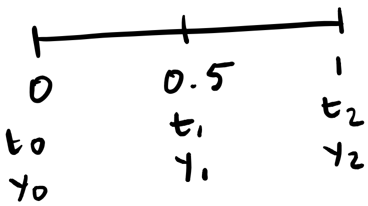
$$f'(t, y) = t \cdot 3e^{3t} + e^{3t} - 2y'$$

$$= 3te^{3t} + e^{3t} - 2\left[te^{3t} - 2y\right]$$

$$f'(t, y) = e^{3t} + te^{3t} + 4y$$

$$y_{i+1} = y_i + hf(t_i, y_i) + \frac{h^2}{2!} f'(t_i, y_i)$$

$$\boxed{y_{i+1} = y_i + h\left[t_i e^{3t_i} - 2y\right] + \frac{h^2}{2!}\left[e^{3t_i} + t_i e^{3t_i} + 4y_i\right]}$$



$h = 0.5$

$(t_0, y_0) \Rightarrow (0, 0)$

$$y_1 = 0 + (0.5)\left[0 \cdot e^{3 \cdot 0} - 2 \cdot 0\right] + \frac{(0.5)^2}{2!}\left[e^{3 \cdot 0} + 0 \cdot e^{3 \cdot 0} + 4 \cdot 0\right]$$

$y_1 = 0.125$

$(t_1, y_1) \Rightarrow (0.5, 0.125)$

$$y_2 = 0.125 + (0.5)\left[0.5 \cdot e^{3 \cdot 0.5} - 2 \cdot 0.125\right] + \frac{(0.5)^2}{2!}\left[e^{3 \cdot 0.5} + 0.5 \cdot e^{3 \cdot 0.5} + 4 \cdot 0.125\right]$$

$y_2 = 2.02324$

ex 2) Using 3rd-order Taylor's m/d w/ $h = 0.5$. "set up" an iteration formula for $\{y_i\}$ to approx. the solution of the ivp

$$\frac{dy}{dt} = t^2 - y \qquad 0 \le t \le 2 \qquad y(0) = 1$$

$$f(t, y) = t^2 - y$$

$$f'(t, y) = 2t - (t^2 - y)$$

$$f''(t, y) = 2 - 2t + y'$$

$$= 2 - 2t + (t^2 - y)$$

$$\boxed{y_{i+1} = y_i + 0.5\left[t_i^2 - y_i\right] + \frac{0.5^2}{2!}\left[2t_i - t_i^2 + y_i\right] + \frac{0.5^3}{3!}\left(2 - 2t_i + t_i^2 - y_i\right)}$$

5.4 Runge-Kutta m/ds (RK-m/ds)

In the Euler's m/d, we use the information on the slope of y at a given time-step to predict the soln to the next time step.

- RK-m/ds are a class of techniques that use the information on the "slope" at "more than one point" to predict the solution to the future time step (y_{i+1})
  - Recall IVP:

$$\frac{dy}{dt} = f(t,y) \qquad \begin{array}{l} a \le t \le b \\ y(a) = y_0 \end{array}$$

The general form of the approximate soln:

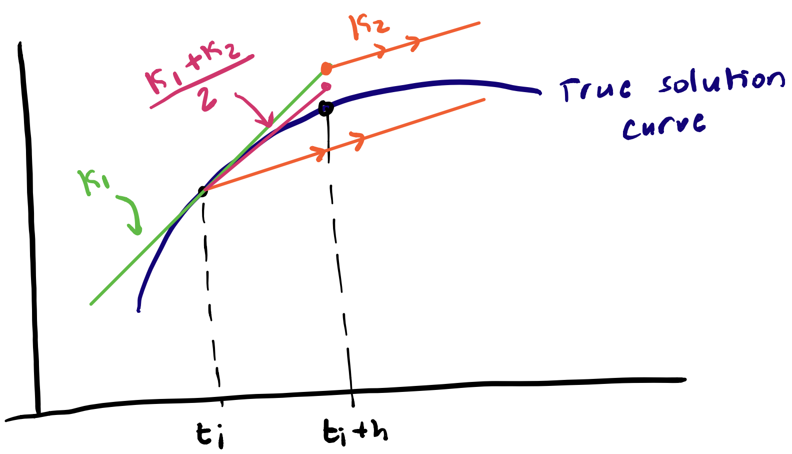
$$y_{i+1} = y_i + h\underbrace{[\overset{\text{psi}}{\varphi}(t_i, y_i)]}_{\text{increment fun}^n}$$

