Order of Convergence

02.08.19

example) Show that the sequence $\{1/2^n\}$ converges to 0 linearly.

$x_n = \frac{1}{2^n}$ $\quad p = 0 \quad$ $x_{n+1} = \frac{1}{2^{n+1}}$

$$* \lim_{n\to\infty} \frac{|x_{n+1} - p|}{|x_n - p|}$$

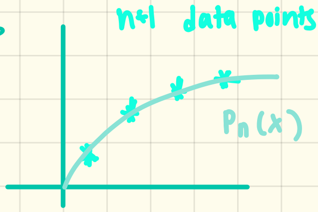
$$= \lim_{n\to\infty} \frac{\left|\frac{1}{2^{n+1}} - 0\right|}{\left|\frac{1}{2^n} - 0\right|^1} \quad (\alpha = 1)$$

$$= \lim_{n\to\infty} \left[\frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}}\right] = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \frac{\cancel{2^n}}{\cancel{2^n}\cdot 2} = \frac{1}{2} = \lambda$$

CHAPTER three:
interpolation & polynomial approximation

Problem: We are given n+1 data points

(x_0, y_0)
(x_1, y_1)
$\vdots$
(x_n, y_n)



Find a polynomial of degree n, $P_n(x)$, such that:

the polynomial fits our data points $\left[\; P_n(x_i) = y_i \qquad i = 0, 1, 2, \ldots n \right.$

$$P_n(x) = \underline{a_n} + \underline{a_1} x + \underline{a_2} x^2 + \ldots + \underline{a_n} x^n$$

n+1 unknowns

example) Given

x_i	y_i
0	1
1	0
2/3	1/2

x_0, x_1, x_2

- 3 unknowns
- 3 equations
- 2nd degree polynomial

$$P_2(x) = a_0 + a_1 x + a_2 x^2$$

* In general - $P_2(x)$

$$P_2(x_0) = a_0 + a_1(0) + a_2(0)^2 = 1$$
$$P_2(x_1) = a_0 + a_1(1) + a_2(1)^2 = 0$$
$$P_2(x_2) = a_0 + a_1(2/3) + a_2(2/3)^2 = 1/2$$

$$\Rightarrow \begin{aligned} a_0 + 0a_1 + 0a_2 &= 1 \\ a_0 + 1a_1 + 1a_2 &= 0 \\ a_0 + 2/3\,a_1 + 4/9\,a_2 &= 1/2 \end{aligned}$$

system of equations

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2/3 & 4/9 \end{bmatrix}}_{X} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1/2 \end{bmatrix}}_{\vec{b}}$$

* Review: Calculating X^{-1}

$$[\,A \mid I\,] \rightarrow [\,I \mid A^{-1}\,]$$

In matlab:
inv(A)

$$\Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = X^{-1}\vec{b} = \begin{bmatrix} 1 \\ -1/4 \\ -3/4 \end{bmatrix}$$

$$\Rightarrow P_2(x) = 1 - \frac{1}{4}x - \frac{3}{4}x^2$$