1-14-19

Thursday, January 17, 2019 5:46 PM

Review of Calculus

1-14-19

- **Continuity of a function**
  - A function (fun^n) is continous at $x=a$ if
    1. $f(a)$ exists
    2. $\lim_{x \to a} f(x)$ exists
    3. $\lim_{x \to a} f(x) = f(a)$

- **Differentiability**
  - Let $f(x)$ be a function defined on open interval containing x_0 Then, the function $f(x)$ is differentiable at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists} \longrightarrow f'(x_0)$$

Ex: If $f(x) = e^x \cos(2x)$ Find $f'(x)$

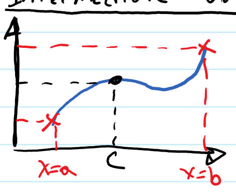
$f(x) = e^x$ $\quad g(x) = \cos(2x)$

$f'(x) = e^x$ $\quad g'(x) = -2\sin(2x)$

$f(x)g'(x) + f'(x)g(x)$

$\boxed{-2e^x \sin(2x) + e^x \cos(2x)}$

- **Intermediate Value Theorem**

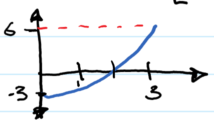


Let $f(x)$ be a continuous function on the interval $[a,b]$
  - If y^* is a $(***)$ between $f(a)$ and $f(b)$, then there exists a number c in (a,b) for which $f(c) = y^*$

Ex: Show that $f(x) = x^3 - 3$ on the interval $[1,3]$ must take on values -2 to 6

$f(1) = -2$

$f(3) = 6$



- **Mean Value Theorem**
  - Let $f(x)$ be a continuously differentiable function on the interval $[a,b]$
    - Then, there exists a number c between a & b such that $\frac{f(b) - f(a)}{b-a} = f'(c)$

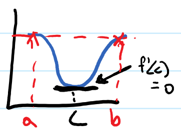
Ex: Find a c value satisfying the MVT for $f(x) = x^2 - 3$ on the interval $[1,3]$

$$f'(c) = \frac{[(3)^2 - 3] - [(1)^2 - 3]}{3-1} \rightarrow \frac{6-(-2)}{3-1} \rightarrow \frac{8}{2} = 4$$

$$f'(x) = 2x \rightarrow 4 = 2x \rightarrow x = \frac{4}{2} \rightarrow \boxed{x=2}$$

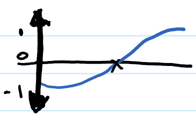
- **Rolle's Theorem**
  - Let $f(x)$ be a continous differentiable function on the interval $[a,b]$ and assume that $f(a) = f(b)$
    - Then, there is a number c between (bc^{n}) a and b such that $f'(c) = 0$

Ex: Show that $x^5 - 2x^3 + 3x^2 - 1$ has a root in the interval $[0,1]$



$f(0) = (0)^5 - 2(0)^3 + 3(0)^2 - 1 \rightarrow f(0) = -1$

$f(1) = (1)^5 - 2(1)^3 + 3(1)^2 - 1 \rightarrow f(1) = 1$



1-16-19 MATLAB Day 1

Thursday, January 17, 2019 8:06 PM

MatLab Review Day

sin (pi/2) = 1

* using 'd' after sin will let you put it a degree

sind (30) = .5

---

a sin (.5) = .5236

a sind (.5) = 30°

* asin gives you the inverse of sin in rads

* asind gives you the inverse of sin in degree

Vectors

Input Vectors

V = [1 2 3 4]

Column Vectors

w = [1; 2; 3; 4]

* To transpose: 1 2 3 4 → 1 2 3 4 (column)

use V' dash

* Row Vector between two numbers with spacing

2 to 17 with 3 number spaces

V = [2 : 3 : 17]

start spacing end

x = [2:3:17]

[2 6 10 14]

x(1) = 2

x(2) = 6

x(3) = 10

x(end) = 14

Find what number is where

Also can use Linspace [2 : 3 : 17]

start end spacing

actually splits it into equal sections

y = linspace (1, 100)

If you leave out last number (spacing) it spaces it by 100

Matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \rightarrow A = [1\ 2\ 3 ; 4\ 5\ 6]$$

How to access a certain element

$$\begin{array}{ll} & 1 \quad 2 \quad 3 \\ \text{2nd row} \rightarrow & 4 \quad 5 \quad 6 \end{array}$$

