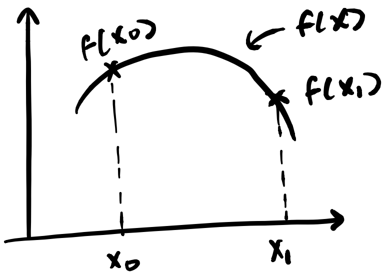
MONDAY JULY 22, 2019

CH. 4 NUMERICAL DIFFERENTIATION AND INTEGRATION

4.3 ELEMENTS OF NUMERICAL INTEGRATION (CONT.)

· The Trapezoidal Rule

* non-composite → one interval



$(x_0, f(x_0))$

$(x_1, f(x_1))$

Lagrange Polynomial of degree 1

$$f(x) = \frac{(x-x_1)}{(x_0-x_1)}[f(x_0)] + \frac{(x-x_0)}{(x_1-x_0)}[f(x_1)] + \frac{f''(\xi)(x-x_0)(x-x_1)}{2!}$$

$$\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_1} \frac{x-x_1}{x_0-x_1} f(x_0)\,dx + \int_{x_0}^{x_1} \frac{x-x_0}{x_1-x_0} f(x_0)\,dx + \int_{x_0}^{x_1} \frac{f''(\xi)(x-x_0)(x-x_1)}{2!}\,dx$$

$$= \frac{f(x_0)}{x_0-x_1}\int_{x_0}^{x_1}(x-x_1)\,dx + \frac{f(x_0)}{x_1-x_0}\int_{x_0}^{x_1}(x-x_0)\,dx + \boxed{\int_{x_0}^{x_1}\frac{f''(\xi)}{2!}\cdot(x-x_0)(x-x_1)\,dx}$$

error term

$$= \frac{f(x_0)}{x_0-x_1}\cdot\frac{(x-x_1)^2}{2}\Big|_{x_0}^{x_1} + \frac{f(x_0)}{x_1-x_0}\cdot\frac{(x-x_0)^2}{2}\Big|_{x_0}^{x_1} + \int_{x_0}^{x_1}(\text{error term})\,dx$$

$$= \frac{f(x_0)}{\cancel{(x_0-x_1)}}\cdot\left[-\frac{(x_0-x_1)^{\cancel{2}}}{2}\right] + \frac{f(x_0)}{\cancel{(x_1-x_0)}}\cdot\left[\frac{(x_1-x_0)^{\cancel{2}}}{2}\right] + \int_{x_0}^{x_1}(\text{error term})\,dx$$

$$= \frac{f(x_0)(x_1-x_0)}{2} + \frac{f(x_1)(x_1-x_0)}{2} + \int_{x_0}^{x_1}(\text{error term})\,dx$$

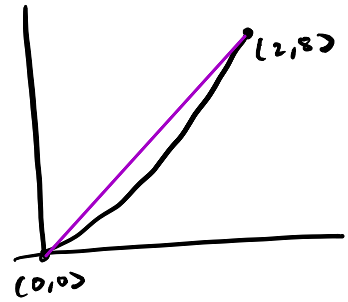
$$= \int_{x_0}^{x_1} f(x)\,dx = \frac{(x_1-x_0)}{2}[f(x_0)+f(x_1)] + \int_{x_0}^{x_1}\text{error term}\,dx$$

Then,

$$\boxed{\int_{x_0}^{x_1} f(x)\,dx \approx \frac{(x_1-x_0)}{2}(f(x_0)+f(x_1))}$$

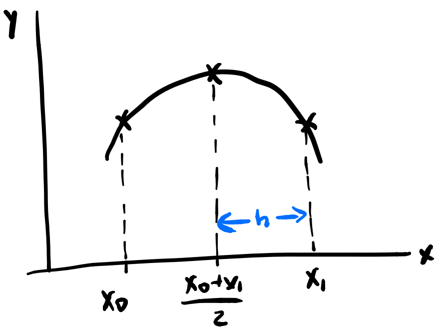
ex.1) Approximate $\int_0^2 x^3 dx$ by the trapezoidal rule

$$\int_{x_0=0}^{x_1=2} x^3\,dx = \frac{(2-0)}{2}(0+8) = \boxed{8}$$



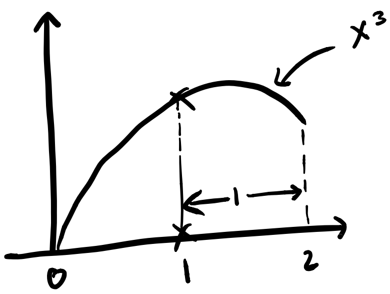
ex.2) Approximate $\int_1^2 e^{x^2} \approx \frac{(2-1)}{2}(e^1+e^4) = 28.65$

