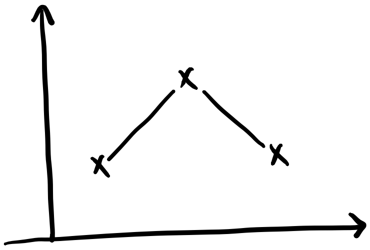
TUESDAY JULY 16, 2019

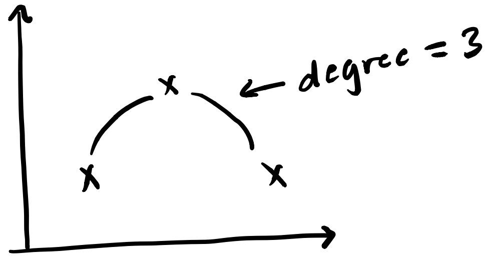
CH. 3 INTERPOLATION AND POLYNOMIAL APPROXIMATION

3.2 CUBIC SPLINES

The idea of the "splines" is to use several polynomials each a lower degree, to pass through the data points.



Linear spline



Cubic spline

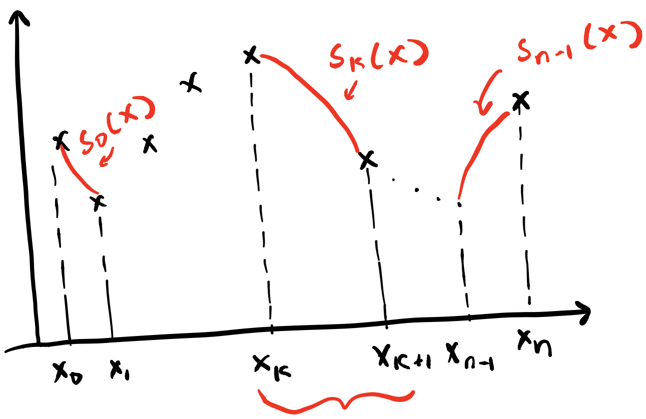
In cubic spline, 3rd degree polynomials are used to interpolate over each interval between data points.

- suppose there $(n+1)$ data points
  $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

NOTE:

- $(n+1)$ data points
- Hence, n-number of intervals

$\Rightarrow$ number of cubic splines



$$S_K(x) = a_K + b_K(x - x_K) + c_K(x - x_K)^2 + d_K(x - x_K)^3$$

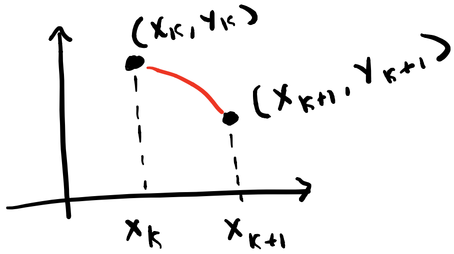
on the interval $[x_K, x_{K+1}]$

$k = 0, 1, 2, \ldots, n-1$

$$= \begin{cases} S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3 \Leftarrow [x_0, x_1] \\ S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3 \Leftarrow [x_1, x_2] \\ \vdots \\ S_{n-1}(x) = a_{n-1} + b_{n-1}(x - x_{n-1}) + c_{n-1}(x - x_{n-1})^2 + d_{n-1}(x - x_{n-1})^3 \Leftarrow [x_{n-1}, x] \end{cases}$$

Property 1 (Left-endpoint)

$S_k(x_k) = y_k \leftarrow n\text{-}eq^{ns}$

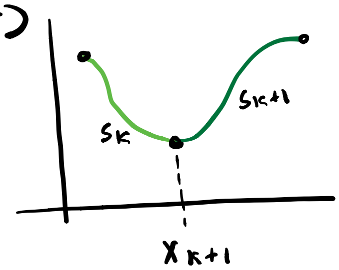


Property 2 (Right-endpoint)

$S_k(x_{k+1}) = y_{k+1} \leftarrow n\text{-}eq^{ns}$

Property 3 (continuity of slope at shared knot)