A = (2,2) = 5

A = (1, :) = 1 2 3

2nd row → 4 5 6

2nd column

A = (2,2) = 5

A = (1,:) = 1 2 3

A = (:,2) = $\begin{matrix} 2 \\ 5 \end{matrix}$

Replace Second Column in Matrix

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $V = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

```
V = [1;0];
A_new = A;
A_new(:,2) = V
```

$A\_new = \begin{bmatrix} 1 & 1 & 3 \\ 4 & 0 & 6 \end{bmatrix}$

Make Matrix with zeros and ones

zeros(5) = $\begin{bmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}$

Can do the same with function ones

zeros(5,2) = $\begin{bmatrix} 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \end{bmatrix}$

row column

Size Function

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ Size(A) = 2 3

row column

Length Function

Ex: x = linspace(1,10,21);
length(x)

ans = 21

Gives you the length of matrix

Ex: length(A) → ans = 3

Square Vectors

x = [1 2 3 4 5]

y = x.^2
y = [1 4 9 16 25]

Plot

$y = x^2$ from $1 \le x \le 10$

```
x = linspace(1,10,20);
y = x.^2;
plot(x,y,'*')
```

creates plot using x as x-values and y as y-values and * as the marker on the plot

k* = black *

R* = Red *

Multiple Plots on 1 Graph

figure
plot(
hold on
plot(
hold on
plot(
grid on

1-18-19 MATLAB Day 2

Friday, January 18, 2019 11:31 AM

MATLAB Day 2

To Plot Multiple lines

* In Script file not Command window *

```
x = linspace (0,2*pi, 50);
y1 = sin(x);
y2 = sin(2*x);
y3 = sin(3*x);
y4 = sin(4*x);
plot= (x,y1,'k',x,y2,'b',x,y3,'g',x,y4,'r')
```

This outputs graph with 4 different graphs

MATLAB Functions

Open New Script → SaveAs: 'FtoC'

```
Function degC = FtoC(degF)
    degC = 5*(degF-32)/9
end
```

NOT Finished look at MATLAB

* Go to old script 'MAT362_Day2.m'

* Delete Program *

FtoC(32)

Ans = 0

You can also do Vectors

FtoC([0 32 40])

Ans = -17.77 0 4.44

'For' Loop

```
for i = 1:2:8
    i
end
```

Ans = 1
3
5
7

Ex: Write a script to generate a 5×5 matrix A with diagonal entries all equal to 1, and super diagonal enteries all equal to 2, while all other entries zero.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

what you want

code

```
A = zeros(4);
for = i = 1:1:4
    A(i,i) = i
end
A
```

output code

1-23-19

Wednesday, January 23, 2019 9:26 AM

Ex: Show that $f(x) = x^3 + x - 1 = 0$ has a root in the interval $[0, 1]$ Using the M.V.T.

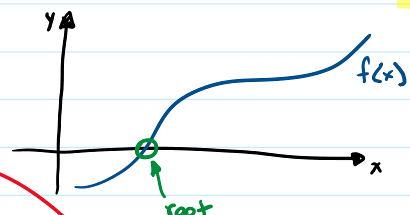
$f(0) = -1$ $\quad (0)^3 + 0 - 1 \to -1$

$f(1) = 1$ $\quad (1)^3 + 1 - 1 \to 1$

Chapter 2: Solution of Equations of one Variable

The focus of this chapter is on numerical solutions of equations in the general form:

$f(x) = 0$



- depending on the nature of the curve of $f(x)$, we may have a unique sol^{n}, multiple sol^{ns}, or no sol^{n}

Quiz: 1-25-19

I.V.T
M.V.T

Know how to state
Know examples of Both

Numerical sol^{n} of $f(x) = 0$
- Bracketing m/ds → Bisection m/d
- Open m/ds
  - Fixed-point iteration
  - Newton's m/d
  - Secant m/d

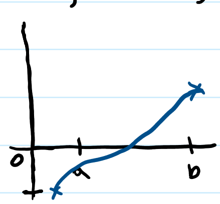
Chapter 2.1: Bisection Method (m/d)

This m/d requires that an initial interval containing the root be identified

* The fun^{n} $f(x)$ has a root $x = r^*$ if $f(r^*) = 0$ *

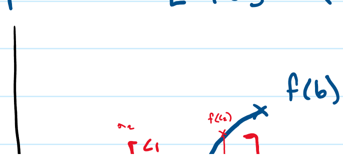
Th^{m}: Let $f(x)$ be a continuous fun^{n} on $[a, b]$ satisfying $f(a) f(b) < 0$

