Group HW #03

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- **ALL DID ALL**

- **Resources used:** We referenced the textbook and the in-class notes, and we also went to office hours.

- **Group goal this week:** To collaborate in a way that is efficient.

Team Contract

All members (Abby, Adi, Antoinette, and Dasha) will meet on Saturday 9am each week to work on group problems. To prepare for these meetings, everyone is expected to attempt each homework problem before the meeting. In addition, one person each week will write up the report to be turned in. The Latex document will be sent to everyone in the group the Sunday morning before the homework is due so that each group member can look over the report before it is submitted. This task of writing up the report will be assigned to people on a rotation basis for each group homework.

Problem 1

Problem: How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29$$

where $x_i, i = 1, 2, 3, 4, 5, 6$, is a non negative integer such that

a) $x_i > 1$ for $i = 1, 2, 3, 4, 5, 6$?
b) $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3, x_4 \geq 4, x_5 > 5$, and $x_6 \geq 6$?
c) $x_1 \leq 5$?
d) $x_1 < 8$ and $x_2 > 8$?

Solution:

a)We can treat the problem as if we were partitioning 29 identical items into 6 distinct groups (it may be the case that a group has 0 items or has the same number of items as another group however). Thus, we can choose x_1 items to be in group one, x_2 items to be in group two, and so on. To that end, we can model this problem using sticks and stones, where the stones represent the 29 items, and there are 5 sticks which are the dividers that separate the x_1 items in group one, x_2 items in group two, x_3 items in group three, x_4 items in group four, x_5 items in group five, and x_6 in group six. Since $x_i > 1$, as stated in the problem, for all 6 groups ($i = 1, 2, 3, 4, 5, 6$), there must be at least 2 stones in each group, leaving $29 - 12 = 17$ items (stones) to sort. Without loss of generality, as all stones are identical, we just need to find the number of ways we can distribute the remaining 17 stones into the six groups. Thus, we want to line up the sticks and stones (17 stones and 5 sticks) and choose 5 of these spots to be the sticks (other spots get the stones which are assumed to be identical). Another way to think about this method is to first line up the 22 symbols (there are 22! ways to do this) then since order does not matter for either the sticks or the stones, we must divide by 5! and 17! to unorder the respective symbols:

$$\binom{22}{5} = \frac{22!}{5!17!} = 26334 \text{ solutions}$$

b) For this problem, we use the same sticks and stones process as for part a, taking into consideration the added constraints that have been posed. We are still partitioning 29 items into six groups, however, since $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3, x_4 \geq 4, x_5 > 5, x_6 \geq 6$, there at least 1 item in group one, 2 items in group two, 3 items of type in group, 4 items in group four, 6 items in group five, and 6 items in group six. Without loss of generality, we can assign one stone to group one, two stones to group two, three stones to group three, four stones to group four, six stones to group five and six stones to group six. This leaves us 7 stones left to partition Using the same sticks and stones logic, we have 7 stones and 5 sticks to line up (12 items total), and of those 12 spots we want to choose 5 of those to be sticks:

$$\binom{12}{5} = \frac{12!}{5!7!} = 792 \text{ solutions}$$

c) For this problem, our condition is that $x_1 \leq 5$. We can find the number of solutions by finding all possible combinations without any restrictions and subtracting all the combinations where $x_1 > 5$.

All combinations: We have 29 stones and 5 sticks (34 items total). We line up the 34 items and choose 5 spots to be sticks, the rest stones:

$$\binom{34}{5} = \frac{34!}{5!29!} = 278256$$

Undesirable Combinations (Combinations where $x_1 > 5$): Under this condition, x_1 must have at least 6 stones, so we have 23 stones to distribute and 5 sticks (28 items total). We line up the 28 items and choose 5 spots to be sticks, the rest stones:

$$\binom{28}{5} = \frac{28!}{5!23!} = 98280$$

$$\text{All Combinations - Undesirable Combinations} = 298256 - 98280 = 179976 \text{ solutions}$$

d) We can solve this problem by finding all the combinations where $x_2 > 8$ and subtracting all combinations where $x_2 > 8$ and $x_1 \geq 8$.

