HW 4.

Problem 1

(a) Please see the notes

b)

$$f(x) = \underbrace{f(x_0) + \frac{f'(x_0)(x-x_0)}{1!} + \frac{f''(x_0)(x-x_0)^2}{2!} + \frac{f'''(x_0)(x-x_0)^3}{3!}}_{\text{degree 3 Taylor Polynomial}} + \underbrace{\frac{f^{(iv)}(c)(x-x_0)^4}{4!}}_{\text{Taylor Remainder.}}$$

$f(x) = \ln x \qquad x_0 = 1$

$f'(x) = \frac{1}{x}$

$f''(x) = \frac{-1}{x^2}$

$f'''(x) = \frac{2}{x^3}$

$f^{(iv)}(x) = \frac{-6}{x^4}.$

$$P_3(x) = \ln(1) + \left(\frac{1}{1}\right)\frac{(x-1)}{1!} + \left(\frac{-1}{1}\right)\frac{(x-1)^2}{2!} + \left(\frac{2}{1}\right)\frac{(x-1)^3}{3!}$$

$$P_3(x) = (x-1) - \frac{(x-1)^2}{2} + 2\frac{(x-1)^3}{3!}.$$

(c) $\ln(1.1) \approx (1.1-1) - \frac{(1.1-1)^2}{2} + 2\frac{(1.1-1)^3}{3!}$

$$\approx 0.1 - \frac{0.01}{2} + \frac{2}{6}(0.001)$$

$$\approx 0.0953$$

(d) Taylor Remainder $= \frac{f^{(iv)}(c)(x-x_0)^4}{4!}$

$$= \frac{-6}{c^4}\frac{(x-1)^4}{4!},$$

where $c \in (1, x)$

HW4

Problem 2

(a) $$P_2(x) = \frac{(x-2)(x-3)}{(0-2)(0-3)}[1] + \frac{(x-0)(x-3)}{(2-0)(2-3)}[3] + \frac{(x-0)(x-2)}{(3-0)(3-2)}[0]$$

(b) is simalar to part (a)

(c) $$P_3(x) = \frac{(x-2)(x-3)(x-5)}{(-1-2)(-1-3)(-1-5)}[0] + \frac{(x+1)(x-3)(x-5)}{(2+1)(2-3)(2-5)}[1] +$$

$$\frac{(x+1)(x-2)(x-5)}{(3+1)(3-2)(3-5)}[1] + \frac{(x+1)(x-2)(x-3)}{(5+1)(5-2)(5-3)}[2]$$

Problem 3.

(a) $$P_3(x) = \frac{(x-1850)(x-1900)(x-2000)}{(1800-1850)(1800-1900)(\cancel{1}800-200)}[280] +$$

$$\frac{(x-1800)(x-1900)(x-2000)}{(50)(-50)(-150)}[283] +$$

$$\frac{(x-1800)(x-1850)(x-2000)}{100\,(50)(-100)}[291] +$$

$$\frac{(x+1800)(x-1850)(x-1900)}{(200)(150)(100)}[370]$$

(b) $P_3(1950)$

$P_3\left(\overset{2050}{\cancel{3110}}\right)$

Problem 4

$$\left| f(x) - P_2(x) \right| = \left| \frac{f^{(3)}(\xi)}{3!} (x+1)\, x\, (x-1) \right| \quad \text{on } [-1, 1]$$

$$f(x) = e^x \qquad f^{(3)}(x) = e^x.$$

$$\left| f(x) - P_2(x) \right| \leq \frac{\left| f^{(3)}(\xi) \right|}{6} \left| x^3 - x \right| \quad \text{on } [-1, 1]$$

$$\max \left| f^{(3)}(\xi) \right| = e^1$$

Let $h(x) = x^3 - x$. Now, let's find the maxi^{m} for $h(x)$

$$h'(x) = 3x^2 - 1 = 0$$

$$x = \pm \frac{1}{\sqrt{3}}$$

$$x = \frac{1}{\sqrt{3}} \Rightarrow \frac{1}{(\sqrt{3})^3} - \frac{1}{\sqrt{3}}$$

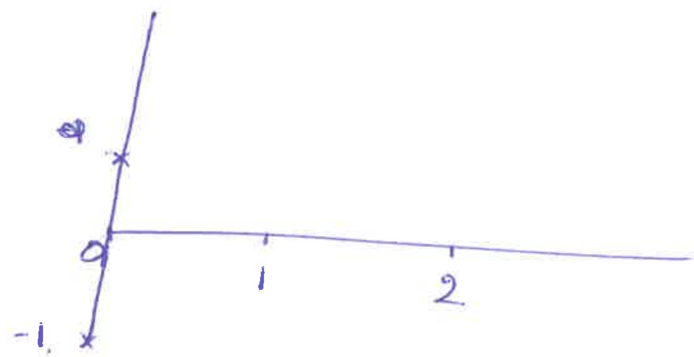
$$x = \frac{-1}{\sqrt{3}} \Rightarrow \frac{-1}{\sqrt{3}^3} + \frac{1}{\sqrt{3}}$$

$$\therefore \max_{x \in [-1,1]} \left| h(x) \right| = \left| \frac{3-1}{3\sqrt{3}} \right| = \frac{2}{3\sqrt{3}}$$

$$\therefore \left| f(x) - P_2(x) \right| \leq \frac{e}{6} \frac{2}{3\sqrt{3}}.$$

Problem 5.

