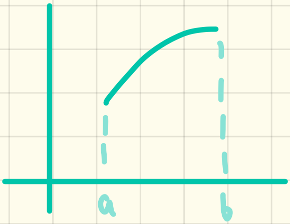
Simpson's Rule

error: $\int_{x_0}^{x_1} f(x)\,dx = \frac{x_1 - x_0}{6}\left[f(x_0) + 4f\left(\frac{x_0+x_1}{2}\right) + f(x_1)\right]$

$$-\frac{(x_1-x_0)^5}{180} f^{4}(c)$$



Weighted Mean Value Theorem for Integrals

Suppose $f(x)$ is continuous at $[x_0, x_1]$ the integral of g exists on $[x_0, x_1]$ and $g(x)$ does not change sign on $[x_0, x_1]$ then there exists $c \in [x_0, x_1] \ni$

$$\int_{x_0}^{x_1} g(x) f(x)\,dx = f(c) \int_{x_0}^{x_1} g(x)\,dx$$

Error of Trapezoidal Rule

$$f(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) + (x - x_0)(x - x_1)\frac{f''(c(x))}{2!}$$

$$\int_{x_0}^{x_1} f(x)\,dx = \underbrace{\int_{x_0}^{x_1} \frac{x - x_1}{x_0 - x_1} f(x_0) + \int_{x_0}^{x_1} \frac{x - x_0}{x_1 - x_0} f(x_1)}_{\cdots} + \int_{x_0}^{x_1} \underbrace{(x - x_0)(x - x_1)}_{g(x)} \frac{f''(c(x))}{2!}\,dx$$

$$= \cdots + \frac{f''(c)}{2} \int_{x_0}^{x_1} (x - x_0)(x - x_1)\,dx \quad \text{by WMVT}$$

$$\Rightarrow \cdots + \frac{f''(c)}{12}(x_0 - x_1)^3 \leftarrow$$ error for trapezoidal Rule

example) Approximate $\int_0^1 x^3 dx$ by mid point, trapezoidal, & Simpson's rule, then estimate the error.

Mid point: $\int_{x_0}^{x_1} f(x)dx \approx f\left(\frac{x_0+x_1}{2}\right)(x_1-x_0)$

$$\approx f\left(\frac{1+0}{2}\right)(1-0) = f\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^3 = \frac{1}{8}$$

trapezoidal: $\int_{x_0}^{x_1} f(x)dx \approx \frac{(x_1-x_0)}{2}\left[f(x_1)-f(x_0)\right]$ + error

$$\approx \frac{(1-0)}{2}\left[(1)^3-(0)^3\right] = \frac{1}{2}[1] = \frac{1}{2}$$

Simpson's: $\int_{x_0}^{x_1} f(x)dx \approx \frac{(x_1-x_0)}{6}\left[f(x_1)-4f\left(\frac{x_1+x_0}{2}\right)+f(x_0)\right]$

$$\approx \left(\tfrac{1}{6}\right)\left[f(1)-4f\left(\tfrac{1}{2}\right)+f(0)\right]$$

$$= \frac{1}{6}\left[1-\frac{1}{2}+0\right] = \frac{1}{6}\left(\frac{1}{2}\right) = \frac{1}{12}$$

Trapezoidal error: $\frac{f''(c)}{12}(x_1-x_0)^3$

c = a number between x_0 & x_1

$f''(x) = 6x$

$$= \frac{f''(c)}{12}(1)^3 = \frac{6c}{12} = \frac{1}{2}c$$

$x_1 = 1$

upper bound error $= \frac{1}{2}(1) = \frac{1}{2}$

Simpson's error: $\frac{(x_1-x_0)^5}{180} f^4(c)$

$= 0$

because $f(x)$ is not greater than degree 3 so $f^4(x) = 0$