Solutions 1-4

Problem 1

Show that the following equations have at least one solution on the given interval.

(a) $x \cos x - 2x^2 + 3x - 1 = 0$, $[0.2, 0.3]$

Using the I.V.T., we find,

$$f(0.2) = x \cos x - 2x^2 + 3x - 1 = (0.2)\cos 0.2 - 2(0.2)^2 + 3(0.2) - 1 = -0.2840$$
$$f(0.3) = x \cos x - 2x^2 + 3x - 1 = (0.3)\cos 0.3 - 2(0.3)^2 + 3(0.3) - 1 = 0.0066$$

Thus, by the I.V.T. this expression is shown to have at least one solution on the given interval.

(b) $x - (\ln x)^x = 0$, $[4, 5]$

Same process as above,

$$f(4) = 4 - (\ln 4)^4 = 0.3066$$
$$f(5) = 5 - (\ln 5)^5 = -5.7987$$

Thus, by the I.V.T. this expression is shown to have at least one solution on the given interval.

Problem 2

Find c satisfying the Mean Value Theorem for $f(x)$ on the interval $[0, 1]$.

To find c, we must use the equation:

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{1}$$

(a) $f(x) = e^x$

What is $f(a)$ and $f(b)$?

$$f(0) = e^0 = 1$$
$$f(1) = e^1 = e$$

Now, plugging into Eq. 1,

$$f'(c) = \frac{e-1}{1-0} = 1.7183$$

$f'(x) = e^x$, and hence, $f'(c) = e^c$.
Therefore, we get

$$e^c = 1.7183 \rightarrow c = \ln 1.7183 = 0.5413$$

So we find that $c = 0.5413$.

(b) $f(x) = x^2$

Same process as above,

$$f(0) = 0$$
$$f(1) = 1$$

Into Eq. 1,

$$f'(c) = \frac{1-0}{1-0} = 1$$

So we have,

$$f'(x) = 2x \rightarrow f'(c) = 2c \rightarrow 2c = 1 \rightarrow c = \frac{1}{2}$$

We find that $c = 0.5000$

Problem 3

Find the 5th iteration (c_5) of the Bisection Method to approximate the root of $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$.

$$M = \frac{a+b}{2}$$

We see that $f(5/8) = -0.0204$ and $f(21/32) = 0.0178$ are both close to zero and on opposite sides of the x-axis; consistent with the desired accuracy of 5 iterations.

i	a_i	$f(a_i)$	c_i	$f(c_i)$	b_i	$f(b_i)$
1	0	$-$	1/2	$-$	1	$+$
2	1/2	$-$	3/4	$+$	1	$+$
3	1/2	$-$	5/8	$-$	3/4	$+$
4	5/8	$-$	11/16	$+$	3/4	$+$
5	5/8	$-$	21/32	$+$	3/4	$+$

Problem 4

Find n for which the n^{th} iteration by the Bisection Method guarantees to approximate the root of $f(x) = x^4 - x^3 - 10$ on $[2, 3]$ with accuracy within 10^{-8}.

Note that we need the number iteration, n so that $|r^* - c_n| \leq 10^{-8}$.

Error analysis formula says:

$$|r^* - c_n| \leq \frac{b_1 - a_1}{2^n}, \text{ where } (n \geq 1)$$

So, if we can find the n value that satisfies $\frac{b_1 - a_1}{2^n} \leq 10^{-8}$, then automatically, we have the $n-$value with required accuracy.

We know that $a_1 = 2$ and $b_1 = 3$. So,

$$\frac{3-2}{2^n} \leq 10^{-8}$$
$$\frac{1}{2^n} \leq 10^{-8}$$
$$2^{-n} \leq 10^{-8}$$

Taking logarithm (I choose base 10) of both sides, we get

$$\log_{10}(2^{-n}) \leq \log_{10}(10^{-8})$$
$$-n \log_{10} 2 \leq -8$$
$$-n \leq \frac{-8}{\log_{10} 2}$$
$$n \geq \frac{8}{\log_{10} 2}$$
$$n \geq 26.5754$$
$$n = 27$$

Here, n must be an integer value, so it is shown that 27 iterations must be performed to obtain the desired accuracy.