THURSDAY AUGUST 1, 2019

# 7.1 HEAT EQUATION

The 1-Dimensional heat eq$^{n}$:

unknown Temperature

$$\frac{\partial u}{\partial t}(x,t) = \alpha \frac{\partial^2 u}{\partial x^2}(x,t)$$

$$0 \le x \le L$$
$$t \ge 0$$

one initial condition:

$u(x,0) = f(x) \quad 0 \le x \le L$

Two boundary conditions:

$u(0,t) = 0$

$u(L,t) = 0 \quad$ for all $t \ge 0$

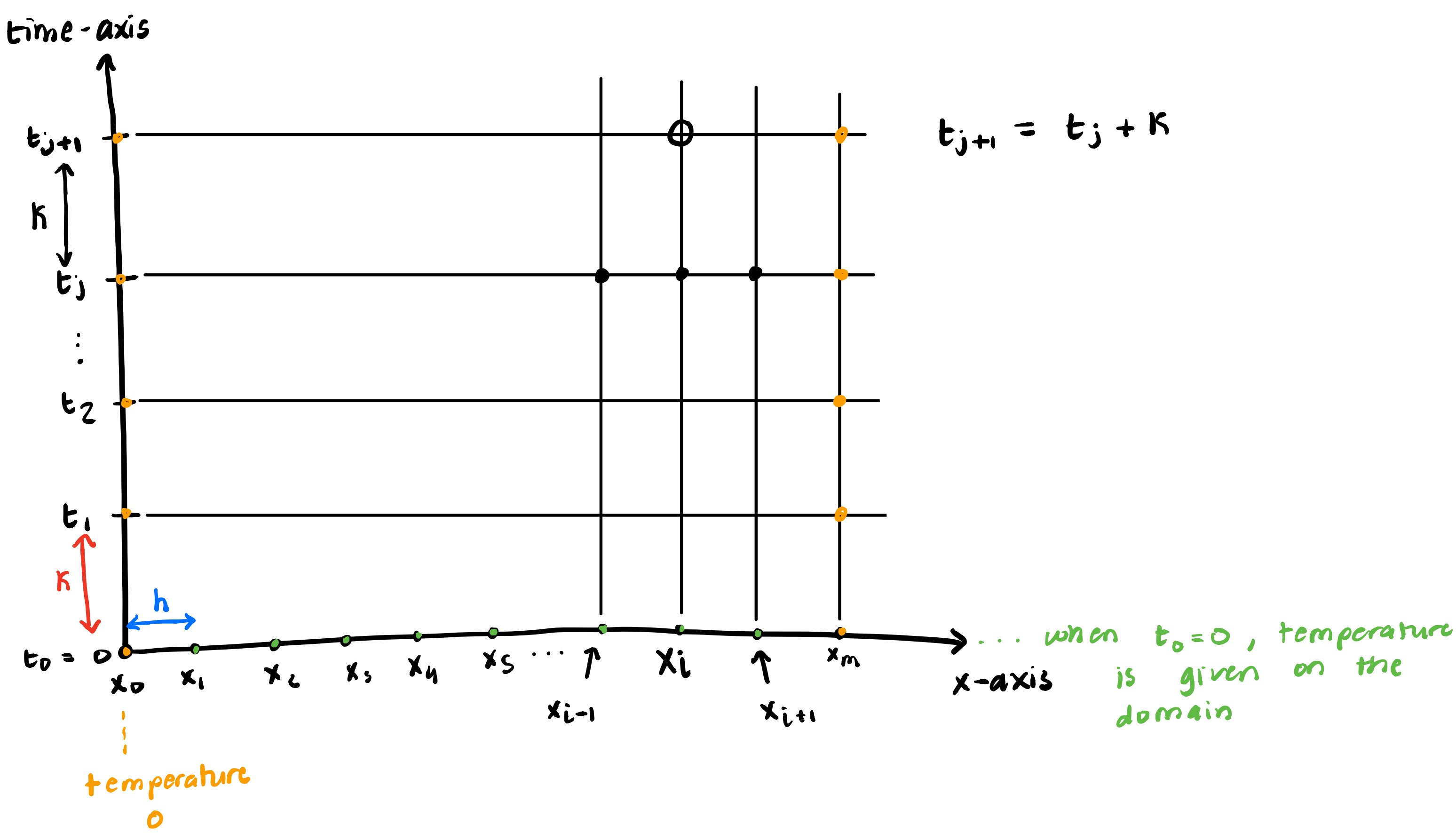
First we make a grid of the domain of $(x,t)$-plane with step size $h$ and $K$, respectively

$x_i = ih \quad i = 0, 1, \dots, m$

$t_j = jK \quad j = 0, 1 \dots$

$m = \frac{(L-0)}{h}$

↑ # of intervals along the x-axis



Difference approximations for derivative terms:

* Forward difference in Time

$$\frac{\partial u}{\partial t}(x_i, t_j) \approx \frac{u(x_i, t_j + k) - u(x_i, t_j)}{k}$$

* Centered - difference formula for $\frac{\partial^2 u}{\partial x^2}$

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) \approx \frac{1}{h^2}\left[u(x_i + h, t_j) - 2u(x_i, t_j) + u(x_i - h, t_j)\right]$$

. . . . . . . .

