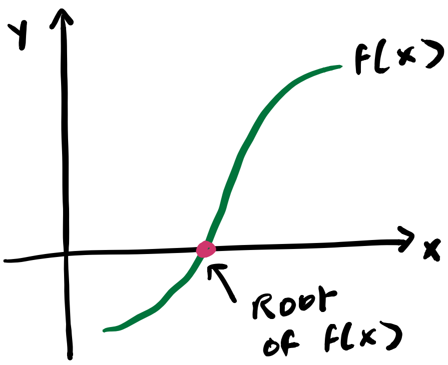
TUESDAY JULY 9, 2019

CH. 2 SOLUTIONS OF EQ$^{\underline{ns}}$ OF ONE VARIABLE

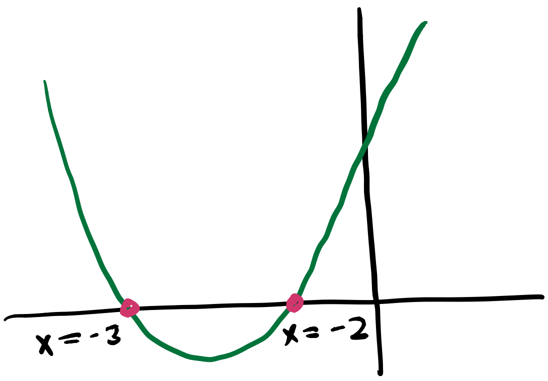
The focus of this chapter is on numerical soln$^{\underline{ns}}$ of eq$^{\underline{ns}}$ in the general form:

$$f(x) = 0 \quad \text{——} \quad (1)$$

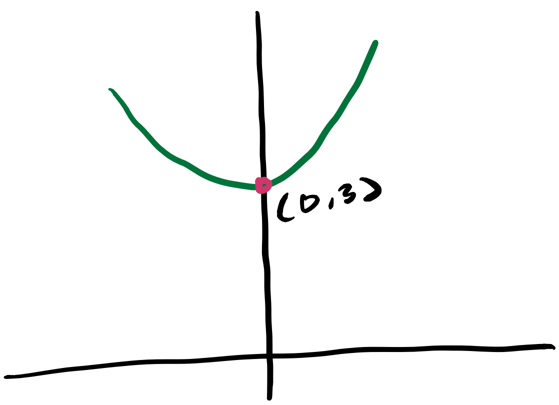
A soln$^{\underline{n}}$ or (a root) of eq$^{\underline{n}}$ (1) refers to the point of intersection of $f(x)$ and x-axis.



ex 1.) $f(x) = x^2 + 5x + 6$

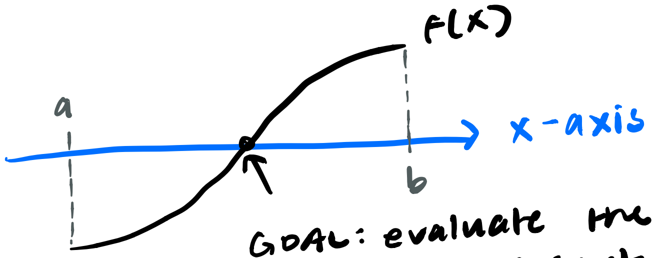


ex 2.) $f(x) = x^2 + 3$



2.1 BISECTION METHOD (BSM)

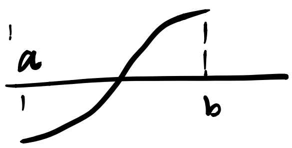
This method requires that an initial interval containing the root to be identified.



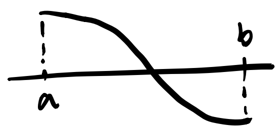
GOAL: evaluate the x-intercept

Th$^{\underline{m}}$: Let $f(x)$ be a continuous fun$^{\underline{n}}$ on $[a,b]$, satisfying $f(a)f(b) < 0$. Then $f(x)$ has a root between a and b.

If the product is a negative # then we definitely have a root btwn $[a,b]$



OR



The bisection method consists in subdividing the interval $[a,b]$ into two and discard the half in which there may not be a root.

