Problem 1

(a) RK4 $h=1$

$$\frac{dy}{dt} = (1-y)\cos t \qquad 0 \le t \le 3 \qquad y(0)=3$$

$$f(t,y) = (1-y)\cos t.$$

$y_0=3$ $y_1?$

0 1 2 3

t_0 t_1 t_2 t_3

$$k_1 = f(t_i, y_i)$$

$$= f(t_0, y_0) = f(0,3) = (1-3)\cos(0) = -2$$

$$k_2 = f\left(t_0+\tfrac{h}{2}, y_0+\tfrac{h}{2}k_1\right) = f(0.5, 2) = (1-2)\cos(0.5)$$

$$= -0.8776.$$

$$k_3 = f\left(t_0+\tfrac{h}{2}, y_0+\tfrac{h}{2}k_2\right) = f(0.5, 2.5612)$$

$$= (1-2.5612)\cos(0.5)$$

$$= -1.3701.$$

$$k_4 = f(t_0+h, y_0+hk_3) = f(1, 1.6299)$$

$$= (1-1.6299)\cos(1)$$

$$= -0.3403$$

$$y_1 = y_0 + \frac{h}{6}\left[k_1+2k_2+2k_3+k_4\right] = \boxed{1.8607}$$

1. (b) $y = 1 + 2e^{-\sin t}$.

True value at $t=1$

$$y\big|_{t=1} = 1 + 2e^{-\sin(1)} = \boxed{1.8622}$$

Absolute error

$$|y(1) - y_1| = |1.8622 - 1.8607|$$

$$= \boxed{0.0015}$$

$A\vec{x} = \vec{b}$

Problem 2.

2.1 $A_{5\times7}$ $B_{5\times7}$ $C_{2\times5}$.

(a) AB Not defined.

(b) A^T defined.

(c) $A+B$ defined.

(d) $B^T C^T$ $(B^T)_{7\times5}\,(C^T)_{5\times2}$ $\Rightarrow$ defined

(e) CB $C_{2\times5}\,B_{5\times7}$ $\Rightarrow$ defined

(f) $B+C$ not defined.

2.2. $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

$$BA^T = \underbrace{\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}}_{2\times2}\underbrace{\begin{bmatrix} 1 & 1 & 2 \\ -2 & 1 & 1 \end{bmatrix}}_{2\times3}$$

$$BA^T = \begin{bmatrix} 5 & 2 & 5 \\ -9 & -3 & -8 \end{bmatrix}$$

Problem 3 Gaussian Elimination

$$2x_1 - 6x_2 + 2x_3 = 4$$
$$x_1 + 2x_3 = 5$$
$$3x_1 - 5x_2 + 3x_3 = 6.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -6 & 2 & 4 \\ 1 & 0 & 2 & 5 \\ 3 & -5 & 3 & 6 \end{array}\right]$$

$$\xrightarrow[E_1 \leftarrow E_1/2]{1}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 1 & 0 & 2 & 5 \\ 3 & -5 & 3 & 6 \end{array}\right]$$

$$\xrightarrow[E_2 \leftarrow E_2 - E_1]{2}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 0 & 3 & 1 & 3 \\ 3 & -5 & 3 & 6 \end{array}\right]$$

$$\xrightarrow[E_3 \leftarrow E_3 - 3E_1]{3}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 0 & 3 & 1 & 3 \\ 0 & 4 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[E_3 \leftarrow \frac{E_3}{4}]{4}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 0 & 3 & 1 & 3 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[E_2 \leftrightarrow E_3]{5}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 3 \end{array}\right]$$

$$\xrightarrow[E_3 \leftarrow E_3 - 3E_2]{6}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$x_1 - 3x_2 + x_3 = 2 \Rightarrow x_1 + 3 = 2$$

$$\boxed{x_2 = 0} \qquad \boxed{x_1 = -1}$$

$$\boxed{x_3 = 3}$$

Problem 4. $\vec{x}^{(0)} = [\ 0, 0, 0\ ]^T$

$$3x_1 + x_2 + x_3 = 6 \quad - (1)$$

$$x_1 + 3x_2 + x_3 = 3 \quad - (2)$$

$$x_1 + x_2 + 3x_3 = 5 \quad - (3)$$

$$x_1 = 2 - \frac{1}{3}x_2 - \frac{1}{3}x_3$$

$$x_2 = 1 - \frac{1}{3}x_1 - \frac{1}{3}x_3$$

$$x_3 = \frac{5}{3} - \frac{1}{3}x_1 - \frac{1}{3}x_2$$

Jacobi Iteration

$$x_1^{(k+1)} = 2 - \frac{1}{3}x_2^{(k)} - \frac{1}{3}x_3^{(k)}$$

$$x_2^{(k+1)} = 1 - \frac{1}{3}x_1^{(k)} - \frac{1}{3}x_3^{(k)}$$

$$x_3^{(k+1)} = \frac{5}{3} - \frac{1}{3}x_1^{(k)} - \frac{1}{3}x_2^{(k)}$$

Iteration #	0	1	2
x_1	0	2	1.1111
x_2	0	1	-0.2222
x_3	0	$5/3$	0.6667

Gauss-Seidel Iteration

$$x_1^{(k+1)} = 2 - \frac{1}{3}x_2^{(k)} - \frac{1}{3}x_3^{(k)}$$

$$x_2^{(k+1)} = 1 - \frac{1}{3}x_1^{(k+1)} - \frac{1}{3}x_3^{(k)}$$

$$x_3^{(k+1)} = \frac{5}{3} - \frac{1}{3}x_1^{(k+1)} - \frac{1}{3}x_2^{(k+1)}$$

Iteration #	0	1	2
x_1	0	2	1.5926
x_2	0	0.3333	0.1728
x_3	0	0.8889	1.0782

Problem 5

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -3 & -5 & 1 \\ 0 & -2 & -3 \end{bmatrix} \qquad \begin{array}{l} |2| > |1| + |0| \\ |-5| > |-3| + |1| \\ |-3| > |0| + |-2| \end{array}$$

$\Rightarrow$ Matrix A is strictly diagonally dominant.

$\therefore$ Both Jacobi and Gauss-Seidel methods will converge.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -3 & -4 & 1 \\ 0 & -2 & 3 \end{bmatrix} \quad |-4| \ngtr |-3| + |1|$$

$\Rightarrow$ Matrix A is not ~~a~~ strictly diagonally dominant.

$\therefore$ Both Jacobi and Gauss-Seidel methods may or ~~May~~ may NOT converge.

Problems 6

Completed yesterday!

See your notes (04/24/2019)