(a). $$S(x) = \begin{cases} S_0(x) = x^3 + x - 1 \\ S_1(x) = 1 + 3(x-1) + 3(x-1)^2 - (x-1)^3 \end{cases}$$



$S_0(1) = 1.$ $S_1(1) = 1.$

$\Rightarrow \boxed{S_0(1) = S_1(1)}$ ok ✓.

$S_0'(1) \overset{?}{=} S_1'(1)$

$S_0'(x) = 3x^2 + 1 \Rightarrow S_0'(1) = 4$

$S_1'(x) = 3 + 6(x-1) - 3(x-1)^2 \Rightarrow S_1'(1) = 3$

$\Rightarrow S_0'(1) \neq S_1'(1)$

Not a cubic spline.

(b)

$$S(x) = \begin{cases} S_0(x) = 2x^3 + x^2 + 4x + 5 & \text{on } [0,1] \\ S_1(x) = 12 + 12(x-1) + 7(x-1)^2 + (x-1)^3 & \text{on } [1,2] \end{cases}$$

* Continuity of $S_0(x)$ and $S_1(x)$ at $x = 1$.

$S_0(1) = 2 + 1 + 4 + 5 = 12$

$S_1(1) = 12$

$S_0(1) = S_1(1)$

* Continuity of the 1st derivatives at $x = 1$.

$S_0'(x) = 6x^2 + 2x + 4$

$S_1'(x) = 12 + 14(x-1) + 3(x-1)^2$

$S_0'(1) = 12 = S_1'(1)$

* Continuity of the 2nd derivative at $x = 1$

$S_0''(x) = 12x + 2$

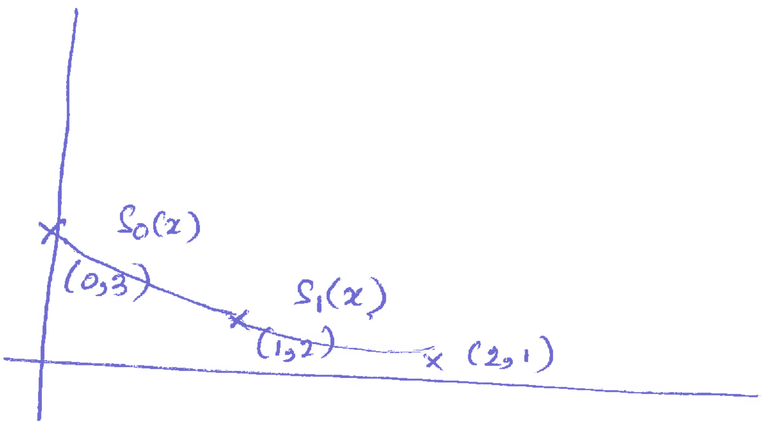
$S_1''(x) = 14 + 6(x-1)$

$S_0''(1) = S_1''(1) = 14$

$\Rightarrow$ Cubic spline!

Problem 6

Natural cubic spline through $(0,3)$, $(1,2)$, and $(2,1)$



$$S_0(x) = a_0 + b_0(x+0) + c_0(x-0)^2 + d_0(x-0)^3 = a_0 + b_0x + c_0x^2 + d_0x^3$$

$$S_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3$$

$S_0(0) = 3$

$$\boxed{a_0 = 3} \quad \text{—} (1)$$

$S_0(1) = 2$

$$\boxed{a_0 + b_0 + c_0 + d_0 = 2} \quad \text{—} (2)$$

$S_1(1) = 2$

$$\boxed{a_1 = 2} \quad \text{—} (3)$$

$S_1(2) = 1$

$$\boxed{a_1 + b_1 + c_1 + d_1 = 1} \quad \text{—} (4)$$

$$S_0'(x) = b_0 + 2c_0x + 3d_0x^2$$

$$S_1'(x) = b_1 + 2c_1(x-1) + 3d_1(x-1)^2$$

Continuity of the 1st derivative

$$S_0'(1) = S_1'(1)$$

$$\boxed{b_0 + 2c_0 + 3d_0 = b_1} \quad \text{— (5)}$$

$$S_0''(x) = 2c_0 + 6d_0 x$$

$$S_1''(x) = 2c_1 + 6d_1(x-1)$$

Continuity of the 2nd derivative

$$S_0''(1) = S_1''(1)$$

$$\boxed{2c_0 + 6d_0 = 2c_1} \quad \text{— (6)}$$

Naturall BC^s

$$S_0''(0) = 0$$

$$\boxed{2c_0 = 0.} \quad \text{— (7)}$$

$$S_1''(2) = 0$$

$$\boxed{2c_1 + 6d_1 = 0} \quad \text{— (8)}$$