Procedure

① Locate the mid-point of $[a,b]$:
That is $\rightarrow$ $c_1 = \frac{a+b}{2}$

② If $f(a)$ and $f(c_1)$ have opposite signs $(f(a)f(c_1) < 0)$, the interval $[a, c_1]$ contains the root and will be retained for further analysis

③ If $f(c_1)$ and $f(b)$ have opposite signs, we continue with $[c_1, b]$.

④ The process is repeated until the length of the most recent interval $[a_k, b_k]$ satisfies the desired accuracy

* (QUIZ CONTENT) *

ex. Show that $f(x) = x^3 + x - 1 = 0$ has the root in the interval $[0,1]$. Then use the bisection m/d to find c_5.

5 iterations

① use IVT to prove $f(x)$ has a root in the interval $[0,1]$

$f(0) = -1$

$f(1) = 1$

$f(0)f(1) < 0$

$\therefore$ we have a root in between 0 and 1 ✓

② Bisection Method

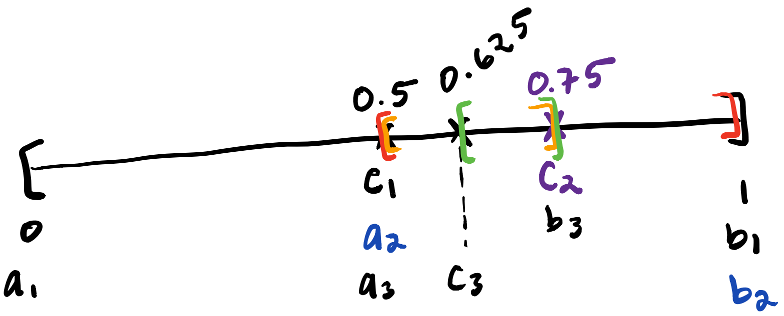
i	a_i	$f(a_i)$	c_i	$f(c_i)$	b_i	$f(b_i)$
1	0	(−)	0.5	(−)	1	(+)
2	0.5	(−)	0.75	(+)	1	(+)
3	0.5	(−)	0.625	(−)	0.75	(+)
4	0.625	(−)	0.6825	(+)	0.75	(+)
5	0.625	(−)	0.65625	(−)	0.6825	(+)

ITERATIONS:

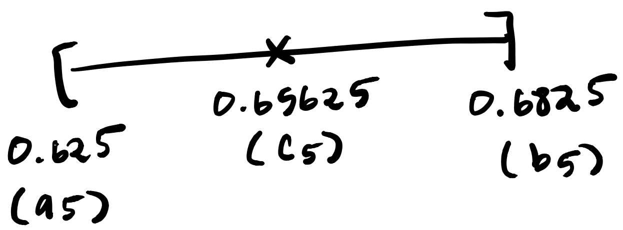
① root btwn 0.5 & 1

② root btwn 0.5 & 0.75

③ root btwn 0.625 & 0.75



From step 5:



SOLUTION:

$$\boxed{f(c_5) = f(0.65625) = -0.06112}$$

In general, at step k, assume that we have obtained an interval $[a_k, b_k]$, such that $f(a_k) < 0$ and $f(b_k) > 0$.

$$\text{Let } c_k = \frac{a_k + b_k}{2}$$

- If $f(c_k) < 0$, then it is the interval $[c_k, b_k]$ which is of interest.

  We set $a_{k+1} = c_k$ and $b_{k+1} = b_k$

- If $f(c_k) > 0$, then it is the interval $[a_k, c_k]$ which is of interest.

  We set $a_{k+1} = a_k$ and $b_{k+1} = c_k$

- If $f(c_k) = 0$, then c_k is a root and we stop the algorithm

<u>NOTE</u>: The advantage of the bisection method is that it is guaranteed to converge to a root by construction

Th^m If $[a_n, b_n]$ is the interval that is obtained in the n^{th} iteration of the BSM, then the limits

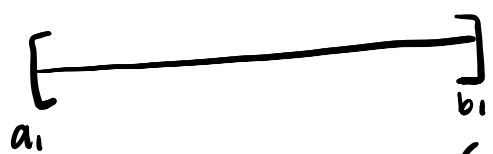
$$\lim_{n\to\infty} a_n \quad \& \quad \lim_{n\to\infty} b_n \quad \text{exist, } \underline{\text{and}}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = r^*, \text{ where } f(r^*) = 0$$