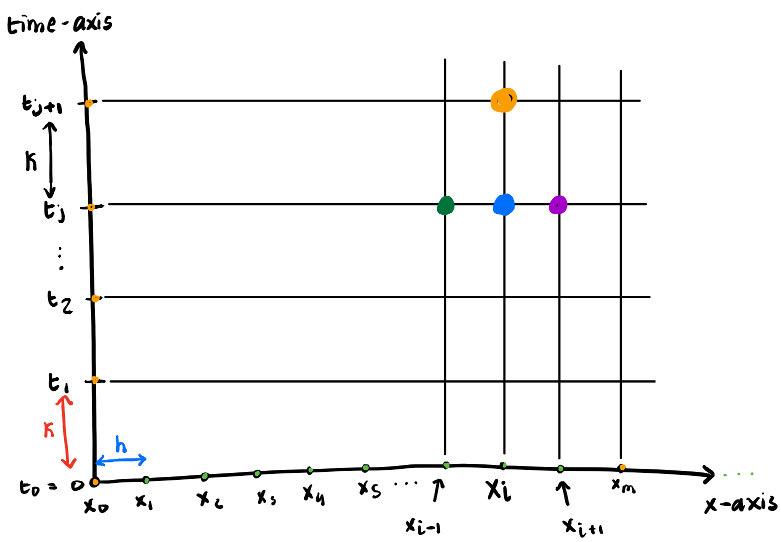
$$\frac{\partial u}{\partial t}(x, t) = \alpha \frac{\partial^2 u}{\partial x^2}(x, t)$$

$$\frac{u(x_i, t_j + k) - u(x_i, t_j)}{k} = \alpha\left(\frac{1}{h^2}\left[u(x_i + h, t_j) - 2u(x_i, t_j) + u(x_i - h, t_j)\right]\right)$$

$$u(x_i, t_j + k) = \frac{k\alpha}{h^2}\left[u(x_i + h, t_j) - 2u(x_i, t_j) + u(x_i - h, t_j)\right] + u(x_i, t_j)$$

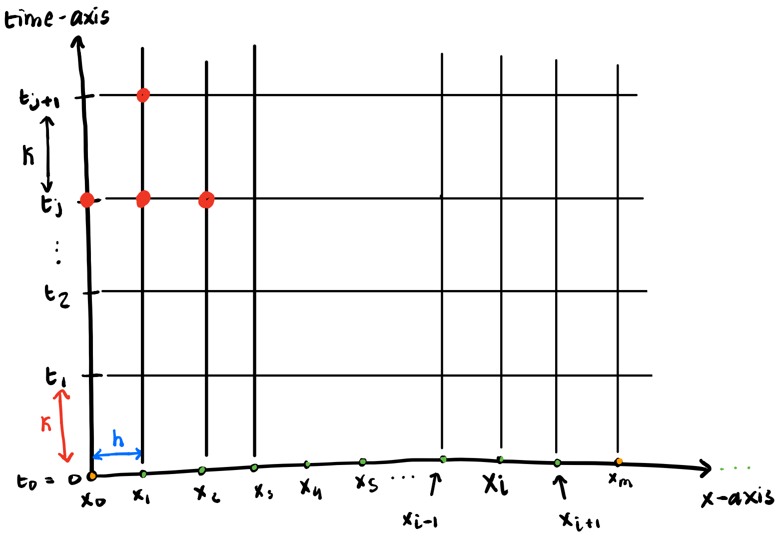
Let $\lambda = \frac{k\alpha}{h^2}$

$$u(x_i, \underbrace{t_j + k}_{\text{temp. @ future node}}) = \lambda u(x_i + h, t_j) - (1 - 2\lambda) u(x_i, t_j) + \lambda u(x_i - h, t_j)$$




$$u_{i,j+1} = \lambda u_{i+1,j} - (1 - 2\lambda) u_{i,j} + \lambda u_{i-1,j}$$

$$u_{1,j+1} = \lambda u_{2,j} - (1-2\lambda) u_{1,j} + \lambda u_{0,j}$$




$$u_{2,j+1} = \lambda u_{3,j} - (1-2\lambda) u_{2,j} + \lambda u_{1,j}$$

$$u_{3,j+1} = \lambda u_{4,j} - (1-2\lambda) u_{3,j} + \lambda u_{2,j}$$

$$\vdots$$

$$u_{m-1,j+1} = \lambda u_{m,j} - (1-2\lambda) u_{m-1,j} + \lambda u_{m-2,j}$$

**NOTE**

$$\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \vdots \\ u_{m-1,j+1} \end{bmatrix} = \begin{array}{c} \begin{array}{ccccc} u_{1,j} & u_{2,j} & u_{3,j} & \cdots & u_{m-1,j} \end{array} \\ \begin{bmatrix} (1-2\lambda) & \lambda & 0 & 0 & 0 \cdots & 0 \\ \lambda & (1-2\lambda) & \lambda & 0 & 0 \cdots & \\ 0 & \lambda & (1-2\lambda) & \lambda \cdots & & \\ 0 & 0 & 0 & (1-2\lambda) \cdots & & \end{bmatrix} \end{array} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ \vdots \\ u_{m-1,j} \end{bmatrix}$$

