WEDNESDAY JULY 17, 2019

CH. 3 INTERPOLATION AND POLYNOMIAL APPROXIMATION

3.2 CUBIC SPLINES (CONT.)

· Theorem: There is a unique natural (or clamped) cubic spline passing through $(n+1)$ data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$

QUIZ

Q1. A natural cubic spline $S(x)$ on $[0,2]$ is defined by

$$S(x) = \begin{cases} S_0(x) = a_0 + 2x - x^3 & \text{on } [0,1] \\ S_1(x) = 2 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3 & \text{on } [1,2] \end{cases}$$

Find: a_0, b_1, c_1 and d_1

P1: $S_K(X_K) = Y_K$

P2: $S_K(X_{K+1}) = Y_{K+1}$

$S_0(0) = a_0$

$S_0(1) = a_0 + 2 - 1 = S_1(1) = \underset{a_1}{2}$

$a_0 + 1 = 2$

$\boxed{a_0 = 1}$

P3: $S_K'(X_{K+1}) = S_{K+1}'(X_{K+1})$

$S_0'(1) = S_1'(1)$

$2 - 3x^2 = b_1 + 2c_1(x-1) + 3d_1(x-1)^2 \Big|_{x=1}$

$\boxed{-1 = b_1}$

P4: $S_K''(X_{K+1}) = S_{K+1}''(X_{K+1})$

$S_0''(1) = S_1''(1)$

$-6x\Big|_{x=1} = 2c_1 + 6d_1(x-1)\Big|_{x=1}$

$\boxed{c_1 = -3}$

P5:

Natural spline (free): $S_0''(X_0) = 0$, $S_{n-1}''(X_n) = 0$

$S_1''(2) = 0$

$2c_1 + 6d_1(x-1) = 0$

$2(-3) + 6d_1(2-1) = 0$

$-6 + 6d_1 = 0$

$\boxed{d_1 = 1}$

Q2. The function defined by:

$$S(x) = \begin{cases} x^3 + x - 1 & \text{on } [0,1] \\ 2 + 3(x-1) + 3(x-1)^2 - (x-1)^3 & \text{on } [1,2] \end{cases} \text{ is a}$$

(a.) Natural Cubic spline

(b.) Clamped " "

(c.) NOT a cubic spline

$S_0(1)$ & $S_1(1)$ yield different y-values @ "shared" knot

CH.4 NUMERICAL DIFFERENTIATION AND INTEGRATION

4.1 NUMERICAL DIFFERENTIATION

In this section, we numerically find $f'(x)$ evaluated at $x = x_0$.

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x)}{h}$$

1. Finite difference formula

(1) Two-point forward-difference formula (FDF)

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h} \qquad h: \text{step-size}$$

ex. Let $f(x) = x^2 e^x$. Approximate $f'(1)$ using FDF with $h = 0.2$.

$$f'(1) \approx \frac{f(1+0.2) - f(1)}{0.2} = \frac{(1.2)^2 e^{1.2} - 1^2 e^1}{0.2}$$

can leave answer in this form for the exam

$$= 10.3134$$

(2) Two-point backward-difference formula (BDF)

$$f'(x_0) \approx \frac{f(x_0) - f(x_0 - h)}{h}$$

$h = 0.01$

$$f'(1) \approx \frac{f(1) - f(1-0.01)}{0.01} = \frac{1^2 e^1 - (0.99)^2 e^{0.99}}{0.01} = 8.0603$$

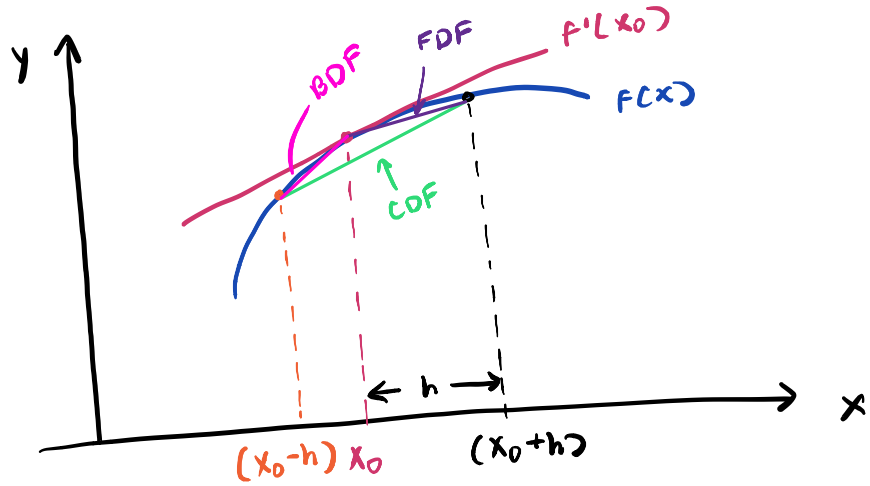
(3) Two-point centered-difference formula (CDF)

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0-h)}{2h}$$

$$f'(1) \approx \frac{f(1.01) - f(0.99)}{2(0.01)} = \frac{(1.01)^2 e^{1.01} - (0.99)^2 e^{0.99}}{0.02}$$

$$= 8.1554$$

Geometric Representation:



ex. Let $f(x) = x^3$; Approximate $f'(2)$ using FDF, BDF, and CDF with $h = 0.1$.

