WEDNESDAY JULY 10, 2019

CH.2 SOLUTIONS OF EQ^{ns} OF ONE VARIABLE

2.2 FIXED-POINT ITERATION (FPI)

Definition: The number p is a fixed-point for a given fun^{n} $g(x)$ if $g(p) = p$

ex 1) If $g(x) = x^2$, then $x = 1$ & $x = 0$ are the fixed-points of $g(x)$.

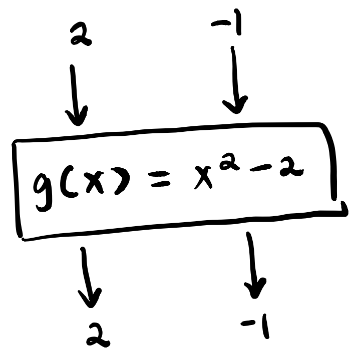
ex 2) Determine any fixed-points of the fun^{n} $g(x) = x^2 - 2$

$x^2 - 2 = x$ ⇐ set fun^{n} = x

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$\boxed{\begin{matrix} x = 2 \\ x = -1 \end{matrix}}$ CHECK:



This FPI technique involves solving the problem $f(x) = 0$ by rearranging $f(x)$ into the form $x = g(x)$.

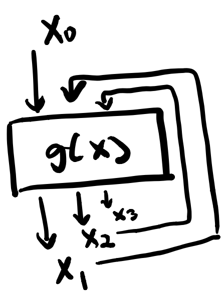
- If the fun^{n} $g(x)$ has a fixed-point at p, then the fun^{n} defined by $f(x) = x - g(x)$ has a root (or zero) at p

Algorithm

① start with an initial guess x_0

② For $n = 1, 2, 3 \ldots$ set $x_n = g(x_{n-1})$

* this an iterative method:



ex) using the FPI, determine the root of the funⁿ $f(x) = x - \left(\frac{x}{2} + \frac{1}{x}\right)$ using $x_0 = 1$.

$$f(x) = x - \underbrace{\left(\frac{x}{2} + 2\right)}_{g(x)}$$

$$x = \frac{x}{2} + 2$$

$$x_n = \frac{x_{n-1}}{2} + \frac{1}{x_{n-1}}$$

$\underline{n=1}$

$$x_1 = \frac{x_0}{2} + \frac{1}{x_0} = 1.5$$

$\underline{n=2}$

$$x_2 = \frac{x_1}{2} + \frac{1}{x_1} = \frac{1.5}{2} + \frac{1}{1.5} = 1.4167$$

$\underline{n=3}$

$$x_3 = \frac{x_2}{2} + \frac{1}{x_2} = \frac{1.4167}{2} + \frac{1}{1.4167} = 1.4142$$

$\underline{n=4}$

$$x_4 = 1.4142$$

$\underline{n=5}$

$$x_5 = 1.4142 \Rightarrow \boxed{\text{root of the fun}^{\underline{n}}: 1.4142}$$

ex) using the FPI, determine the root of the funⁿ $f(x) = x - \frac{\sin x}{2} - \frac{\cos x}{2}$ using $x_0 = 0$.

$$g(x) = \left(\frac{\sin x}{2} + \frac{\cos x}{2}\right)$$

$$x = \frac{\sin x + \cos x}{2}$$

$$x_n = \frac{\sin(x_{n-1}) + \cos(x_{n-1})}{2}$$

ITERATIONS:

$\underline{n=1}$

$$x_1 = \frac{\sin x_0 + \cos x_0}{2} = +\frac{1}{2}$$

$\underline{n=2}$

$$x_2 = \frac{\sin x_1 + \cos x_1}{2} = 0.6785$$

$\underline{n=3}$

$$x_3 = \frac{\sin x_2 + \cos x_2}{2} = 0.7031$$

$\vdots$

$x_4 = 0.7047$

$x_5 = 0.7048$

$x_6 = 0.7048$ ∴ the root is $\boxed{x = 0.7048}$

NOTE 1: Q. Can every eq$^{\underline{n}}$ $f(x)=0$ be turned into a fixed-point problem $x=g(x)$?