In addition, if

$$c_n = \frac{a_n + b_n}{2}, \text{ then } |\underset{\substack{\uparrow \\ \text{actual root}}}{r^*} - \underset{\substack{\uparrow \\ \text{estimated root}}}{c_n}| \le \frac{b_1 - a_1}{2^n} \quad (n \ge 1)$$

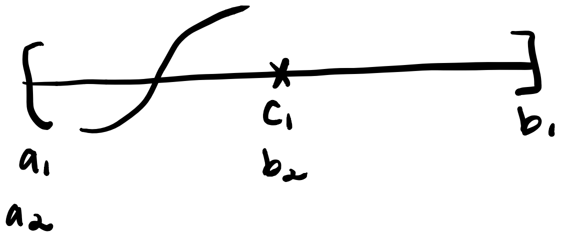
.... <u>PROOF</u> ....



$a_1 \le a_2 \le a_1 \le \dots \le b_1$ $\quad \{a_n\}_{n=1}^{\infty}$ is an increasing seq.

$b_1 \ge b_2 \ge b_2 \ge \dots \ge a_1$ $\quad \{b_n\}_{n=1}^{\infty}$ is a decreasing seq.

From the BMCT, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent



$$b_2 - a_2 = \frac{b_1 - a_1}{2}$$

$$b_3 - a_3 = \frac{b_2 - a_2}{2} = \frac{b_1 - a_1}{2^2}$$

$$\vdots$$

$$\lim_{n\to\infty} b_n - a_n = \lim_{n\to\infty} \frac{b_1 - a_1}{2^{n-1}}$$

$$\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = 0$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = r^*$$

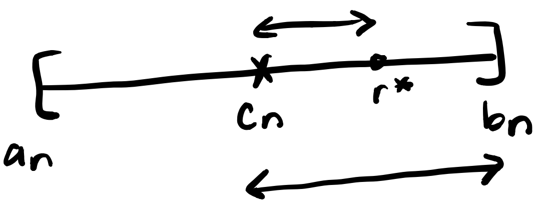
$$f(a_n) f(b_n) \le 0$$

$$\lim_{n\to\infty} f(a_n) f(b_n) \le 0$$

$$f\left(\lim_{n\to\infty} a_n\right) f\left(\lim_{n\to\infty} b_n\right) \le 0$$

$$f(r^*) f(r^*) \le 0$$

$$[f(r^*)]^2 \le 0 \quad \therefore \quad f(r^*) = 0 \Rightarrow r^* \text{ is a root of } f(x)!!!$$



$$|r^* - c_n| \le \frac{b_1 - a_1}{2^n}$$

$$|r^* - c_n| \le b_n - c_n = b_n - \left(\frac{a_n + b_n}{2}\right)$$

$$|r^* - c_n| \leq \frac{(b_n - a_n)}{2} = \frac{b_1 - a_1}{2 \cdot 2^{n-1}}$$

accuracy

$$\boxed{|r^* - c_n| \leq \frac{b_1 - a_1}{2^n}}$$

ex. Determine the number of steps (n) necessary to solve $f(x) = \cos x - x = 0$ with accuracy 10^{-3} using $a_1 = 0$ and $b_1 = 0$.

* we want $|r^* - c_n| \leq 10^{-3}$

* we know that:

$$|r^* - c_n| \leq \frac{b_1 - a_1}{2^n} = \frac{1-0}{2^n} \leq 10^{-3}$$

$\Downarrow$

solve for n

$$\frac{1}{2^n} \leq 10^{-3}$$

$$2^{-n} \leq 10^{-3}$$

$$\log_{10} 2^{-n} \leq \log_{10} 10^{-3}$$

$$(-n) \log_{10} 2 \leq -3$$

$$-n \leq \frac{-3}{\log_{10} 2}$$

$\Leftarrow$ flip inequality since we div. w/ a neg. #

$$n \geq \frac{3}{\log_{10} 2}$$

$$n \geq 9.966$$

$$\boxed{n = 10}$$