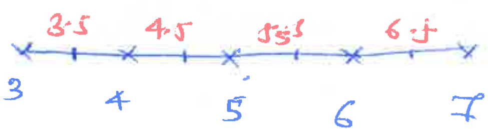
Please read the Instructions and then PRINT your name

- **Show all the steps that you go between the question and the answer. Show how you derived the answer. For your work to be complete, you need to explain your reasoning and make your computations clear.**
- **You will be graded on the readability of your work.**
- **The correct answer with no or incorrect work will earn you NO marks.**
- **Failure to follow these instructions will result in loss points.**
- **Use only four decimal places for all numbers.**
- Name:

$\boxed{\text{56 pts}}$

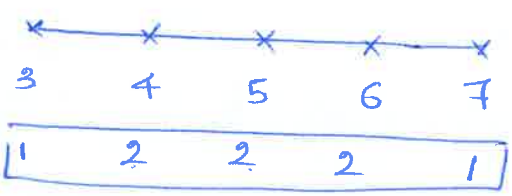
1. (a) SET UP the following composite integrals:

[3] i. Use the Composite Mid-point Rule with $n = 4$ to approximate the integral $\int_3^7 x^2 \ln x \, dx$.



$$\int_3^7 x^2 \ln x \, dx \approx \underset{\uparrow \atop h}{1} \left[(3.5)^2 \ln(3.5) + (4.5)^2 \ln(4.5) + (5.5)^2 \ln(5.5) + (6.5)^2 \ln(6.5) \right]$$

[4] ii. Use the Composite Trapezoidal Rule with $n = 4$ to approximate the integral $\int_3^7 x^2 \ln x \, dx$.



$$\int_3^7 x^2 \ln x \, dx \approx \frac{1}{2} \left[3^2 \ln(3) + 2(4^2 \ln(4)) + 2(5^2 \ln 5) + 2(6^2 \ln 6) + 7^2 \ln 7 \right]$$

5 iii. Use the Composite Simpson's Rule with $n = 4$ to approximate the integral $\int_3^7 x^2 \ln x \, dx$.

x_0	x_1	x_2	x_3	x_4
3	4	5	6	7
1	4	$\boxed{\begin{matrix}1\\1\end{matrix}}$ 2	4	1

$$\int_3^7 x^2 \ln x \, dx \approx \frac{1}{3}\Big[3^2 \ln 3 + 4(4^2 \ln 4) + 2(5^2 \ln 5) + 4(6^2 \ln 6) + 7^2 \ln 7\Big]$$

4 (b) Determine the number of subintervals n required to approximate $\int_2^4 (x^4 + x^2)\, dx$ correct within 3×10^{-4} using the Composite Trapezoidal rule.

$$E_{CTR} = \frac{-h^2}{12}(b-a) f''(\xi) \quad \xi \in [a,b]$$

$$h = \frac{b-a}{n} = \frac{4-2}{n} = \frac{2}{n}$$

$$|E_{CTR}| = \left| \frac{2^2}{n^2\, 12}(4-2) f''(\xi) \right| \leq 3 \times 10^{-4}$$

$$f(x) = x^4 + x^2$$

$$f'(x) = 4x^3 + 2x$$

$$f''(x) = 12x^2 + 2$$

$$= \frac{8}{12 n^2}\left(12 x^2 + 2\right) \leq 3 \times 10^{-4}$$

$$\cancel{\left[\frac{6}{2}(4^2) + 1\right]} \cancel{\leq n^2}$$

$$\cancel{4 \times 3 \times 10^{-4}}$$

$$\cancel{3 \times 10^{+4}} \cancel{\leq n^2}$$

$$\frac{8(12(4^2) + 2)}{12 n^2} \leq 3 \times 10^{-4}$$

$$\sqrt{\frac{8[12 \times 16 + 2]}{12 \times 3 \times 10^{-4}}} \leq n$$

$$656.6 \leq n$$

$$\boxed{n = 657}$$

2. (a) For each of the two following functions, show that it satisfies a Lipschitz condition, with respect to y, on the corresponding domain, and find the Lipschitz constant L.

[2] i. $f(t,y) = \dfrac{3y}{\sqrt{t}}$ for $2 \le t < \infty$

$$\frac{\partial f}{\partial y} = \frac{3}{\sqrt{t}} \qquad \left|\frac{\partial f}{\partial y}\right|_{max} = \boxed{\frac{3}{\sqrt{2}}} = L. = 2.12.$$

[3] ii. $f(t,y) = 2 + t^2 \sin(t\,y)$ for $1 \le t \le 3$

$$\frac{\partial f}{\partial y} = t^2 \cos(ty).t = t^3 \cos(ty)$$

$$\left|\frac{\partial f}{\partial y}\right|_{max} = 3^3(1) = \boxed{27}$$

[2] (b) State the existence and uniqueness theorem for initial value problems (IVPs).

See the Notes!

[2] (c) Prove that the initial value problem:

$$\frac{dy}{dt} = 2 + t^2 \sin(t\,y), \quad y(0) = 1 \quad — (*)$$

has a unique solution for t in $[1, 3]$.

- $f(t,y) = 2 + t^2 \sin(ty)$ is continuous

- $\left|\frac{\partial f}{\partial y}\right|_{m} = 27 = L$ and $f(t,y)$ satisfied the Lipschitz condition

- Hence, from the EUT, the IVP $(*)$ has a unique sol^n.

3.

