Problem 1.

$f(x) = x^2 \ln x$. Approximate $f'(1)$ using two-point FDM, BDF, and CDF with $h = 0.3$

(a) FDM. $f'(x_0) = \dfrac{f(x_0+h) - f(x_0)}{h}$, where $f(x) = x^2 \ln x$

Here, $x_0 = 1$ and $h = 0.3$.

$$f'(1) \approx \frac{f(1.3) - f(1)}{0.3} = \frac{(1.3)^2 \ln(1.3) - 1^2 \ln(1.0)}{0.3}$$

$$= \boxed{1.4780}$$

(b) BDF $f'(x_0) \approx \dfrac{f(x_0) - f(x_0-h)}{h}$

$$= \frac{f(1.0) - f(0.7)}{0.3}$$

$$= \frac{1^2 \ln(1.0) - (0.7)^2 \ln(0.7)}{0.3}$$

$$= \boxed{0.5826}$$

1. (a) MDF

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0-h)}{2h}$$

$$= \frac{f(1.3) - f(0.7)}{2(0.3)}$$

$$= \frac{(1.3)^2 \ln(1.3) - (0.7)^2 \ln(0.7)}{2(0.3)}$$

$$\boxed{= 1.0303}$$

Problem 2

For $f(x) = x^2 \ln x$, find the maximum error in approximating $f'(1)$ by the FDF, BDF, and CDF with $h = 0.3$

(a) ~~FDF~~ Error in FDF

$$E_{FDF} = \frac{h}{2} f''(\xi) \quad \text{where } \xi \in [x_0, x_0+h]$$

$$\Rightarrow \quad E_{FDF} \leq \frac{h}{2} \left| f''(x) \right|_{\substack{max \\ x \in [x_0, x_0+h]}}$$

$$f(x) = x^2 \ln x$$

$$f'(x) = 2x \ln x + x^2 \cdot \frac{1}{x} = 2x \ln x + x.$$

$$f''(x) = 2 \ln x + 2\cancel{x} \cdot \frac{1}{\cancel{x}} + 1 = 2 \ln x + 3$$

$$E_{FDF} \leq \frac{h}{2} \left| 2 \ln x + 3 \right|_{\substack{max \\ x \in [1, 1.3]}} \quad \text{and } h = 0.3.$$

(*) Note that $2 \ln x + 3$ is an increasing function.

$$\therefore \quad E_{FDF} \leq \frac{0.3}{2} \left| 2 \ln(1.3) + 3 \right|$$

(b) Error in BDF

$$E_{BDF} = \frac{h}{2} f''(\xi) \quad \text{where} \quad \xi \in (x_0 - h, x_0)$$

$$E_{BDF} \leq \frac{h}{2} \left| f''(x) \right|_{\substack{max \\ x \in (x_0 - h, x_0)}}$$

$$E_{BDF} \leq \frac{0.3}{2} \left| 2 \ln x + 3 \right|_{\substack{max \\ x \in (0.7, 1)}}$$

$$= \frac{0.3}{2} \left| 2 \ln 1 + 3 \right|$$

$$= 0.4500$$

(c) Error in CDF

$$E_{CDF} = \frac{h^2}{6} f'''(\xi) \quad \text{where} \quad \xi \in (x_0 - h, x_0 + h)$$

$$f'''(x) = \frac{2}{x}$$

$$E_{CDF} \leq \frac{h^2}{6} \left| f'''(x) \right|_{\substack{max \\ x \in (x_0 - h, x_0 + h)}} = \frac{(0.3)^2}{6} \left| \frac{2}{x} \right|_{\substack{max \\ x_0 \in (0.7, 1.3)}}$$

⊛ Note that $\frac{2}{x}$ is a decreasing fun^n. $\therefore E_{CDF} \leq \frac{(0.3)^2}{6} \left| \frac{2}{0.7} \right| \boxed{= 0.0429}$

Problem 3

x	0.4	0.7	1.0	1.3	1.6
$f(x)$	-0.1466	-0.1747	0.000	0.4433	1.2032

(a) 3-point FDF

$$f'(x_0) \approx \frac{-3f(x_0) + 4f(x_0+h) - f(x_0+2h)}{2h}$$

Here, $x_0 = 1$ and $h = 0.3$

$$f'(1) \approx \frac{[-3f(1) + 4f(1.3) - f(1.6)]}{2(0.3)}$$

$$= \frac{[-3[0.000] + 4[0.4433] - 1.2032]}{0.6}$$

$$\boxed{= 0.9500}$$

(b) 3-point BDF

$$f'(x_0) \approx \frac{3f(x_0) - 4f(x_0-h) + f(x_0-2h)}{2h}$$

$$= \frac{3f(1) - 4f(0.7) + f(0.4)}{2(0.3)}$$

$$= \frac{3[0.0000] - 4[-0.1747] - 0.1466}{0.6}$$

$$\boxed{= 0.9203}$$