Group HW #06

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- **ALL DID ALL**
- **Resources used:** We referenced the book and solutions to the classwork.
- **Group goal this week:** To collaborate in a way that is efficient.

Team Contract

All members (Abby, Adi, Antoinette, and Dasha) will meet on Saturday 9am each week to work on group problems. To prepare for these meetings, everyone is expected to attempt each homework problem before the meeting. In addition, one person each week will write up the report to be turned in. The Latex document will be sent to everyone in the group the Sunday morning before the homework is due so that each group member can look over the report before it is submitted. This task of writing up the report will be assigned to people on a rotation basis for each group homework.

Problem 1

Problem: Use mathematical induction to prove that a set with n elements has $n(n-1)(n-2)/6$ subsets containing exactly three elements whenver n is an integer greater than or equal to 3.

Let $P(n)$ be the proposition that any set with n elements has $n(n-1)(n-2)/6$ subsets containing exactly 3 elements. We want to show that $P(n)$ is true for all $n \geq 3$.

Base case: For n = 3, the set has exactly one subset containing exactly three elements, and 3(3-1)(3-2)/6 =1. Therefore, the base case, $P(3)$, is true.

Assume that $P(k)$ is true for some $k \geq 3$, which means that any set with k elements has $k(k-1)(k-2)/6$ subsets of cardinality 3. This is legit because we just showed that this is true at least for $k = 3$.

Consider $P(k+1)$. We want to show that any set with $k+1$ elements has $(k+1)((k+1)-1)((k+1)-2)/6$ subsets containing exactly 3 elements. So, let A be an arbitrary set with $k+1$ elements. We will now show that A has $(k+1)((k+1)-1)((k+1)-2)/6$ subsets containing exactly 3 elements.

Fix an element b in A, and let T be the set of elements of A other than b. There are two varieties of subsets of A containing exactly three elements. First there are those that do not contain

b. These are precisely the three-element subsets of T, and by the inductive hypothesis, there are $k(k-1)(k-2)/6$ of them. Second, there are those that contain b together with two elements of T. Since T has n elements, there are exactly $k(k-1)/2$ of this type (by 25). Therefore the total number of subsets of A containing exactly three elements is $(k(k-1)(k-2)/6) + (k(k-1)/2)$, which simplifies algebraically to $(k+1)((k+1)-1)((k+1)-2)/6$, as desired. See Equation 1 for the algebraic breakdown.

$$\begin{aligned}
(k(k-1)(k-2)/6) + (k(k-1)/2) &= \{(k^2-k)(k-2)/6 + k(k-1)/2\} && \text{(Distribution)}\\
&= \{(k^2-k)(k-2)/6 + (k^2-k)/2\} && \text{(Distribution)}\\
&= \{(k^3-2k^2-k^2+2k)/6 + 3k^2-3k)/6\} && \text{(Distribution)}\\
&= \{(k^3-k)/6\} && \text{(Subtraction)}\\
&= \{k(k^2-1)/6\} && \text{(Distribution)}\\
&= \{k(k+1)(k-1)/6\} && \text{(Distribution)}\\
&= (k+1)((k+1)-1)((k+1)-2)/6 && \text{(Associative Property)}
\end{aligned} \tag{1}$$

We have shown that $P(3)$ is true and we have shown that if $P(k)$ is true for some $k \geq 3$, then $P(k+1)$ is true. Thus, by the principle of mathematical induction, $P(n)$ is true for all $n \geq 3$ and we can conclude that any set with n elements has $n(n-1)(n-2)/6$ subsets containing exactly 3 elements

Problem 2

Problem: Show that $f_0 - f_1 + f_2 - ... - f_{2n-1} + f_{2n} = f_{2n-1} - 1$ whenever n is a positive integer.

Let $P(n)$ be the proposition that $f_0 - f_1 + f_2 - ... - f_{2n-1} + f_{2n} = f_{2n-1} - 1$ whenever n is a positive integer. Let F be the set of integers for which the above identity holds. We want to show that the identity is true for all positive integers n.

Base case: We will show that $1 \in F$ by separately evaluating and comparing the LHS and RHS of the proposed identity when n =1.