FDF: $f'(x) = \dfrac{f(x_0+h) - f(x_0)}{h} \Rightarrow f'(2) = \dfrac{f(2.1) - f(2)}{0.1}$

$$f'(2) = \frac{(2.1)^3 - (2)^3}{0.1} = 12.61$$

BDF: $f'(x) = \dfrac{f(x_0) - f(x_0+h)}{h} \Rightarrow f'(2) = \dfrac{(2)^3 - (1.9)^3}{0.1} = 11.41$

CDF: $f'(x) = \dfrac{f(x_0+h) - f(x_0-h)}{2h} \Rightarrow f'(2) = \dfrac{(2.1)^3 - (1.9)^3}{2(0.1)} = 12.01$

- Errors in finite-difference formulas

By the Taylor's Thm of $f(x)$ about x_0, we get

$$f(x) = f(x_0) + \frac{f'(x_0)(x-x_0)}{1!} + \frac{f''(\xi_1)(x-x_0)^2}{2!} \quad \xi \in [x_0, x_0+h]$$

set $x = x_0 + h$

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(\xi_1)h^2}{2!}$$

$$f(x_0+h) - f(x_0) = f'(x_0)h + \frac{f''(\xi_1)h^2}{2!}$$

$$\frac{f(x_0+h) - f(x_0)}{h} = f'(x_0) + \frac{f''(\xi_1)h}{2!}$$

Error in (FOF) $\Rightarrow$

$$\underbrace{\left| -\frac{f''(\xi_1)h}{2!} \right|}_{\text{error}} = \left| f'(x_0) - \frac{(f(x_0+h) - f(x_0))}{h} \right|$$

(reduce h to decrease error)

$\therefore$ the maximum error in FOF is

$$\frac{h}{2} \max_{\xi \in (x_0, x_0+h)} |f''(\xi_1)|$$

<u>NOTE</u>:
- Error is proportional to h^1.
- FOF is an order-1 approximation $O(h^1)$
- we can make the error small by making h small
- the fact that the formula is $O(h^1)$ means that cutting h in half should cut the error approximating in half

Error in BDF $\Rightarrow$ The maximum error in BDF is

$$\frac{h}{2} \max \left| f''(\xi) \right|_{\xi \in (x_0 - h,\, x_0)}$$

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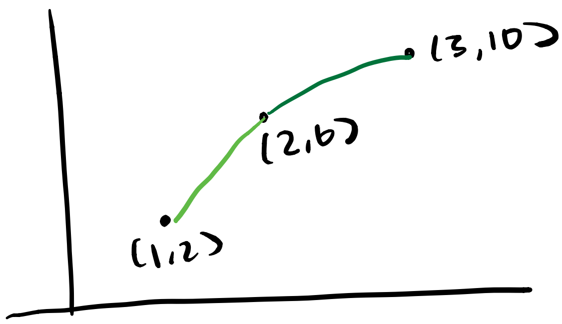
ex. Find the eq$^{\underline{ns}}$ (8) for the cubic splines that interpolate
(1,2) (2,6) (3,10)

Natural



$$s_0(x) = a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3$$

$$s_1(x) = a_1 + b_1(x-2) + c_0(x-2)^2 + d_0(x-2)^3$$

$$s_0(1) = a_0 + b_0\cancel{(1-1)}^{0} + c_0\cancel{(1-1)^2}^{0} + d_0\cancel{(1-1)^3}^{0}$$

$$s_0(1) = a_0 = 2 \quad \text{— } 1$$

$$s_0(2) = a_0 + b_0(2-1) + c_0(2-1)^2 + d_0(2-1)^3$$

$$= a_0 + b_0 + c_0 + d_0 = 6 \quad \text{— } 2$$

$$s_1(2) = a_1 = 6 \quad \text{— } 3$$

$$s_1(3) = a_1 + b_1(3-2) + c_1(3-2)^2 + d_1(3-2)^3$$

$$= a_1 + b_1 + c_1 + d_1 = 10 \quad \text{— } 4$$

$$s_0'(2) = s_1'(2)$$

$$a_0 +$$