Bisection Method

1.25.19

Theorem:

If $[a_n, b_n]$ is the interval that is obtained by the nth iteration of the Bisection Method, then the limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = r^*$, where $f(r^*) = 0$. In addition if $c_n = \frac{a_n + b_n}{2}$ then

$$|r^* - c_n| \leq \frac{b_1 - a_1}{2} \quad (n \geq 1)$$

Proof:

you do not need to know this proof!

$[a_1, b_1]$

$$b_2 - a_2 = \frac{b_1 - a_1}{2}$$

$$b_3 - a_3 = \frac{b_2 - a_2}{2} = \frac{b_1 - a_1}{2^2} \Rightarrow$$

$\vdots$

$$b_n - a_n = \frac{b_1 - a_1}{2^{n-1}}$$

$$\lim_{n\to\infty} b_n - a_n = \lim_{n\to\infty} \frac{b_1 - a_1}{2^{n-1}}$$

$$= 0$$

so...

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = r^*$$

$$f(a_n) f(b_n) < 0$$

$$\lim_{n\to\infty} f(a_n) \lim_{n\to\infty} f(b_n) < 0$$

$$f(\lim_{n\to\infty} a_n) f(\lim_{n\to\infty} b_n) < 0$$

$$f(r^*) f(r^*) < 0$$

$$[f(r^*)]^2 < 0$$

$$f(r^*) = 0$$

$$|r^* - c_n| \leq \frac{b_n - a_n}{2} = \frac{b_1 - a_1}{2^n}$$

$$\boxed{|r^* - c_n| \leq \frac{b_1 - a_1}{2^n} \quad (n \geq 1)}$$

example) Determine the number of iterations necessary to solve $f(x) = \cos x - x = 0$ with accuracy 10^{-3} using $a = 0$ and $b = 1$

$$|r^* - c_n| \leq \frac{b_1 - a_1}{2^n}$$

$$|r^* - c_n| \leq \frac{1 - 0}{2^n}$$

$$\frac{1}{2^n} < 10^{-3} \Rightarrow \frac{1}{2^n} < \frac{1}{10^3}$$

$$\Rightarrow 2^n > 10^3$$

$$\Rightarrow \lg 2^n > \lg 10^3$$

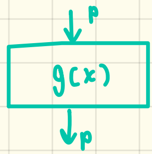
$$n > \lg 10^3$$

$$n > 9.97$$

$$n = 10$$

Fixed Point iteration

Fixed Point iteration: the number P is a fixed point for a given function $g(x)$ if $g(P) = P$



example) Determine any fixed points of the function $g(x) = x^2 - 2$

$$p^2 - 2 = p$$

$$p^2 - p - 2 = 0$$

$$(p-2)(p+1) = 0$$

$$p = 2 \quad p = -1$$

* this technique involves solving the problem $f(x) = 0$ by rearranging $f(x)$ into the form $x - g(x)$ or $x = g(x)$ then finding $x = P$ such that $P = g(P)$ is equivalent to $f(P) = 0$