$S_k'(x_{k+1}) = S_{k+1}'(x_{k+1}) \leftarrow (n-1)\text{-}eq^{ns}$



Property 4

$S_k''(x_{k+1}) = S_{k+1}''(x_{k+1}) \leftarrow (n-1)\text{-}eq^{ns}$

Property 5 Natural spline (free): $S_0''(x_0) = 0$

$S_{n-1}''(x_n) = 0$

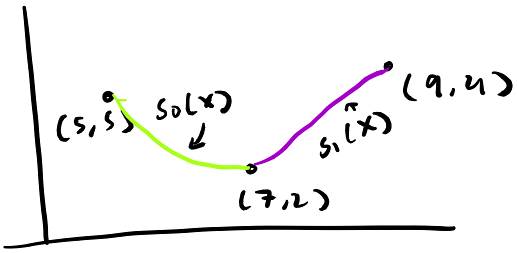
OR

Clamped Cubic spline: $S_0'(x_0) = \alpha_1$

$S_{n-1}'(x_n) = \alpha_2$

α_1 & α_2 are user-specified values

exam #2 problem ↴ (solve for the 8 equations) ex. Construct a piecewise cubic spline interpolant for the curve passing through (5,5), (7,2), (9,4) with natural boundary conditions



$$s_0(x) = a_0 + b_0(x-5) + c_0(x-5)^2 + d_0(x-5)^3$$

$$s_1(x) = a_1 + b_1(x-7) + c_1(x-7)^2 + d_1(x-7)^3$$

$$S_0(5) = a_0 = 5 \quad \text{— ①}$$

$$s_0(7) = a_0 + b_0(7-5) + c_0(7-5)^2 + d_0(7-5)^3 = 2$$

$$= a_0 + 2b_0 + 4c_0 + 8d_0 = 2 \quad \text{— ②}$$

$$s_1(7) = a_1 = 2 \quad \text{— ③}$$

$$s_1(9) = a_1 + b_1(9-7) + c_1(9-7)^2 + d_1(9-7)^3 = 4$$

$$= a_1 + 2b_1 + 4c_1 + 8d_1 = 4 \quad \text{— ④}$$

$\underline{s_0'(7) = s_1'(7)}$

$$s_0'(x) = b_0 + 2c_0(x-5) + 3d_0(x-5)^2$$

$$s_1'(x) = b_1 + 2c_1(x-7) + 3d_1(x-7)^2$$

$$\left.\begin{aligned} s_0'(7) &= b_0 + 4c_0 + 12d_0 \\ s_1'(7) &= b_1 \end{aligned}\right\} \quad b_0 + 4c_0 + 12d_0 = b_1 \quad \text{— ⑤}$$

$\underline{s_0''(7) = s_1''(7)}$

$$s_0''(x) = 2c_0 + 6d_0(x-5)$$

$$s_1''(x) = 2c_1 + 6d_1(x-5)$$

$$@\ x = 7 \Rightarrow 2c_0 + 12d_0 = 2c_1 \quad \text{— ⑥}$$

with natural boundary conditions:

$S_0''(5) = 0$ $\qquad$ $S_1''(9) = 0$

$2c_0 = 0$ — 7 $\qquad$ $2c_1 + 12d_1 = 0$ — 8

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if we solve the system we get:

$a_0 = 5 \quad b_0 = -17/8 \quad c_0 = 0 \quad d_0 = 5/32$

$a_1 = 2 \quad b_1 = -1/4 \quad c_1 = 15/16 \quad d_1 = -5/32$

Final answer:

The natural C.S.

$$S(x) = \begin{cases} S_0(x) = 5 - \frac{17}{8}(x-5) + \frac{5}{32}(x-3)^3 & \text{on } [5,7] \\ S_1(x) = 2 - \frac{1}{4}(x-7) + \frac{15}{16}(x-7)^2 - \frac{5}{32}(x-9)^3 & \text{on } [7,9] \end{cases}$$