All combinations where $x_2 > 8$: Using the same approach as before, we know that 9 stones are already accounted for, leaving 20 stones and 5 sticks (25 items total). We line up the 25 items and choose 5 spots for the sticks, and the rest to be stones:

$$\binom{25}{5} = \frac{25!}{5!20!} = 53130$$

Undesirable Combinations (where $x_2 > 8$ and $x_1 \geq 8$): Here, x_2 is at least 9 and x_1 is at least 8, leaving 12 stones and 5 sticks (17 items total). We line up the 17 items and choose 5 spots for the sticks:

$$\binom{17}{5} = \frac{17!}{5!12!} = 6188$$

All combinations where $x_2 > 8$ - Undesirable Combinations $= 53130 - 6188 = 46942$ solutions

Problem 2

Problem: How many elements are in the union of five sets if the sets contain 10,000 elements each, each pair of sets has 1000 common elements, each triple of sets has 100 common elements, every four of the sets have 10 common elements, and there is 1 element in all five sets?

Solution: Let $A_i, i = 1, 2, 3, 4, 5$ represent 5 sets of 10,000 elements each. To count the number of elements that are in the union of these five sets (A_1, A_2, A_3, A_4, A_5), we can sum the cardinality of each set, and subtract the number of elements that appear in multiple sets, using the Principle of Inclusion-Exclusion, to account for the elements we over count. This can be represented in the following equation

$$\begin{aligned} \mid A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \mid = \sum_{i=1}^{5} \mid A_i \mid - \sum_{1 \leq i < j \leq 5} \mid A_i \cap A_j \mid + \sum_{1 \leq i < j < k \leq 5} \mid A_i \cap A_j \cap A_k \mid \\ - \sum_{1 \leq i < j < k < z \leq 5} \mid A_i \cap A_j \cap A_k \cap A_z \mid + \mid A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \mid \end{aligned} \tag{1}$$

where $\sum_{i=1}^{5} \mid A_i \mid$ sums the cardinality of the sets or counts each element that is in exactly one of the five sets once, elements that are in exactly two of the sets twice, elements that are in exactly three of the sets three times, elements that are exactly in four sets four times, and elements that are in all of the sets five times. To remove the over counted elements, we need to subtract the items in the intersections of each of these sets: $\sum_{1 \leq i < j \leq 5} \mid A_i \cap A_j \mid$. However, all the items that are in three or more sets will be counted zero times with this expression. To fix that, we add the intersections of three sets: $\sum_{1 \leq i < j < k \leq 5} \mid A_i \cap A_j \cap A_k \mid$. However, we end up with over counted elements that appear in four or more sets. To remedy that, we subtract the elements that occur in four of the sets: $\sum_{1 \leq i < j < k < z \leq 5} \mid A_i \cap A_j \cap A_k \cap A_z \mid$. Lastly, we must add the elements that occur in all 5 since the previous expression removes them:$\mid A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \mid$. Since there are 5 unique sets, $\binom{5}{2}$ intersections between two distinct sets, $\binom{5}{3}$ intersections between three distinct sets, $\binom{5}{4}$ intersections between four distinct sets, and 1 intersection between 5 distinct sets, by substituting the given values from the problem, we find that:

$$\mid A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \mid = 5(10000) - \binom{5}{2}(1000) + \binom{5}{3}(100) - \binom{5}{4}(10) + 1 = 40,951 \text{ elements}$$

Problem 3

Problem: Give a Jeopardy-style combinatorial proof of the given equation. Show that if n is a positive integer, then

$$n! = C(n,0)D_n + C(n,1)D_{n-1} + ... + C(n,n-1)D_1 + C(n,n)D_0,$$

where D_k is the number of derangements of k objects.