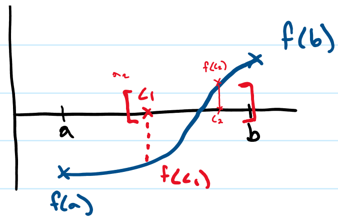
Then $f(x)$ has a root between a and b



* Procedure

1. Locate the mid-point of $[a, b]$ (ie $c_1 = \frac{a+b}{2}$)





$f(a)\, f(c_1) > 0$

$f(c_1)\, f(b) < 0$

2. If $f(a)$ and $f(c_1)$ have opposite signs $\left(f(a)\, f(c_1) < 0\right)$ the interval $[a_1, c_1]$ contains the root and will be retained for further analysis
3. If $f(b)$ and $f(c_1)$ have opposite signs, then we continue with $[c_1, b]$
4. The process is repeated until the length of the most recent interval $[a_k, b_k]$. Satisfied the desired accuracy.

Example: Show that $f(x) = x^3 + x - 1 = 0$ has a root in the interval $[0, 1]$ Then use the Bisection mtd to find c_5

Left interval

right interval

i	a_i	$f(a_i)$	c_i	$f(c_i)$	b_i	$f(b_i)$
1	0 −	-1	.5	-.375 −	1 +	1 +
2	.5	-.375 −	.75	.172 +	1 +	1 +
3	.5	-.375 −	.625	-.131 −	.75 +	.172 +
4	.625	-.131 −	.6875	.012 +	.75 +	.172 +
5	.625	-.131 −	.65625	-.0611 −	.6875 +	.012 +

$$r^* \in [0.6562\,,\ 0.6875]$$

* Look at the pattern, where the root is inbetween

1-25-19

Friday, January 25, 2019 11:35 AM

Theorem: If $[a_n, b_n]$ is the interval that is obtained in the n^{th}-iteration of the Bisection m/d, then the limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = r^*$ where $f(r^*)=0$

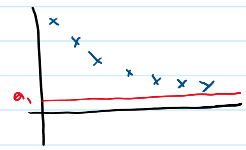
In addition, if $c_n = \frac{a_n+b_n}{2}$, then $|r^* - c_n| \le \frac{b_1-a_1}{2^n}$ $(n \ge 1)$

$[a_n \quad b_n]$

$b_1 \ge b_2 \ge b_3 \ldots \ge b_n \ge a_1$ ← b can't be smaller than a_1

$\{a_n\}_{n=1}^{\infty}$

$a_1 \le a_2 \le a_3 \le a_4 \le \ldots a_n \le b_1$ ← a can't be smaller than b_1



$\{b_n\}_{n=1}^{\infty}$

$b_2 - a_2 = \frac{b_1 - a_1}{2}$

$b_3 - a_3 = \frac{b_2 - a_2}{2} = \frac{b_1 - a_1}{2^2}$

$\vdots$

$b_n - a_n = \frac{b_1 - a_1}{2^{n-1}}$

$\lim_{n\to\infty} b_n - a_n = \lim_{n\to\infty} \frac{b_1 - a_1}{2^{n-1}} = 0$

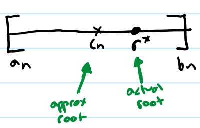
$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = r^*$

$f(a_n)\, f(b_n) < 0$

$\lim_{n\to\infty} f(a_n) \lim_{n\to\infty} (b_n) \le 0$

$f\left(\lim_{n\to\infty} a_n\right) f\left(\lim_{n\to\infty} b_n\right) \le 0$

$f(r^*) \cdot f(r^*) \le 0 \rightarrow [f(r^*)]^2 \le 0 \qquad f(r^*) = 0$



$\boxed{|r^* - c_n| \le \frac{b_1 - a_1}{2^n} \quad (n \ge 1)}$
error

Example:

Determine the number of iterations neccessary to solve $f(x) = \cos x - x = 0$ with accuracy 10^{-3} using $a_1 = 0$ & $b_1 = 1$

