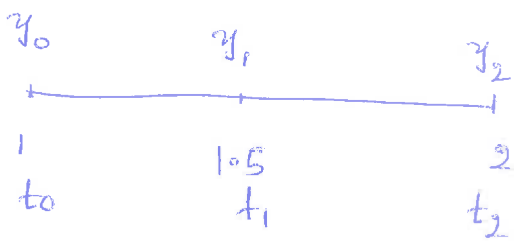
HW 7 - Solutions

Problem 1

$$\frac{dy}{dt} = \frac{e^{y^2}}{t} \qquad 1 \leq t \leq 2 \qquad y(1) = 0$$

$h = 0.5$



Euler's method

$$y_{i+1} = y_i + h f(t_i, y_i) \quad \text{where} \quad f(t,y) = \frac{e^{y^2}}{t}$$

$$y_1 = y_0 + 0.5 f(\underset{t}{1}, \underset{y}{0})$$

$$y_1 = 0 + 0.5 \frac{e^{0^2}}{1} = \boxed{0.5}$$

$$y_2 = y_1 + 0.5 f(t_1, y_1)$$

$$= 0.5 + 0.5 \frac{e^{(0.5)^2}}{(1.5)}$$

$$= 0.5 + 0.5 \frac{e^{0.25}}{(1.5)}$$

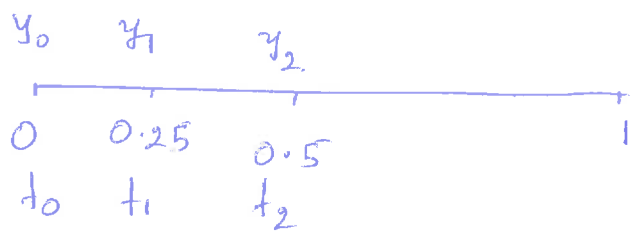
$$\boxed{y_2 = 0.9280}$$

Problem 2.

$$\frac{dy}{dt} = 2ty + t \qquad 0 \leq t \leq 1. \qquad y(0) = 1.$$

(a) $y_{i+1} = y_i + h\, f(t_i, y_i)$

$$y_{i+1} = y_i + h\left[2t_i y_i + t_i\right]$$



$$y_1 = y_0 + 0.25\left[2t_0 y_0 + t_0\right]$$

$$y_1 = 1$$

$$y_2 = y_1 + 0.25\left[2t_1 y_1 + t_1\right]$$

$$= 1 + 0.25\left[2(0.25)(1) + 0.25\right]$$

$$y_2 = 1.1875$$

(b) Exact Sol^n.

$$\frac{dy}{dt} = 2ty + t.$$

$$\frac{dy}{dt} - 2ty = t$$

Integrating factor $= e^{-\int 2t\,dt} = e^{-t^2}$

Then $e^{-t^2}\frac{dy}{dt} - 2te^{-t^2}y = te^{-t^2}$

$$\frac{d}{dt}\left(e^{-t^2}y\right) = te^{-t^2}$$

$$e^{-t^2}y = \int te^{-t^2}dt$$

$$= -\frac{1}{2}\int 2te^{-t^2}dt$$

$$e^{-t^2}y. = -\frac{1}{2}e^{-t^2} + C_1$$

$$y = -\frac{1}{2} + C_1 e^{t^2}.$$

IC $y(0) = 1.$

$$1 = -\frac{1}{2} + C_1$$

$$\frac{3}{2} = C_1$$

$$\boxed{\begin{aligned} y &= -\frac{1}{2} + \frac{3}{2}e^{t^2} \\ y &= \frac{\left(3e^{t^2} - 1\right)}{2} \end{aligned}}$$

(c) Maxi^{m} error.

$$|y_i - y(t_i)| \leqslant \frac{hM}{2L}\left[e^{L(t_i - t_0)} - 1\right]$$

L is the Lipschitz constant.

$t_i = 0.5$. $\quad t_0 = 0$.

$$M = \max_{t \in [0,1]} |y''(x)|$$

$$f(t,y) = 2ty + t$$

$$\frac{\partial f}{\partial y} = 2t$$

$$\boxed{L = \max_{t \in [0,1]} \left|\frac{\partial f}{\partial y}\right| = 2.}$$

Recall:

$$y' = 2ty + t.$$

$$\Rightarrow \quad y'' = 2y + 2ty' + 1$$

$$= 2y + 2t\,[2ty + t] + 1$$

$$= 2y + 4t^2 y + 2t^2 + 1$$

$$y'' = y(2 + 4t^2) + 2t^2 + 1.$$

However, $y = \dfrac{\left(3e^{t^2}-1\right)}{2}$

Then $y'' = \dfrac{\left(3e^{t^2}-1\right)}{2}\left(2+4t^2\right) + 2t^2+1$

y'' is an increasing funⁿ.