LHS: $f_0 - f_1 = 0 - 1 + 1 = 0$

RHS: $f_{2-1} - 1 = f_1 - 1 = 1 - 1 = 0$

0=0 so the proposed identity holds when $n = 1$, and therefore $1 \in F$. This shows that P(1) is true, which means that $f_0 - f_1 = f_{2-1} - 1$.

Assume that $P(k)$ is true for some positive integer k, which means that $f_0 - f_1 + f_2 - ... - f_{2k-1} + f_{2k} = f_{2k-1} - 1$ This is legit because we just showed that this is true at least for $k = 1$.

Consider $P(k+1)$. We want to show that $f_0 - f_1 + f_2 - ... - f_{2(k+1)-1} + f_{2(k+1)} = f_{2(k+1)-1} - 1$.

$$
\begin{aligned}
LHS &= \{f_0 - f_1 + f_2 - ... + f_{(2(k+1)-2} - f_{2(k+1)-1} + f_{2(k+1)}\} && \text{(Expanding)}\\
&= \{f_0 - f_1 + f_2 - ... + f_{2k} - f_{2k+1} + f_{2k+2}\} && \text{(Distribution and Subtraction)}\\
&= \{f_{2k-1} - 1 - f_{2k+1} + f_{2k+2}\} && \text{(Supposed truth of statement for the K case)}\\
&= \{f_{2k-1} - 1 - f_{2k+1} + f_{2k+1} + f_{2k}\} && \text{(Recursive Definition of Fibonacci Numbers)}\\
&= \{f_{2k-1} - 1 + f_{2k}\} && \text{(Subtraction)}\\
&= \{f_{2k+1} - 1\} && \text{(Recursive Definition of Fibonacci Numbers)}\\
&= \{f_{2(k+1)-1} - 1\} && \text{(Associative Property)}\\
&= \text{RHS of the proposed identity for } n = k+1
\end{aligned}
\tag{2}
$$

We have shown that $P(1)$ is true and we have shown that if $P(k)$ is true for some positive integer k, then $P(k+1)$ is true. Thus, by the principle of mathematical induction, $P(n)$ is true for all positive integer n and we can conclude that $f_0 - f_1 + f_2 - ... - f_{2n-1} + f_{2n} = f_{2n-1} - 1$ whenever n is a positive integer.

Problem 3

Problem: Prove using mathematical induction that $1 \cdot 2 + 2 \cdot 3 + ... + n(n+1) = n(n+1)(n+2)/3$ whenever n is a positive integer.

Let $P(n)$ be the proposition that $1 \cdot 2 + 2 \cdot 3 + ... + n(n+1) = n(n+1)(n+2)/3$ whenever n is a positive integer. Let F be the set of integers for which the above identity holds. We want to show that the identity is true for all positive integer n.

Base case: We will show that $1 \in F$ by separately evaluating and comparing the LHS and RHS of the proposed identity when n =1.

LHS: $1 \cdot 2 = 2$

RHS: $1(2)(3)/3 = 2$

2=2 so the proposed identity holds when $n = 1$, and therefore $1 \in F$. This shows that P(1) is true, which means that $1 \cdot 2 = 1(2)(3)/3$.

Assume that $P(k)$ is true for some positive integer k, which means that $1 \cdot 2 + 2 \cdot 3 + ... + k(k+1) = k(k+1)(k+2)/3$. This is legit because we just showed that this is true at least for $k = 1$.

Consider $P(k+1)$. We want to show that $1 \cdot 2 + 2 \cdot 3 + ... + (k+1)((k+1)+1) = (k+1)((k+1)+1)((k+1)+2)/3$.

$$
\begin{aligned}
LHS &= \{1 \cdot 2 + 2 \cdot 3 + ... + (k+1)((k+1)+1)\}\\
&= \{1 \cdot 2 + 2 \cdot 3 + ... + k(k+1) + (k+1)((k+1)+1)\} && \text{(Expanding Equation)}\\
&= \{k(k+1)(k+2)/3 + (k+1)((k+1)+1)\} && \text{(Supposed truth of statement for the K case)}\\
&= \{k(k+1)(k+2)/3 + (k+1)(k+2)\} && \text{(Addition)}\\
&= \{(k(k+1)(k+2) + 3(k+1)(k+2))/3\} && \text{(Creating Common Denominator)}\\
&= \{(k+1)(k+2)(k+3)/3\} && \text{(Distributive Property)}\\
&= \{(k+1)((k+1)+1)((k+1)+2)/3\} && \text{(Associative Property)}\\
&= \text{RHS of the proposed identity for } n = k+1
\end{aligned}
\tag{3}
$$

We have shown that $P(1)$ is true and we have shown that if $P(k)$ is true for some positive integer k, then $P(k+1)$ is true. Thus, by the principle of mathematical induction, $P(n)$ is true for all positive integer n and we can conclude that $f_0 - f_1 + f_2 - ... - f_{2n-1} + f_{2n} = f_{2n-1} - 1$ whenever n is a positive integer.

