Group HW #02

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- **ALL DID ALL**
- **Resources used:** We referenced the book and solutions to the classwork.
- **Group goal this week:** To collaborate in a way that is efficient.

Team Contract

All members (Abby, Adi, Antoinette, and Dasha) will meet on Saturday 9am each week to work on group problems. To prepare for these meetings, everyone is expected to attempt each homework problem before the meeting. In addition, one person each week will write up the report to be turned in. The Latex document will be sent to everyone in the group the Sunday morning before the homework is due so that each group member can look over the report before it is submitted. This task of writing up the report will be assigned to people on a rotation basis for each group homework.

Problem 1

Problem: Let A, B, and C be sets. Show that $(B - A) \cup (C - A) = (B \cup C) - A$.

To answer this question, we proved that the left hand side equals the right hand side using set builder notation:

$$\begin{aligned}
(B - A) \cup (C - A) &= \{x \mid x \in (B - A) \cup (C - A)\} && \text{(Re-Writing)} \\
&= \{x \mid x \in (B - A) \wedge x \in (C - A)\} && \text{(Definition of a Union)} \\
&= \{x \mid x \in B \wedge x \notin A \wedge x \in C \wedge x \notin A\} && \text{(Definition of Difference)} \\
&= \{x \mid x \in B \wedge x \in C \wedge x \notin A\} && \text{(Remove Redundancy)} \\
&= \{x \mid x \in (B \cup C) \wedge x \notin A\} && \text{(Definition of a Union)} \\
&= \{x \mid x \in (B \cup C) - A\} && \text{(Definition of Difference)} \\
&= (B \cup C) - A && \text{(Re-Writing)}
\end{aligned} \tag{1}$$

Problem 2

Problem: If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Justify your answer.

Let g be a function from A to B, f be a function from B to C, and $f \circ g$ be a function from A to C. Let a_n be an element of set A, b_n is an element of set B, and c_n is an element of set C.

If g is not one-to-one, then there exists at least two elements of A that map to a single element of B: $g(a_i) = g(a_j) = b_i$. Since f is stated to be a one-to-one function, each element in set B can only map to one element of set C: $f(b_i) = c_i$. Then, as shown in Figure 1, when g is not a one-to-one function, $f \circ g$ cannot be a one-to-one function because there exists at least two elements of A that map to a single element of C: $f \circ g(a_i) = f \circ g(a_j) = c_i$. Therefore, for both f and $f \circ g$ to be one-to-one functions, g must also be a one-to-one function.

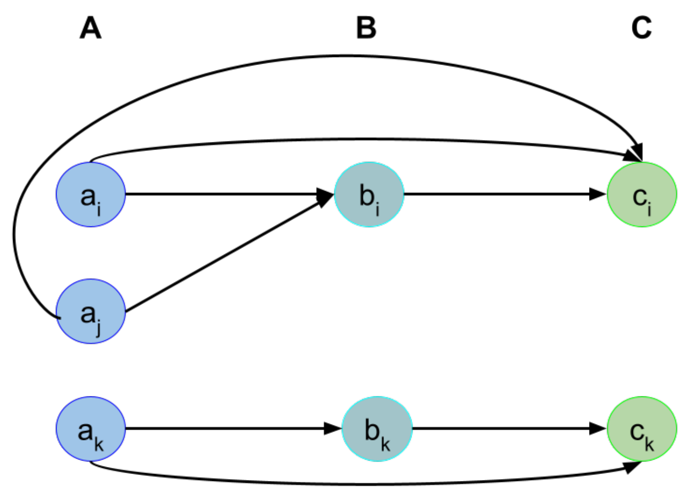

Figure 1: The relationship between g, f, and $f \circ g$ when f is a one-to-one function and g is not.

Problem 3

Problem: How many ways are there for a horse race with four horses to finish if ties are possible? (Note: Two, three, or four horses may tie.)