$\therefore \left|y''\right|_{\substack{max \\ t\in[0,1]}} = \dfrac{(3e-1)(2+4)}{2} + 2+1$

$= 3(3e-1) + 3$

$$\boxed{M = 9e}$$

$$\left|y_2 - y(0.5)\right| \leq \frac{0.25\,(9e)}{2(2)}\left[e^{2[0.5-0]} - 1\right]$$

$$\left|y_2 - y(0.5)\right| \leq 2.6273$$

(d) $\left|y_2 - y(0.5)\right| = \left|1.1875 - \dfrac{\left(3e^{(0.5)^2}-1\right)}{2}\right|$

$$\boxed{= 0.2385}$$

Problem 3 - See class notes!

Problem 4 - 2^{nd} order Taylor's m/d

$h = 0.5$

$$\frac{dy}{dt} = t - y^2 \qquad 1 \leq t \leq 3 \qquad y(1) = 0$$

[*Quiz 5 - Problem 2]

$$y_{i+1} = y_i + \frac{h}{1!} f(t,y) + \frac{h^2}{2!} f'(t,y)$$

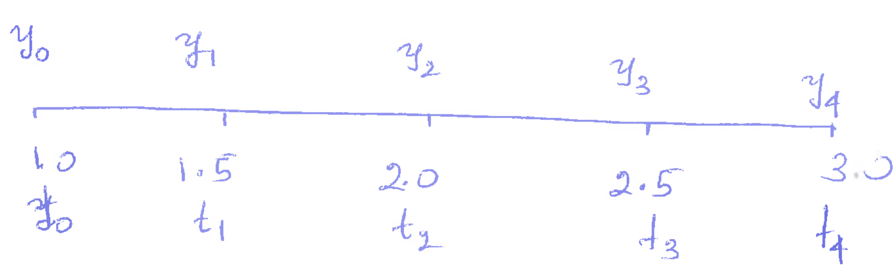
Note $f(t,y) = t - y^2$

$$f'(t,y) = \frac{\partial f}{\partial t} = 1 - 2y \cdot y'$$

$$= 1 - 2y\left[t - y^2\right]$$

$$f'(t,y) = 1 - 2yt + 2y^3$$

Then, $y_{i+1} = y_i + \frac{h}{1!}\left[t_i - y_i^2\right] + \frac{h^2}{2!}\left[1 - 2y_i t_i + 2y_i^3\right]$



$$y_1 = y_0 + 0.5\left(t_0 - y_0^2\right) + \frac{(0.5)^2}{2}\left(1 - 2y_0 y_1 + 2y_0^3\right)$$

$y_1 = 0.6250$

$y_2 = 1.1313$

$y_3 = 1.4127$

$y_4 = 1.6117.$

Problem 5 RK2 - $h = 0.5.$

I'm going to 1st consider the Modified-Euler m/d

$$y_{i+1} = y_i + h\left[\frac{k_1 + k_2}{2}\right], \text{ where}$$

$k_1 = f(t_i, y_i)$ and $k_2 = f(t_i + h, y_i + hk_1)$

where $f(t,y) = \dfrac{e^y}{2t}$ and $1 \leq t \leq 3$, $y(1) = 0.$

y_0	y_1	y_2	y_3	y_4.
1	1.5	2	2.5	3
t_0	t_1	t_2	t_3	t_4

$$y_{i+1} = y_0 + h\left[\frac{k_1 + k_2}{2}\right]$$

$$k_1 = f(t_0, y_0) \qquad \text{and} \qquad k_2 = f\left(t_0 + h, y_0 + h k_1\right)$$

Step	k_1	k_2	y_i
1	0.5	0.4280	0.2320
2	0.4204	0.3890	0.4344
3	0.3860	0.3745	0.6245
4	0.3735	0.3751	0.8116

(*) method 2

Mid-point method

$$y_{i+1} = y_i + h k_2, \qquad k_1 = f(t_i, y_i)$$

$$k_2 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2} k_1\right)$$

Step	k_1	k_2	y_i
1	0.5	0.4533	0.2266
2	0.4181	0.3979	0.4256
3	0.3826	0.3742	0.6127
4	0.3691	0.3680	0.7967