[5] (a) Use Euler's method with step size $h = 0.5$ to compute an approximate solution to the IVP:

$$\frac{dy}{dt} = 1 - t^2 + y \quad \text{for } 0 \le t \le 1.0 \text{ with } y(0) = 1.$$

Be sure to label your approximations for $y(0.5)$ and $y(1.0)$.

$t_0 = 0 \quad y_0 = 1.$

$y_{i+1} = y_i + h\, f(t_i, y_i)$

$t_1 = 0.5 \quad y_1 = 2$

$y_1 = y_0 + h f(t_0, y_0)$

$y_1 = 1 + \frac{1}{2}(1 - 0 + 1) = \boxed{2}$

$y_2 = y_1 + h\, f(t_1, y_1)$

$= 2 + \frac{1}{2}\left[1 - 0.5^2 + 2\right]$

$= 2 + \frac{1}{2}\left[3 - 0.25\right] = \boxed{3.375}$

[6] (b) Euler's method is used to solve the IVP: $y' = 2y + t^2, \quad 0 \le t \le 1, \quad y(0) = 3, \quad h = 0.25$. The actual solution is $y = \frac{13}{4}e^{2t} - \frac{t^2}{2} - \frac{t}{2} - \frac{1}{4}$. Find the maximum error in approximating $y(1)$ by y_4.

$|y(t_i) - y_i| \le \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right]$

$L = \frac{\partial f}{\partial y} = 2.$

$h = 0.25$

$a = 0$

$t_i = 1$

$|y(1) - y_4| \le \frac{0.25\,(13e^2 - 1)}{2(2)}\left[e^{2(1-0)} - 1\right]$

$\le \frac{(13e^2 - 1)}{16}\left[e^2 - 1\right]$

$= \boxed{37.9580}$

$y' = 2y + t^2$

$y'' = 2y' + 2t$

$= 2\left[2y + t^2\right] + 2t$

$y'' = 4y + 2t^2 + 2t$

$y'' = 4\left[\frac{13}{4}e^{2t} - \cancel{\frac{t^2}{2}} - \cancel{\frac{t}{2}} - \frac{1}{4}\right] + \cancel{2}t^2 + \cancel{2}t = 13e^{2t} - 1.$

4.

4 (a) Derive the n^{th}–order Taylor's method for the IVP: $\dfrac{dy}{dt} = f(t,y), \quad a \le t \le b, \quad y(a) = \alpha.$

See the notes!

5 (b) Consider the differential equation given by

$$\frac{dy}{dt} = 16\,t\,y \quad \text{with} \quad y(0) = 1.$$

Do 1-step of Taylor's method of order 2 with step size $h = 0.5$ to approximate the solution $y(0.5)$.

$$y_{i+1} = y_i + hf(t_i, y_i) + \frac{h^2}{2} f'(t_i, y_i)$$

$h = 0.5.$

$$y_1 = y_0 + \frac{1}{2}\left[16\overset{0}{\cancel{(0)}}(1) \right] + \left(\frac{1}{2}\right)^2 \cdot \frac{1}{2}\left[16(1) + 0\right]$$

3

$$= 1 + \frac{1}{2}\left[\frac{16}{8}\right]$$

$$\boxed{y_1 = 3}$$

$f(t,y) = 16\,ty$

$f' = 16\left[1 \cdot y + t\,y'\right]$

$= 16y + 16t \cdot (16ty)$

2

$= 16y + 256t^2 y$

$\boxed{4}$ 5. (a) Consider the IVP:

$$\frac{dy}{dt} = t\,y + 1, \quad t \in [1,7], \quad y(1) = 4. \tag{1}$$

Use Modified Euler's method with $h = 2$ to approximate $y(3)$.

y_0 y_1

1 h 3

$y_{i+1} = y_i + \frac{h}{2}[k_1 + k_2]$

$k_1 = f(t_i, y_i)$

$k_1 = [1(4)+1] = 5$

$k_2 = f(t_i + h, y_i + k_1 h)$

$= f(3, 4 + 5(2))$

$k_2 = f(3, 14)$

$k_2 = 43$

$y_1 = y_0 + \frac{2}{2}[5 + 43]$

$= 4 + 48$

$\boxed{y_1 = 52}$

$\boxed{7}$ (b) Do 1-step of $RK4$ with step size $h = 1$ to approximate the solution to the IVP:

$$\frac{dy}{dt} = 4\,y \quad \text{with} \quad y(0) = 1.$$

Show all your intermediate steps. Find the absolute error in approximating $y(1)$ by y_1 using the actual solution $y = e^{4t}$.

0 $h = 1$ 1

$y_0 = 1$ y_1

$y_1 = y_0 + \frac{h}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 1 + \frac{1}{6}[4 + 24 + 56 + 116]$

(a) $\boxed{y_1 = 34.3333}$ (*)

$k_1 = f(t_0, y_0) = f(0, 1) = \boxed{4}$

$k_2 = f(t_0 + \frac{h}{2}, y_0 + k_1 \frac{h}{2}) = f(0.5, 1 + 4\,\frac{1}{2})$

$= f(0.5, 3) = \boxed{12}$

(b) error $= |34.3333 - e^4|$

$= \boxed{20.2649}$ (*)

$k_3 = f(t_0 + \frac{h}{2}, y_0 + k_2 \frac{h}{2}) = f(0.5, 1 + 12\,\frac{1}{2})$

$= \boxed{28}$

$k_4 = f(t_0 + h, y_0 + k_3 h)$

$= f(1, 1 + 28(1)) = \boxed{116}$