To solve this question, we broke it down into five cases: No horses tie, two tie, three tie, four tie, and two groups of two tie. For the case where there are no horses that tie, there are four options for first place, three options for second (because one horse was already chosen for first place), two options for third (because two horses were chosen for first and second place respectively), and one option for fourth. In other words, there are 4! ways for four horses to finish if there are no ties.

When two horses tie, we first start with the case of no ties between horses. By the previous logic, this means that there are 4! options for the horses to finish. Next, we divided by 2! to remove the options where two horses do not tie, and then divide again by 2! to un-order the groups because we do not care in which order the horses tie. This gives us all the options of choosing two horses to tie, then we must multiply it by 3! to arrange for the three groups (horse one, horse two, and the two that tie). Therefore, the number of ways two horses can tie is $3!\binom{4}{2} = 3!\frac{4!}{2!(2)!}$. For three horses and four horses tying, we can follow the same logic, leaving us with $2!\binom{4}{3}$ and $1!\binom{4}{4}$ ways for the horses to finish respectively. For the case of two groups of two horses tying, we can choose two horses and let the other two also tie, leaving us with $\binom{4}{2}$ ways for this case to occur.

Finally, we can use the sum rule to add these cases together to get the total number of ways four horses can finish a race. This sums to 75.

$$4! + 3!\binom{4}{2} + 2!\binom{4}{3} + 1!\binom{4}{4} + \binom{4}{2} = 75 \tag{2}$$

Therefore, there are 75 ways for a horse race with four horses to finish if ties are possible.

Problem 4

Problem: Suppose that f is a function from a finite set A to finite set B where $|A| = a$ and $|B| = b$, and a>b. Use the Pigeon-hole Principle to prove that f cannot be 1-1. (This should be a pretty short proof!)

Suppose that f is one-to-one, that means that each element in A's domain must take on different elements in B. Let $f(a_n) = b_n$, where a_n is an element of set A and b_n is an element of set B. By the definition of cardinality, there are a elements in set A and b elements in set B. Because $a > b$, there will be more elements for A then there exists in B. By the Pigeon-hole Principle, this means that there must be some element of set B that is taken on more than once, such that $f(a_n) = f(a_m) = b_n$, where a_n and a_m are different elements of set A. This is a contradiction, which means that f cannot be one-to one.

Problem 5

Problem: Provide a "jeopardy style" combinatorial proof of the following equation by asking one question and then answering that one question in two different ways (once for each side of the proposed equation):

$$\binom{n}{t}\binom{t}{c} = \binom{n}{c}\binom{n-c}{t-c} \tag{3}$$

Be sure to **fully** justify both of your answers to your question.

Question: Given a class of n students, how many different project teams of t students can be formed in which c students are chosen as team captains?

Answer One (Left Hand Side): First, we choose t students for the team from the group of n total students. To find the number of ways this can be done, we start with $n!$ total orders of students, then divide by $(n-t)!$, so that there are only there are only t students counted, and then divide by $t!$ to un-order the t students because we only care that they are on the team. This gives us $\frac{n!}{t!(n-t)!} = \binom{n}{t}$, so there are $\binom{n}{t}$ ways this can be done. Then, from the group of t students, we choose c students to be captains which, following the previous logic, can be done $\binom{t}{c}$. Therefore, by the product rule, the total number of different teams is $\binom{n}{t}\binom{t}{c}$.

Answer Two (Right Hand Side): First, we choose c students to be captains of the group from the n students. This can be done $\binom{n}{c}$ ways. Then, from the $n - c$ remaining students, choose $t - c$ students to be assigned to the team with the leaders, so that there is a total of t people on the team. This can be done $\binom{n-c}{t-c}$ ways.

Team Report

Similar to the previous homework, our group goal this week was to collaborate in a way that is efficient. Everyone came to the meeting on time and prepared by attempting all problems in advance. During the meeting, we when around in a circle sharing our answers and methods for solving the problems. If there were different methods for solving a problem, we shared each method, so we can think of the problems in different ways. After the meeting, one person typed up the Latex report, and then the others reviewed it and made changes. This strategy allowed us to efficiently use our group meeting time, while learning from our other group members, so we feel we have met the team goal for this homework.