Problem 4

Problem: Suppose you have a simple planar graph viewed on top of the integer grid of the xy-plane such that the graph has a vertex at each integer point in the xy-plane and has edges connecting vertices that fall on opposite corners of the underlying integer grid squares. In other words, the vertex at the integer point (x,y) is adjacent via an edge to the vertices (x+1, y+1), (x-1, y-1), (x-1, y+1), (x+1, y-1). Suppose this graph describes the possible movements of a metal-detecting robot around a room (the robot moves about the grid by only making diagonal moves). Suppose you place the metal-detecting robot at the origin of your room, corresponding to the grid location (0,0). Will the robot ever walk over the hidden treasure stored underneath the vertex corresponding to grid location (132, 89)? Prove your answer by induction.

Let $P(n)$ be the proposition that if the robot starts at (0,0), for any arbitrary path the robot can take, the nth vertex that the robot visits will have x and y components that are congruent in mod 2.

Base Case: The 0th vertex that the robot visits is by definition the starting vertex, or (0,0). This vertex clearly has both components being congruent mod 2. Which means that the identity holds for P(0).

Assume that $P(k)$ is true; that is, letting the kth vertex that the robot visits be represented by (x_k, y_k), then $x_k \equiv y_k \mod 2$.

Consider $P(k+1)$. We want to show that the $k+1$th vertex on any path has components that are congruent mod 2. Again, letting the kth vertex on the path be represented as (x_k, y_k), from the problem statement, we know that the $k+1$th vertex must be represented by (x_k+1, y_k+1), (x_k-1, y_k-1), (x_k-1, y_k+1), or (x_k+1, y_k-1). Since we are assuming that $P(k)$ is true and thus $x_k \equiv y_k \mod 2$, then it's clear to see in the first two cases that $x_k+1 \equiv y_k+1 \mod 2$ and $x_k-1 \equiv y_k-1 \mod 2$ are true, and thus they have components that are equivalent in mod 2. Furthermore, by modular arithmetic $x_k - 1 \equiv y_k + 1 \mod 2 \Rightarrow x_k \equiv y_k + 2 \mod 2 \Rightarrow x_k \equiv y_k \mod 2$, which we know is true through our inductive assumption and thus by the chain of implication, $x_k - 1 \equiv y_k + 1$ mod 2 is true. Similarly, $x_k+1 \equiv y_k-1 \mod 2 \Rightarrow x_k+2 \equiv y_k \mod 2 \Rightarrow x_k \equiv y_k \mod 2$, which we know is true through our inductive assumption and thus by the chain of implication, $x_k+1 \equiv y_k-1$ mod 2. Since all four cases hold true, it must be the case the the $k+1$th vertex that the robot visits has x and y components that are congruent in mod 2, or in other words that $P(k+1)$ hold true.

We have shown that $P(0)$ is true and we have shown that if $P(k)$ is true for some positive integer k, then $P(k+1)$ is true. Thus, by the principle of mathematical induction, $P(n)$ is true for all positive integer n and we can conclude that the nth vertex that the robot traverses on will have components that are congruent in mod 2. Therefore the robot will not walk over the hidden treasure stored underneath the vertex corresponding to the grid location (132,89) as $132 \not\equiv 89$ mod 2.

Team Report

Similar to the previous homework, our group goal this week was to collaborate in a way that is efficient. Everyone came to the meeting on time and prepared by attempting all problems in advance. During the meeting, we when around in a circle sharing our answers and methods for solving the problems. If there were different methods for solving a problem, we shared each method, so we can think of the problems in different ways. After the meeting, one person typed up the Latex report, and then the others reviewed it and made changes. This strategy allowed us to efficiently use our group meeting time, while learning from our other group members, so we feel we have met the team goal for this homework.