$|r^* - c_n| \le \frac{b_1 - a_1}{2^n} = \frac{1-0}{2^n}$

$\frac{1}{2^n} < 10^{-3}$

$2^{-n} < 10^{-3}$

$\log_{10} 2^{-n} < \log_{10} 10^{-3}$

$(-n)\log_{10} 2 < -3$

$-n < \frac{-3}{\log_{10} 2} \rightarrow n > \frac{3}{\log_{10} 2}$

$n > 9.97$

$\boxed{n = 10}$

* Quiz Monday 1-28 on this stuff

$|r^* - c_n| \le \frac{b_1 - a_1}{2^n} =$

$\frac{1-0}{2^n} \le 10^{-3} \rightarrow \frac{1}{2^n} \le 10^{-3}$

$2^{-n} \le 10^{-3}$

$\log_{10} 2^{-n} \le \log_{10} 10^{-3}$

$(-n)\log_{10} 2 \le (-3) \cancel{\log_{10} 10}$ 1

$(-n)\log_{10} 2 \le -3$

$(-n) \le \frac{-3}{\log_{10} 2}$

$n \ge \frac{3}{\log_{10} 2}$

Chapter 2.2: Fixed-Point Iteration

Definition: The number P is a fixed-point for a given fun$^{=}$

Chapter 2.2: Fixed - Point Iteration

Definition: The number p is a fixed - point for a given fun$^{\underline{n}}$ $g(x)$ if:

$g(p) = p$ $\qquad p \to \boxed{g(x)} \to p$

Example: Determine any fixed - points of the fun$^{\underline{n}}$ $g(x) = x^2 - 2$

$p \xrightarrow{\text{input}} \boxed{g(x)} \xrightarrow{\text{output}} p$

$p^2 - 2 = p$

$p^2 - p - 2 = 0$

$(p-2)(p+1) = 0$

$\boxed{p = 2 \quad p = -1}$

$(2)^2 - 2 = 2 \qquad (-1)^2 - 2 = -1$

Both fixed points

This technique involves solving the problem $\boxed{f(x) = 0}$ by rearranging $f(x)$ into the form $x - g(x)$ [or $x = g(x)$].

Then finding $x = p$ such that $p = g(p)$ is equivalent to $f(p) = 0$

instead of solving this

$f(x) = 0$

solve this

$x - g(x) = 0$

$x = g(x)$

$p = g(p)$

Practice for Quiz 2 (1-28-19)

Determine the number of iterations necessary to solve $f(x) = \cos x - x = 0$ with accuracy 10^{-3} using $a_1 = 0$ & $b_1 = 1$

$$|r^* - c_n| \geq \frac{b_1 - a_1}{2^n} \longrightarrow \frac{1-0}{2^n} \to \frac{1}{2^n} > 10^{-3} \longrightarrow 2^{-n} > 10^{-3}$$

$$\log_{10} 2^{-n} > \log_{10} 10^{-3} \to (-n)\log_{10} 2 > (-3)\cancel{\log_{10} 10}^{\,1} \to (-n)\log_{10} 2 > -3$$

$$(-n) > \frac{-3}{\log_{10} 2} \to n < \frac{3}{\log_{10} 2}$$

number of iterations needed to get 10^{-3} accuracy

1-28-19

Monday, January 28, 2019 11:35 AM

$f(x) = 0$

$f(x) = x - g(x) = 0$

$\boxed{x = g(x)}$

$P = g(P)$

Say $x = P$ is fixed - Point

$P = g(P)$

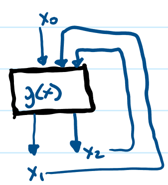
$f(P) = P - g(P) = 0$

Algorithm

(1) Start with an initial guess x_0

(2) For $n = 1, 2, 3, \ldots$

$x_n = g(x_{n-1})$



Example: Using the FPI, determine the root of the fun^n

$f(x) = x - \frac{x}{2} - \frac{1}{x}$ with $x_0 = 1$ $x = g(x)$

$f(x) = 0$

$x - \frac{x}{2} - \frac{1}{x} = 0$

$x = \underbrace{\frac{x}{2} + \frac{1}{x}}_{g(x)}$

$x_1 = g(x_0)$

$x_0 = 1 \rightarrow \boxed{\frac{x}{2} + \frac{1}{x}} \rightarrow x_1 = \frac{1}{2} + 1 = \frac{3}{2}$

$x_1 = 3/2 \rightarrow \boxed{\frac{x}{2} + \frac{1}{x}} \rightarrow x_2 = \frac{3}{4} + \frac{2}{3} = \frac{17}{12} \approx 1.4167$