* $\varphi(t_i, y_i)$ is essentially a suitable slope over the interval $[t_i, t_{i+1}]$

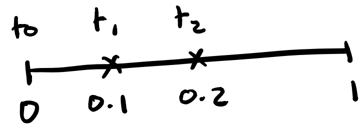
- second-order RK-m/ds (RK-2)

(1) Modified-Euler method

$$y_{i+1} = y_i + h\underbrace{\left[\frac{K_1+K_2}{2}\right]}_{\substack{\text{average of} \\ \text{2 slopes}}} \qquad \text{where } \begin{array}{l} K_1 = f(t_i, y_i) \\ K_2 = f(t_i+h,\ y_i + K_1 h) \\ \quad = f(t_i+h,\ y_i + h\underbrace{f(t_i, y_i)}_{K_1}) \end{array}$$



ex 1) Consider the IVP:



$$\frac{dy}{dt} = 2t^2 + t^2 y \qquad 0 \le t \le 1$$
$$= t^2(2+y) \qquad y(0)=1$$
$$h = 0.1$$

Compute the estimated value of y_1 using the Modified-Euler m/d.

$t_0 = 0,\ y_0 = 1$

$$y_{i+1} = y_i + h\left[\frac{k_1+k_2}{2}\right]$$

$$y_1 = y_0 + 0.1\left[\frac{0+0.03}{2}\right]$$
$$= 1 + 0.1\left(\frac{0.03}{2}\right) = 1.0015$$

$k_1 = f(t_0, y_0) \Rightarrow t^2(2+y)$

$f(0,1) \quad k_1 = 0^2(2+1) = 0$

$k_2 = f(t_i + h, y_i + k_1 h)$

$f(t_0 + h, y_0 + k_1 h)$

$f(0.1, 1) = 0.1^2(2+1) = 0.03$

. . . .

$k_1 = f(t_1, y_1)$

$f(0.1, 1.0015) = 0.1^2(2+1.0015) = 0.030015$

$k_2 = f(0.2, 1.0015 + 0.030015(0.1))$

$f(0.2, 1.0045)$

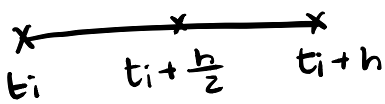
$= 0.2^2(2+1.0045) = 0.12018$

$$y_2 = y_1 + h\left[\frac{k_1+k_2}{2}\right]$$
$$= 1.0015 + 0.1\left[\frac{0.030015+0.12018}{2}\right]$$
$$= 1.009$$

* calculate new k_1, k_2 for each iteration

(2) Mid-Point m/d

$$y_{i+1} = y_i + hk_2 \quad \text{where} \quad k_1 = f(t_i, y_i)$$
$$k_2 = f\left(t_i + \frac{h}{2},\ y_i + \frac{h}{2}k_1\right)$$



. Runge-Kutta m/d of order four (RK4)

$$y_{i+1} = y_i + \frac{h}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

where $k_1 = f(t_i, y_i)$

$k_2 = f\left(t_i + \frac{h}{2},\ y_i + \frac{h}{2}k_1\right)$

$k_3 = f\left(t_i + \frac{h}{2},\ y_i + \frac{h}{2}k_2\right)$

$k_4 = f(t_i + h,\ y_i + hk_3)$