• Simpson's Rule



$$\boxed{\int_{x_0}^{x_1} f(x)\,dx \approx \frac{h}{3}\left[f(x_0) + 4f\left(\frac{x_0+x_1}{2}\right) + f(x_1)\right], \quad \text{where } h = \frac{x_1 - x_0}{2}}$$

ex.1) use Simpson's rule to approximate $\int_0^2 x^3 dx$



$$\int_0^2 x^3 dx \approx \frac{1}{3}\left[f(0) + 4f(1) + f(2)\right]$$

$$\approx \frac{1}{3}\left[0 + 4(1) + 8\right] \approx \underline{\underline{4}}$$

ex.2) use simpson's rule to approximate $\int_2^6 x^2 \ln x\, dx$

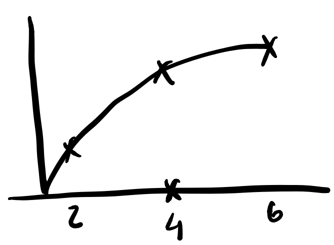
$$\int_2^6 x^2 \ln x\, dx = \frac{2}{3}\left[f(2) + 4(f(4)) + f(6)\right]$$

$h = \frac{6-2}{2}$

$$= \frac{2}{3}\left[2^2\ln 2 + 4\cdot 4^2 \ln 4 + 6^2 \ln 6\right]$$

mid pt: $\frac{6+2}{2} = 4$

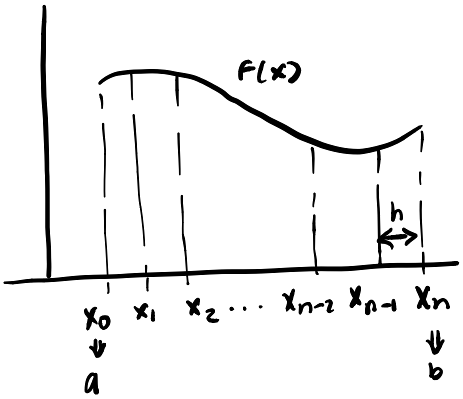
$$= \underline{\underline{103.99}}$$



EXAM #3]

4.4 COMPOSITE NUMERICAL INTEGRATION

The trapezoid and Simpson's rules are limited to operating on a single interval. Since definite integrals are additive over subintervals, we can evaluate an integral by dividing the interval into several subintervals. This strategy is called "composite" numerical integration.



* (n+1) number of points

* (n) - number of intervals

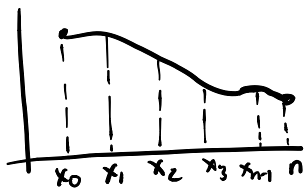
$$\left[h = \frac{b-a}{n}\right]$$

$$\left[x_i = a + ih \;;\; i = 0, 1, \dots, n\right]$$

· Composite Trapezoidal Rule

Apply trapezoidal rule on each subinterval $[x_{i-1}, x_i]$

$$\int_a^b f(x)\,dx \approx \sum_{i=1} \int_{x_{i-1}}^{x_i} f(x)\,dx$$



$$\int_a^b f(x)\,dx \approx \frac{1}{2}h\left[f(x_0) + f(x_1)\right] + \frac{1}{2}h\left[f(x_1) + f(x_2)\right] + \frac{1}{2}h\left[f(x_2) + f(x_3)\right] +$$

$$\frac{1}{2}h\left[f(x_3) + f(x_4)\right] + \dots + \dots + \frac{1}{2}h\left[f(x_{n-2}) + f(x_{n-1})\right]$$

$$+ \frac{1}{2}h\left[f(x_{n-1}) + f(x_n)\right]$$

$$\approx \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_3) + 2f(x_4) + \dots + 2f(x_{n-1}) + f(x_n)\right]$$

$$\approx \boxed{\frac{h}{2}\left[f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f(x_i)\right]}$$

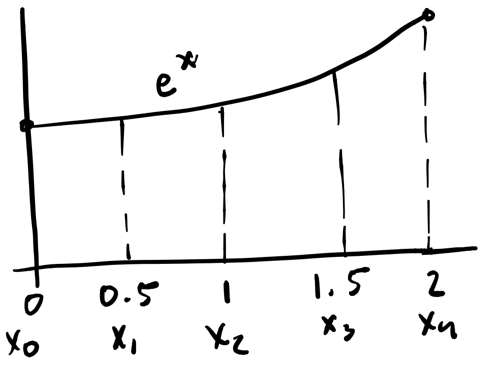
· Composite Midpoint Rule

$$\boxed{\int_a^b f(x)\,dx \approx h\sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)}$$

ex. Approximate $\int_0^2 e^x \, dx$ using 4-subintervals $(n=4)$ in:

(a) composite trapezoidal rule

(b) composite midpoint rule



$n = 4$

$h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$

CTR $\int_0^2 e^x dx \approx \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$

$\approx \frac{0.5}{2}[f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + f(2)]$

$= \frac{0.5}{2}[e^0 + 2e^{0.5} + 2e^1 + 2e^{1.5} + e^2]$