$x_2 = \frac{17}{12} \rightarrow \boxed{\frac{x}{2} + \frac{1}{x}} \rightarrow x_3 = 1.4142$

$x_3 = 1.4142 \rightarrow \boxed{\frac{x}{2} + \frac{1}{x}} \rightarrow x_4 = 1.4142$

same

Fixed Point is 1.4142

[which is the root]

$x - \frac{x}{2} - \frac{1}{x} = 0$

$\frac{x}{2} - \frac{1}{x} = 0$

$\frac{x}{2} = \frac{1}{x}$

$x^2 = 2$

$x = \pm\sqrt{2} \approx \boxed{1.414}$

Example: Using the F.P.I, determine the root of the fun^n

$f(x) = x - \frac{\sin x}{2} - \frac{\cos x}{2}$ with $x_0 = 0$

$x - \frac{\sin x}{2} - \frac{\cos x}{2} \rightarrow x = \frac{\sin x}{2} + \frac{\cos x}{2}$

$x_0 = 0 \rightarrow \boxed{\frac{\sin x}{2} + \frac{\cos x}{2}}$

$x_1 = 1/2 \rightarrow \boxed{\frac{\sin x}{2} + \frac{\cos x}{2}}$

$x_2 = 0.6785 \rightarrow \boxed{\frac{\sin x}{2} + \frac{\cos x}{2}}$

$x_3 = 0.7030 \rightarrow \boxed{\frac{\sin x}{2} + \frac{\cos x}{2}}$

$x_0 = 0$ → $\boxed{\frac{\sin x}{2} + \frac{\cos x}{2}}$ → $x_1 = 1/2$

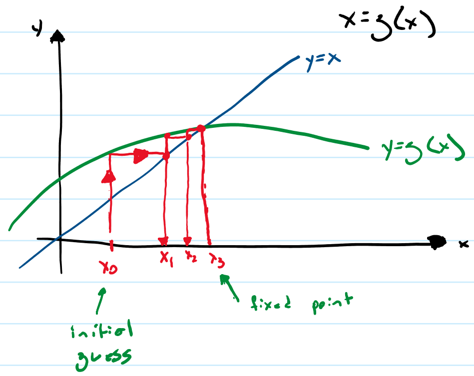
$x_1 = 1/2$ → $\boxed{\frac{\sin x}{2} + \frac{\cos x}{2}}$ → $x_2 = 0.6785$

$x_2 = 0.6785$ → $\boxed{\frac{\sin x}{2} + \frac{\cos x}{2}}$ → $x_3 = 0.7030$

$x_3 = 0.7030$ → $\boxed{\frac{\sin x}{2} + \frac{\cos x}{2}}$ → $x_4 = 0.7047$

$x_4 = 0.7047$ → $\boxed{\frac{\sin x}{2} + \frac{\cos x}{2}}$ → $x_5 = 0.7048$

* Bisection Method always works this section (F.P.I.) doesn't *



Geometric Explination of Fixed-Point Method

Note:

- The $\{x_n\}_{n=1}^{\infty}$ may or may not converge
- However, if the $\{x_n\}_{n=1}^{\infty}$ converges (say to a number) and $g(x)$ is continuous, then P is a fixed-point of $g(x)$

Steps to find:

$f(x) = 0$

$x - g(x) = 0$

$\boxed{x = g(x)}$

Ex:

$f(x) = x - \frac{x^2}{2} - 3x$

$x - \frac{x^2}{2} - 3x = 0$

$x = \frac{x^2}{2} + 3x$

now plug in x_0 until you find the root (x_n = repeating)

1-30-19

Wednesday, January 30, 2019 11:31 AM

Fixed-Point Th^{m}

- Sufficient conditions for existence and uniqueness of a fixed-point.

(existence) Let $g(x)$ be a continous on $[a,b]$ and $a \le g(x) \le b$ for $x \in [a,b]$

$\in$ means in

Then, $g(x)$ has "at least" one fixed-point in $[a,b]$

(uniqueness) Moreover, if $|g'(x)| < 1$ for all $x \in (a,b)$, then $g(x)$ has a unique fixed point in $[a,b]$

$g(x)$

$|g'(x)| < 1$

Then, for any initial point $x_0 \in [a,b]$ the seq. $x_n = g(x_{n-1})$ converges to the unique fixed-point p

Example: Consider the fun^{n} $f(x) = e^x - x - 2$ on the interval $[0,2]$. Find a fun^{n} $g(x)$ that has a unique fixed-point on the interval $[0,2]$

① $0 \le g(x) \le 2$

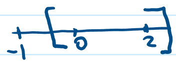
② $|g'(x)| < 1$

Choose correct x

1) $f(x) = e^x - x - 2$

$x = \underbrace{e^x - 2}_{g(x)}$ $\quad [0,2]$

$\cancel{g(0) = -1}$



doesn't work

2) $f(x) = e^x - x - 2$

$\ln(e^x) = \ln(x+2)$

$x = \underbrace{\ln(x+2)}_{g(x)}$

$g(0) = 0.69$

$g(2) = 1.39$ works!

