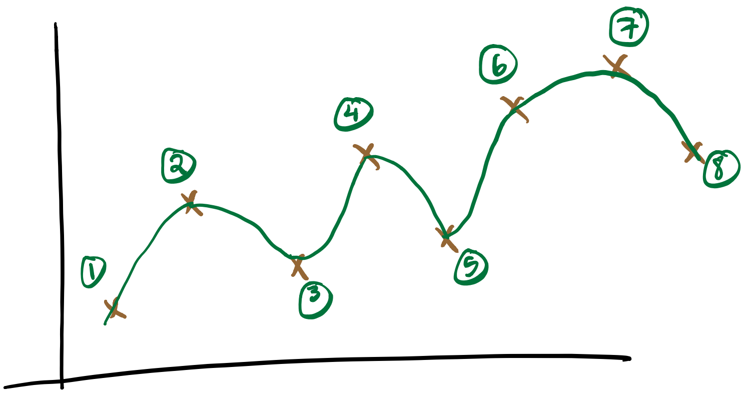
MONDAY JULY 15, 2019

CH. 3 INTERPOLATION AND POLYNOMIAL APPROXIMATION

GOAL: Given $(n+1)$ data points, say (x_i, y_i) $i = 0, 1, 2, \dots, n$, find a polynomial of degree n $(P_n(x))$ such that

$\boxed{P_n(x_i) = y_i}$ for $i = 1:n$



* Polynomial must go through data points

$$P_n(x) = \underline{a_0} + \underline{a_1} x + \underline{a_2} x^2 + \underline{a_3} x^3 + \dots + a_n x^n$$

(n+1) number of unknowns

ex.

x_i	y_i
0	1
1	0
2/3	1/2

$P_2(x) = a_0 + a_1 x + a_2 x^2$

$P_2(0) = a_0 + 0 + 0 = [a_0 = 1]$ — 1

$P_2(1) = [a_0 + a_1 + a_2 = 0]$ — 2

$P_2(2/3) = [a_0 + 2/3\, a_1 + 4/9\, a_2 = 1/2]$ — 3

system of equations:

$$\left.\begin{aligned} a_0 + 0 + 0 &= 1 \\ a_0 + a_1 + a_2 &= 0 \\ a_0 + 2/3\, a_1 + 4/9\, a_2 &= 1/2 \end{aligned}\right\} \quad \begin{aligned} a_0 &= 1 \\ a_1 &= -\tfrac{1}{4} \\ a_2 &= -\tfrac{3}{4} \end{aligned}$$

$$\boxed{P_2(x) = 1 - \frac{1}{4}x - \frac{3}{4}x^2}$$

3.1 LAGRANGE INTERPOLATION

Suppose that we are given 3-points:

(x_0, y_0)

(x_1, y_1)

(x_2, y_2)

Then, the Lagrange polynomial of degree 2 is:

$$P_2(x) = \underbrace{\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0}_{\substack{(x_0,\, y_0) \\ [l_0(x)]}} + \underbrace{\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1}_{\substack{(x_1,\, y_1) \\ [l_1(x)]}} + \underbrace{\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2}_{\substack{(x_2,\, y_2) \\ [l_2(x)]}}$$

General theory: Given $(x_0, x_1, \dots, x_n)$ define the "cardinal fun^{ns}": $l_0, l_1, l_2, \dots, l_n$ polynomials of degree n such that

$$l_i(x_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \Rightarrow \begin{array}{l} l_1(x_1) = 1 \\ l_{\underset{i}{2}}(x_{\underset{j}{1}}) = 0 \end{array}$$

i^{th} cardinal function associated with x_i is:

$$l_i(x) = \frac{(x-x_0)(x-x_1)(x-x_2)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)(x_i-x_1)(x_i-x_2)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)}$$

$$= \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}$$

Lagrange Polynomial:

$$P_n(x) = l_0(x)y_0 + l_1(x)y_1 + l_2(x)y_2 + \dots + l_n(x)y_n$$

....

