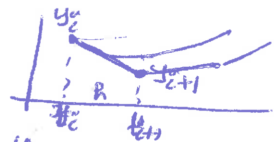
5.4. Runge-Kutta methods

In the Euler method, we used the information on the slope or the derivative of y at a given time step to predict the solu^n^ to the next time-step.



Runge-Kutta methods are a class of ~~methods~~ techniques that we the information on the "slope" at more than one point to predict the solu^n^ to the future time step (y_{i+1}).

Recall the IVP:

$$\frac{dy}{dt} = f(t, y), \qquad y(a) = y_0. \quad a \le t \le b.$$

The general form of the approximate solu^n^:

$$y_{i+1} = y_i + h \underbrace{\varphi(t_i, y_i)}_{\text{increment function}}$$

- $\varphi(t_i, y_i)$ is essentially a suitable slope over the interval $[t_i, t_{i+1}]$.

II Second-order Runge-Kutta methods

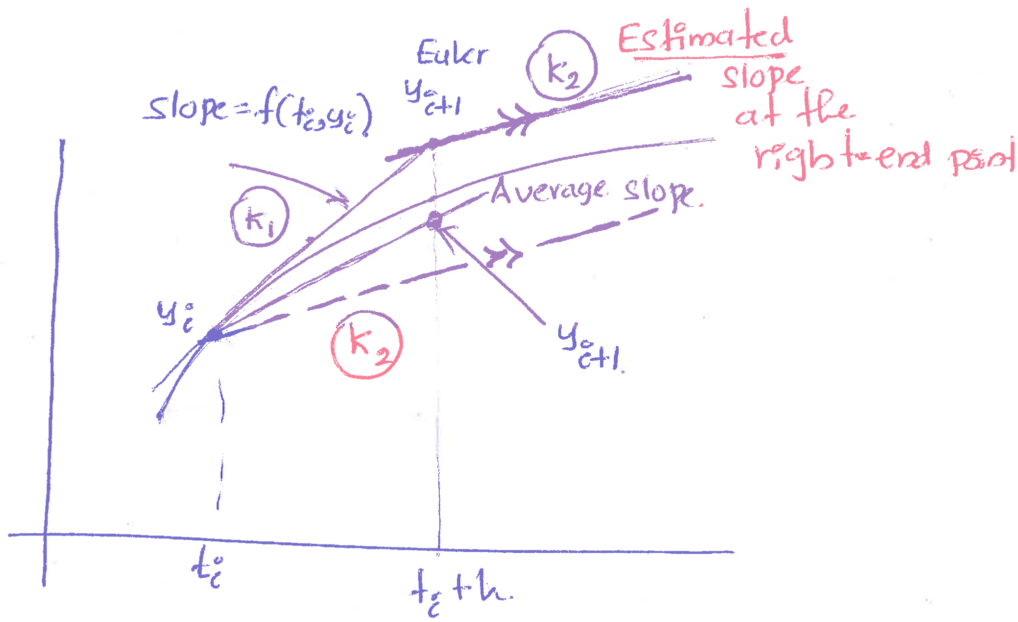
(1) Modified-Euler Method. (Heun's method)

$$y_{i+1} = y_i + h\left[\frac{k_1 + k_2}{2}\right],$$

where $\quad k_1 = f(t_i, y_i)$

$$k_2 = f(t_i + h,\; y_i + k_1 h) = f(t_i + h,\; \underbrace{y_i + h\, f(t_i, y_i)}_{\text{Euler } y_{i+1}})$$

Estimated soln that comes from the Euler's mtd.