$0 \le \ln(x+2) \le 2$

At least one fixed-Point

② $g(x) = \ln(x+2) \longrightarrow g'(x) = \dfrac{1}{x+2}$ $\quad x \in [0,2]$

$g'(0) = 1/2$

$g'(2) = 1/4$

$|g'(x)| < 1$ for all x-values

$\therefore$ we have unique Fixed-point

Maximum Error

Let ε_n be the absolute error for the n^{th} iteration x_n

Then $\varepsilon_n = |p - x_n| \le K^n \max\{x_0 - a,\ b - x_0\}$

where $K = \max |g'(x)|$ $\quad x \in [a,b]$

Example: Given the FPI $x = \dfrac{x^3+3}{5}$ on $[0,1]$

Estimate how many iterations "n" are required to obtain an absolute error less than 10^{-4} when $x_0 = 1$

1 $g(x) = \frac{x^2+3}{5}$

$g(x) = \frac{3}{5}$ $\quad g(1) = \frac{4}{5}$

$0 < 3/5 \leq g(x) \leq 4/5 < 1$

2 $g'(x) = \frac{2x}{5}$ $\quad [0,1]$

max. $|g'(x)| = \frac{2}{5} = k < 1$

$|p - x_n| \leq k^n \max\{x_0 - a, b - x_0\}$

$\leq \left(\frac{2}{5}\right)^n \max\{1, 0\}$

$= \left(\frac{2}{5}\right)^n$

3 we need n such that

$\left(\frac{2}{5}\right)^n < 10^{-4}$

$\ln\left(\frac{2}{5}\right)^n < \ln(10^{-4})$

$n \ln(0.4) < \ln(10^{-4})$

$n > \frac{\ln(10^{-4})}{\ln(0.4)} \longrightarrow n = 10.05 \longrightarrow n \geq 11$

2-1-19

Friday, February 1, 2019 11:30 AM

Chapter 2.3 Newton's Method

Suppose $f(x)$ is a fun$^{\underline{n}}$ with unique root x^* in $[a,b]$

Let x_0 be a "good" initial approximation

Newton's m/d

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Example

Find the Newton's m/d formula for $f(x) = x^3 + x - 1 = 0$

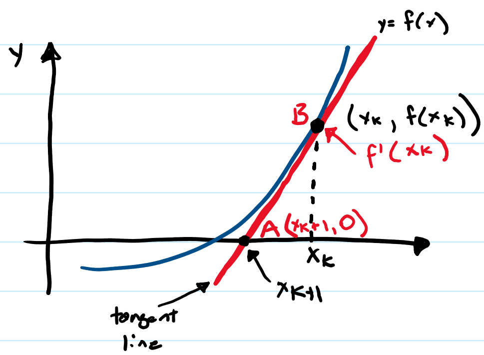
$f'(x) = 3x^2 + 1$

$$x_{k+1} = x_k - \frac{[x_k^3 + x_k - 1]}{3x_k^2 + 1} = \frac{x_k(3x_k^2+1) - x_k^3 - x_k + 1}{(1+3x_k^2)} = \frac{2x_k^3+1}{(1+3x_k^2)}$$

$$= x_{k+1} = \frac{2x_k^3+1}{[1+3x_k]}$$

let $x_0 = 1$

$$x_1 = \frac{2(1)^3+1}{1+3(1)} \rightarrow x_1 = \frac{3}{4}$$



$$\frac{f(x_k) - 0}{x_k - x_{k+1}} = f'(x_k)$$

$$\frac{f(x_k)}{f'(x_k)} = x_k - x_{k+1}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Homework 1

Tuesday, January 29, 2019 11:03 PM

Show that the following equations have at least one solution in the given interval.