$$P_n(x_0) = \underbrace{l_0(x_0)}_{1}y_0 + \underbrace{l_1(x_0)}_{0}y_1 + \underbrace{l_2(x_0)}_{0}y_2 + \dots + \underbrace{l_n(x_0)}_{0}y_n = y_0$$

on exam #1

ex. Find the Lagrange polynomial that goes through the points

x_i	0	1	2	3
y_i	12	15	-6	7

$$P_4(x) = l_0(x)y_0 + l_1(x)y_1 + l_2(x)y_2 + l_3(x)y_3$$

$$= \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)}(12) + \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)}(15) + \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)}(-6)$$

$$+ \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)}(7)$$

Main Theorem

Let $(x_0, y_0)(x_1, y_1), \dots, (x_n, y_n)$ be $(n+1)$ points in the xy-plane with distinct x_i's. Then, there exists one and only one polynomial of degree n ($P_n(x)$) or less that satisfies $P_n(x_i) = y_i$

- **Maximum error**

  Suppose $x_0, x_1, \dots, x_n$ are distinct numbers in the interval $[a,b]$ and $f(x) \in C^{n+1}[a,b]$. Then, for each x in $[a,b]$, a number $\xi(x)$ in (a,b) exists with:

  (unknown)

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi(x))(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n)}{(n+1)!}$$

where $P_n(x)$ is the lagrange polynomial of degree n.

$$|f(x) - P_n(x)| = \left| \frac{f^{(n+1)}(\xi(x))(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n)}{(n+1)!} \right|$$

Ex. Find the maxi^{m} error in approximating $f(x) = 4\ln x$ on $[1, 4]$ by the Lagrange Polynomial $P_2(x)$ using points $x_0 = 1$, $x_1 = 3$, $x_2 = 4$

$f(x) = 4\ln x \Leftarrow$ true Function

x_i	y_i
1	$4\ln 1 = 0$
3	$4\ln 3$
4	$4\ln 4$

$\Leftarrow$ approximate function

$$P_2(x) = \frac{(x-3)(x-4)}{(1-3)(1-4)}(0) + \frac{(x-1)(x-4)}{(3-1)(3-4)}(4\ln 3) + \frac{(x-3)(x-1)}{(4-3)(4-1)}(4\ln 4)$$

calculate max. error:

$$|f(x) - P_n(x)| = \left| \frac{f^{(n+1)}(\xi(x))(x-x_0)(x-x_1)(x-x_2)\dots(x-x_n)}{(n+1)!} \right|$$

$$|4\ln x - P_2(x)| = \left| \frac{f^{(3)}(\xi(x))(x-1)(x-3)(x-4)}{3!} \right|$$

$f(x) = 4\ln x$

$f'(x) = \frac{4}{x}$

$f''(x) = -\frac{4}{x^2}$

$f'''(x) = \frac{8}{x^3}$

$$= \left| \frac{8}{[\xi(x)]^3} \cdot \frac{(x-1)(x-3)(x-4)}{3!} \right| \quad x \in [1, 4]$$

Find max. value for this part

$$\leq \left| \frac{8}{1^3} \frac{(x-1)(x-3)(x-4)}{6!} \right|$$

Next, calculate max value of:

$(x-1)(x-3)(x-4)$

Let $h(x) = x^3 - 8x^2 + 19x - 12$

$h'(x) = 3x^2 - 16x + 19 = 0$

$x_1 = \frac{8+\sqrt{7}}{3}$

$x_2 = \frac{8-\sqrt{7}}{3}$

$h\left(\frac{8+\sqrt{7}}{3}\right) = \frac{20-14\sqrt{7}}{27}$

max value $\Rightarrow h\left(\frac{8-\sqrt{7}}{3}\right) = \frac{20+14\sqrt{7}}{27}$

$$\underset{\text{max}}{\leq} \left| \frac{8}{1} \cdot \frac{20+14\sqrt{7}}{27} \cdot \frac{1}{6} \right| = \boxed{2.8168}$$

this is the max error if we use $P_2(x)$ to approximate $4\ln x$