Solution: To solve this problem, we assume that there are n number of people, each of whom begin with a unique hat. We will show that either side of the equation describes the number of ways in which there are to redistribute the hats.
RHS: Let us look at each of the terms in the sum. First, $C(n,0)D_n$ describes the situation where we pick 0 people who get their own hat back (represented as $C(n,0)$), and everyone else does not get their own hat back (represented as D_n if we consider the original position of a hat to be the person's head who wore the hat in the beginning). $C(n,1)D_{n-1}$ describes the situation where we pick 1 person to keep their original hat, while everyone else does not receive their original hat back. Each time, we pick 1 more person who gets to keep their hat and the remaining people do not get their original hat back, until the last situation $C(n,n)D_0$ where everyone keeps their own hats. Since there are only two states for every individual in the redistribution of hats, either they get their own hat back or they don't, and we have considered every combination of people who get their hat back in this sum, we know that this sum must be equal to the total number of ways to redistribute the hats
LHS: On the left side of the equation ($n!$), the first person gets n choices for a hat, the second person gets $n-1$ choices for a hat, and so on.
Since both sides of the equation count the number of ways to redistribute the hats, we know that both sides of the equation must then be equivalent.

Problem 4

Problem: Suppose that you are making miniature Zen stone gardens. You made five distinctly shaped wooden containers to house these rock gardens. (a) How many ways are there to distribute 20 identical, polished black stones into these five distinct gardens? (It is ok if some gardens get zero of these stones.) (b) How many ways are there to distribute 20 polished stones if they are now all different colors (and it matters which garden gets which colors of stones)? (It is ok if some gardens get zero stones.)

Solution:
a) 20 Identical Stones Since all 20 stones are identical, which garden gets which stone does not matter. To solve this problem, we can use the sticks and stones model where we have 20 stones represented by the dots, and 4 sticks representing the dividers between the 5 gardens:

$$|||| \cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

We have 24 items total, or 24 spots where items can go. To find the number of ways we can arrange the stones, we need to choose which of the 24 spots will be sticks and assign the remaining spots to the stones:

$$\binom{24}{4} = \frac{24!}{4!20!} = 10,626 \text{ ways}$$

b) 20 Unique Stones Since each of the stones are distinct, each time you pick up a stone, there are 5 places it could go. For stone 1, there are 5 choices for the garden, for stone 2 there are also 5 choices, for stone 3 there are also 5 choices, and so on. Thus, we can arrange 20 unique stones in 5^{20} ways.

Problem 5

Problem: Suppose there are five children to whom you would like to give all your fruit. You have nine apples and eight mangoes to distribute, and you would like to ensure that each child gets at least one mango. How many ways can you distribute this fruit? (It matters what each child gets -e.g., Joey getting two mangoes is different from Zoey getting two mangoes - but consider all apples to be identical and all mangoes to be identical.)

Solution: We can also apply the sticks and stones method to this problem. Since there are 2 distinct groups of objects, mangoes and apples, that need to be distributed amongst the children, we can find the number of ways to distribute the mangoes and the number of ways to distribute the apples and multiply them. We use the product rule because the number of ways to distribute mangoes and the number of ways to distribute the apples are independent of each other.

Part 1: Mangoes First, we let the sticks represent the dividers between each of the 5 children, and the stones represent the mangoes. Since each child gets at least one mango, 5 mangoes are already accounted for:

Child 1 | Child 2 | Child 3 | Child 4 | Child 5

$$\cdot \mid \cdot \mid \cdot \mid \cdot \mid \cdot$$

There are $8 - 5 = 3$ mangoes left to distribute. We line up the remaining mangoes (stones) and the sticks and choose where the sticks will divide up the mangoes:

$$\binom{7}{4} = \frac{7!}{3!4!} = 35 \text{ ways}$$

Part 2: Apples Like in part 1, we let the sticks represent the dividers between each of the 5 children, but the stones are now apples. There is no minimum or maximum number of apples a child can get so we line up the 9 stones and 4 sticks to get 13 spots in total, and choose 4 of these spots to be the sticks.

$$\binom{13}{4} = \frac{13!}{4!9!} = 715 \text{ ways}$$

Total number of ways to distribute 8 mangoes and 9 apples:

$$35 \times 715 = 25025 \text{ ways}$$

Problem 6

Problem: There is a pastry display with 15 unique pastries in a line. You want to pick 6 pastries to take home, but you don't want to choose any adjacent pastries. How many choices do you have for this collection of 6 pastries (none of which were originally adjacent on the shelf)? (The order in which you choose the pastries does not matter; it is only which pastries you choose that matters, and you are not lining up your chosen pastries.)