A. Yes ~ in many different ways

ex) $x^3 + x - 1 = 0$

$$x = \underbrace{1 - x^3}_{g_1(x)} \qquad x^3 = 1 - x, \quad x = \underbrace{(1-x)^{1/3}}_{g_2(x)} \qquad x = \underbrace{\frac{1+2x^3}{1+3x^2}}_{g_3(x)}$$

NOTE 2: sometimes, the fun$^{\underline{n}}$ $f(x)=0$ does not lend itself deriving an iteration formula of the form $x = g(x)$

ex) $f(x) = \sin x - \cos x = 0$ ⇐ use: $(+x - x)$

$$\sin x - \cos x + x - x = 0$$

$$\underbrace{\sin x - \cos x + x}_{g(x)} = x$$

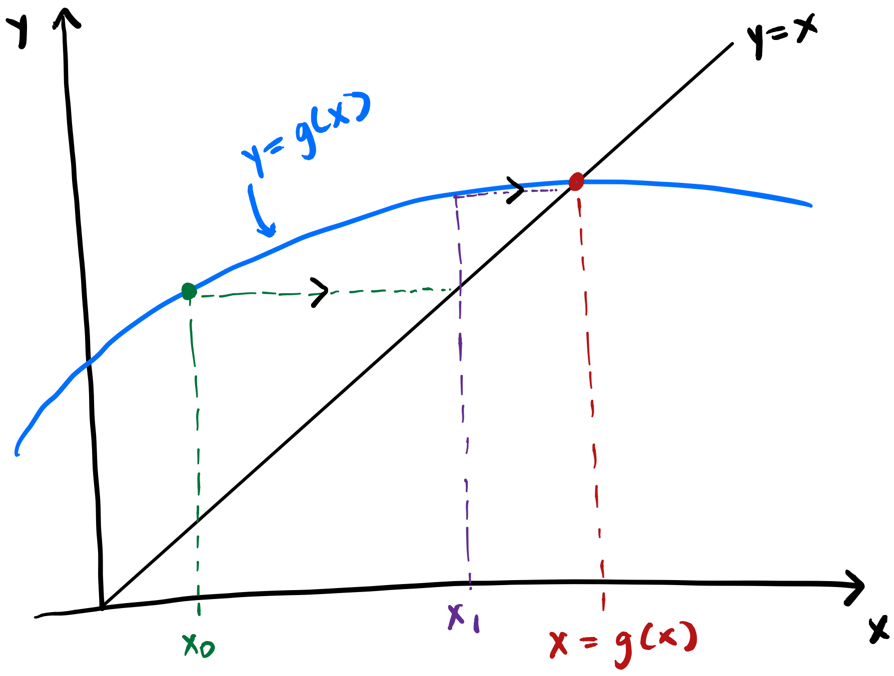
general form: ⇒ $f(x) = 0$

$$x + f(x) - x = 0$$

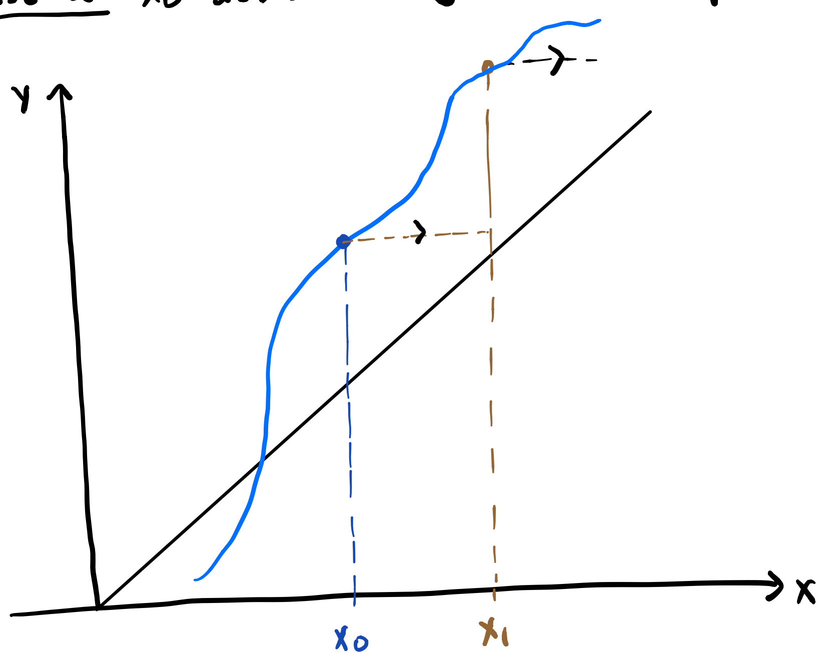
$$\underbrace{x + f(x)}_{g(x)} = x$$

GEOMETRIC REPRESENTATION: FPI

Case 1: what we expect to see using FPI



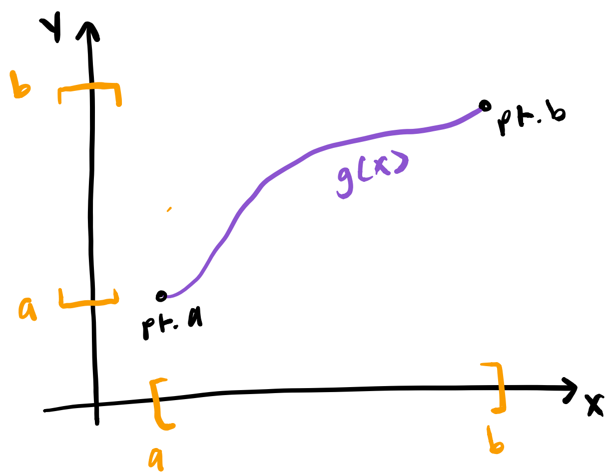
Case 2: x_0 deviates away from fixed point