(2) Mid-point mtd.

$$y_{i+1} = y_i + h k_2,$$

where $\quad k_1 = f(t_i, y_i)$

$$k_2 = f\left(t_i + \frac{h}{2},\; y_i + \frac{h}{2} k_1\right)$$

ex. Consider the IVP.

$$\frac{dy}{dt} = 2t^2 + t^2 y = t^2(2+y).$$

$0 \leq t \leq 1$, and $y(0) = 1$, $h = 0.1$, (step-size)

Compute the estimated values of $y_1 = y(0.1)$ using the modified-Euler method

$$y_{i+1} = y_i + h\left[\frac{k_1+k_2}{2}\right]$$

$$k_1 = f(t_i, y_i)$$

$$k_2 = f(t_i+h, y_i + h k_1) = f(t_i+h, y_i + h f(t_i, y_i))$$

$$f(t,y) = t^2(2+y)$$

$$k_1 = f(t_0, y_0) = t_0^2(2+y_0) \qquad t_0 = 0$$

$$y_0 = 1.$$

$$k_1 = 0$$

$$k_2 = f(t_0+h, y_0 + h k_1)$$

$$= f(0.1, 1)$$

$$= (0.1)^2(2+1) = 0.03$$

$$y_1 = y_0 + h\left[\frac{k_1+k_2}{2}\right] = 1 + 0.1\left(\frac{0+0.03}{2}\right) = \boxed{1.0015}$$

ex. Use the modified-euler method with step size 0.5 to approximate the solu^n of the IVP:

$$\frac{dy}{dt} = t^2 - y, \qquad 0 \leq t \leq 2, \quad y(0) = 1.$$

We have ~~to~~ $h = 0.5$, $t_0 = 0$, $f(t, y) = t^2 - y$.

$$y_{i+1} = y_i + h\left[\frac{k_1 + k_2}{2}\right]$$

$$k_1 = f(t_i, y_i)$$

$$k_2 = f(t_i + h, y_i + h k_1)$$

$$k_1 = t_i^2 - y_i.$$

$$k_2 = f(t_i + h, y_i + h k_1)$$

$$= f\left(\underline{t_i + 0.5}, \underline{y_i + 0.5(t_i^2 - y_i)}\right)$$

$$k_2 = (t_i + 0.5)^2 - \left[y_i + 0.5(t_i^2 - y_i)\right]$$

$$y_{i+1} = y_i + \frac{h}{2}\left[(t_i^2 - y_i) + (t_i + 0.5)^2 - \left[y_i + 0.5(t_i^2 - y_i)\right]\right]$$

$$= \cancel{y_i + \frac{0.5}{2}\left[0.5(t_i^2 - y_i)\right.}$$

2nd-order Runge-Kutta methods (RK-2)

$$\cancel{y_{i+1} = y_i + h f(t_i, y_i)}$$

A general form of the 2nd-order RK method is expressed as

$$y_{i+1} = y_i + h\left[a_1 k_1 + a_2 k_2\right], \quad \text{— (1)}$$

where

$$k_1 = f(t_i, y_i)$$

$$k_2 = \cancel{f(t_i + h, y_i)}\; f(t_i + \alpha h,\; y_i + \beta h k_1).$$

Here a_1, a_2, α and β are constants.

We consider the Taylor series of $y(t)$ about t_i up to degree 2

$$y(t) = y(t_i) + y'(t_i)(t - t_i) + \frac{y''(t_i)(t-t_i)^2}{2!} + \text{H.O.T}\left((t-t_i)^3\right)$$

$t = t_i + h$

$$y(t_i + h) = y(t_i) + h\, y'(t_i) + \frac{h^2}{2!} y''(t_i) + \text{H.O.T}(h^3)$$

$$y_{i+1} = y(t_i) + h f(t_i, y_i) + \frac{h^2}{2} y''(t_i) + O(h^3)$$

$$y''(t_i) = \frac{d}{dt}(y') = \frac{d}{dt}\left[f(t,y)\right] \overset{\text{chain rule}}{=} \frac{\partial f}{\partial t}(t_i, y_i) + \left.\frac{\partial f}{\partial y}\right|_{(t_i, y_i)} f(t_i, y_i)$$

$$y_{i+1} = y_i + h\,f(t_i, y_i) + \frac{h^2}{2}\left[\left.\frac{\partial f}{\partial t}\right|_{(t_i,y_i)} + \left.\frac{\partial f}{\partial y}\right|_{(t_i,y_i)} f(t_i, y_i)\right] + \overset{HOT}{\mathcal{O}(h^3)} \quad (2)$$