Solution: This problem can also be modeled using the sticks and stones model. We let the pastries represent the sticks and the spaces between the pastries are represented by the stones. We have a total of $15 - 6 = 9$ spaces. Since we cannot choose pastries that are adjacent to each other, there is at least 1 space in between each of the sticks:

$$| \cdot | \cdot | \cdot | \cdot | \cdot |$$

Since 5 spaces are already accounted for, we have 4 remaining. We have 10 items in total (4 stones and 6 sticks), and we need to choose where the remaining spaces will go:

$$\binom{10}{4} = \frac{10!}{4!6!} = 210 \text{ choices}$$

Therefore, we have 210 ways of choosing 6 pastries such that no pastries are adjacent to one another.

Problem 7

Problem: How many ways can you distribute identical \$1 bills to bribe your four professors under the following conditions. Linder must get at least one, Somerville must get at least two, Wood must get at least four, and Adams must get at least ten. No professor will get twenty or more, since that might seem overboard. You have sixty bills to distribute, but you don't need to distribute all of your bills.

Solution: We can model this problem with sticks and stones, where the stones represent the \$1 bills, and the sticks represent the dividers between the professors. Since we have the option of not giving some \$1 bills to any professor, we have an additional divider to account for that condition. First, we distribute the number of \$1 bills that are outlined in the conditions:

Linder | Somerville | Wood | Adams | No One

$$\cdot \ | \cdot\cdot \ | \cdot\cdot\cdot\cdot | \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot |$$

There are 47 \$1 bills remaining to distribute, or 43 stones. To find the number of ways to distribute the \$1 bills given that no professor gets more than \$20, we can find the number of ways to distribute the remaining money without any restrictions, subtract the cases where each professor gets \$20 or more, and add back the cases where a pair of professors receive \$20 or more. We don't need to worry about 3 professors having \$20 because there aren't enough \$1 bills for that to happen.

$\binom{47}{4}$ counts the number of ways to distribute 47 \$1 bills among the 4 professors or none of the professors without the \$20 restriction.
$\binom{37}{4}$ counts the number of ways to distribute the remaining bills given that Adams already has at least 20.

$\binom{31}{4}$ counts the number of ways to distribute the remaining bills given that Wood already has at least 20.
$\binom{29}{4}$ counts the number of ways to distribute the remaining bills given that Somerville already has at least 20.
$\binom{28}{4}$ counts the number of ways to distribute the remaining bills given that Linder already has at least 20.
$\binom{21}{4}$ counts the number of ways to distribute the remaining bills given that Adams and Wood both have at least 20.
$\binom{19}{4}$ counts the number of ways to distribute the remaining bills given that Adams and Somerville both have at least 20.
$\binom{18}{4}$ counts the number of ways to distribute the remaining bills given that Adams and Linder both have at least 20.
$\binom{13}{4}$ counts the number of ways to distribute the remaining bills given that Wood and Somerville both have at least 20.
$\binom{12}{4}$ counts the number of ways to distribute the remaining bills given that Wood and Linder both have at least 20.
$\binom{10}{4}$ counts the number of ways to distribute the remaining bills given that Somerville and Linder both have at least 20.

Total number of ways

$$= \binom{47}{4} - \binom{37}{4} - \binom{31}{4} - \binom{29}{4} - \binom{28}{4} + \binom{21}{4} + \binom{19}{4} + \binom{18}{4} + \binom{13}{4} + \binom{12}{4} + \binom{10}{4}$$

Thus, there are 50970 ways to distribute identical \$1 bills to the four professors given the conditions above.

Team Report

Our group goal this week was to work efficiently. Like previous group homeworks, we attempted each problem on our own and met Saturday morning to discuss them. We took turns explaining the problems to each other, going through the methods we used to get to an answer. If any group members used different methods (or had different answers) we talked through all methods until we agreed on the solution. After the meeting, one person typed up the LaTeX report and then others reviewed it and made changes. This strategy allowed us to efficiently complete the group homework and to learn from each other. We feel that we have met our team goal for this week.