(a) $x\cos x - 2x^2 + 3x - 1 = 0, \quad [0.2, 0.3]$

(b) $x - (\ln x)^x = 0, \quad [4, 5]$

a) $f(0.2) = 0.2\cos(0.2) - 2(0.2)^2 + 3(0.2) - 1$

$0.1999 - 0.08 + .6 - 1 = -0.28 < 0$

$f(0.3) = 0.3\cos(0.3) - 2(0.3)^2 + 3(0.3) - 1$

$0.3 - 0.18 + .9 - 1 = 0.2 > 0$

$f(0.2) < f(0.3)$

$0.2 < x < 0.3$

f(x) has atleast one solution in $[0.2, 0.3]$

b) $x - (\ln x)^x = 0 \quad [4, 5]$

$f(4) = 4 - (\ln(4))^4$

$f(4) = .3066 > 0$

$f(5) = 5 - (\ln(5))^5$

$f(5) = -5.799 < 0$

$f(4) > 0 > f(5)$

f(x) has atleast one solution in $[4, 5]$

Problem 2

Find c satisfying the Mean Value Theorem for $f(x)$ on the interval $[0, 1]$.

(a) $f(x) = e^x$

(b) $f(x) = x^2$

a) $f(x) = e^x \quad [0, 1]$

$$f'(c) = \frac{f(b) - f(a)}{b - a} \rightarrow \frac{e^1 - e^0}{1 - 0} \rightarrow \frac{2.718 - 1}{1} = 1.718$$

$f'(x) = e^x \qquad \ln 1.718 = \ln e^x \qquad \boxed{0.5412 = x}$

$f'(x) = e^x$ $\quad \ln 1.718 = \ln e^x$ $\quad \boxed{0.5412 = x}$

b) $f(x) = x^2$ $\quad [0,1]$

$f'(c) = \frac{f(b) - f(a)}{b - a} \rightarrow \frac{(1)^2 - (0)^2}{1 - 0} \rightarrow \frac{1}{1} = f'(c) = 1$

$f'(x) = 2x \rightarrow \frac{1}{2} = \frac{\cancel{2}x}{\cancel{2}} \rightarrow \boxed{x = \frac{1}{2}}$

Problem 3

Find the fifth iteration (c_5) of the Bisection Method to approximate the root of $f(x) = \sqrt{x} - \cos x = 0$ on $[0, 1]$.

i	a	$f(a)$	c	$f(c)$	b	$f(b)$
1	0	-1 −	0.5	$-.1705$ −	1	0.4597 +
2	0.5	-0.1705 −	0.75	0.1343 +	1	0.4597 +
3	0.5	-0.1705 −	0.625	-0.0204 −	.75	0.1345 +
4	0.625	-0.0204 −	0.6875	0.5632 +	.75	0.1345 +
5	0.625	-0.0204 −	0.6563	0.0179 +	0.6875	0.5632 +

$\boxed{5^{th} \text{ iteration} = 0.0179 \quad \text{at} \quad x = 0.6563}$

Problem 4

Find n for which the nth iteration by the Bisection Method guarantees to approximate the root of $f(x) = x^4 - x^3 - 10$ on $[2, 3]$ with accuracy within 10^{-8}.

$|r^* - c^n| \geq \frac{b_1 - a_1}{2^n}$ $\quad \frac{3-2}{2^n} < 10^{-8}$

$\frac{1}{2^n} < 10^{-8} \rightarrow 2^{-n} < 10^{-8} \rightarrow \log_{10} 2^{-n} < \cancel{\log_{10} 10}^{-8}$

$$-n\log_{10}2 < -8 \longrightarrow -n < \frac{-8}{\log_{10}2} \longrightarrow n > \frac{8}{\log_{10}2} \longrightarrow n = 26.575$$

$$\boxed{n \approx 27}$$

Homework 2

Sunday, February 3, 2019 1:12 PM

Problem 1

Use the fixed-point iteration theorem to show that $g(x) = 2^{-x}$ has a unique fixed point on $[\frac{1}{3}, 1]$. Hint: $\frac{d}{dx}a^{-x} = -a^{-x}\ln a$.