at time $t = t_1$

$$\begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ \vdots \\ u_{m-1,1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1-2\lambda & \lambda & & 0 \\ \lambda & 1-2\lambda & & \\ & & \ddots & \\ 0 & & \lambda & 1-2\lambda \end{bmatrix}}_{(m-1)\times(m-1)} \begin{bmatrix} u_{1,0} \\ u_{2,0} \\ u_{3,0} \\ \vdots \\ u_{m-1,0} \end{bmatrix}$$

initial condition
$u(x,0) = f(x)$

ex. Approximate the sol$^n$ of the following heat eq$^n$ at $t = 0.2$ and $t = 0.4$ using $h = 0.25$ and $k = 0.2$.

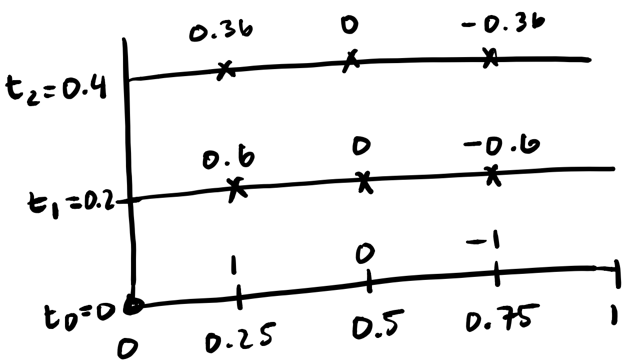
$$\frac{\partial u}{\partial t} = \frac{1}{16} \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 1, \quad t \ge 0 \qquad u(0,t) = 0, \quad u(1,t) = 0 \qquad u(x,0) = \sin(2\pi x)$$

$$m = \frac{b-a}{h} = \frac{1-0}{0.25} = 4$$

$$\lambda = \frac{\alpha k}{h^2} = \frac{1}{16} \cdot \frac{0.2}{0.25^2} = 0.2$$

$t_1 = 0.2$

$$\begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.6 & 0.2 \\ 0 & 0.2 & 0.6 \end{bmatrix}}_{\substack{3\times 3 \\ (m-1)\times(m-1)}} \begin{bmatrix} u_{1,0} \\ u_{2,0} \\ u_{3,0} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.6 & 0.2 \\ 0 & 0.2 & 0.6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0 \\ -0.6 \end{bmatrix}$$




$t_0 = 0$:

$\sin(2\pi \cdot 0.25) = 1$

$\sin(2\pi \cdot 0.5) = 0$

$\sin(2\pi \cdot 0.75) = -1$

$t_2 = 0.4$

$$\begin{bmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \end{bmatrix} = \underbrace{\begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.6 & 0.2 \\ 0 & 0.2 & 0.6 \end{bmatrix}}_{\substack{3\times 3 \\ (m-1)\times(m-1)}} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.6 & 0.2 \\ 0 & 0.2 & 0.6 \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} 0.6 \\ 0 \\ -0.6 \end{bmatrix}}_{3\times 1} = \begin{bmatrix} 0.36 \\ 0 \\ -0.36 \end{bmatrix}$$

(b.) If the exact solu$^{n}$ $u(x,t) = e^{\frac{-\pi^2 t}{4}} \sin(2\pi x)$ compare the results at $t = 0.4$.

at $x = 0.25$:

$$u(0.25, 0.4) = e^{\frac{-\pi^2(0.4)}{4}} \sin(2\pi \cdot 0.25) = 0.3727$$

at $x = 0.5$:

$$u(0.5, 0.4) = e^{\frac{-\pi^2(0.4)}{4}} \sin(2\pi \cdot 0.5) = 0$$

at $x = 0.75$:

$$u(0.75, 0.4) = e^{\frac{-\pi^2(0.4)}{4}} \sin(2\pi \cdot 0.75) = -0.3727$$

ex 2) Consider the heat equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 2 \qquad t \ge 0$$

$$u(0,t) = 0$$
$$u(2,t) = 0$$
$$u(x,0) = \cos\left(\frac{\pi(x-1)}{2}\right)$$

Let $h = 0.4$ and $k = 0.1$

(a.) Approximate the solu$^{n}$ at $t = 0.3$

(b.) Compare the result at $t = 0.3$ using the exact solution

$$u(x,t) = e^{\frac{-\pi^2 t}{8}} \cos\left(\frac{\pi(x-1)}{2}\right)$$

(a.) $m = \frac{b-a}{h} = \frac{2-0}{0.4} = 5$

$$\lambda = \frac{\alpha k}{h^2} = \frac{1}{2} \cdot \frac{0.1}{0.4^2} = 0.3125$$