Note

Taylor's thm for two-variables (about t_i, y_i)

$$f(t,y) = f(t_i,y_i) + (t-t_i)\left.\frac{\partial f}{\partial t}\right|_{(t_i,y_i)} + (y-y_i)\left.\frac{\partial f}{\partial y}\right|_{(t_i,y_i)} + \mathcal{O}\left(\cdot\right)$$

HOT$((t-t_i)^2_j (y-y_i))$

The term k_2 in the proposed RK m/d is a fun^n of two-variables, and can be expanded about (t_i, y_i) as

$$k_2 = f(t_i + \alpha h, y_i + \beta h k_1)$$

$$k_2 = f(t_i, y_i) + \alpha h \left.\frac{\partial f}{\partial t}\right|_{(t_i,y_i)} + \beta h k_1 \left.\frac{\partial f}{\partial y}\right|_{(t_i,y_i)} + \mathcal{O}(h^2) \quad (3)$$

Use eq^{ns} (1), (3) and $k_1 = f(t_i, y_i)$ to get

$$y_{i+1} = y_i + h\left[a_1 f(t_i,y_i) + a_2\left(f(t_i,y_i) + \alpha h \left.\frac{\partial f}{\partial t}\right|_{(t_i,y_i)} + \beta h\, k_1 \left.\frac{\partial f}{\partial y}\right|_{(t_i,y_i)} + \mathcal{O}(h^2)\right)\right] \quad (4)$$

$$y_{i+1} = y_i + h\,(a_1+a_2)\,f(t_i, y_i) + a_2 \alpha h^2 \left.\frac{\partial f}{\partial t}\right|_{(t_i, y_i)} +$$

$$a_2 \beta h^2 f(t_i, y_i) \left.\frac{\partial f}{\partial y}\right|_{(t_i, y_i)} + O(h^3) \quad (5)$$

Equations (2) and (5) represent the same quantity; y_{i+1}.

$\Rightarrow$

$$a_1 + a_2 = 1$$

$$a_2 \alpha = \frac{1}{2}$$

$$a_2 \beta = \frac{1}{2}.$$

4-unknown and 3-equations $\Rightarrow$ There are infinitely many solutions.

(*) If $(a_1, a_2, \alpha, \beta) = \left(\frac{1}{2}, \frac{1}{2}, 1, 1\right) \Rightarrow$ Modified Euler's m/d.

$$y_{i+1} = y_i + \frac{h}{2}\left[k_1 + k_2\right],$$

where $k_1 = f(t_i, y_i)$

$k_2 = f(t_i + h, y_i + k_1 h)$

⊛ If $(a_1, a_2, \alpha, \beta) = (0, 1, \frac{1}{2}, \frac{1}{2}) \Rightarrow$ Midpoint m/d.

$$y_{i+1} = y_i + h\, k_2 , \quad \text{where}$$

$$k_1 = f(t_i, y_i)$$

$$k_2 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2} k_1\right)$$

Runge-kutta method of order four (RK4)

$$y_{i+1} = y_i + \frac{h}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right],$$

where $k_1 = f(t_i, y_i)$

$$k_2 = f\left(t_i + \frac{h}{2}, y_i + k_1 \frac{h}{2}\right)$$

$$k_3 = f\left(t_i + \frac{h}{2}, y_i + k_2 \frac{h}{2}\right)$$

$$k_4 = f\left(t_i + h, y_i + k_3 h\right)$$

ex. Use RK-4 with step $h = 0.5$ to approximate the solu^n of the IVP: $\frac{dy}{dt} = t^2 - y \qquad 0 \leq t \leq 2, \quad y(0) = 1.$

$$k_1 = t_i^2 - y_i$$

$$k_2 = f\left(t_i + \frac{h}{2}, y_i + k_1 \frac{h}{2}\right) = f\left(t_i + 0.25, y_i + 0.25 k_1\right)$$

$$= (t_i + 0.25)^2 - (y_i + 0.25 k_1)$$

$$k_3 = (t_i + 0.25)^2 - (y_i + 0.25 k_2)$$

$$k_4 = (t_i + 0.5)^2 - (y_i + 0.5 k_3)$$

$$y_{i+1} = y_i + \frac{0.5}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$