$g(x) = 2^{-x}$ $[1/3, 1]$

$g(1/3) = 2^{-(1/3)} \rightarrow 0.79$

$g(1) = 2^{-(1)} \rightarrow 0.50$

$\boxed{1/3 \leq 2^{-x} \leq 1}$

$g'(x) = -2^{-x}\ln(2)$

$|g'(x)| < 1$

$g'(1/3) = |-2^{-(1/3)}\ln(2)| \rightarrow \boxed{0.55}$

$g'(1) = |-2^{-(1)}\ln(2)| \rightarrow \boxed{0.35}$

$|g'(x)| < 1$ for all x-values ∴ we have unique Fixed-point

Problem 2

Consider three functions:

$$g_1(x) = \frac{x^2-3}{2}, \quad g_2(x) = \sqrt{2x+3}, \quad \text{and} \quad g_3(x) = \frac{3}{x-2}.$$

(a) Show that fixed points of each $g_i(x)$ are roots of $f(x) = x^2 - 2x - 3$.

(b) Which $g_i(x)$ has a unique fixed point in $[1, 4]$ guaranteed by the Fixed Point Theorem?

a) $g_1(x) = \frac{x^2-3}{2} \rightarrow \frac{x^2-3}{2} = x \rightarrow x^2 - 3 = 2x \rightarrow \boxed{x^2-2x-3}$

$g_2(x) = \sqrt{2x+3} \rightarrow \cancel{\sqrt{2x+3}} = x^2 \rightarrow 2x+3 = x^2 \rightarrow \boxed{x^2-2x-3}$

$g_3(x) = \frac{3}{x-2} \rightarrow \frac{3}{x-2} = x \rightarrow x(x-2) = 3 \rightarrow \boxed{x^2-2x-3}$

b) $g_1'(x) = x$ $[1, 4]$ $\boxed{|g_1'(x)| \nless 1}$

$g_2'(x) = \frac{1}{\sqrt{2x+3}}$ $[1, 4]$ $|g_2'(x)| < 1$ Only one with unique Fixed point between $[1, 4]$

$g_3'(x) = \frac{-3}{(x-2)^2}$ $[1, 4]$ $\boxed{|g_3'(x)| \nless 1}$

$g_3'(x) = \frac{-3}{(x-2)^2}$ $[1,4]$ $\boxed{|g_3'(x)| \nleq 1}$ $\boxed{\text{between } [1,4]}$

Problem 3

Use the fixed point iteration to find the solution of $e^x = 3x$ on $[1,2]$ with $x_0 = 1.5$ correct to roughly within 10^{-3}. Hint: Use the correct $g(x)$ for $f(x) = e^x - 3x$.

$f(x) = e^x - 3x$

$e^x = 3x \rightarrow x = \ln(3x)$

$\ln \quad \ln$

$g(x) = \ln(3x)$

$g(1) = \ln(3(1)) = 1.10$

$g(2) = \ln(3(2)) = 1.79$

$x_0 = 1.5 \downarrow \ln(3x) \downarrow x_1 = 1.504$

$x_1 = 1.504 \downarrow \ln(3x) \downarrow x_2 = 1.507$

$x_2 = 1.507 \downarrow \ln(3x) \downarrow x_3 = 1.509$

$x_3 = 1.509 \downarrow \ln(3x) \downarrow x_4 = 1.510$

$x_4 = 1.510 \downarrow \ln(3x) \downarrow x_5 = 1.511$

$x_5 = 1.511 \downarrow \ln(3x) \downarrow \boxed{x_6 = 1.511}$

accurate to 10^{-3}

Problem 4

Find n for which the nth iteration by the Fixed-point method guarantees to approximate the root of $f(x) = x - \cos x$ on $[0, \frac{\pi}{3}]$ with accuracy within 10^{-8} using $x_0 = \frac{\pi}{4}$.

$f(x) = x - \cos(x)$ $[0, \frac{\pi}{3}]$

$x = \cos(x)$

$g(x) = \cos(x)$

1	$g(\pi/4) = 0.707106781$
2	$g(0.707106781) = 0.76024597$
3	$g(0.76024597) = 0.724667481$
4	$g(0.724667481) = 0.748719886$

3	$g(0.76024597) = 0.724667481$
4	$g(0.724667481) = 0.748719886$
5	$g(0.748719886) = 0.732560845$
6	$g(0.732560845) = 0.743464211$
7	$g(0.74346421\mathbf{1}) = 0.736128257$
8	$g(0.73612825) = 0.741073687$

* Rest done in Excel *

41	$g(0.739085140) = 0.739085129$
42	$g(0.739085129) = 0.739085136$

$0.739085136 - 0.739085129 = 0.7 \times 10^{-8} < 10^{-8}$

$\boxed{n = 41}$