• FIXED-POINT THEOREM

① (Existence) Let $g(x)$ be a continuous function on $[a,b]$, and $a \le g(x) \le b$ for all $x \in [a,b]$. Then, $g(x)$ has at least one fixed-point in $[a,b]$

② (uniqueness) Moreover, if $|g'(x)| < 1$ for all $x \in (a,b)$ then $g(x)$ has a unique fixed-point in $[a,b]$



ex 1) Consider the funⁿ $f(x) = x - \sqrt{2x+3}$ where $x \in [1,4]$ "in". Show that the fixed-point iteration has at least one fixed-point.

$$x - \sqrt{2x+3} = 0$$

$$x = \underbrace{\sqrt{2x+3}}_{g(x)}$$

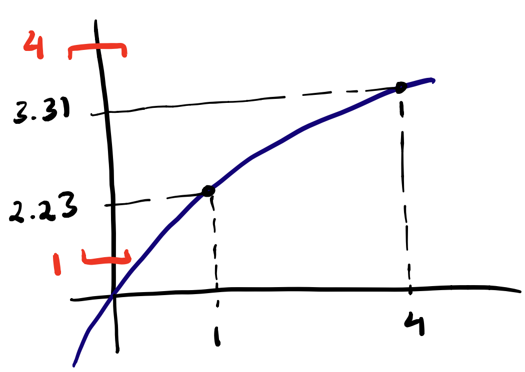
CONDITION 1: EXISTENCE

$g(x) = \sqrt{2x+3}$

$g(1) = \sqrt{2+3} = 2.2361$

$g(4) = \sqrt{8+3} = 3.3166$

visual:



* This supports the (existence) condition ⇒ we can conclude that we have at least one fixed point ✓

CONDITION 2: UNIQUENESS

$g'(x) = \frac{1}{2\sqrt{2x+3}} \cdot 2$

$|g'(x)| = \frac{1}{\sqrt{2x+3}}$, $x \in (1,4)$

$|g'(x)|_{max} = \frac{1}{\sqrt{2+3}} = 0.447$

$|g'(x)| < 1$ for all $x \in (1,4)$

∴ we have a unique fixed-point ✓

* QUIZ CONTENT ⤵ *

ex 2) Consider the fun$^{\underline{n}}$ $f(x) = e^x - x - 2$ on the interval $[0,2]$. Find a fun$^{\underline{n}}$ $g(x)$ that has a unique FP on the interval $[0,2]$.

$$e^x - x - 2 = 0$$

$$x = \underbrace{e^x - 2}$$

$$g_1(x) = e^x - 2$$

EXISTENCE: $a \leq g(x) \leq b$

$g(x) = e^x - 2$

$g(0) = e^0 - 2 = -1 \Leftarrow$ outside the interval

$g(2) = 5.389 \Leftarrow$ outside the interval

$\therefore$ we cannot say that we have at least one fixed-point using this fun$^{\underline{n}}$ $g_1(x)$

. . . . find another fun$^{\underline{n}}$ $g(x)$...

$e^x - x - 2 = 0$

$e^x = x + 2$

$\ln e^x = \ln(x+2)$

$x = \underbrace{\ln(x+2)}_{g_2(x)}$

$g_2(0) = \ln(2) = 0.6931$

$g_2(2) = \ln(4) = 1.3863$

* $g_2(0)$ & $g_2(2)$ are between the interval $[0,2]$ $\therefore$ we have at least one fixed-point ✓

UNIQUENESS: $|g_2'(x)| < 1$ for all $x \in (a,b)$

$$g_2'(x) = \frac{1}{x+2}$$

$|g_2'(x)|_{max}$ occurs @ $x=0$ since our function is decreasing.

$|g_2'(x)|_{max} = \frac{1}{2} < 1$ $\therefore$ we have a unique fixed point ✓

NOTE: Let e_n be the absolute error for the n^{th}-iteration. Then,

$e_n = |x_n - P| \leq k^n \cdot \max\{x_0 - a, b - x_0\}$ where $|g'(x)| \leq k \leq 1$

$k = |g'(x)|_{max}$